COURSE GUIDE

ECO 255
MATHEMATICS FOR ECONOMISTS I

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Printed 2017

ISBN: 978-058-550-0

Introduction

Welcome to ECO: 255 MATHEMATICS FOR ECONOMISTS 1.

ECO 255: Mathematics for Economists 1 is a two-credit unit, first semester undergraduate course for Economics students. The course is made up of four modules (subdivided into twelve study units), spread across fifteen lecture weeks. This course guide tells you about the course material and how you can work your way through it. It also suggests some general guidelines for the amount of time required of you on each unit in order to achieve the course aims and objectives successfully.

Course Content

This course basically emphasizes on the mathematical application of economics theory. The topics covered include the number system; inequalities; exponents and roots; systems of equation; simultaneous equation; quadratic equation; set theory; logarithms; calculus; optimization and linear programming. You will be taken through the meaning of these mathematical concepts and their application to economics problems.

Course Aims

The general aim of this course is to build strong mathematical background needed to aid students develop analytical skills required to solve economics problems. Some of the other aims are to,

- Acquaint students with basic mathematics concepts and operations
- Express economics terms in a more precise manner
- Use Mathematical models to preset simplified versions of how economies work
- Show the difference between the product rule and chain rule of partial derivative as well as showing the difference between differentiation and integration.
- Help train the students' mind to be analytical
- Empirically measure the costs, benefits and effects of competing options when faced with the issue of scarcity and choice.

Course Objectives

To achieve the above aims, there are some overall objectives which the course aims at achieving. Although there are set out objectives for each unit, included at the beginning of the unit- you should read them before you start working through the unit. You may want to refer to them during your study of the unit to check on your progress. You should always look at the unit objectives after completing a unit. This is to assist the students in accomplishing the tasks entailed in this course. This way, you can be sure you have done what is required of you by the unit.

At the end of the course period, the students are expected to be able to:

- Analyze the basic terms of number system and its properties; explain binary, imaginary and complex numbers, and also use the number system to solve problems.
- Explain the concept of inequalities, differentiate between the properties of inequality, solve inequality problems and apply inequality to angle problems.
- Apply the concept of exponent and roots; explain exponents and roots; solve exponent problems using the base ten system and also simplify and approximate roots.
- Discuss the systems of equation, solve problems of linear equation using substitution and the addition/subtraction method, and graph systems of equations solution.
- Explain the concept of Simultaneous and quadratic equation, differentiate between a simultaneous equation and a quadratic equation and solve simultaneous and quadratic equation problem using the elimination and substitution method.
- Differentiate between intersect and union, get acquainted with set properties and symbols, and solve set theory problems using Venn diagram.
- Explain Logarithm, know the difference between an exponent and logarithm, differentiate between the common and the natural logarithm function and understand the properties of logarithm.
- Discuss the concept of differentiation and integration, state the difference between differentiation and integration, solve problems involving higher order derivatives, list the rules of integration, solve definite and indefinite integrals, use both the chain and the product rules to solve differentiation problems, and apply definite integrals to economic problems.
- Apply the concept of Optimization and Linear Programming

Working through the Course

To successfully complete this course, you are required to read the study units, referenced books and other materials on the course.

Each unit contains self-assessment exercises called Student Assessment Exercises (SAE). At some points in the course, you will be required to submit assignments for assessment purposes (please note that at the end of the course there is a final examination). This course should take about 15weeks to complete and some components of the course are outlined under the course material subsection.

Course Material

The major component of the course, what you have to do and how you should allocate your time to each unit in order to complete the course successfully on time are listed follows:

1. Course guide
2. Study units
3. Textbooks
4. CDs
5. Assignment file
6. Presentation schedule

Study Units

There are 12 units in this course which should be studied carefully and diligently.

MODULE 1: NUMBER SYSTEM, INEQUALITITES, EXPONENT AND ROOTS

Unit 1 Number System
Unit 2 Inequalities
Unit 3 Exponents and Roots

MODULE 2: EQUATIONS

Unit 1 Systems of Equation
Unit 2 Simultaneous Equation
Unit 3 Quadratic Equation

MODULE 3: SET THEORY, LOGARITHMS & PARTIAL DERIVATIVES

Unit 1 Set Theory
Unit 2 Logarithms
Unit 3 Partial Derivative

MODULE 4: INTEGRAL CALCULUS, OPTIMIZATION AND LINEAR PROGRAMMING

Each study unit will take at least two hours, and it include the introduction, objective, main content, self-assessment exercise, conclusion, summary and reference. Other areas border on the Tutor-Marked Assessment (TMA) questions. Some of the self-assessment exercise will necessitate discussion, brainstorming and argument with some of your colleges. You are advised to do so in order to understand and get acquainted with the application of mathematics to economic problem.

There are also textbooks under the reference and other (on-line and off-line) resources for further reading. They are meant to give you additional information if only you can lay your hands on any of them. You are required to study the materials; practice the self-assessment exercise and tutor-marked assignment (TMA) questions for greater and in-depth understanding of the course. By doing so, the stated learning objectives of the course would have been achieved.

Textbooks and References

For further reading and more detailed information about the course, the following materials are recommended:

Boates .B and Tamblyn .I (2012). Understanding Math- Introduction to Logarithms (Kindle Edition), Solid Stae Press, Barkeley: CA.

Breuer .J and Howard F.F (2006). Introduction to the Theory of Sets, (Dover Books on Mathematics), Dover Publiations, In.,: New York.

Carter .M (2001). Foundation of Mathematical Economics, The MIT Press, Cambridge, Massachusetts

Chiang .A.C (1967). Fundamental Methods of Mathematical Economics, Third Edition, McGraw-Hill Inc

Chiang A.C and Wainwright .K (2005). Fundamental Methods of Mathematical Economics.4th edition-McGraw-hill

Dorfman .P, Samuelson .P.A and Solow .R.M.(1987). Linear Programming and Economic Analysis.Dover Publications, Inc, :NewYork.

Ekanem .O.T (2004). Essential Mathematics for Economics and Business, Mareh: Benin City

Enderton .H.B (1997). The Elements of Set Theory. Academic Press: San Diego, California.

Franklin .J.N.(2002). Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed-Point Theorems (Classics in Applied Mathematics, 37).Society for Industrial and Applied Mathematics (January 15,2002)

Hands, D. W (2004). Introductory Mathematical Economics, Second Edition, Oxford University Press.

Kamien .M.I and Schwartz .N.L.(1993). Dynamic Optimization, Second Edition: The Calculus of Variations and Optimal Control in Economics and Management. Elsevier Science; 2nd Edition (October 25, 1991), North Holland.

Dowling, E.T.(2001). Schaum's Outline Series. Theory and Problems of Introduction to Mathematical Economics.McGraw-Hill:New York. Third Edition.

Shen, A. and Vereshchagin N.K. (2002). Basic Set Theory. American Mathematical Society (July 9 2002).

Assignment File

Assignment files and marking scheme will be made available to you. This file presents you with details of the work you must submit to your tutor for marking. The marks you obtain from these assignments shall form part of your final mark for this course. Additional information on assignments will be found in the assignment file and later in this Course Guide in the section on assessment.

There are four assignments in this course. The four course assignments will cover:
Assignment 1 - All TMAs' question in Units 1 – 3 (Module 1)
Assignment 2 - All TMAs' question in Units 4 – 6 (Module 2)
Assignment 3 - All TMAs' question in Units 7 – 9 (Module 3)
Assignment 4 - All TMAs' question in Units 10 – 12 (Module 4)

Presentation Schedule

The presentation schedule included in your course materials gives you the important dates of the year for the completion of tutor-marking assignments and attending tutorials. Remember, you are required to submit all your assignments by due dates. You should guide against falling behind in your work.

Assessment

There are two types of the assessment for this course. First are the tutor-marked assignments; second, the written examination.

In attempting the assignments, you are expected to apply information, knowledge and techniques gathered during the course. The assignments must be submitted to your tutor for formal Assessment in accordance with the deadlines stated in the Presentation Schedule and the Assignments File. The work you submit to your tutor for assessment will count for 30 % of your total course mark.

At the end of the course, you will need to sit for a final written examination of two hours' duration. This examination will also count for 70% of your total course mark.

Tutor-Marked Assignments (TMAs)

There are four tutor-marked assignments in this course. You will submit all the assignments. You are encouraged to work all the questions thoroughly. The TMAs constitute 30% of the total score.

Assignment questions for the units in this course are contained in the Assignment File. You will be able to complete your assignments from the information and materials contained in your textbooks, reading and study units. However, it is desirable that you demonstrate that you have read and researched more widely than the required minimum. You should use other references to have a broad viewpoint of the subject and also to give you a deeper understanding of the subject.

When you have completed each assignment, send it, together with a TMA form, to your tutor. Make sure that each assignment reaches your tutor on or before the deadline given in the Presentation File. If for any reason, you cannot complete your work on time, contact your tutor before the assignment is due to discuss the possibility of an extension. Extensions will not be granted after the due date unless there are exceptional circumstances.

Final Examination and Grading

The final examination will be of two hours' duration and have a value of 70% of the total course grade. The examination will consist of questions which reflect the types of self-assessment practice exercises and tutor-marked problems you have previously encountered. All areas of the course will be assessed

Revise the entire course material using the time between finishing the last unit in the module and that of sitting for the final examination. You might find it useful to review your self-assessment exercises, tutor-marked assignments and comments on them before the examination. The final examination covers information from all parts of the course.

Course Marking Scheme

The Table presented below indicates the total marks (100%) allocation.

Assignment	**Marks**
Assignments (Best three assignments out of four that is marked)	30%
Final Examination	70%
Total	**100%**

Course Overview

The Table presented below indicates the units, number of weeks and assignments to be taken by you to successfully complete the course, Mathematics for Economist (ECO 255).

Units	**Title of Work**	**Week's Activities**	**Assessment (end of unit)**
	Course Guide		
MODULE 1	**NUMBER SYSTEM, INEQUALITIES, EXPONENT AND ROOTS**		
1	Number System	Week 1	Assignment 1
2	Inequalities	Week 2	Assignment 1
3	Exponent and Roots	Week 3	Assignment 1
MODULE 2	**EQUATION**		
1	Systems of Equation	Week 4	Assignment 2
2	Simultaneous Equation	Week 5	Assignment 2
3	Quadratic Equation	Week 6	Assignment 2
MODULE 3	**SET THEORY, LOGARITHMS & PARTIAL DERIVATIVES**		
1	Set Theory	Week 7	Assignment 3
3	Logarithms	Week 8 &9	Assignment 3
5	Partial Derivatives	Week 10	Assignment 3
MODULE 4	**INTEGRAL CALCULUS, OPTIMIZATION AND LINEAR PROGRAMMING**		
1	Integral Calculus	Week 11	Assignment 4
2	Optimization	Week 12&13	Assignment 4
3	Linear Programming	Week 14&15	Assignment 4
	Total	**15 Weeks**	

How to Get the Most from This Course

In distance learning the study units replace the University Lecturer. This is one of the greatest advantages of distance learning, you can read and work through specially designed study materials at your own pace and at a time and place that suit you best.
Think of it as reading the lecture instead of listening to a Lecturer. In the same way that a Lecturer might set you some reading to do, the study units tell you when to read your books or other material, and when to embark on discussion with your colleagues. Just as a Lecturer might give you an in-class exercise, your study units provides exercises for you to do at appropriate points.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit.
You should use these objectives to guide your study. When you have finished the unit, you must go back and check whether you have achieved the objectives. If you make a habit of doing this you will significantly improve your chances of passing the course and getting the best grade.

The main body of the unit guides you through the required reading from other sources. This will usually be either from your textbooks or reading sections. Some units may require for you to have a discussion and practical problem solving sections. You will be directed when you need to embark on these and you will also be guided through what you must do.
The purpose of the discussion and practical problem solving sections of some certain mathematical economic problems are in twofold. First, it will enhance your understanding of the material in the unit. Second, it will give you analytical skills to evaluate economics and mathematical problems. In any event, most of the practical problem solving skills you will develop during studying are applicable in normal working situations, so it is important that you encounter them during your studies.

Self-assessments are interspersed throughout the units. Working through these tests will help you to achieve the objectives of the unit and prepare you for the assignments and the examination. You should do each self-assessment exercises as you come to it in the study units.

The following is a practical strategy for working through the course. If you run into any trouble, consult your tutor. Remember that your tutor's job is to help you. When you need help, do not hesitate to call and ask your tutor to provide it.

1. Read this Course Guide thoroughly.
2. Organize a study schedule. Refer to the `Course overview' for more details. Note the time you are expected to spend on each unit and how the assignments relate to

the units. Important information, e.g. details of your tutorials, and the date of the first day of the semester is available from study centre. You need to gather together all this information in one place, such as your dairy or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates for working breach unit.

3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the `Overview' at the beginning of each unit. You will also need both the study unit you are working on and one of your textbooks on your desk at the same time.
6. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit you will be instructed to read sections from your textbooks or other articles. Use the unit to guide your reading.
7. Up-to-date course information will be continuously delivered to you at the study centre.
8. Work before the relevant due date (about 4 weeks before due dates), get the Assignment File for the next required assignment. Keep in mind that you will learn a lot by doing the assignments carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the exam. Submit all assignments no later than the due date.
9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study material or consult your tutor.
10. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.
11. When you have submitted an assignment to your tutor for marking do not wait for its return before starting on the next units. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also written on the assignment. Consult your tutor as soon as possible if you have any questions or problems.
12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this Course Guide).

Tutors and Tutorials

There are some hours of tutorials (2-hours sessions) provided in support of this course. You will be notified of the dates, time and locations of these tutorials. Together with the name and phone number of your tutor, as soon as you are allocated a tutorial group.

Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter, and provide assistance to you during the course. You must mail your tutor-marked assignments to your tutor well before the due date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail, or discussion board if you need help. The following might be circumstances in which you would find help necessary. Contact your tutor if.

- You do not understand any part of the study units or the assigned readings
- You have difficulty with the self-assessment exercises
- You have a question or problem with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should try your best to attend the tutorials. This is the only chance to have face to face contact with your tutor and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating in discussions actively.

Summary

The course, Mathematics for Economist 1 (ECO 255), will expose you to basic concepts in Mathematics and Economics. This course will give you an insight into the use of mathematics in solving economics problems.

On successful completion of the course, you would have developed critical and practical thinking skills necessary for efficient and effective discussion and problem solving in mathematical economic issues. However, to gain a lot from the course please try to apply everything you learnt in the course in order to improve efficiency.

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MODULE 1 NUMBER SYSTEM, INEQUALITITES, EXPONENTS AND ROOTS

UNIT 1 NUMBER SYSTEM

1.0 INTRODUCTION

A numeral system (also known as system of numeration) is a method of representing or writing numbers using digits or symbols in a consistent manner. It can be seen as the context that allows the symbols "10" to be interpreted as the decimal symbol for ten, the binary symbol for two, or a symbol for other numbers in other bases.

A number system will normally,

- Reflect the algebraic and arithmetic structure of the numbers
- Give every number represented a unique/standard representation
- Represent a useful set of numbers (e.g. all integers, or rational numbers).

The most commonly used system of numerals is known as Arabic numerals or Hindu–Arabic numerals. Two Indian mathematicians are credited with developing them. Aryabhata of Kusumapura developed the place-value notation in the 5th century and a century later Brahmagupta introduced the symbol for zero.

The simplest numeral system is the unary numeral system, in which every natural number is represented by a corresponding number of symbols. If the symbol / is chosen, for example, then the number eight would be represented by ////////. Tally marks represent one system still in common use however, it can only be used for small numbers.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Analyze the basic terms of number system and its properties.
- Explain concisely binary, imaginary and complex numbers.
- Use number system to solve problems.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO NUMBER SYSTEM

Whenever we use numbers in our day to day activities, we apply certain conventions. For example we do know that the number 1234 is a combination of symbols which is one thousand, two hundreds, three tens and four units. We also accept that the symbols are chosen from a set of ten symbols: 0,1,2,3,4,5,6,7,8 and 9.This is the decimal number system, it is part of the language of dealing with quantity. When we say a number has a base or radix, we are saying the same thing. These terms both mean how symbols or digits are used to depict or express a number. A base 10 number system has ten digits, and it is called the decimal number system. Sometimes the base of a number is shown as a subscript: 1234_{10}. Here, the 10 is a subscript which indicates that the number is a base 10 number.

Table 1 shows some concepts associated with numbers and the example in Table 1 is a base 10 example. It shows us a few things about the number 6234.125.

Table1. Some key concepts of Number System

Column or Index	3	2	1	0	-1	-2	-3
Digit or Coefficient	6	2	3	4	1	2	5
Base and Index	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
Column weight	1000	100	10	1	0.1	0.01	0.001
Column Value	6000	200	30	4	0.1	0.02	0.005

We can deduce the following from Table 1 above:

- As we move left through the column weight, each symbol increases in value.
- Starting from the left of the decimal point and moving left, the numbers increase positively. At the right of the decimal point the index numbers increase negatively.
- As you can see in the third row, the number base (10) is raised to the power of the column or index. 10^3 means 10 raised to the power 3, or 10 * 10 * 10. As 10 is the base and 3 the index or column.

  We could write the numbers as:

7000 + 500 + 90 + 4 + 0.1 + 0.02 + 0.005

Or as:

$7 * 10^3 + 5 * 10^2 + 9 * 10^1 + 4 * 10^0 + 1 * 10^{-1} + 2 * 10^{-2} + 95 * 10^{-3}$.

SELF ASSESSMENT EXERCISE

Rewrite this base ten number 224.25, showing the base and index representation of each digit.

3.2 TYPES OF NUMBER SYSTEM AND THEIR GENERAL PROPERTIES

The basic types of number system are given below:

Natural Numbers: These are sometimes called the Counting Numbers, as they are used for counting. These numbers are 1, 2, 3, 4,…. (and so on). They do not include zero- we cannot count zero . They also do not have fractions or negative numbers.

Whole Numbers: These are similar to natural numbers just that they include Zero. They are 0,1,2,3,…(and so on). Like the Natural numbers, these numbers do not have fractions or negative numbers.

Integers: These are positive and negative whole numbers. These numbers tell us that in real life, some numbers can fall below zero, like temperature. Some examples of these numbers are,…… --4,-3,-2,-1,0,1,2,3,4…. They do not include fractions.

Rational Numbers: Any number that can be written in fraction / ratio form. They include all integers e.g. 4, 3 (which can also be written as 4/1 and 3/1) and all fractions e.g. (4/5 and 2/6).

Irrational Numbers: These numbers cannot be expressed as fractions. They cannot be represented as terminating or repeating decimals e.g., √3 = 1.7320508…, π = 3.141592654… These numbers are non-ending and non-repeating decimals.

Imaginary Numbers: These are special numbers because when squared, they give a negative result. i.e., $i^2 = -1$. Thus $i = \sqrt{-1}$. Imaginary numbers are important because they are useful in solving problems involving the square roots of negative numbers. Example of such number is 2*i*.

Real Numbers: These are whole, rational, irrational numbers. All numbers except imaginary numbers and infinity numbers are called real numbers. Example, 1, ½, 0, π etc**.**

Complex Number: This is gotten as a result of combination of two sets of numbers (real number and an imaginary number), where each set can be zero.

The General Properties of Number System Are:

1. **Closure:** If you operated on any two real numbers A and B with +, -, ×, or /, you get a real number.
2. **Commutative:** X + Y = Y + X and X × Y = Y × X
3. **Associative:** (X + Y) + Z = X + (Y + Z) and (X × Y) × Z = X × (Y × Z)
4. **Inverse:** X + -X = 0, X × (1/X) = 1
5. **Identity:** X + 0 = X, X × 1 = X
6. **Distributive:** X × (Y + Z) = (X × Y) + (X × Z)

These properties are useful to review and keep in mind because they will help you to simplify more complex calculations.

SELF ASSESSMENT EXERCISE

State the properties of number system.

3.3 IMAGINARY NUMBERS

An imaginary number is a number that can be written as a real number multiplied by the imaginary unit *i*, which is defined by its property $i^2 = -1$ as explained in section 3.2. The square of an imaginary number *yi* is -*z*. For example, 5*i* is an imaginary number, and its square is −25. Except for 0 (which is both real and imaginary), imaginary numbers produce negative real numbers when squared.

An imaginary number *yi* can be added to a real number *x* to form a complex number of the form $x + yi$, where x and yi are called, respectively, the real part and the imaginary part of the complex number. Imaginary numbers can therefore be said to be a complex numbers whose real part is zero. The term "imaginary number" was actually a derogatory word because the number was formally thought to be impossible. However, after much research it became evident that such numbers exist and it is the square root of a negative number, e.g. √-5, √-9, √-17, e.t.c. The imaginary unit denoted by *i,* is the square root of -1, i.e., $i = \sqrt{-1}$. Thus, $i^2 = -1$. It follows that $i^3 = i^2.\ i = -i$**;** $i^4 = i^3.\ i = (-i)(\ i) = (-i^2) = 1$**;** $i^5 = i^4.\ i = 1.\ i = i$ e.t.c.

This shows that when raising *i* to of any value, the answers will always be either of the following values *i*, -1, -*i*, 1 and power will follow a specific pattern as you increase the power by one. This can be seen in the table below.

Table 2. The powers of *i*

i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
i	-1	$-i$	1	i	-1	$-i$	1

With the use of the imaginary units *i,* square roots of negative numbers can be written in a more simplified form, e.g.

$$\sqrt{-7} = \sqrt{7 * (-1)} = \sqrt{7} * \sqrt{-1} = \sqrt{7}\, i \text{ or } i\sqrt{7}$$

$$\sqrt{-5} = \sqrt{5 * (-1)} = \sqrt{5} * \sqrt{-1} = \sqrt{5}\, i \text{ or } i\sqrt{5}$$

$$\sqrt{-16} = \sqrt{16 * (-1)} = \sqrt{16} * \sqrt{-1} = 4i$$

Let us look at another example, this time a more complex problem.

$$2\sqrt{-36} = 2\sqrt{36\ (-1)} = 2\sqrt{36} * \sqrt{-1} = 2 * 6\ i = 12\ i$$

SELF ASSESSMENT EXERCISE

With your understanding of imaginary numbers, simplify this number√-36.

3.4 COMPLEX NUMBERS.

Complex numbers are formed by real numbers and imaginary numbers. A complex number is a number that can be expressed in the form $x + yi$, where x and y are the real parts, and yi is the imaginary part as it has i which is the imaginary unit, that satisfies the equation $i^2 = -1$. In this expression, x is the real part and yi is the imaginary part of the complex number. Complex numbers extend the concept of the one-dimensional number line to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part. For every complex number z, there exists exactly one ordered pair of real numbers (x, y) such that $z = x + yi$. A complex number whose real part is zero is said to be purely imaginary, whereas a complex number whose real part is zero is a real number. In this way the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone.

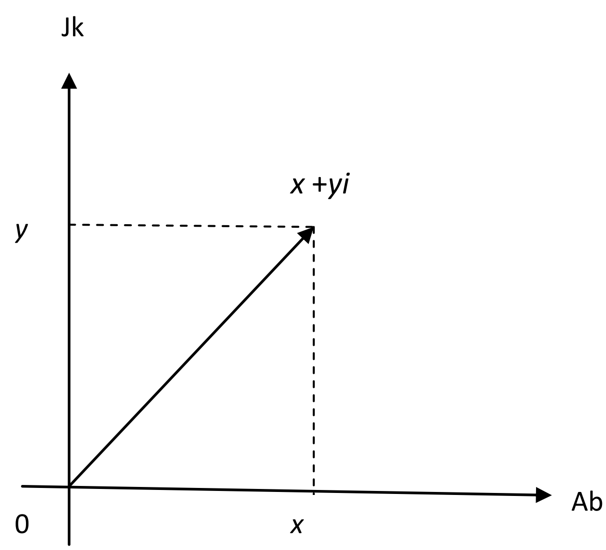

Figure 1.Complex number diagram.

The figure above proves that a complex number can be visually represented as a pair of numbers (x, y) forming a vector on a diagram called an Argand diagram, representing the complex plane. "Ab" is the real axis, "Jk" is the imaginary axis, and i is the imaginary unit which satisfies the equation $i^2 = -1$ this means that $i = \sqrt{-1}$. In the above figure, x and y represents the real numbers, while i represents the imaginary number, thus their combination forms the complex number.

How do we perform operations involving complex numbers? Starting with addition, we shall solve a few problems.

Addition

Example 1: $(3+ 6i)+(4-2i)$

$= (3+4) + (6 -2)i = 7+4i$

Subtraction

Example 1: $(8+ 4i)- (6+2i)$

$= (8-6) + (4-2)i = 2+2i$

Note: Addition or Subtraction of two complex numbers will give a complex number

Multiplication

$(3+2i)(4+5i)$

$=3(4+5i) + 2i(4+5i)$

$= 12+15i +8i+ 10i^2$

$= 12+15i +8i+ 10(-1)$ Recall that $i^2= -1$

$= 12+15i +8i-10$

$= 2 + 23i$

Also, the product of two complex number gives a complex number as seen in the above example.

Division

When dividing two complex numbers, for example $(5+3i) \div (3-2i)$.

You first of all state it in fraction form, so it becomes: $\frac{5+3i}{3-2i}$. Then you divide the numerator and the denominator with the CONJUGATE of the denominator.

What then is a conjugate?

A Conjugate simple means the result gotten when the sign in the middle of two binomials is changed or negated. Conjugates are formed by changing the sign between two terms in a binomial. For example, the conjugate of $2x + 1$ is $2x - 1$ and that of $5x - 1$ is $5x + 1$.

It is important to note that the product of a complex number and its conjugate, gives real number. For example, $(4 + 5i)(4 - 5i) = 4(4 - 5i) + 5i(4 - 5i)$

$= 16 - 20i + 20i - 25i^2$ since $i^2 = -1$ we now have,

$= 16 + 25$ which is same as $4^2 + 5^2$.

Back to our example on division, $\frac{5 + 3i}{3-2i}$. Here, the conjugate of the denominator is $3 + 2i$

When this conjugate multiplies the denominator and numerator, we have -:

$$\frac{5 + 3i}{3-2i} \text{ multiply by } \frac{3 + 2i}{3 + 2i} = \frac{(5 + 3i)(3 + 2i)}{(3 - 2i)\,(3 + 2i)} = \frac{15 + 10i + 9i + 6i^2}{9 + 6i - 6i - 4i^2}$$

$$= \frac{15 + 19i + 6(-1)}{9 - 4(-1)} = \frac{9 + 19i}{13}$$

Taking to a complex form we have, $\frac{9}{13} + \frac{19i}{13}$

SELF ASSESSMENT EXERCISE

Divide: $(3+2i) \div (4-3i)$

3.5 NUMBER BASES

A number can be represented in several ways with it still maintaining its value irrespective of the way it is represented. For example the number 01101 and 13 , are different ways to represent the same number (thirteen) in different bases. Below are some common examples of number base representations.

DECIMAL NUMBERS

These are the regular numbers we are used to and they use digits from 0-9. i.e. 1,2,3,4,5,6,7,8 and 9 and they are base ten numbers.

A number 234_{10} which is a three digit number can be rewritten thus:

$= 2\text{x}10^2 + 3\text{x}10^1 + 4\text{x}10^0 = 2\text{x}100 + 3\text{x}10 + 4\text{x}1 = 200 + 30 + 4 = 234$, which is Two hundred and thirty four.

OCTAL NUMBERS

Another number system is the octal number system which is a base eight number system. This number system uses eight digits from zero to seven, i.e. 0,1,2,3,4,5,6 and 7.

In base eight number system, the digits 0-7 are same as the decimal system but from 8, they assume different values. For example 8 in decimal number system is 10 in the octal system. Below are examples of the base ten numbers and their base eight equivalents.

8_{10}- 10_8

9_{10}-11_8

10_{10} - 12_8 and so on.

You may be required to convert from one base to the other. This is how it is done.

Example 1: Convert 53 from base 10 to base 8.

To convert 53_{10} to a base 8 value,

Step 1: Divide the decimal number by 8 then the quotient and remainder will be 6 and 5 respectively

Step 2: The quotient 6 can't be divided by 8

Step 3: To obtain the resulting number, write down the last quotient first and the remainder(s) from the lower level to upper level. The equivalent Octal number is 65_8

To verify if this is correct, let's convert our result back to base 10.

$65_8 = 6 \times 8^1 + 5 \times 8^0 = 48 + 5 = 53_{10}$

Example 2: Convert 376_8 to base 10 .

$3 \times 8^2 + 7 \times 8^1 + 6 \times 8^0 = 192 + 56 + 6 = 254_{10}$

To check if our answer is correct, we will convert our answer back to base 8.

To convert 254_{10} to base 8 we follow the steps in example 1.

254 divided by 8 gives 31remainder 6

31 divided by 8 gives 3 remainder 7

3 is less than 8 so we cease to divide. To get our result, we write down the last quotient first and the remainder(s) from the lower level to upper level. This will give us 376_8.

BINARY NUMBERS

A binary number is a number expressed in the binary numeral system, or base-2 numeral system, which represents numeric values using two different symbols: typically 0 (zero) and 1 (one) and this binary digits are what the computers use. The usual base-2 system is a positional notation with a radix of 2. e.g.,

Decimal 0 is binary 0

Decimal 1 is binary 1

Decimal 2 is binary 10

Decimal 3 is binary 11

Decimal 4 is binary 100

Each digit "1" in a binary number represents a power of two, and each "0" represents zero. For example,

01 is 2 to the zero power, or $0 \times 2^1 + \mathbf{1 \times 2^0}$ which is $= 1_{10}$

0010 is 2 to the 1st power, or $0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$ which is $= 2_{10}$

0100 is 2 to the 2nd power, or $0 \times 2^3 + \mathbf{1 \times 2^2} + 0 \times 2^1 + 0 \times 2^0$ which is $= 4_{10}$

1000 is 2 to the 3rd power, or $\mathbf{1 \times 2^3} + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$ which is $= 8_{10}$.

When you see a number in base two, you can figure out what the base ten is by adding the powers of 2:

$0101 = 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 0 + 4 + 0 + 1 = 5_{10}$

$1010 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 8 + 0 + 2 + 0 = 10_{10}$

$0111 = 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 0 + 4 + 2 + 1 = 7_{10}$.

DECIMAL COUNTING

Decimal counting uses the ten symbols 0 through 9. Counting primarily involves incremental manipulation of the "low-order" digit, or the rightmost digit, often called the "first digit". When the available symbols for the low-order digit are exhausted, the next-higher-order digit (located one position to the left) is incremented and counting in the low-order digit starts over at 0. Decimal counting proceeds thus:

000, 001, 002, ... 007, 008, 009, (rightmost digit starts over, and next digit is incremented)

010, 011, 012, ... 090, 091, 092, ... 097, 098, 099, (rightmost two digits start over, and next digit is incremented) 100, 101, 102, ... After a digit reaches 9, an increment resets it to 0 but also causes an increment of the next digit to the left.

BINARY COUNTING

In binary, counting follows similar procedure, except that only the two symbols 0 and 1 are used. Thus, after a digit reaches 1 in binary, an increment resets it to 0 but also causes an increment of the next digit to the left:

0000,

0001, (rightmost digit starts over, and next digit is incremented)

0010, 0011, (rightmost two digits start over, and next digit is incremented)

0100, 0101, 0110, 0111, (rightmost three digits start over, and the next digit is incremented)

1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111 ....

Since binary is a base-2 system, each digit represents an increasing power of 2, with the rightmost digit representing 2^0, the next representing 2^1, then 2^2, and so on. To determine the decimal representation of a binary number simply take the sum of the products of the binary digits and the powers of 2 which they represent. For example, the binary number 100101 is converted to decimal form as follows:

$100101_2 = [(1) \times 2^5] + [(0) \times 2^4] + [(0) \times 2^3] + [(1) \times 2^2] + [(0) \times 2^1] + [(1) \times 2^0]$

$100101_2 = [1 \times 32] + [0 \times 16] + [0 \times 8] + [1 \times 4] + [0 \times 2] + [1 \times 1]$

$100101_2 = 37_{10}$

To create higher numbers, additional digits are simply added to the left side of the binary representation

Table 3. Decimal Binary and Octal Counting

Decimal Numbers	Binary Numbers	Octal Numbers
0	0	0
1	1	1
2	10	2
3	11	3
4	100	4
5	101	5
6	110	6
7	111	7
8	1000	10
9	1001	11
10	1010	12

SELF ASSESSMENT EXERCISE

Convert 100231_2 to decimal form.

4.0 CONCLUSION

- Number system is a mathematical notation for representing numbers of a given set, using digits or other symbols in a consistent manner.
- A number or numeral system will represent a useful set of numbers and will give every number represented a unique representation which reflects the algebraic and arithmetic structure of the number.
- Binary numbers are numbers expressed in a binary system or base-2 numeral system which represents numeric values using two different symbols zero (0) and one (1).
- Imaginary numbers are numbers that can be written as a real number multiplied by the imaginary unit i, which is defined by its property $i^2 = -1$.
- Complex number is the combination of real number and imaginary number. It can be expressed in the form $x + yi$, where x is the real numbers, and yi the imaginary unit.

5.0 SUMMARY

This unit focused on number system which is a way of expressing numbers in writing, or the mathematical notation for representing numbers of a given set, using digits or other symbols in a consistent manner. In order to do justice to this topic, the properties of number system was reviewed, also, sub topics such as binary numbers, imaginary numbers and complex numbers were reviewed. It was observed that binary numbers are expressed in base-2 system which represents numeric values using two symbols (0 and 1). Imaginary numbers are numbers that can be written as real numbers multiplied by an imaginary unit *i* which is defined as $i^2 = -1$. Complex number combines both the real and imaginary numbers together, and can be expressed as $x + yi$ where x and y are real numbers and *i* is the imaginary unit.

6.0 TUTOR-MARKED ASSIGNMENT

- Calculate the difference between Z1 and Z2 assuming Z1 = 3 – 7i, and Z2 = 5 + 4i.
- Convert 1793 from decimal to octal number.

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 INEQUALITIES

1. INTRODUCTION

This unit seeks to expose the students to the concept of inequality. In this unit, students will learn how to solve inequalities. Solving inequality means finding all of its solutions. A solution of an inequality is a number which when substituted for the variable, makes the inequality a true statement. Inequality is a statement that holds between two values when they are different.

The notation $x \neq y$ means that x is not equal to y, and not that one is greater than the other, or even that they can be compared in size. If the values in a question are elements of an ordered set, such as the integers or the real numbers, they can be compared in size.

The notation $x<y$ means that x is less than y, while $x>y$ means that x is greater than y, and that both elements are not equal. These relations are known as strict inequalities. The notation $x<y$ may also be read as "x is strictly less than y".

In contrast to strict inequalities, there are two types of inequality relations that are not strict:

- The notation $x \leq y$ means that x is less than or equal to y (or, equivalently, not greater than y, or at most y).
- The notation $x \geq y$ means that x is greater than or equal to y (or, equivalently, not less than y, or at least y).

In this unit, inequalities will be treated extensively.

2. OBJECTIVES

At the end of this unit, you will able to:

- Define the concept of Inequalities
- Differentiate between the properties of inequality
- Solve inequality problems
- Apply inequality to angle problems.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO INEQUALITY

Inequality is the statements that show the relationship between two (or more) expressions with one of the following five signs: $<$, $\leq$, $>$, $\geq$, $\neq$.

Where: $x < y$ means “x is less than y”

$x \leq y$ means " x is less than or equal to y "

$x > y$ means “x is greater than y”

$x \geq y$ means " x is greater than or equal to y "

$x \neq y$ means " x is not equal to y "

Like an equation, an inequality can be true or false.

$40 - 20 > 6 + 4$ is a true statement.

$6 + 10 < 6 - 3$ is a false statement.

$3 + 3 \leq 9 - 3$ is a true statement.

$4 + 3 \neq 5 - 3$ is a true statement.

$-35 < -15$ is a true statement.

To determine whether an inequality is true or false for a given value of a variable, plug in the value for the variable. If an inequality is true for a given value, we say that it holds for that value.

Example 1. Is, $2x + 4 < 9 - x$ true for $x = 2$.

$2(2) + 4 < 9 - 2$

Is $8 < 7$?, the answer is NO.

Example 2. Does $7x - 4 > 2x + 1$ hold for $x = 2$?

$7(2) - 4 > 2(2) + 1$

Is $10 > 5$?, the answer is YES.

Finding a solution set to an inequality, given a replacement set, is similar to finding a solution set to an equation. Plug each of the values in the set, in replacement for the variable. If the inequality is true for a certain value, that value belongs in the solution set.

Example 3: Find the solution set of $x - 5 > 12$ from the replacement set {10, 15, 20, 25}.

$10 - 5 > 12$? False.

$15 - 5 > 12$? False.

$20 - 5 > 12$? True.

$25 - 5 > 12$? True.

Thus, the solution set is {20, 25}.

Example 4: Find the solution set of $-5x \geq 9$ from the replacement set {-4, -3, -2, -1, 0, 1}.

$-5(-4) \geq 9$? True.

$-5(-3) \geq 9$? True.

$-5(-2) \geq 9$? True.

$-5(-1) \geq 9$? False.

$-5(0) \geq 9$?False.

$-5(1) \geq 9$? False.

Thus, the solution set is {-4, -3, -2}.

Example 5. Find the solution set of $x^2 \neq 4x$ from the replacement set {0, 1, 2, 3}.

$0^2 \neq 4(0)$? False (they are, in fact, equal).

$1^2 \neq 4(1)$? True.

$2^2 \neq 4(2)$?True.

$3^2 \neq 4(3)$? True.

Thus, the solution set is {1, 2, 3}.

SELF ASSESSMENT EXERCISE

Are the following inequalities true or false?

1. $2 + 6 > 2 \times 4$
2. $3 + 5 \geq 3 \times 3$

3.2 PROPERTIES OF INEQUALITIES

There are formal definitions of the inequality relations >, <,≥,≤ in terms of the familiar notion of equality. We say x is less than y, written $x<y$ if and only if there is a positive number z such that $x + z = y$. Recall that zero is not a positive number, so this cannot hold if $x = y$. Similarly, we say x is greater than y and write $x>y$ if y is less than x; alternately, there exists a positive number z such that $x = y + z$. The following are the properties of inequalities.

1. TRICHOTOMY AND THE TRANSITIVITY PROPERTIES

TRICHOTOMY PROPERTY

For any two real numbers x and y, exactly one of the following is true: $x<y$, $x = y$, $x>y$.

TRANSITIVE PROPERTIES OF INEQUALITY

If $x<y$ and $y<z$, then $x<z$.

If $x>y$ and $y>z$, then $x>z$.

Note: These properties also apply to "less than or equal to" and "greater than or equal to":

If $x\leq y$ and $y\leq z$, then $x\leq z$.

If $x\geq y$ and $y\geq z$, then $x\geq z$.

2. ADDITION AND SUBTRACTION

ADDITION PROPERTIES OF INEQUALITY

If $x<y$, then $x + z < y + z$

If $x>y$, then $x + z > y + z$

SUBTRACTION PROPERTIES OF INEQUALITY

If $x<y$, then $x - z < y$ - z

If $x>y$, then $x - z > y - z$

These properties also apply to ≤ and ≥:

If $x \leq y$, then $x + z \leq y + z$

If $x \geq y$, then $x + z \geq y + z$

If $x \leq y$, then $x - z \leq y - z$

If $x \geq y$, then $x - z \geq y - z$

3. MULTIPLICATION AND DIVISION

Before examining the multiplication and division properties of inequality, note the following:

Inequality Properties of Opposites

If $x > 0$, then $-x < 0$

If $x < 0$, then $-x > 0$

For example, 4 > 0 and -4 < 0. Similarly, -5 < 0 and 5 > 0.Whenever we multiply an inequality by -1, the inequality sign flips. This is also true when both numbers are non-zero: 6 > 3 and -6 < -3 ; 3 < 7 and -3 > -7 ; -2 < 9 and 2 > -9.

In fact, when we multiply or divide both sides of an inequality by any negative number, the sign always flips. For instance, 5 > 3, so 5(-3) < 3(-3) = -15 < -9.

Also, -2 < 6, so -2/-2>6/-2 = 1 > -3.

This leads to the multiplication and division properties of inequalities for negative numbers.

(3a) Multiplication and Division Properties of Inequalities of positive numbers:

If $x<y$ and $z> 0$, then $xz<yz$ and > ($x/z < y/z$)

If $x >y$ and $z> 0$, then $xz>yz$ and < ($x/z > y/z$)

Note: All the properties apply to ≤ and ≥.

4. PROPERTIES OF RECIPROCAL

Note the following properties:

When we take the reciprocal of both sides of an equation, something interesting happens. If the numbers on both sides have the same sign, the inequality sign flips.

For example, $5 < 7$, but $1/5 > 1/7$.

We can write this as a formal property:

If $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$, and $x < y$, then $1/x > 1/y$.

If $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$, and $x > y$, then $1/x < 1/y$.

Note: All the above properties apply to ≤ and ≥.

SELF ASSESSMENT EXERCISE

What property is demonstrated by the following statement?

1. 5 > 4 and 4 > 2. It follows therefore that 5> 2
2. If 2<6, then 2 – 1 < 6 – 1, name the property or operation used in the inequality.

3.3 SOLVING INEQUALITIES PROBLEMS

Solving linear inequalities is very similar to solving linear equations, except for one small but important detail, you flip the inequality sign whenever you multiply or divide the inequality by a negative value. The easiest way to show this is with some examples:

Example 1: Consider the inequality

$$x - 4 > 5.$$

Solution: When we substitute 10 for x, the inequality becomes 10-4 > 5. Thus, x=10 is a solution of the inequality. On the other hand, substituting -3 for x yields a false statement (-3) - 4 > 5. Thus x = -3 is NOT a solution of the inequality.

Consider the inequality:

$$3x + 4 < 8.$$

The basic strategy for inequalities and equations is the same: isolate x on one side, and put the other value(s) on the other side. Following this strategy, let us move +4 to the right hand side. We accomplish this by subtracting 4 on both sides to obtain:

$$(3x + 4) - 4 < 8 - 4$$

$$3x < 4.$$

Once we divide both sides by +3, we have succeeded in isolating x on the left: $x < \frac{4}{3}$

$x < 1.33$.

Example 2. Find the solution of the inequality $7 - x \leq 9$.

Solution: Let's start by moving the ``7" to the right hand side by subtracting 7 on both sides.

$$(7 - x) - 7 \leq 9 - 7$$

$$-x \leq 2.$$

How do we get rid of the negative sign in front of *x*? Just multiply by (-1) on both sides, changing "≤" to "≥" as you multiply, we will have:

$$(-x)*(-1) \geq 2*(-1),$$

$$x \geq -2.$$

Example 3: Solve the inequality $2(x - 1) > 3(2x + 3)$.

Solution: We need to first simplify the inequality:

$$2x - 2 > 6x + 9$$

There are different ways to proceed, but let us first subtract 2*x* from both sides.

$$(2x - 2) - 2x > (6x + 9) - 2x,$$

$$-2 > 4x + 9.$$

Next, subtract 9 from both sides.

$$-2 - 9 > (4x + 9) - 9$$

$$-11 > 4x.$$

Divide through by 4 to make *x* the subject.

$$\frac{-11}{4} > x$$

-11/4 >*x*. To move *x* to the left hand side, you flip the inequality sign

$$x < \frac{-11}{4}$$

SELF ASSESSMENT EXERCISE

Provide a solution to the inequality problem: $10y + 6 < 4y - 3$.

3.4 SOLVING INEQUALITIES USING INVERSE OPERATIONS

Solving equations using inverse operation is similar to inequalities, with one exception which is the observation of the rules governing multiplication and division by a negative number and its reciprocals, as well as changing the inequality sign when appropriate.

Follow these steps to reverse the order of operations acting on the variable:

1. Reverse addition and subtraction by subtracting and adding stand alone number.
2. Reverse multiplication and division by dividing and multiplying outside parentheses. When multiplying or dividing by a negative number, change the inequality sign. It is of no concern if the number being divided is positive or negative.
3. Eliminate the bracket and reverse the operations

Example 1: $10y + 5 < 13$

$10y + 5 - 5 < 13 - 5$ (Subtracting 5 from both sides of the inequality)

$10y < 8$

$10y/10 < 8/10$

$y < 0.8$

Example 2: $5 – 4y \leq 3y - 7$

$5 – 4y + 4y \leq 3y - 7 + 4y$ (adding 4y to both sides of the inequality)

$5 \leq 7y - 7$

$5 + 7 \leq 7y$

$12 \leq 7y$

$12/7 \leq 7y/7$ (Dividing both sides of the inequality by 7)

$1.71 \leq y$

$y \geq 1.71$

Example 3: $\frac{y + 8}{4} \leq 10$

$\frac{y + 8}{4} \times 4 \leq 10 \times 4$

$y + 8 \leq 40$

$y + 8 - 8 \leq 40 - 8$

$y \leq 32$

Example 4: $20 < 5(y - 3)$

$20 < 5y - 15$(multiplying the right hand side by 5)
$20 + 15 < 5y - 15 + 15$(to eliminate the 15 on the right hand side)
$35 < 5y$(divide both sides by 5 to eliminate 5 on the right hand side)
$7 < y$

Therefore, $y > 7$

SELF ASSESSMENT EXERCISE

Solve for x: (a) $3x – 5 \leq 15$. (b) $–(x + 2) < 4$.

3.5 APPLICATION OF INEQUALITIES TO ANGLES

Inequalities are a significant part of geometry when it comes to angle classification. We have three types of angles which are the right angle, acute angle and the obtuse angle.

Right angle is an angle with 90 degree. An acute angle is an angle from 90 degree downwards, while an obtuse angle is bounded to 90-180 degrees.

Using the three specified angles above, if we assume y to represent right, acute and an obtuse angle independently, we can represent the equation as stated below.

If $y = 90$, then angle y is a right angle.

If $y < 90$, then angle y is an acute angle.

If $y > 90$, but < 180, then angle y is an obtuse angle.

Example1: Identify the angles appropriate for the figures below.

y = 80?, y = 90?, y = 170?

Solution:

Remember, any angle below 90 degree is an acute angle, thus, y = 80 is less than 90, and thus, the angle under question is an acute angle.

For y = 90, since an angle equal to 90 means the angle is a right angle, we have reasonable evidence to conclude that the angle is a right angle.

For y = 170, we can say that the angle in question here is an obtuse angle, because an angle above 90 degree, but below 180 degree, then such an angle is an obtuse angle.

Example 2: Consider an angle with the equation 2y + 12 degrees, with the following figures for y value {15, 5, 25, 40, 12}; determine which angle y is.

2(15) + 12 = 42. Since 42 is less than 90, we say such an angle is an acute angle.

2(5) + 12 = 22. Since 22 is less than 90, we say such an angle is an acute angle.

2(25) + 12 = 62. Since 62 is less than 90, we say such an angle is an acute angle.

2(40) + 12 = 92. Since 92 is greater than 90, but less than 180 degree, we say such an angle is an obtuse angle.

2(12) + 12 = 36. Since 36 is less than 90, we say such an angle is an acute angle.

SELF ASSESSMENT EXERCISE

If angle F measures 3(15x – 45) degrees, for which of the following values of x is F obtuse? {3, 4, 5, 6}.

4.0 CONCLUSION

- Solution of an inequality is a number which when substituted for a variable makes the inequality a true statement.
- Inequality is a statement that holds between two values when they are different.

$x < y$ means " x is less than y ", $x \leq y$ means " x is less than or equal to y ", $x > y$ means " x is greater than y ", $x \geq y$ means " x is greater than or equal to y ", $x \neq y$ means " x is not equal to y ".

- Transitive property of inequality states that if $a < b$ and $b < c$, then $a < c$. Also, if a $> b$ and $b > c$, then $a > c$.
- The addition and subtraction property of inequality states that if $a < b$, then $a + c < b + c$, and if $a > b$, then $a + c > b + c$. Also, if $a < b$, then $a - c < b - c$. If $a > b$, then $a - c > b - c$.
- In solving linear inequality problem is similar to that of linear equation with only flipping the inequality sign whenever the inequality is multiplied or divided by a negative value.
- In order to solve inequalities using inverse operations, attention must be paid to the multiplication and division rules.

5.0 SUMMARY

This unit focused on inequalities. Solving inequality means finding all of the solutions or values which when substituted in the inequality will make it a true statement. In order to do justice to this unit, properties of inequality such as the transitive, addition and subtraction, multiplication and division as well as the reciprocal properties were reviewed. Relationships among variables using the five signs or symbols were analysed. Several solution examples and solution were given on how to solve inequality problem, and the application of inequalities to angles was reviewed as well.

6.0 TUTOR-MARKED ASSIGNMENT

- If angle F measures $y(y - 3) + 90$ degree, for which of the following values of y is F right angle given the following figures for y $\{0, 1, 2, 3\}$.
- Solve for x: $-2x - 5 \neq -3 - x$.

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UNIT 3 EXPONENTS AND ROOTS

1. INTRODUCTION

Exponents and roots are one of the most important topics in mathematics, and they are used in virtually all aspects of mathematics. This unit will focus on the introductory part of exponents and roots, making sure learners have basic understanding in which they will build on to apply the concept to economic issues.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Explain the concept of exponents and roots.
- Solve exponents problems using the base 10 system.
- Simplify and approximate roots with ease.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO EXPONENTS AND ROOTS

An exponent is a mathematical notation that implies the number of times a number is to be multiplied by itself.

For example: In the expression x^9, the exponent is 9, and x is the base.

In 3^4, 4 is the exponent. It indicates that 3 is to be multiplied by itself 4 times.

$3^4 = 3 \times 3 \times 3 \times 3 = 81$.

The root of a number (say x) is another number, which when multiplied by itself a given number of times, equal x.

For example: The second (square) root of 9 is 3, because 3 x 3 = 9.

The second root is usually called the square root, while the third root is usually called the cube root.

SELF ASSESSMENT EXERCISE

Differentiate between exponents and roots, and give two examples each.

3.2 EXPONENTS

As we early indicated, an exponent involves the number of times in which a number is multiplied by itself. For example, 7^2, here, 7 is the base, and 2 is the square (indicating 7 multiplied by itself). Based on our definition of exponent, when we multiply 7 by itself (7*x*7), we get 49. Similarly, the square root of 49 that is ($\sqrt{49}$) is 7.

Let us try with another number. Take 5 for instance; let us find the cube root of 5, meaning 5^3. Here, what we simply mean is 5*x*5*x*5 which will give us 125. Conversely, the cube root of 125 is 5.

Below is a list of the powers of base two system:

$$2^0 = 1$$
$$2^1 = 2$$
$$2^2 = 2\times2 = 4$$
$$2^3 = 2\times2\times2 = 8$$
$$2^4 = 2\times2\times2\times2 = 16$$
$$2^5 = 2\times2\times2\times2\times2 = 32$$

And so on.

Let us preview exponents with base five system.

$5^0 = 1$

$5^1 = 5$

$5^2 = 5\times5 = 25$

$5^3 = 5\times5\times5 = 125$

$5^4 = 5\times5\times5\times5 = 625$

$5^5 = 5\times5\times5\times5\times5 = 3{,}125$

And so on.

5^0 is 1 (a 1 in the one place), 5^1 is 1 five (a 1 in the 5 places), etc. This is the meaning of base five. A "1" in each place represents a number in which the base is 5 and the exponent is the number of zeros after the 1. The place value is the number that is multiplied by this number.

Also, we can preview an exponent with base ten system.

$10^0 = 1$

$10^1 = 10$

$10^2 = 10\times10 = 100$

$10^3 = 10\times10\times10 = 1{,}000$

$10^4 = 10\times10\times10\times10 = 10{,}000$

$10^5 = 10\times10\times10\times10\times10 = 100{,}000$

And so on.

In same vein, four thousand is equivalent to 4×1000, or 4×10^3.

We can write out any number as a sum of single-digit numbers times powers of ten. The number 584 has a 5 in the hundreds place (5×10^2), a 8 in the ten places (8×10^1) and a 4 in the one place (4×10^0). Thus, $584 = 5\times10^2 + 8\times10^1 + 4\times10^0$.

Examples: Write out the following numbers as single-digit numbers multiplied by the powers of ten.

$875 = 8\times10^2 + 7\times10^1 + 5\times10^0$

$55, 426 = 5\times10^4 + 5\times10^3 + 4\times10^2 + 2\times10^1 + 6\times10^0$

$9, 145 = 9\times10^3 + 1\times10^2 + 4\times10^1 + 5\times10^0$.

3.2.1 FORMS OF EXPONENTS

1. SQUARE

The square of a number is that number multiplied by itself. 4 squared, denoted 4^2, which is equal to 4×4, or 16. 3 squared is $3^2 = 3×3 = 9$. One way to remember the term "square" is that there are two dimensions in a square (height and width) and the number being squared appears twice in the calculation. In fact, the term "square" is no coincidence; the square of a number is the area of the square with sides equal to that number.

A number that is the square of a whole number is called a perfect square. $5^2 = 25$, so 25 is a perfect square. 25 and 4 are also perfect squares. We can list the perfect squares in order, starting with: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, ...,.

2. CUBES

When we are talking about the cube of a number, we simply mean the number will multiply itself three times. We can represent the cube root of 3 as 3^3, meaning 3×3×3 which will give a result of 27.

We can look at the cube root of different number as listed below:

10^3 = 10×10×10 = 1,000

15^3 = 15×15×15 = 3,375

20^3 = 20×20×20 = 8,000

25^3 = 25×25×25 = 15,625

30^3 = 30×30×30 = 27,000

35^3 = 35×35×35 = 42,875

Based on the above, we can easily find the cube root of any number by simply multiplying the number by itself three times to arrive at the final figure.

SELF ASSESSMENT EXERCISE

Provide answers to the following: (a) 8^3, (b) 4^7, (c)100^3, (d) 1500^3

3.3 EXPONENTS OF SPECIAL NUMBERS

3.3.1 EXPONENTS OF NEGATIVE NUMBERS

As we have seen, an exponent on a number involves multiplying that same number by itself. In the case of a negative number, the same procedure is followed; that is, we will multiply the negative number by itself to arrive at the final figure.:

$(-10)^3 = -10\times -10\times -10 = -1000$

$(-7)^3 = -7\times-7\times-7 = -343$

$(-9)^3 = -9\times-9\times-9 = -729$

$(-15)^2 = -15\times-15 = 225$.

Notice that in the examples above with the cube root of negative number gives a final figure of a negative number. This is as a result of three negative number multiplying itself.

Remember, - × - = +, and - × - × - = - . Also, - × - × - × - = +, etc.

This simply means when two negative numbers multiply itself, it gives a positive sign, while reverse is the case with three negative numbers.

Put more clearly, an odd number (3 negative numbers multiplying each other) of negative numbers multiplied together will give a negative number and an even (2 negative numbers) number of negative numbers multiplied together will yields a positive number, a negative number with an odd exponent will always be negative and a negative number with an even exponent will always be positive. So, to take a power of a negative number, take the power of the (positive) opposite of the number, and add a negative sign if the exponent is odd.

Example 1: $(-7)^6$

1. Take the power of the positive opposite. $7^6 = 7\times7\times7\times7\times7\times7 = 117{,}649$
2. The exponent (6) is even, so $(-7)^6 = 117{,}649$

Example 2: $(-3)^9$

1. Take the power of the positive opposite. $3^9 = 3\times3\times3\times3\times3\times3\times3\times3\times3 = -19{,}683$
2. The exponent (9) is odd, so $(-3)^9 = -19{,}683$.

3.3.2 EXPONENTS OF DECIMAL NUMBERS

The exponent of decimal numbers involved multiplying the decimal number by itself based on the power. When we square 0.18, we must remember that we are multiplying 0.18 by 0.18 (0.18×0.18) which is equal to 0.0324. Similarly, following the procedure indicated earlier, $0.76^5 = 0.76\times0.76\times0.76\times0.76\times0.76 = 0.2536$.

When taking the power of a decimal, first count the number of decimal places in the base number, as when multiplying decimals. Next, multiply that number by the exponent. This will be the total number of decimal places in the answer. Then, take the power of the base number with the decimal point removed. Finally, insert the decimal point at the correct place, calculated in the second step.

Example 1: 4.9^3

1. There is 1 decimal place and the exponent is 3, thus 1×3 = 3.

2. 49^3 = 117,649.

3. Now, insert the decimal point 3 places to the right. 4.9^3 = 117.649.

Example 2: 0.52^4

1. There are 2 decimal places and the exponent is 4. 2×4 = 8.

2. 52^4 = 7,311,616

3. Now, insert the decimal point 8 places to the left to arrive at the final answer.

$$0.52^4 = 0.07311616$$

3.3.3 EXPONENTS OF FRACTIONS

There is a special way of presenting an exponent when it has to do with fraction. It entails breaking the numbers down into fraction and multiplying it with itself based on the exponent. Let us for instance examine $(5/3)^4$. In order to solve this problem, we have to multiply the observation by itself four times (the figure of the exponent), thus yielding (5/3)×(5/3)×(5/3)×(5/3). In order to multiply the fraction together in the correct way, we multiply the numerators (figures being divided) together, and also we multiply the denominators together to get a single figure. Continuing with our example, it means we have to multiply 5 by itself four times (5×5×5×5) which will yield 625. We will do the same to the denominator by multiplying 3 by itself four times (3×3×3×3). This will give us 81. The final step is to divide the numerator figure (625) by the denominator figure (81): 625/81 = 7.716.

Another way to go about is to take the 4th root of 5, and divide it by the 4th root of 3. This process is presented below.

$$(5^4)/(3^4) = 625/81 = 7.716.$$

Examples:

- $(9/4)^5 = (9^5)/(4^5) = 59{,}049/1{,}024 = 57.6650$
- $(-6/5)^2 = ((-6)^2)/(5^2) = 36/25 = 1.44$
- $(3/-9)^3 = (3^3)/(-9^3) = 27/(-729) = -0.0370.$

SELF ASSESSMENT EXERCISE

Provide answers to the following: (a) $(-18/5)^3$, (b) $(7/-4)^7$, (c) $(0.21/0.05)^9$.

3.4 THE NEGATIVE EXPONENTS

Negative exponent is a little different from positive exponent. The only difference is the sign. When we take a negative exponent, it does not necessary mean that we are going to have a negative answer, and in fact, we are likely to have either a positive or a negative answer. Taking a base number to a negative exponent is equivalent to taking the base number to the positive opposite of the exponent (the exponent with the negative sign removed) and placing the result in the denominator of a fraction whose numerator is 1.

For example, $4^{-2} = 1/4^2 = 1/16 = 0.0625$.

Similarly, $5^{-3} = 1/5^3 = 1/125 = 0.008$.

NEGATIVE EXPONENTS AND THE BASE TEN SYSTEM

A list of negative powers of ten is presented below.

10^{-1} = $1/10^1 = 1/10 = 0.1$

10^{-2} = $1/10^2 = 1/100 = 0.01$

10^{-3} = $1/10^3 = 1/1, 000 = 0.001$

10^{-4} = $1/10^4 = 1/10, 000 = 0.0001$

10^{-5} = $1/10^5 = 1/100, 000 = 0.00001$

And so on.

Now we can write out any terminating decimal as a sum of single-digit number multiplied by the power of ten. The number 45.45 has a 4 in the tens place (4×10^1) , a 5 in the unit place (5×10^0) , a 4 in the tenths place (4×10^{-1}) and a 5 in the hundredths place (5×10^{-2}). Thus, $44.59 = 4\times10^1 +4\times10^0 +5\times10^{-1} +9\times10^{-2}$.

Examples: Write out the following numbers as single-digit numbers multiplied by powers of ten:

$723.63 = 7\times10^2 +2\times10^1 +3\times10^0 +6\times10^{-1} +3\times10^{-2}$

$6.232 = 6\times10^0 +2\times10^{-1} +3\times10^{-2} +2\times10^{-3}$

$17.924 = 1\times10^1 +7\times10^0 +9\times10^{-1} +2\times10^{-2} +4\times10^{-3}$

SELF ASSESSMENT EXERCISE

Provide answers to the following: (a) $8^{-3}$and, (b) $(2/5)^{-2}$

3.5 ROOTS (SQUARE and CUBE ROOTS)

The root of a number "*a*" is another number, which when multiplied by itself a given number of times, gives "*a*". For example, the square root of a number is the number that, when squared (multiplied by itself), is equal to the given number. The symbol used to denote the square root is "√". For example, the square root of 25, is denoted $25^{1/2}$ or √25, which gives a result of 5. When we find the square of 5, i.e., 5^2, it simply means 5×5, which gives an answer of 25. The square root of 121, denoted √121, is 11, because 11^2 = 121. √25/4 = 5/2, because $(5/2)^2$ = 25/4. √81 = 9, because 9^2 = 81.

In the event that we are dealing with a fraction , we go about it just the same way we went about dealing with fraction numbers earlier by taking the root of both the nominator and denominator separately and dividing the resulting figures to get the final answer.

All perfect squares have square roots that are whole numbers. All fractions that have a perfect square in both numerator and denominator have square roots that are rational numbers. For example, √36/9 = 6/3. All other positive numbers have squares that are non-terminating, non- repeating decimals, or irrational numbers. For example,√2 = 1.41421356 and √53/11 = 2.19503572.

Usually when people say root of a number, what readily comes to mind is the square root of that number, however, there are other roots like the cube root, forth roots, eight roots etc. These roots are represented by the square root symbol "√" but with the root number in front of the symbol e.g. $^3\surd$ for cube root, $^4\surd$ for fourth root, $^8\surd$for eighth root etc.

3.5.1 SQUARE ROOTS OF NEGATIVE NUMBERS

Based on the residual knowledge we have gained of the negative exponent, similar process with slight modification can be followed here. When a positive number is multiplied by itself, it produces a positive figure as the answer, and if two negative numbers is used to multiply one another, it will produce a positive number. Remember the sign relationship shown earlier where: - × - = +, and - × - × - = -, e.t.c. One unique thing about negative number is that we cannot take its square root.

Taking a square root is almost the inverse operation of taking a square. Squaring a positive number and then taking the square root of the result does not change the number: $\surd 7^2$ = √49 = 7. However, squaring a negative number and then taking the square root of the result is equivalent to taking the opposite of the negative number: $\surd(-7)^2$ = √49 = 7 . Thus, we conclude that squaring any number and then taking the square root of the result is equivalent to taking the absolute value of the given number. For example, $\surd 6^2$ = | 6| = 6, and $\surd(-7)^2$ = | - 7| = 7 .

Taking the square root first and then squaring the result yields a slightly different case. When we take the square root of a positive number and then square the result, the number does not change: $(\surd 121)^2 = 11^2$ = 121.

3.5.2 CUBE ROOTS AND HIGHER ORDER ROOTS

A cube root is a number that, when cubed, is equal to the given number. It is denoted with an exponent of "1/3". For example, the cube root of 35 is $35^{1/3} = 3.27$. Similarly, the cube root of 90/25 is $(90/25)^{1/3} = (90^{1/3})/(25^{1/3}) = 4.48/2.92 = 1.534$.

Roots can also extend to a higher order than cube roots like the 4th, 5th root and above. The 4th root of a number is a number that, when taken to the fourth power, is equal to the given number. The 5th root of a number is a number that, when taken to the fifth power, is equal to the given number, and so on. The 4th root is denoted by an exponent of "1/4", the 5th root is denoted by an exponent of "1/5"; every root is denoted by an exponent with 1 in the numerator and the order of root in the denominator.

SELF ASSESSMENT EXERCISE

What is the 7th root of (25/12)?

3.6 SIMPLIFYING AND APPROXIMATING ROOTS

3.6.1. SIMPLIFYING ROOTS

One of the ways to simply roots is by removing all factors that are perfect square from inside the square root, and place their square roots outside the sign. In order to ensure this is done correctly, follow the steps highlighted below.

1. Factor the number inside the square root sign.
2. If a factor appears twice, cross out both and write the factor one time to the left of the square root sign. If the factor appears three times, cross out two of the factors and write the factor outside the sign, and leave the third factor inside the sign. Note: If a factor appears 4, 6, 8, etc. times, this counts as 2, 3, and 4 pairs, respectively.
3. Multiply the numbers outside the sign. Multiply the numbers left inside the sign.
4. Check: The outside number squared times the inside number should equal the original number inside the square root.

To simplify the square root of a fraction, simplify the numerator and simplify the denominator. Let us proffer solution to some example to facilitate quick understanding.

Example 1: Simplify $8^{1/2}$.

$$\sqrt{8} = \sqrt{2x2x2}$$

$$\sqrt{2x2x2} = 2x\sqrt{2}$$

Now, let us square both 2 and √2 to arrive at our final answer.

$= 2^2 \text{ x } \sqrt{2}^2 = 4 \text{ x } 2 = 8.$

Note: the square of √2 simply eliminates the square root (√) sign.

Example 2: Simplify √108.

$\sqrt{108} = \sqrt{3x3x3x4}$

$= \sqrt{3x3} \text{ x } \sqrt{3} \text{ x } \sqrt{4}$

$= 3 \text{ x } \sqrt{3} \text{ x } 2$

Squaring all figures in order to eliminate the square root sign:

$3^2 \text{ x } \sqrt{3}^2 \text{ x } 2^2 = 9 \text{ x } 3 \text{ x } 4$

$= 108.$

Example 3: Simplify √810

$\sqrt{810} = \sqrt{2x3x3x3x3x5}$

$\sqrt{2x3x3x3x3x5} = 3x3x\sqrt{2x5}$

$3x3x\sqrt{2x5} = 9x\sqrt{10}$

$9^2x10 = 810.$

Similarly, to simplify a cube root, factor the number inside the "(3√)" sign. If a factor appears three times, cross out all three and write the factor one time outside the cube root sign.

Example4: Find the cube root of 8.

$^3\sqrt{8}$

$^3\sqrt{2*2*2}$

Since 2 appear three times, we cross it out and write 2 as our answer. So the cube root of 8 is 2.

Example 5: Find the cube root of 216.

$^3\sqrt{216}$

$^3\sqrt{6*6*6}$

Thus giving us an answer of 6.

SELF ASSESSMENT EXERCISE

Simplify √343

3.6.2 APPROXIMATING SQUARE ROOTS

Sometimes, identifying square root is difficult; however, following the method and steps highlighted below, we will be able to make our work less cumbersome. In order to use this method, it is paramount to memorize the square roots.

To use the formula, follow the steps below:

1. Pick a perfect square that is close to the given number. Take its square root.
2. Divide the original number by this result.
3. Take the arithmetic mean of the result of (1) and the result of (2) by adding the two numbers and dividing by 2 (this is also called "taking an average").
4. Divide the original number by the result of (3).
5. Take the arithmetic mean of the result of (3) and the result of (4)
6. Repeat steps IV-VI using this new result, until the approximation is sufficiently close.

Example 1: Approximate $\sqrt{22}$.

25 is close to 22. $\sqrt{25} = 5$

$22/5 = 4.4$

$(5 + 4.4)/2 = 4.7$

$22/4.7 = 4.68$

$(4.7 + 4.68)/2 = 4.69$

$22/4.69 = 4.69$

$\sqrt{22} = 4.69$.

Example 2: Approximate $\sqrt{71}$.

71 is close to 64. $\sqrt{64} = 8$

$71/8 = 8.9$

$(8 + 8.9)/2 = 8.45$

$71/8.45 = 8.40$

$(8.45 + 8.40)/2 = 8.425$

$71/8.425 = 8.427$

$(8.425 + 8.427)/2 = 8.426$

$71/8.426 = 8.426$

$\sqrt{71} = 8.426$

Example 3: Approximate $\sqrt{56}$.

$\sqrt{56}$ can be simplified: $\sqrt{56} = \sqrt{2x2x2x7} = 2\times\sqrt{2x7} = 2\times \sqrt{14}$

Approximate $\sqrt{14}$:

14 is close to 16. $\sqrt{16} = 4$

$14/4 = 3.5$

$(4 + 3.5)/2 = 3.75$

$14/3.75 = 3.73$

$(3.75 + 3.73)/2 = 3.74$

$14/3.74 = 3.74$

$\sqrt{14} = 3.74$

Thus, $\sqrt{56} = 2\times\sqrt{14} = 2\times 3.74 = 7.48$

SELF ASSESSMENT EXERCISE

Simplify $\sqrt{180}$, $\sqrt{189}$ and $\sqrt{150}$.

4.0 CONCLUSION

- An exponent is a mathematical notation that implies the number of times a number is to be multiplied by itself.
- The square of a number is that number multiplied by itself, while cube of a number is that number multiplied by itself three times.
- An exponent on a negative number is simply the negative number multiplied by itself in a certain number of times.
- To take the power of a fraction, take the power of the numerator to get the numerator, and take the power of the denominator to get the denominator.
- To simplify the square root of a fraction, simplify the numerator and the denominator.

5.0 SUMMARY

This unit focused on Exponent and Roots. Exponent implies the number of times a number is to be multiplied by itself, while the root of a number is another number which when multiplied by itself a given number of times equal that same number. Sub-topics such as the square, cubes and higher order exponents were reviewed. Similarly, powers of negative numbers, decimals, negative exponent as well as square roots were reviewed.

6.0 TUTOR-MARKED ASSIGNMENT

- Approximate √70, √50 and √99 to two decimal places.
- Simplify √291.
- Write out3728 as a sum of powers of ten.

7.0 REFERENCES/FURTHER READINGS

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MODULE 2 EQUATIONS

UNIT 1 SYSTEMS OF EQUATION

2.0 INTRODUCTION

When we talk about equation, we simply mean two different numbers are equal to one another. We have different types of equation such as the linear equation, exponential equation, polynomial equation, quadratic equation, trigonometric equation and radical equation. Solving equations means finding the value (or set of values) or unknown variables contained in the equation.

Let us consider a single equation: $7x - 4 = 8 + 4x$.

We can represent it in another form by combining it as:

$$7x - 4x = 8 + 4$$
$$3x = 12$$
$$x = 4.$$

Our attention has been on single variable equation, and now we will focus more on two system equations. Linear equation can be zero, one, two, or an infinite number of solutions, but this will depend on whether the equations are dependent on one another or independent. The first sub-unit will cover the introductory aspect of systems of linear equations, while the second and third sub-units will focus on the different methods of solving systems of linear equations (substitution, addition and subtraction). Substitution is useful when one variable in an equation of the system has a coefficient of 1 or a coefficient that easily divides the equation. If one of the variables has a coefficient of 1, substitution is very useful and easy to do. However, many systems of linear equations are not quite so neat (not easy to calculate) and substitution can be difficult, thus an alternative method for solving systems of linear equations (the Addition/Subtraction method) is introduced.

2.0 OBJECTIVES

At the end of this unit, learners should be able to:

- Explain broadly, the meaning of equation system
- Solve problems of linear equation using substitution method.
- Solve problems of linear equation using the addition/subtraction method.
- Graph systems of equations solution.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO SYSTEMS OF EQUATIONS AND GRAPHING

As mentioned above, we have single variable and multi-variable equations. The solution to a single variable is limited, while that of multiple variable equations is infinite. This is because equations with more than one variable are underdetermined, and have more variables than the equation itself.

A system of equations is a set of two or more equations with the same variables. A solution to a system of equations is a set of values for the variable that satisfy all the equations simultaneously. To solve problems on systems of equation, we must first find the values for all the variables constituting the equation.

Example: Which of these set of figures is a solution to the equations below?

{(4, 5),(8, 3),(6, 4),(4, 6),(7, 2)}

$$y + 2x = 14 \qquad (1)$$

$$xy = 24 \qquad (2)$$

(4, 5) is a solution of the first equation, but not the second.

We need to find out why (4,5) satisfy the first equation, but failed to satisfy the second equation.

First, let 4 represent y, and 5 for x, in this context, our equation becomes:

$$4 + 2(5) = 4 + 10 = 14.$$

The above equation gives an answer of 14, thus it satisfy the first equation.

Now, let us see if it satisfies the second equation;

$$xy = 24.$$

Using the same representation of y = 4, and x = 5, let us insert the variables at the appropriate places to see if it satisfy the equation:

$$4 \text{ x } 5 = 20.$$

The above result of 20 is lower than the result of 24 which is the true outcome of the equation. Thus, (4,5) is not the solution to our equation (2).

Following the same trend, we got the following outcome for the other observations.

(8, 3) is a solution of the both equations.

(4, 6) is a solution of the second equation, but not the first.

(6, 4) is a solution of both equations.

(7, 2) is not a solution of either equation.

Thus, the solution set of the system is {(8, 3), (6, 4)}.
The option (8, 3) and (6, 4) were choosing because they fit into the equations above. For instance, consider y = 8 and x = 3 in the first and second equation, the resulting outcome will be:

$$8 + 2(3) = 8 + 4 = 14.$$

And

$$8 \times 3 = 24.$$

Thus, the option gave us the solution for both equations.

a. SOLVING SYSTEMS OF LINEAR EQUATIONS BY GRAPHING

We can also use graph to solve problems on linear equations with two variables.
A line graph is employed here, and all the points on the graph correspond to ordered pairs that satisfy the equation. Thus, when we graph two equations, all the points of intersection (the points which lie on both lines) are the points which satisfy both equations.
To solve a system of equations by graphing, we need to graph all the equations in the system. The point(s) at which all the lines intersect are the solutions to the system.
Example: Solve the following system by graphing:

$$4x - 6y = 12 \qquad (1)$$
$$2x + 2y = 6 \qquad (2)$$

From equation (1),

$$4x - 6y = 12$$
$$6y = 4x - 12$$
$$y = (4x/6) – 12/6$$
$$y = 2x/3 - 2$$

From equation (2),

$$2x + 2y = 6$$
$$2y = -2x + 6$$
$$y = -x + 3$$

In the above solutions, the slope for the first and second equation are 2/3 and -1 respectively, while the y-intercept are -2 and 3 respectively.

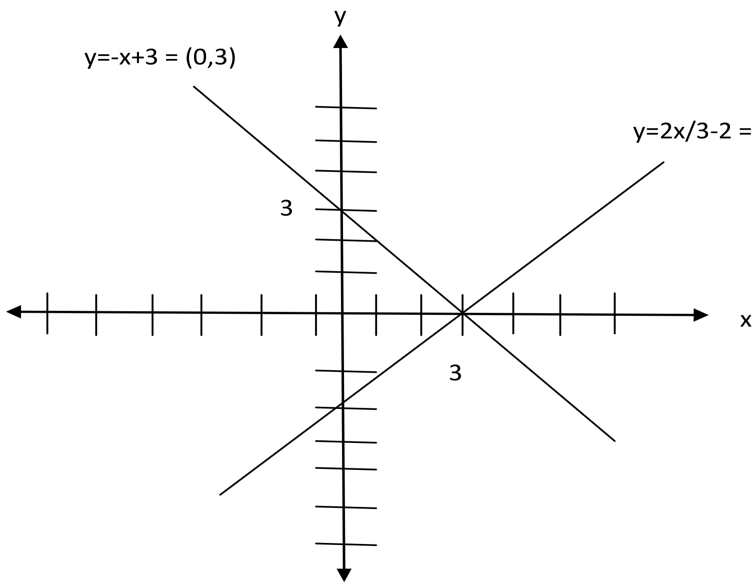

Figure1. Systems of Equation Graph

Since the two lines intersect at the point (0, 3), this point is a solution to the system. Thus, the solution set to the system of equations is {(0, 3)}.

To check, plug (0, 3) with y = 0, and x = 3 in to the equations y = 2x/3 – 2 and y = -x + 3.

y = -x + 3 = -3 + 3? Yes.

y = 2x/3 – 2 = 2(3)/3 – 2 = 6/3 – 2 = 2 – 2=0

b. CLASSIFICATION OF SYSTEMS

Graph of two linear equations could meet at different points and manner. The lines could intersect once, not intersect at all (be parallel), or intersect an infinite number of times (the two lines are the same).

1. If the two equations describe the same line, and the lines intersect in an infinite number of times, the system is dependent and consistent.
2. If the two equations describe lines that intersect once, the system is independent and consistent.
3. If the two equations describe parallel lines, and thus lines that do not intersect, the system is independent and inconsistent.

A system is consistent if it has one or more solutions. A system of two equations is dependent if all solutions to one equation are also solutions to the other equation.

SELF ASSESSMENT EXERCISE

Which ordered pairs in the set {(2, 5),(2, -9),(5, 2),(- 5, -2),(- 5, 12)} satisfy the following system of equations?

$$x + y = 7$$
$$x^2 + 3x = 10.$$

3.2 SOLVING SYSTEMS OF LINEAR EQUATION BY SUBSTITUTION

Solving systems of equation using substitution method is a quicker way of solving systems of equation problem, and it involves the isolation of one variable in one equation, and substitutes the resulting expression for that variable in the other equation.
Consider the following example below.

Example 1: Solve the following system, using substitution:

$$6x + 2y = 14 \quad (1)$$
$$4x = 16 - 6y \quad (2)$$

The easiest variable to isolate is y in the first equation; thus, from (1):

$$2y = -6x + 14$$
$$y = -3x + 7 \quad (3)$$

In the second equation, substitute y for its equivalent expression:

$$4x = 16 - 6(-3x + 7)$$

Solve the equation:

$$4x = 16 + 18x - 42$$
$$18x - 4x = 42 - 16$$
$$14x = 26$$
$$x = 26/14$$
$$x = 1.86 \quad (4)$$

Now substitute this x -value into the "isolated equation" to find y:

$$y =- 3(1.86) + 7 = -5.88 + 7 = 1.42. \quad (5)$$

Example 2: Solve the following system, using substitution:

$$2x + 4y = 36 \quad (1)$$
$$10y - 5x = 0 \quad (2)$$

It is easier to work with the second equation, because there is no constant term:

From (2), $5x = 10y$

$$x = 2y \quad (3)$$

In the first equation, substitute for x with its equivalent expression:

$$2(2y) + 4y = 36$$
$$4y + 4y = 36$$
$$8y = 36$$
$$y = 4.5 \quad (4)$$

Plug this y -value into (3) to find x:

$$x = 2y = 2(4.5) = 9 \quad (5)$$

Thus, the solution to the system is (9, 4.5).

Example 3: Solve the following system, using substitution:

$$2x - 4y = 12 \quad (1)$$
$$3x = 21 + 6y \quad (2)$$

It is easiest to isolate x in the second equation, since the x term already stands alone:

$$x = \underline{21 + 6y}$$

3

$x = 7 + 2y$ (3)

In the first equation, substitute for x with its equivalent expression:

$2(7 + 2y) - 4y = 12$

$14 + 4y - 4y = 12$

$14 = 12$

Since $14 \neq 12$, the system of equations has no solution. It is inconsistent (and independent). The two equations describe two parallel lines.

Example 4: Solve the following system, using substitution:

$10x = 4y - 68$ (1)

$2y - 5x = 34$ (2)

Either equation can be used to isolate the variable. We will isolate y in the second equation:

$4y - 10x = 68$

$2y - 5x = 34$

$y = (68 + 10x) \div 4$

$y = 17 + \frac{5}{2}x$

$2(17 + \frac{5}{2}x) - 5x = 34$

$34 + 5x = 34 + 5x$

$0 = 0$

Since 0 = 0 for any value of x, the system of equations has infinite solutions. The system is dependent (and consistent). The two equations describe the same line.

SELF ASSESSMENT EXERCISE

Solve the following system of equations:

(a) $2x + 2y = 4$ (b) $4y + 2x = 5$.
$5y – 3x = 6$ $x = 8 – 2y$.

3.3 SOLVING SYSTEMS OF EQUATION BY ADDITION OR SUBTRACTION

Another way of solving system of equation problem is the addition and subtraction method. This method is a little simpler and straight forward unlike the substitution method which although direct, but has a disadvantage of solving the equation by isolating a variable which involves dealing with exhaustive results.

In the addition or subtraction method, we add or subtract the two equations from one another in order to create a new equation with only one variable. In order for the new equation to have only one variable, the other variables must cancel out. In other words, we must first perform operations on each equation until one term has an equal and opposite coefficient as the corresponding term in the other equation.

We can produce equal and opposite coefficients simply by multiplying each equation by an integer.

Example 1: Add and subtract to create a new equation with only one variable:

$2x + 4y = 3$ (1)

$x + 3y = 13$ (2)

Here, we can multiply equation (2) by -2, and equation (1) by 1, and then add.

$2x + 4y = 3$

$-2x - 6y = -26$

Adding these two equations yields $-2y = -23$.

Example 2: Add and subtract to create a new equation with only one variable:

$4x - 2y = 16$ (1)

$7x + 3y = 15$ (2)

Here, we can multiply the first equation by 3 and the second equation by 2, and then add them together:

$12x - 6y = 48$

$14x + 6y = 30$

Adding these two equations yields $26x = 78$

We can add and subtract equations by the addition property of equality, since the two sides of one equation are equivalent, we can add one to one side of the second equation and the other to the other side.

Here are the steps to solving systems of equations using the addition/subtraction method:

1. Rearrange each equation so the variables are on one side (in the same order) and the constant is on the other side.
2. Multiply one or both equations by an integer so that one term has equal and opposite coefficients in the two equations.
3. Add the equations to produce a single equation with one variable.
4. Solve for the variable.
5. Substitute the variable back into one of the equations and solve for the other variable.
6. Check the solution; it should satisfy both equations.

Example 1: Solve the following system of equations:

$2y - 3x = 7$

$5x = 4y - 12$

Rearrange each equation:

$-3x + 2y = 7$ (1)

$5x - 4y = -12$ (2)

Multiply the first equation by 2:

$-6x + 4y = 14$

$5x - 4y = -12$

Add the equations:

$-x = 2$

Solve for the variable:

$x = -2$

Plug x = -2 into one of the equations and solve for y:

$-3(-2) + 2y = 7$

$6 + 2y = 7$

$2y = 1$

$y = 1/2$

Thus, the solution to the system of equations is (-2,1/2) .

Check:

$2(1/2) - 3(-2) = 7$

$5(-2) = 4(1/2) - 12$

Example 2: Solve the following system of equations:

$4y - 5 = 20 - 3x$

$4x - 7y + 16 = 0$

Rearrange each equation:

$3x + 4y = 25$ (1)

$4x - 7y = -16$ (2)

Multiply the first equation by 4 and the second equation by -3:

$12x + 16y = 100$

$-12x + 21y = 48$

Add the equations:

$37y = 148$

Solve for the variable:

$y = 4$

Plug y = 4 into one of the equations and solve for x:

$3x + 4(4) = 25$

$3x + 16 = 25$

$3x = 9$

$x = 3$

Thus, the solution to the system of equations is (3, 4).

Check:

$4(4) - 5 = 20 - 3(3)$? Yes.

$4(3) - 7(4) = - 16$? Yes.

Example 3: Solve the following system of equations:

$2x - 5y = 15$

$10y = 20 + 4x$

Rearrange each equation:

$2x - 5y = 15$ (1)

$-4x + 10y = 20$ (2)

Multiply the first equation by 2:

$$4x - 10y = 30$$
$$-4x + 10y = 20$$

Add the equations:

$$0 = 50.$$

Since $0 \neq 50$, this system of equations has no solutions. It is inconsistent (and independent). The equations describe two parallel lines.

Example 4: Solve the following system of equations:

$$6x + 14y = 16$$
$$24 - 9x = 21y$$

Rearrange each equation:

$$6x + 14y = 16 \qquad (1)$$
$$-9x - 21y = -24 \qquad (2)$$

Multiply the first equation by 3 and the second equation by 2:

$$18x + 42y = 48$$
$$-18x - 42y = -48$$

Add the equations:

$$0 = 0$$

Since 0 = 0 for any value of x, the system of equations has infinitely many solutions. Every ordered pair (x, y) which satisfies $6x + 14y = 16$ (or $-9x - 21y = -24$) is a solution to the system. The system is dependent (and consistent), and the two equations describe the same line.

SELF ASSESSMENT EXERCISE

Solve the following system of equations using the addition/substitution method:

(a) $3x + 4y = 12$ (b) $2x + 9y = 32$.
$18 - 2x = 6y$ $4x + y = -12$.

4.0 CONCLUSION

- A system of equation is a set of two or more equations with the same variables.
- System of linear equations can have zero, one or an infinite number of solutions, depending on whether they are consistent or inconsistent, and whether they are dependent or independent.
- When a linear equation of two variables is graphed as a line in the plane, all the points on the line correspond to ordered pairs that satisfy the equation.
- A quicker way to solving systems of linear equation is by substitution through the isolation of one variable in one question, and substituting the resulting expression for that variable in the other equation.
- Since substitution method isolates a variable, the addition/ subtraction method provides a better approach to solving systems of equation by adding the two equations in the system to create a new equation with only one variable.

5.0 SUMMARY

This unit focused on system of equations which is a set of two or more equations with the same variables. Sub-topics such as the graphing of equations which emphasize that all the points of intersection are the points which satisfy the equations were reviewed. The substitution method to solving systems of equations was also reviewed, and we said that it is a major advantage over the graphing method, as it isolates one variable in one equation, and substituting the resulting expression for that variable in the other equations. The problem with this approach is the isolation of variables which often involves dealing with exhausting results. The addition/subtraction method solves this caveat of substitution method as it involves the addition or subtraction of the two systems from each other in order to create a new equation with only one variable.

6.0 TUTOR-MARKED ASSIGNMENT

- Solve the following using the addition or subtraction method:

$$8x + 9y = 24$$
$$7x + 8y = 21$$

- Solve by graphing:

$$y - 12x = -4$$
$$y - 3 = -2(x + 6)$$

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 SIMULTANEOUS EQUATION

1.0 INTRODUCTION

The purpose of this unit is to look at the solution of elementary simultaneous linear equations. The term simultaneous equation refers to conditions where two or more unknown variables are related to each other through an equal number of equations. A simultaneous equation is two (or more) equations which contain more than one letter term. We will look at the steps and different methods involved in solving simultaneous equations.

2.0 OBJECTIVES

At the end of this unit, you will be able to

- Explain the concept of Simultaneous equations
- Differentiate between a simultaneous equations and a quadratic equation
- Solve simultaneous problem using the elimination and substitution method
- Graph the solution to a simultaneous equations

3.0 MAIN CONTENT

3.1 INTRODUCTION TO SIMULTANEOUS EQUATIONS

As defined above, simultaneous equations refer to a condition where two or more unknown variables are related to each other through an equal number of equations.

To solve a simultaneous equation problem, follow these two highlighted steps:

First, eliminate one of the letter terms and find the value of the remaining letter.

Second, substitute this value into the original equations to find the value of the other letter term.

A common way of solving simultaneous equations is by equating the coefficients. Before we do that, let us just have a look at a relatively straightforward single equation. The equation we are going to look at is $4x - y = 3$.

The above expression is a linear equation. It is a linear equation because there are no terms involving x^2, y^2 or xy, or any higher powers of x and y, thus making our equation a linear equation.

We can rearrange it so that we obtain y on its own on the left hand side. We can add y to each side so that we get:

$$4x - y + y = 3 + y.$$
$$4x = 3 + y$$

Now let's take 3 away from each side.

$$4x - 3 = y$$

This gives us an expression for y:

$$y = 4x - 3.$$

Suppose we choose a value for x, say x = 1, then y will be equal to:

$$y = 4(1) - 3 = 1$$

Suppose we choose a different value for x, say x = 2.

$$y = 4(2) - 3 = 5$$

Suppose we choose another value for x, say x = 0.

$$y = 4(0) - 3 = -3$$

For every value of x we can generate a value of y.

Similarly, consider the example below.

$$5x - y = 6$$
$$2x + y = 8$$

For the first equation, the solution becomes:

$$5x - y = 6$$
$$y = 5x - 6.$$

In the above equation, the y-intercept is -6, while the slope is 5.

For the second equation, the solution is:

$$2x + y = 8$$
$$y = -2x + 8.$$

The y-intercept in the above result is 8, while the slope is -2.

For this set of equations, there is but a single combination of values for x and y that will satisfy both. Either equations considered separately has an infinitude of valid (x,y) solutions, but together there is only one solution. Plotted on a graph, this condition becomes obvious:

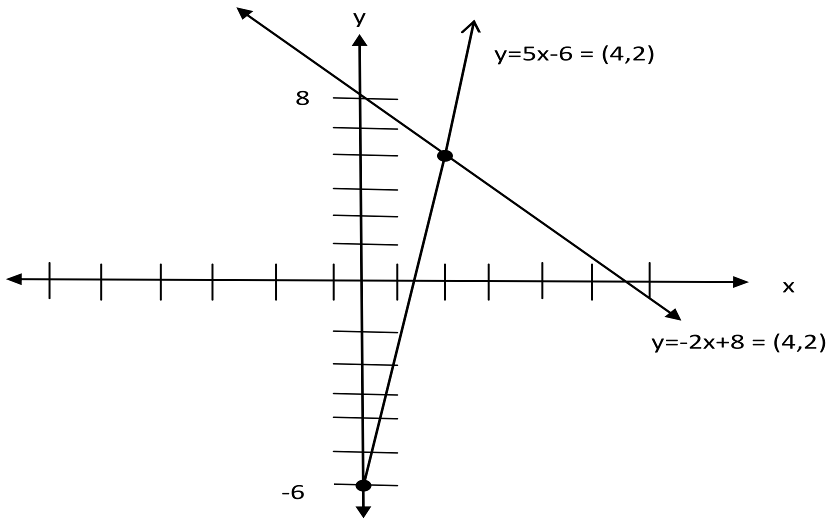

Figure1. Simultaneous Equation Diagram

Each line is actually a continuum of points representing possible x and y solution pairs for each equation. Each equation, separately, has an infinite number of ordered pair (x,y) solutions. There is only one point where the two linear functions 5x - y = 6 and 2x + y = 8 intersect (where one of their many independent solutions happen to work for both equations), and that is the point where x is equal to a value of 2 and y is equal to a value of 4.

Usually, though, graphing is not a very efficient way to determine the simultaneous solution set for two or more equations. It is especially impractical for systems of three or more variables.

SELF ASSESSMENT EXERCISE

Graph the following simultaneous equation problem

$$3x + 7y = 27$$
$$5x + 2y = 16.$$

3.2 SUBSTITUTION METHOD OF SIMULTANEOUS EQUATION

Substitution method is described as one of the simplest methods of solving simultaneous equation. Take, for instance, our two-variable example below:

$$x + 2y = 20 \qquad (1)$$
$$2x – 2y = -4 \qquad (2)$$

In this method, we can manipulate one of the equations such that one variable is defined in terms of the other, i.e, defining *y* in terms of *x*. From there, we take this new definition of one variable and substitute it for the same variable in the other equation.

Let us continue with our equation above:

Making y the subject of the formula:

$$2y = 20 – x.$$
$$y = 10 – 0.5x$$

Now, substitute y = 10 – 0.5x into the equation (2)

$$2x - 2(10 - 0.5x) = -4$$

Now that we have an equation with just a single variable (x), we can solve it using "normal" algebraic techniques:

$$2x - 20 + x = -4$$

Combining the like terms gives:

$$3x - 20 = -4$$

Adding 20 to both sides:

$$3x = 16$$

Dividing both side by 3

$$x = 5.3.$$

Now that x is known, we can plug this value into any of the original equations and obtain a value for y. Or, to save us some work, we can plug this value (5.3) into the equation we just generated to define y in terms of x, being that it is already in a form to solve for y:

$$x = 5.3$$

Substitute it into the equation $y = 10 - 0.5x$

$$y = 10 - 0.5(5.3)$$
$$y = 10 - 2.7$$
$$y = 7.3$$

Example: Solve this simultaneous equations:

$$2x + 5y = 27 \qquad (1)$$
$$2x + 2y = 12 \qquad (2)$$

Solution:

In both equations, notice we have the same number of x.

Make x the subject of the formula in equation 1:

$$2x = 27 - 5y$$
$$x = 27/2 - 5y/2$$
$$x = 13.5 - 2.5y \qquad (3)$$

Substitute this value into equation (2) to get the value for y:

$$2(13.5 - 2.5y) + 2y = 12$$
$$27 - 5y + 2y = 12$$

Collecting like terms yielded:

$$-3y = -15$$

Dividing both sides by -3:

$$y = 5.$$

Now, we impute $y = 5$ into our equation (1) to get the actual value for x or we can impute it in our equation (3) to make things faster. In order to facilitate easy understanding, we will consider both options.

Impute $y = 5$ into equation (1).

$$2x + 5(5) = 27$$
$$2x + 25 = 27$$
$$2x = 27 - 25$$
$$2x = 2$$
$$x = 1.$$

Imputing y = 5 in equation (3).

$$x = 13.5 – 2.5(5)$$
$$x = 13.5 – 12.5$$
$$x = 1.$$

Both options gave us the same value for x.

SELF ASSESSMENT EXERCISE

Solve this pair of simultaneous equations using the substitution method:

$$3a + 3b = 21$$
$$6a – 4b = 22.$$

3.3 ELIMINATION (ADDITION AND SUBTRACTION) METHOD OF SOLVING SIMULTANEOUS EQUATIONS

The addition or subtraction method for solving simultaneous equation is also called the elimination method. It involves adding or subtracting equations from one another and even multiplying each equation by a constant if need be. While the substitution method may be the easiest to grasp on a conceptual level, there are other methods of solution available to us. One of such method is the addition or subtraction method, whereby equations are added or subtracted from one another for the purpose of canceling variable terms. This is done when the coefficients of y or x in each equation are the same and the signs of y or x are opposite, then add or subtract each side of the equations to eliminate y or x.

Referring to algebra rule that state that we can add any number to an equation so long as we add equivalent number to both side of the equation, and thus this is incapable of altering the outcome of the equation. An option we have, then, is to add the corresponding sides of the equations together to form a new equation. Since each equation is an expression of equality, the left-hand side of one equation corresponds to the left-hand side of the other. The equation is valid so long as we add the two equations' right-hand sides together as well.

An example of additive simultaneous problem is giving below.

Example: $4x + 2y = 14 \qquad (1)$

$2x – 2y = -6 \qquad (2)$

Notice that y in both equations have opposite signs, thus adding the two equations together eliminates y, thus resulting in 6x = 8.

We can solve the above equation completely:

$$6x = 8$$
$$x = 1.33 \qquad (3)$$

To find y, we need to impute x = 1.33 into equation (1)

$$4(1.33) + 2y = 14$$
$$5.32 + 2y = 14$$
$$2y = 14 – 5.32$$
$$2y = 8.68$$
$$y = 4.34$$

The solution to the above problem is thus x = 1.33, y = 4.34.

Example: $4x - 4y = 12$ (1)

$3x + 4y = 8$ (2)

Subtracting equation (2) from (1) gives:

$x = 4$ (3)

We impute x = 4 into equation (1) in order to find y.

$4(4) - 4y = 12$

$16 - 4y = 12$

$-4y = 12 - 16$

$-4y = -4$

Dividing through by -4:

$y = 1.$

We impute y = 1 into equation 1 to confirm the value for x.

$4x - 4y = 12$

$4x - 4(1) = 12$

$4x - 4 = 12$

$4x = 12 + 4$

$4x = 16$

$x = 4.$

We can see that the value for our x corresponds to the initial value of 4, thus confirming that our result is in line.

Another style of solving using the addition/ subtraction method is presented below.

Example: $7x + 2y = 47$ (1)

$5x - 4y = 1$ (2)

Notice that the coefficients of y in equation (1) and (2) are not the same, thus making it impossible to eliminate them through addition or subtraction. The technique used in this case is multiplying equation (1) with the value of y in equation (2), and multiplying equation (2) with the value of y in equation (1).

Based on the question above, we multiply equation (1) with 4 which belongs to y in equation (2), and we multiply equation (2) with 2 which belongs to y in equation (1) to get a uniform model.

$(7x + 2y = 47) \times 4$

$(5x - 4y = 1) \times 2$

$28x + 8y = 188$ (3)

$10x - 8y = 2$ (4)

Eliminating equation (3) and (4) by adding the two equations together yielded:

$38x = 190$

$x = 5.$

Now that we have a value for x, we can substitute this into equation (2) in order to find y.

Substituting:

$5x - 4y = 1$

$5(5) - 4y = 1$

$25 - 4y = 1$

$-4y = -24$

Dividing both side by -4

$y = 6.$

SELF ASSESSMENT EXERCISE

Solve this pair of simultaneous equations using the substitution method:

$3x + 7b = 27$

$5x + 2y = 16.$

4.0 CONCLUSION

- Simultaneous equation refers to a condition where two or more unknown variables are related to each other through an equal number of equations.
- To solve a simultaneous equation, first eliminate one of the letter terms and find the value of the remaining letter.
- Substitution method involves the transformation of one of the equations such that one variable is defined in terms of the other.
- The elimination method (Addition/subtraction) is done by adding or subtracting the equations from one another for the purpose of cancelling variable terms.
- When the coefficient of y or x in each equations are the same, and the signs of y and x are opposite, then adding or subtracting each side of the equation will eliminate y or x.

5.0 SUMMARY

This unit focused on simultaneous equation, which is a condition where two or more unknown variables are related to each other through an equal number of equations. In order to explain this topic, the different methods of solving simultaneous equation problems were utilized. The substitution method defines one variable in terms of the other, while the elimination method (addition/subtraction) involves adding or subtracting the equations from one another in order to form a unified equation.

6.0 TUTOR-MARKED ASSIGNMENT

Solve the following equations using the substitution and the elimination method:

(a) $4x + 5y = 1$

$-4x - 5y = -1$

(b) $5y - 2z = 5$

$-4y + 5z = 37$

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UNIT 3 QUADRATIC EQUATION

1.0 INTRODUCTION

This unit deals with equations involving quadratic polynomials (an expression of more than two algebraic terms). Polynomials of degree two Quadratic equations are equations of the form y = $ax^2 + bx + c$. Usually, they are arranged so that the square part goes first, then the part with the variable, and some constant, while the right side is equal to zero.
The first sub-unit introduces us to the concept of quadratic equation, while the second sub-unit focuses on factoring quadratic equation. Here, we factor equations of the form $x^2 + bx + c = 0$, splitting the expression into two binomials (an expression of the sum or the difference of two terms) and using the zero product property of quadratic equation to solve the problem. Not all equations $ax^2 + bx + c = 0$ can be easily factored. Thus, we need a formula to solve for x. This is the quadratic formula, and it is the focus of sub- unit three. Finally, in the last sub-unit, we will learn how to graph quadratic equations of the form $y = ax^2 + bx + c$ by completing the square: adding and subtracting a constant to create a perfect square trinomial (algebraic expression consisting of three terms) within our equation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Explain the concept of quadratic equation.
- Solve quadratic equation problems using the factorization and quadratic formula.
- Graph the solution of the quadratic equation.
- State the steps involved in solving quadratic equation.
- Identify the number of discriminate quadratic solution.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO QUADRATIC EQUATION

The name quadratic comes from "quad", which means square; this is because the variables get squared (like x^2). It is also called "Equation of Degree 2" because of the "2" on the *x*. Any equation of the form $ax^2 + bx + c = 0$ is referred to as a quadratic equation; where x represents an unknown, and *a*, *b*, and *c* are constants with *a* not equal to 0. However, in the event that $a = 0$, then we can say the equation is linear, and thus not quadratic. The parameters *a, b,* and *c* are referred to as the quadratic coefficient, the linear coefficient, the constant or free term respectively.

The quadratic equation only contains powers of *x* that are non-negative integers, and thus, a polynomial equation, and in particular it is a second degree polynomial equation since the greatest power is two. Quadratic equations can be solved through factorization (the resolution of an entity into factors such that when multiplied together, they provide the original entity).

STEPS INVOLVED IN SOLVING QUADRATIC EQUATION

Many quadratic equations with one unknown variable may be solved by using factoring techniques in conjunction with the Zero Factor Property as described below:

1. Write the quadratic equation in standard form.
2. Factor the quadratic polynomial into a product of linear factors.
3. Use the Zero Factor Property to set each factor equal to zero.
4. Solve each of the resulting linear equations.

The resulting solutions are solutions of the original quadratic equation.

For example, Solve for x: $x^2 + 8x + 16 = 0$.

Stage 1: $x^2 + 8x + 16 = 0$

Stage 2: $(x + 4)(x + 4)$

Stage 3: $(x + 4)(x + 4) = 0$

$x + 4 = 0$ or $x + 4 = 0$

Stage 4: $x = -4$.

SELF ASSESSMENT EXERCISE

Why is quadratic equation referred to as equation of degree two?

3.2 FACTORISING QUADRATIC EQUATION

Product of two binomials (expression of the sum or difference of two terms) is a way we can factor quadratic equations. After factoring, we are left with an equation of the form $(x + a)(x + b) = 0$, where *a* and *b* are integers.

The zero product property states that, if the product of two quantities is equal to 0, then at least one of the quantities must be equal to zero. Thus, if $(x + a)(x + b) = 0$, either $(x + a) = 0$ or $(x + b) = 0$. Consequently, the two solutions to the equation are $x = -a$ and $x = -b$.

Example 1: Solve for x: $x^2 – 5x – 14 = 0$

$x^2 – 5x – 14$

$(x – 7)(x + 2) = 0$

Note: we picked -7 and +2 because when we add the two values together, we get -5, and when we multiply the two values together, we get -14, thus the two values give the solution to the problem.

$x - 7 = 0$ or $x + 2 = 0$

$x = 7$ or $x = -2$.

Example 2: Solve for x: $x^2 + 6x + 5 = 0$

$x^2 + 6x + 5 = 0$

$(x + 1)(x + 5) = 0$

$x + 1 = 0$ or $x + 5 = 0$

$x = -1$ or $x = -5$.

Example 3: Solve for x: $2x^2 – 16x + 24 = 0$

$2x^2 – 16x + 24 = 0$

$2(x^2 – 8x + 12) = 0$

$2\{(x – 2)(x – 6)\} = 0$

$x – 2 = 0$ or $x – 6 = 0$

$x = 2$ or $x = 6$.

The above examples worked out easily. However, sometimes, it gets harder to solve some quadratic equations; in such situations, solving the quadratic equations using the quadratic formula is recommended.

SELF ASSESSMENT EXERCISE

Solve for x given: $4x^2 – 16x - 20 = 0$

3.3 SOLVING USING THE QUADRATIC FORMULA

Trinomials (algebraic expression consisting of three terms) are not always easy to factor. In fact, some trinomials cannot be factored. Thus, we need a different way to solve quadratic equations. Herein lays the importance of the quadratic formula:

Given a quadratic equation $ax^2 + bx + c = 0$, the solutions are given by the equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1: Solve for x: $x^2 + 8x + 15.75 = 0$.

We need to assign the numbers in the equation to the letters in the formula above.

Here, $a = 1$, since it's a number assigned to the first element of the question " i.e. x^2".

$b = 8$, since it is assigned to the next figure in the question, and $c = 15.75$.

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(15.75)}}{2(1)}$$

$$x = \frac{-8 \pm \sqrt{64 - 63}}{2}$$

$$x = \frac{-8 \pm \sqrt{1}}{2}$$

Since the square root of 1 is 1, we continue with the solution.

$$x = \frac{-8 + 1}{2}, \text{ or } x = \frac{-8 - 1}{2}$$

$$x = \frac{-7}{2} \text{ or } x = \frac{-9}{2}$$

x = -3.5 or x = -4.5.

Example 2: Solve for x: $3x^2 - 10x - 25 = 0$

a = 3, *b* = -10, and *c* = -25.

$$x = \frac{-(-10) \pm \sqrt{-10^2 - 4(3)(-25)}}{2(3)}$$

$$x = \frac{10 \pm \sqrt{100 + 300}}{6}$$

$$x = \frac{10 \pm \sqrt{400}}{6}$$

Since the square root of 400 is 20, we continue with the solution.

$$x = \frac{10 + 20}{6}, \text{ or } x = \frac{10 - 20}{6}$$

$$x = \frac{30}{6} \text{ or } x = \frac{-10}{6}$$

x = 5 or x = - 1.7.

Example 3: Solve for x: -3x2 – 24x – 48 = 0

Where a = -3, b = -24, and c = -48.

$$x = \frac{-(-24) \pm \sqrt{-24^2 - 4(-3)(-48)}}{2(-3)}$$

$$x = \frac{24 \pm \sqrt{576 - 576}}{-6}$$

$$x = \frac{24 \pm \sqrt{0}}{-6}$$

Since the square root of 0 is 0, we continue with the solution.

$$x = \frac{24 + 0}{-6}, \text{ or } x = \frac{24 - 0}{-6}$$

$$x = \frac{24}{-6} \text{ or } x = \frac{24}{-6}$$

$$x = -4.$$

Example 4: Solve for x: $2x^2 - 4x + 7 = 0$.

Where $a = 2$, $b = -4$, and $c = 7$.

$$x = \frac{-(-4) \pm \sqrt{-4^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 - 56}}{4}$$

$$x = \frac{4 \pm \sqrt{-40}}{4}$$

Since we cannot take the square root of a negative number, there are no solutions.

THE DISCRIMINANT

As we have seen, there can be 0, 1, or 2 solutions to a quadratic equation, depending on whether the expression inside the square root sign ($b^2 - 4ac$) is positive, negative, or zero. This expression is specially referred to as "The Discriminant".

- If the discriminant is positive, that is; if $b^2 - 4ac > 0$, then the quadratic equation has two solutions.
- If the discriminant is zero, that is; if $b^2 - 4ac = 0$, then the quadratic equation has one solution.
- If the discriminant is negative, that is; if $b^2 - 4ac < 0$, then the quadratic equation has no solutions.

Example: How many solutions does the quadratic equation $2x^2 + 5x + 2$ have?

Where $a = 2$, $b = 5$, and $c = 2$.

$b^2 - 4ac = 5^2 - 4(2)(2) = 25 - 16 = 9 > 0$.

Thus, the quadratic equation has 2 solutions.

SELF ASSESSMENT EXERCISE

Solve the quadratic equation $4x^2 + x - 20 = 0$

Solve the quadratic equation using the quadratic formula: $12x^2 + 9x + 17 = 0$.

3.4 GRAPHING QUADRATIC FUNCTIONS

A quadratic function is a function of the form $y = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are real numbers.

1. INTERCEPTS OF A QUADRATIC FUNCTION

The y-intercept is given by $x = 0$: $y = a(0^2) + b(0) + c = c$, since any variable multiplied by zero becomes zero. Thus, the y-intercept is $(0, c)$.

The x-intercept is given by $y = 0$: $0 = ax^2 + bx + c$. Thus, the x-intercept(s) can be found by factoring or by using the quadratic formula.

In addition, the discriminant gives the number of x-intercepts of a quadratic function, because it gives us the number of solutions to $ax^2 + bx + c = 0$. If $b^2 - 4ac > 0$, there are 2 solutions to $ax^2 + bx + c = 0$ and consequently 2 x-intercepts. If $b^2 - 4ac = 0$, there is 1 solution to $ax^2 + bx + c = 0$, and consequently 1 x-intercept. If $b^2 - 4ac < 0$, there are no solutions to $ax^2 + bx + c = 0$, and consequently no x-intercepts. The graph of the function does not cross the x-axis; either the vertex (the highest or lowest point of an equation) of the parabola (a U shaped curve) is above the x-axis and the parabola opens upward, or the vertex is below the x-axis and the parabola opens downward.

2. COMPLETING THE SQUARE

A quadratic function in the form $y = ax^2 + bx + c$ requires some level of concentration to graph (it is not always simple to graph). To make the function easier to graph, we should convert the above equation to the form $y = a(x - h)^2 + k$. This is done through the completing the square method: adding and subtracting a constant to create a perfect square trinomial (an equation consisting of three terms) within our equation.

A perfect square trinomial is of the form $x^2 + 2dx + d^2$. In order to "create" a perfect square trinomial within our equation, we must find d. To find d, divide b by $2a$. Then square d and multiply by a, and add and subtract ad^2 to the equation (we must add and subtract in order to maintain the original equation). We now have an equation of the form $y = ax^2 + 2adx + ad^2 - ad^2 + c$. Factor $ax^2 + 2adx + ad^2$ into $a(x + d)^2$, and simplify $-ad^2 + c$.

Here are the steps for completing the square, given an equation $ax^2 + bx + c$:

1. Compute $d = b/2a$. Cross multiply, and you get $b = 2ad$. Substitute this value for b in the above equation to get $ax^2 + 2adx + c$.
2. Add and subtract ad^2 to the equation. This produces an equation of the form $y = ax^2 + 2adx + ad^2 - ad^2 + c$.
3. Factor $ax^2 + 2adx + ad^2$ into $a(x + d)^2$. This produces an equation of the form $y = a(x + d)^2 - ad^2 + c$.
4. Simplify $ad^2 + c$. This produces an equation of the form $y = (x - h)^2 + k$.
5. Check by plugging the point (h, k) into the original equation. It should satisfy the equation.

Example1: Complete the square: $y = x^2 + 8x – 14$.

$a = 1$, $b = 8$ and $c = -14$.

1. $d = 8/2(1) = 4$.

2. $ad^2 = 16$, so $y = (x^2 + 8x + 16) - 16 - 14$.
3. $y = (x + 4)^2 - 16 - 14$.
4. $y = (x + 4)^2 - 30$.
5. Check: $-30 = (-4)^2 + 8(-4) - 14$. → $-30 = -30$.

Example 2: Complete the square: $y = 4x^2 + 16x$.
$a = 4$ and $b = 16$, while $c = 0$.

1. $d = 16/2(4) = 2$.
2. $ad^2 = 16$. $y = (4x^2 + 16x + 16) - 16$.
3. $y = 4(x + 2)^2 - 16$.
4. $y = 4(x + 2)^2 - 16$.
5. Check: $-16 = 4(-2)^2 + 16(-2)$. → $-16 = -16$.

Example 3: Complete the square: $y = 2x^2 - 28x + 100$.
$a = 2$. $b = -28$ and $c = 100$

1. $d = -28/2(2) = -7$.
2. $ad^2 = 98$. $y = (2x^2 - 28x + 98) - 98 + 100$.
3. $y = 2(x - 7)^2 - 98 + 100$.
4. $y = 2(x - 7)^2 + 2$.
5. Check: $2 = 2(7)^2 - 28(7) + 100$. → $2 = 2$.

3. GRAPHING QUADRATIC EQUATION

Let us plot the quadratic equation diagram using the model: $x^2 + 3x - 4 = 0$.
Solving this problem using factorization, we have:

$$x^2 + 3x - 4 = (x + 4)(x - 1) = 0.$$

$$x = -4 \text{ or } x = 1.$$

We already know that the solutions are $x = -4$ and $x = 1$. How would our solutions look in the quadratic formula? Using $a = 1$, $b = 3$, and $c = -4$.

$$x = \frac{-(3) \pm \sqrt{3^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9 + 16}}{2}$$

$$x = \frac{-3 \pm \sqrt{25}}{2}$$

Square root of 25 is 5, we continue with the solution.

$$x = \frac{-3 + 5}{2}, \text{ or } x = \frac{-3 - 5}{2}$$

$$x = \frac{2}{2} \text{ or } x = \frac{-8}{2}$$

$$x = 1 \text{ or } x = -4.$$

Then, the solution is $x = -4$ and $x = 1$.

Let us assume we have $ax^2 + bx + c = y$, and we want to plug zero in for y, the corresponding x-values are the x-intercepts of the graph. So solving $ax^2 + bx + c = 0$ for x means, among other things, that you are trying to find x-intercepts. Since there were two solutions for $x^2 + 3x – 4 = 0$, there must then be two x-intercepts on the graph. Graphing, we get the curve below:

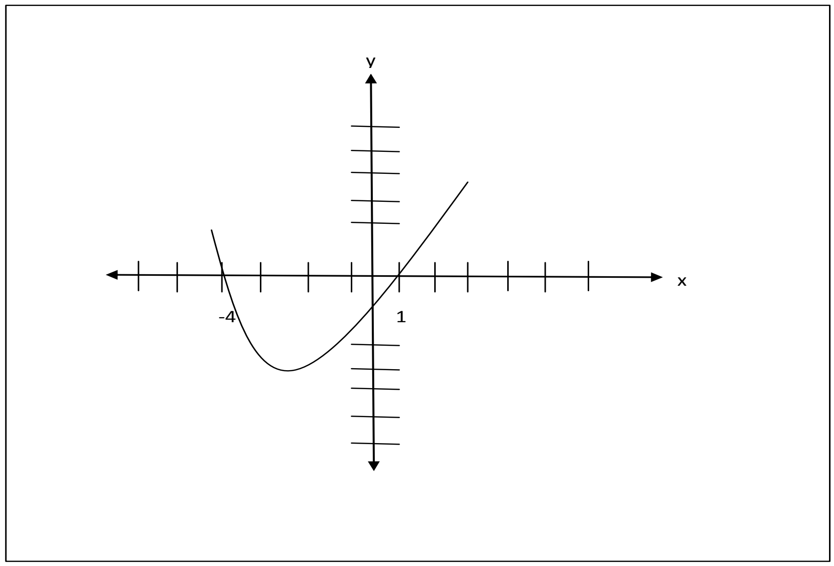

Figure1. Quadratic Equation Diagram for $x = -4$ and $x = 1$.

As you can see, the x-intercepts match the solutions, crossing the x-axis at $x = –4$ and $x = 1$. This shows the connection between graphing and solving.

SELF ASSESSMENT EXERCISE

Solve and graph $2x^2 – 4x – 3 = 0$.

4.0 CONCLUSION

- The word "quad" which means "square" is the origin of the name quadratic equations because the variables are squared. Quadratic equation is also called equation of degree 2.
- Quadratic equation is any equation having the form $ax^2 + bx + c = 0$.
- Quadratic equation is called "univariate because it involves only one unknown.
- The zero product property states that if the product of two quantities is equal to zero (0), then at least one of the quantities must be equal to zero. That is, if $(x + d)(x + e) = 0$, either $(x + d) = 0$ or $(x + e) = 0$.

5.0 SUMMARY

This unit focused on quadratic equations. Quadratic equation implies any equation having the form $ax^2 + bx + c = 0$; where x represents an unknown, and a, b and c are constants with $a \neq 0$. Sub-topics such as the steps involved in solving quadratic equations, the factorization system as well as the graphical representation of quadratic equation solution were all reviewed.

6.0 TUTOR-MARKED ASSIGNMENT

- Complete the square for the equation:
  $y = -x^2 + 10x - 1$, assuming $y = 0$.

- Find the y-intercept of:
  $y = 2x^2 + 4x - 6$
  $y = 6x^2 - 12x - 18$.
- Using the quadratic formula, solve for x:
  $3x^2 - 14x + 8 = 0$
  $4x^2 - 1 = 0$
- Using the factorization method, solve for x:
  $x^2 - x - 6 = 0$
  $x^2 - 9x + 20 = 0$

7.0 REFERENCES/FURTHER READINGS

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MODULE 3: SET THEORY, LOGARITHMS & PARTIAL DERIVATIVES

UNIT 1 SET THEORY

3.2 INTRODUCTION

The purpose of this unit is to discuss set theory which is a branch of mathematics that studies the collection of objects usually called sets. In order to discuss this topic, sub-topic such as Venn diagram which was introduced in 1880 will be reviewed. Venn diagram is attributed to John Venn, and it is a set diagram that shows all possible logical relations between finite collections of sets. Also, the properties of set operations and symbols used in set operations will be discussed extensively in order to facilitate easy understanding.

2.0 OBJECTIVES

At the end of this unit, you will be able to:

- Explain the concept of set theory
- Differentiate between intersect and union.
- Be familiar with different set symbols.
- State the properties of set operations
- Expalin the concept of Venn diagram and its application to set theory.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO SET THEORY

A set is a collection of things of same kind that belong to or are used together. For example we could say a Chess set, this set is made up of chess pieces and a chess board used for playing the game of chess. We often hear people say set of cloths, Jewelry set, set of furniture etc. All these are collection of objects grouped together as a common singular unit. In mathematics, the same concept applies, as set is regarded as a collection of objects(usually numbers) that can stand alone in their own right but when collected together, they form an item e.g. 1, 2, 3 are separate numbers, however, they can be grouped together and placed in braces "{}" (which is the set notation), so that we have {1, 2, 3} which forms a single set of size 3 because they are 3 objects or members called "elements" in the braces and they are separated by comas.

Set theory is a branch of mathematics which deals with the formal properties of sets as units (without regard to the nature of their individual constituents) and the expression of other branches of mathematics in terms of sets. Set theory is also the branch of mathematics that studies sets, which are collections of objects. Set theory begins with a fundamental binary relation between an object o and a set X. If o is a member (or element) of X, we write $o \in X$.

A binary correlation between two sets is the subset relation, which we can also denote as "set inclusion". If all the members of set X are also members of set Y, then X is a subset of Y, and we denote it as $X \subseteq Y$. For example, {4,5} is a subset of {4,5,6}, but {4,7} is not. From this definition, we can deduce that a set is a subset of itself.

However, for cases where one wishes to rule this out, the term proper subset is used. If X is called a proper subset of Y, then this can only happen if and only if X is a subset of Y, but Y is not a subset of X.

Let us briefly look at the different operations involved in set theory.

- **Union**: The union of sets X and Y is denoted as $X \cup Y$. This is the set of objects that are members of X, or Y, or both. The union of {9,10,11} and {10,11,12} is the set {9,10,11,12}.
- **Intersection**: The intersection of the set X and Y is denoted as $X \cap Y$, which is the set of all objects that are members of both X and Y. The intersection of {9,10,11} and {10,11,12} is the set {10,11}.
- **Set Difference**: This is the set of U and X, and it's denoted as $U \setminus X$, which is the set of all members of U that are not members of X. The set difference {9,10,11}\{10,11,12} is {9}, while conversely, the set difference {10,11,12}\{9,10,11} is {12}. When X is a subset of U, the set difference $U \setminus X$ is also called the "complement" of X in U. In this case, if the choice of U is clear from the context, the notation X^c is sometimes used instead of $U \setminus X$, particularly if U is a "universal set" as the study of Venn diagram.
- **Symmetric Difference**: The symmetric difference of sets X and Y is denoted by $X \triangle Y$ or $X \ominus Y$ which is the set of all objects that are a member of exactly one of X and Y (elements which are in one of the sets, but not in both). For instance, for the sets {9,10,11} and (10,11,12}, the symmetric difference set is {9,12}. It is the set difference of the union and the intersection, $(X \cup Y) \setminus (X \cap Y)$ or $(X \setminus Y) \cup (Y \setminus X)$.

- **Cartesian Product**: The Cartesian product of X and Y is denoted by X × Y, which is the set whose members are all possible ordered pairs (*x*,*y*) where *x* is a member of X and *y* is a member of Y. The Cartesian product of {11, 12} and {orange, mango} is {(11, orange), (11mango), (12, orange), (12, mango)}.
- **Power Set**: Power set of a set X is the set whose members are all possible subsets of X. For example, the power set of {11, 12,13} is {11}, {12}, {13}, {11,12,13}.

SELF ASSESSMENT EXERCISE

What is the intersection between X = {10,15,20,25) and Y = {20,25,30,40}?

3.2 SYMBOLS AND PROPERTIES OF SET OPERATION

3.2.1 COMMON SET THEORY SYMBOLS

The symbols that are used in set theory are presented in the table below with examples where necessary.

Table1. Set Theory Symbols

Symbol	Symbol Name	Meaning/Definition	Example
{ }	Set	a collection of elements	X = {3,7,9,14}, Y = {9,14,28}
|	Such that	So that	X = {x | x∈R, x<0}
X ∩ Y	Intersection	objects that belong to set X and set Y	X ∩ Y = {9,14}
X ∪ Y	Union	objects that belong to set X or set Y	X ∪ Y = {3,7,9,14,28}
X ⊆ Y	Subset	subset has fewer elements or equal to the set	{9,14,28} ⊆ {9,14,28}
X ⊂ Y	Proper subset / strict subset	subset has fewer elements than the set	{9,14} ⊂ {9,14,28}
X ⊄ Y	Not subset	left set not a subset of right set	{9,66} ⊄ {9,14,28}
X ⊇ Y	Superset	set X has more elements or equal to the set Y	{9,14,28} ⊇ {9,14,28}
X ⊃ Y	Proper superset / strict superset	set X has more elements than set Y	{9,14,28} ⊃ {9,14}
X ⊅ Y	Not superset	set X is not a superset of set Y	{9,14,28} ⊅ {9,66}
X = Y	Equality	both sets have the same members	X={3,9,14}, Y={3,9,14}, X=Y
X^c	Complement	all the objects that do not belong to set X	
X \ Y	Relative complement	objects that belong to X and not to Y	X = {3,9,14}, Y = {1,2,3}, X \ Y = {9,14}

X – Y	Relative complement	objects that belong to X and not to Y	X = {3,9,14}, Y = {1,2,3}, X - Y = {9,14}
X ∆ Y	Symmetric difference	objects that belong to X or Y but not to their intersection	X = {3,9,14}, Y = {1,2,3}, X ∆ Y = {1,2,9,14}
X ⊖ Y	Symmetric difference	objects that belong to X or Y but not to their intersection	X = {3,9,14}, Y = {1,2,3}, X ⊖ Y = {1,2,9,14}
x∈X	Element of	set membership	X={3,9,14}, 3 ∈ X
x∉X	Not element of	no set membership	X={3,9,14}, 1 ∉ X
(x,y)	ordered pair	collection of 2 elements	
X×Y	Cartesian product	set of all ordered pairs from A and B	
|X| or #X	Cardinality	the number of elements of set A	X={3,9,14}, |X|=3 or #X=3
Ø	Empty set	Ø = { }	Z = {Ø}
{U}	Universal set	set of all possible values	

3.2.2. PROPERTIES OF SET OPERATIONS

The properties of set theory are similar to that of arithmetic operations. Addition and union of sets are both operations of putting things together, and they turn out to have similar properties. The same can also be said about multiplication and intersection of sets. These properties are very useful to simplify expressions and computations.

Examples of properties in arithmetic are **commutative** ($x+y = y+x$ or $x \times y = y \times x$) and **distributive** {3× (2+4) = (3×2)+(3×4)} properties. Addition and multiplication are commutative because “x + y = y + x, and x × y = y × x” for all numbers. Multiplication is distributive over addition because “x × (y + z) = x × y + x × z” for all numbers.

The numbers 0 and 1 are important in arithmetic because they are identity elements; they do not change numbers that they operate on. For example, in addition, we have “0 + x = x”, and in multiplication, “1 × x = x”. The null set ∅ plays a role like 0 since X∪∅ = X, and the universal set **U** or the sample space **S** plays a role like 1 since X∩**U** = X.

Parentheses or brackets are used to clarify an expression involving set operations in the same way that they are used to clarify arithmetic or algebraic expressions. The expression 4 + 8 × 9 is ambiguous because you do not know whether to perform the addition first or the multiplication first. By putting in parentheses, you can distinguish between (4 + 8) × 9 = 108 and 4 + (8 × 9) = 76. In set theory, X∪Y∩Z can be different, depending on whether union or intersection is done first, so parentheses or brackets are necessary.

By convention, complementation is done before the other operations. It is not necessary to use parentheses in an expression like $X \cup Y^C$ to mean the intersection of the set X with the complement of the set Y. However, it is necessary to use parentheses or brackets in the expression $(X \cup Y)^C$ to mean that the union must be performed before the complementation.

The most important properties of set operations are given below. The events X, Y, and Z are all taken from the same universal set **U**.

1. **The Identity Laws**: In addition (+), 0 acts as an identity since "0 + x = x", and in multiplication (×), 1 acts as identity since "1 × x = x". In set theory, the empty set acts as identity for union, and the sample set **S** or universal set **U** acts as identity for intersection.
   $X \cup \emptyset = X$
   $\mathbf{U} \cap \emptyset = X$

2. **The Involution Laws:** The operation of complementation in set theory behaves somewhat like finding the additive or multiplicative inverse of a number (the inverse of the inverse of a number is the number itself). For addition, – (– x) = x, and for division, 1 / (1 / x) = x.
   $(X^C)^C = X$.

3. **Idempotent Laws**: A number is an idempotent if the number operated on itself gives the number back again. In addition, 0 is idempotent since 0 + 0 = 0 and in multiplication 1 is idempotent since 1 × 1 = 1. In set theory, every set is idempotent with respect to both union and intersection.
   $X \cup X = X$
   $X \cap X = X$

4. **Commutative Laws**: This law says that the order of the operation doesn't matter.
   $X \cup Y = Y \cup X$
   $X \cap Y = Y \cap X$

5. **Associative Laws**: This law says that in a sequence of two or more instances of the same operation, it does not matter which one is done first. In arithmetic, addition and multiplication are associative since "(x + y) + z" is the same as "x + (y + z)" and "(x × y) × z" is the same as "x × (y × z)".
   $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
   $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

6. **Distributive Laws**: In arithmetic, multiplication is distributive over addition, "x × (y + z) = x × y + x × z" for all numbers, but addition is not distributive over multiplication since, for example, 3 + (2 × 5) is not the same as (3 + 2) × (3 + 5). Set theory is different, and we have two distributivity laws: intersection is distributive over union, and union is distributive over intersection.
   $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
   $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

7. **Complements Law**: These laws show how sets and their complements behave with respect to each other.
   $X \cup X^C = U$
   $X \cap X^C = Ø$
   $U^C = Ø$
   $Ø^C = U$.

8. **DeMorgan's Laws**: These laws show how complementation interacts with the operations of union and intersection.
   $(X \cup Y)^C = X^C \cap Y^C$
   $(X \cap Y)^C = X^C \cup Y^C$

SELF ASSESSMENT EXERCISE
List and explain five properties of set operations.

3.3 VENN DIAGRAM

Venn diagrams show relationships between sets in picture form. Put differently, a Venn diagram or set diagram shows all possible logical relations between a finite collection of sets. The mathematical invention was named after its inventor a British logician John Venn in 1880. A rectangle is used to represent the universe of all the elements we are looking at, and circles are used to represent sets.
Consider the Venn diagram below, let us assume you have twelve(12) friends having the following names, Kaychi, Farouk, Gbemi, Mary, Caro, Shola, Bisi, Chinyere, Aishatu, Lade, Itse and David. Among your friends, you have the first five of them in your football team; then you have another five counting from the back (from David) as members of your handball team. However, you have two of these friends (Farouk and David) in both teams, i.e., they play both football and handball; and you also have two others (Shola and Bisi) who do not belong to any team. The twelve (12) friends represent the universal set

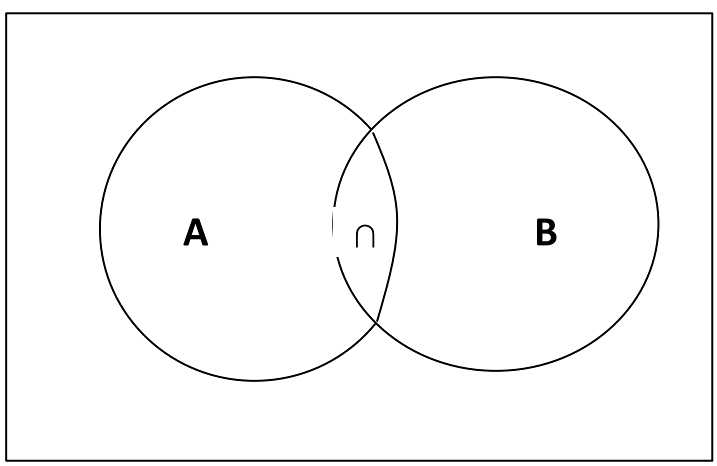


Figure1. Venn diagram
Let A represent the football team and B the handball team.

The combine area of set A and B is called the union (U) of A and B. The union in this case is made up of all ten of your friends mentioned above. Then the area in both A and B, where the two sets overlap (∩) is called the intersection of A and B, which is denoted by A∩B. For example, the intersection of the two sets is not empty, because there are points that show the combination of two people who participate in both games i.e. Farouk and David.

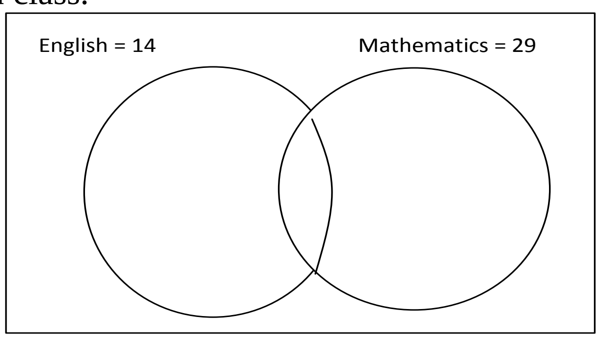
Venn diagrams normally comprise overlapping circles. The interior of the circle symbolically represents the elements of the set, while the exterior represents elements that are not members of the set.

Example: Out of 40 students, 14 are taking English, and 29 are taking Mathematics.

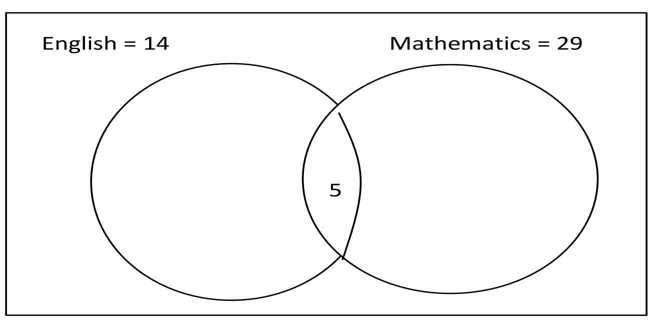
a. If 5 students are in both classes, how many students are in neither class?
b. How many are in either class?
c. What is the probability that a randomly chosen student from this group is taking only the Mathematics class?

Solution: There are two classifications in this universal set of 40 students: English students and Mathematics students.

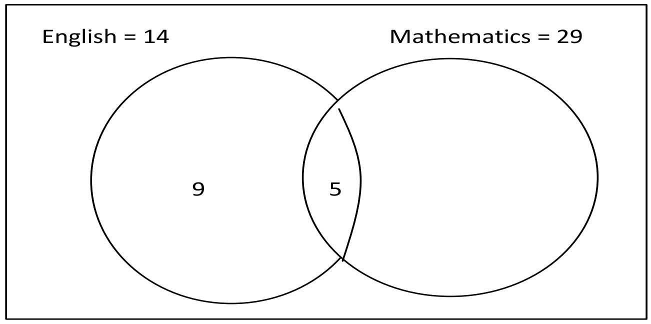
First, we draw the universe for the forty students, with two overlapping circles labeled with the total in each class.



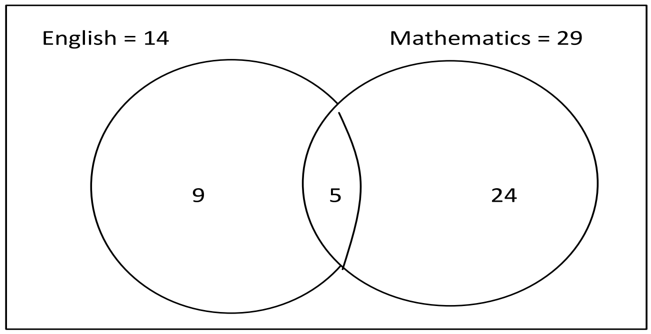
Since 5 students are taking both classes, we put 5 in the overlap, and the diagram becomes:



Now, we have accounted for 5 of the 14 English students, leaving 9 students taking English but not Mathematics. The next step is to try and put 9 in the English only part of the English circle.



Now, we have also accounted for 5 of the 29 Mathematics students, leaving 24 students taking Mathematics but not English. Now, we need to put 24 in the Mathematics only part of the Mathematics circle.



This tells us that the total of 9 + 5 + 24 = 38 students are in either English or Mathematics or both. This leaves two students unaccounted for, so they must be the ones taking neither class.

From the above Venn diagram, we can deduce the following:

- 2 students are taking neither of the subjects.
- There are 38 students in at least one of the classes.
- There is a 24/40 (60%) probability that a randomly-chosen students in this group is taking Mathematics but not English.

SELF ASSESSMENT EXERCISE

Using Venn diagram, show the relationship between A = {1,2,3,4,5} and B = {3,4,5,6,7}.

4.0 CONCLUSION

- Set theory is the branch of mathematics that studies sets, which are collections of objects.
- A derived binary relation between two set is the subset relation, which is also called set inclusion.
- A is called proper subset of B if and only if A is a subset of B, but B is not a subset of A.
- Venn diagram shows the relationship between sets in picture form.
- The combination of set A and B is called union (U) of A and B.
- The overlapping of set A and B is called the intersection of A and B.

- The interior of a set circle symbolically represents the elements of the set, while the exterior represents elements that are not members of the set.

5.0 SUMMARY

This unit focused on the set theory. Set theory is a branch of mathematics which deals with the formal properties of sets as units (without regard to the nature of their individual constituents) and the expression of other branches of mathematics in terms of sets. Sub-unit 3.1 introduced us to set theory, the next sub-unit, focused on the symbols and properties of set operations, explaining the different properties such as the identity law, the involution laws, idempotent laws, e.t.c., while the last sub-unit introduced the concept of Venn diagram which uses diagram to show the relationship between sets.

6.0 TUTOR-MARKED ASSIGNMENT

Out of 100 students in a class, 65 are offering economics as a course while 55 are offering mathematics

a. If 20 students are offering both courses, how many students are offering neither of the courses?
b. How many are offering either of the courses?
c. What is the probability that a randomly chosen student from the class is offering only economics?

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UNIT 2 LOGARITHMS

3.0 INTRODUCTION

Logarithm proffer solution to the question of how many of a particular number must be multiplied to get another number; i.e., how many 4's do we multiply together to get 256?, the answer to this question is 4 *x* 4 *x* 4 *x* 4 = 256. This simply means we have to multiply 4 by itself 4 times in order to get 256. This kind of logarithm is base 4, and we write it as $\log_4 256 = 4$. Like many types of functions, the exponential (a number representing the power to which a number is to be raised, i.e., 2^3; 3 is the exponent) function is the inverse of logarithm function, and that is the focus of this unit.

In order to do justice to this unit, the first sub-unit introduces us broadly to the meaning of the logarithmic function. This sub-unit also addresses the domain and range of a logarithmic function, which are inverses of those of its corresponding exponential function. The second sub-unit presents the two special logarithmic functions (the common logarithmic function and the natural logarithmic function). The common logarithm is $\log_{10}x$, while the natural logarithm is $\log_e x$. Sub-unit three deals with the properties of logarithms. The eight properties discussed in this section are helpful in evaluating logarithmic expressions by hand or using a calculator. They are also useful in simplifying and solving equations containing logarithms or exponents. The last sub-unit focuses on solving exponential and logarithm functions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Explain the concept Logarithm
- Know the difference between an exponent and logarithm
- Differentiate between the common and the natural logarithm function
- State and explain the properties of logarithm.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO LOGARITHMS

Whenever we are talking of logarithms, we can see it as the opposite of exponents. For example, consider two variables *x* and *y*, such that $x = a^y$, this is an exponential function, which can be written in log form $\log_a x = y$. In general, if $a^y = x$, then $\log_a x = y$.

From the above specification, $x = a^y$ is the exponential function, while $\log_a x = y$ is the equivalent logarithm function. The value of "*a*" is the base of the logarithms, just as "*a*" is the base in the exponential expression "$_a x$". And, just as the *"a"* in an exponential is always positive (greater than zero) and not equal to one (1), so also is the base "*a*" for a logarithm is always positive and not equal to 1. Whatever is inside the logarithm is called "the argument" of the log.

The "log" denotes a common logarithm (base = 10), while "*ln*" denote a natural logarithm (base = e).

For example, $\log_2 64 = 6$ because $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

$\log_5 1/125 = -3$ because $5^{-3} = 1/125$ (using calculator).

To evaluate a logarithmic function, we have to determine what exponent the base must be taken to in order to yield the number *x*. Sometimes the exponent will not be a whole number. If this is the case, the use of logarithm table or a calculator is recommended.

Examples:

$y = \log_3 27 = 3$. Since the cube root of 3 = 27, this solves the problem. Using similar approach to other problems, we provide a solution to logarithm problems.

$y = \log_5 1/625 = -4$.

$y = \log_7 2401 = 4$.

SIMPLIFYING THE LOGS

The log is normally equal to some number, which we can call *y* like some of the example cited above. Naming this equation makes it easy for us to solve.

For example, given $\log_2 16$. In order to simplify it, we need to re-write the model as:

$\log_2 16 = y$.

$2^y = 16$

This means $\log_2 16$ (also known as *y*), is that power by which 2 can be raised to in order to get 16, and the power that does this is 4.

$2^4 = 2 \times 2 \times 2 \times 2 = 16$.

We can apply similar rule to solving the following problems.

Simplify $\log_5 125$.

The relationship says that, since $\log_5 125 = y$, then $5^y = 125$. This means that the given $\log_5 125$ is equal to the power *y* that, when put on 5, turns 5 into 125. The required power is 3, because:

$5^3 = 5 \times 5 \times 5 = 125$.

Then, $\log_5 125 = 3$.

Simplify $\log_7 7$.

The Relationship says that, since $\log_7 7 = y$, then $7^y = 7$. But $7 = 7^1$, so $7^y = 7^1$, and $y = 1$. That is: $\log_7 7 = 1$

SELF ASSESSMENT EXERCISE

Simplify $\log_3 2187$.

3.2 TWO SPECIAL LOGARITHM FUNCTIONS

In both exponential functions and logarithms, any number can be the base. However, there are two bases that are used so frequently; these are the common and natural logarithms. On our scientific calculators, special keys are included denoting the common and the natural logs (the "log" and "*ln*" keys).

A common logarithm is any logarithm with base 10. Recall that our number system is base 10; there are ten digits from 0-9; and place value is determined by groups of ten. The common log is popular, and is usually written as "log*x*"

The function $f(x) = \log_{10} x$ is called the common logarithmic function. The common log function is often written as $f(x) = \log x$. When log is written without a base, the base is assumed to be 10.

Examples: Simplify log1000.

Since $1000 = 10 \text{ x} 10 \text{ x } 10 = 10^3$

$\text{Log}1000 = \log 10^3 = 3$, because:

$\text{Log}1000 \rightarrow 10^y = 1000 = 10^3$

$y = 3$.

Natural logarithms are different from common logarithms. While the base of a common logarithm is 10, the base of a natural logarithm is the special number *e* which is mostly denoted as "$ln(x)$". Although this looks like a variable, it represents a fixed irrational number approximately equal to 2.718281828459. (It continues without a repeating pattern in its digits.) The natural log function is often abbreviated to $f(x) = lnx$.

Example: Simplify *ln*100.

Since 100 is a whole number and *e* is not a whole number, then it is likely that 100 is going to give a cumbersome analysis, and thus we use the calculator to calculate the value of *e* instead, which gives us 4.6052, rounded to four decimal places.

Similarly, *ln*230 and *ln*0.04 is equivalent to 5.4381 and -3.2189 respectively.

SELF ASSESSMENT EXERCISE

1) Differentiate between a common and the normal logarithm.
2) Simplify $\log_{50} 62500000$.

3.3 PROPERTIES OF LOGARITHMS

Logarithms have the following properties:

1. Since $a^0 = 1$, and $a^1 = a$:

   Property A: $\log_a 1 = 0$.

   Property B: $\log_a a = 1$.

2. Since a^x and $\log_a x$ are inverse:

   Property C: $\log_a a^x = x$.

   Property D: $a^{\log_a x} = x$.

3. Since $a^p a^q = a^{p+q}$ and $a^p/a^q = a^{p-q}$:

   Property E: $\log_a(pq) = \log_a p + \log_a q$

   Property F: $\log_a(p/q) = \log_a p - \log_a q$

4. Since $\log_a(M^n) = \log_a(M+M+M\ldots M) = \log_a M + \log_a M + \log_a M + \ldots + \log_a M = n\log_a M$.

   Property G: $\log_a(M^n) = n\log_a M$

5. Logarithms have an additional property called property H, and a property H_1, which is a specific case of property H.

   Property H: $\log_a M = \log_b M/\log_b a$, where b is any base.

   Property H_1: $\log_a M = \log M/\log a$.

Application of Properties

The above listed properties can be used to evaluate logarithm functions. Property H_1 is especially useful when evaluating logarithm with a calculator: since most calculators only evaluate logarithms with base 10, we can evaluate $\log_a M$ by evaluating $\log M/\log a$.

Example 1: Solve $\log_5 10 + \log_5 20 - \log_5 8 =?$

$= \log_5((10 * 20)/8)$

$= \log_5(200/8)$

$= \log_5 5^2$

$= 2.$

SELF ASSESSMENT EXERCISE

Using the property H_1, solve for $\log_7 2401 + \log_7 343 - \log_7 49$.

3.4 SOLVING EXPONENTIAL AND LOGARITHM FUNCTIONS

To solve an equation containing a variable exponent, we need to isolate the exponential quantity. Then we take a logarithm, to the base of the exponent, of both sides.

For example: Solve for x:

$5^x = 20$
$\log_5 5^x = \log_5 20$
$x = \log_5 20.$
$x = \log 20/\log 5$
$x = 1.8614$.(using calculator).

Example 2: Solve for x:

$5(5^{2x}) = 80$

Dividing both sides by 5 gives:

$5^{2x} = 16$
$\text{Log}_5 5^{2x} = \log_5 16$
$2x = \log_5 16$
$2x = \log 16/\log 5$
$2x = 1.7227$
$x = 0.8614.$

Solving Equations Containing Logarithms

In order to solve an equation containing a logarithm, the use of the logarithms properties which involves combining the logarithmic expressions into one expression is needed. We then convert it to exponential form and evaluate. After this, we check the solution(s). Consider the following examples.

Example 1: Solve for x in the following equation,
$\log_3 3x + \log_3(x - 2) = 2.$

Using the logarithm property E: $\log_a(pq) = \log_a p + \log_a q$
We can go ahead and solve this problem:

$\log_3 3x + \log_3(x–2) = 2$
$\log_3(3x * (x-2)) = \log_3 3^2$
$3x * (x- 2)=3^2$
$3x^2 -6x= 9$
$3x^2 - 6x – 9 = 0$
$3(x^2 – 2x – 3) = 0$
$3(x – 3)(x+ 1) = 0$
$x = 3$ or $x = -1$.

Remember, we cannot take a log value of a negative number, thus, -1 is not a solution, but 3.

To check this answer, let us plug $x = 3$ into the equation:

$\log_3 3(3) + \log_3(3 - 2) = 2.$
$\log_3 9 + \log_3 1 = 2.$
$\text{Log}_3 9/1 = \log_3 9 = \log_3 3^{2} = 2.$

Example 2: Solve for x: $2\log_{(2x+1)}2x + 4 – \log_{(2x+1)}4 = 2.$

$\log_{(2x+1)}(2x + 4)^2 – \log_{(2x+1)}4 = 2.$

$\log_{(2x+1)}(2x + 4)^2/4 = 2.$
$(2x+1)^2 = (2x + 4)^2/4$
$(2x+1)^2 = 4x^2 + 16x + 16)/4$
$4x^2 + 4x + 1 = x^2 + 4x + 4$
$3x^2 – 3 = 0.$
$3(x^2 – 1) = 0$
$3(x + 1)(x – 1) = 0.$
$x = 1$ or $x = -1$.

Similarly, since we cannot find the log value of a negative number, our answer is 1. We can confirm this by plugging 1 into the equation:

$x = 1$in :$2\log_{(2(1)+1)}2(1) + 4 – \log_{(2(1)+1)}4 = 2.$

$2\log_{(2+1)}(2 + 4) – \log_{(2+1)}4 = 2.$

$2\log_3 6 – \log_3 4 = 2.$

$\log_3 6^2 – \log_3 4 = 2.$

$\log_3 36 – \log_3 4 = 2.$

$\mathrm{Log}_3 36/4 = \log_3 9 = 2.$

Thus 1 is the solution to the above equation.

SELF ASSESSMENT EXERCISE

Solve for x: $\log_2(x) + \log_2 (x-2) = 3$

4.0 CONCLUSION

Logarithm answers the question of how many of one variable is to be multiplied together to get another number.

Logarithm function is the inverse of exponential function, i.e., if $a^y= x$, then $\log_a x = y$.

Simplifying the logs makes it easy for us to solve.

The two special logarithm functions comprises of the common logarithm function and the natural logarithm function.

The common logarithm function is any logarithm with base 10, while the base of a natural logarithm function is denoted by "*e*", which is mostly written as"*lnx"*.

In order to solve an equation containing an exponent variable, exponential quantity of the model must be isolated, then taking logarithm to the base of the exponent of both sides.

In order to solve an equation containing a logarithm, we need to consider the properties of logarithm which compresses the expressions into one.

5.0 SUMMARY

This unit focused on the logarithm function. Logarithm implies a quantity representing the power to which a fixed number (the base) must be raised to produce a given number. It is also the inverse of an exponent. The sub-topics focuses on the introduction, properties and the two special logarithms function respectively, while the last sub-unit presented detailed information on how to solve logarithm and exponential problem.

6.0 TUTOR-MARKED ASSIGNMENT

- Solve for x: $\log_8 x + \log_8 7x - 9$
- Evaluate log36 and $ln62$

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UNIT 3 PARTIAL DERIVATIVES

1.0 INTRODUCTION

The purpose of this unit is to discuss partial derivative which is the solution for differentiation (i.e., the differentiation of variable x with respect to variable y). The term partial derivation refers to the solution or the outcome of differentiation. Differentiation entails differencing an equation with respect to a particular variable while holding others constant. For example, z = 2xy; the partial derivative of z with respect to x is 2y, while the partial derivative for z with respect to y is 2x. The first section(3.1) of this unit, handles the introductory part of the topic, while the second section (3.2) treats the higher order partial derivatives; the third and fourth sections (3.3 and 3. 4) of this unit, focuses on the chain and product rule of partial differentiation respectively.

2.0 OBJECTIVES

At the end of this unit, you will be able to

- Discuss the concept of differentiation.
- State the difference between differentiation and derivation.
- Solve problems involving higher order derivatives.
- Use both the chain and the product rule to solve differentiation problem.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO PARTIAL DERIVATIVES

Derivatives (something based or dependent on another source) are the outcome of differentiation, that is, differentiation is a process, while derivative is the result obtained from process.

For instance consider the differentiation of the single variable model $y = 5x^3$.

$$ðy/ðx = 15x^2.$$

$15x^2$ is the derivative obtained by differentiation.

Partial derivative is a function of several variables, and it is its derivative with respect to one of those variables, with the others held constant.

The introductory analysis is the partial differentiation of a single variable, and it is worthwhile moving to the analysis of more than one variable where there are more than one variable with all other variables constrained to stay constant when a variable is differentiated. The derivative is carried out in the same way as ordinary differentiation.
For example, given the polynomial in variables x and y, that is: $f(x,y) = ax^2 + by^2$
The partial derivative with respect to x is written as:

$$ðf(x,y) /ðx = 2ax.$$

The partial differentiation with respect to y is written as:

$$ðf(x,y) /ðy = 2by.$$

The problem with functions of more than one variable is that there is more than one variable. In other words, what do we do if we only want one of the variables to change, or if we want more than one of them to change? Even if we want more than one of the variables to change they are then going to be an infinite amount of ways for them to change. For instance, one variable could be changing faster than the other variables in the function.

Partial Derivatives with One Variable
In order to differentiate with respect to one variable, we will differentiate partially by allowing one variable to vary at the expense of the other.
Consider $f(x,y) = 3x^2 2y^3$ and lets determine the rate at which the function is changing at a point, (a,b), if we hold y fixed and allow x to vary and if we hold x fixed and allow y to vary.
In order to solve the above problem, we'll start by looking at the case of holding y fixed and allowing x to vary.
We can get the partial derivative of x:

$$ð(x,y)/ðx = 6x2y^3.$$

Thus, the function of the model is denoted as: $fx(x,y) = 6x2y^3$.
Now, let's do it the other way. We will now hold x fixed and allow y to vary. We can do this in a similar way.

$$ð(x,y)/ðy = 6y^2 3x^2.$$

Note that these two partial derivatives are called the first order partial derivatives. Just as with functions of one variable we can have derivatives of all orders. We will be looking at higher order derivatives later in this unit.
Let's look at some examples.

Example 1: $f(x,y) = x^3 + 2\sqrt{y} - 8$
The first thing to do is to take the derivative with respect to x, and thus rendering y constant. The partial derivative with respect to x is:

$$fx(x,y) = 3x^2.$$

Notice that the second and the third variables $(2\sqrt{y} – 8)$ differentiate to zero in this case. This is because those variables have been made constant and constants always differentiate to zero. Remember that since we are differentiating with respect to x here,

we are going to treat all our *y*'s as constants. This means that terms that only involve *y*'s will be treated as constants and hence will differentiate to zero.
Second step therefore is to take the derivative with respect to *y*. Just like the first instance where we took the derivatives of *x* and making *y* redundant; we will re-apply the same concept here on *y* by making *x* redundant and equal to zero. Here is the partial derivative with respect to *y*.

$$fy(x,y) = 1/\sqrt{y}.$$

Example 2: $v = 2x^2y - 20y^2z^4 + 40x - 7\tan(4y)$
With this function we have got three first order derivatives to compute. We will do the partial derivative with respect to *x* first. Since we are differentiating with respect to *x* we will treat all *y*'s and all *z*'s as constants. This means that the second and fourth terms will differentiate to zero since they only involve *y*'s and *z*'s.
This first term contains both *x*'s and *y*'s and so when we differentiate with respect to *x* the y will be thought of as a multiplicative constant and so the first term will be differentiated just as the third term will be differentiated.
The partial derivative with respect to x is given as:

$$ðv/ðx = 4xy + 40.$$

Applying the same principles, we can find the partial derivative with respect to *y*:

$$ðv/ðy = 2x^2 - 40yz^4 - 28\sec^2(4y).$$

Applying the same principles, we can find the partial derivative with respect to *z*:

$$ðv/ðz =- 80y^2z^3.$$

Thus, we are have successfully differentiated the three variables in the equation.

SELF ASSESSMENT EXERCISE
Differentiate $y = 2a^48b^2 - 12c3b^3 + 5bc$.

3.2 HIGHER ORDER PARTIAL DERIVATIVES

We have higher order derivatives with functions of one variable; we also have higher order derivatives of functions of more than one variable. In this sub-unit, we will be focusing on more than once variable higher order analysis.
The derivative of a function $a = f(b)$ is itself a function $a^1 = f^1(b)$. The second order derivative of $f(b)$ can be written as $a^1 = f^{11}(b)$ or f^2(b).
Let us look at a function of two variables $f(x,y)$, since both of the first order partial derivatives are also functions of *x* and *y* we could in turn differentiate each with respect to *a* or *b*. This means that for the case of a function of two variables there will be a total of four possible second order derivatives.

$$(f_a)_a = f_{aa} = ð/ða(ðf/\mathrm{d}a) = ð^2f/ða^2$$

$$(f_a)_b = f_{ab} = ð/ðb(ðf/\mathrm{d}a) = ð^2f/ðbða$$

$$(f_b)_a = f_{ba} = ð/ða(ðf/\mathrm{d}b) = ð^2f/ðaðb$$

$(f_b)_b = f_{bb} = ð/ðb(ðf/db) = ð^2f/ðb^2$

The second and third order partial derivatives can be referred to as mixed partial derivatives since we are taking derivatives with respect to more than one variable. Note as well that the order that we take the derivatives is given by the notation in front of each equation. If we are using the subscripting notation, e.g. f_{ab}, then we will differentiate from left to right. In other words, in this case, we will differentiate first with respect to *a* and then with respect to *b*. With the fractional notation, e.g. $ð^2f/ðbða$, it is the opposite. In these cases we differentiate moving along the denominator from right to left. So, again, in this case we differentiate with respect to x first and then y.

Example 1: Compute the second order partial derivatives of $f(a,b) = a^2b + 5a\sin(b)$.
We find the first order derivatives in order to find the higher order derivatives.

$$f_a(a,b) = ð(a,b)/da = 2ab + 5\sin(b).$$

$$f_b(a,b) = ð(b,a)/db = a^2 + 5a\cos(b).$$

The second order derivative is given as:

$$F_{aa}(a,b) = 2b$$
$$f_{ab}(a,b) = 2a + 5\cos(b)$$
$$f_{ba}(a,b) = 2a + 5\cos(b)$$
$$f_{bb}(a,b) = -5x\sin(b).$$

Notice that in the two examples, $f_{ab}(a,b) = f_{ba}(a,b)$. Indeed, this is typically always the case. Thus it does not matter if we take the partial derivative with respect to *a* first or with respect to *b* first.

Example 2: Find all the second order derivatives for $f(a,b) = \cos(10a) – a^4e^{5b} + 4b^2$
First, we need to find the first order derivatives:

$$F_a(a,b) = ð(a,b)/da = -10\sin(10a) – 4a^3e^{5b}.$$

$$F_b(a,b) = ð(a,b)/db = -5a^4e^{5b} + 8b.$$

Now, we are done with the first order derivative, we need to find the second order derivatives:

$$f_{aa}= -20\cos(10a) – 4e^{5b}$$
$$f_{ab} = -10ae^{5b}$$
$$f_{ba} = -10ae^{5b}$$
$$f_{bb} = -25a^4e^{5b} +8.$$

SELF ASSESSMENT EXERCISE

Compute the first and the second order derivative of $f(x,y) = 10xy^2 – 15y^3x8^5 + 15y\sin(x)$

3.3 THE CHAIN RULE OF PARTIAL DIFFERENTIATION

The chain rule is a rule for differentiating compositions of functions. It is a method of computing a derivative of the functional disposition of two of more functions.

So far, we have been using the standard chain rule for functions of one variable throughout the last two sub-units, let us look at the extension to functions of more than one variable.

If a function *G* depends on a variable *a*, which also depends on another variable *b*, ($G = f(a(b))$, then the rate of change of *G* with respect to *b* can be referred to as the rate at which *f* changes with respect to *b*, multiplied by the rate of change of *f* with respect to *b*.

Let us review the notation for the chain rule for functions of one variable before diving deeper.

The chain rule states formally that:

$D\{f(g(x))\} = f^t(g(x))\ g^t(x)$. (Don't get confused by the notation, it is to enhance retention).

The above expression is rarely used when applying the chain rule to specific problems. Instead, an intuitive approach is favored. For example, it is sometimes easier to think of the functions *f* and g as "layers" of a problem. Function *f* is the "outer layer"" and function *g* is the "inner layer". Thus, the chain rule tells us to first differentiate the outer layer, leaving the inner layer unchanged (the term $f^t(g(x))$, then differentiate the inner layer (the term $g^t(x)$). This process will become clearer as we apply them to some problems.

Example 1: Differentiate $y = (3x + 1)^2$.

The outer layer is "the square" and the inner layer is (3*x*+1). Differentiate the square first, leaving (3*x*+1) unchanged. Then differentiate (3*x*+1).) Thus:

$Đ(3x + 1)^2$

$= 2(3x + 1)^{2-1}Đ(3x + 1)$

By differentiating further, we get:

$= 2(3x + 1)*(3)$

$= 6(3x + 1)$.

Example 2: Differentiate $y = (2 – 4x + 5x^4)^{10}$.

The outer layer is the 10th power, and the inner layer is $(2 – 4x + 5x^4)$. Differentiate the 10th power first, leaving $(2 – 4x + 5x^4)$ unchanged. Then differentiate $(2 – 4x + 5x^4)$.

$Đ(2 – 4x + 5x^4)^{10} = 10(2 – 4x + 5x^4)^{10-1} * Đ(2 – 4x + 5x^4)$

$= 10(2 – 4x + 5x^4)^{10-1} * (-4 + 20x^3)$

$= 10(2 – 4x + 5x^4)^9 * (20x^3 - 4)$.

Example 3: Differentiate y = sin(5*x*)

The outer layer is the "sin function" and the inner layer is (5*x*). Differentiate the "sin function" first, leaving (5*x*) unchanged. Then differentiate (5*x*).

$Đ\{\sin(5x)\} = \cos(5x) * Đ(5x)$

$= \cos(5x) * (5)$
$= 5\cos(5x)$.

SELF ASSESSMENT EXERCISE

Using the chain rule, differentiate $y = \cos 2(x^3)$.

3.4 THE PRODUCT RULE OF PARTIAL DIFFERENTIATION

The difference between the chain and the product rule is that the chain rule deals with a function of a function, that is, $ð/ðx\{f(g(x))\}$, while the product rule deals with two separate functions multiplied together; that is, $ð/ðx\{f(x)*g(x)\}$.

The product rule is a formal rule for differentiating problems where one function is multiplied by another. The rule follows from the limit definition of derivative and is given by:

$$Đ\{f(x)g(x)\} = f(x)g^t(x) + f^t(x)g(x).$$

Let us apply this rule to some problems in order to see how it works.

Example 1: Differentiate $y = (2x^3 + 5x – 1)*(4x + 2)$

$y^t= (2x^3 + 5x – 1)* Đ(4x + 2) + Đ(2x^3 + 5x – 1)*(4x + 2)$
$= (2x^3 + 5x – 1)* (4) + (6x^2 + 5)*(4x + 2)$
$= 8x^3 + 20x – 4 + 24x^3 + 20x + 12x^2 + 10.$
$= 32x^3 + 12x^2+40x + 6.$

Example 2: Differentiate $x^{-4}(5 + 7x^{-3})$.

$y^t= x^{-4}* Đ(5 + 7x^{-3}) + Đ(x^{-4}) * (5 + 7x^{-3})$
$= x^{-4}*(-21x^{-4}) + (-4x^{-5})*(5 + 7x^{-3})$
$= -21x^{-8} + (-20x^{-5} – 28x^{-8})$
$= -21x^{-8} – 20x^{-5} – 28x^{-8}$
$= -49x^{-8} – 20x^{-5}$
$= -49/x^8 – 20/x^5$

Example 3: Differentiate $y = 10x^2 + \{\sin(x)*(\cos(x))\}$.

$y^t= Đ10x^2 + \sin x*Đ(\cos x) + Đ(\sin x)*\cos x$
$= 20x +\sin x*(-\sin x) + (\cos x)*\cos x$
$= 20x + \cos^2x – \sin^2x.$

An alternative answer can be given using the trigonometry identity: $\cos^2x – \sin^2x = \cos 2x$. Thus, our answer changes to $20x + \cos 2x$.

SELF ASSESSMENT EXERCISE

Using the product rule, differentiate $y = (x + 5)^4* \cos 3x$.

4.0 CONCLUSION

- Differentiation is the process while derivatives are the outcome of the process.
- Partial derivative is the situation where there is more than one variable with all the other variables constrained to stay constant with respect to the differentiated one.
- Higher order derivatives are the second or third order derivative, which is the derivative of the derivative of a function.
- Mixed derivative involves taking the derivative of a variable while holding several others constant. It is often referred to as the second and the third order derivatives.
- The chain rule is a rule for differentiating composition of functions, while the product rule is a rule for differentiating problems where one function is multiplied by another.
- The difference between the chain rule and the product rule is that the chain rule deals with a function of a function, while the product rule deals with two separate functions multiplied together.

5.0 SUMMARY

This unit focused on partial derivative which is the outcome of differentiation. In order to analyze the topic lucidly, sub-topics such as the higher order partial derivatives was reviewed, with focus on the second order partial differentiation. Similarly, the chain rule and the product rule for differentiation were reviewed. Both rules like we found out,are rules used for differentiating compositions of functions. The difference between them is that the chain rule deals with a function of a function, while the product rule deals with differentiation where one function is multiplied by another.

6.0 TUTOR-MARKED ASSIGNMENT

- Using the product rule, differentiate $y = (4x^2 + 17 + 12x)*(2 + 17x)$
- Find the partial derivative of $f(x,y) = 5x^4 + 12\sqrt{y} + 21$

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MODULE 4 INTEGRAL CALCULUS, OPTIMIZATION AND LINEAR PROGRAMMING (LP)

UNIT 1: INTEGRAL CALCULUS

1.0 INTRODUCTION

In this unit, the topic integral calculus will be discussed extensively. Basically, integral is the inverse of derivative. Recall that in the last unit of module 3, we were taught that derivative measures the sensitivity to change of a quantity which is determined by another quantity. Integral is the function of which a given function is the derivative. Put differently, it is the original function gotten from a derivative as a result of reversing the process of differentiation. To enable students understand this concept better, the main content of this unit has been divided into four sections, with the first section focusing on the introductory part of the topic, while the second and third sections will look at the rules guarding integration and the definite and indefinite integrals. Lastly, the last section will focus on the application of definite integrals to economic problems.

2.0 OBJECTIVES

At the end of this unit, you will be able to:

- Explain the concept of integration.
- Master the rules of integration.
- Explain the difference between differentiation and integration
- Solve definite and indefinite integrals problems.
- Apply definite integrals to economic problems.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO INTEGRAL CACULUS

Calculus is the branch of mathematics that deals with the findings and properties of derivatives and integrals of functions by methods originally based on the summation of infinitesimal differences. The two main types of calculus are differential calculus and integral calculus. Integral calculus is concerned with the determination, properties and application of integrals.

An integral is a function of which a given function is the derivative which yield that initial function when differentiated. Integration is the inverse of differentiation, and the two are one of the main operations in calculus.

Finding an integral is the reverse of finding a derivative. A derivative of a function represents an infinitesimal change in the function with respect to one of its variables. The simple derivative of a function f with respect to a variable x is denoted either by $f'(x)$ or df/dx.

Integral is made up of definite and indefinite integral. Definite integral is an integral expressed as the difference between the values of the integral at specified upper and lower limits of the independent variable. Given a function f of a real variable x and an interval (a, b) of the real line, the definite integral $\int_a^b f(x)dx$ is defined informally to be the signed area of the region in the xy-plane bounded by a graph of f, the x-axis, and the vertical lines $x – a$ and $x = b$, such that area above the x-axis adds to the total, and below the x-axis subtracts from the total.

An indefinite integral is an integral expressed without limits, and so containing an arbitrary constant. Indefinite integral for $\int 2x dx$ is $x^2 + c$; assuming x takes the sum of 2, then the indefinite integral for the equation will be $2^2 + c$.

SELF ASSESSMENT EXERCISE

In your own words, define calculus.

3.2 RULES OF INTEGRATION

Rule 1: The Constant Rule:

The integral of a constant k is,

$$\int k dx = kx + c$$

Example 1: Determine $\int 4dx$.

Using the above rule, the answer is $4x + c$.

Rule 2: The Power Rule:

The integral of a power function x^n,where $n \neq -1$ is,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, \quad n \neq -1.$$

Example 2: Evaluate $\int 12x^3 dx$

$$= 12\int x^3 dx = 12\left(\frac{1}{3+1}x^{3+1} + C\right) = 12x^4/4 + c = 3x^4 + c.$$

Example 3: Evaluate $\int (x-7)^5 dx$.

Let us represent our u as $x - 7$, and u^5 as $(x-7)^5$.

Since $u = x - 7$,

$du/dx = 1$, and $du = dx$.

$$\int (x-7)^5 dx = \int u^5 dx$$

$= u^6/6 + c$. (Recall rule #2)

Rule 3: The Difference Rule

$$\int (u \pm v)dx = \int u dx \pm \int v dx$$

Where $u = f(x)$, and $v = h(x)$.

Integration by Substitution

This method of integration is useful when the function $f(x)$ is difficult to solve or when simpler methods have not been sufficient. Integration by substitution allows changing the basic variable of an integrand.

This formula takes the form $\int f(g(x))g^!(x)dx$.

Let us consider $\int \cos(2x^2)3xdx$.

From the above equation, our $f = \cos$, and our $g = 2x^2$, while its derivative $g^!(x) = 3x$.

Once we are able to set up our integral in the form: $\int f(g(x))$ and $g^!(x)dx = \int f(u)$, and du respectively, then we can integrate $f(u)$ and finish by putting $g(x)$ back as a replacement for u.

So, let us complete our example.

$$\int \cos(2x^2)3xdx$$

From the above, let us substitute $u = 2x^2$, the integral will then contain the much simpler form $\cos u$ which we will be able to integrate.

$\frac{du}{dx} = 4x$ therefore

$du = 4xdx$ and $du/4 = xdx$

Now Substituting the values gotten above in the question we have,

$\int \cos u \frac{3}{4}du$ which is

$\frac{3}{4}\int \cos u\, du$

Now, we integrate:

$$\frac{3}{4}\int \cos(u)du = \frac{3}{4}\sin(u) + c.$$

And finally put $u = 2x^2$ back in the equation and the resulting answer is:

$$\frac{3}{4}\sin(2x^2) + c.$$

Rule 4:Logarithm Rule

Given that $y = lnx$, then $dy/dx = 1/x$.

$\int 1/x dx = lnx + c$.

$\int_z^y \frac{1}{x} dx = \int_x^y lnx = lny - lnz$

In general, $\int 1/x dz = lnz + c$.

Example 4: Evaluate $\int \frac{5}{2x+3} dx$

Here, we let $u = 2x + 3$

Then $du/dx = 2$, and $du = dx$.

We can still go further:

$\int \frac{5}{2x+3} dx = \int \frac{5}{u} du = 5lnu + c$

Since $u = 2x + 3$,

$5lnu + c = 5ln(2x + 3) + c$.

Rule 5: The Exponential Rule

If $z = f(x) = a^x + c$,

Then, $dz/dx = a^x$.

And $\int z dx = \int a^x dx = a^x + c$.

Example 5: Evaluate $\int a^{5x} dx$

To solve this, let:

$u = 5x$.

$du/dx = 5$.

$du = 5dx$.

$dx = du/5$

$\int a^{5x} dx = \frac{1}{5} \int a^u du = \frac{1}{5} a^{5x} + c$.

Integration by Parts

Integration by parts is a technique for performing indefinite integration of the form $\int u dv$ or definite integration of $\int_a^b u dv$ by expanding the differential of a product of functions $d(uv)$ and expressing the original in terms of a known integral $\int v du$.

A simple integration by parts starts with:

$d(uv) = u dv + v du$, and integrates both sides:

$\int d(uv) = uv = \int u dv + \int v du$.

Rearranging gives:

$\int u dv = uv - \int v du$.

Example 6: Consider the integral $\int x\cos x dx$ and let:

$u = x$, and $dv = \cos x\, dx$, $du = dx$, and $v = \sin x$;

Integration by parts gives:

$\int x\cos x dx = x\sin x - \int \sin x dx$

$= x\sin x + \cos x + c$.

Where c is a constant of integration (we earlier discussed this under partial differentiation).

Example 7: Integrate $\int x^5\cos(x^4)dx$

Note that: $\int x^5\cos(x^4)dx = \int x^4 x\cos(x^4)dx$

Let: $u = x^4$, $dv = x\cos(x^4)$, $du = 4x^3dx$ and $v = (1/4)\sin(x^4)$.

Based on the above, $\int x^5\cos(x^4)dx = x^4(1/4)\sin(x^4) - \int(1/4)\sin(x^4)\,dx$.

$= (1/4)x^4\sin(x^4) - \int x\sin(x^4)dx$

$= (1/4)x^4\sin(x^4) – (1/4)(-\cos(x^4)) + c$

$= (1/4)x^4\sin(x^4) + (1/4)\cos(x^4) + c$.

SELF ASSESSMENT EXERCISE

Determine the following integral.

Evaluate $\int(x – 4)^3dx$.

3.3 DEFINITE AND INDEFINITE INTEGRALS

In the last unit of module three, we talked about function of x ($f(x)$). Here, we are going to look at things the other way round. Now, the question is "what function are we going to differentiate to get the function $f(x)$.

DEFINITE INTEGRAL

Definite integral is an integral expressed as the difference between the values of the integral at specified upper and lower limits of the independent variable. It is an integral of the form $\int_a^b f(x)dx$ with upper and lower limits. In the formula above, a, b and x are complex numbers and the path of integration from a to b is referred to as the contour.

Remember (from the power rule #2), in order to integrate, we add 1 to the exponent, and use it to divide the numerator. We will be applying the rule here.

Let us consider the following examples.

Example 1: Evaluate $\int x^5 + x^{-4}dx$.

Here, we are going to be integrating a negative exponent.

The integration of $\int x^5 + x^{-4}dx = (1/6)x^6 – (1/3)x^{-3} + c$.

Example 2: Integrate $\int \frac{4x^{10} - 2x^4 + 15x^2 dx}{x^3}$

In the question above, we need to break down the equation to make it easy.

$\int 4x^{10}/x^3 - 2x^4/x^3 + 15x^2/x^3 dx$

$\int 4x^7 – 2x + 15/x) dx$

Now, we will integrate the equation.

$4(1/8)x^8 – 2(1/2)x^2 + 15/x\ dx$

$\int 15/x dx = 15\int 1/x\ dx$

$= 15ln|x| + c.$

Thus, our solution becomes:

$(1/2)x^8 – x^2 + 15ln|x| + c.$

Example 3: Integrate $\int 20x^3 – 35x^{-6} + 7dx.$

$20(1/4)x^4 – 35(1/5)x^{-5} + 7x + c.$

$= 5x^4 – 7x^{-5} + 7x + c.$

INDEFINITE INTEGRALS

An indefinite integral is an integral expressed without limits, and it also contains an arbitrary constant. An integral of the form $\int f(x)dx$ without upper or lower limit is called an anti-derivative. The first fundamental theorem of calculus allows definite integrals to be computed in terms of indefinite integrals. This theorem states that if F is the indefinite integral for complex function $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$ emerges. Remember in our earlier coverage, we talked about the derivative of a constant being zero, thus we can go further to prove that any constant may be added to an anti-derivative and will still correspond to the same integral.

Let us consider some examples.

Example 1: What function did we differentiate to get the following?

$F^!(x) = 2x^2 + 4x – 10?$

To solve this problem, we follow these steps.

Since the first term of the question is $2x^2$. This means that when we differentiate a function, the answer became $2x^2$, so we need to find what we differentiated to get $2x^2$.

Since we drop the exponent by one point after differentiating, then we must have differentiated x^3. If we assume the figure in front of the value to be 1, we would have x^2 which is different from $2x^2$; therefore the figure is $2/3x^3$. When differentiated gave $6/3x^2$ and when divided gave $2x^2$.

For the second term, we have $4x$ after differentiating $4/2x^2$. And lastly, the third term of -10 is a constant, and we simply differentiated of $-10x$.

Putting all this together gives the following equation: $F(x) = 2/3x^3 + 4/2x^2 – 10x + c$.

Given a function, $f(x)$, an anti-derivative of $f(x)$ is any function $F(x)$ such that $F!(x) = f(x)$.

If $F(x)$ is any anti-derivative of $f(x)$, then the most general anti-derivative of $f(x)$ is called an indefinite integral.

Example 2: Evaluate the indefinite integral: $\int 2x^2 + 4x - 10dx$.
This question is asking for the most general anti-derivative; all we need to do is to undo the differentiation.
If you recall the basic differentiation rules for polynomial as earlier discussed, this shouldn't be difficult.
To solve this problem, we need to increase the exponent by one after differentiation
The indefinite integral for $\int 2x^2 + 4x - 10dx = (2/3)x^3 + (4/2)x^2 - 10x + c$.

One common mistake students make with integrals (both indefinite and definite) is to drop the dx at the end of the integral. With integrals, think of the integral sign as an "open parenthesis" and the dx as a "close parenthesis". If you drop the dx it won't be clear where the integration ends. Consider the following variations of the above example.

$$\int 2x^2 + 4x - 10dx = 2/3x^3 + 4/2x^2 - 10x + c.$$
$$\int 2x^2 + 4x\,dx - 10 = 2/3x^3 + 4/2x^2 + c - 10$$
$$\int 2x^2\,dx + 4x - 10 = 2/3x^3 + c + 4x - 10.$$

We can only integrate what is between the integral sign and the dx. Each of the above integrals end in a different place and so we get different answers because we integrate a different number of terms each time. In the second integral the "-10" is outside the integral and so is left alone and not integrated. Likewise, in the third integral the "$4x$–10" is outside the integral and so is left alone.
Knowing which terms to integrate is not the only reason for writing the dx down.

Example 3: Integrate $\int 40x^3 + 12x^2 - 9x + 14dx$.

$$\int 40x^3 + 12x^2 - 9x + 14dx = 40(1/4)x^4 + 12(1/3)x^3 - 9(1/2)x^2 + 14x + c.$$
$$10x^4 + 4x^3 - 4.5x^2 + 14x + c.$$

To confirm your answer, you just simply differentiate it, and it should give you the original question.
Let us differentiate our answer $10x^4 + 4x^3 - 4.5x^2 + 14x + c$, and see if it will give us back our integrand.
Let us represent $10x^4 + 4x^3 - 4.5x^2 + 14x + c$ as U.

$$dU/dx = 40x^3 + 12x^2 - 9x + 14dx.$$

We can see that our answer gave back the integrand, and thus we can be satisfied.

SELF ASSESSMENT EXERCISE
Integrate $\int 0.5x^{-4} - 5x^{-15} + 12dx$.

Integrate $\int 15x^6 - 4x^4 + 18x^2 - 12dx$.

3.4 APPLICATION OF INTEGRAL TO ECONOMICS PROBLEMS

We will be applying definite integral to solve consumer surplus (CS) and producer surplus (PS) for any given demand and supply function.

Consumer surplus is the deference between the price consumers are willing to pay for a good or service and the actual price they paid. Put differently, consumer surplus is the monetary gain obtained by consumers because they are able to purchase a product for a price that is less than the highest price that they would be willing to pay. Producer's surplus on the other hand is the difference between the amount that a producer of a good receives and the minimum amount that he or she would be willing to accept for the good. The difference or surplus amount is the benefit that the producer receives for selling the good in the market. Consumers' surplus and producers' surplus are calculated using the supply and demand curves/ functions. The supply function or curve gives the quantity of items that producers will supply at any given price, while the demand function or curve gives the quantity that consumers will demand at any given price. The point of intersection of the supply and demand curves is called the market equilibrium and it is the point where quantity demanded is equals to quantity supplied.

Assuming that the equilibrium price is P_0 and the equilibrium quantity is Q_0 then the formulae for consumer surplus and producer surplus are:

$$CS = \int_0^{Qo} D(Q)dq - PoQo$$

$$PS = PoQo - \int_0^{Qo} S(Q)dq$$

Where D (Q) represents the demand function and S(Q), the supply function.

Let us consider some examples.

Example 1: Given the demand function P = 30 − Q and the supply function P = 15 + 2Q, and assuming pure competition, calculate the consumer's surplus and the producer's surplus.

First, we need to find the equilibrium quantity.

$$30 - Q = 15 + 2Q$$

$$30 - 15 = 2Q + Q$$

$$15 = 3Q$$

$$5 = Q.$$

Using Q = 5, and plugging it into either the supply or the demand function we find P, the equilibrium price .

$P_o = 30 - Q = 30 - 5 = 25.$

Therefore, $Q_0 = 5$ and $P_0 = 25$.

Based on the above findings, the consumer's surplus is calculated as:

$$\int_0^5 (30 - Q)dq - (5)(25)$$

$$\left(30Q - \frac{Q^2}{2}\right)\int_0^5 -125$$

$\left(30(5) - \frac{25}{2}\right) - 125 = \frac{25}{2}$

Consumer Surplus (CS) = 12.5.

The producer's surplus is:

$5(25) - \int_0^5 (15 + 2Q)dq$

$125 - \int_0^5 (15Q + 5^2)$

$125 - (15(5) + 5^2)$

$125 - (75 + 25)$

Producer Surplus (PS) = 25.

SELF ASSESSMENT EXERCISE

Find the consumer and the producer surplus given the demand function 2P = 60 – Q, and the supply function P = 12 + 4Q.

4.0 CONCLUSION

- Given a function *f* of a real variable *x* and an interval (*a*, *b*) of the real line, the definite integral $\int_a^b f(x)dx$ is defined informally to be the signed area of the region in the *xy*-plane bounded by the graph of *f*, the *x*-axis, and the vertical lines *x* – *a* and *x* = *b*, such that area above the *x*-axis adds to the total, and below the *x*-axis subtracts from the total.
- Definite integral is an integral of the form $\int_a^b f(x)dx$ with upper and lower limits
- Integration by substitution method is suitable or useful when the function *f*(*x*) is hard or when simpler methods have not been sufficient.
- Integration by parts is a technique for performing indefinite integration of the form $\int u dv$ or definite integration of $\int_a^b u dv$ by expanding the differential of a product of functions d(*uv*) and expressing the original in terms of a known integral $\int v du$.
- An integral of the form $\int f(x)dx$ without upper or lower limit is called an antiderivative.
- Consumer surplus is the difference between the price consumers are willing to pay for a good or service and the actual price.
- Producer's surplus on the other hand is the difference between the amount that a producer of a good receives and the minimum amount that he or she would be willing to accept for the good.

5.0 SUMMARY

This unit focused on integral calculus which is the derivative that yields the function when differentiated. Sub-unit 3.1 exposed us to the concept of integration which is the inverse of differentiation. Sub-unit 3.2 reviewed the rules of integration which are the constant rule, the power rule, the difference rule, integration by parts and substitution, the logarithm rule, e.t.c. Sub-unit 3.3 focused on definite and indefinite integral, and we learnt that definite integral is an integral expressed as the difference between the values of the integral at specified upper and lower limits of the independent variable, while indefinite integral is an integral expressed without limits and also containing an arbitrary constant. The last section focused on how definite integral can be applied to economics problem.

6.0 TUTOR-MARKED ASSIGNMENT

- Find the consumer and the producer surplus given the demand function 4P = 60 – 8Q, and the supply function 2P = 8 + Q.
- Using the indefinite integral method, Integrate $\int 15x^6 + 22x^2 - 9x^5 + 5dx$.

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UNIT 2 OPTIMIZATION

1.0 INTRODUCTION

The purpose of this unit is to discuss the optimization theory which is the selection of a best element (with regards to some criteria) from some set of available alternatives. The sub-unit one will introduce broadly the concept of optimization, while sub-unit two will consider the optimization problems. Finally, sub-unit three will focus on solving optimization problem using matrix.

2.0 OBJECTIVES

At the end of this unit, you will be able to:

- Explain the concept of Optimization
- List and explain the assumptions of optimization
- Apply optimization formula to economic problems
- Solve optimization problem using matrix.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO OPTIMIZATION

Decision-makers (e.g. consumers, firms, governments) in standard economic theory are assumed to be "rational". That is, each decision-maker is assumed to have a preference ordering over the outcomes to which their actions lead, and to choose an action among those feasible, that is most preferred according to this ordering. We usually make assumptions that guarantee that a decision-makers preference ordering is represented by a payoff function (sometimes called utility function), so that we can present the decision-makers problem as one of choosing an action, among those feasible, that maximizes the value of this function or minimizes the cost. That is, we write the decision-makers problem in the form:

$\max_a u(a)$ subject to $a \in S$, or $\min_a u(a)$ subject to $a \in S$,

Where u is the decision-makers payoff function over their actions and S is the set of their feasible actions.

If the decision-maker is a classical consumer, for example, then *a* is a consumption bundle, *u* is the consumer's utility function, and *S* is the set of bundles of goods the consumer can afford. If the decision-maker is a classical firm then *a* is an input-output vector, *u*(*a*) is the profit the action *a* generates, and *S* is the set of all feasible input-output vectors (as determined by the firm's technology).
Even outside the classical theory, the actions chosen by decision-makers are often modeled as solutions of maximization problems. A firm, for example, may be assumed to maximize its sales, rather than its profit; a consumer may care not only about the bundle of goods he/she consumes, but also about the bundles of goods the other members of her family consume, maximizing a function that includes these bundles as well as his/her own, a government may choose policies to maximize its chance of reelection.
In the case of minimization, we can assume, for example, that firms choose input bundles to minimize the cost of producing any given output; an analysis of the problem of minimizing the cost of achieving a certain payoff greatly facilitates the study of a payoff-maximizing consumer.

SELF ASSESSMENT EXERCISE
Explain briefly, the concept of optimization.

3.2 SOLVING OPTIMIZATION USING LAGRANGIAN MULLTIPLIER

We are going to consider the constraint on consumption before introducing income.
First, we want to look at the optimization of a single variable without constraint.
For example, given $TU = 25x + 5x^2 - 2/3x^3 = 0$, calculate the saturation rate of consumption at the point of diminishing marginal utility.
In order to solve this problem, we differentiate the total utility equation, as the differentiation of total utility with respect to *x* gives the marginal utility.

$$MU = dTU/dx = 25 + 10x - 6/3x^2 = 0$$
$$MU = 25 + 10x - 2x^2 = 0.$$

Since the marginal utility defines the slope of the total utility, and the slope of a function is zero at its maximum or minimum point, we set MU = 0.
Similarly, the marginal utility of: $TU = 10x^4 + 17x^2 - 16 = 0$ is the differentiation of the total utility function.

$$MU = dTU/dx = 40x^3 + 34x = 0.$$

Now, let us consider multiple commodities. Here, we consider not only x_1, but $x_2 \ldots x_n$. i.e., $TU = f(x_1, x_2, \ldots x_n)$.
In order for us to find the marginal utility of a function similar to the one above, we need to differentiate partially and hold other *x* constant.

Example: Determine the maximum or the minimum level of satisfaction from the following two commodities. $TU = 10x_1 + 1.5x_1x_2^2 + 2x_1^2x_2 + 5x_2$.

For us to solve this problem, we need to solve for the marginal utility of each x while holding others constant.

$$MUx_1 = dTU/dx_1 = 10 + 1.5x_2^2 + 4x_1x_2 = 0.$$
$$MUx_2 = dTU/dx_2 = 3x_1x_2 + 2x_1^2 + 5 = 0.$$

Now, let us determine the utility maximizing combination subject to income constraint.
Here, price will be introduced, thus our equation looks like this:

$$P_1x_1 + p_2x_2 + p_3x_3\ldots + p_nx_n = Y.$$

Where P_1 represents the price for good x_1, and Y represents the income.
For us to solve this kind of problem, we will introduce the Lagrange multiplier (λ).
Given $Z = f(x_1, x_2, x_3\ldots x_n) + \lambda(1 - p_1x_1 - p_2x_2 - p_3x_3\ldots - p_nx_n)$, we proceed to differentiate the function partially.

$$ðZ/ðx_1 = ðTU/ðx_1 - \lambda p_1 = 0 \quad (1)$$
$$ðZ/ðx_2 = ðTU/ðx_2 - \lambda p_2 = 0 \quad (2)$$
$$ðZ/ðx_3 = ðTU/ðx_3 - \lambda p_3 = 0 \quad (3)$$
$$ðZ/ðx_n = ðTU/ðx_n - \lambda p_n = 0 \quad (4)$$
$$ðZ/ð\lambda = 1 - p_1x_1 - p_2x_2 - p_3x_3\ldots - p_nx_n = 0 (5).$$

We have successfully solved equation 1 to 5 simultaneously in order to determine the consumption level of the commodities that would maximize the total utility (TU) function.

From equation (1), $ðTU/ðx_1 - \lambda p_1 = 0$

$$ðTU/ðx_1 = \lambda p_1$$
$$\lambda = ðTU/ðx_1/p_1.$$
$$\lambda = MUx_1/p_1$$

Note, $ðTU/ðx_1 = MUx_1$, while P_1 is the price for x_1 good.
Similar condition applies to all other equations simultaneously.

For equation (2), $ðTU/ðx_2 - \lambda p_2 = 0$

$$ðTU/ðx_2 = \lambda p_2$$
$$\lambda = ðTU/ðx_2/p_2.$$
$$\lambda = MUx_2/p_2.$$

For equation (3), $ðTU/ðx_3 - \lambda p_3 = 0$

$$ðTU/ðx_3 = \lambda p_3$$
$$\lambda = ðTU/ðx_3/p_3$$
$$\lambda = MUx_3/p_3.$$

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For equation (4) $\partial TU/\partial x_n - \lambda p_n = 0$

$\partial TU/\partial x_n = \lambda p_n$

$\lambda = \partial TU/\partial x_n / p_n$

$\lambda = MUx_n / p_n.$

In mathematics, the three dots mean we solve until we get to the last equation which is equation (4) in our case.

In summary, $\lambda = \partial TU/\partial x_1/p_1 = \partial TU/\partial x_2/p_2 = \partial TU/\partial x_3/p_3 = \ldots = \partial TU/\partial x_n/p_n$.
Since $\partial TU/\partial x_1 = MUx_1$, $\partial TU/\partial x_2 = MUx_2$, $\partial TU/\partial x_3 = MUx_3 \ldots \partial TU/\partial x_n = MUx_n$
So, our equations thus become:

$$\lambda = MUx_1/p_1 = MUx_2/p_2 = MUx_3/p_3 = \ldots = MUx_n/p_n.$$

The above solution is known to be the necessary condition for consumer utility maximization.

Now, let us apply the Lagrangian Multiplier (λ) to numerical problem.

Example 1: Maximize the consumer utility:
$TU = 12x_1 + 18x_2 - 0.5x_1^2 - 0.5x_2^2$
Subject to the income constraint (Y) = 60.
$px_1 = 2$, and $px_2 = 5$.
Where px_1 is the price of commodity x_1, and px_2 is the price for commodity x_2.

<u>Solution</u>
For us to solve this equation, we will introduce the Lagrange multiplier and also the prices and income.

$TU = 12x_1 + 18x_2 - 0.5x_1^2 - 0.5x_2^2 + \lambda(60 - 2x_1 - 5x_2)$
$TU = 12x_1 + 18x_2 - 0.5x_1^2 - 0.5x_2^2 + \lambda 60 - \lambda 2x_1 - \lambda 5x_2$
$\partial TU/\partial x_1 = 12 - x_1 - 2\lambda = 0$
$12 - x_1 = 2\lambda$
$\lambda = 6 - 0.5x_1$ (1)

$\partial TU/\partial x_2 = 18 - x_2 - 5\lambda$
$18 - x_2 = 5\lambda$
$\lambda = 3.6 - 0.2x_2$ (2)

$\partial TU/\partial \lambda = 60 - 2x_1 - 5x_2 = 0$ (3).

From equation (1) and (2), let us equate $\lambda = \lambda$ at equilibrium; thus, our equation becomes:

$6 – 0.5x_1 = 3.6 – 0.2x_2$
$6 – 3.6 + 0.2x_2 = 0.5x_1$
$2.4 + 0.2x_2 = 0.5x_1$

Making x_1 the subject of the formula:

$x_1 = 4.8 + 0.4x_2$ (4)

We impute equation (4) into equation (3) to get the value for our x_2.

$60 – 2x_1 – 5x_2 = 0$
$60 – 2(4.8 + 0.4x_2) - 5x_2 = 0$
$60 – 9.6 – 0.8x_2 - 5x_2 = 0$
$50.4 – 5.8x_2 = 0$
$50.4 = 5.8x_2$.
$x_2 = 8.7$. (5)

Now, since we have the value for our x_2, we can continue by imputing this value which is our equation (5) into equation (4) in order to get the actual value for x_1.

From equation (4): $x_1 = 4.8 + 0.4x_2$
$x_1 = 4.8 + 0.4(8.7)$
$x_1 = 4.8 +3.5$
$x_1 = 4.8 +3.5$
$x_1 = 8.3$.

This means that for the consumer to maximize his utility giving the price of good $x_1 = 2$ and good $x_2 = 5$ and income = 60, he must consume 8.3 units of good x_1 and 8.7 units of good x_2.

Example 2: Maximize $Z = 4 – (x_1 – 4)^2 – 3(x_2 – 1)^2$
Subject to: $x_1 + 4x_2 = 3$

Solution:

$Z = 4 – (x_1 – 4)^2 – 3(x_2 – 1)^2 + \lambda(3 – x_1 – 4x_2)$
$Z = 4 – (x_1 – 4)^2 – 3(x_2 – 1)^2 + \lambda 3 – \lambda x_1 – \lambda 4x_2$
$Z = 4 – x_1^2 – 16 – 3x_2^2 – 3 + \lambda 3 – \lambda x_1 – \lambda 4x_2$

$ðZ/ðx_1 = 2x_1 - \lambda = 0$
$\lambda = 2x_1$ (1)
$ðZ/ðx_2 = -6x_2 – \lambda 4 = 0$
$\lambda 4 = -6x_2$
$\lambda = \frac{-6x_2}{4}$
$\lambda = -1.5x_2$ (2)
$ðZ/ð\lambda = 3 – x_1 – 4x_2 = 0$ (3)

From equation (1) and (2), let us set $\lambda = \lambda$:

$2x_1 = -1.5x_2$

Making x_1 the subject of the formula:

$$x_1 = \frac{-1.5x_2}{2}$$

$x_1 = -0.75x_2$ (4)

Input equation (4) into (3) to get x_2:

$3 - x_1 - 4x_2 = 0$

$3 - (-0.75x_2) - 4x_2 = 0$

$3 + 0.75x_2 - 4x_2 = 0$

$3 - 3.25x_2 = 0$

$3.25x_2 = 3$

$x_2 = 0.92$ (5)

We insert equation (5) into equation (4) to get x_1:

$x_1 = -0.75x_2$

$x_1 = -0.75(0.92)$

$x_1 = -0.69.$

SELF ASSESSMENT EXERCISE

Maximize $U = 10x_1^2 + 15x_2^4 - 21x_1^2x_2$

Subject to: income = 100, $px_1 = 4$, $px_2 = 3$.

3.3 SOLVING OPTIMIZATION USING MATRIX

Matrix can also be used to solve optimization problem.

Let us consider the same problem of maximizing $TU = 12x_1 + 18x_2 - 0.5x_1^2 - 0.5x_2^2$

Subject to the income constraint (Y) = 60, $px_1 = 2$, and $px_2 = 5$.

Solution:

For us to solve this equation, we will also introduce the Lagrange multiplier.

$TU = 12x_1 + 18x_2 - 0.5x_1^2 - 0.5x_2^2 + \lambda(60 - 2x_1 - 5x_2)$

$TU = 12x_1 + 18x_2 - 0.5x_1^2 - 0.5x_2^2 + \lambda 60 - \lambda 2x_1 - \lambda 5x_2$

Here, after differentiating the total utility equation partially with respect to the particular commodity, what we need to do next is simply to rearrange it in the following way.

$ðTU/ðx_1 = 12 - x_1 - 2\lambda = 0$

$x_1 + 2\lambda = 12$ (1)

$ðTU/ðx_2 = 18 - x_2 - 5\lambda = 0$

$x_2 + 5\lambda = 18$ (2)

$ðTU/ð\lambda = 60 - 2x_1 - 5x_2 = 0$

$2x_1 + 5x_2 = 60$ (3)

We can arrange the above equations into the matrix box using the Crammer's rule.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 2 & 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ 60 \end{pmatrix}$$

The above matrix diagram is formed by the coefficients of the equation.
Take note that the right hand side values of the equation are depicted in the right hand side of the diagram as well. These values are important, as they will be used to replace the column of the matrix A.
Now, we can calculate the determinants of the matrix A to get |A| = -29, How did we get this? Follow the steps below.

1. In order to evaluate the determinants of A, we first start with making up a check-board array of + and – signs.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

2. We will start our analysis from the upper left with a sign +, and alternate signs going in both directions.
3. Choose any row or any column. In the equation above, let us choose the third row, i,e,. 2, 5 and 0.
4. We choose 2 first, and we use it against (0:2) and (1:5). Also using 5 against (2:1) and (5:0), similarly using 0 against (1:0) and (0:1).

$$2 \quad x \begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix}, \quad 5 \quad x \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}, \quad 0 \quad x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Remember the check-board sign and assign it to the figures when calculating.
6. Remember, the 2 outside the first diagram in (4) above is with a positive sign, while the next is with a negative sign and the last with a positive sign. If the entry comes from a positive position in the checkerboard, add the product. If it comes from a negative position, subtract the product.
7. $$2 \quad x \begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix} - 5 \quad x \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix} + 0 \quad x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

8. The above gives the following: 2(0 – 2) – 5(0 – 5) + 0(1 – 0)
9. The above gives us -4 – 25 + 0 = -29. Thus |A| = -29.

Even if we choose another row or column, we will still get the same answer for our determinant |A|.
Now that we have the bottom in the computation for all three unknown, let us expand the top determinants in each case by replacing the first column by the resource column: i.e.,

$$x_1 = \begin{pmatrix} 12 & 0 & 2 \\ 18 & 1 & 5 \\ 60 & 5 & 0 \end{pmatrix}$$

x_1 is calculated by dividing determinants of $x_1\{|x_1|\}$ by the equation determinant |A|.
We will follow the same step as above to get the determinants of x_1.
Now, let us choose the upper row of (12, 0 and 2). Remember, the sign +, - and + is assigned to them based on the checkerboard.

$$12 \times \begin{pmatrix} 1 & 5 \\ 5 & 0 \end{pmatrix} - 0 \times \begin{pmatrix} 18 & 5 \\ 60 & 0 \end{pmatrix} + 2 \times \begin{pmatrix} 18 & 1 \\ 60 & 5 \end{pmatrix}$$

12(0 – 25) – 0(0 – 300) + 2(90 – 60)

$|x_1|$ = -300 + 60 = -240.

Thus x_1 is calculated as $|x_1| \div |A|$.

x_1 = -240 ÷ -29

x_1 = 8.28.

This same approach was applied to get our x_2 and λ.
For x_2, we have this:

$$x_2 = \begin{pmatrix} 1 & 12 & 2 \\ 0 & 18 & 5 \\ 2 & 60 & 0 \end{pmatrix}$$

Where $|x_2|$ = -252, and remember, to get our x_2, we need to divide $|x_2|$by |A|**.**
Thus, x_2 = -252 ÷ -29

x_2 = 8.69.

For λ, we have:

$$\lambda = \begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 18 \\ 2 & 5 & 60 \end{pmatrix}$$

Where |λ| = -54. Our λ value is derived from |λ| ÷ |A|.

$$\lambda = -54 \div -29$$

$$\lambda = 1.86.$$

SELF ASSESSMENT EXERCISE

Using the Crammer's rule and Matrix, Maximize U = $5x_1 + 12x_2^4 - 11x_1^3 2x_2^2$
Subject to: income = 20, $px_1 = 3$, $px_2 = 5$.

4.0 CONCLUSION

- According to the cardinal theory, utility can be quantified in terms of the money a consumer is willing to pay for it, thus MUx=Px.
- $ðTU/ðx_1 = MUx_1$, $ðTU/ðx_2 = MUx_2$, $ðTU/ðx_3 = MUx_3$…$ðTU/ðx_n = MUx_n$. This means that partial differentiation of the total utility function with respect to x_1, x_2, x_3, …, x_n gives us the marginal utility for that variable.
- The checkerboard is used to assign signs to the figures in the matrix.
- If we choose any row or column, we will get the same determinants of the matrix |A|.

5.0 SUMMARY

This unit focused on partial optimization which is the selection of a best element from some set of available alternative. Optimization is therefore the process of choosing something better out of the lot. In order to analyze the topic, sub-units such as the Lagrangian method for solving optimization and the application of matrix to solving optimization problem were reviewed. The Lagrange multiplier was introduced to factor in the constraints, while the matrix method using Crammer's rule is another way of calculating the optimization problem.

6.0 TUTOR-MARKED ASSIGNMENT

- Using Lagrangian multiplier, maximize U = $2x_1 + 5x_2 - 16x_1^3 10x_2^3$ subject to Y = 50, $px_1 = 5$, $px_2 = 7$.
- Solve the above question using the matrix approach.

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UNIT 3 LINEAR PROGRAMMING (LP)

1.0 INTRODUCTION

The term linear programming consists of two words, linear and programming. Linear programming considers only linear relationship between two or more variables. By linear relationship, we mean that relations between the variables can be represented by straight lines. Programming means planning or decision-making in a systematic way. Linear programming is the technique for maximizing or minimizing a linear function of several variables such as output or cost. Linear programming can also be referred to as optimization of an outcome based on some set of constraints using a linear mathematical model. LP involves linear function of two or more variables which are to be optimized subject to a set of linear constraints at least one of which must be expressed as inequality. The sub-unit one focuses on the introduction to LP, while sub-unit two and three focus on the assumptions, merits and demerits as well as the LP calculation using simplex algorithm.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Explain the concept of linear programming
- State the assumptions, merits and demerits of linear programming
- Solve linear programming problem using simplex algorithm.

3.0 MAIN CONTENT

3.1 INTRODUCTION TO LINEAR PROGRAMMING

The linear programming problem is that of choosing non-negative values of certain variables so as to maximize or minimize a given linear function subject to a given set of linear inequality constraints. It can also be referred to as the use of linear mathematical relations to plan production activities.

Linear programming is a resource allocation tool in production economics. Put differently, linear programming is a tool of analysis which yields the optimum solution for the linear objective function subject to the constraints in the form of linear inequalities. This is to say that linear programming aims at the maximization or minimization of an objective, subject to a constraint. For instance, a company may want to determine the quantity of good *x* and *y* to be produced in order to minimize cost subject to price of those goods and the company's budget. In order to determine the optimal or best combination, optimization can be used. Before going further, it is necessary to look at the terms of linear programming.

Terms in Linear Programming

1. Objective Function

Objective function, also called criterion function, describe the determinants of the quantity to be maximized or to be minimized. If the objective of a firm is to maximize output or profit, then this is the objective function of the firm. If the linear programming requires the minimization of cost, then this is the objective function of the firm. An objective function has two parts- the primal and dual. If the primal of the objective function is to maximize output then its dual will be the minimization of cost.

2. Technical Constraints

The maximization of the objective function is subject to certain limitations, which are called constraints. Constraints are also called inequalities because they are generally expressed in the form of inequalities. Technical constraints are set by the state of technology and the availability of factors of production. The number of technical constraints in a linear programming problem is equal to the number of factors involved in it.

3. Non-Negative Constraints

This express the level of production of the commodity cannot be negative, i.e, it is either positive or zero.

4. Feasible Solutions

Feasible solutions are those which meet or satisfy the constraints of the problem and therefore it is possible to attain them.

5. Optimal Solution

The best of all feasible solutions is the optimum solution. In other words, of all the feasible solutions, the solution which maximizes or minimizes the objective function is the optimum solution. For instance, if the objective function is to maximize profits from the production of two goods, then the optimum solution will be that combination of two products that will maximize the profits of the firm. Similarly, if the objective function is to minimize cost by the choice of a process or combination of processes, then the process or a combination of processes which actually minimizes the cost will represent the optimum solution. It is worthwhile to repeat that optimum solution must lie within the region of feasible solutions.

SELF ASSESSMENT EXERCISE

List and explain the terms in linear programming.

3.2 ASSUMPTIONS, ADVANTAGES AND LIMITATIONS OF LINEAR PROGRAMMING (LP)

3.2.1 Assumptions of Linear Programming

The linear programming problems are solved on the basis of some assumptions which follow from the nature of the problem.

1. Linearity: The objective function to be optimized and the constraints involve only linear relations.
2. Non-negativity: The decision variable must be non-negative.
3. Additive and Divisibility: Resources and activities must be additive and divisible.
4. Alternatives: There should be alternative choice of action with a well defined objective function to be maximized or minimized.
5. Finiteness: Activities, resources, constraints should be finite and known.
6. Certainty: Prices and various coefficients should be known with certainty.

3.2.2 Advantages of Linear Programming.

1. It helps decision-makers to use their productive resources effectively.
2. The decision-making approach of the user becomes more objective and less subjective.
3. In a production process, bottlenecks may occur. For example, in a factory, some machines may be in great demand, while others may lie idle for some time. A significant advantage of linear programming is highlighting such bottlenecks.

3.2.3 Limitations of Linear Programming

1. Linear programming is applicable only to problems where the constraints and objective function are linear i.e., where they can be expressed as equations which represents straight lines. In real life situations, when constraints or objective functions are non-linear, this technique cannot be used.
2. Factors such as uncertainty and time are not taken into consideration.

3. Parameters in the model are assumed to be constant but in real life situations, they are not constant.
4. Linear programming deals with only single objectives, whereas in real life situations, we may have multiple and conflicting objectives.
5. Mostly, linear programming models present trial and error solutions and it is difficult to find out really optimal solutions to the various economic complexities.

SELF ASSESSMENT EXERCISE 2
What are the assumptions, advantages and limitations of linear programming?

3.3 SOLVING LP USING SIMPLEX ALGORITHM

The simplex algorithm proceeds by performing successive pivot operations which each give an improved basic feasible solution. The choice of pivot element at each step is largely determined by the requirement that this pivot improves the solution.
Let us look at how simplex algorithm method is applied to linear programming problem.

Example: Consider the production function:

Max $Z= 200x_1 +240x_2$

Subject to

$30x_1 + 15x_2 \leq 2400$

$20x_1 + 30x_2 \leq 2400$

First, we need to determine the stages that maximize our equation and representing it in the simplex algorithm table.
In the table, Pj is the representation of the maximizing equation, as 200 and 240 correspond to x_1 and x_2 respectively, while there are no figures for s_1 and s_2, thus represented by zero (0).
Zj represents the available resources, and it is calculated by multiplying Pj value with the resources value and adding them together. i.e. For the stage 1 in our table, the Pj value for S_1 and S_2 are both 0, while the resources value are both 2400 each, thus the Zj value is (0*2400) + (0*2400) = 0.

<u>Stage 1</u>: First thing to do is to get rid of (≤) by introducing the slack element.
Thus, we rewrite the model as:

$$30x_1 + 15x_2 + s_1 + 0S_2 = 2400 \qquad (1)$$

$$20x_1 + 30x_2 + 0s_1 + S_2 =2400 \qquad (2)$$

Equation (1) and (2) above fills the stage one of the figure below.

Table1. Simplex Algorithm

Stage 1							
Pj	-	-	200	240	0	0	Ɵ
	Activity	Resources	X_1	X_2	S_1	S_2	-
0	S1	2400	30	**15**	1	0	160
0	**S2**	**2400**	**20**	**[30]**	**0**	**1**	**80**
	Zj	0	0	**0**	0	0	
	Pj - Zj	-	200	**240**	0	0	
Stage 2							
0	**S1**	**1200**	**[-19.95]**	**0**	**1**	**0.45**	**60.15**
240	X2	80	**-0.67**	1	0	-0.03	119.4
	Zj	19,200	**-160.8**	240	0	-7.2	
	Pj - Zj	-	**360.8**	0	0	7.2	
Stage 3							
200	X1	60.2	1	0	0.05	-0.02	
240	X2	39.7	0	1	-0.03	0.04	
	Zj	21568	200	240	2.8	5.6	
	Pj - Zj	-	0	0	-2.8	-5.6	

Source: Authors Computation.
Note: stage three (3) maximizes the equation.

The **bolded** row and column are both our pivot row and column, while the pivot element is acquired by observing the highest value in row Pj – Zj and the lowest values in column Ɵ. The pivot value is the value in block bracket (**[30]**) in stage 1.
Since the pivot element belong to column X_2 and S_2 row, we replace S_2 with X_2 in stage 2, and the same principle applies to stage 3.

Stage 2: Here, we make x_2 the subject of the formula in equation 2.

$$20x_1 + 30x_2 + 0s_1 + s_2 = 2400$$
$$30x_2 = 2400 - 20x_1 - 0s_1 - s_2$$

Dividing through by 30 gives:

$$x_2 = 80 - 0.67x_1 - 0s_1 - 0.03s_2 \qquad (3)$$

Equation (3) above is imputed in the X_2 row in stage 2.
In order to get our S_1 row values, we impute equation (3) into (1) which gives:

$$30x_1 + 15(80 - 0.67x_1 - 0s_1 - 0.03s_2) + s_1 + 0s_2 = 2400$$
$$30x_1 + 1200 – 10.05x_1 – 0s_1 – 0.45s_2 + s_1 + 0s_2 = 2400$$
$$19.95x_1 + s_1 – 0.45s_2 = 1200 \qquad (4)$$

Making s_1 the subject of the formula:

$$s_1 = 1200 – 19.95x_1 + 0.45s_2 \qquad (5)$$

Stage 3: Here, the pivot element is identified, and it belongs to the row S1, thus we will replace S_1 with X_1 in stage3. Here, we will make x_1 the subject of the formula in (4):

$$19.95x_1 + s_1 – 0.45s_2 = 1200$$

$$x_1 = 60.2 – 0.05s_1 + 0.02s_2 \qquad (6)$$

Equation (6) above is imputed in the X_1 row in stage 3.
Imputing (6) into (3) gives:

$$x_2 = 80 - 0.67x_1 - 0s_1 - 0.03s_2$$

By rearranging the above equation gives:

$$0.67x_1 + x_2 + 0s_1 + 0.03s_2 = 80$$

Thus our model gives:

$$0.67(60.2 – 0.05s_1 + 0.02s_2) + x_2 + 0.03s_2 = 80$$

$$40.3 – 0.03s_1 + 0.01s_2 + x_2 + 0.03s_2 = 80$$

$$x_2 – 0.03s_1 +0.04s_2 = 39.7$$

Making X_2 the subject of the formula:

$$x_2 = 39.7 + 0.03s_1 – 0.04s_2 \qquad (7)$$

We impute equation (7) into X_1 row in stage 3.
The equation is maximize when the values for X_1,X_2,S_1 and S_2 are either zero (0) or negative.
The maximizing equation is thus:

$$200x_1 + 240x_2$$

$$200 (60.2) + 240 (39.7) = 21,568.$$

This maximizes the value Z. i.e., revenue is maximize at the production of $60.2x_1$ and $39.7x_2$ goods.

SELF ASSESSMENT EXERCISE
Using the simplex algorithm method,
Maximize $Z = 2x_1 + 4x_2$
Subject to: $15x_1 + 25x_2 \leq 10$
$10x_1 + 17x_2 \leq 15$

4.0 CONCLUSION

From our discussion so far on linear programming, we can infer the following.

- The objective function according to the linear programming assumption is that the objective function is linear and that there is an additivity of resources and activities.
- The terms of linear programming include the objective function, technical constraints, non-negative constraint, feasible solutions and optimal solution.
- Linear programming problems are solved based on some assumptions which are; Linearity, Non-negativity, additive and divisibility, alternatives, finiteness and certainty.

- Linear programming helps decision makers to use their production resources effectively.
- Linear programming is applicable to only problems where the constraints and the objective function are linear.
- Simplex algorithm performs successive pivot operations with each giving an improved basic feasible solution, and the choice of pivot element at each step is largely determined by the requirement that the pivot improve the solution.

5.0 SUMMARY

The concept of linear programming is that of choosing non-negative values of certain variables so as to maximize or minimize a given linear function, subject to a given set of linear inequality constraints. In order to construct the model, transformation of the constraints using slack variable is required. The standard form of presenting a linear problem is by using the simplex algorithm. The simplex algorithm proceeds by performing successive pivot operations which each gives an improved basic feasible solution, the choice of pivot element at each step is largely determined by the requirement that this pivot improves the solution.

6.0 TUTOR-MARKED ASSIGNMENT

- max $Z = 10x_1 + 9x_2$
  Subject to: $x_1 + 22x_2 \leq 50$
  $3x_1 + 5x_2 \leq 70$
  $4x_1 + 4x_2 \leq 32$
- max Z = 5x1 + 7x2 + 2x3
  Subject to: $15x_1 + 2x_2 + 2x_3 \leq 80$
  $4x_1 + 8x_2 + 2x_3 \leq 90$
  $3x_1 + 2x_2 + 5x_3 \leq 40$

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