NATIONAL OPEN UNIVERSITY OF NIGERIA

FACULTY OF SOCIAL SCIENCES

COURSE CODE: ECO 153

COURSE TITLE: Introduction to Quantitative Method I

COURSE GUIDE

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Introduction

Introductory Mathematics for Economists I is a three-unit course for 100Level students in National Open University of Nigeria. It comprises of 28 study units, subdivided into six modules. The materials have been developed with the Nigerian context in view by using simple and local examples. This course guide gives you an overview of the course. It also provides you with organisation and the requirement of the course.

Course Competencies

This lecture notes gave the students in undergraduate programs in economics, business, finance and social science in general the understand of mathematical reasoning, fundamental concepts in mathematical economics and the extend mathematical toolbox. The only assume prerequisite need to follow the topics covered in this lecture note is patience. Mathematics has become the language of economics as it relates to business and finance. Most undergraduate programs in social science uses geometric tools in exposition of relationships and theories. However, the inherent limitations of geometric tools call for the use of general algebra. Thus, necessitate training in the use of mathematics techniques and tools. From experience, large numbers of undergraduate students of social science have "math-phobia" and "math-aversion." Due to the unclear presentation of course. This lecture notes have been developed with the Nigerian context in view by using simple and local examples to help replace the feelings of phobia and aversion with passion and appreciation of the subject matter and the understanding and applications of mathematics techniques and tools to real life situation.

Course Objectives

The main objective of this course is to present the students an understanding of mathematics concepts, techniques, tools and their applications in real life situation. And the course has its specific objective to;

- To give an understanding of the stages and processes involved in identify business problems.
- To give an understanding of what constitutes Business and Economics problems.
- To help students relate mathematics techniques and tools to different business and Economics problems analysis. And
- To generate, access and work with quantitative idea in solving economics problems

Working Through this Course

To complete this course, you are required to go through the study units in this lecture note and other related materials. You will also need to undertake practical exercises for which you need a pen, a notebook and other materials that will be listed in this guide. The exercises are to help you in the understanding of the basic concept and principles in the lecture note. At the end of each unit, you will be required to submit written assignments for assessment purpose. At the end of the course, you will write a final examination.

Study Units

There are six modules in this course broken into 28 study units.

References and Further Readings

- Fabayo J.A (1996) Mathematical analysis in Economics: Obafemi Awolowo University Press Limited. Ile-Ife, Nigeria
- Agbadudu A.B. (1998) Mathematical methods in Business and Economics: United City press, Benin City, Nigeria.

- Eke,C.E.(2010) Solution Oriented on Business Mathematics :Justice Jeco Printing & publishing Global (recommended).
- Adibe,O.R(1995) Introduction to management Mathematics for Business students :Model publishing ltd Aba.

Presentation Schedule

1. Week 1-2: Set Theory
2. Week 3: Series, Sequence and progression
3. Week 4: Elementary Geometry
4. Week 5-6: Intro to matrix Algebra
5. Week 7: General Arithmetic: Approximation and percentage errors
6. Week 8: inequalities
7. Week 9: Intro. to probability and Trigonometric function and their inverse
8. Week 10: permutation and combination
9. Week 11: Differentiation, calculus, logarithmic, implicit functions
10. Week 12: Revision

The presentation schedule included in your course materials gives you the important dates for this year for the completion of tutor-mark assignments and attending tutorials. Remember, you are required to submit all your assignments by due date. You should guide against falling behind in your work.

Set theory; relations and algebraic operations and economic applications of various types of algebraic functions; series and sequences and their applications in economics; Matrix Algebra and special matrices; introduction to calculus of algebraic functions of single variable with applications to marginal analysis and optimization in Economics.

Assessment

The assessments are of two folds for this course; The tutor marked assignments; The second, the written examination.

In attempting the assignments, you are expected to apply information, knowledge and techniques gathered during the course. The assignments must be submitted to your tutor for formal assessment in accordance with the deadlines stated in the Presentation Schedule and theAssignments File. The work you submit to your tutor for assessment will count for 30 % of your total course mark. At the end of the course, you will need to sit for a final written examination of three hours' duration. This examination will also count for 70% of your total course mark.

Assignments & Grading

Academic Honesty: All class work should be done independently, unless explicitly stated otherwise on the assignment.

You may discuss general solution strategies, but must write up the solutions yourself.

If you discuss any problem with anyone else, you must write their name at the top of your assignment, labeling them “collaborators”.

How to get the Most from the Course

In open distance learning (OPL), the study units replace the university classroom lectures. This is one of the merits of distance learning; you can read and work through the outlined study materials at your own pace, time and place of your choice. The idea is that you are reading the lecture note rather than listening to it. In the same way that a lecturer might give you some reading to do, the study units contain instructions on when to read your set of books or other materials and practice some practical questions. Just as a lecturer might give you an in- class exercise or quiz, your study units provide exercises for you to do at appropriate point in time. Each of the study units follows a common format. The following is a practical strategy for working through the course. Always remember that your tutor's job is to help you. When you need his assistance, do not hesitate to call and ask your tutor to provide it. Follow the under-listed pieces of advice are fully: -

1) Organise a Study Schedule: refer to the course overview for more details. Note the time you are expected to spend on each unit and how the assignments relate to the units.
2) Having created your personal study schedule, ensure you adhere strictly to it. The major reason that students fail is their inability to work along with their study schedule and thereby getting behind with their course work. If you have difficulties in working along with your schedule, it is important you let your tutor know.
3) Assemble the study material. Information about what you need for a unit is given in the 'overview' at the beginning of each unit. You will almost always need both the study unit you are working on and one of your set books on your desk at the same time.
4) Work through the study unit. The content of the unit itself has been arranged to provide a sequence you will follow. As you work through the unit, you will be instructed to read sections from your set books or other articles. Use the unit to guide your reading.
5) Review the objectives for each unit to be informed that you have achieved them. If you feel uncertain about any of the objectives, review the study material or consult your tutor.
6) When you are sure that you have achieved the objectives of a unit, you can then start on the next unit. Proceed unit by unit through the course and try to space your study so that you keep yourself on schedule.
7) When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. You are strongly advised to consult your tutor as soon as possible if you have any challenges or questions.
8) After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the objectives of the units (listed at the beginning of each unit) and the course objective (listed in this Course Guide).
9) After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the objectives of the units (listed at the beginning of each unit) and the course objective (listed in this Course Guide).
10) Keep in touch with your study centre. Up-to-date course information will be constantly made available for you there.

Online Facilitation

There will be ten hours of tutorials to support of this course. The dates and times of these tutorials will be communicated together with the name and phone number of your tutor as soon as you are allocated a tutorial group. There be need to call your tutor for comment on relating assignments, progress and any difficulties you might encounter so as to provide assistance to you during the course. You must mail your tutor done assignment to your tutor well before the due date (at least two working days are required). The assignments will be marked by your tutor and returned to you as soon as possible. Do not hesitate to contact your tutor by telephone, e-mail for personal discussions if you need help. The following might be circumstances in which you would find help necessary:

i. You do not understand any part of the study unit
ii. You have difficulty/difficulties with the self-assessment exercise(s)
iii. You have a question or problem with your tutor's comments on any assignment or with the grading of an assignment.

You are advised to ensure that you attend tutorials regularly. This is the only opportunity to have a face-to-face contact with your tutor and ask questions. You can raise any problem encountered in the course of study. To gain the maximum benefit from course tutorials, prepare a question list before attending them and ensure you participate maximally and actively.

Course Information

Course Code: ECO 153

Course Title: Introduction to Quantitative Method I

Credit Unit: 3

Course Status: Compulsory

Course Blub:

Semester: First Semester

Course Duration: 32 hours in course of 12 weeks Required Hours for Study

MODULE 1 MATHEMATICAL ECONOMICS, NUMBER AND NUMERATION

Unit 1 Nature of Mathematical Economics

Unit 2 Set Theory

Unit 2 Fraction, Ratio, Proportion and Percentages

Unit 3 Multiple and Lowest Common Multiples (LCM)

Unit 4 Factors, Highest Common Factors (HCF) and Factorisation

Unit 5 Indices, Logarithms and Surds

Unit 1 Nature of Mathematical Economics

Unit Structure

1.0 Introduction

2.0 Learning Outcomes

3.0 Concept of Mathematical Economics

3.1 Language of Mathematical Economic

3.2 Economic Model.

3.3 Concepts in Economic Model Building

1.0 Introduction

Mathematical economics is not a basic branch of economics like public finance or international. But it is a tool for economic analysis, which economist use to analyse a statement of the problem mathematically as it relates to mathematical theorems to aid economics reasoning be it microeconomic or macroeconomic theory, public finance, urban economics, and other branches of economic. In using mathematical economics, most textbook of economics uses geometrical methods to derive theoretical inferences. Classically, mathematical economics use mathematical techniques that go beyond simple geometry, like algebra, differential and integral calculus, differential equations, difference equations, etc. So, this unit introduce the most fundamental of mathematical methods daily encountered in economic literature.

1.2 Learning Outcomes

After reading this unit, students will be able to:

- Understand the basic mathematical notations as it relates to economics
- Understand the basic mathematical methods and techniques use in economics.
- Apply mathematical methods and techniques to economic assumption and economic theories.

3.0 Concept of Mathematical Economics

Mathematical economics is a tool for economic problems analysis and it does not differ from non-mathematical tool of economic problems analysis. The aim of any economic theory analysis, regardless of the tool use in the analysis is to arrival at a set of inferences or theorems from a given set of assumptions or postulates via human reasoning. The major difference between mathematical economics and non-mathematical economic sometime call "literary economics" is that MATHEMATICAL ECONOMICS uses mathematics symbols and in equations for analysis and the NON-MATHEMATICAL uses words and sentences, that literary logic. It does not matter which tool is use in economic, however, without argument, symbols and equations are better fit for inferences with high precision. So, mathematical economics has the following advantages;

- it uses a more concise and precise language for analysis;
- existences of mathematical theorems for use in the analysis;
- help to state explicitly all prerequisite assumption for the use of mathematical theorems reduces pitfall from unintentional adoption of unwanted implicit assumptions; and
- it allows us to treat the general n-variable.

The advantages of mathematical economics are obvious and the possibility of introduce all mathematical tools used by economists in a single volume low. So, need to concentrate on those fundamental mathematic to economics. Sometime, mathematical economics is compared to econometrics because of the term "metric" in econometrics. But econometrics is concern mainly with the measurement of economic data for empirical observations using statistical methods of estimation and hypothesis testing. Mathematical economics here apply mathematics to pure theories of economic analysis, with no concern about such statistical problems as the errors of measurement of the variables under study.

Self-Assessment Exercises 1

- What are the tools for economic problems analysis
- List three advantages and two disadvantages of using mathematics economics tools for economic problems analysis.

3.1 Language of Mathematical Economic

The mathematical economic is generally built-up by series of definitions consists of propositions. Which include;

- Axiom is a statement that is assumed to be true. Axioms define basic concepts like sets, natural numbers or real numbers: A family of elements with rules to manipulate.
- Theorem is a statement that describes properties of an object that need to be proof to be true. If the theorem is true, theorem become a new definition.

Of course, the development in mathematics is not straightforward as indicate. It is rather a tree with some additional links between the branches.

3.2 Economic Model.

Economic theories are abstraction from the real world. The complexity of the real economy makes it impossible to understand all the interrelationships and the interrelationships of equal importance of understanding economic phenomenon to be study. So, there is need on the primary factors and relationships relevant to the phenomenon. This deliberate simplification of analytical framework is called economic model. That is, the skeletal and rough representation of the actual phenomenon. It is important to know that Economists, like scientists. build models in order to increase the understanding of the real-world economic phenomenon. An Economic model is a theoretical framework that organized set of relationships to describes the functioning of an economic agent under a set of assumptions from which a conclusion or a set of conclusions is logically inferred. The economic agent may be a household, a single industry, a region, an economy or the world as a whole. Sometimes economists use the term theory instead of model. Models is more ideas and situations, while theory combine the ideas and situations in an abstract manner. That is, Model is use to test a theory's validity and Theories is are derived from real-life events. An economic model can also be defined as a set of economic relationships which is generally expressed with help of mathematical equations. To deliberate simplify analytical framework economist do two things; One, leaves out many elements that operate in reality, second, use falsifies evidence in a number of respects. Instead of representing a real situation, economist explains the necessary relationships that are sufficient for analysing and explain the main attribute of a particular situation at hand.

Concluding, economic models are simplified representations of the real-world economic phenomenon that is used to analyse the underlying economic principles at work. Economic models are built on assumptions about how economic agent (people, firms, and markets) behave using mathematical and statistical techniques to make predictions and test hypotheses about economic phenomena.

Economic assumptions are statements about how people, firms, or markets are expected to behave in a given situation. Like, assumption that people always act in their own self-interest or that firms always maximise profits. Those are economic assumptions simplifications of reality which may not always hold in all situations. Ceteris paribus "other things being equal." use in economics has the assumption that all other factors are held constant in order to isolate the effect of a single variable on an economic outcome. It is often used in building economic models. To build a model for the effect of change in taxes on consumer spending, the economist holds all other factors constant, such as income, prices, and consumer confidence to get effect of the tax change on consumer spending, without the influence of other variables.

3.3 Concepts in Model Building

Economist analysis an economic problem quantitatively by first building mathematical model that simplified and describes the real-world economic phenomenon intent to explain and the related assumptions.

A mathematical model has three (3) elements

- Dependent variable, i.e., the economic phenomenon to be explained.
- Independent variable. there could be one or more independent variables. (The factors that determine the variation in the dependent variables.
- The behavioural assumptions which explain the nature of the causal relationships between independent variable (explanatory) and dependent variables.

VARIABLES: - A variable is anything, number, or quantity that can be counted or measured. Age, sex, business income and expenses, country of birth, capital expenditure, class grades, eye colour and vehicle type are all variables. A variable magnitude or quantity can change over a specified time period under consideration. There two major types of variables; Numeric variables and Categorical variables. Numeric variables have values that are measurable in quantity of number, numeric variables are quantitative variables. Numeric variables are either continuous or discrete:

- A continuous variable can take any value between a certain set of real numbers. Examples of continuous variables include height, time, age, and temperature.
- A discrete variable can only take a value based on count of distinct whole values but cannot take the value of a fraction. Discrete variables include the number of student and number of class room.

While categorical variables are qualitative variables and tend to be represented by a non-numeric value. Categorical variables are either ordinal or nominal:

- An ordinal variable can take a value that can be logically ordered or ranked. Examples of ordinal categorical variables include academic grades (i.e. A, B, C), clothing size (i.e. small, medium, large, extra-large) and attitudes (i.e. strongly agree, agree, disagree, strongly disagree).
- A nominal variable can take a value that is not able to be organised in a logical sequence. Examples of nominal variables include sex, business type, eye colour, religion and brand.

Dependent and Independent Variable: - A dependent variable depends on other variable(s) that are measured and the variable(s) are expected to change the value in dependent variable or variables. It is the presumed effect. For example, other things being equal, demand varies inversely with price, demand is the dependent variable and price is the independent. So, Independent Variable is stable and unaffected by the other variables you are trying to measure. It the presumed cause variables.

ENDOGENOUSVARIABLES AND EXOGENOUSVARIABLES: - Endogenous variables are those whose values are determined from within the model. On the other hand, there may be certain variable in the model whose values are determined by external forces. Such variables are exogenous variables. Endogenous variables frequently used in model building are demand, supply, national income, consumption, saving, investment, etc. while exogenous variables are price, export, import, revenue, labour force, and investments. etc.

Flow Variables and Stock Variables: - Flow variable is a quantity that can be measured in terms of specified period of time and a stock variable is a quantity that can be measured at a specified point in time. The market demand and supply schedules are flow variables while the supply of a commodity available in the market at specific point in time is a stock variable.

Constants: - A constant is the variable in model whose value does not change. Thus, it is the opposite of dependent variable and independent variable. When a constant is joined to a variable. it is called the coefficient of that variable.

Parameters: - Variables are quantities which vary from individual to individual.

By contrast, parameters do not relate to actual measurements or attributes but to quantities defining a theoretical model

A parameter is a symbol which is constant for the purpose of any particular problem but may assume different values in different problems. Although a parameter may be assigned different values, still it is regarded as a constant in the model. That is why, it is called a parametric constant. Parameters are normally represented bv such symbols a, b, and cord, or y.

Functional Relationship: - A functional relationship between two variables exists when a change in the value of one variable determines the change in the value of the other variable. If, for instance, we assign a single. definite Y - value to each X - value. then Y is a function of X which may be written as Y=f (x).

EQUATIONS: - There three types of equations that is use to express economic models: definitional, behavioral, and equilibnum.

a. DEFINITIONAL EQUATIONS: This equation specifies a relation between two alternate expressions having the same meaning. The sign = (equal to) is used for such an equation. E.g., total profit (Π) may be defined as the excess of total revenue (R) over total cost (C), which can be written as $\Pi = R – C$.

b. BERAVIOURAL EQUATIONS: A behavioural equation specifies how a variable behaves in response to changes in other variables. For instance, if we are to build a model to analyse the demand for tea, the behavioural assumption is the hypothesis that consumers always try to maximize their satisfactions in deciding how much tea to buy. Consider the demand functions.

   Qd = 800 - l 6p (1)

   Qd = 800 - 8p (2)

   Where; Qd is quantity demanded

   P is price

c. EQUILIBRIUM CONDITION: - equilibrium is when the demand is equal to the available supply. When a model involves the study of equilibrium, the equation that explains the attainment of equilibrium is called the equilibrium condition. The equilibrium condition for the market model is: Qd = Qs.

$Qd = 36 - 4p$ (1)

$Qs = -12 + 12p$ (2)

At eqmlibrium

$Qd = Qs$ (3)

Substituting (1) and (2) in (3) we have $36 - 4p = -12 + 12p$

$-4p - 12p = -12 - 36$

$-16 = -48$

$P = \frac{-48}{-16} = 3$

Putting the value of P in equations (1) and (2). we obtain the equilibrium condition

$Qd = 36 - 4(3)$

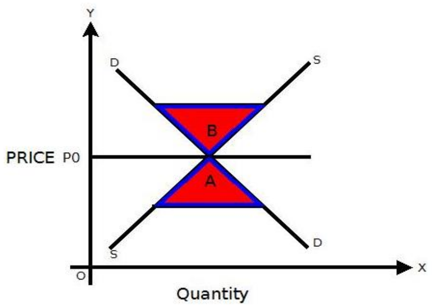
$Q_d = 36 - 12$

$Q_d = 24$

$Q_S = -12 + 12(3)$

$Q_S = -12 + 36$

$Q_S = 24$



Qd = Os = 2

The assumptions of equilibrium condition are:

1. The quantity demanded is a decreasing function of price.
2. The quantity supplied is an increasing function of price.
3. Quantity demanded and quantity supplied are stock variables
4. The market is in equilibrium when excess demand is zero, i.e., $Q_d - Qs = 0$.

Simply put, the equilibrium condition is $Q_d = Os$.

4.0 Summary

This study unit exposes the following;

- The tools for economic problems analysis
- The two economic problems analysis tool (MATHEMATICAL ECONOMICS and the NON-MATHEMATICAL)
- Mathematical economics is a tool for economic problems analysis
- Economics study is an observe real world phenomena

- Economics develop theories relating to real world economic phenomena
- Economics formulate and test hypothesis to ascertain the theories predictions
- Economics compare theories predictions with real world economic phenomena, to Accept, or Reject, or Modify the theories
- Compare mathematical economic analysis advantages and nonmathematical economic analysis advantages.
- Language of Mathematical Economic

5.0 References/Further Readings/Web Resources

- Chiang, A.C. (1984) Fundamental Methods of Mathematical Economics. 3rd Edition, McGraw-Hill, Singapore.
- Akihito Asano (2012) An Introduction to Mathematics for Economics. Cambridge University Press. doi.org/10.1017/CBO9781139035224
- Karl, J. S. (1990). Mathematics for Business. US: Win C. Brown Publishers.
- Lucey, T. (1988). Quantitative Techniques: An Introductional Manual. (3rd ed.). London: ELBS/DP **Publications**

6.0 Self-Assessment Exercise(s)

1. True or False Market equilibrium refers to a
    - Condition where a market price is established through competition such that the amount of goods or services sought by buyers is equal to the amount of goods or services produced by sellers.
    - Condition where a market price is established through competition such that the amount of goods or services sought by buyers is not equal to the amount of goods or services produced by sellers.

- If an economist explains economic issues verbally, such is called _____________.
    A. Non-mathematical economics
    B. Mathematical economics
    C. Statistic economics
    D. Behavioural economics
- Observably, the mathematical approach has the follow
    A. All of mention
    B. concise and precise use of language
    C. keeps economists from unintentional adoption of assumptions; and
    D. allows economists treat the general n-variable

1. What is your understanding of equilibrium condition
2. What are the assumptions of equilibrium condition

Unit 2 SETS, NUMBERS, AND FRACTIONS

Unit Structure

1.0 Introduction

In this unit, there is the need to introduction the notion of a set, the concept of real number system and the various component of the real number system which are; integers, rational number, irrational numbers, fractions, decimals and imaginary numbers. Equations and variables are the essential ingredients of mathematical expressions or models. The values of economic variables are usually in number. Numbers are relevant to mathematical analyses so are the different forms of number that make up the real number system.

2.0 Learning Outcomes

After reading this unit, students will be able to:

- Understand Notion of a Set

- explain the concept “numbers”
- identify the relevance of number system in quantitative analyses
- outline various forms of numbers in the real number system with examples
- draw the chart showing the number system.

3.0 S Set Theory

3.1 Introduction to Notion of a Set

The collection of any defined objects is called a set. That is, the possibility to determine readily whether or not a given object belongs to a collection. Any objects in a given set are called elements or members. A set is stated by a capital letters and elements or members in a set is stated by small letters. So, x is an element in the set A, the equation is written as $x \in$ A and if y is not an element of set A, the equation is written as y $\notin$ A. The number of elements in set A is stated by *n* (A). There are two class of set; *finite set* and *infinite set*; the finite set has a definite number of elements while an infinite set has no definite elements, it does not end.

A set define by listing its elements;

$$A = | \ 2, 4, 6, 8 \ |$$

That is, set A has 2, 4, 6, and 8 as its elements or members. Again, set can be defined by stating the properties in the elements of the set. like set $A = | 2, 4, 6, 8 |$ can be written $A = | \ 2n{:}\ 1 \leq n \leq 4$, and n is an integer $|$.

3.1.2 The empty set

There is time that a set contains no elements. Like, the set of triangles with four sides contains no elements, since there is no triangle with four sides. A set without an element is called the *empty* or *null* set and it is stated by Φ.

3.1.3 Subsets

If all elements in set A are again the elements in set B, it can be stated that set A is a *subset* of set B or that A is contained in B. The equation is written as $A \sqsubseteq B$ or $B \supset A$. The equation is written as $| a, b | \subset | a, b, c |$ since each of the two elements in $| a, b |$ belongs to $| a, b, c |$. From this definition, all sets are subset of itself; that is, for all <u>A, A</u> $\sqsubseteq A$. Any subset of a set that is not the set itself is called a *proper* subset of that set. (Some time the symbol $\sqsubseteq$ is use for subset and the symbol $\subset$ for a real subset). An empty set by definition is a subset of every set. The subsets of $| a,b |$ are $| a,b | \ | a | \ b |$ and Φ. It can be shown that a set of n elements has 2^n subsets. Note; there need for carefulness in distinguishing between the elements of a set and subsets of a set. For example, if $C = | a,p |$; then a is an element of C, again $| \ a \ |$ is a set that has a as an element. So, it can be written $a \in C$, and $| \ a | \subset C$.

3.1.4 Equality of sets

Two sets are side to be equal when they contain same elements. That is, set A is equal to set B, it is written as A = B and when set T is not equal to set S, it is written as $T \neq S$. Therefore, $| 2,3 | = | 3,2 |$ and $| 2,3 | \neq | 2,4 |$.

Note; the order of arrangement or listed of elements in a set does not matter. So, can take from $B = D \Leftrightarrow B \subset D$ and $D \subset B$. This provide a way to show that two sets are equal. So, all empty sets are equal, for empty sets to be unequal there is need for one the sets to contain an element, which is impossible since the sets are empty sets without any element.

3.1.5 Laws of Set Operations

(a) Idempotent Properties: This property states that the union and intersection of the same set gives the set.

AUA = A

AnA = A

For example, given that A = {1, 2, 3, 4, 5, 6,}

B = {1, 2, 5, 7, 10}

AUA = {1, 2, 3, 4, 5, 6} U {1, 2, 3, 4, 5, 6}

= {1, 2, 3, 4, 5, 6} = A

Likewise, BnB = {1, 2, 5, 7, 10} n {1, 2, 5, 7, 10}

BnB = {1, 2, 5, 7, 10} = B

(b) Commutative Properties: The commutative law states that

AnB = BnA and AUB = BUA.

This implies that the order of the set does not matter in their union or intersection.

For example:

If A = {1, 2, 3, 4, 5, 6} and

B = {3, 4, 5, 7, 10}

Then,

AnB = {3, 4, 5}

Likewise, BnA = {3, 4, 5}

Hence, AnB = BnA = {3, 4, 5}

Similarly, AUB = BUA

A = {1, 2, 3, 4, 5, 6}, B = {3, 4, 5, 7, 10}

∴ AUB = {1, 2, 3, 4, 5, 6, 7, 10} and

BUA = {1, 2, 3, 4, 5, 6, 7, 10}

Hence, AUB = BUA and AnB = BnA

(c) Associative Properties: This state that

AU (BUC) = (AUB) UC and An(BnC) = (AnB)nC

For example,

let A = {1, 2, 3, 4, 5, 6}

B = {2, 4, 6, 8, 10}, C = {3, 6, 9, 10}

Then, BUC = {2, 3, 4, 6, 8, 9, 10}

AU (BUC) = {1, 2, 3, 4, 5, 6} U {2, 3, 4, 6, 8, 9, 10}

= {1, 2, 3, 4, 5, 6, 8, 9, 10}

Likewise,

AUB = {1, 2, 3, 4, 5, 6, 8, 10}

C = {3, 6, 9, 10}

∴ (AUB) UC = {1, 2, 3, 4, 5, 6, 8, 9, 10}

∴ AU (BUC) = (AUB) UC

In the same manner,

An(BnC) = (AnB)nC

A = {1, 2, 3, 4, 5, 6}

BnC = {6, 10}

.

∴ An(BnC) = {6}

AnB = {2, 4, 6}

C = {3, 6, 9, 10}

∴ (AnB) nC = {6}

Hence, (AnB) n C = {6} = An (BnC) = {6}

(d) Distributive Law: This property states that

AU(BUC) = (AUB) n (AUC) and An(BUC) = (AUB) U (AnC)

From the example above A = {1, 2, 3, 4, 5, 6}

B = {2, 4, 6, 8, 10}

C = {3, 6, 9, 10}

AnC = {6, 10

A = {1, 2, 3, 4, 5, 6}

∴ AU(BnC) = {1, 2, 3, 4, 5, 6, 10)

But AUB = {1, 2, 3, 4, 5, 6, 8, 10}

AUC = {1, 2, 3, 4, 5, 6, 9, 10}

(AUB)n(AUC) = {1, 2, 3, 4, 5, 6, 10}

Hence,

AU(BnC) = (AUB) n (AUC) = {1, 2, 3, 4, 5, 6, 10}

Likewise, we can proof that An(BUC) = (AnB)U(AnC)

BUC = {2, 3, 6, 8, 9, 10}

A = {1, 2, 3, 4, 5, 6}

An(BUC) = {2, 3, 6}

AnC = {3}, (AnB) = {2, 6}

∴ (AnB)U(AnC) = {2, 3, 6}

Hence, An(BUC) = (AnC)U(AnC) = {2, 3, 6}

(e) Identity Properties: This states that the union or intersection of a given set with respect to an empty set gives the setback.

For instance, AUØ = A, AnØ = A If A = {1, 2, 3, 4, 5, 6,}   Ø = { }

∴ AUØ = {1, 2, 3, 4, 5, 6} = A

AnØ = {1, 2, 3, 4, 5, 6} = A

Similarly, the union of a given set with respect to the universal set gives the universal set while the intersection of a given set with respect to the universal set, gives the setback i.e. AUμ = μ and An μ = A

E.g. if A = {1, 2, 3, 4, 5}

μ = {1, 2, 3, 4 …10}

Then $AU\mu = \{1, 2, 3\ldots10\} = \mu$

$An\mu = \{1, 2, 3, 4, 5\}\ n\ \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\therefore\ An\mu = \{1, 2, 3, 4, 5\} = A$

(f) Complement Properties: This states that

$AUA^1 = \mu$

$AnA^1 = \emptyset$ or { } or empty set $(A^1)^1 = A$

$\mu = \emptyset$

For example,

$A = \{1, 2, 3, 4, 5\}$

$\mu = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\therefore A^1 = \{6, 7, 8, 9, 10\}$

$\therefore AnA^1 = \{1, 2, 3, 4, 5\}\ U\ \{6, 7, 8, 9, 10\} = \{\}$

$AUA^1 = \{1, 2, 3, 4, 5\}\ U\ \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \mu$

$A^1 = \{6, 7, 8, 9, 10\}$

$\therefore (A^1)^1 = \{1, 2, 3, 4, 5\} = A$

$\mu = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\mu^1 = \{\ \}$ or $\emptyset$ (Empty set)

(g) De Morgan's Law This states that $(AUB)^1 = A^1nB^1$ and that $(AnB)^1 = A^1UB^1$

For example, $\mu = \{1, 2, 3 \ldots 10\}$

$A = \{1, 2, 3, 4, 5, 6\}$

$B = \{4, 6, 7, 8\}$

Then

$AUB = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$(AUB)^1 = \{9, 10\}$

$A^1 = \{7, 8, 9, 10\}$, $B^1 = \{1, 2, 3, 5, 9, 10\}$

$A^1nB^1 = \{9, 10\}$

Hence $(AUB)^1 = A^1nB^1$

Similarly $(AnB) = A^1UB^1$

From the example above $AnB = \{4, 6\}$

$(AnB)^1 = \{1, 2, 3, 5, 7, 8, 9, 10\}$

$A^1 = \{7, 8, 9, 10\}$, $B^1 = \{1, 2, 3, 5, 9, 10\}$

$A^1UB^1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9. 10\}$

Hence $(AnB) = A^1UB^1$

SELF-ASSESSMENT EXERCISE

i. Which of the following statement(s) is/are valid?

(a) $AUA = A$

(b) AnA = A

(c) AUØ = A

(d) AUµ = U

(e) AnØ = Ø

(f) Anµ = A

(g) The complement of A1 is A (i.e. (A1)1=A)

(h) (BnC) = B1n C1

(i) AU(BUC) = An(BnC)

(j) An (BUC) = (AnB)U(AnC)

ii. Given A = {4, 5, 6}, B = {3, 4, 6, 7} and C = {2, 3, 6}, verify:

(a) Associative law

(b) Distributive law

(c) De Morgan's law

3.1.6 The universal set

A set that has all objects or elements and all the other sets are subsets is call universal set. Universal set is denoted by U, it has elements of all the related sets and the elements of related sets are repeated. Where; C and D are two sets, such that as C has 1, 2, 3 as elements and D has 1, a, b, c, as elements, and the universal set is given as U = {1, 2, 3, a, b, c}. Note; for any set it is assume that there is a universal set, all sets and consideration are subsets of U. Again, different sets may have difference universal sets. So, there is need to restrict the elements to the relevant elements to the problem under study. That is, the universal set contain all relevant elements. If there is need study the average score of English students in Nigeria secondary school, the universal set is all students in secondary school be it junior or senior secondary school.

3.1.6.1 Venn diagrams

Venn diagram is use to show the relationships among objects or finite groups of things. Venn diagrams visually show the similarities and differences between two or more objects or elements. It is a usefulness educational tools to show elements contain in universal set.

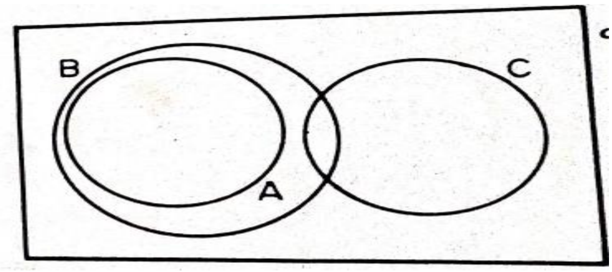


Fig. 1.1 Venn Diagram (universal Set)

SELF-ASSESSMENT EXERCISE

i. What is Venn diagram? What is its relevance to set theory?

ii. Draw a Venn diagram illustration the relationship between sets X and Y if:

(i) Some elements are common to X and Y

(ii) No element is common to X and Y

(iii) Y contains every element in X.

Simple Set Problems involving Venn Diagrams. The use of Venn diagram is applicable to all problems involving sets operations. However, it is very important to commence its application to simple set problems involving the use of numbers to indicate the elements or membership of a set.

Example 1

If A = {Prime factors of 30}

B = {Prime factors of 70}

C = {the prime factors of 42}

(a) List the elements of A, Band C

(b) Show their relationship in a Venn diagram

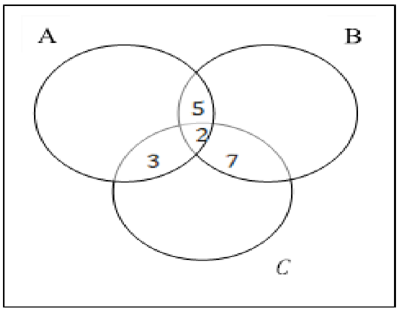
Solution

(a) A = {2, 3, 5}

B = {2, 5, 7}

C = {2, 3, 7}

(b)



It should be noted that '2' is a common element to all the three sets so it takes the centre stage i.e., a point of intersection of A, B and C '5' is common to only A and C. The only number common to sets B and C is 7. Having considered the sets common to two or more sets, we observe sets are left in each of the three sets to occupy their independent spaces. Hence, the spaces were left vacant. It should also be noted that the universal set is not given, so there is no element found outside the three oval shapes representing sets A, B and C.

Example 2

Given the universal set as μ= {1, 2, 3, 4 …10}

A = {2, 3, 4, 5, 6}

B = {1, 2, 5, 7, 9}

C = {1, 3, 5, 9, 10}

Required:

(a) Present the information using Venn diagram.

(b) On different Venn diagrams shade the region which satisfy the following

(i) AnBnC

(ii) B^1

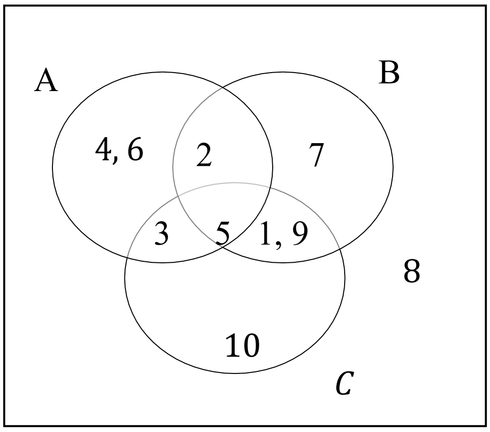
(iii) $(BUC)^1$

(iv) AUBUC

(v) $(AnBnC)^1$

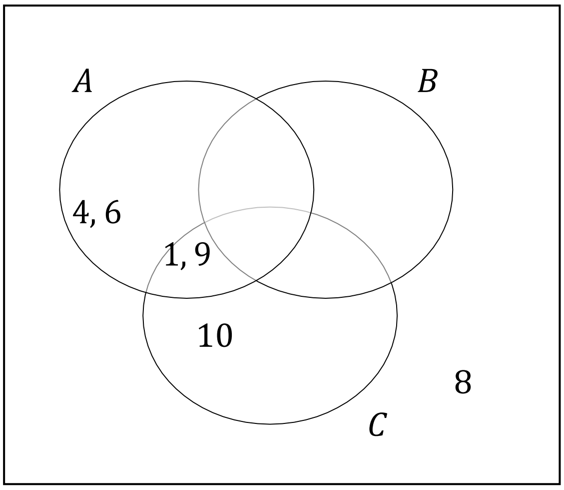
(c) Use the diagram in (b) above to list the elements of sets obtained from b (i)-(v) above.

Solution



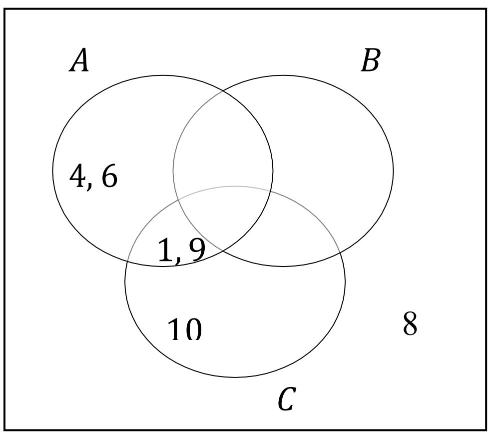
Note: '8' is an element of the universal set but not present in any of sets A, B and C. So, it is written outside the three oval shapes representing sets A, B, C.

b. (i) AnBnC



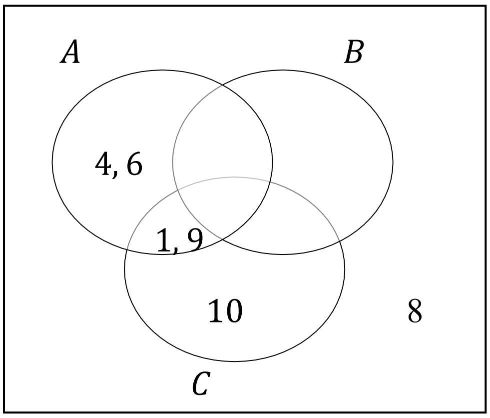
Note: The shaded area represents the region of intersection of A, B and C i.e. elements common to the three sets.

(ii) B^1



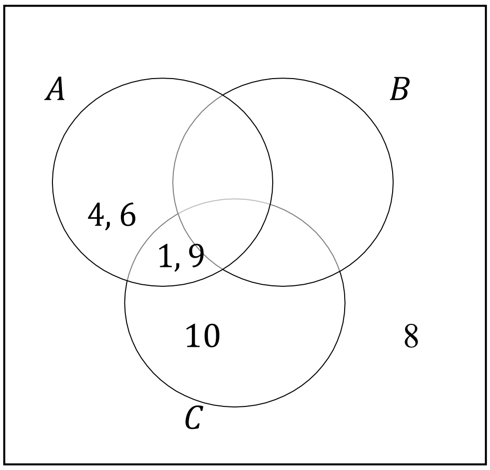
Note: B^1 is represented by the elements found outside set B i.e., all the areas set B are shaded.

(iii) (BUC)^1



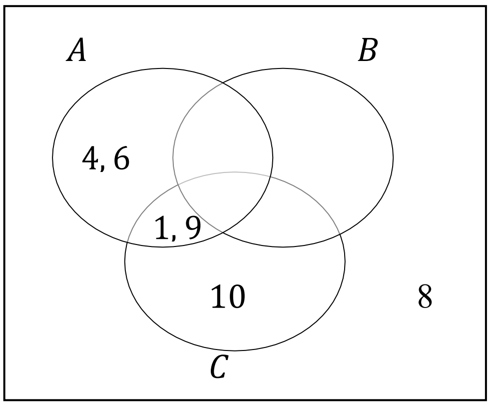
Note: (BUC)^1 implies the area outside the enclosure of B and C. So, all the spaces outside (BUC) are shaded.

(iv) AUBUC



Note: (AUBUC) is the totality of elements in sets A, B and C. Hence, all the set spaces are shaded.

(v) (AnBnC)^1



Note: (AnBnC)^1 implies the area outside (AnBnC). Hence the entire space is shaded with the exception of (AnBnC).

C (i) AnBnC = {5}

(ii) B^1 = {3, 4, 6, 8, 10}

(iii) (BUC)^1 = {4, 6, 8}

(iv) AUBUC = {1, 2, 3, 4, 5, 6, 7, 9, 10}

(v) $(AnBnC)^1$ = {1, 2, 3, 4, 6, 7, 9, 10}

SELF-ASSESSMENT EXERCISE

i. Given that

A = {all prime numbers between 10 and 30}

B = {all even numbers between 9 and 29}

C = {all multiples of 3 between 10 and 29}

Required: (a) List the elements of each set and draw a Venn diagram showing the relationship among the sets.

(b) In separate Venn diagrams, show the region that satisfy each of

(i) AU(BnC)

(ii) (AnB)UC

(c) List the elements of b(i) and b(ii) above

(d) Is AU(BnC) = (AnB)UC? What do you think is responsible for this?

ii. If P and Q are sets, using a Venn diagram, shade the following relationship on different diagrams

(a) PnQ

(b) PUQ

(c) P^1

(d) Q^1

This unit exposes us to the origin of Venn diagram, the relevance of Venn diagram in set theory and analyses as well as the practical application of using Venn diagram in presenting elements of sets and their interrelationship as well as using the same diagram to solve problems involving set notations. Representation of sets and their elements with the use of diagram (Venn diagram), is not only meant to show the interaction among the elements or among the sets, it is also an alternative way of solving problems involving set notations. This is illustrated in Example 2 of this unit.

3.1.7 Application of Sets Theory to Managerial and Economic Problems

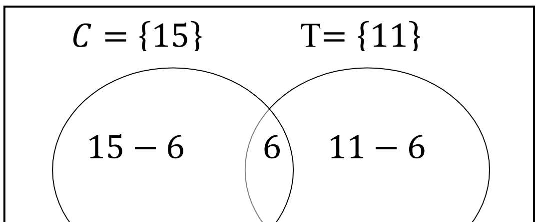
In our day-to-day activities, we are confronted with categorisation of people or objects in different classes or division. In the course of doing this, there may be overlapping of one element appearing in two or more sets. For instance, students in a class may be able to speak English language or French fluently. Among these set of students, some may be able to speak the two languages and some may not be able to speak any of them fluently. In the course of analysing this, there may be a need to introduce Venn diagram.

Another example is the elective courses taken by students in higher institutions. For example, if course A, B and C are made available as elective courses. Some students may offer one of the courses; some offer combination of two courses while some may offer the three courses. It is equally possible for some students not to offer any of the three courses. To be able to carry out mathematical analyses on the distribution of the students that offer courses and combination of courses, the use of appropriate Venn diagram may be required.

Example 1

A survey in a class shows that 15 of the pupils play cricket, 11 play tennis and 6 play both cricket and tennis. How many pupils are there in the class, if everyone play at least one of these games?

Solution



$\mu = x$

n(C) = 15; No of Pupils that play Cricket

n(T) = 11; No of Pupils that play Tennis.

Note: If 6 pupils play both games, recall that, the pupils are part of the entire pupil that play cricket so the remaining pupil who play cricket and not tennis equals to 15-6 = 9. Likewise, for Tennis, some of the 11 students are already part of the 6 pupils that play both games. Therefore, those who play tennis alone equals to 11-6 = 5.

Since everybody play at least one of these games, it implies that there are no pupils that play neither of the two games.

∴ Number of students in the class = 6 + 9 + 5 = 20 pupils

∴x = n(CUT) = 20

Example 2

A survey conducted for 50 members of staff in an organisation shows that 30 of them have degree while 35 possess Diploma. If 8 of the employees possess neither of the certificates.

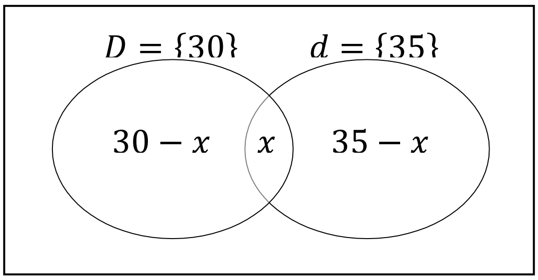
Required

(a) How many have Degree but not Diploma

(b) How many have both Degree and Diploma

Solution

$\mu = 50$



"D" represents the employees who have Degree

"d" represents the employees who have Diploma

"x" represents the employees that have both degree and diploma

μ = Universal set or set of all employees

∴ x = n (D n d)

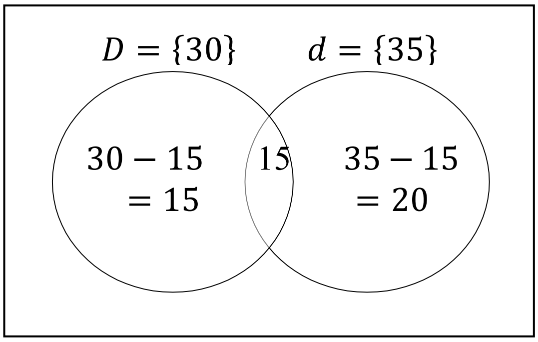
∴ 30 – x + x + 35 – x = 50 30 + 35 – x = 50

65 – x = 50

65 – 50 + x

x = 15

The Venn diagram can now be redrawn as thus:



$\mu = 50$

Note: 15 + 15 + 20 = μ

μ = 50

(a) No of employees that have Degree but not Diploma = 30 – x = 30– 15 = 15

(b) No of employees that have both Degree and Diploma x = 15

Example 3

Given that n (X) = 24, n(Y) = 28, n(XUY) = 40, n(XUY)1 = 5. Use Venn diagram to find

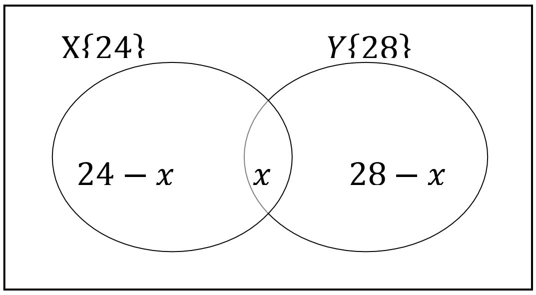
(i) n(XnY)

(ii) n(X1)

(iii) n(XnY)1

(iv) μ

Solution



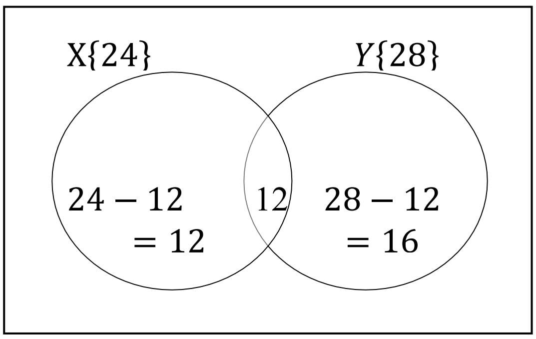
Note: n(XnY) = 24 – x + x + 28 – x = 40

5 2- x = 40

52 – 40 = x

x = 12

The Venn diagram can now be redrawn as



From the Venn diagram

(i) n(XnY) = 12

(ii) n(X1) = 16 + 5 = 21

Note: n(X1) represents values outside the oval shape of X i.e. 16 and 5.

(iii) n(XnY)1 = 12 + 16 + 5 = 33

Note: $n(XnY)^1$ represents the values outside the intersection of X and Y. These values are 12, 5, and 16.

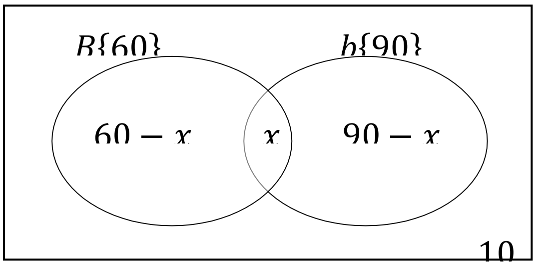
(iv) μ = 12 + 12 + 16 + 5 = 45

Note: μ is the totality of all the values in the Venn diagram.

Example 4

In a school of 140 pupils, 60 have beans for breakfast and 90 have bread. How many students have both beans and bread for breakfast if 10 pupils took neither beans nor bread?

$\mu = 140$



n(B) = no of pupils that take beans = 60

n(b) = no of pupils that take bread = 90

x = no of pupils that have both beans and bread for breakfast.

60 – x + x + 90 – 10 = 140

160 – x = 140

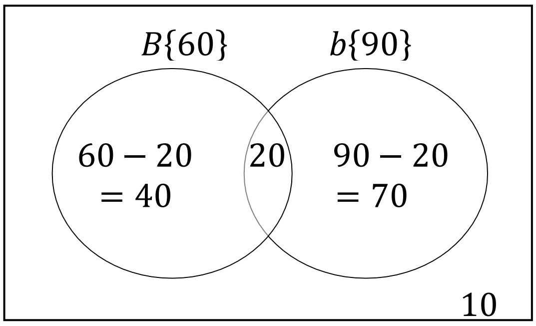
160 – 140 = x

20 = x

x = 20

` The Venn diagram is redrawn as follows:

$\mu = 140$



Note: 10 + 40 + 20 + 70 = 140

40 + 20 = 60

20 + 70 = 90

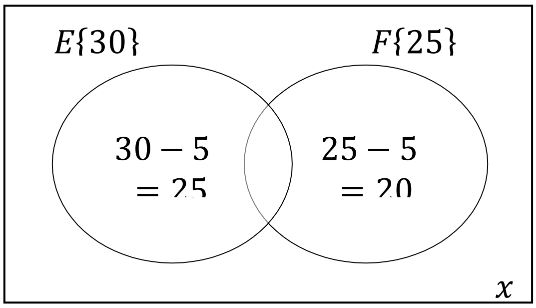
∴ 20 pupils have both beans and bread for breakfast

Example 5

In a class of 60 students, 30 speak English Language fluently while 25 speak French language fluently. If 5 students can speak both languages, how many cannot speak either of the languages? How many speak only one language?

Solution

$\mu = 60$



n(E) = No of student that speak English language fluently = 30

n(F) = No of student that speak French Language fluently = 25

n(EnF) = no of student that can speak both English Language and French = 5

x = No of students who cannot speak any of the Languages

$\therefore$ x + 25 + 5 + 20 = 60

x + 50 = 60

x = 60 – 50

x = 10

Therefore, those who speak only one language are those who speak French only and those who speak English only.

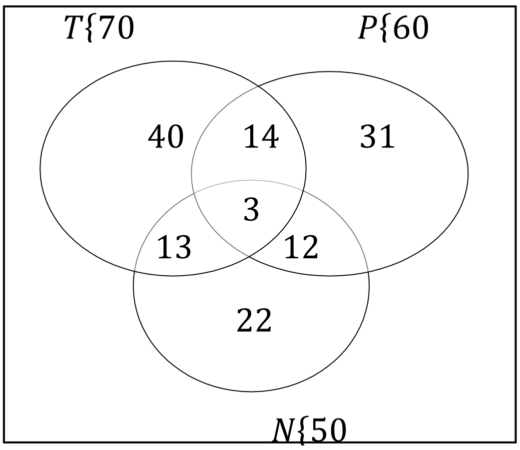
= (30 – 5) + (25 - 5)

25 + 2 = 4

Example 6

A newspaper agent sells three papers, the Times the Punch and the Nations. 70 customers by the Times, 60 the Punch and 50 the Nations. If 17 customers buy both Times and Punch, 15 buy Punch and the Nations and 16 the Nation and the Times. How many customers have the agent if 3 customers buy all the papers?

Solution $\mu = x$



n(T) = no of customers that buy the Times

n(P) = no of customers that buy the Punch

n (N) = no of customers that buy the Nation.

∴ Those who buy only times = 70 – [14 + 3 + 13]

= 70 – [30] = 40

Those who buy only Punch = 60 – [14 + 3 + 12]

= 60 – [29]

= 31

Those who buy the Nation only = 50 – [13 + 3 + 12]

= 50 – [16 + 12]

= 50 – 28 = 22

∴ The total number of customers

= 40 + 14 + 31 + 12 + 22 + 13 + 3 = 135 Customers.

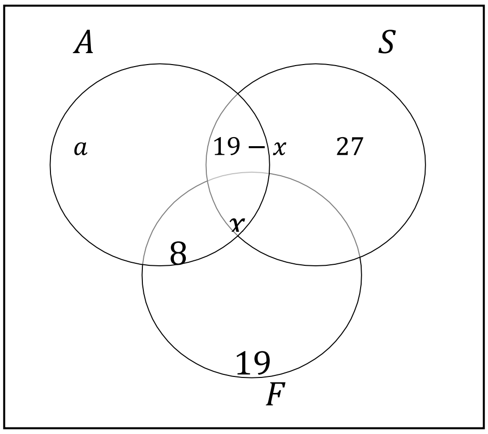
Example 7

Of the 112 students from college of Education, 19 read both Accountancy and Sociology, 70 read Accountancy or Sociology but not French, 27 read Sociology but not Accountancy or French, 53 read Sociology or French but not Accountancy, 19 read French but not Accountancy or Sociology and 8 read Accountancy and French but not Sociology. Assume that each student reads at least one of these courses. How many of the students read:

(i) All three courses?

(ii) Only one course?

(iv) Only two courses?



$\mu = 112$

n(A) = no of students who read Accountancy

n(S) = no of students who read Sociology

n(F) = no of students who read French

n = no of students who read Accountancy, Sociology and French

i.e. those who read all the three courses.

n(AUS) = 70

∴ a + 19- x + 27 = 70 a – x + 46 = 70

a = 70 – 46 + x

a = 24 + x (i)

n (S U F) = 53

b + 19 + 27 = 53

b + 46 = 53

b = 53 – 46 = 7

Since n(AUSUF) = 112

∴ a – 19 – x + 8 + x + 19 + b + 27 = 112

a – 19 – x + 8 + x + 19 + 7 + 27 = 112

a + 80 = 112 (ii)

But from equation (i) a = 24 + x

Substitute (i) into (ii) 24 + x + 80 = 112

x + 104 = 112

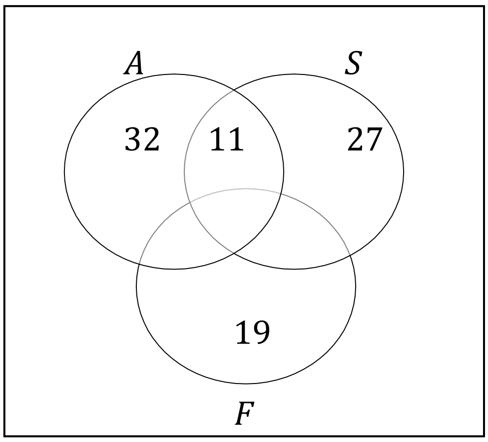
x = 112 - 104

x = 8

∴ a = 24 + 8 = 32

Then, the Venn diagram is drawn as thus:

$\mu = 35$



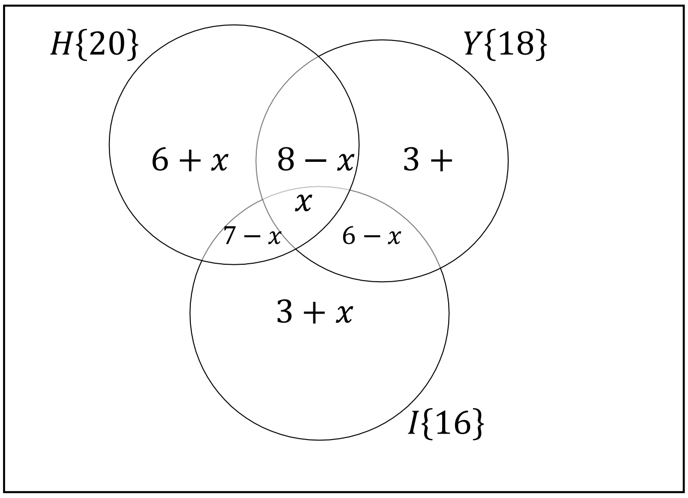
(i) Those who did all the three courses = 8

(ii) Those who did one course = 32 + 27 +19 =76

(iii) Those who did two courses = 8 + 7 + 11 = 26

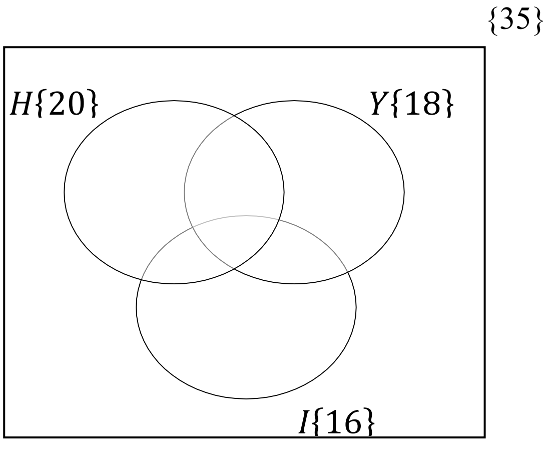
Example 8

In a class of 35 students, 20 speak Hausa, 16 speak Igbo and 18 speak Yoruba. If 8 speak Hausa and Yoruba, 7 speak Hausa and Igbo and 6 speak Yoruba and Igbo. How many speak all the three languages?



Those who speak Hausa only = 20 – [8 – x + x + 6 – x]
=20 – [14 – x]
= 6 + x
Those who speak Igbo only = 16 – [6 – x + x + 7 – x]
=16 – [13 – x] = 3 + x
Those who speak Yoruba only = 18 – [8 – x + x + 7 –x]
= 18 – [15 - x] = 3 + x
∴ H U Y U I = 50
6 + x + 8 − x + x + 6 − x + 3 + x + 7 − x + 3 + x = 35
33 + x = 35
x = 35 − 33 = 2
∴ 2 Student speak all the three languages.

The Venn diagram can be drawn as thus: -



The roles of Venn diagram in solving practical words interrelationship among elements of a set and sets in a universal set can best be illustrated with the use of Venn diagrams. The use of Venn diagram is not only a compact way of presenting set information, it is also an alter nature approach of solving practical problems involvin**g sets.**

4.0 Sets of Real Numbers

The real number system is the set of all the number that are quantifiable in real terms. Whole numbers such as 1, 2, 3, ... are called positive integer or natural numbers; these are the numbers most frequently used in counting. Their negative counterparts -1, -2, -3 ... are called negative integers. The negative integers are used to indicate sub-zero quantities e.g., temperature (degree). The number 0 (Zero), on the other hand is neither a positive or a negative integer and, in that sense, unique. Sometimes, it is called a neutral number. A set of negative and positive whole numbers (integers) can be presented with the use of number line as follows:

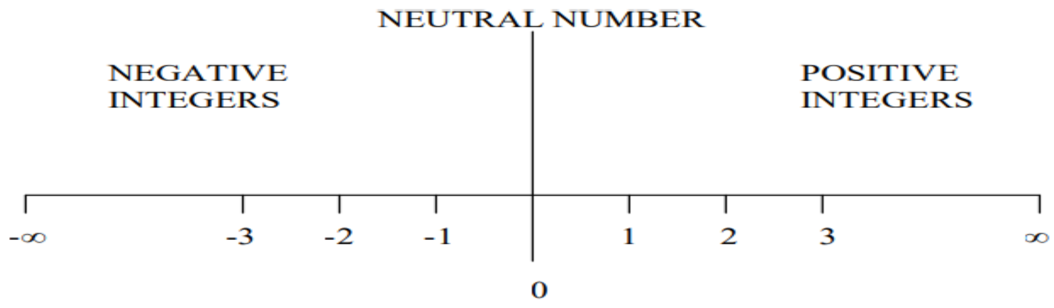


Fig. 1.2: Real Number System

Sometimes, positive integers, negative integers and zero are lumped together into a single category, referring to them collectively as the set of all integers.

The real-number system comprises of rational and irrational numbers. The set of all integers and set of all fractions form the set of all rational numbers while numbers that cannot be expressed as ratio of a pair of integers form the class of irrational numbers. Figure 1.3 gives a summary of the structure of the real-number system.

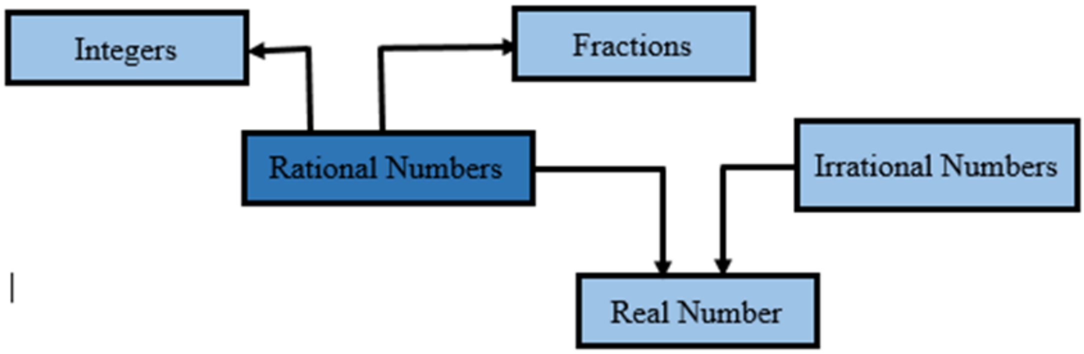


Fig. 1.3: Summary of the Structure of the Real-Number System

Integers, of course, do not exhaust all the possible whole number. Factors such as $\frac{2}{3}, \frac{5}{4}, \frac{7}{8}, \frac{1}{2}$, also fall within the number line showing the members of integers. Negative fractions such as $-\frac{1}{2}, -\frac{7}{8}, -\frac{5}{4}$ are also components of negative integers. In other words, integers are set of positive or negative whole numbers as well as the set of positive or negative fraction, zero inclusive.

The common property of all fractional numbers is that each is expressible as a ratio of two integers; thus, fractions qualify for the class of rational numbers (in this usage, rational means rational). However, integers are also rational because any integer, n can be considered as the ratio $\frac{n}{1}$. The set of all integers and the set of all fractions together form the set of all rational numbers.

Numbers that are not rational are termed irrational numbers. They are numbers that cannot be expressed as a ratio of a pair of integers. One example of irrational number is $\sqrt{2} = 1.4142...$, which is a non-repeating, non-terminating and recurring decimal. Another common example is $\prod = \frac{22}{7} = 3.1415$ (representing the ratio of the circumference of any circle to its diameter).

Each irrational number can also be placed in a number line lying between two rational numbers, just as fraction fills the gaps between the integers. The result of this filling – in the process gives a large quantum of numbers; all of which are so called "real numbers". The set of real numbers is denoted by symbol R. When the set of R is displaced on a straight line (an extended ruler), we refer to the line as the real line.

At the extreme opposite of real numbers are the set of number are the set of numbers that are not real. They are called imaginary numbers. The common examples of imaginary numbers are the square root of negative numbers e.g.,$\sqrt{-7}$, $\sqrt{-4}$, $\sqrt{-25}$, $\sqrt{-140}$ etc.

SELF-ASSESSMENT EXERCISE

i. Identify and discuss the various components of real number system.

ii. Of what relevance is 'number' to mathematics and quantitative analyses?

5.0 Summary

Numbers are generally quantifiable variables used in mathematics to describe the magnitude of quantities. There are two broad classifications of numbers namely: real numbers and imaginary numbers. While real number comprises of positive and negative whole numbers, positive and negative fractions and irrational numbers, the imaginary numbers are the extreme opposite of real numbers. They are number that cannot be placed on the real line. Basic mathematical skills and technique starts from numbers and numerations since numbers especially figures are the basis of counting, quantifying and carrying out mathematical operations. In mathematics and quantitative analysis, the roles play by numbers in all its ramifications cannot be over-emphasised. For this reason, every arithmetic class starts with the counting and writing of whole numbers. As students of arithmetic moves to mathematics class, other forms of numbers such as negatives numbers, fractions, decimals, irrational numbers and, imaginary numbers are introduced. Therefore, as the basis of numerical and quantitative analyses, numbers and numeration take the foremost and essential attention at all levels.

6.0 References/Further Readings/Web Resources

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7.0 Self-Assessment Exercise(s)

Classify each of the following numbers into integers, irrational number and imaginary numbers:

1. (i) $\frac{5}{8}$ (ii) $\sqrt{-9}$ (iii) $\sqrt{9}$ (iv) $\sqrt{23}$ (v) -20 (vi) $\frac{32}{5}$ (vii) e = 2.71828… (viii) -7 (ix) -(√8) (x)) $\frac{17}{7}$.
2. What are numbers?
3. What are the relevances of numbers to mathematics and quantitative analyses?
4. What are real numbers?
5. What are the components of real numbers?
6. Give two examples of each of the following types of numbers:
   a) Positive integers
   b) Negative integers
   c) Fractions
   d) Irrational numbers
   e) Imaginary numbers.

UNIT 3 FRACTION, RATIO AND PROPORTION

1.0 Introduction

In this unit, the lecture notes breakdown whole numbers into fractions, ratio and proportion. The division of a whole component into divisions is a day-to-day phenomenon. Individual's income is divided into ratio, fraction or proportion to ensure that all the basic needs are met, the same thing applies to the business organisation's profit and the governments resources, income and wealth. Therefore, it is important to study the fractional parts of a whole as it affects sharing scares resources in terms of ratio and proportion. In the previous unit, fractions have been identified as a component of real numbers which forms between the whole number lines. Therefore, in this unit, we shall examine the components of rational number known as fraction which is also expressed in ratio, proportion or percentages.

2.0 Learning Outcomes

After reading this unit, students will be able to:

- identify the relationship between fraction, ratio, proportion and percentages
- convert fractions to ratio, proportion and percentages and vice versa
- perform simple mathematical problems involving fraction, ratio, proportion and percentages
- solve some practical problems involving fraction, ratio, proportion and percentages.

3.0 Fractions

3.1 Forms of Fraction

Fractions are of different types. Some of the types are:

Proper Fraction: - This is a fraction which has its numerator smaller than the denominator e.g., $\frac{1}{2}, \frac{3}{4}, \frac{3}{5}$ etc.

Improper fraction: - This is a fraction which has its denominator smaller than the numerator e.g., $\frac{3}{2}, \frac{7}{4}, \frac{9}{5}$ etc.

Mixed Number: - A mixed number is a fraction that has two portions; the whole number portion and the proper fraction portion e.g., $1\frac{3}{5}$, $1\frac{2}{5}$, $3\frac{2}{7}$ etc. it should be noted that mixed numbers can be expressed as improper fraction e.g. $2\frac{1}{4} = \frac{9}{4}$ (the value of 9 is obtained by multiplying 4 by 2 and adding 1), $1\frac{2}{5} = \frac{7}{5}$, $3\frac{2}{7} = \frac{23}{7}$ etc.

Complex Number: -This is one in which the numerator or the denominator or both are either proper fraction, improper fraction or mixed numbers e.g., $\frac{\frac{2}{3}}{\frac{4}{6}}, \frac{2\frac{2}{3}}{4\frac{4}{6}}, \frac{\frac{2}{3}X\frac{5}{6}}{\frac{5}{4}X2\frac{1}{2}}$ etc.

SELF-ASSESSMENT EXERCISE

With appropriate examples, demonstrate the relationship between fraction, ratio, proportion and percentages.

3.2 Equivalent Fractions

When both numerator and denominator of a fraction are multiplied by the same number (except zero), another fraction is obtained. Likewise, when both the numerator and the denominator of a fraction are divided by the same number (except zero), another fraction of equal value is obtained. The fraction obtained in each of the cases is called the Equivalent Fraction.

Example: Express the following fractions in their lowest possible equivalent/term. (a) $\frac{3}{9}$ (b) $\frac{8}{12}$ (c) $\frac{4xy}{12xy}$

(d) $\frac{16_x5_y}{32xy^2}$

(a) $\frac{3}{9} = \frac{1}{3}$ (3 is the highest factor common to both numerator and denominator. Each of them is divided by 3 to obtain 1 /3).

(b) $\frac{8}{12} = \frac{2}{4}$ (4 is the highest common factor)

(c) $\frac{4xy}{12xy} = \frac{x}{3z}$ (4y is the highest common factor of both numerator and denominator, diving each by 4y gives $\frac{x}{3z}$).

(d) $\frac{32_x5_y}{64xy^2} = \frac{x}{2y}$ (32 is the higher common factor)

From the above;

$$\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{4xy}{12xy} = \frac{4\ X\ X\ X\ y}{3\ X\ 4\ x\ y\ x\ z} = \frac{x}{3z}$$

$$\frac{16_x5_y}{32xy^2} = \frac{2\ x\ 2\ x\ 2\ x\ 2\ x\ x\ x\ x\ y}{2\ x\ 2\ x\ 2\ x\ 2\ x\ 2\ x\ x\ x\ y\ x\ y} = \frac{x}{2y}$$

3.3 Addition and Subtraction of Fractions

When fractions have the same number as their denominators, their numerators can simply be added together or subtracted from one another, keeping their denominator constant.

a) $\frac{3}{7} + \frac{1}{7} = \frac{4}{7}$

b) $\frac{5}{9} - \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$

c) $\frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$

d) $2\frac{2}{3} + 3\frac{1}{3} = 5\frac{4}{6} = 5\frac{3}{3} = 5 + 1 = 6$

Where the denominators are different, the lowest common multiple is found first and the expression would then be simplified.

e) $\frac{1}{3} + \frac{3}{2} + \frac{1}{4} = \frac{4+8+3}{12} = \frac{25}{12} = 2\frac{1}{12}$

f) $2\frac{1}{4} + 1\frac{3}{3} = 3\frac{9\_-4}{12} = 3\frac{13}{12} = 3 + \frac{13}{12} = 3 + 12\frac{13}{12} = 3 + 1\frac{1}{12} = 4\ \frac{1}{12}$

g) $3\frac{1}{2} - 2\frac{7}{12} = 1\ \frac{6-7}{12} = \frac{(6+12)-7}{12} = \frac{18-7}{12} = \frac{11}{12}$

Note: In the example (g) above, 6 minus 7 appears impossible, so the whole number 1 is borrowed, where 1 in this context equals to $\frac{12}{12}$.

3.4 Multiplication of Fractions

In multiplication, the numerators are multiplied by each other while the denominators are also multiplied with each other. It should be noted that when numbers are multiplied by each other, the result is a bigger number but when fraction are multiplied by each other, the result is a smaller number e.g.

2 X 2 = 4, 3 X 3 = 9 4 X 8 = 23

$\frac{1}{2}\ X\ \frac{1}{2} = \frac{1}{4}$, $\frac{1}{3}\ X\ \frac{1}{3} = \frac{1}{9}$, $\frac{1}{2}\ of\ \frac{1}{6} = \frac{1}{2}\ X\ \frac{1}{6} = \frac{1}{12}$

It should be noted that, when multiplication is to be carried out for mixed numbers, they (the mixed number) are converted to improper fraction before the multiplication.

$$1\frac{2}{5}\ X\ 2\frac{1}{2}\ X\ 4\frac{1}{2} = \frac{7}{\not{5}}\ X\ \frac{\not{5}}{2}\ X\ \frac{9}{2} = \frac{63}{4} = 15\frac{3}{4}$$

(Note '5' in the numerator and the denominator cancels each other out)

3.5 Division of Fractions

In division of one fraction by other, the divisor is inverted (turned upside down) and then multiplied by individual i.e., the dividend is multiplied by the reciprocal of the divisor

$\frac{1}{3} \div \frac{2}{3}$ ($\frac{1}{3}$ is the dividend while $\frac{2}{3}$ is the divisor)

$\frac{1}{3} X \frac{2}{3} = \frac{1}{3}$

$\frac{7}{8} \div 5 = \frac{7}{8} X \frac{1}{5} = \frac{7}{40}$

When dealing with mixed numbers, the dividend and the divisor are first expressed in improper fraction and the principle of division is then applied. e.g.

$7\frac{1}{2} \div 3\frac{1}{3} = \frac{15}{2} X \frac{10}{3} = \frac{15}{2} X \frac{3}{10} = \frac{9}{4}$

$\frac{4ab}{2xy} \div \frac{3x}{2a} = \frac{4ab}{2xy} X \frac{3x}{2a} = \frac{3b}{y}$

3.6 Mixed Operation

By mixed operation, we mean a fraction that involves the mixture of operations (addition, subtraction, multiplication, division). In such situation, the rule of precedence is followed and this is summarised by the word "BODMAS" which gives the order as follows:

B = Bracket

O = Of

D = Division

M = Multiplication

A = Addition

S = Subtraction

This implies that in mixed operation, attention should first be based on bracket, followed by 'of', then division before multiplication. Addition signs are evaluated before the subtraction.

Simplify:

(i) $1\frac{6}{9} X \frac{2}{3} - 3\frac{1}{5} + 2\frac{1}{2} \div \frac{1}{2}$

(ii) $\frac{2}{3}\, of \left(2\frac{1}{3} - 1\frac{5}{7}\right)$

(iii) $\frac{1\frac{3}{4} + 2\frac{1}{2}}{4\frac{3}{5} + 1\frac{1}{3}}$

Solution:

(i) $1\frac{6}{9} X \frac{2}{3} - 3\frac{1}{5} + 2\frac{1}{2} \div \frac{1}{2} = \left[\frac{15}{9} X \frac{2}{3}\right] - \frac{16}{5} + \left[\frac{5}{2} X \frac{2}{1}\right] = \frac{10}{9} - \frac{16}{5} + \frac{5}{1} = \frac{50+144+225}{45} = \frac{181}{45} = 2\frac{41}{45}$

(ii) $\frac{2}{3}\, of \left(2\frac{1}{3} - 1\frac{5}{7}\right) = \frac{2}{3}\, of \left(1 - \frac{7-10}{14}\right) = \frac{2}{3}\, of \left(\frac{21-10}{14}\right) = \frac{2}{3}\, of \left(\frac{11}{14}\right) = \frac{2}{3} X \frac{11}{14} = \frac{11}{21}$

(iii) $\frac{1\frac{3}{4} + 2\frac{1}{2}}{4\frac{3}{5} + 1\frac{1}{3}} = \frac{\frac{7}{4} + \frac{5}{2}}{\frac{23}{5} - \frac{4}{3}} = \frac{\frac{7+10}{4}}{\frac{69-20}{15}} = \frac{17}{4} X \frac{49}{15} = \frac{17}{4} X \frac{15}{49} = \frac{225}{196} = 1\frac{59}{196}$

3.7 Change of Fraction to Decimal, Ratio and Percentage

Fractions are converted to decimal by dividing the numerator by the denominator while fraction are converted to percentage by multiplying the fraction by 100 and presenting the answer in percentage (%). The conversion of fraction to ratio only requires the presentation of the numerator as the ratio of the denominator.

E.g., Convert (a) $\frac{2}{5}$

(b) $2\frac{1}{4}$

to (i) Decimal (ii) Ratio (iii) Percentage

(a) $\frac{2}{5} = 2 \div 5 = 0.4$ (*Decimal*)

$\frac{2}{5} = 2 : 5$ (*ratio*)

$$\frac{2}{5} = \frac{5}{2} = X\frac{100}{1} = 40\% \ (Percentage)$$

(b) $2\frac{1}{4} = \frac{9}{4} = 2.25$ (*Decimal*)

$2\frac{1}{4} = \frac{9}{4} = 9 : 4$ (*ratio*)

$2\frac{1}{4} = \frac{9}{4}\ X\ \frac{100}{1} = 225\%\ (Percentage)$

3.8 Application of Fraction to Business Management/Economics

The addition, subtraction, multiplication and division of fraction can be applied in business management and economics in areas involving the determination of total cost of items, discounts, deductions from salaries, tax liabilities, unit cost of manufactured products or service rendered, return inwards, return outwards etc.

Example 1: The total production cost of XYZ Company was N 3 million. Factory overhead amounted to N1 million and prime cost is N1.5 million. Other items of the cost amounted to N 50,000. Calculate the fraction of the total spent on other items.

Solution:

Factory overhead $= \frac{N\ 1m}{N\ 3m} = \frac{1}{3}$

Prime cost $= \frac{N\ 15m}{N\ 3m} = \frac{1}{2}$

Other items $= 1 - \left(\frac{1}{3} + \frac{1}{2}\right) = 1 - \left(\frac{2+3}{6}\right) = 1 - \frac{5}{6} = \frac{1}{6}$

Example 2: The total expenses of Ade and Olu Nigeria Limited in the year ended 31 December, 2004 was N180, 000. This consists of: Wages and salaries N 50, 000 Selling and distributing expenses N 20,000 Insurance N 5,000 Rent N 30,000 Electricity N 35,000 Traveling and fueling N 35,000 Miscellaneous expenses N 5,000 What fractional part of the total is:

a) Rent
b) Insurance and electricity
c) Wages, salaries and rent

Solution:

a) Rent $= \frac{N\ 30{,}000}{N\ 180{,}000} = \frac{1}{6}$
b) Insurance and Electricity $= \frac{N\ 5{,}000 + N\ 35{,}000}{N\ 180{,}000} = \frac{2}{9}$
c) Wages, salaries and rent $= \frac{N\ 50{,}000 + N\ 30{,}000}{N\ 180{,}000} = \frac{N\ 80{,}000}{N\ 180{,}000} = \frac{4}{9}$

3.8.1 Ratio

A ratio expresses the functional relationship between two or more magnitude. It can be described as a quotient stated in a linear lay out. For example: $\frac{a}{b}$ = a: b. it should be noted that only quantities in the same unit or of the same kind can be expressed in ratio. Ratios are normally reduced to lowest terms, like fractions.

Example 1

a) What ratio of N20 is N5?
b) Express 3 hours as a ratio of 3 days.
c) What ratio of 2 hours is 45 minutes?
d) Express 200g as a ratio of 4kg.

Solution:

a) N5: N20 = 1:4
b) 3hrs: 3days
   3hrs: (3 x 24) hrs
   3hrs: 72 hrs 1: 24
c) 45mins: 2 hours

   45mins: (2 x 60) mins

   45mins: 120mins

   9: 24

   3: 8
d) 200g: 4kg

   200g: (4 x 1000)g

   200g: 4000g

   1: 20

Example 2

Johnson Construction Company uses metals to produce its products. In a year, the company purchased an alloy with which to construct a frame for a part of a certain building. The ratio by weight (kg) of zinc, tin, copper and lead in the alloy was 1:4:3:2. if the work requires 840kg of the alloy, what is the composition by weight of each of these constituents in the alloy?

Solution:

	Zinc	:	Tin	:	Copper	:	Lead
=	1	:	4	:	3	:	2

Total ratio = 1 + 4 + 3 + 2 = 10

Therefore:

The required weight of zinc = 1 /10 x 840kg = 84kg

The required weight of tin = 4 /10 x 840kg = 336kg

The required weight of copper = 3 /10 x 840kg = 252kg

The required weight of lead = 2 /10 x 840kg = 168kg

Example 3

Oyo, Ogun and Lagos invested N300m, N200m and N500m in a joint venture business to produce large quantity of cocoa for exports. They agreed to share gains or loss based on the capital contribution. If the gains realised amounted to N1.5billion, how much should each receive?

Solution:

Oyo	:	Ogun	:	Lagos
3	:	2	:	5

Therefore, share of gain is as follows:

Oyo: $\frac{3}{10}$ x N1, 500,000,000 = N450, 000,000

Ogun: $\frac{2}{10}$ x N1, 500,000,000 = N300, 000,000

Lagos: $\frac{5}{10}$ x N1, 500,000,000 = N750, 000,000

3.8.2 Proportion

A proportion is simply an expression of the equivalence or equality of two ratios. It is a method used to divide a given quantity in a given ratio. Proportion techniques are useful in finding the missing value of a given set of ratios.

Example 1

2: 3 = x : 12 = 10 : y = 6 : z. Find x, y and z

Solution:

To find the value of x, y and z, there is need to use a complete ratio of 2:3 in order to find the missing values.

2: 3 = x: 12

3 is multiplied by 4 to get 12, the same 4 should multiply 2 to get x

Therefore, 2: 3 = (2 x 4): 12

2: 3 = 8: 12, therefore x = 8.

Likewise,

2: 3 = 10: y

2: 3 = (2 x 5): (3 x 5)

2: 3 = 10: 9, therefore y = 15.

Likewise,

2: 3 = 6: z

2: 3 = (2 x 3): (3 x 3)

2: 3 = 6: 9, therefore z = 9.

Example 2

It costs a company N200 to purchase 1 liter of fuel for the company delivery van use, 30 liters can travel 300km, how much would this company need to spend on fuel if only 100km is required to be traveled?

Solution:

1 gallon cost N200 300km uses 30 liters

Therefore, 1km uses $(\frac{30}{300})$ liters A journey of 100km uses $(\frac{30}{300}$ x 100) gallons = 10 gallons.

Since a gallon cost N200 Therefore, 10 liters cost N200 x 10 = N2,000.

Example 3

It took 5 men 20 hours to clean up the warehouse of a certain company. How much time will be taken by 10 men to do the same work at the same rate?

Solution:

5 men take 20 hours to do the work 1 man takes (20 x 5) hours = 100 hours to do the work.

Therefore, it will take 10 men $\frac{20 \, x \, 5}{10}$ = 10 hours to do the work.

Example 4

If 250 labourers are needed to clean up a manufacturing company having 50 machines. How many labourers will be needed if additional 40 similar machines are required?

Solution:

250 labourers needed for 50 machines i.e., $\frac{250}{50}$ labour is needed for 1 machine

Additional 40 machines make the machine 90 in number i.e., Number of labourers required to clean up 90 machines = $\frac{250 X 90}{50}$ = 5 x 90 = 450 labourers.

3.8.3 Percentages

Percentages refer to a ratio that equates the second number to 100. It is actually the number of parts that are taken out of a hundred parts. For instance, 60% (60 per cent) means 60 out of a hundred or $\frac{60}{100}$ or $\frac{6}{10}$ or $\frac{3}{5}$ or 3:5. Percentages can be easily converted to fraction, decimal or ratio.

Example 1

Convert the following to percentages

a. $\frac{1}{3}$ b. $\frac{1}{4}$ c. $\frac{2}{5}$

Solution:

a. $\frac{1}{3} = \frac{1}{3} \text{ x } \frac{100}{1} = 33.33\%$

b. $\frac{1}{4} = \frac{1}{4} \text{ x } \frac{100}{1} = 25\%$

c. $\frac{2}{5} = \frac{2}{5} \text{ x } \frac{100}{1} = 40\%$

Example 3

Convert the following ratio to percentages.

(a) 2: 5 (b) 3: 4 (c) 15: 25

Solution:

(a) 2: 5 = $\frac{2}{5}$ x 100 = 40%

(b) 3: 4 = $\frac{3}{4}$ x 100 = 75%

(c) 15: 25 = $\frac{15}{25}$ x 100 = 60%

Example 4

A man bought an article for N20, 000 and sold it for N25,000. What is the percentage profit?

Solution:

Profit = Selling Price minus Cost price %

Profit = $\frac{Profit}{Cost\ Price} \text{ x } \frac{100}{1} = \frac{N25{,}000 - N20{,}000}{N20{,}000} \text{ X } \frac{100}{1}\% = \frac{N5{,}000}{N20{,}000} \text{ X } \frac{100}{1} = 25\%$

Example 5

The population of a country was 1.5 million in 1998 and in 2008, the population dropped to 1.2 million. What is the percentage reduction in population?

Solution:

Percentage Reduction = $\frac{Reduction}{Initial\ Population} \text{ x } \frac{100}{1}\% = \frac{(1.5 - 1.2)\ million}{1.5\ million} \text{ x } \frac{100}{1}\% = \frac{0.3\ million}{1.5\ million} \text{ x } \frac{100}{1}\% = 20\%$

4.0 Summary

Fractions, ratio, decimal and percentages are basically the same but different ways of expressing one variable in the proportion of the other. Mathematical and quantitative analyses require sound knowledge of fractions, ratio, percentages and proportions to able to quantify variables and draw relative proportionality among the variables or magnitudes of interest. Conversion of fractions to ratio, decimals and percentages and vice versa involves less rigorous mathematics. Day to day business require proper understanding of the concepts of fraction, ratio, proportion, decimal and percentages to be able to make appropriate relative compression among the variables.

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6.0 Self-Assessment Exercise(s)

1. With appropriate examples, demonstrate the relationship between fraction, ratio, proportion and percentages.
2. Express:

   (a) 2 weeks as a ratio of 1 year

   (b) 5 days as a ratio of 5 weeks

   (c) 350g as a ratio of 5kg

   (d) 720 seconds as a ratio 1 hour

   (e) Half a million as a ratio of 2 billion
3. Danladi and Sons Limited made total sales of N50, 200 on a certain market day. Of this sales figure, sales of beans accounted for N32, 600. What is the ratio of beans sales to the total sales?
4. Ade, Olu and Dayo share profits in a partnership business in the ratio 2:5:8. If the total profit realised is N30, 000,000. How much should each of them share?
5. Hasikye Flour Mill uses raw materials labelled A, B and C. A cost 50k per kg, B cost N1, 000 per kg and C cost N50, 000 per kg.
   a. Express the cost of these raw materials in the simplest ratio.
   b. Express the weights in the simplest possible ratio.
6. Change the following fractions to percentages (i)
   i. 17/20 (ii) 3 /8 (iii) 5 /12 (iv) 17/25

   Change the following to decimal (
   I. 15: 22 (ii) 13/4 (iii) 125% (iv) 35%
7. Jamila Industries Ltd produces body products. In the year ended 30th June 2002, it produced 25,500 bottles of Body Pears. By the year ended 30th June 2003, it produced 30,000 bottles of the product.
   a. Calculate the percentage increase in production.

8. Obiageli Industries Ltd earned N100, 000 as return on investment of N500, 000. How much would be earned at the same rate of return on an investment of N50, 000?
9. John Coy Limited produces certain products, each of which combines 12kg of lead with 15kg of copper. How many kg of lead will this product combine with 20kg of copper?
10. Andrew and Sons is an oil servicing company. It takes 9 engineers of the company 8 days to complete the servicing of an oil drilling equipment at the rig offshore. How many days will it take 6 engineers to complete the same work if working at the same rate?
11. For selling an item for N900 a trader makes a profit of 25%. What should the selling price be to make a profit of 30%?

UNIT 4 MULTIPLES AND LOWEST COMMON MULTIPLES (LCM) AND HIGHEST COMMON FACTORS (HCF) AND FACTORISATION

1.0 Introduction

Counting at a particular interval such as the multiplication tables, gives the multiples of a particular number. Therefore, it is possible for two or more numbers to have some multiples common to them. The lowest of such common multiples is called the Lowest Common Multiples (LCM). In identifying multiples of a number as to the product of the number with natural whole numbers. For instance, the multiples of 4, 8, 12, 16, 20, 24, 28, 32, 36… Similarly, 4 is a factor of each of the multiples because 4 is divisible by each of the multiples leaving no remainder. Therefore, factors can be defined as numbers that can divide a given number in which there is no remainder in the process of division. For example, the factors of 24 are the numbers that divide 24 and leave no remainder. Therefore, the factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24. It should be noted that, 1 is a universal factor of every number. Therefore, the factor of all numbers starts with 1 and ends with the number itself. This implies that every number has at least two factors (except 1, 1 has only one factor which is 1); which are 1 and itself. When a number has exactly two factors, such number is called a Prime Number. Example includes 2 (its factors are 1 and 2), 3 (its factors are 1 and 3), 5 (its factors are 1 and 5). 4 is not a prime number because it has more than 2 factors. Factors of 4 are 1, 2 and 4. Therefore, 4 is not a prime factor.

1.2 Learning Outcome

After reading this unit, students will be able to:

- explain the term “multiples”
- state the multiples of some numbers and algebraic terms
- determine the lowest common multiples (LCM) of numbers or algebraic expression.
- define the terms “factors” with examples
- state factors of numbers
- obtain the common factors for a set of number
- calculate the highest common factor (HCF) for a set of numbers and algebraic expression
- factorise algebraic expressions.

SELF-ASSESSMENT EXERCISE

i. What are multiples?
ii. State the multiples of 5
iii. State the multiples of 3b

3.0 Multiples and Factorisation

3.1 Common Multiples

Multiples are the results obtained when a constant number multiplies a set of natural numbers. For example: The multiples of 2 are 2, 4, 6, 8, 10, 12… Multiples of -4 are -4, -8, -12, -16, -20… Multiples of 2a are 2a, 4a, 6a, 8a, 10a, 12a… In the example above, the multiples of 2 are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28… while that of 3 are 3, 6, 9, 12, 15, 18, 21, 24… Therefore, it is possible for two numbers to have some multiples in common. In the example above, the common multiples of 2 and 3 are 6, 12, 18, 24…

SELF-ASSESSMENT EXERCISE

i. List the six common multiples of 5 and 7

ii. List four common multiples of 2a and 3a

3.2 Lowest Common Multiples (LCM)

The lowest common multiple is the lowest value among the set of common multiples. For example, the common multiples of 2 and 3 are 6, 12, 18, 24…, but the least of the common multiples is 6. Therefore, the LCM of 2 and 3 is 6.

Example 1 Find the LCM of 2, 4 and 3

Solution:

Multiples of 2 are: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26…

Multiples of 4 are: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44…

Multiples of 3 are: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36…

Therefore, common multiples are: 12, 24…

Lowest Common Multiple (LCM) = 12

Alternatively,

2	2, 3, 4
2	1, 3, 2
3	1, 1, 1

therefore, LCM = 2 x 2 x 3
= 12

therefore, LCM = 2 x 2 x 3 = 12

Note The prime factors are used to divide each of the numbers. When the prime numbers is not a factor of any of the number being considered, the number is left unchanged and the process continues until we get 1, 1, 1, 1…

2	3, 4, 5
2	3, 2, 5
3	3, 1, 5
5	1, 1, 5
	1, 1, 1

i.e. LCM of 3, 4 and 5 = 2 x 2 x 3 x 5
= 60

OR 3 = 3 x 1

2 = 2 x 1

2 = 2 x 2

5 = 5 x 1

Therefore, LCM = 2 x 2 x 3 x 5 = 60

Example 3

Find the LCM of $4a^3b$, $2a^2c^3$ and $5a^3b^2$

$4a^3b$ = 2 x 2 x a x a x a x b

$2a^2c^3$ = 2 x a x a x c x c x c

$5a^3b^2$ = 5 x a x a x a x b x b

Therefore, LCM = 2 x 2 x 5 x a x a x a x b x b x c x c x c = $20a^3b^2c$

Note To obtain the LCM, we find the product of the most occurring factors of all the numbers.

Example 4

Find the LCM of 24, 36 and 60

24 = 2 x 2 x 2 x 3

36 = 2 x 2 x 3 x 3

60 = 2 x 2 x 2 x 3 x 3 x 5

Therefore, LCM = 2 x 2 x 2 x 3 x 3 x 5 = 360

OR

2	24	36	60
2	12	18	30
2	6	9	15
3	3	9	15
3	1	3	5
5	1	1	5
	1	1	1

Therefore, LCM = 2 x 2 x 2 x 3 x 3 x 5
= 360

SELF-ASSESSMENT EXERCISE

i. Find the LCM of 24, 36 and 48.

ii. Find the LCM of $24a^3b^2$, $18c^3$ and $36a^3bx^3$

3.3 Highest Common Factor (HCF)

Among a given set of common factors, the highest is called the Highest Common Factor (HCF). From the previous example,

The Factors of 42 are 1, 2, 3, 6, 7, 21, and 42.

The Factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72.

The common factors o 72 and 42 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72.

The common factors are 2, 3 and 6.

Therefore, the highest factor common (HCF) = 6

Example 2 Find the highest common factor of 14ab and 28bc

Solution:

Factors of 14ab are 1, 2, 7, 14, a, 2a, 7a, 14a, b, 2b, 7b, 14b, ab, 2ab, 7ab and 14ab.

Factors of 28bc are 1, 2, 4, 7, 14, 28, b, 2b, 4b, 7b, 14b, 28b, c, 2c, 4c, 7c, 14c, 28c, bc, 2bc, 4bc, 7bc, 14bc and 28bc.

Therefore, common factors are 2, 7, 14, b, 2b, 7b and 14b. Then, the highest common factors 14b

OR

$42 = 2 \times 3 \times 7$

$72 = 2 \times 2 \times 2 \times 3 \times 3$

Therefore, HCF = 2 x 3 = 6.

The HCF in the above approach is obtained by finding the products of the common prime factors.

Likewise,

$14ab = 2 \times 7 \times a \times b$

$28bc = 2 \times 2 \times 7 \times 8 \times b \times c$

HCF = 2 x 7 x b = 14b.

Example 3 Find the HCF of $144a^3b^2$ and $54a^2bc^2$

Solution:

$144a^3b^2 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times a \times a \times a \times b \times b$

$54a^2bc^2 = 2 \times 3 \times 3 \times 3 \times a \times a \times b \times c \times c$

HCF = 2 x 3 x 3 x a x a x b =$18a^2b$.

Example 4 Find the HCF of $16a^2b$, 8b, $24bc^2$

$16a^2b = 2 \times 2 \times 2 \times 2 \times a \times a \times b$

$8b = 2 \times 2 \times 2 \times b$

$24bc^2 = 2 \times 2 \times 2 \times 3 \times b \times c \times c$

HCF = 2 x 2 x 2 x b = 8b.

3.4 Factorisation

Factorisation of algebraic expression requires getting the highest common factors of the numbers or the algebraic expression first. For instance: 20 + 35 = 5 (4 + 7) = 5 (11) = 55. The factors of 20 are: 1, 2, 4, 5, 10, and 20. The factors of 35 are: 1, 5, 7, and 35. The common factors are 1, 5 while the highest factor is 5. It should be noted, 5 is put outside the bracket while each of the number is divided by 5 to obtain the numbers in the bracket. This is a simple technique of factorisation.

3.1 Common Factors

Common factors are the factors common to a pair of numbers or algebraic terms. 1 is a common to all set of numbers but sometimes; it may not be the only common factor except the numbers are prime numbers.

Example:

(i) Find the common factors of 42 and 72.

(ii) Find the common factors of 24a and $16a^2$.

Solution:

(i) Factors of 42 are 1, 2, 3, 6, 7, 21, and 42.

Factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72.

Therefore, the common factors are 2, 3 and 6.

(ii) The factors of 24a are: 1, 2, 3, 4, 6, 8, 12, 24, a, 2a, 3a, 4a, 6a, 8a, 12a and 24a

The factors of $16a^2$ are: 1, 2, 4, 8, 12, a, 2a, 4a, 8a, 16a, a2 , 2a2 , 4a2 , 8a2 , and 16a2 .

Therefore, the common factors of 24a and $16a^2$ are 1, 2, 4, 8, a, 2a, 4a and 8a

3.5 Factorisation of Polynomial

Factorisation or factoring is the decomposition of an object or a number or a polynomial into a product of other object or factors which when multiplied together gives the original. Factorisation is a simplified form of an expression by dividing through by the highest common factor. For example, factorise each of the following: (i) 4a - 6b (ii) $24a^2y + 12aby$ (iii) $48x^2 – 16xy + 12 x y^2$

Solution

(i) 4a - 6b = 2(a - 3b)

The HCF of 4a and 6b is 2, then, we put the 2 outside bracket and divide each of the components by the HCF.

$$i.e.,\ 4a – 6b = 2\left[\frac{4a}{2} - \frac{6b}{2}\right] = 2[2a - 3b]$$

(ii) $24a^2y + 12aby = 12ay\left[\frac{24a^2y}{12ay} - \frac{12aby}{12ay}\right] = 12ay\left[\frac{24\ X\ a\ X\ a\ X\ y}{12\ X\ a\ X\ y} - \frac{24\ X\ a\ X\ b\ X\ y}{12\ X\ a\ X\ y}\right] = 12ay$ $[2a - b]$

(iii) $48x^2 – 16xy + 12 x y^2 = 4[12x - 4y + 3y^2]$

4.0 Summary

Multiples are figures or expression obtained when a particular number or algebraic expression are multiplied continuously by natural numbers. The concept of multiples enables us to know the set of numbers common as multiples of two or more numbers. Such common multiples are therefore divisible by the numbers from which the multiples are obtained. Multiples enable us to know the interrelationship, divisibility and common multiples which exist between a set of numbers. The concept of multiples and common multiples are essential in obtaining the lowest common multiples of numbers or algebraic expressions. Again, the concept of factors of numbers and algebraic expression is very useful in mathematics not only in the process of factorisation but also in reducing algebraic terms to the lowest terms. Therefore, further algebraic exercise requires good knowledge of factorisation. Such areas include solving of quadratic expression and simultaneous linear equations. Factors of a number enable us to know the products of the numbers or expression. It equally enables us to express the product of the prime number that makes up the number. For instance, 144 = 2 x 2 x 2 x 2 x 3 x 3 = 2^4 x 3^2. It should be noted that only the prime factors of a number can be used to express the products of the number. More importantly, factors need to be obtained first before proper factorisation exercises could be carried out. Only the highest common factor is used for the process of factorisation and not just any common factors. If any factor is used, the expression in the bracket will still be factorisable.

e.g. 20xy – 12xy
= 2xy (10x – 6)
= 2xy [2(5x – 3)]
= 4xy (5x -3).

The first factor used, is not the HCF, hence the expression in the bracket is not in its lowest term. With the use of the highest common factor (HCF), i.e. 4xy, the expression in the bracket cannot be further factorised.

5.0 References/Further Readings/Web Resources

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6.0 SELF-ASSESSMENT EXERCISE

1. Find the multiples of 6.
2. Find the common multiples of 3, 4 and 6.
3. Find the Lowest Common Multiples (LCM) of: (a) 36, 48, 54 (b) $12ax^2$, $18b^3xy$ and $24xy^3$.
4. Find the LCM of 24, 36 and 48.
5. Find the LCM of $24a^3b^2$, $18c^3$ and $36a^3bx^3$
6. Define the term "factor'?
7. What are prime numbers?
8. List the factors of 72
9. Simplify by factorisation [63 + 108].
10. Find the common factors of 84 and 144.
11. Find the common factors of 36b2 and 4ab.
12. Find the HCF of 72, 144 and 120.
13. Find the HCF of $4a^2b$, 8ab and 24bc.
14. Factorise:
    (a) $4x^3 – 12xz$
    (b) $25ab^2 -15ab + 35ax^2$
15. Factorise the following completely
    (a) 45x3 y – 30abx2
    (b) $72ab^2x + 42abx – 54xy^2$
16. Find the HCF of
    (a) 120xy and 150axy
    (b) 144, 120 and 72

UNIT 5 INDICES, LOGARITHM AND SURDS

1.0 Introduction

In mathematics, indices are the little numbers that show how many times one must multiply a number by itself. Thus, given the expression 4 3 , 4 is called the base while 3 is called the power or the index (the plural is called indices). Logarithm (log) of a number is another number that is used to represent it such as in this in this equation 102 = 100, 10 is called the base while 2 is the power (just like indices). In the case of logarithm, the power i.e. 2 is called the log. Note that the base of a log can be any positive number or any unspecified number represented by a letter. Surd is a mathematical way of expressing number in the simplest form of square roots. Perfect square (numbers that have square roots e.g. 9, 16, 25, 100, 144) cannot be expressed in surd form. However, square roots of non-perfect square numbers are expressed in surd form to give room for addition, subtraction, multiplication and division.

1.2 Learning Outcome

After reading this unit, students will be able to:

- solve some problems involving indices
- simplify problems involving logarithms
- simplify problems involving addition, subtraction, multiplication and division of surds.

3.0 Indices, Logarithm and Surds.

3.1 Indices

This concise and compact form of expressing product of numbers or mathematical expressions is known Indices. Sometimes, the product of a number is often long. For instance, 288 can be expressed as the product of its prime factor as: 288 = 2 x 2 x 2 x 2 x 2 x 3 x 3. Hence, in indices form 288 can be expressed as 2^5 x 3^2. The application of indices involves the usage of some properties, which apply to any base. These properties are called rules of indices. Type equation here.Using "a" as a general base, the following rules of indices hold

Rule 1: $a^m \text{ x } a^n = a^{m+n}$

e.g., $2^2 \text{ x } 2^3 = 2^{2+3} = 2^5$ or 32.

Rule 2: $a^m \div a^n = a^{m-n}$

e.g., $3^5 \div 3^2 = 3^{5-2} = 3^3$ or 27.

Rule 3: $(a^m)^n = a^{m \text{ x } n}$

e.g. $((32)^{\frac{-2}{5}})^4 = 2^{3 \text{ x } 4} = 2^{12}$

Rule 4: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

e.g., $(125)^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$.

Note $\sqrt[3]{125}$ require for the number that must be multiplied by itself three times to give 125.

Rule 5: $a^{-m} = \frac{1}{a^m}$

e.g. $4^{-2} = \frac{1}{4^2} = \frac{1}{16} = (32)^{\frac{-2}{5}} = \frac{1}{(32)^{\frac{2}{5}}} = \left(\sqrt[3]{32}\right)^2 = \frac{1}{(2)^2} = \frac{1}{4}$

Rule 6: $a^0 = 1$

e.g. $4^0 = 1$, $(2n)^0 = 1$

Rule 7: $a^m \text{ x } y^m = (ay)^m$

e.g. $4^2 \text{ x } 3^2 = (4 \text{ x } 3)^2 = (12)^2 = 144$.

Examples 1:

Evaluate the following:

(i) $9\,\frac{-3}{2}$ (ii) $(x-3)^0$ (iii) $\frac{a^2b^3}{ab}$ (iv) $32^{0.8}$ (v) $(2x^2y)\,(3xz)^2$ (vi) $(0.001)^{1/3}$ (vii) $\frac{(2a^3)(a^{-2}b)}{2ab^{-2}}$ (viii) $32\frac{-2}{5}$

(ix) $\frac{a^2b^3}{ab}$ (x) $\frac{x^3}{x^{\frac{2}{3}}}$

Solution

(i) $9^{-3/2} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{(\sqrt{9})^3} = \frac{1}{((3)^3} = \frac{1}{27}$

(ii) $(x-3)^0 = 1$

(iii) $\frac{a^2b^3}{ab} = \frac{a\ x\ a\ x\ b\ x\ b\ xb}{ab} = ab^2$

Or

$\frac{a^2b^3}{ab} = a^{2-1}b^{3-1} = ab^2$

(iv) $32^{0.8} = (32)^{\frac{8}{10}} = (32)^{\frac{4}{5}}\ 32^{\frac{-2}{5}} = (2)^4 = 16$

(v) $(2x^2y)\ (3xz)^2 = (2x^2y)\ (3xz)\ (3xz) = (2x^2y)\ (9x^2z^2) = 18x^{2+2}yz^2 = 18x^4yz^2$

(vi) $(0.0001)^{1/3} = \left(\frac{1}{1000}\right)^3 = \sqrt[3]{\frac{1}{1000}} = \frac{1}{10}$

(vii) $\left(\frac{200}{72}\right)^{\frac{-3}{2}} = \frac{1}{\frac{200}{72}}^{\frac{3}{2}} = \left(\frac{72}{200}\right)^{\frac{3}{2}} = \left(\frac{36}{100}\right)^{\frac{3}{2}} = \left(\frac{18}{50}\right)^{\frac{3}{2}} = \left(\frac{9}{25}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{9}{25}}\right)^3 = \left(\frac{3}{5}\right)^3 = \frac{27}{125}$

(viii) $\frac{(2ab^2b^2)(a^{-2}b)}{(2ab)^{-2}} = \frac{(2ab^2b^2)(a^{-2}b)}{\frac{1}{(2ab)^{-2}}} = (2ab^2b^2)(a^{-2}b) \div \frac{1}{(2ab)^{-2}} =$

$(2ab^2b^2)(a^{-2}b)\ x\ \frac{(2ab)^{-2}}{1}$

$= (2ab^2b^2)(a^{-2}b)(4a^2b^2) = (2ab^3)(4a^2b^2) = 8a^{1+2}b^{3+2} = 8a^3b^5.$

(ix) $32^{\frac{-2}{5}} = \frac{1}{32^{\frac{-2}{5}}} = \frac{1}{\left(\sqrt[5]{32}\right)^2} = \frac{1}{2^2} = \frac{1}{4}.$

(x) $\frac{x^3}{x^{\frac{2}{3}}} = x^{3^{-\frac{2}{3}}} = x^{\frac{3}{1}-\frac{2}{3}} = x^{2\frac{1}{3}}$ or $x^{\frac{7}{3}}$

Example 2

Solve for x in each of the following:

(a) $3(3^x) = 27$

(b) $(0.125)^{x+1} = 16$

(c) $3x = \frac{1}{81}$

(d) $(3^x) + 2(3^x) - 3 = 0$

(e) $2x + y = 8,\ 3^{2x-y} = \frac{1}{27}$

`

(f) $27^x = \sqrt{3}$

Solution

(a) $3(3x) = 27$

Divide both sides by 3

$\frac{3(3^x)}{3} = \frac{27}{3} = 3^x = 9 = 3^x = 3^2$

$x = 2$

(b) $(0.125)^{x+1} = 16 = \left(\frac{125}{1000}\right)^{x+1} = 16 = \left(\frac{1}{8}\right)^{x+1} = (2^{-3})^{x+1} = 2^4$

$= 2^{-3x-3} = 2^4 = -3x - 3 = 4$

Collet like term;

$-3x = 4 + 3 = -3x = 7$

divide both sides by 3

$x = -\frac{7}{3}$

(c) $3x = \frac{1}{81} = 3x = \frac{1}{3^4} = 3x = 3^{-4} = x = -4$

(d) $(3^x)^2 + 2(3^x) - 3 = 0$

Let $3x = p$

$\therefore p^2 + 2p - 3 = 0$

Using factorization

$P(p + 3) - 1(p + 3) = 0$

$(p - 1)(p + 3) = 0 = p - 3 = 0$ or $p + 3 = 0$

$p = 1$ or $p = -3$.

Recall $3^x = p \therefore 3^x = 1$

$3^x = 3^0$

$x = 0$

(e) $2x + y = 8$ (i)

$3^{2x - y} = \frac{1}{27}$ (ii)

From equation (i) $2x+y = 23; x + y = 3$

From equation (ii) $32x-y = 3-3; 2x - y = -3$

$x + y = 3$

$2x - y = -3$

Using elimination method

$x + y = 3$

$2x - y = -3$

$x + 2x = 3 + (-3)$

$3x = 3 - 3$

$3x = 0$

$x = \frac{0}{3} = 0$

(f) $27x = \sqrt{3} = (3^3)^x = 3^{\frac{1}{2}}$

$= 3x = \frac{1}{2}$; *divide both sides by* 3

$= x = \frac{1}{2} x \frac{1}{3} = \frac{1}{6}$.

3.2 Logarithms

Similar to indices, simplification of logarithms involves some basic rules. Some of the rules are:

(i) Log (a + b) = log a + log b

e.g., log 6 = log (2 x 3) = log 2 + log 3

(ii) $\text{Log}\left(\frac{a}{b}\right)$ = log a – log b

e.g., $\log\left(\frac{12}{3}\right)$ = log 12 – log 3

(iii) $\text{Log}\ (a)^m = m \log a$

e.g., $\log 2^3 = 3\log 2$

(iv) $\text{Log}\ a^0 = 1$

e.g., $\log 100^0 = 1$, $\log 10000^0 = 1$

(v) $loga \sqrt[n]{x} = 1\ log_a\, x$

e.g., $log10\ \sqrt[6]{2} = \frac{1}{6}\ log_{10}\ 2$

(vi) $log_a\ a = 1$

e.g., $log_3\ 81$

$= log_3\ 3^4$

$= 4\ log_3\ 3 = 4 \text{ x } 1 = 4$

Note: If a logarithm is given without a base, it is assumed to be a natural log i.e., in base 10.

Example 1 Simplify without using tables:

(a) Log$_3$ 27 (b) $\frac{log_3\ 27}{log_3\ \sqrt{3}}$ (c) Log 81 – Log 9 (d) Log$_2$ 16 (e) Log$_{64}$ 4 (f) Log$_3$ 27 + 2Log$_3$ 9 - Log$_3$ 54

Solution

(a) $log_3\ 27 = log_3\ 3^3 = 3log_3\ 3 = 3 \text{ x } 1 = 3$

(b) $\frac{log_3\ 27}{log_3\ \sqrt{3}} = \frac{log_3\ 3^3}{log_3\ 3^{\frac{1}{2}}} = \frac{3log_3\ 3}{\frac{1}{2}log\ _3\ 3}$

$= 3 \div \frac{1}{2} = 3\ x\ 2 = 6$

(c) $log\ 81 - \log 9 = \log\left(\frac{81}{9}\right) = \log 9$

(d) $log_2\ 16 = log_2\ 24 = 4log_2\ 2 = 4 \text{ x } 1 = 4$

(e) $log_{64}\ 4 = log_{64}\ 64 = \frac{1}{3}log_{64}\ 64 = \frac{1}{3} \text{ x } 1 = \frac{1}{3}$

(f) $log_3\ 27 + 2log_3\ 9 - log_3\ 54 = log_3\ 27 + log_3\ 92 - log_3\ 54$

$= log_3\ 27 + log_3\ 81 - log_3\ 54 = log_3\left(\frac{27\ x\ 81}{54}\right)$

$= log_3\ 27 = log_3\ 3^3 = 3log_3\ 3 = 3\ x\ 1 = 3$

Example 2

Given that log 3 = 0.4771 and log2 = 0.3010

Find (a) log 6 (b) log 16 (c) log 18

Solution

(a) Log 6 = log (3 x 2) = log 3 + log 2

= 0.4771 + 0.3010

= 0.7781

(b) Log 16 = log (2 x 2 x 2 x 2) = log 2 + log 2 + log 2 + log 2

= 0.3010 + 0.3010 + 0.3010 + 0.3010

= 1.2040

(c) Log 18 = log (2 x 3 x 3) = log 2 + log 3 + log 3

= 0.3010 + 0.4771 + 0.4771

= 0.3010 + 0.9542

= 1.2552

Example 3

Given that if $\log_a x = b$, $x = a^b$. Find the value of x in each of the following:

(a) Log 9 = 2 (b) Log$_{10}$ x = 4 (c) Log$_3$ $\frac{1}{81}$ = x

Solution

(a) $\log_x 9 = 2 = 9 = x^2$

$3^2 = x^2$

$x = 3$

(b) $log_{10}\, x = 4 = x = 10^4$

$x = 10\ x\ 10\ x\ 10\ x\ 10 = 10000$

(c) $\text{Log}_3 \frac{1}{81} = x = \frac{1}{81} = 3^x$

$= \frac{1}{3^4} = 3^x$

$= 3^{-4} = 3^x$

$= x = -4$

3.3 Surds

With the exception of perfect squares, square roots of numbers are expressed in their simplest surd forms e.g.

$\sqrt{75} = \sqrt{25 \text{ x } 3} = \sqrt{25} \text{ x } \sqrt{3} = 5\sqrt{3}$

$\sqrt{48} = \sqrt{16 \text{ x } 3} = \sqrt{16} \text{ x } \sqrt{3} = 4\sqrt{3}$

$\sqrt{72} = \sqrt{36 \text{ x } 2} = \sqrt{36} \text{ x } \sqrt{2} = 6\sqrt{2}$

$\sqrt{24} = \sqrt{4 \text{ x } 6} = \sqrt{4} \text{ x } \sqrt{6} = 2\sqrt{6}$

Note: To express the root of a number in the simplest surd form, it is important to find two products of the number, one of which is expected to be a perfect square. Surds can be added together, multiply one another, subtracted from one another or even divide one another. Below are some illustrated examples.

Simplify: (a) $3\sqrt{12} - 4\sqrt{75} + \sqrt{48}$

(b) $\sqrt{200} - \frac{1}{4}\sqrt{72} + \frac{3}{4}\sqrt{98}$

Solution

(a) $3\sqrt{12} - 4\sqrt{75} + \sqrt{48}$

$= 3\ \sqrt{3 \text{ x } 4} - 4\ \sqrt{25 \text{ x } 3} + \sqrt{16 \text{ x } 3}$

$= 3 \text{ x } 2\sqrt{3} - 4 \text{ x } 5\sqrt{3} + 4\ \sqrt{3}$

$= 6\ \sqrt{3} - 20\ \sqrt{3} + 4\ \sqrt{3}$

$= 6\ \sqrt{3} + 4\ \sqrt{3} - 20\ \sqrt{3}$

$= 10\ \sqrt{3} - 20\sqrt{3}$

$= -10\ \sqrt{3}$

Note: It should be that each of the roots is expressed in the product of their common square root.

(b) $\sqrt{200} - 1/4\ \sqrt{72} + 3/4\ \sqrt{98}$

$= \sqrt{100 \text{ x } 2} - \frac{1}{2}\sqrt{36 \text{ x } 2} + \frac{3}{4}\sqrt{49}$

$= 10\sqrt{2} - \frac{1}{2}\, 6\sqrt{2} + \frac{3}{4}\, 7\sqrt{2}$

$= 10\sqrt{2} - 3\sqrt{2} + \frac{21}{4}\sqrt{2}$

$= 7\sqrt{2} + \frac{21}{4}\sqrt{2}$

$= 7\sqrt{2} + 5\frac{1}{4}\sqrt{2} = 12\,\frac{1}{4}\sqrt{2}$ or $\frac{49}{4}\sqrt{2}$

2. (a) $\sqrt{\frac{9}{7}} = \frac{\sqrt{9}}{\sqrt{7}} = \frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \text{ x } \frac{\sqrt{7}}{\sqrt{7}} = \frac{3}{\sqrt{7}}$

Note: = [$\frac{3}{\sqrt{7}} \text{ x } \frac{\sqrt{7}}{\sqrt{7}} = \frac{3}{\sqrt{7}}$] i.e., multiplying both numerator and denominator by the denominator. This process is called rationalising the denominator.

(b) $\frac{4-2\sqrt{3}}{2+\sqrt{3}} = \frac{4-2\sqrt{3}}{2+\sqrt{3}} \text{ x } \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= \frac{4(2)-4\sqrt{3}-(2\sqrt{3})2+2\sqrt{3}(\sqrt{3})}{2(2)-2\sqrt{3}-(2\sqrt{3})+2\sqrt{3}(\sqrt{3})}$

$= \frac{8-4\sqrt{3}-4\sqrt{3})+2 x 3}{4-2\sqrt{3}+2\sqrt{3}-3}$

$= \frac{8-8\sqrt{3}+6}{4-3}$

$= \frac{8+6-8\sqrt{3}}{4-3}$

$= 14 - 8\sqrt{3}$

Note: To rationalise the denominator in this case, we multiply both the numerator and denominator by the denominator but change the sign between the number and the square root.

(C) $\frac{\sqrt{27}\ X\ \sqrt{50}}{\sqrt{54}} = \frac{\sqrt{9 x 3}\ X\ \sqrt{25 x 2}}{\sqrt{9 x 6}} = \frac{3\sqrt{3}\ X\ 5\sqrt{2}}{3\sqrt{6}} = \frac{15\sqrt{6}}{3\sqrt{6}} = 5$

4.0 Summary

Simplification of numbers in indices, logarithms and surds requires strict compliance to some set of rules. Once these rules are adhered to, it becomes easy to write expanded products in index form simplify logarithm and express roots in their simplest surd forms. Indices, logarithm and surds are very important tools of simplification in quantitative analyses. They are equally important mathematical techniques necessary to simplify some seemingly difficult problems. Indices for instance, is used in our daily lives as we often make decisions that have to do with economic variables due to changes in time and places e.g., the analysis of price indices

4.0 References/Further Readings/Web Resources

- Black, T. & Bradley J. F. (1980). Essential Mathematics for Economists. (2nd ed.). Chicherter: John Wiley and Son.
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Self-Assessment Exercise(s)

I. Simplify each of the following

(a) x^4 x x^{15}

(b) x^a x x^b x x^c

(c) x^3 x y^3 x z^3

(d) $\frac{x^3}{x^{-3}}$

(e) $(x^{\frac{1}{2}} \text{ x } x^{\frac{1}{3}}) \div x^{\frac{2}{3}}$

II. Evaluate the followings without using calculator

(a) $27^{2/3}$

(b) $\frac{(3ab)^2(2c)^2}{12ac^2}$

(c) $28^{1/2} \text{ x } 7^{1/2}$

(d) $8^{-2/3}$

(e) $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$

III. Simplify and solve for x

(a) $8^{x-1} = \frac{1}{32}$

(b) $(0.125)^{x+1} = \frac{1}{64}$

(c) $(0.5)^x = (0.25)^{1-x}$

(d) $5^{2x-y} = \frac{1}{25}$

(e) $27 = 3^{x-2y}, \frac{1}{8} = 64^{x-y}$

IV. Simplify each of the following:

(a) $\frac{1}{2}\log_4 8 + \log_4 32 - \log_4 2$

(b) $^4\log 2 + \log 5 - \log 8$

(c) $\log_5 12.5 + \log_5 2$

(d) log5 $\sqrt{125}$

(e) log3 $3\sqrt{81}$

(f) Given that log 12 = 1.0792 and log 24 = 1.3802. Find

(i) Log 6 (ii) log72

(g) Find x in each of the following

$\log_x 32 = 5$

$\log_x 3 = 4$

$\log_2 64 = x$

(h) Evaluate the following in their simplest surd form:

(a) $\frac{6}{\sqrt{2}}$ (b) $\sqrt{\frac{16}{7}}$ (c) $\sqrt{48} - 3\sqrt{75} + \sqrt{12}$ (d) $\frac{(\sqrt{2})^6}{\sqrt{6} \, x \, \sqrt{2}}$ (e) $\frac{3-\sqrt{2}}{2+\sqrt{3}}$

Change the following logarithm to their equivalent exponential forms:

(a) $log_2 8 = 3$

(b) $log_{36} 6 = ½$

(c) $log_x y = r$

(d) $log_{16} 2 = ¼$

(e) $log_2 y = 5x$

(Hint: if $log_a x = b$, the exponential form is that $x = a^b$).

MODULE 2 EQUATIONS AND FORMULAE

UNIT 1 EQUATIONS, FUNCTIONS AND CHANGE OF SUBJECT OF FORMULAE

Unit Structure

1.0 Introduction

Most mathematical problems involve writing and solving equations. The use of letters and numbers are very important in establishing mathematical relationship. In the course of doing this, an equation which requires two sides: left hand side and the right-hand side, is required. The two sides are made equal with an equal sign (=). For example: Simple Interest (SI) = $\frac{PxRxT}{100}$

Where SI is simple interest, P = principal, R = rate and T = time. Equation enables us to provide a link or relationship between or among variables. It shows how one variable (dependent variable) is related to other set(s) of variables(s), known as independent variables. In the example of simple interest cited, the simple interest is the dependent variable while principal rate and time are independent variable.

2.0 Learning Outcomes

After reading this unit, students will be able to:

- explain the concept of equation and its relevance in mathematics
- make a clear distinction between equations and functions
- make a particular variable in an equation or relation the subject of the relation.

3.0 Equations and Functions

Equations and functions are closely related concepts. An equation establishes a mathematical relationship between two and more variables with the use of equality sign showing the equivalence or equality between one side and the other. A function on the other hand expresses quantitative or qualitative relationship between or among variables. A function may be stated explicitly in terms of equation or may be stated with the use of some other signs of relationship e.g., Qd = 20 – 4p. This is an equation as well as a function establishing functional relationship between quantity demanded and price. However, the functional form is Qd = f(P). This implies that all equations are functions but not all functions are equations. An equation tells us in a clear term the nature of relationship between one variable and the other variable(s) but a function may not be explicit enough. The example above Qd = f(P) is basically a function and not an equation because it does not state categorically the quantitative relationship in terms of magnitude and direction between the dependent variable (Qd) and the independent variable (P). The equation above, Qd = 20 – 4P shows that interrelationship between quantity demanded (Qd) and the price (P) in terms of both the magnitude and direction. Another important distinction between an equation and a function is that for an equation each of the values of independent variable should give a corresponding value of dependent variable, this is not compulsory for a function.

In solving day to day problems in mathematics, it is essential to condense the problem into a functional relation with the use of equations. An equation is a mathematical expression that tells us the equality of one side to the other. An equation requires that one side of mathematical expression equals the other. What makes an equation different from a mathematical expression is the "equal to" sign (=). For instance: x + 2y is a mathematical expression but x + 2y = 7 is a mathematical equation. This implies that an equation has three main features namely the right-hand side (RHS), the left-hand side (LHS) and the "equal to" sign (=) that breaks the sides into two.

SELF-ASSESSMENT EXERCISE

i. What is an equation? How is it different from a mathematical expression?

ii. Of what relevance are equations in solving day to day mathematical problem?

iii. What are the components of an equation?

3.1 Functions and Relation

A function is a relationship or expression involving one or more variables. A function is used in showing the functional relationship between two or more variables which has to do with dependent and independent variables. A function can be explicit and implicit.

An explicit function is a function in which the dependent variable can be written explicitly. The following are explicit functions:

$y= x^2 - 3$, $f(x) = \sqrt{}+7$ and $y = \log 2x$

An implicit function is a function in which the dependent variable is not isolated on one side of the equation. Also, it is a function in which the dependent variable has not been given explicitly in terms of the independent variable. An example of implicit function equation is given as $x^2 + xy - y = 1$, $3x^2 - 8xy – 5y\ 4a = 0$

SELF-ASSESSMENT EXERCISE

i. "Not all functions are equations". Discuss.

ii. Distinguish between dependent and independent variables in an equation or function.

Given two sets X and Y, a function (mapping) f from X to Y. $f: X : Y$, associates each element of X with only one element of Y (element of X determines the element of Y you get) e.g., Consumption is a function of income

Where:

- X: domain
- For each $x \in X$, $y = f(x) \in Y$: the image of x (value of f at x).→
- $f(X) = \{ y \in Y : y = f(x) \}$: the range

Examples

1. X: set of countries, $Y \subset R$, f: 'the GDP of"

   f(Ghana) = 21; 000

2. $X = R, Y = R, y = f(x) = 2x + 3$

3. $y^2 = 2x + 3$: *association of* $x \in X$ *and* $y \in Y$,

   but y is not a function of x

- Different x 2 X may have the same image
  e.g., $y = f(x) = x$
- If each x has a different image: the function is one-to-one
  can be inverted: $x = f$
  $f^{\frac{-1}{(p)}}$: inverse function
  (The rule that associates the image of x with x)

e.g.,

$y = f(x) = 2x + 3 \rightarrow x = f^{\frac{-1}{(p)}} = \frac{y-6}{4}$

The Composite function of two functions $f : X \rightarrow Y$ *and* $g : Y \rightarrow Z$

is

$g^{\,o} f : X \rightarrow Z$

or

$z = g(f(x))$

- The range of f must be a subset of the domain of g.

Examples

1. Consumption is a function of income.
   income is a function of age.
   Consumption is a function of age.
2. $y = f(x) = 2x + 3;\ z = g(y) = y^2$
   $z = g(f(x)) = g^{\,o} f(x) = (2x + 3)^2$

Types of $R \rightarrow R$ functions

$f : R \rightarrow R\ CR \times R$

Graph of f:$\{ (x; f(x)) : x \in R; f(x) \in \} R \quad C\, R^2$

- Identity function: $f(x) = x$
- Constant functions: $f(x) = a$
- Linear functions: $f(x) = ax + b$
- Quadratic functions: $f(x) = ax^2 + bx + c$
- Power functions: $f(x) = ax^b$
- Exponential functions: $f(x) = b^x$, b: base
  If $b = c \approx 2{:}718$ (Napier's constant) $f(x) = e^x = exp(x)$.
  If $y = b^x$, x: the logarithm of y to base b: $x = log_b y$
- Logarithmic functions: $f(x) = log_b x$
  If $b = c$, $f(x) = \ln x$: natural logarithm
- Absolute Value: $f(x) = |x| = x$ if x ¸ 0

  $= -x$ *if* $x < 0$

  Implicit Functions

  $y = f(x) = 3x^2$: an explicit function.

  $y - 3x^2 = 0$: an implicit function

  General form: F (x; y) = 0

e.g., utility function: $U(x; y)$

$\rightarrow U(x; y) = c$ (constant): an indifference curve

Not all equations of the form $F(x; y) = 0$ are implicit functions.

$F(x; y) = x^2 + y^2 = 9$

3.2 Change of Subject of Formulae

In any mathematical relation or function, there are at least two variables. One of the variables is the dependent variable on which other variable(s) i.e., independent variables depend. For example, the volume of a cone, $V = \frac{1}{3}\pi r^2 h$ where V (volume) is the dependent variable because the value of V depends on the value of r (radius) and the height (h) of cone. is a constant (approximately$\frac{22}{7}$). Therefore, r and h are independent variable. The relationship which is established between or among variable is bilateral i.e., one variable can be obtained given the information required of all other variables. For instance, the radius of a cone can be obtained given the volume and the height of the cone. This requires that the radius (r) be made the independent variable or made the subject of the relation. In the process of making any variable the subject of relation, a number of diverse mathematical exercises are required to ensure that the variable to be made the subject of relation is taken from the right-hand side of the equation to the left-hand side of the equation. It is also required that the variable to be made the subject of relation is the only variable that remains on the left-hand side.

Example

For each of the following equations, make x the subject of the relation:

(a) $a = b(1 - x)$

(b) $a\sqrt{x} - 1 = bT$

(c) $\frac{a}{bx} = \frac{a}{a+x}$

(d) $R = \sqrt{\frac{ax-p}{Q+bx}}$

(e) $S = \frac{wd}{x} = \left(x - \frac{d}{2}\right)$

(f) $V = \frac{1}{3}\pi x^2 h$

(g) $\frac{k}{x+1} - \frac{b}{x+1} = \frac{1}{x}$

Solution

(a) $a = b(1 - x)$

Step I: Open the bracket to make the equation linear equation. a = b – bx

Step II: Take the component of x to the lefthand side and to the righthand side. bx = b − a.

(Note: the signs of "bx" and of "a" change as we move them across the equality sign)bx = b − a)

Step III: Divide both sides by b to ensure that only x remains in the left-hand side as the dependent variable.

$$\frac{bx}{b} = \frac{b-a}{b} = x = \frac{b-a}{b} = \frac{b}{b} - a \therefore x = 1 - \frac{b}{a}$$

(b) $a\sqrt{x} + 1 = bT$

Step I: Square all the variables to remove the square root from x.

$$a^2(\sqrt{x} + 1)^2 = (bT)^2$$

$$a^2(x + 1) = b^2T^2$$

Step II: open the bracket to make the equation linear without bracket

$$a^2x + a^2 = b^2T^2$$

Step III: Take a^2 to the other side

$$a^2x = b^2T^2 - a^2$$

Step IV: Divide both sides by a^2 to have only x left on the left-hand side

$$\frac{a^2x}{a^2} = \frac{b^2T^2 - a^2}{a^2} = x = \frac{b^2T^2}{a^2} - \frac{a^2}{a^2} = x = \frac{b^2T^2}{a^2} -$$

(c) $\frac{a}{bx} = \frac{a}{a+x}$

Step I: Cross multiply to make the fractional equation completely linear

$$a(a + x) = b(b - x)$$

Step II: Open up the brackets to make the equation linear without brackets

$$a^2 + ax = b^2 - bx$$

Step III: Collect like terms by bringing variables that have 'x' to the left-hand side.

$$ax + bx = b^2 - a^2$$

Step IV: Divide both sides by (a + b) to leave only x on the left-hand side

$$= \frac{b^2 - a^2}{(a+b)} = x = \frac{b^2 - a^2}{(a+b)}$$

(d) $R = \sqrt{\frac{ax-p}{Q+bx}}$

Step I: Square both sides to remove the square root signs from the RHS

$$R^2 = \sqrt{\frac{ax-p}{Q+bx}} = \frac{R^2}{1} = \frac{ax-p}{Q+bx}$$

Step II: Cross multiply to make the equation linear.

$$R^2(Q + bx) = ax - P$$

Step III: Open the brackets

$$R^2Q + R^2bx = ax - P$$

Step IV: Collect like terms by separating components of x to one side

$$R^2Qbx - ax = -P + R^2Q$$

Step V: Factor out x

$$x(R^2b - a) = R^2Q - P$$

Step VI: Divide both side by $R^2b - a$; to leave x on LHS.

$$\frac{x(R^2b - a)}{R^2b - a} = \frac{R^2Q - p}{R^2b - a} = x = \frac{R^2Q - p}{R^2b - a}$$

(e) $S = \frac{wd}{x} = \left(x - \frac{d}{2}\right)$

Step I: Rewrite the equation properly to enable you do some mathematical exercise

$$\frac{S}{1} = \frac{wd}{x} = \left(\frac{x}{1} - \frac{d}{2}\right)$$

Step II: Find the LCM and simplify

$$\frac{S}{1} = \frac{wd}{x} = \left(\frac{x}{1} - \frac{d}{2}\right) = \frac{S}{1} = \frac{wd}{x} = \left(\frac{2x-d}{2}\right) = \frac{S}{1} = \frac{wd(2x-d}{2x}$$

Step III: Cross multiply

$$2Sx = wd\,(2x - d)$$

Step IV: Open the bracket and make the equation linear without brackets

$2Sx = 2wdx - wd2$

Step V: Collect like terms by bringing all components of x to the LHS

$2Sx - 2wdx = -wd2$

Step VII: Divide both sides by 2(S – wd) to make only x remains on the LHS

$$\frac{2Sx - 2wdx}{2(S - wd)} = \frac{-wd2}{2(S - wd)} \therefore x = \frac{-wd2}{2(S - wd)}$$

(f) $V = \frac{1}{3}\pi x^2 h$

Step I: Cross multiply to make the equation linear

$3V = \pi x^2 h$

$\pi x^2 h = 3v$ (Note a = b is the same as b = a)

Step II: Divide both sides πh to make x^2 left in the LHS

$$\frac{\pi x^2 h}{\pi h} = \frac{3v}{\pi h} = x^2 = \frac{3v}{\pi h}$$

Step III: Finding the square root of both sides to make x^2 reduce to x

$$\sqrt{x^2} = \sqrt{\frac{3v}{\pi h}} \therefore x = \frac{3v}{\pi h}$$

(g) $\frac{k}{x+1} - \frac{b}{x+1} = \frac{1}{x}$

Step I: Find the LCM and multiply throughout by LCM

$\frac{k}{x+1} - \frac{b}{x+1} = \frac{1}{x}$ the LCM is $x(x+1)$

$\frac{k}{x+1} . [x(x+1)] - \frac{b}{x+1} . [x(x+1)] . = \frac{1}{x} x(x+1) = kx - bx = x + 1$

Step II: Collect like terms by bringing the components of x to the LHS

$kx - bx = 1$

Step III: Factorise out 'x'

$x(k - b - 1) = 1$

Step IV: Divide both sides by (k – b – 1) to leave only x on the LHS

$$\frac{x(k-b-1)}{(k-b-1)} = \frac{1}{(k-b-1)} \therefore x = \frac{1}{(k-b-1)}$$

SELF-ASSESSMENT EXERCISE

Make x the subject of the relation in each of the following:

i. $b = \frac{1}{2}\sqrt{a^2 - x^2}$

ii. $a\sqrt{x + y} = bM$

iii $L = \frac{xh}{a(x+p)}$

4.0 Summary

Equations and functions are very essential in business, economics and mathematics to solve some day-to-day problems. Functional relationship among variables enables us to establish interdependence among the variables by identifying the direction and magnitude of the relationship. Equation and function are usually written in two sides [left hand side (LHS) and right-hand side (RHS)]. The two sides are separated by equal to sign. The component of the LHS is the dependent variable or subject of the relation. Some techniques are involved in making any variable in the RHS, the subject of the relation. The principle of change of subject of relation provides us with the intuition that any variable in a stated equation could be made the dependent variable. This is important because the variable in an equation can be obtained given the value of other variables.

5.0 References/Further Readings/Web Resources

- Black, T. & Bradley J. F. (1980). Essential Mathematics for Economists. (2nd ed.). Chicherter: John Wiley and Son.
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- Usman, *et al.* (2005). Business Mathematics. Lagos: Apex Books Limited.

6.0 Tutor-Marked Assignment

1. $T = \frac{\pi x}{b} + kx$
2. $M = \sqrt{\frac{Qx+T}{Qx-T}}$
3. $D = \frac{0}{360}\ X\ 2\pi x$
4. $A = x + (1 + r)\, x$
5. $D = \frac{c-x}{N}$
6. $E = xm - F$
7. $\frac{1}{x} + \frac{1}{y} = \frac{1}{Z}$

UNIT 2 LINEAR EQUATIONS AND LINEAR SIMULTANEOUS EQUATION

Unit Structure

1.0 Introduction

A linear equation is the simplest form of equation. It is an equation that has a highest power of 1 and when graphed on a plane scale, it produces a straight-line graph. A typical linear equation has the form of y = mx + c, where:

y = dependent variable

m = slope of the graph

c = intercept of the graph

x = independent variable.

Most often than none, because of its simplicity, linear equations are used in economics and business to establish functional relationship among variables. Examples include relationship between quantity demanded and price e.g., Qd = 40 – 3p relationship between consumption and disposable income e.g. C = 40 + 0.6Yd, etc.

2.0 Learning Outcomes

After reading this unit, students will be able to:

- explain the components of a linear equation
- solve linear equations
- solve simultaneous linear equations.

3.0 Linear Equations and Linear Simultaneous Equation

3.1 Linear Equations

Most relationship in economics and business are expressed in single linear terms, not only for simplicity purpose but also for easy assignment in terms of the magnitude and direction of the variables concerned. A linear equation is a mathematical statement which shows that two algebraic terms or expressions are equal. The algebraic terms should have the highest power of terms equal to 1. For example:

(a) $2x + 9 = 27$

(b) $x + 4 = 5 - 3x$

(c) $9 = 4 - 2q$

(d) $2x + 2y = 1$

In examples a, b and c above, the unknown in each of the equations is *x*. *x* has the highest power of 1 in each case. In example (d), we have a linear equation with two unknowns. In equation (a), (b) and (c), the unknown can easily be obtained by collecting like terms and dividing by the coefficient of the unknown. In the case of example (d), there are two unknowns in a single linear equation. The value unknowns cannot be obtained in (d) unless another similar equation is provided, thereby, making the two equations a pair of simultaneous linear equation.

SELF-ASSESSMENT EXERCISE

i. Explain the term “linear equation”? How is it different from other forms of equations?

ii. Why are linear equations commonly used in establishing relationship between or among variables?

3.1.1 Solving Linear Equation

In solving a linear equation to obtain the value of the unknown – three major steps are required:

a) collect like terms

b) add/subtract the like terms to/from the one another

c) divide throughout by the coefficient of the unknown.

Example: Solve for the unknown in each of the following equations:

i. $20 - 3y = 17$

ii. $20 - 8x = 2 + x$

iii. $25 - 5x = 5x$

iv. $6x + 9 = 63 - 3x$

v. $2x + 19 - 5x = x - 5$

Solution

i. $20 + 3y = 17$

Collecting like terms

$20 - 17 = 3y$

$3 = 3y \ or \ 3y = 3$

Divide through by 3 (the co-efficient of y)

$\frac{3y}{3}$ ∴ y = 1

Note: To confirm the correctness of the value obtained, you can substitute the value back to the equation and see whether the LHS equals the RHS. If it does not, then the solution obtained is wrong. For example:

$20 - 3y = 17$

Substituting y = 1

$20 - 3(1) = 17$

$20 - 3 = 17$

17 = 17. Hence, y = 1 is absolutely correct.

ii. $20 - 8x = 2 + x$

Collecting like terms

$20 - 2 = x + 8x$

$18 = 9x$

Divide both sides by 9

$\frac{9x}{9} = \frac{18}{9} \therefore x = 2$

iii. $25 - 5x = 5x$

Collecting like terms,

$25 = 5x + 5x$

$25 = 10x \ or \ 10x = 25$

Dividing both sides by 10

$\frac{10x}{10} = \frac{25}{10} \therefore x = \frac{5}{2}$ or $2\frac{1}{2}$

iv. $6x + 9 = 63 - 3x$

Collecting like terms

$6x + 3x = 63 - 9$

$9x = 54$

Divide both sides by 9

$\frac{9x}{9} = \frac{54}{9} \therefore x = 6$

v. $2x + 19 - 5x = x - 5$

Collect like terms

$2x - 5x - x = 15 - 19$

$-4x = 24$

Divide both sides by -4

$\frac{-4x}{-4} = \frac{-24}{-4} \therefore x = 6$

SELF-ASSESSMENT EXERCISE

i. Solve for x in each of the following equations:

(a) $42 = 7x - 14$ (b) $18 - 2x = 4x - 12$ (c) $5x + 17 = 2x + 41$ (d) $2x + 5 = 13$ (e) $2(2x + 9) = 3(4x - 10)$

ii. Given that the quantity demand of orange $Qd = 40 - 5P$

(a) What quantity of orange is demanded if price is N5?

(b) At what price will 25 units of oranges demand?

iii. Given the quantity demand and quantity supplied as $Qs = -20 + 3P$ and $Qd = 220 - 5P$. Determine the equilibrium price for the market (Hint: equate Qs and Qd and solve for P i.e. $-20 + 3P = 220 - 5P$.

3.2 Simultaneous Linear Equation

A simultaneous linear equation is a set of linear equations in which the numbers of the equations equal the number of unknown. The two commonest mathematical approaches of solving simultaneous equation are:

1. Elimination method
2. Substitution method

Elimination Method: This involves multiplying the coefficient of the unknown by some constant values so as to make one of the unknowns have the same coefficient and therefore eliminated leaving only one unknown that could be solved using simple linear equation technique.

Example 1

$2x + 5y = 2$

$x - 2y = 10$

Solution

$2x + 5y = 2$ (1)

$x - 2y = 10$ (2)

$-2x + 5y = 2$ (1) x 1

$2x - 4y = 20$ (2) x 2

$5y - (-4y) = 2 - 20$

$5y + 4y = -18$

$9y = -18$

Divide both sides by 9

$\frac{9y}{9} = \frac{-18}{9}$ ∴ y = -2

From equation (2)

$x - 2y = 10$

$x - 2(-2) = 10$

$x + 4 = 10$

$x = 10 - 4 = 6$

Example 2

$a + b - c = 0$ (i)

$3a - 2b = 4c = 11$ (ii)

$5a - b - c = 0$ (iii)

From (i) and (ii)

$a + b - c = 0$ (i) x-3

$3a - 2b = 4c = 11$ (ii) x1

$-3a + 3b - 3c = 0$

$3a - 2b = 4c = 11$

$3b - (-2b) - 3c - (4c) = 0 - 11$

$5b - 7c = -11$ (iv)

From (ii) and (iii)

$3a - 2b = 4c = 11$ (ii) x -5

$5a - b - c = 0$ (iii) x 3

$-15a - 10b + 20c = 55$

$15a - 3b - 3c = 0$

$-10b - (+3b) + 20c - (-3c) = 55 - 0$

$10b - 3b + 20c + 3c = 55$

$7b + 23c = 55$ (v)

Solving (iv) and (v) simultaneously

$5b - 7c = -11$ (iv) x -7

$-7b + 23c = 55$ (v) x 5

$-35b + 49c = 77$

$-35b + 115c = 275$

$49c - 115c = 77 - 275$

$-66c = -198$

$C = \frac{-198}{-66} \therefore c = 3$

From equation (iv)

$5b - 7c = -11$

$5b - 7(3) = -11$

$5b - 21 = -11$

$5b = -11 + 21$

$5b = 10$

$C = \frac{10}{5} \therefore c = 2$

From equation (i)

$a + b - c = 0$

$a + 2 - 3 = 0$

$a - 1 = 0$

$a = 1$

Substitution Method: Substitution method requires writing one of the unknown of the equation in terms of the unknown. The equation obtained is therefore substituted to the other equation.

$2x - 4y = -4$ (i)

$3x + 2y = 18$ (ii)

From equation (i)

$2x = -4 + 4y$

$x = \frac{-4 + 4y}{2}$

Substitute x in equation (ii)

$3x + 2y = 18$ (ii)

$3\frac{-4 + 4y}{2} + 2y = 18 = \frac{-12 + 12y}{2} + \frac{2y}{1} = \frac{18}{1}$

Multiply through by 2

$-12 + 12y + 4y = 36$ $12y + 4y = 36 + 12$

$16y = 48$

$y = \frac{48}{16} \therefore y = 3$

From equation (i) $2x - 4y = -4$

$2x - 4(3) = -4$

$2x - 12 = -4$

$2x = -4 + 12$

$2x = 8 \therefore x\ \frac{8}{2} = 4$

SELF-ASSESSMENT EXERCISE

i. Solve the following pairs of simultaneous equation using elimination method:

(a) $2x - y = 8$

$3x + y = 17$

(b) $a - b + c = 2$

$2a - 2b + c = 3$

$4a - 3b + 2c = 7$

ii. Use substituting method to solve for p and q if:

$2p - 3q = 1$

$3p + 2q = 21$

iii. The equilibrium conditions of two markets, butter and margarine, where P_b and P_m are the prices of butter and margarine respectively, are given as:

$8P_b - 3P_m = 7$ and $-P_b + 7P_m = 19$. Find the prices of butter and margarine.

4.0 Summary

Linear equation may have one unknown or more than one unknown. When a linear equation has only one unknown, it is a simple linear equation and it can be solved by collecting like terms and dividing by the coefficient of the unknown after simplification. When it has more than one unknown, for instance, two equations of the same type and number of unknown for the unknowns to be solved. Any of the two techniques of solving simultaneous equation (elimination or substitution method) could be used to solve a pair of a simultaneous linear equation. The procedures for solving simple linear equation and simultaneous linear equation should be able to produce answer(s) such that when substituted back to the equation, give the same answer at both the right-hand side and left-hand side of the equation. This is a quick way of testing for the correctness of the answer(s) obtained.

5.0 References/Further Readings/Web Resources

- Black, T. & Bradley J. F. (1980). Essential Mathematics for Economists. (2nd ed.). Chicherter: John Wiley and Son.
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6.0 Tutor-Marked Assignment

1. Solve for x in each of the following equation:

   a) $10x + 60 = 10 + 5(3 - x)$

   b) $3(2x - 5) = 9$

   c) $21 - (6x + 7) = 8(3 - 2x)$

   d) $43 - 3(7 - x) = 4 - 3x$

2. Solve for x and y in each of the following equation:

   i. $3x + 4y = 12$ $9x + 2y = -9$

   ii. $2x - 3y = 10$
   $x - 3y = 8$

   iii. $3x + y = 10$
   $3x - 2y = -2$

   iv. $2x - 3y = 6$
   $x - 2y = 5$

3. 8 Choco and 3 pens cost N288 while 5 Choco and 2 pens cost N184,

   (a) Find the cost of a Choco

   (b) Find the cost of a pen.

[Hint: let Choco be x and pen be y i.e., equation 1 is 8x + 3y = 288]

UNIT 3 QUADRATIC EQUATION

Unit Structure

1.0 Introduction

In Unit 2 of this module, we defined a linear equation as an equation which has its highest power as 1 e.g., 3x + 12, 3x =16, etc. it is possible for an equation to have its highest power as 2 such as $x^2 - 4x + 5 = 0$, $x^2 = 25$, $x^2 + x = 5$ etc. These are called quadratic equations. It is an equation as a result of the equal sign, if it is written without equal to sign, such as $x^2 + 4x - 5$, $x^2 - 16$, $x^2 - x$, it is called quadratic expression.

2.0 Learning Outcomes

After reading this unit, students will be able to:

- identify quadratic equation and expression
- factorise quadratic expressions
- solve for the unknown in a quadratic equation
- identify perfect squares and make equations perfect squares.

3.0 Quadratic Equations

Quadratic equations can be solved using different approaches or methods. Common among these methods are factorisation, completing the square, formula method and the graphical method. Recall that, in solving a linear equation, a single answer is obtained. In solving a quadratic equation, two different answers are obtained. The general form of a quadratic expression is $y = ax^2 \pm bx \pm c$ while the general form of a quadratic equation is $y = ax^2 \pm bx \pm c = 0$

SELF-ASSESSMENT EXERCISE

Distinguish clearly between quadratic equations and quadratic expressions.

3.1 Factorisation

A quadratic expression of the form $y = ax^2 + bx + c$ can be factorised following the following steps.

Step I: Multiply a by c (ac)

Step II: Find two products of "ac" such that when the products are added to it gives "b".

Step III: Replace b with the two products obtained in step II above.

Step IV: Break the expression obtained in step III into two parts and factorise each of the two parts.

Example

Factorise each of the following quadratic expressions

I. $x^2 + 5x + 6$

II. $3x^2 - 7x - 6$

III. $2y^2 - 11y + 5$

IV. $x^2 - 36$

Solution

I. $x^2 + 5x + 6$

Note: The co-efficient of x 2 = 1 and the value of c = 6

Therefore, two products of 6 that add up to 5 are +3 and +2 Replacing 5 with the products, we have

$x^2 + 3x + 2x + 6$

Factorising, we have

$(x^2 + 3x) + (2x + 6)$

$(x + 3) + 2(x + 3)$

II. $3x^2 - 7x - 6$

$-18 + 2x - 9 = -18$ ($+2$, -9)

$+2 - 9 = -7$

$3x^2 - 9x + 2x - 6$

$3x(x - 3) + 2(x - 3)$

$(x - 3)(3x + 2)$

III. $2y^2 - 11y + 5$

$10 - 10x - 1 = -10$ (-10, -1)

$-10 + (-1) = -11$

Therefore, $2y^2 - 10y - 1y + 5$

$2(y - 5) - 1(y - 5)$ $(y - 5)(2y - 1)$

Note: when it seems that no factor is common between the terms, 1 (universal factor) is used to factorise the expression.

IV. $x^2 - 36$

This can better be written as

$x^2 + 0x + 36$

$36 - 6 + 6 = 0$ (-6, $+6$)

$-6x + 6 = -36$

Therefore, $x^2 + 6x - 6x - 36$

$(x + 6) - 6(x + 6)$

$(x + 6)(x - 6)$

It should be noted that factorisation can be used to solve quadratic expression. For example, use factorisation to solve the following quadratic equations.

(a) $y2 - 2y - 35 = 0$

(b) $3x2 + 10x - 8 = 0$

(c) $5t2 + 21t - 20 = 0$

(d) $x2 - 169 = 0$

(e) $3x2 - 75 = 0$

(f) $m2 + 6m + 9 = 0$

solution

(a) $y2 - 2y - 35 = 0$

$y2 - 7y + 5y - 35 = 0$

$(y - 7) + 5 = 0$ $(y - 7)(y + 5) = 0$

$(y - 7) = 0; y = 7$

or

$(y + 5) = 0; \quad y = -5$

Therefore, $y = 7 or - 5$

(b) $3x2 + 10 - 8 = 0$

$3x2 + 12x - 2x - 8 = 0$

$3x\ (x + 4) - 2(X + 4) = 0$ $(3x - 2)\ (x + 4) = 0$

$3x - 2 = 0\ or\ x + 4 = 0$

$3x = 2\ or\ x = 0 - 4$

$x = \frac{2}{3}$ or x = - 4

(c) $5t^2 + 21t - 20 = 0$

$5t^2 + 25t - 4t - 20 = 0$

$(5t^2 + 25t) - (4t - 20) = 0$

$5(t + 5) - 4(t + 5) = 0$

$(5t - 4)(t + 5) = 0$
$5t - 4 = 0 \ or \ t + 5 = 0$
$5t = 0 + 4 \ or \ t + 5 = 0$
$5t = 4 \ or \ t = 0 - 5$
$t = \frac{5}{4}$ or t = - 5

(d) $x^2 - 169 = 0$
$x^2 = 0 + 169$
$x^2 = \sqrt{169} \ = x = \pm 13$
$x = 13 \ or - 13$
By factorising, $x^2 - 0x - 169 = 0$
$x^2 - 13x + 13x - 169 = 0$
$x\ (x - 13) + 13(x - 13) = 0\ (x + 13)\ (x - 13) = 0$
$x + 13 = 0 \ or \ x - 13 = 0$
$x = 0 - 13 \ or \ x = 0 + 13$
$x = -13 \ or \ x = 13$
$x = -13 \ or \ 13$

(e) $3x^2 - 75 = 0$
$Divide\ through\ by\ 3$
$x^2 - 25 = 0$
$x^2 + 0x - 25 = 0$
$x^2 + 5x - 5x - 25 = 0$
$x\ (x + 5) - 5(x + 5) = 0\ (x + 5)(x - 5) = 0$
$x + 5 = 0 \ orx - 5 = 0$
$x = -5 \ orx = 5$
$x = -5 \ or \ 5$

(f) $m^2 + 6m + 9 = 0$
$m^2 + 3m + 3m + 9 = 0$
$m\ (m + 3) + 3(m + 3) = 0\ (m + 3)\ (m + 3) = 0$
$m + 3 = 0 \ orm + 3 = 0$
$m = 0 - 3 \ orm = 0 - 3$
$m = -3 \ orm = -3$
$m = -3\ (twice)$

SELF-ASSESSMENT EXERCISE

i. Factorise the following quadratic expressions:
(a) $3x2 - 6x + 9$
(b) $5p2 + 12p - 9$

ii. Solve for x in each of the following by using factorisation method.
(a) $5x2 + 12x + 7 = 0$
(b) $6x2 = 96$
(c) $x2 - 121 = 0$
(d) $x2 - x - 2 = 0$
(e) $x2 - 2x + 15 = 0$

3.2 Completing the Square Method

Given a general quadratic equation as $ax^2 + bx + c = 0$, here the steps to follow in solving the equation using the completing the square method.

Step I: Divide throughout by the coefficient of x^2 i.e.,

a. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step II: Take ‘c’ to the RHS.

$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$

Step III: Find ½ coefficient of x, square it and add it to both sides i.e.

$$\left(\frac{1}{2}\ x\ \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right) = x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\ c + \left(\frac{b}{2a}\right)^2$$

Step IV: Complete the squares

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$= \left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

To obtain x, find the square root of both sides

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= x + \frac{b}{2a} = \sqrt{\frac{b^2-4ac}{4a^2}} = \therefore x = \frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{4a^2}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{4a^2}$$

It should be denoted that completing square method is often applicable when the quadratic equation cannot be factorised e.g.,$3x^2 + 6x - 8 = 0$

This expression is not factorisable because there are no factors of –24 that add up to +6. Therefore, we can be use completing the squares to solve as follows:

$3x^2 + 6x - 8 = 0$

Divide through by 3.

$x^2 + 2x - \frac{8}{3} = 0$

Take constant to the other side

$x^2 + 2x = \frac{8}{3}$

Find ½ coefficient of x, square it and add it to both sides $\left(\frac{1}{2}\ x\ 2\right)^2 = (1)^2$

$x^2 + 2x + (1)^2 = \frac{8}{3} + (1)^2$

Complete the squares

$(x + 1)^2 = \frac{8}{3} + \frac{1}{1} = (x + 1)^2 = \frac{8+3}{3} = (x + 1)^2 = \frac{11}{3}$

Find the square root of both sides

$\sqrt{(x + 1)2} = \sqrt{\frac{11}{3}} = x + 1 = \sqrt{\frac{11}{3}}$

$\therefore x = -1 \pm \sqrt{\frac{11}{3}} = x^2 = -1 + \sqrt{3.67}\ or - 1 - \sqrt{3.67}$

SELF-ASSESSMENT EXERCISE

Use completing square to find the value of x in each of the following

i. $5x^2 + 12x - 7 = 0$

ii. $4x^2 - 7x + 9 = 0$

3.3 Formula Method

The formula method is obtained from the completing the squares method. From the final answer obtained in the completing the squares method,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a^2}$$

The formula method is applicable in solving any form of quadratic equation, whether the equation is factorisable or not.

Example: Given that: $3x^2 + 6x - 8 = 0$, use the formula method to obtain the value of x.

Solution

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a^2}$$

a = 3, b = 6, c = -8

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-8)}}{2 \ x \ 3}$$

$$= \frac{-6 \pm \sqrt{36 + 96}}{6}$$

$$= \frac{-6}{6} \pm \frac{\sqrt{132}}{6} = \quad = \frac{-6}{6} \pm \frac{\sqrt{4 \ x \ 33}}{6} = -1 \pm \frac{2\sqrt{33}}{6}$$

$X = -1 + \frac{\sqrt{33}}{3}$ or $-1 - \frac{\sqrt{33}}{3}$

Note: $b^2 - 4ac$ is called Discriminant (D), if D = 0 or $b^2 = 4ac$ - the quadratic equation has two equal ` answers.

If $b^2 > 4ac$ or D > 0, we have two distinct answers; and

If $b^2 < 4ac$ or D < 0, we have complex number as answers.

SELF-ASSESSMENT EXERCISE

Use formula method to solve the unknown in each of the following equations.

i. $5x^2 - 12x - 9 = 0$

ii. $3m^2 - 4m + 2 = 0$

3.4 Perfect Square

A perfect square is a quadratic expression that has the same term after factorisation. A quadratic equation is said to be perfect square when it has two equal answers or roots. This implies that; $b^2 - 4ac = 0 \ or \ b^2 = 4ac$

Examples include:

$x^2 + 4x + 4 = 0$

$x^2 + 2x + 2x + 4$

$x(x + 2) + 2(x + 2) \ (x + 2)(x + 2)$

Example 1: Find the value of K for which each of the following to be a perfect square:

(a) $12x^2 - 6x + K$

(b) $2x^2 + 8x + K$

Example 2: Find the quadratic equation whose roots are:

-1 and 4

3 and 2

-3 twice

Solution

(a) $12x^2 - 6x + K$

$b^2 = 4ac$

$(-6)^2 = 4(12) \ K$

$36 = 48K$

$K = \frac{36}{48} = \frac{3}{4}$

$\therefore 12x^2 - 6x + \frac{3}{4}$ *is a perfect square*

$48x^2 - 24x + 3$ is a perfect square

$Check$: $48x^2 - 12x - 12x + 3$

$= 12x(4x - 1) - 3(4x - 1)$

$= (4x - 1)(12x - 3)$

$= (4x - 1)(4x - 1)$ $Note$: $12x - 3 = 4x - 1$

(b) $2x^2 + 8x + K$

Where $b^2 = 4ac$

$(8)^2 = 4(2) \ K$

$64 = 8K$

$K = \frac{64}{8} = 8$

Therefore, $2x^2 + 8x + 8 = 0$ is a perfect square.

Check: $2x^2 + 4x + 4x + 8 \ 2x(x + 2) + 4(x + 2)$

$(2x + 4)(x + 2)$

$(x + 2)(x + 2)$

Example 2

(a) $(x + 1)(x - 4) = 0$ Note: there is a change in the signs of the roots.

$x(x + 1) - 4(x + 1) = 0$

$x2 + x - 4x - 4 = 0$

$x2 - 3x - 4 = 0$

(b) $(x - 3)(x - 2) = 0$

$x(x - 2) - 3(x - 2) = 0$

$x2 - 2x - 3x + 6 = 0$

$x2 - 5x + 6 = 0$

(c) $(x + 3)(x + 3) = 0$

$x(x + 3) + (x + 3) = 0$

$x2 + 3x + 3x + 9 = 0$

$x2 + 6x + 9 = 0$

SELF-ASSESSMENT EXERCISE

i. Find the value of K for each of the following to be a perfect square.

(a) $4x^2 - 8x + K$

(b) $x^2 + 4x + K$

ii. Check whether each of the following equations is perfect squares or not.

(*Hint*: use $b^2 = 4ac$)

(a) $3x^2 - 12x + 6$

(b) $2x^2 + 8x + 8$

(c) $2x^2 - 8x + 16$

iii. Find the quadratic equation whose roots are:

(i) $-\frac{1}{2}\ and\ \frac{3}{2}$

(ii) $4\ and - 3$

(iii) 2 twice

4.0 Summary

Quadratic equations are non-linear equation in which the highest power of the terms is 2. When quadratic expressions are factorised, two distinct or different terms are obtained. Likewise, when a quadratic equation is solved, two different or distinct answers are obtained depending on the value of the Discriminant (D). When the roots or answer are the same, the expression or equation becomes a perfect square. Quadratic equation can be simply solved using factorisation method. When the equation is not factorisable, completing the square method or formula method is used.

5.0 References/Further Readings/Web Resources

- Black, T. & Bradley J. F. (1980). Essential Mathematics for Economists. (2nd ed.). Chicherter: John Wiley and Son.
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6.0 SELF-ASSESSMENT EXERCISE

1. Factorise the following quadratic expressions:

(a) $p2 - 5pq - 6q2$

(b) $4t2 - 12t + 5$

2. Solve for x in each of the following using factorisation method.

   (a) $3x2 - 23x - 8 = 0$

   (b) $x2 - 8x - 9 = 0$

3. The following equations are not factorisable, use any method of your choice to solve for x:

   (i) $3x2 + 9x - 8 = 0$

   (ii) $5x2 + 14x + 6 = 0$

4. Find the quadratic equation whose roots are:

   (a) -4 twice

   (b) -1 and ½

   (c) +3 and -6

5. Find the value of K required to make each of the following a perfect square.

   (i) $4x2 - 12x + K$

   (ii) $5x2 - 14x + K$

   (iii) $3x2 + 6x + K$

UNIT 4 SIMULTANEOUS LINEAR AND QUADRATIC EQUATIONS

Unit Structure

1.0 Introduction
2.0 Learning Outcome
3.0 Simultaneous Linear and Quadratic Equations
4.0 Summary
5.0 Tutor-Marked Assignment
6.0 References/Further Reading

1.0 Introduction

In Units two and three of this module, we discussed linear equations linear simultaneous equation and quadratic equations. We discussed the approaches of solving two linear simultaneous equations (substitution and elimination method). Sometimes, there could be simultaneous equation comprising of two equations; one linear and the other quadratic. These equations are solved simultaneously by using a combination of substitution method and factorisation techniques. Examples of such pairs of simultaneous equation are:

$$x^2 + y^2 = 70; \quad x + y = 8$$
$$x + y = 7; x^2 + 2x^2 - 3 = y$$

It should be noted that, in each of the examples cited above, at least two values are obtained for each of the unknowns (x and y).

2.0 Learning Outcome

After reading this unit, students will be able to:

Solve pairs of simultaneous linear equation and quadratic equations.

3.0 Simultaneous Linear and Quadratic Equation

Linear equations and quadratic equations have different traits or characteristics. Yet, they could be solved simultaneously. The tendency of the quadratic component to give two roots or two answers will generate two answers/roots to each of the unknown except in few cases of repeated roots. In most cases, we factorise the quadratic equation to obtain the roots of the equation which are substituted to the other equation. In some cases, we write one of the unknowns in a linear equation and substitute it into the quadratic equation. Whichever the approach employed, multiple values of each of the unknowns are most likely to be the result.

SELF-ASSESSMENT EXERCISE

Explain how a linear and a quadratic equation could be solved simultaneously.

Example of Simultaneous Linear and Quadratic Equation

Solve for x and y if:

(a) $x - y = 4$
$4y^2 + 5y - 51 = 0$

(b) $x + y = 10$
$xy = 21$

(c) $2x - y = 6$
$x^2 + y^2 = 41$

Solution

(a) $x - y = 4$ (i)
$4y^2 + 5y - 51 = 0$ (ii)
From (ii), $4y^2 + 5y - 51 = 0$
$4y^2 + 17y - 12y - 51 = 0$
$y\,(4y + 17) - 3(4y + 17) = 0\ (y - 3)\,(4y + 17) = 0$
$y - 3 = 0;\quad y = 3$
$or\quad 4y + 17 = 0;\ 4y = -17;\ y = -\frac{17}{4}$

From equation (i), $x - y = 4$
$x = 4 + y$
When y = 3
$x = 4 + 3 = 7$
When y $= -\frac{17}{4}$
$x = 4 + -\frac{17}{4} = 4 - 5\frac{2}{3} = -1\frac{2}{3}$ or $-\frac{5}{3}$
$\therefore\ x, y = [7, 3]\left[-\frac{5}{3}, -\frac{17}{4}\right]$

(b) $x + y = 10$ (i)
$xy = 21$ (ii)
From (i), $x = 10 - y$ (iii)
Substitute equation (iii) into equation (ii)
$(10 - y)\,y = 21$
$10y - y2 = 21$
$-y2 + 10y - 21 = 0,$
$multiply\ through\ by - 1$
$y2 - 10y + 21 = 0$
$y2 - 7y - 3y + 21 = 0$
$y\,(y - 7) - 3(y - 7) = 0\ (y - 3)\,(y - 7) = 0$
$y - 3 = 0,\ y = 3\ or\ y - 7 = 0,\ y = 7$
$From\ equation$ (iii), $x = 10 - y$
When $y = 3,\ x = 10 - 3 = 7$
When $y = 7,\ x = 10 - 7 = 3$
$\therefore x, y = [(3, 7)\ (7, 3)]$

(c) $2x - y = 6$ (i)
$x2 + y2 = 41$ (ii)
From (i), $2x - 6 = y$ (iii)
Substitute equation (iii) into equation (ii)
$x2 + (2x - 6)2 = 41$
$x2 + (2x - 6)\,(2x - 6) = 41$
$x2 + 2x\,(2x - 6) - 6(2x - 6) = 41$
$x2 + 4x2 - 12x - 12x + 36 = 41$
$x2 + 4x2 - 24x + 36 - 41 = 0$
$5x^2 - 24x - 5 = 0$
$5x^2 - 25x + x - 5 = 0$

$5x\,(x-5)+1(x-5)=0\;(5x+1)(x-5)=0$

$5x+1=0\;or\;x-5=0$

$5x=-1\;or\;x=5$

$x=-\frac{1}{5}$ or $x=+5$

From equation (iii), $2x-6=y$

When $x=-\frac{1}{5}$, $\;y=2\left(-\frac{1}{5}\right)-6$

$=-\frac{2}{5}-6$

$y=-6\frac{2}{5}$ or $-\frac{32}{5}$

When $x=5$, y = 2(5) – 6

y = 10 -6 = 4

$\therefore x, y=[(5, 4)\;(-\frac{1}{5}, -\frac{32}{5}]$

4.0 Summary

Simultaneous linear and quadratic equation involves getting the solution to two unknowns in two different forms of equation – one linear equation and the other quadratic equation. The nature of such equation gives room for obtaining multiple answers. Although linear and quadratic equation has different features, the use of substitution and factorisation provide means of solving the pair of simultaneous equation simultaneously.

5.0 References/Further Readings/Web Resources

- Black, T. & Bradley J. F. (1980). Essential Mathematics for Economists. (2nd ed.). Chicherter: John Wiley and Son.
- Channon, et al. (1990). New General Mathematics for West African. (4th ed.). United Kingdom: Longman Group.
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6.0 Self-Assessment Exercise

Solve for x and y in each of the following:

1. $x2-y2=12$
   $x-2y=3$
2. $3x2-xy=0$
   $2y-5x=1$

UNIT 5 INEQUALITIES

Unit Structure

1.0 Introduction

The first four Units of this module discussed various forms of equations. All equations are characterised with "equal to" sign which affirms that one side of the equation equals to the other. Sometimes, one side of an equation may not be equal to the other. In that case, one side may be less or greater than the other. All these constitute the concept known as Inequalities. The basic signs in inequalities are:

$<$	Less than Greater than
$>$	Greater than
$\leq$	Less than or equal to
$\geq$	Greater than or equal to

2.0 **Learning Outcome**

After reading this unit, students will be able to:

- define the word "inequalities"
- solve inequality problems
- present inequality problems in line graphs
- solve for inequality problems involving range of values.

3.0 Inequalities.

An inequality compares two unequal or unbalanced expressions or quantities. For example; 3 is not equal to 8 ($3 \neq 8$), 3 is less than 8 ($3 < 8$), 4 is greater than 1 ($4 > 1$), etc. the inequality sign remains unchanged when the same term is added or subtracted from both sides.

For example; $8 > 3$

$$\therefore 8 + 2 > 3 + 2 \text{ i.e., } 10 > 5$$
$$8 - 1 > 3 - 1 \text{ i.e., } 7 > 2$$

The inequality sign also remains unchanged when the same term (positive umber) is multiplies by both sides or divided by both sides.

For example; $7 > 4$

$$\therefore 7 \times 4 > 4 \times 4 \text{ i.e., } 28 > 16$$
$$24 < 30$$
$$24 \div 2 < 30 \div 2;\ 12 < 15$$

However, the inequality sign changes as a when both sides of inequality are divided by equal negative constant or multiplied by a constant negative value. For instance,

$$7 < 10$$
$$7 \times -2 > 10 \times -2$$
$$-14 > -20$$

It should be noted that the sign changes as a result of multiplying both sides by a constant negative number. Also, consider: $10 > 4$

$$\text{But } \frac{+10}{-2} > \frac{+4}{-2}$$
$$-5 < -2$$

SELF-ASSESMENT EXERCISE

i. What is an inequality?
ii. Under what conditions will the signs of inequalities change?

3.1 Solving Inequality Problems

Solution to inequality problems resembles that of linear equations. The procedures involved are similar. They are:

a. Collect the like terms.
b. Divide both sides by the co-efficient of the unknown terms. The only difference is that equations have equal to signs as the balancing symbol while inequality has inequality signs.

Example

Find the values of x that satisfy the following inequalities, hence, present the solution in a line graph.

a. $x - 4 < 3 - x$
b. $x + 3 < 3x - 5$
c. $3x - 4 \geq 4$
d. $x < \frac{1}{2}x + 3$

Solution

a. $x - 4 < 3 - x$

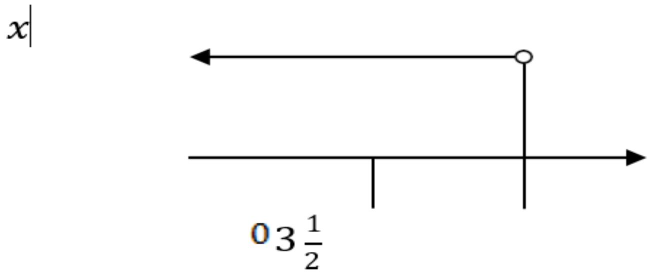
$Collecting\ like\ terms$

$x + x < 3 + 4$

$2x < 7$

$Divide\ throuh\ by\ 2$

$x < \frac{7}{2}$ or $x < 3\frac{1}{2}$



b. $x + 3 < 3x - 5$

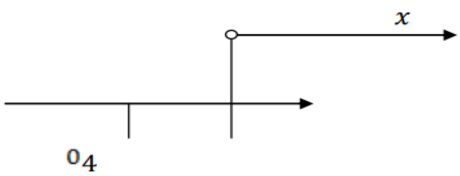
$Collect\ like\ terms$

$x - 3x < -5 - 3$

$-2x < -8$

$Divide\ both\ sides\ by - 2$

$\frac{2x}{-2} > \frac{8}{-2} \therefore x > 4$



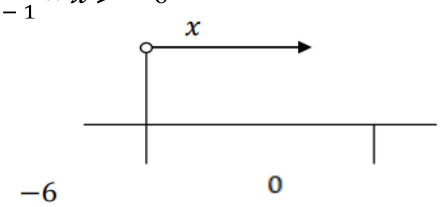
c. $x - 5 < 2x + 1$

$x - 2x < 1 + 5$

$-x < 6$

$Divide\ both\ sides\ by - 1$

$\frac{-x}{-1} > \frac{6}{-1} \therefore x > -6$



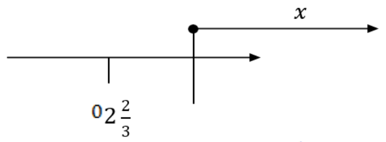
d. $3x - 4 \geq 4$

$3x \geq 4 + 4$

$3x \geq 8$

$Divide\ both\ sides\ by\ 3$

$\frac{3x}{3} \geq \frac{8}{3} \therefore x \geq \frac{8}{3}\ or\ 2\frac{2}{3}$



Note: For greater than or equal to $(\geq)$, the line graph node is shaded

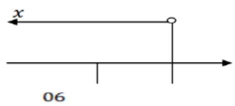
e. $x < \frac{1}{2}\ x + 3$.

$x - \frac{1}{2}\ x < 3$

$\frac{1}{2}\ x < 3$

$Multiply\ both\ sides\ by\ 2$

$\frac{1}{2}\ x\ X\ 2 < 3 \text{ x } 2 \therefore x < 6$



SELF-ASSESSMENT EXERCISE

Solve each of the following inequality problems and present the solution on a line graph.

i. $5x - 1 > 4$

ii. $4(2 - x) < 3(3 - 2x)$

iii. $4(x - 3) \leq 5$

iv. $4x - 2 \geq 3x - 1$

v. $x < \frac{1}{3} x + 4$

3.2 Range of Values of Inequalities

The domain and range of a linear inequality is always all real numbers, regardless of the sign of inequality. The range of absolute value inequalities will depend on the vertex. With polynomial inequalities, intervals of the x values created from zeros will determine the domain and range.

Example 1

Solve the inequalities:

$|1 - x| \leq 3$

Solution: The above inequality is equivalent to:

$-3 \leq 1 - x \leq 3$

$-3 \leq 1 - x \text{ and } 1 - x \leq 3$

$-3 - 1 \leq - x \text{ and } - x \leq 3 - 1$

$-4 \leq -x - x \leq 2$

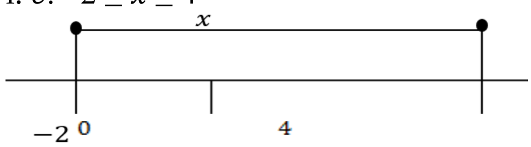
$-x \geq -4$ or $x \geq -2$

$x \leq 4$

$This\ implies\ that\ x \geq -2\ and\ the\ same\ x$ is $less\ than$ 4

$x \geq -2, x \leq 4$

i. $e. -2 \geq x \leq 4$



Example 2

If $2x - 6 \leq 8 < 2x + 4$. Find the range of values of x

$2x - 6 \leq 8$ $\quad and \quad$ $8 < 2x + 4$

$2x \leq 8 + 6$ $\qquad$ $8 - 4 < 2x$

$2x \leq 14$ $\qquad$ $4 < 2x$

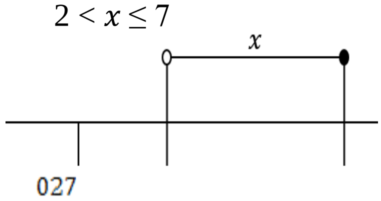
Divide through by 2 $\qquad$ Divide both sides by 2

$\frac{2x}{2} \leq \frac{14}{2}$ $\qquad$ $\frac{4}{2} < \frac{2x}{2}$

$x \le 7$ $\quad\quad\quad\quad$ $x < 2$

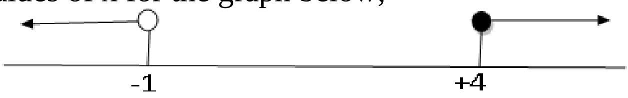
Combining the solution, we have:

$2 < x \le 7$



Example 3

1. State the range of values of x for the graph below;



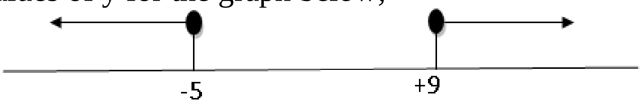
Solution

x < -1 $\quad\quad$ or $\quad\quad$ x ≥ +4

Therefore, the range of values of x will be;

-1 > x ≥ +4.

2. State the range of values of y for the graph below;



Solution

x ≤ -5 $\quad\quad$ or $\quad\quad$ x ≥ 9

Therefore, the range of values of x will be;

-5 ≥ x ≥ 9.

3. If 3 + x ≤ 5 and 8 + x ≥ 5, what range of values of x satisfies both inequalities?

Solution

3 + x ≤ 5 $\quad\quad$ or $\quad\quad$ 8 + x ≥ 5

(Making x the subject)

x ≤ 5 – 3 $\quad\quad$ or $\quad\quad$ x ≥ 5 – 8

x ≤ 2 $\quad\quad$ or $\quad\quad$ x ≥ - 3

Therefore, the range will be;

2 ≥ x ≥ -3.

SELF-ASSESSMENT EXERCISE

i. Solve for the following:
   (a) $|x + 1| < 6$
   (b) $|4 - 3x| < 2$

ii. Find the range of values of x that satisfy the followings:
   (a) $10 < 3x + 4 \le 30$
   (b)
   $-5 \le 3x - 14$
   ≤ 0
   (c) 3
   $< 11 - x < 11$
   (d) 6
   $< 4x - 2 < 12$

iii. Find he largest possible value of x that satisfy the followings:
   (a) $x + 7 \le 13$
   (b) $3x - 7 \le 6$

iv. Find the smallest possible of x that satisfy the followings:
   (a) $x + 6 > 4$
   (b) $2 \ge 14 - x$

v. a. State the range of values of x for the graph below;

a. If 6x < 2 – 3x and x – 7 < 3x, what range of values of x satisfies both inequalities?

b. What is the range of values of x for which 3(1 – x) > 3 and 3(1 + x) ≥ 0 are both satisfied?

c. State the range of values of x for the graph below;

d. What is the range of values of y for which 4y – 7 ≤ 3y and 3y ≤ 5y + 8 are both satisfied?

4.0 Summary

Inequalities involve the use of signs to compare two unequal expression or numbers. Solutions to inequality problems or equation follow similar approach with the linear equation. An inequality expression having more than one inequality sign is bound to have more than one range of value. The use of signs in mathematics is meant to compare the magnitude of one expression or quantity with another. Mathematicians use both equality and inequality signs to compare and contrast values so as to be able to arrive at a conclusion.

5.0 References/Further Readings/Web Resources

- Black, T. & Bradley J. F. (1980). Essential Mathematics for Economists. (2nd ed.). Chicherter: John Wiley and Son.
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- Usman, *et al.* (2005). Business Mathematics. Lagos: Apex Books Limited.

6.0 TUTOR-MARKED ASSIGNMENT

1. Assume x is a whole number and x is an integer. For each of the following list, the set of value of x.
   (a) $4 < x \leq 7$
   (b) $-5 < x < 1$
   (c) $5 < x + 9$
   (d) $6x > 4$
   (e) $-9 \leq x < 4$
2. Find the range of values of x that satisfy each of the following:
   (a) $-1 \leq 4x + 15 < 23$
   (b) $0 \leq 15 - x \leq 10$
   (c) $2x - 3| \leq 5$

MODULE 4 SEQUENCE AND SERIES

Unit 1 Meaning and Types of Sequence and Series
Unit 2 Arithmetic Progression (AP)
Unit 3 Geometric Progression (GP)

Unit 4 Application of Series and Sequences to Economics, Business and Finance

UNIT 1 MEANING AND TYPES OF SEQUENCE AND SERIES

Unit Structure

1.0 Introduction

In the first module of this book, you were introduced to the real number system. You will observe that integers are set of negative and positive whole numbers including zero (0) as a neutral number. A set of integers may be written following a particular order e.g.

(a) 1, 2, 3, 4, 5 …………… ………….
(b) – 1 – 2 – 3 – 4 – 5 – 6 – 7 ………………
(c) 2, 4, 6, 8, 10, 12 ……………………
(d) 1, 4, 9, 16, 25, 36 ………………
(e) - 10, - 20, - 30, - 40, - 50, - 60, -70 …………….
(f) 2, 4, 8, 16, 32, 64, 128 ………………….

You observe that each of these set of number has its own peculiar pattern or order of construction. For example,

(a) Is a set of positive integers (increasing by a unit i.e., adding (1) one to the previous)
(b) Is a set of negative integers (reducing by a unit i.e., subtracting (1) from the previous
(c) Is a set of positive even numbers (obtained by continuous addition of 2)
(d) Is set of squares of numbers i.e., perfect squares
(e) Is obtained by subtracting 10 from the subsequent set of numbers
(f) Is obtained by multiply the subsequent term by 2

These set of numbers are written based on a specific rule and they are collectively known as sequence. It should be noted that sequence can also be a set of fractions, decimals or even indices. The essential requirement for a set of number to form a sequence is that there must be a defined pattern or rule(s) in forming the set of numbers or terms.

2.0 Learning Outcome

After reading this unit, students will be able to:

- define the term "sequence" and "series"
- identify the basis types of sequence series
- solve problem involving arithmetic progression and geometric progression
- apply the basic principles of sequence and series to solve practical question relating to sequence and series.

3.0 Sequence and Series

Every magnitude (figure) has the tendency to increase, decrease, be multiplied or remain constant over time. The ability of a positive number to increase is subject to be added to another positive number (except 1). Similarly, a positive number decrease when it is added to another negative number or a positive number is taken from it or divided by a positive number (except 1). A number remains the same if multiplied by 1, divided by 1 or added to zero or zero subtracted from it.

A sequence is formed when a set of number is increasing or decreasing by a constant value as the term of the sequences are being formed. For example:

a. -2, -4, -6, -8, -10, ...
b. 4, 8, 10, 11, 15, 17, 19, ….
c. 10, 20, 30, 40, 50, 60, ……

d. 2, 6, 8, 54, 162, ……….…

e. 90, 80, 75, 65, 50, 40, ….

It should be noted that only sets of numbers in (a), (b) and (d) form sequence of number because the pattern of the decrease or increase is uniform over time. Although set of increases but not by a constant value, so also the set of numbers in (e). Therefore, set of numbers in (a), (c) and (d) form a sequence, while those in (b) and (e) do not.

SELF-ASSESMENT EXERCISE

For each of the following set of number, state whether it is a sequence or not.

i. 10, 9, 8, 7……..........

ii. 8, 9, 11, 14, 18, 23…………

iii. 0, -2, -4, -6, -8…………

iv. 144, 72, 36, 18………...

v. 1/144, 2/121, /100, 4/81, 5/64…………

3.1 Definition and Types of Sequence and Series

A sequence or progression of numbers is a set of number arranged in some order or rules with a constant pattern of growth or decline in the set of figures which will guide us to determine successive terms from their predecessors. Example of see include:

2, 4, 6, 8, 10, …………...

2, -1, -4, -7, -10, …………...

1, 2, 4, 8, 16, 32, …………...

It could be observed that in the first sequence, each successive term is greater than its predecessor by 2. In the second sequence each successive term is smaller than its predecessor by 3 while in the third sequences each successive term is twice its predecessor. A series is obtained when the terms of the sequence are connected by positive or negative signs e.g.

2 + 4 + 6 + 8 + 10…

2 + (-1) + (-4) + (-7) + (-10) …

144 – 72 – 36 -18 ….

A series in which successive terms have a common different or common ratio known as progressive. These are two major types of progressive namely:

i. Arithmetic progression

ii. Geometric progression

A sequence or progression of numbers where terms have constant difference is arithmetic sequence or progressive. The sum of the terms is called arithmetic series. This implies that if a sequence of the term (set of numbers) is such that the different between any terms and the one immediately preceding it is a constant, such terms are said to form arithmetic progression. Examples of the arithmetic progression are:

5, 8, 11, 14…

3, -1, -5, -9……

$-2, \frac{-3}{4}, \frac{1}{2}, \frac{13}{4}$ …...

a, a + d, a + 2d, a + 3d …

a – 2d, a – d, d, a+ d, a + 2d ….

An arithmetic progression is therefore described as a progression or series formed by continuous addition or subtract of a constant value called the "common difference" (d). In the example above, the common different of 5, 8, 11, 14… is 3, that of 3, - 1, -5, - 9, is – 4, the common different of -2, $\frac{-3}{4}, \frac{1}{2}, 1, 1\frac{3}{4} \ldots .. is\ 1\frac{1}{4}$

while that of at a – 2d, a- d, a, a + d… is d. The common difference can be obtained as thus:

$T_2 - T_1 = T_3 - T_2 = T_4 - T_3 \ldots \ldots T_n - (T_n - 1)$

Where T_2 = second term, T_3 = third term, T_1 = first term, T_4 = fourth term, T_n = *n*th term and T_{n-1} is the term that preceded the *n*th term.

If, in a sequence of terms, each term is a constant multiple of the preceding the terms are said to be in geometric progression. In a geometric progression (or sequence), there is a common ratio between each term and the previous term. The sum of the term in such progression is known as geometric

series. A geometric progression has the same ratio of all consecutive terms. It can also be defined as a sequence which is formed by a continuous multiplication or division by a constant value known as a common ratio (r). The common ratio (r) is given as $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \cdots \frac{T_n}{T_{n-1}}$. where T_1, T_2, T_3, T_4, and T_{n-1} are as defined under the geometric progression.
Examples of geometric progression (GP) are:

3, 6, 12, 24 ….
8, 4, 2, 1, ½ …….
2 – 10, 50 …….
a, ar, ar^2, ar^3 ……

In the first example successive terms obtained by multiplying to preceding term by 2 the common rate is $\frac{3}{6} = 2$

In the second example, the successive terms are obtained by multiply the preceding term by $\frac{1}{2}$ or by dividing the preceding term by 2 (the common ratio is $= \frac{4}{8} = \frac{1}{2}$). In the third example, common ratio $= \frac{ar}{a} = \frac{ar^2}{a} = r$

It should be noted that a special sequence or progression is obtained by certain manipulation that is not uniform. Successive terms are obtained by reasoning and following the trend or pattern of the sequence. Such sequences are not generated by a specific formula. Example includes:

1, 4, 9, 16, 25………..
$\frac{1}{121}, \frac{2}{100}, \frac{3}{81}, \frac{4}{64}, \frac{5}{49}$ ……..
8, 9, 1, 14, 18, 23 …

The first sequence is a set of perfect squares i.e., square of numbers. You will observe that the differences between each of the successive terms is not constant (the first is 3, then 5, then 7, then 9) but the difference themselves form an arithmetic progression. The second example is a sequence obtained by fraction of increasing whole number numerators and decreasing perfect square denominator. Although, other terms in the sequence can be generated but by general reasoning. The last example under the special sequence is obtained by an increasing difference. The difference between 8 and 9 is 1 that of 9 and 11 is 2, and so on. Hence, the differences, not the term form the AP of 1, 2, 3, 4, 5……….

SELF-ASSESSMENT EXERCISE

i. Distinguish clearly between arithmetic progression and geometric progression. Give example of each.
ii. In what ways is special sequence different from arithmetic progression and geometric progression?

4.0 Summary

Sequence can broadly be classified into two; arithmetic progressions (formed by continuous addition or subtraction of a common difference) and the geometric progression (formed by a continuous multiplication or division by a common difference). At the extreme are the special sequences which are generated by aptitude reasoning.

Progression or sequence are not just sets of increasing or decreasing numbers but a set of numbers formed following a defined rule. This makes it difference from special sequence which are formed by manipulations obtainable from the trend or pattern of the existing terms.

5.0 References/Further Readings/Web Resources

Hardwood, C. L. (1978). *Ordinary Level Mathematics*.(6th ed.). London: Heinemann Educational Books.

Robert, S. & Geoff, B. (2007). *Excellence in Mathematics for Senior Secondary Schools*. Lagos: Macmillan Publishers Limited.

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6.0 SELF-ASSESSMENT EXERCISE

For each of the following progression,

1. Identity the type of sequence
2. State the common difference or the common ratio (if it is AP or GP)
3. Generate the next four terms of the sequence:
   - (i) 1, 2, 3, 4, 5 …………….
   - (ii) 1, 4, 9, 16, 25 ………….
   - (iii) 8, 9, 11, 14, 18, 23 ………….
   - (iv) 6, 4, 2, 0, -2 ………….
   - (v) 144, 72, 36, 18 …………….
   - (vi) 2, 4, 8, 16, 32 …………….
   - (vii) $\frac{1}{144}, \frac{1}{121}, \frac{2}{100}, \frac{3}{81}, \frac{4}{64}, \ldots\ldots$
   - (viii) $\frac{1}{10}, \frac{4}{9}, \frac{9}{8}, \frac{16}{7}, \frac{25}{6}, \frac{36}{5} \ldots\ldots$

UNIT 2 ARITHMETIC PROGRESSION

Unit Structure

1.0 Introduction

In the preceding unit, you have learnt about two major types of sequence namely, arithmetic progression (AP) and the geometric progression (GP). You are aware that arithmetic progressions are generated by a continuous addition of a common difference and that the pattern can continue over a set of terms. Some mathematical formula can be used to generate large numbers of terms. Manual generation of such terms by following the rules may be time consuming and hectic when some terms such as 50th terms, 100th term, etc. are to be generated. More so, sum of terms no matter how large the number of terms is, can be obtained using some mathematical formula. The formula generating the term is called the n^{th} term formula while the one generating the sum is the sum of the arithmetic progression to the n^{th} term.

2.0 Learning Outcome

After reading this unit, students will be able to:

- generate the n^{th} tern of the progression
- generate the sum of a given arithmetic progression
- obtain the common difference or the first term of an AP given some sets of information.

3.0 Arithmetic Progression

3.1 nth Term of Arithmetic Progression

The nth term of an arithmetic progression (AP) is the mathematical formula used to generate any term in a given arithmetic progression. Assuming the first term of an AP is given as ‘a’ and the common difference is given as ‘d’. Then the nth term of an AP is mathematically given as:

$$T_n = a + (n - 1) d, \text{ where } T_n \text{ is the nth term.}$$

For instance,

4^{th} term = $T_4 = a + (4 - 1) d = a + 3d$

18^{th} term = $T_{18} = a + (18 - 1) d = a + 17d$, and so on.

Examples

1. Find 127th term of the AP given as 5, 8, 11….

Solution

$T_n = a + (n-1) d$

a = 1st term = 5

d = common difference = $T_2 - T_1 = 8 - 5 = 3$

$T_{127} = a + (127-1) 3$

$T_{127} = 5 = (126) 3$

$T_{127} = 5 + 378$

$T_{127} = 383$

2. How many terms has the arithmetic progression 8, 11, 14, 17, …, 380

Solution

Note: $T_n = a + (n - 1) d$

The question requires the position of 380 in the arithmetic progression i.e., the value of n, given that

$T_n = 380$, a = 8 and d = 11- 8 = 3 380 = 8 + (n – 1)3

380 = 8 + 3n – 3

380 – 8 + 3 = 3n

372 + 3 = 3n

$\frac{372+3}{3} = \frac{3n}{3} \therefore n = \frac{375}{3} = 125$

Therefore, the AP has 125 terms.

3. Insert 6 arithmetic terms between -3 and 18.

Solution

Let the AP be -3, a, c, d, e, f, g, 18

i.e., b, c, d, e, f, and g are the terms to be inserted.

$T_n = a + (n – 1) d$ 18 = -3 + (8 – 1) d

Note: 18 now becomes the 8th term and the difference is not yet known. Opening the brackets, we have 18 = -3 + 7d

$18 + 3 = 7d$

$21 = 7d \therefore d = \frac{21}{7} = 3$

Since the difference = 3 and a = -3

b = 2nd term = a + d = -3 + (-3) = 0

c = 3rd term = a + 2d = -3 + 2(3) = 3

d = 4th term = a + 3d = -3 + 3(3) = 6

e = 5th term = a + 4d = -3 + 4(3) = 9

f = 6th term = a + 5d = -3 + 5(3) = 12

g = 7th term = a + 6d = -3 + 6(3) = 15

Hence, the terms inserted are b, c, d, e, f, and g which are 0, 3, 6, 9, 12, 15,

Then the AP becomes: 0, 3, 6, 9, 12, 15, 18.

4. The 21st term of an AP is 50,000, if the first term is 20,000. Find the common difference.

Solution

$T_n = a + (n – 1) d$

50000 = 20,000 + (21 – 1) d

50,000 = 20,000 + 20d

50,000 – 20,000 = 20d

$30\ 000 = 20d \therefore d = \frac{30000}{20} = 1500$

3. The fifth and eighth term of an AP are 9 and 27 respectively.

Find

(a) The common difference;

(b) The first term;

(c) The nth term; and

(d) The 21st term.

Solution

$T_n = a + (n – 1) d$ $T_5 = a + (5 – 1) d$

$9 = a + 4d$ (i)

Likewise, $T_8 = a + (8-1) d$ $27 = a + 7d$ (ii)

Solving (i) and (ii) simultaneously,

$- a + 4d = 9$ (i)

$a + 7d = 27$ (ii), subtracting (i) from (ii)

$7d – 4d = 27 – 9$

$3d = 18 \therefore d = \frac{18}{3} = 6$

$\therefore$ Common difference = 6

From equation (i) $9 = a + 4d$ $9 = a + 4(6)$

$9 = a + 24$

$9 - 24 = a$

$a = -15$

The AP is -15, (-15 + 6), (-15 + 6 + 6), (-15 + 6 + 6 + 6) ……….

= -15, -9, -3, 3, 9, 15, 21, 27

The n^{th} term if the AP, $T_n= a + (n – 1) d$

$= -15 + (n – 1) d$

$-15 + (n – 1) 6$

$-15 + 6n – 6$

$= -21 + 6n.$

:. 21st term = $T_{21} = a + (21 – 1) d$ $T_{21} = -15 + (20)d$

$T_{21}= - 15 + 20(6)$

$T_{21} = -15 +120 =10$

SELF-ASSESSMENT EXERCISE

i. Find the 10th term of the following AP

(a) 1, 4, 7 …….

(b) 2, 21/2, 3 ……

(c) 10, 8, 6 …….

ii. The 4th term of an AP is 13 and the second term is 3 find the common different and the first term

iii. The 4th and the 7th term of an arithmetic sequence are 6 and 15 respectively. Find the nth term of the sequence

iv. The 9th term of an arithmetic sequence is 12 and the 17th term is 28. Find the 4th term.

v. If 3, x, y, 18 are in AP. Find x and y

3.2 Sum of an Arithmetic Progression

Term in an arithmetic progression can be added together using either of the formula below.

$S_n = \frac{n\{a+1\}}{2}$ where a = first term and *l* = Last term n = nth term. i.e., the number of terms to be added.

OR

$S_n = \frac{n\{2a+(n-1)d\}}{2}$ where a = first term d = common different and n = nth term.

The two formulae are proved the same as thus: Recall, AP is in the form:

a, a + d, a + 2d……….

Therefore, their sums become

a + (a + d) + (a + 2d) + (i)

Recall that the nth term = a + (n – 1) d, call this l.

Writing the series from backward we have

l + (l - d) + (l - 2d) (ii)

Each term of series in (i) added to the corresponding term of series in (ii) gives (a + l).

There are n terms and so the sum is n (a + l). This is the sum of two equal series and so the sum of each is $\frac{1}{2}\ n(a+1)$

Recall that $l = a + (n-1)\,d$

$\frac{1}{2}\ n(a+l) = \frac{1}{2}\ n\{a + a\ (n-1)\ d\}$

$= \frac{1}{2}\ n\{2a + (n-1)\ d\}$ or $\frac{n}{2}\{2a + (n-1)\ d\}$ And so

Example

1. Find the sum of 28 term of the arithmetic progression 3 + 10 + 17 +….

Solution

a = 3, d = 7, n = 28, $S_n = \frac{n\{2a+(n-1)d\}}{2}$

$= \frac{28}{2}\{2 \times 3 + (28 - 1)\ 7\}$

$S_{28} = 14\{6 + 27(7)\} = 14\{195\} = 2730$

2. An arithmetic sequence has first term as 3 and the common difference of 2. How many terms are needed to make sum equal to 99?

Solution

Recall: $\frac{1}{2}\ n\{2a + (n-1)\ d\}$

Where: $S_n = 99$, n = Unknown, a = 3, and d = 2

$99 = \frac{1}{2}\ n\{2 \times 3 + (n-1)\ 2\}$

$99 = \frac{1}{2}\ n\{6 + 2n - 2\}$

cross multiply

$198 = n\ \{6 + 2n - 2\}$

$198 = n\ \{4 + 2n\}$

$198 = 4n + 2n^2$

$\therefore 2n^2 + 4n - 198 = 0$ (divide through by 2)

$n^2 + 2n - 99 = 0$

Using factorisation method,

$n^2 + 11n - 9n - 99 = 0$

$n\ (n + 11) - 9\ (n + 11) = 0$

$(n - 9)\ (n + 11) = 0$

$n - 9 = 0$ $n = 9$ or $n + 11 = 0$, $n = -11$

Therefore 9 terms are needed to make sum equal to 99.

3. Find the sum of the AP given as: $1 + 3^1/_2 + 6 + \ldots\ldots + 101$

Solution

a = 1

$d = 3\frac{1}{2} - 1 = 2\frac{1}{2}$ or 2.5 nth term =101

$\therefore n^{th}$ term = a + (n - 1) d

$101 = 1 + (n - 1)2.5$

$101 = 1 + 2.5n - 2.5$

$101 - 1 + 2.5 = 2.5n$

$102.5 = 2.5n \therefore n\ \frac{102.5}{2} = 41$

: The AP has 41 terms

: Sum of AP = $S_n = \frac{1}{2}\ n\{2a + (n-1)\ d\}$

$S_{41} = \frac{41}{2}\ n\{2 \times 1 + (41 - 1)\ 2.5\}$

$S_{41} = 20.5\{2 + 40 \times 2.5\}$

$= 20.5\{102\} = 2091$

4. The first and the last term of an AP are 5 and 100 respectively. Find the sum of the AP, if the AP has 20 terms.

Solution

$S_n = \frac{n}{2}\ n(a+l)$

$$S_{20} = \frac{20}{2}\{5 + 100\} = 10\{105\} = 1050$$

4.0 Summary

Sum of arithmetic can be obtained using two approaches depending on the nature of arithmetic progression. The sum of AP = $S_n = \frac{n\{2a+(n-1)d\}}{2}$ when the common difference can be obtained, however, when only the first term and the last terms are given the sum of the AP is given as $S_n = \frac{n\{a+1\}}{2}$. Two formulas have been proved to be the same.

5.0 References/Further Readings/Web Resources

Hardwood, C. L. (1978). Ordinary Level Mathematics. (6th ed.). London: Heinemann Educational Books Limited.

Koop, J. S. (1990). Mathematics for Business. US: Win C. Brown Publishers.

Selah, U. A. (2003). Quantitative Techniques in Business. Maiduguri: Compaq Publishers.

6.0 SELF-ASSESSMENT EXERCISE

1. An arithmetic sequence has its first term as 2 and the common difference 3 the sum of the *nth* term is 77. Find n.
2. The first term of an arithmetic sequence is 8 and the sum of the first 20 terms is 445. Find the common difference
3. The third term of an Arithmetic Progression is 8 and the 16th term is 47. Find the first term and the common difference of this AP.
4. The 14th term of an AP is -15 and the common difference is -5. Find the first term and write the first six terms.
5. If -5, p, q, 16 are in AP. Find p and q.
6. Find n, given that 697 is the nth term of the AP.4, 11, 18….
7. How many terms have the series 1, 4, 7 ……. (6n – 2)?
8. The sum of the 3rd and 9th term of an AP is 6 and their product is
9. Find the sum of the first fifteen term of the sequence.
10. Given that 3 + 7 + 11 + 15 +…. is an AP find
    a. the nth term;
    b. the 11th term (T11); and
    c. the sum of the first 120 natural numbers divisible by 6.

UNIT 3 GEOMETRIC PROGRESSION

Unit Structure

1.0 Introduction

In the last unit, you learnt about arithmetic progression which is a sequence of numbers generated by continuous addition or subtraction of a constant number called common difference. Sequences are of two main types namely arithmetic progression and geometric progression. Geometric progression unlike the arithmetic progression, are sequence of numbers generated by continuous multiplication or division by a constant value called the common ratio. Some mathematical formula can be used to generate large volume of term of geometric progression. Geometric progression has the tendency to generate extremely large or small numbers such that it could be extremely tasking and time

consuming to obtain terms like 50th term, 100th term, etc.
More importantly, sum of geometric progression, no matter the quantity of the term being considered can be easily obtained with the use of mathematic formulas depending on size of the common ratio the formulas generating the terms in the geometric progression is called the nth term of the GP while the one used to compute the sum of the GP is the sum of GP to the nth term.

2.0 Learning Outcome

After reading this unit, students will be able to:

- generate the n^{th} term of geometric progression
- generate the sum of any given geometric progression
- obtain the common ratio or the first term of a GP, provided some adequate information are given.

3.0 Geometric Progression

3.1 n^{th} Term of Geometric Progression

The n^{th} term of a geometric progression (GP) is the mathematical formula used to generate any term in a given geometric progression. Assuming the first term of a GP is 'a' and the common ratio is 'r' then the nth term of a GP = $T_n = ar^{n-1}$, where T_n is the nth term e.g., 6th term = $T_6 = ar^{6-1} = ar^5$
12th term $= T_{12} = ar^{12-1} = ar^{11}$, and so on.

Example

1. Given a series as follow 2 + 4 + 8 + 16 + …. Find the value of the 20th term of the sequence.

Solution

The sequence is a GP because the proceeding term is being multiplied by a constant value.

To generate the subsequent terms.

$T_n = ar^{n-1}$

$T_{20} = ar^{n-1}$

$r = \frac{T_2}{T_1} = \frac{4}{2} = 2$

$T_{20} = 2\,(2)^{20-1}$

$T_{20} = 2^1 . 2^{19}$

2. Insert four geometric terms between 2 and 486.

Solution

Let the GP be: 2, w, x, y, z. 486. Therefore 486 becomes the 6^{th} term.

$\therefore T_6 = ar^{n-1}$

$T_6 = ar^{6-1}$

$486 = ar^5$

$486 = 2r^5$

Divide both sides by 2

$243 = r^5 \therefore r = \sqrt[5]{243} = 3$

w = 2 x r = 2 x 3 = 6

x = 6 x r = 6 x 3 = 18

y = 18 x r = 18 x 3 = 54

z = 54 x r = 54 x r = 162.

Therefore, the four terms inserted in the GP are 6, 18, 54, and 162.

Example 3

The 6^{th} and 9^{th} term of a GP are 96 and 768 respectively. Find

i. The common ratio;
ii. The first term; and
iii. The nth term.

Solution

$T_6 = ar^{6-1} : 96 = ar^5$ (i)

$T_a = ar^{9-1}$ 768 = ar^8 (ii)

Solving (i) and (ii) simultaneously:

$ar^5 = 9$ (i)

$ar^8 = 768$ (ii), dividing (ii) by (i)

$\frac{ar^8}{ar^5} = \frac{768}{96} = \frac{r^8}{r^5} = 8$

$r^{8-5} = 8 = r^3 = 8$

$\therefore r = \sqrt[3]{8} = 2$

From equation (i), $ar^5=96$

$a(2)^5 = 96$

a (32) = 96

32a = 96

Dividing both side by 32

$\frac{32a}{32} = \frac{96}{32} \therefore a = 3$

The n^{th} term = $ar^{n-1} = 3(2)^{n-1}$

SELF-ASSESMENT EXERCISE

i. Insert 2 geometric terms between 7 and 189

ii. If the 3rd and the 6th term of and GP are 18 and 486 respectively. Find the common ratio, the first term and the10th term

iii. Find the 9th term of the GP: 40, 20, 10 …

iv. The third term of a geometric sequence is 36 and the sixth term is 121.5. Find the first four terms of the sequence

v. The second term of a geometric progression is 6 and the fifth term is 162. Find the third term

vi. Find the sixth term of the following GPs

(a) 3, 6, 12….

(b) -4, 2, -1 …

(c) 3, 2, $\frac{4}{3}$

vii. Find the 40th term of the GP 2, 14, 98….

viii. Find n given that 1024 is the nth term of the GP 4, 8, 16….

ix. If 5, x, y, 40 are in GP. Find x and y

x. Find the geometric mean of 35 and 140 (hint geometric mean of x and y =$\sqrt{xy}$

3.2 Sum of Geometric Progression

Suppose S denotes the sum of n terms of the GP, whose first is a and common ratio = r, then

S = a + ar + ar2 + ar3 + …. + arn-2 + arn-1 (1)

Multiplying both sides by r.

R = ar + ar^2 + ar^3 + ……. + ar^n -1 + ar^n (2)

Subtracting (ii) form (i)

S – rs = a – ar^n

S (1 – r) = a (1 – r^n)

$S = \frac{a(1-r^n)}{1-r}$, this is applicable when r < 1 (both numerator and denominator of the fraction are positive and this is the most convenient form of S. if however, r > 1, then $S = \frac{a(r^n-1)}{r-1}$

Note: The sum of GP to infinity is given as $S_\infty = \frac{a}{1-r}$, this is more applicable when r < 1

Examples

1. Find the sum of eight terms of the GP 2, 6, 18 …….

Solution

$S_n = \frac{a(r^n-1)}{r-1}$ a = 2, r = $\frac{6}{2}$ = 3

$\therefore S_8 = \frac{2(3^8-1)}{3-1} = 3^8 - 1 = 6570$

2. Find the sum of infinity of series: 2 + (-1) + $\left(\frac{1}{2}\right)$+ $\left(\frac{-1}{4}\right)$ + $\left(\frac{1}{8}\right)$....

Solution

$$a\ \frac{t_1}{t_2} = -\ \frac{1}{2}$$

Sum of GP to infinity = $S_\infty = \frac{a}{1-r}$

$$\frac{2}{1-\left(-\frac{1}{2}\right)} = \frac{2}{1+\frac{1}{2}} = 2 \div 1\frac{1}{2} = 2 \div \frac{3}{2}$$

$$= 2 \text{ x } \frac{3}{2} = \frac{4}{3}$$

3. Given the G.P: 144, 72, 36, 18 ……. Find the sum of the first eight terms

Solution

$$S_8 = \frac{a(r^n-1)}{r-1} = \text{r} = \frac{T_2}{T_1} = \frac{72}{144} = \frac{1}{2}$$

$$S_8 = \frac{144\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} = S_8 = \frac{144\left(1-\frac{1}{256}\right)}{\frac{1}{2}}$$

$$S_8 = \frac{144\left(\frac{255}{256}\right)}{\frac{1}{2}} = S_8 = 144\ x\ \frac{\left(1-\frac{1}{256}\right)}{\frac{1}{2}} \text{ x } \frac{2}{1} \therefore S_8 = \frac{2295}{8}$$

SELF-ASSESSMENT EXERCISE

i. Find the sum of the first nine terms of the Geometrical Progression below:
   - a. 5 + 10 + 20 + 40 +
   - b. 2 + 4 + 8 + 16 +…...

ii. Find the sum of the first 10 terms of the series 1 + 2 + 4 + 8 +…. How many terms of the series are needed for the sum to exceed 2000?

iii. The first term of a geometric sequence is 64, and the common ratio is -1/4. Find the sum of the first five terms.

iv. Find the sum to infinity of the following series: -
   - a. 2 + 1 + ½ + ¼ +……
   - b. 3 – 1+ $^1/_3$ – $^1/_9$ + ……

4.0 Summary

The nth term of a GP is given as

$T_n = ar^{n-1}$, while the sum of a GP is given as

$S = \frac{a(1-r^n)}{1-r}$ where r < 1 or $S = \frac{a(r^n-1)}{r-1}$ where r < 1

However, the sum of GP to infinity is given as $S_\infty = \frac{a}{1-r}$,

The nth term of a geometric progressions and the sum of geometric progressions enable us to generate any term in a GP as well as find the sum of GP with ease.

5.0 References/Further Readings/Web Resources

Hardwood, C. L. (1978). Ordinary Level Mathematics. (6th ed.). London: Heinemann Educational Books Limited.

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6.0 SELF-ASSESSMENT EXERCISE

1. The third term of a geometric progression is 5, and the fifth term is 20. Find the possible values of the second term.
2. The third term of a geometric progression is 36 and the sixth term is 121.5. Find the common difference and the first term
3. Find the value of the 20th term and the sum to the 8th term for the sequence: 2, 4, 8, 16, 32….
4. The 5th and the 8th term of an arithmetic progression is 9 and 27 respectively. Find

a. The common difference

b. The first term

c. The AP
d. The sum of the AP to the 20th term
5. Find four geometric terms between 2 and 486
6. lf the 3rd and 6th term of a GP are 18 and 486 respectively. Find the GP the 10th term and the sum of the first ten terms.
7. Find the T10 and S20 for each of the following sequence
   (a) -3, 6, -12
   (b) -2, -4, 8
8. Find the value of n given that -49 is the nth term of the AP: 11, 8, 5…
9. Find the sum of the first 50 odd numbers
10. Find the nth term of the GP: x, 2xy, 4xy2

UNIT 4 APPLICATION OF SEQUENCE AND SERIES TO ECONOMICS, BUSINESS AND FINANCE

Unit Structure

1.0 Introduction
2.0 Learning Outcome
3.0
  3.0 Main Content
  3.1 Application of Arithmetic Progression to Economics, Business and Finance
  3.2 Application of Geometric Progression to Economics, Business and Finance
4.0 Summary
5. References/Further Readings/Web Resources
6.0 SELF-ASSESSMENT EXERCISE

1.0 Introduction

The purpose of studying mathematical techniques and principles of mathematics in economics and related disciplines is not only to know them but to apply them solve some problems relating to economics business and finance. You will recall that, after studying the basic principles and laws in set theory, we concluded the topic by applying the principles and law to day-to-day problems.

In a similar manner, the basic formulae and principles learnt in sequence and series are not meant to solve mathematical and quantitative problems only, we should be able to adopt these tools in solving practical problems especially those that have to do with economics, business and finances. It is only when this is done, that we can really appreciate the mathematical formulae.

2.0 Learning Outcome

After reading this unit, students will be able to:

- apply basic principles and formulae of arithmetic progression to solve day to day economics and business-related problems
- solve practical problems in economics and business using the basic principles and formulae of geometric progression

3.0 Application of Arithmetic Progression to Economics, Business and Finance

The use of n^{th} term and the sum of arithmetic progression is applicable in solving practical problem which include savings, income accumulation, future projection, etc.

Example

1. Mr. Johnson is an employee of engineering firm. His initial annual salary is N20, 000. If his salary increases by N1, 500 annually, required:
   (a) Construct the trend of his annual salary for the next six years;
   (b) What will be his annual salary in ten yea's time?
   (c) In how many years' times will his salary be N50, 000?
   (d) If he spends 30 years in service before retirement, what is the cumulative salary he received for the period he served the company?

Solution

(a) The first term = a = N20, 000
(b) The difference = d = N1, 500
(c) Therefore, the trend is as follows:

Year	1	2	3	4	5	6
	a	a + d	a + 2d	a + 3d	a + 4d	a + 5d
Salary	N20,000	N20,000 + N1,500 = N21,500	N20,000 + N3,000 = N23,000	N 20.000 + N4,500 = N24,500	N20,000 + N6,000 = N26,000	N20,000 + N 7,500 = N 27,500

(b) $T_n = a + (n-1) d$

T_{10} = #20,000 + (10-1) 1,500

= N20, 000 + 9 (N 1,500)

= N 20,000 + N13, 500

= N 33,500.

In ten years', time his salary will be N33, 500 per annum.

(c) Tn = a + (n-1) d

50,000 = 20,000 + (n-1) d

50,000 = 20,000 + (n -1) 1,500

50,000 = 20,000 + (1500n -1500)

50,000 = 20,000 + 1500n -1500

50,000 - 20,000 + 1500 = 1,500n

31,500 = 1500n $\therefore n = \frac{31500}{1500}$ = 21 years

Therefore, this salary will be N 50,000 in 21years time

(d) $S_n = \frac{n}{2}\{2a + (n+1)\,d\}$

$S_{30} = \frac{30}{2}\{2\ x\ 20{,}000 + (30+1)\ 15{,}000\}$

$S_{30} = 15{,}000\ \{40{,}000 + 29\ x\ 15{,}000\}$

$S_{30} = 15{,}000\ \{40{,}000 + 43{,}500\}$

$S_{30} = 15{,}000\ \{83{,}500\}$

$S_{30} = 15{,}000\ \{83{,}500\}$

S_{30} = N1, 252,500

Therefore, if he spent 30 years in service his cumulative or total salary earned is N1, 252,500

2. Business entities earn a profit of N1m in the first year of operation N 2m in the second year of operation, N 3m in the third year of operation. How much altogether in the first 20 years of operation?

a = N1m

d = N2m – 1m = 3m – 2m = 1m

$S_n = \frac{n}{2}\{2a + (n+1)\,d\}$

$S_n = \frac{20}{2}\{2\ x\ 1 + (20+1)\ 1\}$

$S_n = 10\ \{2 + 19\}\ = S_n = 10\ \{21\}\ \therefore S_n = 210$

In the first 20 years of operation, the business entity earns a cumulative profit of N210m.

SELF-ASSESSMENT EXERCISE

i. Aminat starts a job at an annual salary of N600, 000. Every year she receives a pay rise of 50,000. What is the total amount she has earned in 8 years?

ii. Chiamaka started a job saved N40, 000 in her first year. For each year, she was able to saveN10, 000 more than in the pronoun year. How many years will it take her to save a total ofN600,000?

iii. A man saves N1000 in the first month and increases the savings by N200 each month. What amount does he save by the fourteenth month?

3.1 Application of Geometrical Progression to Economics, Business and Finances

The nth term of a geometric progression and the sum of geometric series is found applicable to a number of areas in economics, finance and business. Among the common application of GP are compound interest, present value, growth rate estimations etc.

Examples

A student borrows 600 at 7% interest compounded annually. He pays off the loan at end of 3 years. How much does he pay?

Solution

7% of N 600 = N42

2nd term of the GP is N600 + N42 = N642 Hence the GP is 600 + 642+…. a = 600

$\therefore r = \frac{642}{600} = 1.07$

At the end of the 3rd year the GP will have 4 terms;

4th term = ar3

600 (1.07)3 = 600 (1.2250)

= N735 ∴ He pays back N735

2. Olawale saved N30, 000 in the first year of a new job. In each subsequent year, he saved 10% more than in the previous year.

a. How much in total had he saved in 5 years?

b. How many years did he take to save a total of more than N330, 000?

Solution

a. a = N30,000, $r = 1 + \frac{10}{100} = 1.1$

$S_n = \frac{30,000(1.1^n - 1)}{1.1 - 1} = 30,000\ (1.1 - 1) \div 0.1$

$S_n = 1.1 - 1$

$= 30,000(1.1^n - 1)$

$P_{nt} = 5$

$\therefore 300000\ (1.1^5 - 1) = 300000\ (1.611 - 1)$

=300000 x 0.61 = N183, 300

b. If he saved a total of N330, 000 after n years.

$330,000 = 300\,000\ (1.1^n - 1)$

Divide both sides by 300,000.

$\frac{330,000}{300,000} = \frac{300\,000\ (1.1^n - 1)}{300,000}$

$1.1 = 1.1^n - 1$

$1.1 + 1 = 1.1^n$

$2.1 = 1.1^n$

Taking the logarithm of both sides

Log 2.1 = n log 1.1

$n = \frac{Log\ 2.1}{log\ 1.1} \therefore n = 7.8$

It takes about 8 years to saved more than N330, 000.

2. A man deposit 10,000 at 8% per annum. Find the compound amount at the end of 10 years if (i) Interest is compound annually

(i) Quarterly

(ii) Monthly

Solution

i. P = 10,000 = a

$S_n = a\ (1 + r)^n$

$S_{10} = 10,000\ [1 + 8\%]^{10}$

$= 10,000\ [1 + 0.08]^{10}$

= 10,000 [2.159]

= N21, 589

ii. P = 10,000

$L = \frac{8}{4}\ \% = 2\%$

n = 10 x 4 = 40 quarters

Note: There are four quarters in a year, so the interest rate is divided by 4 and the number terms (n) is multiple by 4:

$S_n = a\ (1 + r)^n$

$S_{40} = 10,000\ (1 + 2\%)$

$= 10,000\ (1+0.02)^{40}$

$= 10.000\ (1.02)^{40}$

$= 10,000 (2.208) = N22,080.$

3. Find the present value of N722 receivable in 5 years if the money is worth 12% per annum compounded quarterly.

Solution

$$PV = \frac{p}{(1+1)^n} = p\,[1 + r]^{-n}$$

n = 5 X 4 = 20 quarters

$$PV = 722\,[1 + 0.12]^{-20}$$
$$= 722\,[1.12]^{-20}$$
$$= 722\,[0.5536] = N4000$$

4. The population census in 1960 was 95 million. Ten years later the census gave a total population of 115million. Find the annual growth rate.

Solution

Base year t = 0 and P0 = 95, e^{rt}, = 1

Then S0 = 95 = $P^{er\,(0) =}$ P

After ten years

$S_{10} = 115 = Pe^{rt} = Pe^{r\,(10)}$

$S_{10} = 115 = 95e^{10r}$

$115 = 95e^{10r}$

Divide both sides by 95

$$\frac{115}{95} = \frac{95er^{10}}{95}$$

$= 1.21 = e^{10r}$

Finding the exponential log of both sides

In 1.21 = 10r

0.191 = 10r

$$\frac{0.191}{10} = \frac{10r}{10} \therefore r = 1.9\%$$

The annual growth rate is 1.9%

SELF-ASSESSEMENT EXERCISE

i. Bolarinwa starts a job at an annual salary of N800, 000. At the end of each year, his salary increases by 15%.

a. Find his salary during his fourth year in the job.

b. What is the total amount earned in 5 years?

c. After how many years will he have earned a total of N800, 000?

ii. Suppose a dropped ball re-bounces 1/8 of the height when it falls. How far has it travelled when it reaches the top of the 8th bounce?

iii. Find the compound amount and interest on N 10.000 for 3 years at 8% per annum compounded.

(i) Annually

(ii) Quarterly

(iii) Monthly.

iv. Find the present value of N 10,000 receivable 5 years from now if money is worth 10% per annum

v. The population of Nigeria in 1997 was estimated at 100 million people. The population is expected to grow at 3.2% every year. What is the expected population of Nigeria in the year 2015?

4.0 Summary

Just like many other concepts in mathematics, sequence and series are of great relevance in solving practical problems. The use n^{th} term of AP and GP as well as their summation or series are useful in estimating future value of sequential and practical oriented problems. Most often problems relating to interest, compounding is easily synthesised into either arithmetic or geometric progression to be able to provide solution to them. It is important to note that no sequence can be arithmetic progression and at the same time be a geometric progression. Hence, the foremost step to know whether the sequence is arithmetic progression or geometric progression after this, it is important to know what the question is interested in testing, while some questions are interested in forecasting future values some are essentially focused on getting the sum of the sequence. The use of appropriate formula is therefore important in solving practical questions involving sequence

Basically, both arithmetic progression (AP) and the geometric progression (GP) have diverse application to economic, business and financial problems. For instance, AP is relevant to income and savings accumulation projection and forecasting while GP is relevant to compound interest, present value analysis and the growth rates

5.0 References/Further Readings/Web Resources

Hardwood, C. L. (1978). *Ordinary Level Mathematics.* (6th ed.). London: Heinemann Educational Books Limited.

Robert, S. & Geoff, B. (2007). *Excellence in Mathematics for Senior Secondary Schools.* Ibadan: Macmillan Publishers.

Usman, A. S. (2005). *Business Mathematics.* Lagos: Apexs Book Limited.

6.0 SELF-ASSESSEMENT EXERCISE

1. An employee started on an annual salary of N1m. every year, he received a constant pay rise. After six years, he has earned a total of 8, 250,000. What was the constant pay rise?
2. A contractor for a construction job specifies penalty for delay of completion beyond a certain date as follow: N15, 000 for the first day N16, 000 for the second day etc. How much does a 30 – day delay in completion cost the contractor if the penalty for each day is N1000 more than the previous day?
3. If a person were offered a job N400 the first day, N800 the second day N1600 the third day etc. each day wage being double that for the preceding days, how much would he received at the end of 7^{th} days?
4. Find the compound amount and the compound interest on N20, 000 for 5 years at 10% per annum compounded (i) annual (ii) quarterly (iii) monthly.
5. The total enrolment at a state polytechnic is expected to grow at the rate of 10% each year. If the initial enrolment is 120,000 students, what is the expected number of students enrolled at the end of the 5^{th} year?

MODULE 5 POLYNOMIAL AND BINOMIAL THEOREMS

Unit 1 Meaning and Scope of Polynomials
Unit 2 Remainders' and Factors Theorem
Unit 3 Partial Fractions
Unit 4 Binomial Expansions
Unit 5 Factorials, Permutation and Combination

UNIT 1 MEANING AND SCOPE OF POLYNOMIALS

Unit Structure

1.0 Introduction
2.0 Objectives
3.0 Main Content
    3.1 General Overview
    3.2 Algebra of Polynomial
4.0 Summary
5.0 References/Further Readings/Web Resources
6.0 SELF-ASSESSEMENT EXERCISE

1.0 Introduction

You recall that in Module 2, you were introduced to two major types of equations – linear equations and quadratic equations. A linear equation is described as a mathematical equation whose highest power of the unknown is 1 while a quadratic equation is the one that has the highest power of the unknown as 2. For example, $3x - 2$ and $4x2 - 3x + \frac{1}{5}$ are linear expression and quadratic equation respectively. To make them equations, there is need to put the 'equal to' sign e.g., $3x - 2 = 0$ or $4x2 - 3x + 1 = 0$. These types of equations are examples of 5 polynomials. A polynomial in x consists of positive integer power of x, multiplied by constant and added. These constants are called coefficients

2.0 Learning Outcome

After reading this unit, students will be able to:

- define the term "polynomial' with examples
- identify the functions/equations that are polynomials and those that are not
- carry out basic algebra of polynomials.

3.0 Polynomial

3.1 Concepts of Polynomial

A polynomial is a mathematical expression or equation comprising of the sums or difference of terms, each term being a product of constant and non-negative or zero power of variable. For example: Given an expression such as $5x3$, x is called a variable because it can assume any number of given values, and 5 is referred to as the coefficient of x. expression consisting simply of a real number or of a co-efficient times one or more variables raised to the power of a positive integer are called Monomials. Monomials can be added or subtracted to form polynomials. Each of the monomials comprising of a polynomial is called a Term. Terms that have the same variables and exponents are called Like-Terms. Examples of polynomials are; $x3 - 5x2 + 7$, $2 - 3x - \frac{1}{2}x^4$, $\pi x^5 + 2x^3 - \frac{1}{8}x$. Not all mathematical expressions are polynomials. Mathematical expressions that do not have all the properties of polynomial such as sin x, $\sqrt{x}$, $\frac{2x}{1+x}, log x$ are not polynomials. Every polynomial is expected to have a degree. The degree of a polynomial is the highest power of x. All linear expressions like $3x - 2$, $4x + 3$, etc. are in degree 1 because the highest power of the variable is 1. Quadratic expression such as $4x^2 - 3x + \frac{1}{5}$ has degree 2 while each of $x^3 + 4x^2 - 3x + 1$ and $1 - 3x + 4x^2 + x^3$ has a highest
power of 3 and hence the degree is 3.

It should be noted that if the degree of a polynomial is n, the number of terms is almost $n + 1$. For example, the polynomial given as $x^3 + 4x^2 - 3x + 1$, is of degree 3 and has 3 + 1 = 4 terms (x^3, $4x^2$, $-3x$ and $+ 1$). There may be fewer than $(n + 1)$ terms if some of the coefficients are zero. For example, the polynomial $3x^4 - 5x^2 - 7$ is of degree 4 but has only three terms. This is, $3x^4 - 5x^2 - 7$ can also be written as $3x^4 - 0x^3 + 5x^2 + 0x - 7$.

We often write a polynomial as p(x) and p(x) can generally be written as:

$p(x) = a_nt^n \pm a_{n-1}t^{n-1} \pm \cdots \pm at \pm a_0$, where a_n, a_{n-1}, a, a_0 are constant of a variable, sometimes called its coefficient. A constant function is actually a "degenerate" case of what are known as polynomial functions. The word polynomial means "multi term" and a polynomial function of a single variable x has a general form $y = a_n + a_1x + a_2x^2 + \cdots + a_nx^n$.

SELF-ASSESSMENT EXERCISE

i. Define each of the following with appropriate examples:
   a. Polynomials
   b. Monomials
   c. Coefficients
   d. Degree of polynomials
   e. Terms
   f. Like terms

ii. Which of the followings are polynomials?
   a. $x^2 - 2x + 7$

b. $6 - x^3$

c. $x^3 + 2\sqrt{x-3}$

d. $\sin(x^3 - 2x^2)$

e. $\sqrt{x+3}$

f. $\pi x^2 + \frac{1}{5}x - \sqrt{2}$

iii. For each of the expression in question 2 above that are polynomials, write down the degree.

iv. State the degree of each of the following polynomials.

a. $y = 4x^3 - 5x^2 + x - 5$

b. $y = 5x^3 + x^5 - 2x^2 + 8$

c. $y = 5x^4 + 3x^3 + 1$

d. $F(x) = 10x^4 - 25x^3 - 3x^2 - x^7 + 24$

e. $y = f(x) = x^3 + 3x^2 - 2x$

3.2 Algebra of Polynomial

Addition and Subtraction of a Polynomials: Addition or subtraction of terms in two or more polynomials require the summing or subtraction of their like terms. Recall that, from your knowledge of indices, two powers x are like terms if the indices are the same. The like terms can be added or subtracted.

Example 1

1. Let $f(x) = x^2 + 3x - 1$ *and* $g(x) = 2x^3 + x^2 + 7$, Find f(x) + g(x).

Solution

$f(x) = x^2 + 3x - 1$

$g(x) = 2x^3 + x^2 + 7$

To do the addition effectively, there is a need to pay attention to the like and unlike terms. There are no x^3 terms in f(x), and no x term in g(x). You are expected to leave blank spaces or write as $0x^3$ and $0x$. $0x^3 + x^2 + 3x - 1 + \frac{2x^3 + x^2 + 0x + 7}{2x^3 + 2x^2 + 3x + 6}$

Example 2

Simplify

a. $(7x^3 + 5x^2 - 8x) + (11x^3 - 9x^2 + 2x)$

b. $(24x - 17y) + (6x + 5z)$

c. $(4x^3 - 6x^2 + 9) - (5x^2 - 4x^3 + 3x - 18)$

Solution

a. $(7x^3 + 5x^2 - 8x) + (11x^3 - 9x^2 + 2x)$

Collect like terms

$7x^3 + 11x^3 + 5x^2 - 9x^2 - 8x + 2x$

$= 18x^3 - 4x^2 - 6x$

b. $(24x - 17y) + (6x + 5z)$

Collecting like terms

$24x + 6x - 17y + 5z$

$30x - 17y + 5z$

c. $(4x^3 - 6x^2 + 9) - (5x^2 - 4x^3 + 3x - 18)$

Opening the brackets;

$4x^3 - 6x^2 + 9 - 5x^2 + 4x^3 - 3x + 18$

Collecting like terms,

$4x^3 + 4x^3 - 6x^2 - 5x^2 + 3x + 9 + 18$

$= 8x^3 - 11x^2 - 3x + 27$

Example 3

Given that $f(x) = 5x^3 - 3x^2 + x + 7$

$g(x) = 6x^2 + 5x - 4$

$h(x) = 8x^3 + 5x - 2$

Required:

a. f (3) + g (- 4)

b. f(x) + 2g(x) – 3h(x); at x = 2

Solution

a. $f(x) = 5x^3 - 3x^2 + 4x + 7$

$f(3) = 5(3)^3 - 3(3)^2 + 4(3) + 7$

$= 5(27) - 3(9) + 4(3) + 7$

$= 135 - 27 + 12 + 7 = 127$

$g(-4) = 6(-4)^2 + 5(-4) - 4$

$= 6(16) + (-20) - 4$

$= 96 - 20 - 4 = 72$

Hence, $f(3) + g(-4) = 127 + 72 = 199$

b. $f(x) + 2g(x) - 3h(x) = 5x3 - 3x2 + 4x + 7 + 2[6x2 + 5x - 4] - 3[8x3 + 5x - 2]$

$= 5x3 - 3x2 + 4x + 7 + 12x2 + 10x - 8 - 24x3 - 15x + 6$

$= 5x3 - 24x3 - 3x2 + 12x2 + 4x + 10x - 15x + 7 - 8 + 6$

$= -19x3 + 9x2 - x + 5$

$\therefore f(x) + 2g(x) - 3h(x); at\ x = 2$

Then, substitute $x = 2$

$-19(2)3 + 9(2)2 - (2) + 5$

$= -19(6) + (9 \text{ x } 4) - 2 + 5$

$= -144 + 36 - 3 = -111$

Multiplication of Polynomials: Unlike in addition and subtraction of polynomials, like and unlike terms can be multiplied by multiplying both the coefficients and variables. When there exists one or more mathematical expression (linear, quadratic or otherwise), a proper expansion of terms may be required.

Examples

1. Simply each of the following

a. $(5x)(13y^2)$

b. $(2x^3y)(17y^4z^2)$

c. $(6x + 7y)(4x + 9y)$

d. $(2x + 3y)(8x - 5y - 7z)$

2. Given that $f(x) = x^2 + 3x - 1$ and $g(x) = 2x^3 + x^2 + 7$.

find $f(x)$ x $g(x)$

3. If

$f(x) = (x - 6)$ and $g(x) = x + 6$.

find $\{[g(x)]^2. f(x)\}, at\ x = 2$.

Solution

1 a) $(5x)(13y^2) = (5 \text{ x } 13) \, . \, y^2 = 65xy^2$

b) $(2x^3y)(17y^4z^2) = (2 \text{ x}17)\, x^3 . \, y \, . \, y^4 \, . \, z^2 = 34x^3y^5z^2$

c) $(6x + 7y)(4x + 9y)$

$6x(4x + 9y) + 7y(4x + 9y)$

$= 24x^2 + 54xy + 28xy + 63y^2$

$= 24x^2 + 82xy + 63y^2$

d) $(2x + 3y)(8x - 5y - 7z)$

$2x(8x - 5y - 7z) + 3(8x - 5y - 7z)$

$= 16x^2 - 10xy + 14xz + 24xy - 15y^2 - 21yz$

Collecting the like terms

$= 16x^2 - 10xy + 24xy + 14xz - 15y^2 - 21yz$

$= 16x^2 + 14xy + 14xz - 15y^2 - 21yz$

2. $f(x) = x^2 + 3x - 1$

$g(x) = 2x^3 + x^2 + 7$

$f(x)$ x $g(x)$

$= (x^2 + 3x - 1)(2x^3 + x^2 + 7)$

$Expanding$

$x^2(2x^3 + x^2 + 7) + 3x\,(2x^3 + x^2 + 7) - 1(2x^3 + x^2 + 7)$

$2x^5 + x^4 + 6x^4 + 3x^3 + 21x - 2x^3 - x^2 - 7$

$Collecting\ like\ terms$

$2x^5 + x^4 + 6x^4 + 3x^3 - 2x^3 + 7x^2 - x^2 + 21x - 7$

$= 2x^5 + 7x^4 + x^3 + 6x^2 + 21x - 7$

3. $[(x)]^2 = g(x)\,.\,g(x)$

$= (x + 6)\,(x + 6)$

$= x\,(x + 6) + 6(x + 6)$

$= x^2 + 6x + 6x + 36$

$= x^2 + 12x + 36$

$\therefore [g(x)]^2.\ f(x) = (x^2 + 12x + 36)\,(x - 6)$

$= x^2(x - 6) + 12x\,(x - 6) + 36(x - 6)$

$= x^3 - 6x^2 + 12x^2 - 72x + 36x - 216$

$= x^3 + 6x^2 - 36x - 216\ [(x)]^2\,.\,f(x);\ at\ x = -2$

$= (-2)^3 + 6(-2)^2 - 36(-2) - 216$

$= -224 + 96 = -128$

SELF-ASSESSMENT EXERCISE

i. Let $f(x) = 2x^2 + 3x + 1$ and $g(x) = 3x - 2$, find the following;

a) $f(x) + g(x)$

b) $f(x) - g(x)$

c) $f(x)\,.\,g(x)$

ii. Given $f(x) = x^2 + 4x - 5$; find $f\,(2) - f\,(-3)$.

iii. Let $m(x) = x^2 - x + 1$ and $n(x) = x^2 + x + 3$

Find

(i) $m(x) + n(x)$

(ii) $m(x) - n(x)$

(iii) $(x)\,.\,n(x),\ at\ x = 4$

4.0 Summary

Polynomials are essential in economics, business and finances because sometimes functions are given in higher degree. Some functions may be of degree four or more. To handle such function, a good understanding of polynomials above that of linear equation and quadratic expressions are required. The just concluded unit introduced you to the basic components of polynomials as well as some elementary algebraic operation (addition, subtraction and multiplication). Subsequent units shall focus on some advanced areas such as remainders theorem, factor theorem, partial fraction, binomial expansion and so on. Polynomials are set of mathematical expressions written in the form of sums or differences of monomials. A collection of monomials joined by addition or subtraction signs gives a long chain of equation called *polynomials*. Polynomials can be added or subtracted from one another by collecting like terms. Multiplication of polynomials involves multiplication of the coefficient and variables or expansion of one or more expression.

5.0 References/Further Readings/Web Resources

Edward, T. D. (2006). *Introduction to Mathematical Economics.* (3rd ed.). Tata-McGraw Hill.

Robert, S. & Geoff, B. (2007). *Excellence in Mathematics for Senior Secondary Schools.* Ibadan: Macmillan Publishers.

Saleh, U. A. (2003). *Quantitative Techniques in Business*. Maiduguri: Compaq Publishers.

6.0 SELF-ASSESSMENT EXERCISE

1. Let $p(x) = x2 - 3x - 1\ and\ q(x) = 2x - 3$. Find the following:

a. p(x) + q(x)

b. p(x) – q(x)

c. p(x) . q(x)

2. Given that $f(x) = x3 + 1$ and $(x) = x2 - 1$. Find f(x) . g(x) at x = -3.
3. Given $f(x) = 2x3 - 5x2 + 8x - 20$. Find f (5) and f (-4).
4. If h(x) = $13x2 + 35x$ and $p(x) = 4x2 + 17x - 49$. Find (i) h(x) + p(x) (ii) 3h(x) + [p(x)]2.

UNIT 2 REMAINDERS AND FACTOR THEOREM

Unit Structure

1.0 Introduction
2.0 Objectives
3.0 Main Content
3.1 Remainder Theorem
3.2 Factor Theorem
4.0 Summary
5.0 Tutor-Marked Assignment
6.0 References/Further Reading

1.0 Introduction

In Unit 1, you learnt about the meaning and scope of polynomials as well as the basic algebra in polynomials i.e., addition, subtraction and multiplication. It is equally possible to divide one polynomial by another. Suppose we want to divide one polynomial *P(x)* by another polynomial *D(x)*, where the degree of *D(x)* is less than the degree of *P(x)*. The process is similar to long division of numbers. There will be a quotient *Q(x)*, the result of the division and they may also be a remainder *R(x)*. Therefore, $P(x) \div D(x) = Q(x) + \frac{R(x)}{Q(x)}$. The degree of the remainder *R(x)* is less than the degree of *D(x)*. if not, we cannot continue the division. In particular, if *D(x)* is linear of the form *ax* + *b*, then *P(x)* is constant. In other words, $P(x) = D(x) + R(x)$.

2.0 Learning Outcome

After reading this unit, students will be able to:

- perform operation of division between two proper polynomials
- obtain the remainder and quotient when a polynomial divides theother
- show that a polynomial is a factor of the other or not.

3.0 Polynomial Theorem

3.1 Remainder Theorem

Suppose we want to find the remainder when a polynomial P(x) is divided by another polynomial Q(x). There are two approaches of obtaining the remainder, namely;

(i) Long Division
(ii) Remainder Theorem
(iii) Long Division

Example 1

Find the quotient and remainder when:

a. $x3 - 2x2 + 4x - 7$ is divided by $x - 3$
b. $3x4 + 2x3 - x + 7$ is divided by $x2 - 3$

Solution

a. Set out the division as:

$$x-3 \;\overline{\big)\, x^3 - 2x^2 + 4x - 7}$$

The highest power of *x* is x^3. Divide this by *x*, gives x^2. Write this above the bar and multiply it by the quotient *(x -3)*. Alternatively, think of what you will use to multiply *x* to gives x^3 i.e., you multiply *x* by x^2 to give x^3.

$$\begin{array}{r l}
 & x^2 \\
x-3 \,\big)\!\! & \overline{x^3 - 2x^2 + 4x - 7} \\
 & -\underline{(x^3 - 3x^2)} \\
 & x^2
\end{array}$$

Note: $-2x^2 - (-3x^2) = -2x^2 + 3x^2 = x^2$

Now bring down the next term in the polynomial i.e., $4x$, to continue the process

$$\begin{array}{r l}
 & x^2 + x \\
x-3 \,\big)\!\! & \overline{x^3 - 2x^2 + 4x - 7} \\
 & -\underline{(x^3 - 3x^2)} \\
 & x^2 + 4x \\
 & -\underline{(x^2 - 3x)} \\
 & 7x
\end{array}$$

Note: $4x - (-3x) = 4x + 3x = 7x$. Now bring down the -7 and repeat.

$$\begin{array}{r l}
 & x^2 + x + 7 \\
x-3 \,\big)\!\! & \overline{x^3 - 2x^2 + 4x - 7} \\
 & -\underline{(x^3 - 3x^2)} \\
 & x^2 + 4x \\
 & -\underline{(x^2 - 3x)} \\
 & 7x - 7 \\
 & -\underline{(7x - 21)} \\
 & 14
\end{array}$$

Note: $-7 - (-21) = -7 + 21 = 14$

The degree of 14 is less than the degree of the original divisor. We cannot divide any further. Therefore, when $x^3 - 2x^2 + 4x - 7$ is divided by $(x - 3)$, the quotient is $x^2 + x + 7$ and the remainder is 14.

$$\begin{array}{r l}
 & 3x^2 + x + 9 \\
x^2-3 \,\big)\!\! & \overline{3x^4 + 2x^3 - x + 7} \\
 & -\underline{(3x^4 - 9x^2)} \\
 & 2x^3 + 9x^2 - x + 7 \\
 & -\underline{(2x^2 - 0x^2 - 6x)} \\
 & 9x^2 + 5x + 7 \\
 & -\underline{(9x^2 + 0x - 27)} \\
 & 5x + 34
\end{array}$$

Note: the polynomial has no coefficient for x^2 so it is taken to be $0x^2$ i.e., $0x^2 - (-9x^2) = 9x^2$.

The degree of $(5x + 34)$ is less than the degree of $x^2 - 3$, so the division stops. Therefore, when $3x^4 + 2x^3 - x + 7$ is divided by $x^2 - 3$, the quotient is $3x^2 + 2x + 9$ and the remainder is $5x + 34$.

(ii) Remainder Theorem

Suppose, we want to find the remainder when a polynomial $P(x)$ is divided by a linear expression $(x - a)$, the method of the long division is long and it is liable to error. The remainder theorem provides a quicker way to find the remainder, it should be noted that the Remainder Theorem becomes more appropriate when the divisor has a degree of 1. When a divisor has a higher degree, remainder theorem may become more cumbersome.

Example 2

Find the remainder when $x^3 - 2x^2 + 4x - 7$ is divided by $(x - 3)$.

Solution

Equate the divisor to zero

$(x - 3) = 0$

$x = 0 + 3 \therefore x = 3$

Substitute the value of x into the polynomial

$x^3 - 2x^2 + 4x - 7$

$(3)^3 - 2(3)^2 + 4(3) - 7$

$27 - 18 + 12 - 7$

$9 + 5 = 14$

The remainder is 14.

Note: The remainder obtained here is the same as the remainder obtained when using a long division method (for the same question). It should also be noted that, the remainder theorem can only be used to obtain the remainder and not the quotient.

SELF-ASSESSMENT EXERCISE

i. In each of the following divisions, find the quotient and the remainder.

a. $(x3 + 5x2 + 4x - 17) \div (x - 14)$

b. $(x4 + x3 + x2 + x + 1) \div (x2 + 1)$

c. $(8x3 + 5x2 - 3x - 1) \div (2x + 1)$

d. $(x4 + 2x3 + 7) \div (x2 + 2)$

ii. Use the Remainder Theorem to find the remainder:

a. $(x3 - 8x2 - 4x + 5) \div (x - 2)$

b. $(x4 - 4x3 + 2x2 + 3x + 2) \div (x - 3)$

c. $(2x3 + 4x2 - 6x + 1) \div (x + 3)$

3.2 Factor Theorem

A special case of the Remainder Theorem is when P(a) = 0. In this case, the remainder is zero (0) and hence the divisor (x – a) divides exactly into P(x). Hence, (x – a) is a factor of P(x). Therefore, (x – a) is a factor of P(x) if and only if P(a) = 0.

Example 3

Show that (x + 3) is a factor of x3 – 3x2 +7x +75.

Solution

(x – a) = x + 3

x – a = 0 i.e., x + 3 = 0

x = 0 – 3 = -3

Substituting x = -3 into the polynomial,

$P(-3) = (-3)3 - 3(-3)2 + 7(-3) + 75$

$= -27 - (3 \text{ x } 9) - 21 + 75$

$= -27 - 27 - 21 + 75$

$= -75 + 75 = 0$

Since P (-3) = 0, then (x + 3) is a factor of x3 – 3x2 +7x +75.

Example 4

Let P(x) = x3 + x2 +ax + b, (x – 2) is a factor of P(x) and the remainder when P(x) is divided by (x – 3) is 11. Find the value of a and b.

Solution

Using Factor Theorem, (x – 2) = 0, x =2 P (2) = 0

P(x) = x3 + x2 +ax + b

$0 = (2)3 + (2)2 + a(2) + b$

$0 = 8 + 4 + 2a + b$

$\therefore 2a + b = -8 - 4$

$2a + b = -12$ (i)

For the other divisor, (x – 3)

x – 3 = 0, x = 3

P (3) = 11 (Note: Here there is a remainder when (x – 3) divides P(x), so P(3) is set to 11 and not zero.

$11 = (3)3 + (3)2 + a(3) + b$

$11 = 27 + 9 + 3a + b$

$11 = 36 + 3a + b$

$3a + b = 11 - 36$

$3a + b = -25$

(ii) Solving (i) and (ii) simultaneously;

$2a + b = -12$ (i)

$3a + b = -25$ (ii)

Using elimination method

$2a + b = -12$ (i)

$3a + b = -25$ (ii)

Subtracting equation (ii) from (i)

$2a - 3a = -12 - (-25)$

$-a = -12 + 25 = 13$

Multiply both sides by -1

$a = -13$

From equation (i)

$2a + b = -12$

$2(-13) + b = -12$

$-26 + b = -12$

$b = -12 + 26$

$b = 14$

... $\therefore a = -13 \; and \; b = 14$

Example 5

Let P(x) = $x^3 + x^2 - 10x + 8$. Show that (x - 2) is a factor of P(x). Hence, factorise P(x). Solve the equation P(x) = 0.

Solution

(x – 2) = 0

x = 2

P (2) = 23 + 22 – 10(2) + 8

= 8 + 4 – 20 + 8 = 0

Hence, by factor theorem, (x – 2) is a factor of P(x). Now, find the quotient when P(x) is divided by (x – 2)

$$\begin{array}{r|l} & x^2 + 3x - 4 \\ x - 2 & x^3 + x^2 - 10x + 8 \\ & \underline{x^3 - 2x^2} \\ & 3x^2 - 10x + 8 \\ & \underline{3x^2 - 6x} \\ & -4x + 8 \\ & \underline{-4x + 8} \\ & - \quad - \end{array}$$

Note: The remainder is 0, confirming that (x – 2) is a factor of P(x).

Therefore, P(x) = x3 + x2 – 10x + 8 = (x – 2) (x2 + 3x – 4) (x2 + 3x – 4) can be factorised as thus:

x2 +4x – x – 4

x(x + 4) – 1(x + 4) = (x – 1) (x + 4)

Therefore, all the factors of the polynomial are (x – 2), (x + 4) and (x – 1).

.∴ $x^3 + x^2 - 10x + 8 = (x - 1)(x + 4)(x - 1)$

If *(x – 2)*, *(x + 4)* and *(x – 1)* are each factor of *P(x)*, *P(x)* = 0.

x – 2 = 0 = *2*

or

x + 4 = 0 = -4

or

x – 1 = 0 = *1*

`` Example 6

Factorise completely x3 – 6x2 + 11x – 6.

Solution

We need to use trial method to obtain the first factor

x + 1 = 0 = -1

Substitute x = -1

(-1)3 – 6(-1)2 +11(-1) – 6

-1 – 6 – 11 – 6 ≠ 0 ... (x +1) is not a factor.

Assume (x – 1) is a factor

x – 1 = 0 = 1

P(x) = x3 – 6x2 +11x – 6

= (1)3 – 6(1)2 + 11(1) – 6
= 1 – 6 + 11 – 6
= 1 + 11 – 6 – 6 = 12 – 12 = 0, (x – 1) is a factor.

$$
\begin{array}{r|l}
 & \quad x^2 - 5x + 6 \\
\hline
x - 1 & \;- \; x^3 - 6x^2 + 11x - 6 \\
 & \quad x^3 \underline{- x^2} \\
 & \quad - \; -5x^2 + 11x \\
 & \qquad -5x^2 \underline{+ 5x} \\
 & \qquad\quad - \; 6x - 6 \\
 & \qquad\qquad \underline{6x - 6} \\
 & \qquad\qquad - \quad -
\end{array}
$$

Hence, x3 – 6x2 + 11x – 6 = (x – 1) (x2 – 5x + 6)

Then, factorise x2 – 5x + 6

x2 – 3x – 2x + 6

x(x – 3) -2(x – 3) i.e. (x – 3) (x – 2)

∴ x3 – 6x2 + 11x – 6 = (x – 1) (x – 3) (x – 2)

SELF-ASSESSMENT EXERCISE

i. In these questions, show that the linear expression is a factor of the polynomial:

a. (x – 1), x3 – x2 + 3x – 3

b. (x + 2), x3 + 4x2 – x – 10

ii. Which of the following are factor of x3 + 3x2 – 5x – 10:

a. x – 1b. x – 2 c. x + 5

iii. (x + 1) is a factor of x3 – 8x2 – ax + 4. Find the value of a.

iv. The remainder when x3 – 4x2 – ax + 5 is divided by (x – 3) is 8. Find a?

v. When x3 + 3x2 + ax + b is divided by (x +1), the remainder is 5, and when it is divided by (x – 2), the remainder is 8. Find a and b?

vi. (x – 1) and (x – 2) are both factors of x3 + ax2 + bx + 4. Find a and b?

vii. In question a – c below, show that the linear expression is a factor of the polynomial. Hence, factorise the polynomial:

a. (x – 2), x3 + 3x2 – 4x – 12

b. (x – 4), x3 – 9x2 + 26x – 24

c. (x + 2), 3x3 + 5x2 – 4x – 4

viii. Use the results obtained in 7(a), (b) and (c) to solve for the following equations:

a. x3 + 3x2 – 4x – 12 = 0

b. x3 – 9x2 + 26x – 24 = 0

c. 3x3 + 5x2 – 4x – 4 = 0

4.0 Summary

Division in polynomial can take two forms namely long division and the Remainder Theorem. The long division is used to obtain both the quotient and the remainder while remainder theorem is specifically used to the remainder. The long division is used for division of two polynomials whether the divisor is of degree 1 (linear expression) or more than degree. However, the remainder theorem is best applied when the divisor is of degree 1. The divisor is set to zero to obtain the value of the variable and thereafter substitute the value into the polynomial. The value obtained is the remainder from the division of the polynomial by the linear expression (the divisor). When no remainder is left, the divisor is said to be a factor of the polynomial. Other factors of the polynomial can be obtained by factorising the quotient obtained.

A polynomial may be expressed in terms of one or two other unknowns apart from the main variable. To obtain these unknown, the factors (with or without remainder) is substituted into the polynomial to obtain either a linear equation or a set of simultaneous equation. This equation or equations are solved simultaneously to obtain the values of the unknown.

Polynomials can also divide one another as much as the polynomial has a higher power relative to the quotient. In diving one polynomial by the other, it is likely there exist a remainder. Hence, given

P(x) as the polynomial, *D(x)* as divisor and *Q(x)* as the quotient while *R* is the remainder. The relationship which exists among the four terms can be expressed as:

P(x) = *D(x)* . *Q(x)* + *R*, just as we have in normal division of positive whole number. For instance, 100 ÷ 7 gives 14 remainder 2; which implies that: 100 = (7 x 14) + 2

When a divisor divides a polynomial without a remainder, that divisor is a factor of the polynomial, hence *P(x)* = *D(x)* + *Q(x)*, since *R* = 0. Such factor if equated to zero and a value is obtained, if the value obtained is substituted to the polynomial, we get a value of zero.

5.0 References/Further Readings/Web Resources

Edward, T. D. (2006). *Introduction to Mathematical Economics*. (3rd ed.). Tata–McGraw – Hill.

Robert, S. & Geoff, B. (2007). *Excellence in Mathematics for Senior Secondary Schools.* Ibadan: Macmillian Publishers Limited.

Saleh, U. A. (2003). *Quantitative Techniques in Business*. Maiduguri: Compaq Publisher Limited.

Stroud, K. A. (1995). *Engineering Mathematics*. England: Graham Burn Publisher.

6.0 SELF-ASSESSMENT EXERCISE

1. Given that $f(y) = \frac{2(y-1)}{y-1}$. Find the value of f(1).

   (Hint: you may have to factorise and perform necessary division before you substitute).

2. Given that f(x) = 5x3 – 3x2 +4x + 7, g(x) = 6x2 + 5x – 4 and h(x) = 8x3 + 5x – 2.
   Find
   (i) f(x) + 2g(x) – 3h(x)
   (ii) f(x) . g(x) – 3h(x)
   (iii) [g(x)]2 ÷ h(x)
   (iv) f(x) ÷ g(x)

3. a) Find the remainder when 2x2 – 5x + 6 is divided by (x – 3).
   b) Find the quotient and the remainder when f(x) = 3x3 – 2x2 + 4x + 5 is divided by (x – 1).
   c) Factorise completely x3 – 6x2 +11x – 6.

4. Let p(x) = x2 – 3x – 1 and q(x) = 2x – 3. Find the following, simplifying your answers:
   (a) p(x) + q(x) (b) p(x) – q(x) (c) p(x) . q(x)

5. Find the following; simplifying your answer
   a. (2x – 3) (x + 7)
   b. $(x^2 + 2x + 1)(x - 8)$
   c. $(2x - 3)^3$
   d. $(2x - 3)^2 (x - 1)$
   e. $(x^3 + 1)(x^2 - 1)$

6. In each of the following divisions, find the quotient and the remainder.
   a. (x3 – 3x2 + 4x + 1) ÷ (x – 2)
   b. (x3 + 4x – 3) ÷ (x + 2)
   c. (16x4 + x – 3) ÷ (2x – 3)
   d. (2x4 + 4x – 3) ÷ x2 + 2x)
   e. (3x5 + 2x3 – 8) ÷ (x2 + 1)
   f. (2x5 + 8x2 + 7x – 3) ÷ (2x2 + 3)

7. In these divisions, use the remainder theorem to find the remainder.
   a. (x3 – 2x2 + 7x – 3) ÷ (x + 1)
   b. (x6 – x3 + 3) ÷ (x + 1)
   c. (x3 + x2 – x + 1) ÷ (x + 5)

8. a) The remainder when (x3 – 6x2 + 5x + a is divided by (x + 2) is 3. Find a.
   b) The remainder when x4 – ax + 3 is divided by (2x – 1) is 2. Find a.

9. (x – 2) s a factor of P(x) = x3 + ax2 +bx + 6 and the remainder when P(x) is divided by (x + 3) is 30. Find a and b.

10. In question a – c below show that the linear expression is a factor of the polynomial, hence, solve the equation that follows:
    (i) (x + 1), x3 + x2 – 4x – 4

(ii) (x + 1), x3 – 6x2 + 5x + 12
(iii) (x – 1), 2x3 - x2 – 2x + 1. Hence, solve for the following equations:
a) x3 + x2 – 4x – 4 = 0
b) (x + 1), x3 – 6x2 + 5x + 12 = 0
c) 2x3 - x2 – 2x + 1 = 0

11. Find the values of p and q if (x – 1) and (x + 2) are factor of $2x^2 + px – x + q$.
12. Determine the remainder if x – 4 is a divisor of x4 + 2x3 – 6x2 + 3.
13. Find the remainder and the quotient if 4x3 + 2x2 – 6x + 9 is divided by (2x + 1).
14. Find the other factors if (x + 1) is a factor of x3 + 4x2 – x – 4.
15. If f(x) = 3x + 2 and g(x) = 2x – 1.
Find (a) f(x) + g(x)
(b) {[f(x)]2 – 2g(x)} if x = 3.
16. Given that f(x) = -3x3 – x2 + 3x – 100. Evaluate
(a) f(-3) (b) f(5) (c) f(0)

UNIT 3 Partial Fraction

Unit Structure

1.0 Introduction
2.0 Objectives
3.0 Partial Fraction
3.1 Resolving Proper Rational Expression to Partial Fraction
3.2 Resolving Improper Rational Expression to Partial Fraction
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Reading

1.0 Introduction

Sometimes, a polynomial divides the other not to know the quotient or the remainder but to find a simpler alternative of expressing the rational algebraic function (function of one polynomial dividing the other). A rational algebraic fraction is a quotient fraction is a quotient of two polynomial i.e., $\frac{P(x)}{Q(x)}$. This could either be proper or improper. A proper rational expression is an algebraic function whose degree of numerator is less than that of the denominator e.g., $\frac{x+5}{x^2+x}, \frac{5}{x^2+3x}, \frac{x^2+4x}{(x+4)(x^2+1)}$ etc.
An improper rational expression is the one in which the degree of the denominator is less than the degree of the numerator e.g., $\frac{3x^2+2x^2+3}{x^2+6}, \frac{(x^2+4)(x+1)}{x^2-3x+4}$. Both the proper and improper rational expression can be resolved to partial fraction.

2.0 Learning Outcome

After reading this unit, students will be able to:

- resolve proper rational expression into partial fraction
- resolve improper rational expression into partial fraction.

3.0 Partial Fraction

3.1 Resolving Proper Rational Expression into Partial Fraction

The process involved in resolving proper rational expressions to partial fraction depends on the nature of the denominator of the expression to be resolved. The four examples below are some of the simplest forms of proper rational expressions.

Example 1

Resolve into partial fraction $\frac{5}{x^2+x-6}$

Solution

$$\frac{5}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$$

Note: $(x + 3)$ and $(x – 2)$ are obtained when $x^2 + x – 6$ is factorised.

i.e., $x^2 + x – 6$

$x^2 + 3x – 2x – 6$

$x (x + 3) – 2(x + 3)$

$(x + 3) (x – 2)$

Since $(x – 2)$ and $(x + 3)$ are linear terms, their numerator is expressed in coefficient (A and B).

$\frac{5}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$, the LCM is $(x + 3) (x – 2) = x^2 + x – 6$

$$\frac{5}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{5A(x-2) + B(x+3)}{(x+3)(x-2)}$$

This is as good as multiplying through by (x + 3) (x – 2) which is the same thin as $x^2 + x – 6$

$5 = A\,(x - 2) + B\,(x + 3)$

$5 = Ax - 2A + Bx + 3B$

Separating the terms

$5 = -2A + 3B$ (Coefficients of constants)

$0 = A + B$ (Coeffeicients of x) Solving simultaneously, we have

$A + B = 0$ ………………………… (i) x 3

$-2A + 3B = 5$ ………………………. (ii) x 1

3A + 3B = 0

$\underline{-2A + 3B = 5}$

3A – (-2A) = 0 – 5

3A + 2A = -5

5A = -5 $\therefore A = \frac{-5}{5} = 1$

From equation (i)

A + B = 0

-1 + B = 0

B = 0 + 1

B = 1

$\therefore$ The expression $\frac{5}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$

Substituting A and B as -1 and 1 respectively,

Then, $\frac{5}{x^2+x-6} = \frac{-1}{x+3} + \frac{1}{x-2}$ Or $\frac{5}{x^2+x-6} = \frac{1}{x-2} - \frac{1}{x+3}$

Example 2

Resolve $\frac{3x^2-4x+5}{(x+1)(x-3)+(2x-1)}$

Solution

$$\frac{3x^2-4x+5}{(x+1)(x-3)+(2x-1)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{B}{2x-1}$$

The LCM is (x + 1) (x – 3) (2x – 1)

Diving through by the LCM, we have:

$$\frac{3x^2-4x+5}{(x+1)(x-3)+(2x-1)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{B}{2x-1}$$

$$\frac{3x2 - 4x + 5 = A(x-3)(2x-1) + B(x+1)(2x-1) + C(x+1)(x-3)}{(x+1)(x-3)(2x-1)}$$

Expanding the numerator and ignoring the denominator, we have: -

$3x^2 - 4x + 5 = A\,(2x^2 - 7x + 3) + B\,(2x^2 + x - 1) + C\,(x^2 - 2x - 3)$

Note: You may revisit multiplication of polynomials if you do not know how the expansion in the brackets is obtained.

$3x^2 - 4x + 5$

$= 2Ax^2 - 7Ax + 3A + 2Bx^2 + Bx – B + Cx^2 - 2Cx - 3C$

Separating like terms

3 = 2A + 2B + C (coefficients of) (i)

-4 = 7A+ B – 2C (coefficients of x) (ii)
5 = 3A – B – 3C (the constants) (iii)

Solving equations (i), (ii) and (iii) simultaneously, we have

2A + 2B + C = 3 (i)
7A + B – 2C = -4 (ii)
3A – B – 3C = 5 (iii)

Using elimination method, From equation (i) and (ii)

2A + 2B + C = 3 (i) x 1
7A + B – 2C = -4 (ii) x 2
_ 2A + 2B + C = 3
14A + 2B – 4C = -8
2A – (-14A) + C – (-4C) = 3 – (-8)
2A + 14A + C + 4C = 3 + 8
16A + 5C = 11 (iv)

From equation (ii) and (iii), eliminate (B)

7A + B – 2C = -4 (ii)
3A – B – 3C = 5… (iii)
-7A + 3A – 2C + (-3C) = -4 + (5)
-4A -2C – 3C = -4 + 5
-4A – 5C = 1 (v)

Solving equation (iv) and (v) simultaneously to obtain A and B

+ 16A + 5C = 11
-4A – 5C = 1
16A + (-4A) = 11 + 1
16A – 4A = 12
12A = 12

Divide through by 12

$\frac{12A}{12} = \frac{12}{12} \therefore A = 1$

From equation (v)

-4A – 5C = 1
-4(1) – 5C = 1
-4 – 5C = 1
-5C = 1 + 4
-5C = 5

$\frac{-5c}{-5} = \frac{5}{-5} \therefore c = -1$

From any of the equations (i), (ii) or (iii), B can be obtained. From equation (i)

2A + 2B + C = 3
2(1) + 2B + (-1) = 3
2 + 2B – 1 = 3
1 + 2B = 3
2B = 3 – 1

2B = 2 $\therefore B = \frac{2B}{2} = \frac{2}{2} = 1$

Therefore, $\frac{3x^2 - 4x + 5}{(x+1)(x-3)+(2x-1)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{B}{2x-1}$

Substituting A, B and C, we have:

$$\frac{3x2 - 4x + 5}{(x+1)(x-3)(2x-1)} = \frac{1}{x+1} + \frac{1}{x-3} + \frac{-1}{2x-1}$$

$$= \frac{1}{x+1} + \frac{1}{x-3} - \frac{1}{2x-1}$$

Example 3

Resolve $\frac{16x}{x^4 - 16}$ into partial fraction

Solution

x^4 – 16, has to be completely factorised, let (x – 2) be a factor

$(x - 2) = 0$

$x = 2$

$2^4 - 16 = 0 \therefore (x - 2)$ is a factor.

$$\begin{array}{r|l} & x^3 + 2x^2 + 4x + 8 \\ \hline x - 2 & x^4 - 16 \\ & -\underline{(x^4 - 2x^3)} \\ & 2x^3 - 16 \\ & -\underline{(2x^3 - 4x^2)} \\ & 4x^2 - 16 \\ & -\underline{(4x^2 - 8x)} \\ & 8x - 16 \\ & -\underline{(8x - 16)} \\ & - \quad - \end{array}$$

Therefore, $x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$

Note: $(x^4 - 16)$ cannot be factorised further

$$\frac{16x}{x^4 - 16} = \frac{1}{x - 2} + \frac{1}{x + 2} + \frac{-1}{x^2 - 4}$$

Note: The numerator of the third component on the RHS is Cx + D. This is because when the denominator is a quadratic expression, the numerator is expressed as a linear expression.

$$\frac{16x}{x^4 - 16} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 - 4}$$

Multiplying through by the LCM = $(x - 2)(x + 2)(x^2 + 4)$,

we have $16x = A(x + 2)(x^2 + 4) + B(x - 2)(x^2 + 4) + Cx + D(x - 2)(x + 2)$

$16x = (x^3 + 4x + 2x^2 + 8) + (x^3 + 4x - 2x^2 - 8) + Cx + D(x^2 - 4)$

$16x = Ax^2 + 4Ax + 8Ax^2 + Bx^2 + 4Bx - 2Bx^2 - 8B + Cx^3 - 4Cx + Dx^2 - 4D$

Separating the terms

$0 = A + B + C$ (coefficients of x^3) (i)

$0 = 2A - 2B + D$ (coefficients of x^2) (ii)

$16 = 4A + 4B - 4C$ (coefficients of x) (iii)

$0 = 8A - 8B - 4D$ (constants) (iv)

From equation (iii); dividing through by 4

$4 = A + B - C$

$\therefore A + B = 4 + C$ *(v)*

From (i), $0 = A + B + C$

Substitute $A + B = 4 + C$

$\therefore 0 = 4 + C + C$

$0 - 4 = 2C$

$-4 = 2C$

$C = \frac{-4}{2} = -2$

From equation (v)

$A + B = 4 + C$

$A + B = 4 + (-2)$

$A + B = 4 - 2$

$A + B = 2$ (vi)

Eliminating C from equation (i) and (iii)

$+ \; 0 = A + B + C$

$4 = A + B - C$

$0 + 4 = 2A + 2B$

$4 = 2A + 2B$

$2 = A + B$

(Note, this is the same as equation (vi), so no need to number it again)

Eliminating D from equations (ii) and (iv)

$0 = 2A - 2B + D$ x4

$0 = 8A - 8B - 4D$ x1

$+ \; 0 = 8A - 8B + 4D$

0 = 8A – 8B – 4D

16A – 16B = 0

A – B = 0 (vii)

Solving (vi) and (vii) simultaneously, we have

_ A + B = 2 …………………… (vi)

A – B = 0 …………………… (vii)

B – (-B) = 2 – 0

B + B = 2

2B = 2

B = $\frac{2B}{2}$ = 1

From equation (vi),

A + B = 2

A + 1 = 2

A = 2 -1 ∴ A = 1

Then, D can be obtained from equation (ii) since A, B and C are already known

0 = 2A – 2B + D

0 = 2(1) – 2(1) + D

0 = 2 – 2 + D

0 = 0 + D

D = 0 ∴ A = 1, B = 1, C = -2 and D = 0.

Hence, the partial fraction is resolved as thus: -

$$\frac{16x}{x^4-16} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2-4}$$

Substituting A, B, C and D, we have

$$\frac{16x}{x^4-16} = \frac{1}{x-2} + \frac{1}{x+2} + \frac{(-2)x+0}{x^2-4}$$

This can be finally expressed thus: -

$$\frac{16x}{x^4-16} = \frac{1}{x-2} + \frac{1}{x+2} + \frac{2x}{x^2-4}$$

Example 4

Resolve into partial fraction+ $\frac{x-4}{(x+3)^2}$

Solution

$$\frac{x-4}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

Note: The order of the denominator, if the denominator of the partial fraction is $(x + a)^n$, then the resolved side (RHS) becomes:

$$\frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{c}{(x+a)^3} + \ldots\ldots + \frac{B}{(x+a)^n}$$

For instance, if you want to resolve $\frac{x-8}{(x+a)^4}$, it is simplified as

$$\frac{x-8}{(x+a)^4} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{c}{(x+4)^3} + \frac{D}{(x+a)^4}$$

Now, we can go back to example 4;

$$\frac{x-4}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}, \textit{multiplying through by LCM} = (x+3)^2, \textit{we have}$$

x – 4 = A (x + 3) + B (1)

x – 4 = Ax + 3A + B

Separating the term,

1 = A (coefficient of x) …………………… (i)

-4 = 3A + B (constants) …………………… (ii)

From equation (i) A = 1

Substituting A = 1 into equation (ii) to obtain 'B'

-4 = 3A + B

-4 = 3(1) + B

-4 = 3 + B

-4 – 3 = B

-7 = B or B = -7

∴ The partial fraction becomes

$$\frac{x-4}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

$$\therefore \frac{x-4}{(x+3)^2} = \frac{A}{x+3} + \frac{-7}{(x+3)^2}$$

Hence, $\frac{x-4}{(x+3)^2} = \frac{A}{x+3} - \frac{7}{(x+3)^2}$

SELF-ASSESSMENT EXERCISE

I. Resolve into partial fraction $\frac{3x^2-4x-7}{(x-1)(x+2)(x-3)}$

II. Resolve $\frac{7x-10}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5}$ into partial fraction.

III. Given that $\frac{7x-9}{(x+1)(x+5)} = \frac{A}{x+1} + \frac{B}{x+5}$ Find A and B

3.2 Resolving Improper Rational Expression into Partial Fraction

The process of resolving an improper rational function into partial fraction involves the use of long division first.

Example

Resolve $\frac{t^3+t^2+4t}{(t+2)(t-1)}$ into partial fraction.

Solution

Since the degree of numerator is greater than that of the denominator, the rational function is improper; so long division should be carried out first. $\frac{t^3+t^2+4t}{(t+2)(t-1)} = \frac{t^3+t^2+4t}{t(t-1)+2(t-1)} = \frac{t^3+t^2+4t}{t^2+t+2}$

```
                 t
t² + t – 2 ⟌   t³ + t² + 4t
           –   t³ + t² – 2t
                         6t
```

$\therefore \frac{t^3+t^2+4t}{t^2+t+2}$ = Quotient + Resolution of $\frac{R(x)}{D(x)}$ into partial fraction.

$$\therefore \frac{t^3+t^2+4t}{t^2+t+2} = t + Resolution\ of\ \frac{6t}{t^2+t+2}$$

To resolve, $\frac{6t}{t^2+t+2} = \frac{6t}{(t+2)(t-1)}$

$$\frac{6t}{t^2+t+2} = \frac{A}{(t+2)} + \frac{B}{(t-1)}$$

Multiplying through by (t + 2) (t – 1)

6t = A(t – 1) + B(t + 2)

6t = At – A + Bt + 2B

Sorting out like terms

6 = A + B (coefficient of t) (i)

0 = -A + 2B (coefficient of constant) (ii)

Solving (i) and (ii) simultaneously

Adding 1 to 2

6 + 0 = B + 2B

6 = 3B

divide both sides by 3

B = $\frac{6}{3}$ = 2

From equation (i)

6 = A + B

6 = A + 2

A = 6 – 2

A = 4

Hence, $\frac{t^3 + t^2 + 4t}{t^2 + t + 2} = t + \frac{A}{(t+2)} + \frac{B}{(t-1)}$

Example 2

Resolve $\frac{2x^3 - 2x^2 + 2x - 7}{x^2 + x - 2}$ into partial fraction.

Solution

$$\begin{array}{r} 2x \qquad\qquad \\ x^2 - x - 2 \overline{\big)\ 2x^3 - 2x^2 - 2x - 7} \\ -\ 2x^3 - 2x^2 \underline{- 4x} \qquad \\ 2x - 7 \end{array}$$

Hence, $\frac{2x^3 - 2x^2 + 2x - 7}{x^2 + x - 2} = 2x = \frac{2x - 7}{x^2 + x - 2}$

Resolving the rational part of RHS,

$$\frac{2x-7}{x^2+x-2} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

Multiplying through by (x – 2) (x + 1)

2x – 7 = A (x + 1) + B (x – 2)

2x – 7 = Ax + A + Bx – 2B

Solving the like terms

2 = A + B (coefficient of x)

-7 = A – 2B (coefficient of constants)

Solving (i) and (ii) simultaneously

_2 = A + B

-7 = A – 2B

2 – (-7) = B – (-2B)

2 + 7 = B + 2B

9 = 3B ∴ B = $\frac{9}{3}$ = 3

From equation (i),

2 = A + B

2 = A + 3

A = 3 – 2 = 1

$\frac{2x^3 - 2x^2 + 2x - 7}{x^2 + x - 2} = 2x + \frac{-3}{x-2} + \frac{5}{x+1} = = 2x - \frac{3}{x-2} + \frac{5}{x+1}$

SELF-ASSESSMENT EXERCISE

i. Resolve into partial fraction $\frac{x^2 + 3x - 5}{(x-2)\,(x+1)^2}$

ii. Resolve $\frac{t^3 + 2t^2 + 5t}{(t-2)\,(t+7)}$

4.0 Summary

Both proper and improper rational expression can be resolved into partial fraction. The major difference in the approach lies on the procedure involved. The resolution of a rational fraction to partial fraction can be obtained in two folds namely proper and improper fraction. For example: $\frac{ax^2 + bx + c}{(x+a)\,(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} = \frac{ax^2 + bx + c}{(a+x)^n}$ $= \frac{A}{(a+x)} + \frac{B}{(a+x)^2} + \frac{c}{(a+x)^3} + \ldots\ldots \frac{N}{(a+x)^n}, \frac{ax^2 + bx + c}{ax^2 + bx}$.

For improper fraction $\frac{x^{n-1} + x^n + c}{x^2 + x}$ = Quotient + Resolution $of\ \frac{R}{x^n + x}$ quotient is obtained by dividing $x^{n-1} + x^n + c$ by $x^n + x$. This is an improper fraction because the power of the numerator is greater than the power of denominator.

5.0 References/Further Readings/Web Resources

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6.0 SELF-ASSESSMENT EXERCISE

1. Resolve each of the following into partial fraction
   a. $\frac{2x + 3}{(2x + 1)(x^2 + x - 1)}$
   b. $\frac{2x + 8}{(x + 7)(2x+3)}$
   c. $\frac{x^2 + 3x - 7}{(3x + 1)(x^2+1)}$
   d. $\frac{2x - 7}{x^2 - 9}$
   e. $\frac{1}{x^2+1}$
2. Resolve $\frac{4x-7}{(x + 1)(2x+5)}$ into partial fraction.
3. Given that $\frac{7x + 9}{(x + 5)(x-1)} = \frac{A}{(x - 1)} + \frac{B}{(x+5)}$ find x?
4. Resolve $\frac{3x^2 + 4x - 7}{(x-1)(x+2)(2x-3)}$ into partial fraction.
5. Resolve into partial fraction
   a. $\frac{(2x+1)(x-7)}{2x+3}$
   b. $\frac{2x+3}{(2x+1)(x^2+x-1)}$
   c. $\frac{2x}{(x-7)(2x+3)}$
   d. $\frac{x^2+3x-54}{5x-6}$
   e. $\frac{x^2+3x-7}{(3x+1)(x^2+1)}$

UNIT 4 BINOMIAL THEOREM

Unit Structure

1.0 Introduction
2.0 Learning Outcomes
  3.0 Binomial Theorem
  3.1 Factorials
  3.2 Binomial Expansion
4.0 Summary
5.0 References/Further Readings/Web Resources
6.0 SELF-ASSESSMENT EXERCISE

1.0 Introduction

In Unit 1 of this module, you were introduced to expansion of algebraic terms and polynomials. Then, $(a + b)^2$ is expanded to $(a + b)(a + b)$ which gives $a(a + b) + b(a + b) = a^2 + ab + ab + b^2$, which finally gives $a^2 + 2ab + b^2$. The coefficient of a^2 is 1, the coefficient of ab is 2 and the coefficient b^2 is 1. Sometimes, expansion of algebraic terms or polynomials may be raised to higher indices such as $(a + b)^4$, $(x - y)^8$, $(2a - 6d)^{10}$. In such exercises, simple expansion may be too cumbersome to apply, there is need to apply the binomial expansion.

2.0 Learning Outcome

After reading this unit, students will be able to:

- evaluate factorials
- carry out expansion of higher indices using binomial expansion

- state the coefficients of the terms in a binomial expansion.

3.0 Binomial Theorem

3.1 Factorials

Factorials are useful concepts in economics and business-related issues. The factorial of a number (n) is the product of the consecutive positive integers from n to 1. For example,

$n! = (n-1)(n-2)(n-3) \ldots (3)(2)(1)$

4! = 4 x 3 x 2 x 1

7! = 7 x 6 x 5 x 4 x 3 x 2 x 1

Note: 7! = 7 x 6 x 5!

or 7 x 6 x 5 x 4!

or 7 x 6 x 5 x 4 x 3!

Hence, $\frac{12!}{3!5!} = \frac{12 \ x \ 11 \ x \ 10 \ x \ 9 \ x \ 8 \ x \ 7 \ x \ 6 \ x \ 5!}{3 \ x \ 2 \ x \ 1 \ x \ 5!}$

$= 12 \ x \ 11 \ x \ 10 \ x \ 9 \ x \ 8 \ x \ 7 = 665, 280$

It should be noted that 1! = 1 and 0! = 1

SELF-ASSESSMENT EXERCISE

Simplify each of the following factorials:

i. 8!

ii. 3! x 4!

iii. $\frac{8!}{2!4!}$

iv. $\frac{7!}{3!4!}$

v. $\frac{15!}{4! \, 3!2!}$

3.2 Binomial Expansions

A binomial expression is any algebraic expression which contains two terms. Example is (a + b). Higher powers of a binomial expression can be obtained by expanding the expression $(a + b)^n$, while n is a nonnegative integer, The general theorem of expansion of $(a + b)^n = a^n + na^{n-1} b + \frac{n(n-1)a^{n-2} \, b^2}{2!} + \frac{n(n-1)a^{n-3} \, b^3}{2!} + \cdots + \frac{((n-r+1)(a^{n-r} \, b^r}{r!} + \cdots + b^n$

The above is the binomial theorem for any integer value of n. Based on the above formula, it can be shown that

$(a + b)^0 = 1$ (Recall $n^0 = 1$)

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Note: $(a + b)^3 = (a + b) \, (a + b)^2 = (a + b)(a^2 + 2ab + b^2) \, (a + b)^4 = (a + b)^2 + (a + b)^2$

$= (a^2 + 2ab + b^2) \, (a^2 + 2ab + b^2) \, (a + b)^5 = (a + b)^2 + (a + b)^3$

$= (a^2 + 2ab + b^2) \, (a^3 + 3a^2b + 3ab^2 + b^3)$

$\therefore (a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

You will observe that the result of the expansion gives an increase power of one of the terms (i.e. b) and a decreasing power in other term *(a)*. You equally observe that the coefficients of the terms keep changing as we increase the power of the expansion. The coefficient can be soughed out from the triangle below

$(a+b)^0, n=0$ 1
$(a+b)^1, n=1$ 1 1
$(a+b)^2, n=2$ 1 2 1
$(a+b)^3, n=3$ 1 3 3 1
$(a+b)^4, n=4$ 1 4 6 4 1
$(a+b)^5, n=5$ 1 5 10 10 5 1
$(a+b)^6, n=6$ 1 6 15 20 15 5 1
$(a+b)^7, n=7$ 1 7 21 35 35 20 6 1
$(a+b)^8, n=8$ 1 8 28 56 70 56 28 7 1

∴ and so on.

You observe that the triangle numbers corresponding to the coefficient of terms in each expansion. For instance $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the coefficients of the terms are: -

1 for a^4
4 for a^3b
6 for a^2b^2
4 for ab^3 and
1 for b^4, the set of the coefficient is 1 4 6 4 1, which corresponds to the value at n = 4 from the triangle.

Also, $(a+b)^5$ gives the coefficients of 1 for a^5, 5 for a^4b, 10 for a^3b^2, 10 for a^2b^3, 5 for a^2b^4 and 1 for b^5.

This gives the set of coefficients as 1, 5, 10, 10, 5, 1 which correspond to the values obtained when n = 5 from the triangle.

It should be noted that the values in the triangle are obtained starting with 1 adding up the preceding coefficients and ending up with 1. It should be noted from the expansion that as one component of the term increases, the other reduces. For instance, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ i.e. $(a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4ab^3 + a^0b^4$. The powers of a decrease along the expansion (from 4 to 3 to 2 to 1 to 0) while the power of b increases (from 0 to 1 to 2 to 3 to 4). The coefficients 1, 4, 6, 4 and 1 can easily be obtained from the triangle. The triangle is known as *Pascal's Triangle*.

Examples

1. Evaluate (a) $(x - y)^7$, stating the coefficient of the terms in the expression.
2. Find $(2a + 3c)^5$ and state the coefficient of a^3c^2.

Solution

a) Recall the Pascal triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 5 1
1 7 21 35 35 20 6 1

$\therefore (x-y)^7 = 1(x^7(-y)^0) + 7(x^6(-y)^1) + 21(x^5(-y)^2) + 35(x^4(-y)^3) + 35(x^3(-y)^4) + 21(x^2(-y)^5) + 7(x^1(-y)^6) + 1(x^0(-y)^7)$

$= x^7 + 7(x^6.-y) + 21(x^5.y^2) + 35(x^4.-y^3) + 35(x^3.y^4) + 21(x^2.-y^5) + 7(x.y^6) + 1(1.-y)$

$= x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

Coefficient of $x^7 = 1$
Coefficient of x^6 y = -7
Coefficient of x^5 y^2 = 21
Coefficient of x^4 y^3 = -35
Coefficient of x^3 y^4 = 35
Coefficient of x^2 y^5 = -21
Coefficient of xy^6 = 7 and
Coefficient of y^7 = 1

b) $(2a + 3c)^5 = 1(2a)^5 + 5(2a)^4(3c) + 10(2a)^3(3c)^2 + 10(2a)^2(3c)^3 + 5(2a)^1(3c)^4 + 1(2a)^0(3c)^5$

$= 32a^5 + 5(16a^4.3c) + 10(8a^3.9c^2) + 10(4a^2 27c^3) + 5(2a.81c^4) + 1(1.243c^5)$

$= 32a^5 + 240a^4c + 720a^3c^2 + 1080a^2c^3 + 810ac + 243c^5$

Therefore, the coefficient of a^3c^2 is 720.

SELF-ASSESSMENT EXERCISE

i. Expand $(3x – 5y)^8$ and state the coefficient of each of the terms.
ii. Simplify $(x + y)^8 + (x – y)^4$ and state the coefficient of each of the terms obtained.

4.0 Summary

Factorial of any number given is the product of the consecutive positive integer from that number to 1. High powered expansions are usually cumbersome to carry out using simple expansion procedures. Such problems are easily carried out with the use of binomial expansion theorem which is guided by the Pascal’s triangle in determining the coefficients of the terms involved. Factorials and binomial expansion are closely related. This is because the concept of factorial is important in binomial expansion. Binomial expansion of high powers can easily be obtained by using the Pascal theorem.

5.0 References/Further Readings/Web Resources

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6.0 SELF-ASSESSMENT EXERCISE

1. Find the approximate value of
   a. $(1.06)^4$
   b. $(0.98)^4$; by using the first four terms of the binomial expansion.
   (Hint: $(1.06)^4 = (1 + 0.6)^4$ while $(0.98)^4 = (1 – 0.02)^4$.
2. Expand $(2x + 5y)^6$, hence state the coefficient of x^4y^2.
3. Simplify a. $\frac{15!}{(10-5)!}$
   b. $\frac{15!}{7!3!5!}$
   c. $\frac{10!}{5!}$
   d. 3! X 6!
4. Expand (a) $(4x – 7y)^7$
   (b) $(2 – 4x)^5$
   (c) $(3y + 6x)^4$; hence state the coefficient of x^3y^4, x^5 and x^4 for the expression obtained.

UNIT 5 PERMUTATION AND COMBINATION

Unit Structure

1.0 Introduction

The concept of factorial is very useful in the process of arranging a number of different objects. For example, the four-letter a, b, c and d can be arranged in different forms or ways. The first letter can be chosen in four ways i.e., can take the first or second or third or fourth position. The second letter can be chosen in three ways, the third letter in two ways and the fourth letter in one way. So, the number of ways is 4 x 3 x 2 x 1 = 24. You recall that, in the previous unit, 4 x 3 x 2 x 1 = 4! In general, the number of ways of arranging n – different objects is n (n – 1) (n – 2) … 3 x 2 x 1 = n! ways. The concept of factorial is not only useful in arranging non – repeated objects, it can also be used to know the number of ways repeated objects could be arranged as well as to provide a tool for the computation of permutations and combinations problems.

2.0 Learning Outcomes

After reading this unit, students will be able to:

- determine the number of ways a non-repeated object could be arranged
- determine the number of ways a repeated object could be arranged
- apply the concept of factorials in solving Permutations and Combination problems.

3.0 Permutation and Combination

3.1 Arranging Objects

There are two forms of arrangements of objects that the concept of factorial can assist us to obtain. These are (i) arranging non-repeated objects and arranging repeated objects. For a non-repeated object; the number of ways the objects can be arranged is simply n!, where n = number of distinct objects.

3.1.1 Non-Repeated Objects

Example 1

In how many ways can the letters of the word "DANGER be written?"

Solution

Note; The word "DANGER" has no repeated object i.e. none of the six letter is repeated.

Hence, the number of ways the letters of the word can be written = 6!

= 6 x 5 x 4 x 3 x 2 x 1

= 720 ways.

Example 2

In how many ways can we arrange the figures in the number: 97452018

Solution

There is no repetition in the eight figures. Therefore, the number of ways it could be arranged is 8!

= 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

= 40,320 ways.

3.1.2 Repeated Objects

Suppose some of the objects are repeated. If there are r repeated objects, then these r objects can be arranged in r! ways, so we divide n! by r! to obtain the number of ways repeated objects can be arranged. This means that, given the number of objects as n and the number of repeated objects as r, the number of ways the objects can be arranged is given as $\frac{n!}{r!}$.

Example 3

In how many ways can the letters of the words "OSOGBO" be arranged?

Solution

There are six letters but there are three Os. These Os can be arranged in 3! different ways.

∴ The number of ways "OSOGBO" can be arranged

$= \frac{6!}{3!}$

$= \frac{(6 \times 5 \times 4 \times 3!)}{3!}$

Note: If there is more than one repetition, keep on dividing with the factorial of the different repetitions you can see.

Example 4

In how many ways can the word “DAMATURU” be arranged?

There are 8 letters and 2 different repetitions ‘A’ and ‘U’, each of the repetition occur twice.

Therefore, number of ways of arrangement is

$$\frac{8!}{2!2!}$$

(8 x 7 x 6 x 5 x 4 x 3 x 2!)/(2 x 1 x 2!)

= 10,080 ways.

Example 5

In how many ways can the word ORGANISATIONAL be arranged?

Solution

n = 14

O is written twice

A is written thrice

N is written twice

I Is written twice

R, G, S, T and L are written once

Therefore, numbers of the way

$$= \frac{14!}{2!3!2!2!1!1!1!1!1!} = \frac{14!/}{2!3!2!2!} \text{ Note } 1! = 1$$

$$= \frac{(14\ x\ 13\ x\ 12\ x\ 11\ x\ 10\ x\ 9\ x\ 8\ x\ 7\ x\ 6\ x\ 5\ x\ 4\ x\ 3!)}{(2\ x\ 1\ x\ 2\ x\ 1\ x\ 2\ x\ 1\ x\ 2\ x\ 1)}$$

$= 1, 816, 214, 400\ ways$

Example 6

Eda, Ngozi, Viola, Oluchi, Chinyere and Adamu are to sit on a bench. In how many ways can this be done if:

(a) Eda and Ngozi insist on sitting next to each other?

(b) Eda and Ngozi refuse to sit next to each other?

Solution

(a) Consider Eda and Ngozi as a single unit. There are now 5 units instead of 6 units. The 5 units can be arranged in 5! ways since Eda and Ngozi can sit in either two ways (Eda, Ngozi) or (Ngozi, Eda). Therefore, the number of possible sitting arrangement.

= 5! X 2!

= (5 x 4 x 3 x 2 x 1) x 2

= 240 ways

(b) Without any restrictions, the number of ways is 6! = 720 ways. If Eda and Ngozi refuse to sit together, then, the number of ways possible = Total number of ways without restriction minus ways possible if they insist to sit together

= 720 ways – 240 ways

= 480 ways.

Example 7

In how many ways can 4-digit numbers greater than 3000 be formed using the digits 0, 1, 2, 3, 4, 5 if

(a) There is no restriction?

(b) If there is repetition?

Solution

(a) 3 x 5 x 4 x 3 = 180 ways

Note: The first figure is 3 and not 6 because three figures can start the figure.

(b) 3 x 5 x 5 x 5 = 375 ways

Note: Other figures in the products are 5 all through because any of the five numbers can take the place since repetition is allowed.

SELF-ASSESSMENT EXERCISE

i. Simplify the following by writing as single factorial expression.

a. 11! X 12 b. 13 x 12! c. 10! ÷ 10 d. 12! ÷ 132 e. 8! ÷ 56

ii. A competition lists seven desirable properties of a car and asks the entrants to put than in order of importance. In how many ways can this be done?

iii. How many five-digit numbers can be made using 1, 2, 3, 4, 5, if no digit is repeated?

iv. Five men sit around a circular table. In how many ways can this be done?

v. How many arrangements are there of the letters from the following:

a. AFRICA
b. AMERICA
c. COTTON
d. MACMILLAN
e. BUCKWELL
f. MATHEMATICS
g. REWARD

3.2 Permutation

Permutation is a way of sharing n-objects among N population in a given order. Therefore, it is mathematically written as N Pn where n ≤ N. For example, suppose there are 20 children who are to receive three prizes of ₦1000, ₦500 and ₦200. In how many ways can the prize winners be chosen if no children win more than one price?

The winner of ₦1000 can be chosen in 20 ways.
The winner of ₦500 can be chosen in 19 ways.
The winner of ₦200 can be chosen in 18 ways.

(Note: Once someone has been chosen as the winner of ₦1000 he can no longer be chosen as the winner of ₦500 or ₦200).

Therefore, the total number of ways is 20 x 19 x 18 = 6840.

By decision, permutation is the number of ways of picking n items out of N sample space. In general, ${}^{n}P_{r}$ is the number of picking r objects out of n sample space, in a particular order. The formula is

$${}^{n}P_{r} = n\,(n-1)\,(n-2)\,\ldots\,(n-r+1)$$

This is written as: $\frac{(n(n-1)\,(n-2)\,\ldots(n-r+1)\,(n-r)\,\ldots 3\ x\ 2\ x\ 1)}{((n-r)\,\ldots 3\ x\ 2\ x\ 1)}$

$$= \frac{n!}{(n-!)!)}$$

Example 8

A committee has ten members. In how ways can the chairman, the secretary and the treasurer be selected?

Solution

We need to choose three out of ten. As the number of posts are different, the order in which they are matters.

Hence, this is a permutation.

∴ The number of ways is 10P3

$$= \frac{10!}{(n-3)!} = \frac{10!}{7!} = \frac{(10\ x\ 9\ x\ 8\ x\ 7!)}{7!} = = 720\ way$$

Example 9

In how many different ways can seven different books be arranged in four spaces on four spaces on a book shelf?

Solution

$${}^{n}P_{r} = 7P4 = \frac{7!}{(7-4)!} = \frac{7!}{3!}$$

$$= \frac{7x\ 6\ x\ 5\ x4\ x\ 3!}{3!} = 7x\ 6\ x\ 5\ x4 = 840 \text{ ways}$$

Example 10

How many five letter code words can be formed from the word BAKERY?

Solution

${}^{6}P_{5}$ ways

$$\frac{6!}{6-5!} = \frac{6!}{1!} = 6! = 6\ x\ 5\ x\ 4\ x\ 3\ x\ 2\ x\ 1 = 720\ ways$$

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SELF-ASSESSMENT EXERCISE

i. In how many ways can 4-digit numbers greater than 4000 be formed from 0, 1, 2, 3, 4, 5, 6 if:
   a. There is no repetition.
   b. There is repetition.
ii. How many telephones number of 6-digit can be formed from all the digit 0 to 9 if:
   a. No telephone number must begin with zero and repetition is not allowed.
   b. Repetition of digits is not allowed.
   c. Repetition of digits is allowed.
iii. How many other telephone numbers starting with 0 can be formed from Bola's phone number which is 08035725368?
iv. Evaluate the following without using a calculator a. ${}^{4}P_{3}$ b. ${}^{5}P_{2}$ c. ${}^{9}P_{2}$
v. A class has 18 pupils. In how many ways can a 1st, 2nd, 3rd and 4th prizes be awarded?
vi. There are 16 books on a library shelf. How many ways can a student borrow four of them?
vii. How many three-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8 if no digit is repeated.
viii. From a class of 19 students, six are to be selected as a member of a committee. In how many ways can this be done?

3.3 Combination

Permutation and combination are closely related concepts. While permutation deals with the number of ways of picking r objects out of n in a particular order, combination is concerned with the number of ways of choosing r objects out of n objects in any order.

Suppose that the prizes given to students are of equal values. That is no one is considered better than the other, which means it does not matter in which order the prize winners are named. Hence, the number of ways is divided by the number of ways of ordering the 3 prize winners. For instance, suppose there are 20 children to receive 3 prizes, in how many ways can the prizes be won?

This can be ${}^{120}C_3 = {}^{20}P_3 \div 3$

That is ${}^{n}C_r = {}^{n}P_r \div r!$

$${}^{n}P_r = \frac{n!}{(n-r)!}$$

$$\therefore {}^{n}C_r = \frac{n!}{(n-r)!} \div r = \frac{n!}{(n-r)!} \, x \, \frac{1}{r!}$$

$$\therefore {}^{n}C_r = \frac{n!}{r(n-r)!}$$

Example 11

A committee has ten members. There is to be a subcommittee of three. In how many ways can this be chosen?

Solution

The three sub-committee members are equal. So, the order in which they are chosen does not matter.

Hence, this is a Combination example.

$\therefore$ The number of ways = ${}^{10}C_3$

$$\therefore {}^{10}C_3 = \frac{10!}{3!(10-3)!} = \frac{10 \, x \, 9 \, x \, 8 \, x \, 7!}{3 \, x \, 2 \, x \, 1 \, x \, 7!} = 120 \text{ ways}$$

Example 12

A school committee is to be formed. There are eight eligible girls and six eligible boys. In how many ways can the committee be formed if there are four girls and three boys?

Solution

The number of ways of selecting the girls is ${}^{8}C_4$

$$\therefore {}^{8}C_4 = \frac{8!}{4!(8-4)!} = \frac{8 \, x \, 7 \, x \, 6 \, x \, 5 \, x \, 4!}{4 \, x \, 3 \, x \, 2 \, x \, 1 \, x \, 4!} = 28 \text{ ways}$$

The number of ways of selecting the boys is $^6C_3$

$$\therefore {}^6C_3 = \frac{6!}{3!(6-3)!} = \frac{6 \times 5 \times 4 \times 3!}{3 \times 2 \times 1 \times 3!} = 20 \text{ ways}$$

∴ The committee can be chosen in (28 x 20) ways = 1400 ways

Example 13

A committee of five is to be chosen from ten men and eight women. In how many ways can this be done if there must be at least one man and at least one woman?

Solution

Without any restriction, the number of ways is $^{18}C_5$

If there is at least one man and at least one woman, the committee cannot be all-women or all-man.

Number of all women committee = $^8C_5$

Number of all men committee = $^{10}C_5$

With the consideration of the restriction, the number of ways is

= 18C5 – 8C5 – 10C5

= 8,568 – 56 – 252 = 8260 ways

SELF-ASSESSMENT EXERCISE

i. Evaluate the following without a calculator a. $^4C_2$ b. $^5C_3$ c. $^6C_3$
ii. How many ways can a committee of two men and three women be selected from groups of eight men and seven women?
iii. A library shelf contains ten novel and eight biographies. In how many ways can a borrower choose three novels and two biographers?
iv. A football tournament is to be arranged between teams from Africa and Europe. There will be four African teams chosen from eight national sides and four European teams chosen from ten national sides. How many possible selections are there?
v. A committee of six is to be formed by a state governor from nine state commissioners and three members of the State's House of Assembly. In how many ways can the members be chosen so as to include one member of the House of Assembly?

4.0 SUMMARY

The concept of factorial, permutation and combination are interwoven and interrelated. They are specifically used to determine the number of way an arrangement or selection can be done. The number of ways of arranging n different objects is n! If a particular object is repeated k times, the number of ways is divided by k! If an object is arranged in a circle rather than a row, the number of ways (n – 1)! The number of ways of picked r objects out of n objects in a particular order is $^nP_r = \frac{n!}{(n-r)!}$ while the number of ways of picking r objects out of n objects, in any order is $^nC_r = \frac{n!}{(n-r)!r!}$. . If the selections are to be made from two different sets, then we multiply together the relevance permutations and combinations.

5.0 References/Further Readings/Web Resources

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Stroud, K. A. (1995). *Engineering Mathematics*. England: Graham Bum Publishers.

6.0 SELF-ASSESSMENT EXERCISE

1. Simplify (a) 11! (b) $\frac{8!}{3!4!}$ (c) $\frac{12!}{12 \times 9!}$
2. Simplify the following by writing as single factorial expressions.
   a) 10 x 9! b) 72 x 7! c) 110 x 9! d) 13! ÷ 156

3. How many arrangements are there of the letters from the following:
a) LAGOS b) NIGERIA c) AUSTRALIA d) HORROR e) SOLOMON f) EGYPTIAN

4. Eight people are to sit in a row. In how many ways can this be done in the following cases?
a) Two particular people must sit next to each other.
b) Two particular people must not sit next to each other.

5. Evaluate the following without calculator
a) ${}^{6}P_{2}$ b) ${}^{7}C_{3}$ c) ${}^{7}P_{5}$ d) ${}^{10}C_{4}$ e) ${}^{8}P_{3}$

6. In a competition, the entrants have to list the three most important properties of a computer from a list of eight properties. In how many ways can this be done?
a) If the three properties must be in the correct order?
b) If the order of the three properties does not matter?

7. A tennis team is to contain three men and two women. In how many ways can the team be selected from a club with 15 men and 9 women?

8. A committee of five is chosen from A, B, C, D, E, F and G. In how many ways can this be chosen?
a) If both A and B must be on the committee.
b) If neither A or B must be on it, but both.

9. An examination consists six-part A questions and seven part B questions. A candidate must attempt five questions. In how many ways can this be done?
a) Without any restriction?
b) If there must be two questions from part A and three question from part B.

10. A box containing 4 black, 3 white and 5 yellow balls are taken out of the box and arranged in a row from left to right. In how many ways can this arrangement be made?

MODULE 6 MATRICES

UNIT 1 MEANING AND TYPES OF MATRICES

Unit Structure

1.0 Introduction

Sometimes, there may be a need to present numerical or non-numerical information in a succinct, compact and simplified way. One of the means of doing this in mathematics is the used of matrices. Information such as price of garment, the score in the test, sales outlets etc. can be presented in a compact rectangular form called Matrices (singular matrix).

Linear algebra permits expression of a complicated system of equations in a compact form in an attempt to provide a shorthand method to determine whether the solutions exist or not. Matrices provide a way of solving linear equations. Therefore, most economic relationship especially linear relationship can be solved using matrices.

Examples of information that can be displayed in matrices are:

$$\begin{matrix} HighSt.Garage \\ parkRdGarage \\ CentralGarage \end{matrix} \begin{pmatrix} 3 & 1 & 02 \\ 1 & 2 & 33 \\ 1 & 3 & 01 \end{pmatrix}$$

The above show the number of stocks of fuel the transporter has in different garage.

Another example is a company with different products, provided by matrix below.

Outlet	Skis	Poles	Bindings	Outfits
1	120	110	90	150
2	200	180	210	110
3	175	190	160	80
4	140	170	180	40

The matrix above presents in a concise way to track the stocks of the businessman. By reading across the horizontal axis the firm can determine the level of stock in any of outlets. By reading down the vertical axes, the firm can determine the stock of any of its products. Therefore, matrices enable us to present information in a compact array so as to provide a summary of the information at a glance. Beyond this, set of equations can be present in a matrix form and can also be solved using some techniques, that shall later be discussed in the subsequent units of this module.

2.0 Learning Outcome

After reading this unit, students will be able to:

- define the term "matrices"
- state the order or the dimension of a given matrix
- identify the position of any element in a matrix
- list and explain the various types of matrices.

3.0 Matrices

3.1 Meaning of Matrices

A matrix can be defined as a rectangular array of numbers, parameters or variables, each of which has a carefully ordered place within the matrix. The plural of matrix is matrices.

Examples of matrices are:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} = B = \begin{bmatrix} 1 \\ 6 \\ 7 \\ x \end{bmatrix} = C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} = D = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} E = \begin{bmatrix} a, b, & c, d, \; e \ldots & z \\ b & \cdots & \vdots \\ \vdots & \cdots & b \\ z, y, & x, w, & \ldots a \end{bmatrix}$$

$$F = \begin{bmatrix} 4 & 2 \\ 1 & 9 \end{bmatrix} = G = \begin{bmatrix} 1 & 5 \\ 9 & 6 \\ 1 & x \end{bmatrix}$$

The number or variables in a horizontal line are called Rows, while the number variables in the vertical lines are called columns. The number of rows(r) and columns(c) defines the order or the dimensions of the matrix. That is, the order or a matrix is given by the number of rows and columns the matrix has. The dimension or order of a matrix is given by (r × c), which is read (r) by (c). The row number always precedes the column number. The order of a matrix is also called the size of the matrix. In the examples given earlier, A is a 2 by 3 matrix i.e. (2×3) therefore it has 6 elements. B is a 4 by 1 matrix, C is a 1 by 3 matrix, D is a 3 by 3 matrix, E is a 26 by 26 matrix, F is a 2 by 2 matrix while G is a 3 by 2 matrix. Each element in a matrix is strategically or orderly positioned to occupy a given row and a given column. This implies that no single element has less than 1 row or column; neither can any element claim more than a row or column. For example, given that:

$$X = \begin{bmatrix} 4 & 6 & 2 & 5 \\ 1 & 8 & 9 & 3 \\ a & b & c & d \end{bmatrix}$$

Then, 4 is the element in row 1 and column 1, therefore its position is given as a_{11}, 9 is the element in the 2nd row and the 3rd column, so its position is given as a_{34} . The position of elements in matrix X above can be presented as thus: $= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

SELF-ASSESSMENT EXERCISE

i. (a) State the number of rows and column each of the following matrices has;

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 & 9 \\ 1 & 6 & 3 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} x & \gamma & \emptyset \\ \alpha & y & z \\ q & r & w \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad D =$$

$$\begin{bmatrix} 1, & 2, & 3, & 4, & \ldots \ldots .. & 20 \\ 2 & & & & & \vdots \\ 3 & & & & & \vdots \\ \vdots & & & & & 2 \\ 20, & 19, & 18, & 17, \ldots & & 1 \end{bmatrix}$$

$E = \begin{bmatrix} 4 & 6 \end{bmatrix}$

(b) State the order of the matrix in each case above.

ii. State the position of (x) in each of the following matrices.

$$P = \begin{bmatrix} 2 & 0 & 6 \\ 1 & x & 4 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 \\ 6 \\ 7 \\ x \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 6 & 2 & 0 \\ 1 & 8 & 9 & 0 \\ a & b & c & d \end{bmatrix} \qquad V = \begin{bmatrix} 3 & 3 \\ 4 & 2 \\ 1 & x \\ 0 & 9 \end{bmatrix} R = \begin{matrix} 2 & 1 & 6 \\ 4 & x & 2 \end{matrix} \begin{bmatrix} 6 & 9 & 5 & 4 \\ 8 & & & \\ 9 & & & \\ 3 & 2 & 0 & 4 \end{bmatrix} T =$$

4 6 1

3.2 Types of Matrices

1) **Vector**

A vector is a matrix that has either one row or one column. When it has one column, e.g. = $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ It is called a column matrix or a column vector. Another example is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ The order of the first example is 3 by 1 while the order of the last example is 4 by 1. Generally, the order of a column vector or column matrix is n by 1 or n × 1. It should be noted that when a matrix has only one row, it is called a line matrix or a line vector. Examples are; $[a \quad b \quad c]$, $[4 \quad x \quad 3 \quad 2 \quad 7]$. In first example, the order of the matrix is 1 by 3 while in second example the order is 1 by 5. Generally, the order of a line matrix or line vector is 1 by n.

2) Square Matrix

A square matrix is a matrix that has the same number of rows and column. That is, a matrix is a square matrix when the number of rows it possesses equals the number of columns it has. A square matrix is any matrix whose order is m by n where m = n. Examples of a square matrix are:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 6 & 2 & 0 \\ 1 & 8 & 9 & 0 \\ a & b & c & d \end{bmatrix} \quad C = \begin{bmatrix} a & b & c & d & e \\ 1 & 2 & 3 & 4 & 5 \\ x & e & l & 8 & 6 \\ v & s & n & 0 & 5 \\ z & f & a & 9 & 3 \end{bmatrix} \quad D = \begin{bmatrix} x & c & v & b \\ y & f & m & h \\ 7 & h & 5 & g \\ j & e & 6 & f \end{bmatrix} \quad E =$$

$$\begin{bmatrix} 1, & 2, & 3, & 4, & \ldots\ldots.. & 20 \\ 2 & & & & & \vdots \\ 3 & & & & & \vdots \\ \vdots & & & & & 2 \\ 20, & 19, & 18, & 17, \ldots & & 1 \end{bmatrix}$$

In each of the examples above, the number of rows equals the number of columns. For instance, in matrix A, the number of rows is 2 and number of columns is 2. In matrix D, the number of row is 4 and the number of column is 4. In summary, A = 2 by 2 matrix B = 3 by 3 matrix C = 4 by 4 matrix D = 26 by 26 matrix Therefore, each of A, B, C, D and E are examples of a square matrices

3) Null or Zero Matrixes

This is a matrix that has all its elements as zero e.g., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [0 \quad 0 \quad 0]$

4) Diagonal Matrix

A diagonal matrix is a square matrix that has all its elements as zero with the exception of the elements in the leading diagonal which take other values outside zero. The elements in the leading diagonal can take other values outside zero. The elements in the leading diagonal of a square matrix are the elements in the north east direction or from left to right while the elements in the North West direction from the minor diagonal. Examples of diagonal matrices are:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Minor diagonal

Principal or Leading diagonal

Principal or Leading diagonal

5) Identity or Unit Matrix

This is a square matrix that has all its elements as zero with the exception of the elements in the leading or principal diagonal which take the value of 1. E.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6) Triangular Matrix

This is a matrix that has elements outside the triangular entries as zero. For example:

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ This is a lower triangular matrix

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ This is an upper triangular matrix.

7) Symmetrical Matrix

A matrix is said to be symmetrical if after being transposed, we obtained the original matrix back. To transpose a matrix, we replace the row elements by the column elements.

For example, if $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 9 \\ 1 & 5 & 2 \\ 1 & 6 & 7 \end{bmatrix}$

A^T (pronounced as A transposed) or A^1 = $\begin{bmatrix} 1 & -2 & 3 \\ -2 & -1 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

This is obtained by writing the elements on the horizontal axis e.g., 1, -2, 3 in the vertical spaces

Since A = A^T, A is said to be symmetrical. Similarly, BT = $\begin{bmatrix} 2 & 1 & 1 \\ 6 & 5 & 6 \\ 9 & 2 & 8 \end{bmatrix}$

You observe that B≠B^T, hence we say that B is not symmetrical.

8) Markov Matrix

This is a matrix obtained when the column or row elements adds up to 1. It should be noted that either each of the rows or each of the columns should add up to 1 and not all the rows and the columns at the same time. Examples of Markov Matrices are:

$$\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \quad \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -6 \\ 3 & 2 & -4 \\ 4 & -1 & -2 \end{bmatrix} \begin{matrix} 4 + 3 - 6 = 1 \rightarrow \\ 3 + 2 - 4 = 1 \rightarrow \\ 4 - 1 - 2 = 1 \rightarrow \end{matrix}$$

$$\downarrow \quad \frac{1}{4} + \frac{3}{4} = 1 \qquad \downarrow \quad \frac{2}{3} + \frac{1}{3} = 1$$

9) Scalar Matrices

A Scalar is a single number used to multiply each element of a matrix e.g. If X = $\begin{bmatrix} 2 & 4 & 3 \\ -1 & 7 & 8 \end{bmatrix}$

$2X = \begin{bmatrix} 4 & 8 & 6 \\ -2 & 14 & 16 \end{bmatrix}$ Hence '2' is a scalar. A scalar is therefore a 1 by 1 matrix.

10) Equal Matrices

Two matrices are said to be equal if they have the same order the same elements, elements occupy the same position and the elements have the same value. For example: If $A = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 8 \\ 0 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$ $E = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$

It should be noted that B is obtained by multiplying each element of A by 2. This does not make A to be equal to B. You should also note that A, C and D has the same order and the same elements with matrix A but the elements were not put in the same position. It should be noted that only matrices A and E are equal in all respects. Therefore, we say that A = E.

11) Singular Matrix

A singular matrix is a matrix whose determinant is zero. The concept of determinant shall be treated in full in the third unit of this module.

4. 0 Summary

A Matrix is a rectangular block of numbers arranged in rows and columns. Mathematical equations (linear) can be represented in matrix form. Various types of matrices include square matrix, null matrix, diagonal matrix, identity matrix, symmetrical matrices etc. All the types of matrices have their peculiar features in terms of the nature of the element, the order of the matrix and so on. Matrix is a compact way of presenting information in a rectangular array. Every matrix has an order which tells us the number of rows by the number of columns it has. Members of a matrix are its element. Each element has its own position in a matrix.

5.0 References/Further Readings/Web Resources

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6.0 SELF-ASSESSMENT EXERCISE

1 Define the term "matrix".

2 With the aid of examples, explain each of the following types of matrices:
(a) Line matrix (b) Diagonal matrix (c) Triangular matrix (d) Symmetrical matrix
(e) Square matrix

3 State 'True' or 'False'
(a) All identity matrices are square matrices (b) All square matrices are identity matrices
(c) The order of a column matrix is generally given as nx1. (d) The order of a matrix is the same as the position of elements in a matrix (e) All square matrices are symmetrical

4 Given the size of each of the following matrices

(a) A = [1 0 0 2] (b) = $\begin{bmatrix} 3 & 3 \\ 4 & 2 \\ 1 & x \\ 0 & 9 \end{bmatrix}$ C = [1] D = $\begin{matrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{matrix}$

5 Classify each of the following matrices as square, diagonal or identity, as appropriate

$A = \begin{bmatrix} 3 & 4 \\ 2 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 0 \end{bmatrix}$ $E = [1\ 2\ 3\ 4]$ $F = \begin{bmatrix} 9 & 0 \\ 0 & 8 \end{bmatrix}$

$G = \begin{bmatrix} 0 & 8 \\ 9 & 0 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6 How many elements are there in a matrix whose size is
(a) 3 × 1 (b) 1 × 3 (c) m × n (d) 5 × 4 (e) n × n.

7 State the order of each of these matrices and state the position of x in each case

(a) $\begin{bmatrix} 4 & 6 & 2 & 0 \\ 1 & 8 & X & 0 \\ a & b & c & d \end{bmatrix}$ (b) $_1\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & \\ 5 & x \end{bmatrix}$ (c) $4x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & x \end{bmatrix}$

UNIT 2 MATRIX OPERATIONS

Unit Structure

1.0 Introduction

Elements in matrices are usually presented in quantifiable terms. Most often, we use numbers, variables or parameters to represent them. As a result of this basic mathematical operations such as addition, subtraction and multiplication are easily carried out. However, in carrying out these matrix operations, certain rules principles and guidelines must be properly adhered to. This unit addresses the basic rules involved in adding, subtracting and multiplying matrices as a well as examining the basic identities in matrix operation.

2.0 Learning Outcome

After reading this unit, students will be able to:

- carry out addition and subtraction of matrices
- multiply compatible matrices
- state and show the basic identities in matrix operation.

3.0 Matrix Operations

3.1 Addition and Subtraction of Matrices

Two or more matrices can be added together or subtracted from one another if and if only they have the same number of rows and column. This implies that addition and subtraction of matrices is only possible. If the matrices are to be added together or subtracted from one another, they must have the same order.

Example 1

$$\text{If } A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 3 \\ 4 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ -1 & 2 \end{bmatrix}$$

Find (a) A + B

(b) A – C

Solution

$$\text{(a)}\quad A + B = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 6 & -1 \end{bmatrix}$$

The answer is obtained simply by adding up the corresponding terms in the rows and the columns.

$$\text{(b)}\quad A - C = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ -1 & 2 \end{bmatrix}$$

This is not possible because A and C are not in the same order. While A is a 2 by 2 matrix, C is a 3 by 2 matrix. Hence, addition or subtraction of matrices A and C is not possible.

Example 2

$$\text{Given that } A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & -9 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 7 & 8 \\ 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 & 6 \\ 7 & 8 & 9 \\ -1 & 2 & 3 \end{bmatrix}$$

Find (i) A + 2C

(ii) $\frac{1}{2}$A – 3C

(iii) A + 4B

Solution

$$\text{(i)}\qquad 2C = 2\begin{bmatrix} 3 & 2 & 6 \\ 7 & 8 & 9 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 12 \\ 14 & 16 & 18 \\ -2 & 4 & 6 \end{bmatrix}$$

$$\therefore A + 2C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & -9 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 12 \\ 14 & 16 & 18 \\ -2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 16 \\ 19 & 22 & 25 \\ 6 & -5 & 7 \end{bmatrix}$$

$$\text{(ii)}\qquad \frac{1}{2}A - 3C = \frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & -9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & 2 \\ \frac{5}{2} & 3 & \frac{7}{2} \\ 4 & -9/2 & 1/2 \end{bmatrix}$$

$$3C = 3\begin{bmatrix} 3 & 2 & 6 \\ 7 & 8 & 9 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 18 \\ 21 & 24 & 27 \\ -3 & 6 & 9 \end{bmatrix}$$

$$\therefore \tfrac{1}{2}A - 3C = \begin{bmatrix} 1 & \frac{3}{2} & 2 \\ \frac{5}{2} & 3 & \frac{7}{2} \\ 4 & -{}^{9}/_{2} & {}^{1}/_{2} \end{bmatrix} - \begin{bmatrix} 9 & 6 & 18 \\ 21 & 24 & 27 \\ -3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -\frac{9}{2} & -16 \\ -\frac{21}{2} & -21 & -\frac{57}{2} \\ 7 & -\frac{21}{2} & -\frac{17}{2} \end{bmatrix} = \begin{bmatrix} -8 & -4.5 & -16 \\ -10/5 & -21 & -28.5 \\ 7 & -10.5 & -8.5 \end{bmatrix}$$

(iii) A + 4B

$$4B = 4\begin{bmatrix} 3 & 2 \\ 7 & 8 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 28 & 32 \\ 8 & 12 \end{bmatrix}$$

$$\therefore A + 4B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & -9 & 1 \end{bmatrix} + \begin{bmatrix} 12 & 8 \\ 28 & 32 \\ 8 & 12 \end{bmatrix}$$

The addition of A and 4B is not possible simply because matrix A and matrix 4B are not in the same order while A is a 3 by 3 matrix, 4B is a 3 by 2 matrix.

SELF-ASSESSMENT EXERCISE

i. Given that $A = \begin{bmatrix} 3 & 1 & 6 \\ 1 & 0 & -7 \\ 2 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & -2 \\ 5 & 4 & -4 \\ -6 & 8 & -3 \end{bmatrix}$. Find A + B, A – B, B – A.

ii. Find the value of $\begin{bmatrix} 4 & 7 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 5 & 7 \end{bmatrix}$

iii. Simplify $\begin{bmatrix} 1 & -3 & 2 \\ 5 & 1 & -3 \end{bmatrix} - 3\begin{bmatrix} 2 & 5 & -3 \\ 1 & 2 & 1 \end{bmatrix}$

iv. Given that $A = \begin{bmatrix} 5 & 1 & 6 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{bmatrix}$. Find the trace of A.

(Hint: Trace of A is the sum of the diagonal element).

v. Given that $A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$.

Evaluate each of the following where possible:

(a) A + 2B (b) 2C + 3D (c) 2C + 3D (d) A – B + C – D

vi. Let $M = \begin{bmatrix} 3 & 1 & 6 \\ 1 & 0 & -7 \\ 2 & 4 & 2 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & 1 & 6 \\ 1 & 0 & -7 \\ 2 & 4 & 2 \end{bmatrix}$. Evaluate (a) M – N (b) 2M + 3N (c) 3M – 2N

3.2 Multiplication of Matrices

Two matrices A and B are conformable for multiplication to obtain A.B if and only if the number of columns of A is equal to the number of rows of B. Similarly, A and B are conformable for multiplication to obtain B.A if and only if the number of columns of B equals to the number of rows of A. therefore, is not equal to B.A.

Example 1

Consider the following matrices

$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 8 \\ 2 & 6 \\ 5 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 5 \\ 1 & 3 & -3 \end{bmatrix}$

(a) Which of these operations are possible?

(i) A.B (ii) B.A (iii) C.D (iv) B.C (v) D.A (vi) A.C (vii) B^2 (viii) C^2 (ix) D^2 (x)

A.D

(b) For each of the possible operations, obtain the results.

Solution

(a) (i) A.B is not possible because number of columns of A is not equal to number of rows of B ($C_A = 3$, $R_B = 2$ ∴ $C_A \neq R_B$).

(ii) B.A is possible because number of columns of B is equal to number of rows of A ($C_B = 2$, $R_A = 2$ ∴ $C_B = R_A$).

(iii) C.D is not possible because number of columns of C is not equal to number of rows of D. ($C_C = 2$, $R_D = 3$ ∴ $C_C \neq R_D$).

(iv) B.C is not possible because number of columns of B is not equal to number of rows of C ($C_B = 2$, $R_C = 3$ ∴ $C_B \neq R_D$).

(v) D.A is not possible because number of columns of D is not equal to number of rows of A. ($C_D = 3$, $R_A = 2$ ∴ $C_D \neq R_A$).

(vi) A.C is possible because number of columns of A is equal to the number of rows of C ($C_A = 3$, $R_C = 3$∴ $C_A = R_C$).

(vii) B^2 = B.B, it is possible because number of columns of B equals the number of rows of B ($C_B = R_B$ i.e., $C_B = 2$, $R_B = 2$).

(viii) C^2 is not possible i.e., C.C is not conformable for multiplication because number of columns of C is not equal to the number of rows of C ($C_C = 2$, $R_C = 3$ ∴ $C_C \neq R_C$).

(ix) D^2 = D.D is possible because the number of columns of D equals to the number of rows of D ($C_D = R_D$ i.e., $C_D = 3$, $R_D = 3$).

(x) A.D is possible because the number of columns of A equals to the number of rows of D ($C_A = 3$, $R_D = 3$ ∴ $C_A = R_D$).

Note: After finding out whether multiplication is possible or not, the result of the possible multiplication is obtainable following the steps below:

Take the first element in the row of the first matrix to multiply the first element in the column of the second matrix, put an addition sign, then, use the second element in the row of the first matrix to multiply the second element in the column of the second matrix and continue in that order, until the process is completed. After that, take the first element in the row of the first matrix to multiply the first element in the second column of the second matrix, put an addition sign and then use the second element in the row of the first matrix to multiply the element in the second column of the second matrix and so on until the multiplication is exhausted.

After exhausting the elements in the first row of the first matrix, the same thing is done to the elements in the second row of the first matrix. From question (a), the following operations are possible:

(i) B.A

(ii) A.C

(iii) B.B = B^2

(iv) D.D = D^2

(v) A.D

The solutions to each of them are as follow:

(i) $B.A = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

$$\begin{bmatrix} 2 \times 2 + 4 \times 5, & 2 \times 3 + 4 \times 6, & 2 \times 4 + 4 \times 7 \\ -3 \times 2 + 5 \times 5, & -3 \times 3 + 5 \times 6, & -3 \times 4 + 5 \times 7 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 30 & 36 \\ 19 & 21 & 23 \end{bmatrix}$$

(ii) $A.C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} -1 & 8 \\ 2 & 6 \\ 5 & 9 \end{bmatrix}$

$$\begin{bmatrix} 2X & -1 + 3 \times 2 + 4 \times 5, & 2 \times 8 + 3 \times 6 + 4 \times 9 \\ 5X & -1 + 6 \times 2 + 7 \times 5 & 5 \times 8 + 6 \times 6 + 7 \times 9 \end{bmatrix} = \begin{bmatrix} 24 & 70 \\ 42 & 139 \end{bmatrix}$$

(iii) B.B = $B^2 = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix} . \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 2 + 4 \times -3, & 2 \times 4 + 4 \times 5 \\ -3 \times 2 + 5 \times -3, & -3 \times 4 + 5 \times 5 \end{bmatrix} = \begin{bmatrix} -8 & 28 \\ -21 & 13 \end{bmatrix}$

(iv) D.D = $D^2 = \begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 5 \\ 1 & 3 & -3 \end{bmatrix} . \begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 5 \\ 1 & 3 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1 \times 1 + 6 \times 4 + 1 \times 8, & 1 \times 6 + 6 \times 2 + 8 \times 3, & 1 \times 8 + 6 \times 5 + 8 \times -3 \\ 4 \times 1 + 2 \times 4 + 5 \times 1, & 4 \times 6 + 2 \times 2 + 5 \times 5, & 4 \times 8 + 2 \times 5 + 5 \times -3 \\ 1 \times 1 + 3 \times 4 + -3 \times 1, & 1 \times 6 + 3 \times 2 + -3 \times 3, & 1 \times 8 + 3 \times 5 + -3 \times -3 \end{bmatrix}$$

$$= \begin{bmatrix} 33 & 42 & 14 \\ 17 & 53 & 27 \\ 10 & 3 & 32 \end{bmatrix}$$

(vi) A.D = $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} . \begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 5 \\ 1 & 3 & -3 \end{bmatrix}$

$$\begin{bmatrix} 2 \times 1 + 3 \times 4 + 4 \times 1, & 2 \times 6 + 3 \times 2 + 4 \times 3, & 2 \times 8 + 3 \times 5 + 4 \times -3 \\ 5 \times 1 + 6 \times 4 + 7 \times 1, & 5 \times 6 + 6 \times 2 + 7 \times 3, & 5 \times 8 + 6 \times 5 + 7 \times -3 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 30 & 19 \\ 36 & 63 & 49 \end{bmatrix}$$

SELF-ASSESSMENT EXERCISE

i. Let A, B and C matrices be defined as $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Find (i) A.B (ii) B.C (iii) A.C

ii. Given that $A = \begin{bmatrix} 10 & 12 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 4 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$. Find (i) A.B (ii) B.A

iii. Let A = $\begin{bmatrix} 2 & 4 \\ -6 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 4 & -1 \\ 0 & 3 \end{bmatrix}$

(a) State whether each of the following operations is/are conformable. (i) X.Y (ii) Z.Z (iii) Y.Z (iv) Z.Y (v) X2 (vi) Y2 (vii) Z.X (viii) X.Z

(b) Find the results of the conformable operations.

Find the results of each of the following operations.

$(a) = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$ $(b) = \begin{bmatrix} 4 & -4 & 0 \\ 1 & -2 & 6 \\ 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 6 & 13 \\ 0 & 7 & 0 \\ -4 & 8 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 3 & 1 \end{bmatrix}$

v. Evaluate the matrix products

(a) $\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 7 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 5 & 7 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0.5 & 0.3 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 0.2 & 0,6 \\ -0.3 & 0.4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 8 & 5 \\ 1 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 6 \\ 6 & 3 & 1 \end{bmatrix}$

(vi) Given that $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$ B = $\begin{bmatrix} 5 & 3 \\ 4 & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ -1 & 2 \end{bmatrix}$

Find (a) BA (b) CA (c) CB

3.3 Basic Identity in Matrix Operation

Addition, subtraction and multiplication of matrices exhibit some properties. These properties are equally exhibited when such matrix operation are carried out with some special types of matrices. Among the commonest identity in matrix operations are:

(i) Commutative law in matrix
(ii) Associative law in matrix
(iii) Distributive law in matrix
(iv) Identity and Null matrices relations
(v) Unique properties of matrices

(i) Commutative Law in Matrix

Matrix addition is commutative (that is, A + B = B + A) since matrix addition merely involves, he summing of corresponding elements of two matrices of the same order. Hence, the order in which the addition takes place is inconsequential (immaterial).

Similarly, A – B = -B + A

For example, if $A = \begin{bmatrix} 4 & 11 \\ 17 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 7 \\ 6 & 2 \end{bmatrix}$

$$A + B = \begin{bmatrix} 4+3 & 11+7 \\ 17+6 & 6+2 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 23 & 8 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 3+4 & 7+11 \\ 6+17 & 2+6 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 23 & 8 \end{bmatrix}$$

$$\therefore\ A + B = B + A$$

Likewise, $A - B = \begin{bmatrix} 4-3 & 11-7 \\ 17-6 & 6-2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 11 & 4 \end{bmatrix}$

In conclusion, $A + B = B + A$, $A - B = -B + A$

(ii)Associative Law of Matrix

Since (A – B) can be converted to matrix addition A + (-B), matrix subtraction is also commutative and associative. Matrix multiplication with few exceptions is not commutative (that is A.B ≠ B.A). Scalar multiplication however is commutative for multiplication. If three matrices X, Y and Z are conformable for multiplication, (XY)Z = X(YZ). Subject to this condition A(B + C) = AB + AC. The associative law of matrices state that (A + B) + C = A + (B + C).

For example, if $A = \begin{bmatrix} 6 & 2 & 7 \\ 9 & 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 9 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}$ *and* $C = \begin{bmatrix} 7 & 5 & 1 \\ 10 & 3 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 15 & 3 & 10 \\ 13 & 7 & 9 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 15 & 3 & 10 \\ 13 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 7 & 5 & 1 \\ 10 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 22 & 8 & 11 \\ 23 & 10 & 17 \end{bmatrix}$$

$$(B + C) = \begin{bmatrix} 16 & 6 & 4 \\ 14 & 5 & 14 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 6 & 2 & 7 \\ 9 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 16 & 6 & 4 \\ 14 & 5 & 14 \end{bmatrix} = \begin{bmatrix} 22 & 8 & 11 \\ 23 & 10 & 17 \end{bmatrix}$$

Therefore, $(A + B) + C = A + (B + C) = \begin{bmatrix} 22 & 8 & 11 \\ 23 & 10 & 17 \end{bmatrix}$

This illustrates that matrix addition is associative. It should be noted that matrix multiplication is associative, provided the proper order of multiplication is maintained (AB) C = A(BC).

(iii) Distributive Law

Matrix multiplication is distributive A(B + C) = AB + AC

For example, let $A = \begin{bmatrix} 4 & 7 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 9 \\ 5 \\ 8 \end{bmatrix}$

$$B + C = \begin{bmatrix} 6+9 \\ 5+5 \\ 1+8 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 9 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 4 & 7 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 10 \\ 9 \end{bmatrix} = 4 \times 15 + 7 \times 10 + 2 \times 9 = 148$$

$$AB = \begin{bmatrix} 4 & 7 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \\ 8 \end{bmatrix} = 4 \times 9 + 7 \times 5 + 2 \times 8 = 87$$

$\therefore$ [AB + AC] = A [B + C]

[61 + 87] = 148

Thus, for the set of matrix A, B, C, distributive law of matrix multiplication holds.

(iv) Identity and Null Matrices

An identity matrix I is a square matrix which has 1 for every element on the principal or leading diagonal from left to right and zero everywhere else. Then, In denotes n by n identity matrix. The multiplication of an identity matrix with any other matrix gives the matrix back. AI = IA = A Multiplication of the identity with itself gives the identity as a result I. $I = I^2 = I$

Therefore, any matrix for which $A = A^T$ or $A = A^1$ is said to be symmetrical. A symmetrical matrix for which A × A = A is termed an idempotent matrix. Therefore, an identity matrix is symmetric and idempotent.

It should be noted that given a square null matrix N, when N multiplies another matrix N, gives N i.e., BN = N. However, when a null matrix is added or subtracted from another matrix (B), gives the matrix B as the result i.e., B + N = B – N = N.

Illustration

Let $A = \begin{bmatrix} 7 & 10 & 14 \\ 9 & 2 & 6 \\ 1 & 3 & 7 \end{bmatrix} B = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} and\ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$AI = \begin{bmatrix} 7 & 10 & 14 \\ 9 & 2 & 6 \\ 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 \times 1 + 10 \times 0 + 14 \times 0, & 7 \times 0 + 10 \times 1 + 14 \times 0, & 7 \times 0 + 10 \times 0 + 14 \times 1 \\ [\,9 \times 1 + 2 \times 0 + 6 \times 0, & 9 \times 0 + 2 \times 1 + 6 \times 0, & 9 \times 0 + 2 \times 0 + 6 \times 1 \\ 1 \times 1 + 3 \times 0 + 7 \times 0, & 1 \times 0 + 3 \times 1 + 7 \times 0, & 1 \times 0 + 3 \times 0 + 7 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 & 14 \\ 9 & 2 & 6 \\ 1 & 3 & 7 \end{bmatrix} \therefore AI = A$$

$$BN = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 \times 0 + 12 \times 0 & 5 \times 0 + 12 \times 0 \\ 20 \times 0 + 4 \times 0 & 20 \times 0 + 4 \times 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore BN = N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B + N = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} = B$$

$$B - N = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} = B$$

$$I\,.\,I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \times 1 + 0 \times 0 + 0 \times 0, & 1 \times 0 + 0 \times 0 + 0 \times 0, & 1 \times 0 + 0 \times 0 + 0 \times 1 \\ 0 \times 1 + 1 \times 0 + 0 \times 0, & 0 \times 0 + 1 \times 1 + 0 \times 0, & 0 \times 0 + 1 \times 0 + 0 \times 1 \\ 0 \times 1 + 0 \times 0 + 1 \times 0, & 0 \times 0 + 0 \times 1 + 1 \times 0, & 10 \times 0 + 0 \times 0 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(v) Unique Properties of Matrices

When two singular matrices multiply themselves, the result may be a null matrix or a non-null matrix but another singular matrix. A singular matrix is the one in which a row or column is a multiple of another row or column.

Example

$$A = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix} B = \begin{bmatrix} 5 & 12 \\ 20 & 4 \end{bmatrix}$$

$$\text{AB} = \begin{bmatrix} 6 \times 12 & + & -12 \times 6, & 6 \times 6 & + & -12 \times 3 \\ -3 \times 12 & + & 6 \times 6, & -3 \times 6 & + & 6 \times 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A and B are both singular matrices but their product is a null matrix. Sometimes, the product of two singular matrices may not give a null matrix but another singular matrix. For example,

$$A = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$$

$$\text{AB} = \begin{bmatrix} 6 \times 12 + 12 \times 6 & 6 \times 6 + 12 \times 3 \\ 3 \times 12 + 6 \times 6 & 3 \times 6 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 144 & 72 \\ 72 & 36 \end{bmatrix}$$

$$\text{A.C} = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}$$

A.B gives another singular matrix. When a singular matrix is used to multiply two other different matrices independently, the result obtained may be the same which may neither be a null matrix nor a singular matrix.

For example, if $A = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} B = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} C = \begin{bmatrix} -2 & 1 \\ 4 & 2 \end{bmatrix}$

$$\text{A.B} = \begin{bmatrix} 4 \times 2 + 8 \times 2 & 4 \times 1 + 8 \times 2 \\ 1 \times 2 + 2 \times 2 & 1 \times 1 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 24 & 20 \\ 6 & 5 \end{bmatrix}$$

$$\text{A.C} = \begin{bmatrix} 4 \times -2 + 8 \times 4 & 4 \times 1 + 8 \times 2 \\ 1 \times -2 + 2 \times 4 & 1 \times 1 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 24 & 20 \\ 6 & 5 \end{bmatrix}$$

Even though B ≠ C, AB = AC. Unlike algebra, where multiplication of one number by two different numbers cannot give the same product in matrix algebra, multiplication of one matrix by two different matrices may and may not, produce the same result. In this case, A is a singular matrix.

SELF-ASSESSMENT EXERCISE

i. Given that $A = \begin{bmatrix} 4 & 11 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -11 \\ -1 & 4 \end{bmatrix}$. Verify that AB = I where I is a 2 X 2 identity matrix.

ii. Calculate $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Comment upon the effect of multiplying a matrix by $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

iii. Given the matrices: $A = \begin{bmatrix} 6 & 2 & 7 \\ 9 & 5 & 4 \end{bmatrix} B = \begin{bmatrix} 9 & 1 & 3 \\ 4 & 2 & h \end{bmatrix}$ *and* $C = \begin{bmatrix} 7 & 5 & N \\ 10 & 3 & 8 \end{bmatrix}$. Show that (A + B) + C = A + (B + C).

4,0 Summary

Matrix operation in terms of addition, subtraction and multiplication is strictly based on some rules. Once these rules are adhered to, matrices can be added together, subtracted from one another and also multiply one another. Some basic identity may evolve in the course of performing matrix operations. These identities are proven results of certain matrix operation which may be general or peculiar to some circumstances.

Two matrices can be added together or subtracted from one another if they have the same order. Multiplication of two matrices A and b to produce A.B is only possible if the number of columns of A equals the number of rows of B. However, A.B ≠ B.A. Like in the set theory, commutative law, associative law and distributive laws apply to all matrices provided the matrices are conformable and/or addition

A + B = B + A = Commutative law

A + (B + C) = (A + B) + C = Associative law

(AB)C = A(BC) = Associative law

A (B + C) = AB + AC = Distributive law

The product of an identity matrix with another matrix gives the other matrix while the product of an identity matrix with itself gives the identity matrix as the result. When a null matrix is added or subtracted from another matrix of the same order, the result obtained is the matrix which the null matrix is added to or subtracted. The product of two singular matrix may give a null matrix or a non-null matrix but another singular matrix. When a singular matrix multiplies two or other different matrices independently, the result obtained may be the same.

5.0 References/Further Readings/Web Resources

Anthony, C. & Roberts, D. (2006). Foundation Mathematics. (4th ed.). London: Pearson Hall Publishers.

Edwards, T. D. (2004). Introduction to Mathematical Economics. (3rd ed.). Delhi: Tata Mc-Graw Hills Publishing Limited.

Robert, S. & Geoff, B. (2008). Excellence in Mathematics for Senior Secondary School. Ibadan: Macmillan Publishers Limited.

6.0 SELF-ASSESSMENT EXERCISE

1. Given that $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $d = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix}$ $e = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ $d = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$

   Which of these matrices is it possible to add together? a. Using the matrices given above, find where possible, each of the following

   (i) B + E (ii) E – C (iii) D + F (iv) 4A (v) BD –E (vi) F – 2D (vii) ½ A (viii) B – ½ C (ix) B + C + E

2. Find x, y, z, w if: $\begin{bmatrix} 2 & 4 \\ 3 & z \end{bmatrix} + \begin{bmatrix} x & y \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ w & 0 \end{bmatrix}$

3. Given that $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}$. Find A.B, B.C, A.C, A^2, B^2, C^2, A^3.

4. Given that $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $d = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ e = $\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$. Find A.C, B.D, E.D, B.E.

5. If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}$. Find A^1, B^1, A.B, $A^1.B^1$, B.A.

6. (a) $\begin{bmatrix} a & b \\ x & t \end{bmatrix} + \begin{bmatrix} y & q \\ a & b \end{bmatrix}$ (b) $\begin{bmatrix} 5 & x-3 \\ x^2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2-x^2 & -1 \end{bmatrix}$. Solve.

7. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

   (i) Which of the following can be evaluated: AB, BA, AC, CA, BC and CB?

   (ii) Evaluate those which are possible

8. Let $A = \begin{bmatrix} 3 & 2 \\ 0 & -3 \end{bmatrix}$. (a) Find A2 +A (b) Find B if B + A2 = 3A

9. Let $M = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$. Evaluate the following (a) M2 + N2 (b) MN +NM (c) 2MN + 3NM

10. Let $A = \begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix}$ $and B = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$. Find A + B, A – B, 7A, -½B, 3A + 2B

11. Find (a) $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$ (c) $\begin{bmatrix} 14 & 9 \\ 20 & 1 \end{bmatrix} \begin{bmatrix} 19 \\ 87 \\ -73 \end{bmatrix}$

12. If $M = \begin{bmatrix} 7 & 8 \\ 2 & -1 \end{bmatrix}$ $and N = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$. Find NM and MN. Comment on your answer.

13. If $A = \begin{bmatrix} 1 & -7 \\ -1 & 2 \end{bmatrix}$ B = $\begin{bmatrix} 3 & 7 & 2 \\ -1 & 0 & 10 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & -1 & 1 \end{bmatrix}$ d = $\begin{bmatrix} 7 \\ 9 \\ 8 \\ 1 \end{bmatrix}$ e = $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

    Find if possible: 3A, A + B, A + C, 5D, BC, CD, AC, CA, 5A – 2C, D – E, BA, AB, A^2.

14. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} x \\ y \end{bmatrix}$ $C = \begin{bmatrix} 3 & 9 & 5 \\ 4 & 2 & 8 \end{bmatrix}$. Find where possible IA, AI,

BI, IB, CI, IC. Comment on your answers.

15. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}$. Find A + B, A – B, AB, BA?

16. Find (a) $\begin{bmatrix} -7 & 5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ (b) $\begin{bmatrix} 9 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

17. Given $A = \begin{bmatrix} 23 & -1 \\ 40 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$

18. a) Given that $A = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$

b) Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$, State and proof any two-identity familiar with A, B and C (you may introduce other matrices to conduct your proof).

19. Given that $A = \begin{bmatrix} 7 & 2 & 6 \\ 5 & 4 & 8 \\ 3 & 1 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 2 \\ 5 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 11 \\ 4 \\ 13 \end{bmatrix}$ $D = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$ $E = [8 \quad 1 \quad 10]$ $F = [13 \quad 2]$.

Determine for each of the following whether the products are defined i.e., conformable for multiplication. If so, indicate the dimensions of the product matrix.
(a) AC (b) BD (c) EC (d) DF (e) CA (f) DE (g) BD (h) CF (i) EF (j) B^2

20. Crazy Teddies ells 700 CDs, 400 cassettes and 200 CD players each week. The selling price of CD is \$4, cassette \$6 and CD players \$150. The cost to the shop is \$3.25 for a CD, \$4.75 for a cassette and \$125 for a CD player. Find the weekly profit.

Hint: $Q = \begin{bmatrix} 700 \\ 400 \\ 200 \end{bmatrix}$ $P = \begin{bmatrix} 4 \\ 6 \\ 150 \end{bmatrix}$ $C = \begin{bmatrix} 3.25 \\ 4.25 \\ 125.0 \end{bmatrix}$

UNIT 3 DETERMINANT OF MATRICES

Unit Structure

1.0 Introduction

Determinants are usually obtained for square matrices. It is the numerical value of the matrix. The determinant of $|A|$ of a 2 by 2 matrix called a second order determinant is derived by taking the product of the two elements on the principal diagonal and subtracting from it the product of the two element of the principal (minor diagonal).

Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = [a_{11}.\, a_{22}] - [a_{12}.\, a_{21}]$$

The determinant is a single number or scalar and it is found for square matrices. If the determinant of a matrix is equal to zero, the determinant is said to vanish and the determinant is termed singular.

The determinant of a 3 by 3 matrix

Given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. is called a third order determinant and it is obtained using a more elaborate procedure compared to a 2 by 2 matrix. Details on how the determinants of 3 by 3 matrices are obtained shall be discussed later in this unit.

2.0 Learning Outcome

At the end of this unit, you should be able to:

- obtain the determinant of 2 by 2 matrices
- obtain the determinant of 3 by 3 matrices
- obtain given value(s) in a matrix, provided determinant and other required values are given
- state the properties of determinants and prove them.

3.0 Determinant of Matrices

3.1 Determinant of 2 by 2 Matrices

To evaluate the determinant of a 2 by 2 matrix given as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the following procedures are followed:

(i) Multiply the elements in the leading or principal diagonal (i.e., a × d = ad).

(ii) Multiply the elements in the minor diagonals (i.e., c × d = cd).

(iii) Subtract (cd) from (ad) i.e. (ad) – (cd) gives the determinant of A, written as $|A|$.

Examples 1. Find the following (a) $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$

2. Find the value of x for which) $\begin{bmatrix} x & 2 \\ 3 & 8 \end{bmatrix} = 0$. which type of matrix is $\begin{bmatrix} x & 2 \\ 3 & 8 \end{bmatrix}$? How do you know?

Solution

1(a) $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ = (2 x 2) – (-4 x -1) = 4 − (4) = 0

(b) $\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix} = (-2 \times 1) - (3 \times 5) = -2 - 15 = -17$

2. $\begin{bmatrix} x & 2 \\ 3 & 8 \end{bmatrix}$ = 0 Therefore, 8x – (3 × 2) = 0 = 8x – 6 = 0 = 8x = 6 ∴ x = $\frac{8x}{6} = \frac{3}{4}$

The matrix is a singular matrix because its determinant is zero.

SELF-ASSESSMENT EXERCISE

i. Find the determinant of each of the following matrices

$A = \begin{bmatrix} 7 & 2 \\ 4 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ 9 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 8 & 7 \\ 9 & 2 \end{bmatrix}$

ii. In each of the following, find the value of x

(a) = $\begin{bmatrix} x & 1 \\ 2 & 7 \end{bmatrix} = 4$. (b) = $\begin{bmatrix} 3x & 2 \\ 2x & 3 \end{bmatrix} = \begin{bmatrix} 5x & 2 \\ 4 & -2 \end{bmatrix}$ (c) = $\begin{bmatrix} 5x & 2 \\ 3x & 1 \end{bmatrix} = 11$

3.2 Determinants of 3 by 3 Matrices

There are a number of approaches adopted in obtaining the determinant of a 3 by 3 matrix. Prominent among the approaches are:

(i) Diagonal method

(ii) Expansion method

(i) Diagonal Method

When a determinant is of the size 3 by 3, the elements of the first two columns are reproduced as if we put a mirror on the third column. This results into additional two columns to the already existing matrix. Hence, the determinant of the matrix is given as the difference between the addition of the products of the elements in the principal diagonals and the addition of the products of the elements in the non-principal diagonal for example, finds the determinant of

$$A = \begin{bmatrix} 4 & 6 & 3 \\ 3 & 5 & 6 \\ 2 & 8 & 4 \end{bmatrix}, \text{ using the diagonal method}$$

$$|A| = \begin{bmatrix} 4 & 6 & 3 & 4 & 6 \\ 3 & 5 & 6 & 3 & 5 \\ 2 & 8 & 4 & 2 & 8 \end{bmatrix}$$

$\therefore |A| = (4 \times 5 \times 6 + 6 \times 6 \times 2 + 3 \times 3 \times 8) - (2 \times 5 \times 3 + 8 \times 6 \times 4 + 4 \times 3 \times 6)$
$- (2 \times 5 \times 3 + 8 \times 6 \times 4 + 4 \times 3 \times 6) = 224 - 294 = -70$

(ii) Expansion Method:

The expansion method is used in the evaluation of 3 by 3 matrices and other higher determinants. Using this method, the following steps are followed:

Given that $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step I: Take the first element of the first row, and mentally delete the row and the column in which it appears. Then, multiply by the determinant of the remaining elements.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ i.e., } 11 \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = a_{11}[a_{22}.\, a_{33} - a_{32}.\, a_{23}]$$

Step II: Take the second element of the first row and mentally delete the row and column in which it appears. Then, multiply by times determinant of the remaining elements.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ i.e., } 12 \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = a_{12}[a_{21}.\, a_{33} - a_{31}.\, a_{23}]$$

Step III: Take the third element of the first row, a_{13} and mentally delete the row and the column in which it appears. Then, multiply a_{13} by thedeterminant of the remaining elements.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ i.e., } 12 \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = a_{13}[a_{21}.\, a_{32} - a_{22}.\, a_{31}]$$

Step IV: Obtain the sum of the results derived from step I, II, and III above.

Hence, $|A| = a_{11}[a_{22}.\, a_{33} - a_{32}.\, a_{23}] - a_{12}[a_{21}.\, a_{33} - a_{31}.\, a_{23}] + a_{13}[a_{21}.\, a_{32} - a_{22}.\, a_{31}]$

Note: The sign alternates as follows $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$ and so on.

Example 1

Given that $A = \begin{bmatrix} 4 & 6 & 3 \\ 3 & 5 & 6 \\ 2 & 8 & 4 \end{bmatrix} = 4 \begin{bmatrix} 5 & 6 \\ 8 & 4 \end{bmatrix} - 3 \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix} + 2 \begin{bmatrix} 6 & 3 \\ 5 & 6 \end{bmatrix}$

$= 4(20 - 48) - 3(24 - 24) + 2(36 - 15)$

$= 4(-28) - 3(0) + 2(21)$

$= -112 - 0 + 42 = -70$

A is a non-singular matrix because $|A| \neq 0$.

Note: The same solution is obtained using the two methods for the same question.

Example 2

Solve for x given that $\begin{bmatrix} x & x & x \\ 0 & x-2 & 2 \\ 0 & 2 & x-1 \end{bmatrix}$ is singular matrix.

Solution

If $\begin{bmatrix} x & x & x \\ 0 & x-2 & 2 \\ 0 & 2 & x-1 \end{bmatrix}$ is singular matrix.

$\begin{bmatrix} x & x & x \\ 0 & x-2 & 2 \\ 0 & 2 & x-1 \end{bmatrix} = 0 \therefore x \begin{bmatrix} x+2 & 2 \\ 2 & x-1 \end{bmatrix} - x \begin{bmatrix} 0 & 2 \\ 0 & x-1 \end{bmatrix} + x \begin{bmatrix} 0 & x-2 \\ 0 & 2 \end{bmatrix}$

$x\,[(x + 2)\,(x - 1) - 4] - x\,[0(x - 1) - (2 \times 0)] + x\,[0.2 - (0.\,x + 2)]$

$= x\,[x^2 + x - 2 - 4] - x\,[0 - 0] + x\,[0 - 0] = 0$

$x\,[x^2 + x - 6] = 0$

Factoring $x\,(x2 + 3x - 2x - 6) = 0$

$x\,(x\,(x + 3) - 2(x + 3)) = 0$

$x\,(x + 3)\,(x - 2) = 0$

Hence, x = 0, or x + 3 = 0

x = 0 – 3 = -3; or x – 2 = 0

x = 0 + 2 or x = +2

Therefore, for $\begin{bmatrix} x & x & x \\ 0 & x-2 & 2 \\ 0 & 2 & x-1 \end{bmatrix}$, x can either be 0, -3 or +2.

SELF-ASSESSMENT EXERCISE

i. Find the determinants of these matrices (a) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 4 \\ 1 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & 2 \\ 5 & 6 & 0 \\ 0 & 3 & 7 \end{bmatrix}$

ii. Find $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 0 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

iii. Solve for x in the matrix. $\begin{bmatrix} 0 & x & 0 \\ x-1 & 1 & 0 \\ 5 & 6 & x-1 \end{bmatrix} = 0$

iv. Use expansion method to obtain the determinant of B, given that

$$B = \begin{bmatrix} 4 & -2 & 6 & 3 \\ 1 & 0 & 2 & 4 \\ 0 & 2 & 4 & 1 \\ 2 & 6 & -1 & -3 \end{bmatrix}$$

3.3 Properties of Determinants

The following properties of determinants provide the ways in which a matrix can be manipulated to simplify its elements or reduced part of them to zero, before evaluating the determinant.

(i) Adding or subtracting any non-zero multiple of one row (or column) from another row (or column) will have no effect on the determinant.

$$\text{e.g., } A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = |A| = a_1b_2 - a_2b_1$$

$$\text{Also, } \begin{bmatrix} a + ka_2 & b_1 + kb_2 \\ a_2 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1 = |A|$$

(ii) Interchanging any two rows of columns or a matrix will change the sign, but not the absolute value of the determinant.

$$\text{e.g., } A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix}$$

$$|A| = (8 \times 2) - (6 \times 4) = 72 - 24 = 48$$

$$|B| = (4 \times 6) - (8 \times 9) = 24 - 72 = -48$$

(iii) Multiplying the elements of any row or column by a constant will cause the determinant to be multiplied by the same constant.

$$\text{e.g., } X = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \quad Y = \begin{bmatrix} 2\ x\ 5 & 1 \\ 4\ x\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 20 & 3 \end{bmatrix}$$

$$|X| = (2 \times 3) - (4 \times 1) = 6 - 4 = 2$$

$$|Y| = (10 \times 3) - (20 \times 1) = 30 - 20 = 10$$

Note: As a result of multiplication of the elements of the first column of X by 5 to obtain Y, the determinant of Y is 5 times the determinant of X.

(iv) The determinant of a triangular matrix i.e., a matrix with zero elements everywhere above or below the principal diagonal equals to the products of the elements in the principal diagonal.

e.g., If $R = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 8 & 2 \end{bmatrix}$ ∴ R is a lower triangular matrix, $|R| = 1 \times 4 \times 2 = 8$ or

$$|R| = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 8 & 2 \end{bmatrix} = 1\begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} - 0\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} + 0\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}$$

$$= 1(8 - 0) - 0(4 - 0) + 0(16 - 12)$$

$$= 1(8) - 0(4) + 0(4) = 8 - 0 + 0 = 8$$

(v) The determinant of a matrix equals to the determinant of its transpose. e.g., If $R = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 8 & 2 \end{bmatrix}$

$R^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{bmatrix}$ $\therefore |R^T| = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{bmatrix} = 1 \begin{bmatrix} 4 & 8 \\ 0 & 2 \end{bmatrix} - 2\ 1 \begin{bmatrix} 0 & 8 \\ 0 & 2 \end{bmatrix} + 31 \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

$= 1(8 - 0) - 2(0 - 0) + 3(0 - 0)$

$= 1(8) - 2(0) - 3(0) = 8 - 0 + 0 = 8$

$\therefore |R| = |R^T|$

(vi) If all the elements of any row or column are zero, the determinant is zero.

e.g., $K = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$ $\therefore |K| = (3 \times 0) - (4 \times 0)$

$M = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 3 \\ 6 & -8 & 4 \end{bmatrix}$ $\therefore |M| = 0 \begin{bmatrix} 1 & 3 \\ -8 & 4 \end{bmatrix} - 01 \begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix} + 0\ 1 \begin{bmatrix} 2 & 1 \\ 6 & -8 \end{bmatrix} =$

$= 0[4 - (-24)] - 0(8 - 18) + 0(-16 - 6)$

$= 0(4 + 24) - 0(-10) + 0(-22)$

$= 0 + 0 - 0 = 0$

(vii) If two rows or columns are identical or proportional, i.e., linearly dependent, the determinant is zero.

e.g., if $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ $\therefore |A| = (2 \times 6 - 4 \times 3) = 12 - 12 = 0$

if $B = \begin{bmatrix} 7 & 8 \\ 7 & 8 \end{bmatrix}$ $\therefore |B| = (7 \times 8) - (7 \times 8) = 0$

4.0 Summary

Determinants are applicable to square matrices only. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A written as $|A| = (a.\ d) - (c.\ d)$. There are alternative approaches of obtaining the determinant of a 3 by 3 matrices. Common among them are the diagonal method and the expansion method. The diagonal method involves addition to the original matrix, the elements in the first two column of a 3 by 3 matrix. Then, the sum of the products of elements in the non-leading diagonal is subtracted from that in the

leading diagonal. For the expansion method, $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ a $\begin{bmatrix} e & f \\ h & i \end{bmatrix}$ – b $\begin{bmatrix} d & f \\ g & i \end{bmatrix}$ + c $\begin{bmatrix} d & e \\ g & h \end{bmatrix}$. A number of properties of determinants provide the ways in which square matrices can be manipulated to simplify its element to give a specific result. This unit has exposed you to the techniques involved in obtaining determinant of square matrices. A good attempt has been made to expose you to the number of properties of determinants with adequate illustrations and examples.

5.0 References/Further Readings/Web Resources

Anthony, C. & Robert, D. (2006). Foundation Mathematics. (4th ed.). London: Pearson Hall Publisher.

Edward, T. D. (2004). Introduction to Mathematical Economics. (3rd ed.). Delhi: Tata, Mc-Graw Hills Publishers.

Elayn, M. (2003). Introductory Algebra. (2nd ed.). New Jersey: Prentice-Hall Inc.

Usman, A. S. (2005). Business Mathematics. Lagos: Apex Books Limited.

6.0 SELF-ASSESSMENT EXERCISE

1. Given that $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 0 \\ 5 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 & 1 \\ 5 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$. Find $|3A - 2B|$

2. Given that $\begin{bmatrix} 0 & 0 & x+1 \\ 1 & x-2 & 6 \\ x-3 & 6 & 7 \end{bmatrix} = 0$

3. Solve for x in the following (a) $\begin{bmatrix} 1 & x \\ -2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} x & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & x \end{bmatrix}$

4. Given that $\begin{bmatrix} 2 & 3 \\ x & x^2 \end{bmatrix}$ = 12. Find the possible values of x

5. Given that $\begin{bmatrix} \beta + 1 & -10 \\ 3 & \beta - 10 \end{bmatrix}$, is a singular matrix. Find the value of β

6. Given that $P = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$. Find $|P^2 + 2Q - 2PQ^2|$.

7. Outline any four properties of determinants with appropriate illustrations.

8. Given that $\begin{bmatrix} x - 4 & 5 \\ 6 & x + 3 \end{bmatrix}$ = $5x$ + 15. Find the value(s) of x

UNIT 4 INVERSE OF MATRICES AND LINEAR PROBLEMS

Unit Structure

1.0 Introduction

In the earlier units, we have added, subtracted and multiplied matrices. To divide matrices, we need some definitions. The identity matrix I is such that IA = AI = A for all matrices A which have the same size or order as I.

For 2 by 2 matrices, I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and for 3 by 3 matrices, I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

To divide by a number is equivalent to multiplying it by its reciprocal. For example, to divide by 2 is the same as multiply by ½. Hence, we define the inverse of a matrix A, so that dividing by A is the same as multiplying by its inverse. Recall that the inverse of 2 is ½ and the inverse of ½ is 2. Similarly, if we invert a matrix twice, we get the original matrix. So, the inverse of A-1 is A itself. Here, the superscript '- 1' should not be read as power but as a notation for the inverse of matrix.

It should be noted that an inverse A-1, which can be found only for a square, non-singular matrix is a unique matrix satisfying the relationship.

AA-1 = I = A-1A

Multiplying a matrix by its inverse reduces it to an identity matrix. Thus, the inverse matrix in linear algebra performs much the same function as the reciprocal in ordinary algebra. The formula for deriving the inverse is A-1 = $\frac{1}{|A|} \cdot Adj A$. To obtain the adjoint (Adj) of A, it may be necessary to obtain the minors and the co-factor first. This shall be treated in the later part of this unit.

2.0 Learning Outcome

After reading this unit, students will be able to:

- obtain the inverse of 2 by 2 matrices
- obtain the inverse of 3 by 3 matrices
- use the techniques of matrix inversion to solve linear simultaneous equations
- apply Crammer's rules as an alternative to matrix inversion in solving simultaneous linear equations.

3.0 Inverse Matrices and Linear Problems

Given that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse of A written as $A^{-1} = \frac{1}{|A|} \cdot Adj A$. Here, $=A = (ad) -$

(cb) while adj A in a 2 by 2 matrix requires that the elements of the leading diagonal are interchanged while the elements of the minor diagonals are multiplied by -1. Therefore, A^{-1} = $\frac{1}{ad-cb} \cdot (AdjA)$ = A^{-1} = $\frac{1}{ad-cb} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 1

Find the inverse of $A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$, Verify that AA-1 = I.

Solution

A-1 = $\frac{1}{|A|} \cdot AdjA$

$|A| = (3 \times 2) - (4 \times 1) = 6 - (-4) = 6 + 4 = 10$

$adjA = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = A{-}1 = \frac{1}{10} \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{10} & \frac{-1}{10} \\ \frac{4}{10} & \frac{3}{10} \end{bmatrix}$ or $\begin{bmatrix} 0.2 & -0.1 \\ 0.4 & 0.3 \end{bmatrix}$

$AA^{-1} = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 0.2 & -0.1 \\ 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 3 \times 0.2 + 1 \times 0.4 & 3 \times (-0.1) + 1 \times 0.3 \\ -4 \times 0.2 + 2 \times 0.4 & -4 \times 0.1 + 2 \times 0.3 \end{bmatrix}$

$= \begin{bmatrix} 0.6 + 0.4 & -0.3 + 0.3 \\ -0.8 + 0.8 & 0.4 + 0.6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as required.

SELF-ASSESSMENT EXERCISE

i. Find the inverse of the following matrices. (a) $\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 5 & 4 \end{bmatrix}$

ii. A matrix P has an inverse $P^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ Find the matrix P?

(Hint: Obtain$(P^{-1})^{-1} toget P$)

iii. Find the inverse of the matrix. $A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$, hence, show that A. A^{-1} = I

iv. Show that the matrix $P = \begin{bmatrix} 6 & -2 \\ -24 & 8 \end{bmatrix}$ has no inverse.

v. Find the inverse of each of the following matrices if they exist:

$(a) \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$ $(d) \begin{bmatrix} 4 & -2 \\ 9 & 5 \end{bmatrix}$ $(e) \begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$

3.2 Inverse of 3 by 3 Matrices

To obtain the inverse of a 3 by 3 matrix, the following steps are required

(i) Find the determinant of the matrix

(ii) Find the adjunct of the matrix. To obtain the adjunct of a matrix, we have to obtain the minor, the co-factor matrix and transpose of the co-factors.

(a) **The Minors**: The elements of a matrix remaining after the deletion process as described under expansion method of determinants form a sub-determinant of the matrix called a ***minor***.

(b) **The Co-factors Matrix**: This is a matrix in which every element formed from the determinants of the minor are multiplied by the alternative sins given as: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

(c) The transpose of a co-factor is automatically the adjoin of the matrix.

(iii) The product of the reciprocal of the determinant and the adjunct of the matrix gives the inverse of the matrix i.e. $A^{-1} = \frac{1}{|A|} \cdot AdjA$.

Example 1

Find the inverse of the matrix $A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Solution

$|A| = -3\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} - 0 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$= -3(3 - 1) - 0(1 - 2) + 2(1 - 6)$

$= -3(2) - 0(-1) + 2(-5) = -6 + 0 - 10 = -16$

Minors are obtained as follows

$a_{11}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$ $a_{12}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$ $a_{13}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$

$a_{21}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$ $a_{22}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$ $a_{23}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$

$a_{31}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$ $a_{32}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$ $a_{33}=\begin{bmatrix}-3 & 0 & 2\\ 1 & 3 & 1\\ 2 & 1 & 1\end{bmatrix}$

$\therefore a_{11}=\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}$ $a_{12}=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$ $a_{12}=\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$

$a_{21}=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$ $a_{22}=\begin{bmatrix}-3 & 2\\ 1 & 1\end{bmatrix}$ $a_{22}=\begin{bmatrix}-3 & 0\\ 1 & 3\end{bmatrix}$

$a_{31}=\begin{bmatrix}0 & 2\\ 3 & 1\end{bmatrix}$ $a_{32}=\begin{bmatrix}-3 & 2\\ 1 & 1\end{bmatrix}$ $a_{33}=\begin{bmatrix}-3 & 0\\ 1 & 3\end{bmatrix}$

The co-factor of gives the determinants of each minor arranged in matrix form.

$$\begin{bmatrix}2 & -1 & -5\\ 2 & -7 & 3\\ -6 & -5 & -9\end{bmatrix} - \text{This is obtained by using}$$

$$\begin{bmatrix}|a_{11}| & |a_{12}| & |a_{13}|\\ |a_{21}| & |a_{22}| & |a_{23}|\\ |a_{31}| & |a_{32}| & |a_{33}|\end{bmatrix}$$

Introducing alternative signs $\begin{bmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{bmatrix}$ to the cofactors of the minors' determinants, we obtain the matrix of the co-factors as: $\begin{bmatrix}2 & -1 & -5\\ 2 & -7 & 3\\ -6 & -5 & -9\end{bmatrix} = C_f$

To get the adjoin, we transpose the matrix of the co-factor C_f^T or C_f^1 = $\begin{bmatrix}2 & -2 & -6\\ 1 & -7 & 5\\ -5 & 3 & -9\end{bmatrix}$. This is the adjunct of matrix A. Therefore, $A^{-1} = \frac{1}{|A|} \cdot AdjA = \frac{\begin{bmatrix}2 & -2 & -6\\ 1 & -7 & 5\\ -5 & 3 & -9\end{bmatrix}}{-16} = \frac{-1}{16}\begin{bmatrix}2 & -2 & -6\\ 1 & -7 & 5\\ -5 & 3 & -9\end{bmatrix} = \begin{bmatrix}\frac{-2}{16} & \frac{-2}{16} & \frac{6}{16}\\ \frac{-1}{16} & \frac{-7}{16} & \frac{5}{16}\\ \frac{-5}{16} & \frac{3}{16} & \frac{-9}{16}\end{bmatrix}$

SELF-ASSESSMENT EXERCISE

i. Given $A = \begin{bmatrix}4 & 1 & -5\\ -2 & 3 & 1\\ 3 & -1 & 4\end{bmatrix}$. Find the inverse of A.

ii. a. Obtain the minor and matrix of the co-factor of the matrix given as: $P = \begin{bmatrix}-4 & 4 & 2\\ 2 & 1 & 1\\ 0 & 0 & 4\end{bmatrix}$

b. Find the inverse of the matrix from the minor and the cofactor obtained in (a) above.

iii. Find the determinant and the inverse of $A = \begin{bmatrix}3 & 2 & 1\\ 2 & 1 & -4\\ 1 & -1 & -1\end{bmatrix}$

3.3 The Use of Matrix Inversion in Solving Linear Equation

The concept of inverse matrix can be used in solving linear equations. The examples below illustrate how inverse matrix can be used to solve two linear simultaneous equation and three linear simultaneous equations.

Example

Use inverse matrix to solve the unknown in each of the following pairs of simultaneous equations.

(1) 4x – y = 13

2x – 5y = -7

(2) Ajao bought 3 pens, 2 books and 4 pencils at ₦19. In the same market, Tola bought 6 pens, 2 books and a pencil at ₦37 while Oye bought 1 pen, 2 books and 3 pencils at ₦10.

Required

(a) Represent the information in a matrix system.

(b) Use inverse method to obtain the unit price of a pen, a book and a pencil.

Solution

(1) 4x – y = 13

2x – 5y = -7

Writing in matrix form:

$$\begin{bmatrix} 4 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$A \qquad B \qquad C$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 13 \\ -7 \end{bmatrix}}{\begin{bmatrix} 4 & -1 \\ 2 & -5 \end{bmatrix}} = \begin{bmatrix} 4 & -1 \\ 2 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

Recall, $(A^{-1}) = \frac{AdjA}{|A|}$

$= \begin{bmatrix} 4 & -1 \\ 2 & -5 \end{bmatrix} = (4 \times 5) - (2 \times -1)$

$\therefore = -20 - (-2)$

$= -20 + 2 = -18$

$adj \begin{bmatrix} 4 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & 4 \end{bmatrix}$

Hence, $\begin{bmatrix} 4 & -1 \\ 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & -1 \\ -2 & 4 \end{bmatrix} \cdot \frac{1}{-18} = \begin{bmatrix} \frac{-5}{18} & \frac{-1}{18} \\ \frac{-2}{18} & \frac{4}{18} \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-5}{18} & \frac{-1}{18} \\ \frac{-2}{18} & \frac{4}{18} \end{bmatrix} \begin{bmatrix} 13 \\ -7 \end{bmatrix}$$

$x = \left[-\frac{5}{18} \quad X \quad 13 \quad + \quad \frac{1}{-18} \quad X \quad -7\right] = \frac{-65}{-18} - \frac{7}{-18} = \frac{-72}{-18} = +4$

$y = \left[\frac{-2}{18} \quad X \quad 13 \quad + \quad \frac{4}{18} \quad X \quad -7\right] = \frac{-26}{-18} - \frac{28}{-18} = \frac{-54}{-18} = 3$

$\therefore$ x = 4 and y = 3

Check: 4x – y = 13

4(4) – 3 = 16 – 3 =13

2x – 5y = -7

2(4) – 5(3) = 8 – 15 = -7

2. Let X_1 = unit price of pen

X_2 = unit price of book

X_3 = unit price of pencil

Ajao: $3X_1 + 2X_2 + 4X_3 = 19$

Tola: $6X_1 + 2X_2 + X_3 = 37$

Oye: $X_1 + 2X_2 + 3X_3 = 10$

Presenting this in a compact form, we have:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 19 \\ 37 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 6 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 19 \\ 37 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix}3&2&4\\6&2&1\\1&2&3\end{bmatrix} = 3\begin{bmatrix}2&1\\2&3\end{bmatrix} - 2\begin{bmatrix}6&1\\1&3\end{bmatrix} + 4\begin{bmatrix}6&2\\1&2\end{bmatrix}$$

$= 3(6-2) - 2(18-1) + 4(12-2)$

$= 3(4) - 2(17) + 4(10) = 12 - 34 + 40 = 52 - 34 = 18$

But, $\begin{bmatrix}3&2&4\\6&2&1\\1&2&3\end{bmatrix}^{-1} = \dfrac{adj\begin{bmatrix}3&2&4\\6&2&1\\1&2&3\end{bmatrix}}{\begin{vmatrix}3&2&4\\6&2&1\\1&2&3\end{vmatrix}}$

To obtain the $adj\begin{bmatrix}3&2&4\\6&2&1\\1&2&3\end{bmatrix}$, we need to obtain.

The minor first

$a_{11}=\begin{bmatrix}2&1\\2&3\end{bmatrix}$ $a_{12}=\begin{bmatrix}6&1\\1&3\end{bmatrix}$ $a_{13}=\begin{bmatrix}6&2\\1&2\end{bmatrix}$

$a_{21}=\begin{bmatrix}2&4\\2&3\end{bmatrix}$ $a_{22}=\begin{bmatrix}3&4\\1&3\end{bmatrix}$ $a_{22}=\begin{bmatrix}3&2\\1&2\end{bmatrix}$

$a_{31}=\begin{bmatrix}2&4\\2&1\end{bmatrix}$ $a_{32}=\begin{bmatrix}3&4\\6&1\end{bmatrix}$ $a_{33}=\begin{bmatrix}3&2\\6&2\end{bmatrix}$

$= \begin{bmatrix}4&17&10\\-2&5&4\\-6&-21&-6\end{bmatrix}$. The co-factor $\begin{bmatrix}4&17&10\\-2&5&4\\-6&-21&-6\end{bmatrix}$

Then, the adjunct = Transpose of co-factor which gives

$$adj = \begin{bmatrix}4&2&-6\\-17&5&21\\10&-4&-6\end{bmatrix} \therefore \begin{bmatrix}3&3&4\\6&4&1\\1&2&3\end{bmatrix}^{-1} = \frac{\begin{bmatrix}4&2&-6\\-17&5&21\\10&-4&-6\end{bmatrix}}{18} = \begin{bmatrix}\frac{4}{18}&\frac{2}{18}&\frac{-6}{18}\\\frac{-17}{18}&\frac{5}{18}&\frac{21}{18}\\\frac{10}{18}&\frac{-4}{18}&\frac{-6}{18}\end{bmatrix}$$

$$\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix} = \begin{bmatrix}\frac{4}{18}&\frac{2}{18}&\frac{-6}{18}\\\frac{-17}{18}&\frac{5}{18}&\frac{21}{18}\\\frac{10}{18}&\frac{-4}{18}&\frac{-6}{18}\end{bmatrix}\begin{bmatrix}19\\37\\10\end{bmatrix}$$

$X_1 = \frac{4}{18}$ X 19 + $\frac{2}{18}$ X 37 − $\frac{6}{18}$ X 10 = $\frac{76}{18} + \frac{74}{18} - \frac{60}{18} = \frac{90}{18} = 5$

$X_2 = \frac{-17}{18}$ X 19 + $\frac{5}{18}$ X 37 + $\frac{21}{18}$ X 10 = $\frac{-323}{18} + \frac{185}{18} + \frac{210}{18} = \frac{72}{18} = 4$

$X_3 = \frac{10}{18}$ X 19 + $\left(-\frac{4}{18}\ X\ 37\right) + \left(-\frac{6}{18}\ X\ 10\right) = \frac{190}{18} - \frac{148}{18} - \frac{60}{18} = -\frac{18}{18} = -1$

This implies that a pen was sold for ₦5, a book for ₦4 and ₦1 was given to the customer for buying a pencil.

Note: An alternative and simpler way of solving simultaneous linear equation with the aid of matrices is the use of Crammer's Rule.

Example 1: $2x_1 + 6x_2 = 22$

$-x_1 + 5x_2 = 53$.

Use crammer's rule to solve for x_1 and x_2.

Solution

For Crammer's rule, $x_i = \frac{|A_i|}{A}$.

The equation is written in matrix form as follows: - $\begin{bmatrix}2&6\\-1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}22\\53\end{bmatrix}$

$|A| = \begin{bmatrix}2&6\\-1&5\end{bmatrix}$ = (2 X 5) – (-1 X 6) = 10 – (-6) = 10 + 6 = 16

Replacing the first column of the coefficient matrix $\left(i.e., \begin{bmatrix}2&6\\-1&5\end{bmatrix}\right)$ with the column of constant, we have: $\begin{bmatrix}22&6\\53&5\end{bmatrix} = Ax_1$

$|Ax_1| = 22(5) - 6(53) = -208$. Thus, $x_1 = \frac{|A_i|}{A} = \frac{-208}{16} = 13$.

Likewise, replacing the second column of the original coefficient matrix with the column of constant,

we have: $\begin{bmatrix} 22 & 6 \\ 53 & 5 \end{bmatrix} = Ax_1 \therefore\ Ax_1 = 22(5) - 6(53) = -208$

Thus, $x_1 = |\frac{|Ax_1|}{|A|} = \frac{-208}{16} = -13$

Likewise, replacing the second column of the original coefficient matrix with the column of constant, we have: $|Ax_2| = \begin{bmatrix} 2 & 22 \\ -1 & 53 \end{bmatrix}$

$|Ax_2| = 2(53) - (-1 \times 22) = 106 + 22 = 128 \therefore x_2 = \frac{|Ax_2|}{|A|} = \frac{128}{16} = 8$

Example 2 Given that 5x – 2y + 3z =16

2x + 3y – 5z = 2

4x – 5y + 6z = 7

Use Crammer's rule to solve for x, y and z. we have: $\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 7 \end{bmatrix}$

$$x = \frac{\begin{bmatrix} 16 & -2 & 3 \\ 2 & 3 & -5 \\ 7 & -5 & 6 \end{bmatrix}}{\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix}} = \frac{16(18-25) + 2(12+35) + 3(-10-21)}{5(18-25) + 2(12+20) + 3(-10-21)} = \frac{-111}{-37} = 3$$

$$y = \frac{\begin{bmatrix} 5 & -16 & 3 \\ 2 & 2 & -5 \\ 4 & 7 & 6 \end{bmatrix}}{\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix}} = \frac{5(12+35) - 16(12+20) + 3(14-8)}{5(18-25) + 2(12+20) + 3(-10-21)} = \frac{-259}{-37} = 7$$

$$z = \frac{\begin{bmatrix} 5 & -2 & 16 \\ 2 & 3 & 2 \\ 4 & -5 & 7 \end{bmatrix}}{\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix}} = \frac{5(21+10) + 2(14-8) + 16(-10-21)}{5(18-25) + 2(12+20) + 3(-10-21)} = \frac{-185}{-37} = 5$$

SELF-ASSESSMENT EXERCISE

1. Use crammer's rule to solve for Q1, Q2 and π if $\begin{bmatrix} 0 & 1 & -10 \\ 1 & 0 & -2 \\ -10 & -2 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -240 \end{bmatrix}$

2. Use matrix algebra to solve for x and y in each of the following

(a) $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix}$

(b) 3x + y = 1

x + 2y = 2

(c) 4x + 5y = 2

2x + 3y = 0

3. Use inverse matrix to solve for x_1, x_2 and x_3. $x_1 + 4x_2$

$x_1 + 4x_2 + 3x_3 = 1$

$2x_1 + 4x_3 + 5x_3 = 4$

$x_1 - 3x_2 = 5 + 2x_3$

(Hint: Rearrange the equations properly first).

4.0 Summary

The unit has successful examined the procedures and techniques involved in obtaining the inverse of square matrices (2 by 2 and 3 by 3 matrices). The unit equally examines the alternative techniques of solving linear equations (2 variables and 3 variables) using both matrix inversion as well as the Crammer's rules.

The inverse of a square matrix is mathematically given as the quotient of the adjoin of the matrix and the determinant of the matrix or the product of the adjoin of the matrix and the reciprocal of the determinant.

$$A^{-1} = \frac{AdjA}{|A|} = \frac{1}{|A|} \text{ X } AdjA$$

Simultaneous linear equations could be solved using matrix inversion and the Crammer's rule. While the matrix inversion is a long process of solving linear equations, the Crammer's rule is shorter and it employs only the concept of determinants.

5.0 References/Further Readings/Web Resources

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6.0 SELF-ASSESSMENT EXERCISE

1. Solve the following systems of linear equation using determinant (Crammer's rule).

(a) $2x + 5y = 5$
$3x + 2y = 6$

(b) $x_1 + x_2 + x_3 = 6$
$2x_1 + x_2 + x_3 = 3$
$2x_1 + 3x_2 – 2x_3 = 2$

2. Use the matrix inverse to solve for x and y in each of the following

(a) $2x + 5y = 5$
$3x + 2y = 6$

(b) $4x + 9y = 18$
$6x + 4y = 2$

3. Solve for x, y and z using any approach of your choice.

$3x1 - x2 – 2x3 = 2$
$2x2 – x3 = -1$
$6x1 – 5x2 = 3$

(Hint: Rearrange the functional system first).

4. Use matrix inversion approach to solve for x, y and z.

$2x + y + 2z = 1$
$-x + 4y + 2z = 0$
$3x + y + z = 2$

5. Find, if it exists the inverse of each of the following

(a) $\begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 1 \\ 5 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} -4 & 6 & 8 \\ 2 & 1 & 7 \\ 1 & 5 & 9 \end{bmatrix}$

6. By method of co-factor expansion, find the inverse of $A = \begin{bmatrix} 4 & -3 & 0 \\ 2 & 2 & -4 \\ 1 & 2 & -2 \end{bmatrix}$

7. Amina, Usman who is a merchant dealer in Yobe state deals in three products namely Gold, Ball pen and Diamond. In her book of account were found sales for the month of August 2002 in the following order.

3 Gold, 5 Ball pens and 6 Diamond were sold at Damaturu.
5 Gold, 7 Ball pens and 4 Diamond were sold in Potiskum

Total products sold at Gashua were in the ratio 1: 2: 2 for God, Ball pen and Diamond respectively.
If the sales in Damaturu summed up to N6, 000, that of Potiskum summed up to N7, 000 and that of Gashua summed up to N6, 000.

Required

(i) Present the information using matrix algebra.
(ii) At what rate was each product sold.

8. The table below shows the number of hours required to make one of each of three products X, Y and Z in each of the three departments A, B and C. For example, it takes 1 hour of department A's time, 2 hours of department B's time and 3 hours of department C's time to make I unit of product X. similar relation as shown for other entries

Department	**Hours to make 1 unit of product**		
	X	Y	Z
A	1	2	3
B	2	3	4
C	3	4	4

Hours available in department A, B and C are 140, 200 and 230 respectively.

Required

(i) Set up a system of linear equations available department are to be used
(ii) Determine the product schedule using a matrix method