COURSE GUIDE

MTH 402
GENERAL TOPOLOGY II

Course Team

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Introduction

General Topology, MTH402 is a 3-credit unit. The course is a core course in second semester of 400L. It will take you 15 weeks to complete the course. You are to spend 45 hours of study for a period of 13 weeks while the first week is for orientation and the last week is for end of semester examination. You will receive the course material which you can read online or download and read off-line. The online course material is integrated in the Learning Management System (LMS). All activities in this course will be held in the LMS. All you need to know in this course is presented in the following sub-headings.

Course Competencies

By the end of this course, you will gain competency in the:

- basics of topology and how to apply them.

Course Objectives

The course objectives are to:

- Recognise and understand the basic concepts of topology.
- Be able to use these results to analyze concrete examples.
- Apply the concepts of topology to other fields of Mathematics.

Working through this Course

The course is divided into modules and units. The modules are derived from the course competencies and objectives. The competencies will guide you on the skills you will gain at the end of this course. So, as you work through the course, reflect on the competencies to ensure mastery. The units are components of the modules. Each unit is sub-divided into introduction, intended learning outcome(s), main content, self – assessment exercise(s), conclusion, summary, and further readings. The introduction introduces you to the unit topic. The intended learning outcome(s) is the central point which help to measure your achievement or success in the course. Therefore, study the intended learning outcome(s) before going to the main content and at the end of the unit, revisit the intended learning outcome(s) to check if you have achieved the learning outcomes. Work through the unit again if you have not attained the stated learning outcomes. The main content is the body of knowledge in the unit. Self-assessment exercises are embedded in the content which helps you to evaluate your mastery of the competencies. The conclusion gives you the takeaway while the summary is a brief of the knowledge presented in the unit. The final part is the further readings. This takes you to where you can read more on the topic presented in the unit. The modules and units are presented as follows:

Module 1

Unit1 Concepts of Topological Spaces
Unit 2 Separation Axioms

Module 2

Unit 1 Category and Separability
Unit 2 Compact Sets and Spaces
Unit Connectedness

References and Further Readings

1. Sidney A. Morris(2007), Topology Without Tears, https://www.topologywithouttears.net/topbook.pdf
2. Munkres,J. R. (1999), Topology, second edition, Pearson.
3. Freiwald, R. C. (2014), An Introduction to Set Theory and Topology, Washington University, St. Louis Saint Louis, Missouri.
4. Bourbaki, N. (1996), General topology, Part I, Addison Wesley, Reading, Mass.
5. Englking, R.(1989), Outline of general topology, Amsterdam.
6. Willard, S. (1970), General topology, Addison Wesley Publishing Company, Inc, USA.
7. Michael, S. (1972), Elementary Topology, Second edition, Gemidnami.

Presentation Schedule

The activities for each week are as presented in Table 1 while the required hours of study and the activities are presented in Table 2. Spend time to complete each unit hence module.

Table 1: Weekly Activities

Week	**Activity**
1	Course Orientation and Guide
2	Module 1 Unit 1
3	Module 1 Unit 1
4	Module 1 Unit 2
5	Module 1 Unit 2

6	Module 1 Unit 2
7	Module 2 Unit 1
8	Module 2 Unit 1
9	Module 2 unit 2
10	Module 2 Unit 2
11	Module 2 Unit 3
12	Module 2 Unit 3
13	Revision

The activities in Table I include facilitation hours.

Assessment

Table 2: Assessment

S/N	**Method of Assessment**	**Score(%)**
1	**Tutor Marked Assignment**	**30**
2	**Final Examination**	**70**
	Total	**100**

Assignment

This is tutor marked assignment you will be asked to do for assessment.

Examination

Finally, the examination will help to test the cognitive domain. The test items will be mostly application, and evaluation test items that will lead to creation of new knowledge/idea.

How to get the Most from the Course

To get the most in this course, you:

- need a personal laptop. The use of mobile phone only may not give you the desirable environment to work.
- must have regular and stable internet.
- have to install the recommended software.

- to work through the course step by step starting with the programme orientation.
- must do all the assessments following given instructions.
- must create time daily to attend to your study.

Online Facilitation

There will be two forms of facilitation – synchronous and asynchronous. The synchronous will be held through video conferencing according to weekly schedule. During the synchronous facilitation:

- There will be one hour of online real time contact per week making a total of 13 hours for thirteen weeks of study time.
- At the end of each video conferencing, the video will be uploaded for view at your pace.
- You are to read the course material and do other assignments as may be given before video conferencing time.
- The facilitator will concentrate on main themes.
- The facilitator will take you through the course guide in the first lecture at the start date of facilitation.

For the asynchronous facilitation, your facilitator will:

- Present the theme for the week.
- Direct and summarise forum discussions.
- Coordinate activities in the platform.
- Score and grade activities when need be.
- Support you to learn. In this regard personal mails may be sent.
- Send you videos and audio lectures, and podcasts if need be.

Read all the comments and notes of your facilitator especially on your assignments, participate in forum discussions. This will give you opportunity to socialise with others in the course and build your skill for teamwork. You can raise any challenge encountered during your study. To gain the maximum benefit from course facilitation, prepare a list of questions before the synchronous session. You will learn a lot from participating actively in the discussions.

Course Blub: This course presents the concepts of topology, which include separability, compactness and connectedness. Various results were proved with sufficient examples to guide learners.

Semester: Second
Course Duration: 13 Weeks
Required Hours for Study: 65Hours

Ice Breaker: Topology is the mathematical study of those properties of geometric forms that remain invariant under certain transformations, as bending or stretching.Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness, and continuity. They appear in virtually every branch of modern mathematics, which is an indication of the importance of studying this course.

MAIN COURSE

CONTENTS

MODULE 1 TOPOLOGICAL SPACES

The background needed for this study is explained in this module. Topology studies spaces by asking questions from a qualitative perspective. For example, some topological questions include: Is a space connected? Is a space simply connected? This question provides a technique for distinguishing between a sphere and a torus. For on the torus, there exist closed curves which cannot be ‘shrunk’ to a point. Is a space oriented? For example, the regular cylinder is oriented (as it has two sides), while the Möbius space is not (it has only one side). Note that there are easier ways to distinguish these two, namely by examining their boundaries. A topological space is an abstraction of metric spaces. In short, a topological space is a set equipped with the additional data necessary to make sense of what it means for points to be ‘close’ to each other. This will allow us to develop notions of limits and continuity.

Unit1 Concepts of Topological Spaces
Unit 2 Separation Axioms

UNIT 1 CONCEPTS OF TOPOLOGICAL SPACES

Unit structure

1.1 Introduction
1.2 Intended Learning Outcomes (ILOs)
1.3 Main Content
 1.3.1 Definitions and Examples of Basic Concepts
 1.3.2 Basis for Topology
 1.3.3 The Subspace Topology
 1.2.4 Closed Sets and Limit Points
1.4 Self-Assessment Exercise(s)
1.5 Conclusion
1.6 Summary
1.7 References/Further Reading

1.1 Introduction

In your study of metric spaces, you defined several important concepts, like limit point, closure of a set, etc. In each case, the definition is based on the idea of a neighborhood—or, to put it another way, on the idea of an open set. The concepts of neighborhood and open set were then defined by using the metric (or distance) in the specified space. However, you can approach things differently by defining a system of open sets in a given set X with sufficient properties, as opposed to adding a metric to the specified set X. This idea leads to the introduction of the notion of a topological space. Metric spaces are a special type of topological space that is particularly significant.

1.2 Intended Learning Outcomes (ILOs)

At the end of this unit, you shall be:

- able to define a topological space
- familiar with some important topological notions

1.3 Main Content

1.3.1 Definitions and Examples of Basic Concepts

Definition 1.2.1.1: Suppose X is a set. A topology on X is a collection τ of subsets of X such that the following properties are satisfied:

i. $\exists \emptyset \in \tau$ (The set X it self and the empty set $\emptyset$ arein τ)
ii. $X \in \tau$ (The set X is in τ)
iii. If $u \in \tau$ and $v \in \tau$ that is if $u_i \in \tau$ for all $i \in I$ then $\bigcup_{i \in I} u_i \in \tau$, i.e., $u_i \in \tau \Longrightarrow \bigcup_{i=1} y_i \in \tau, \bigcap_{i=1} k_i \in \tau$ (Arbitrary unions and finite intersections of elements of τ are in τ).

In theory, a topology τ on X is a set of subsets that can be closed by arbitrary union and finite intersection. The complement of an open set is a closed set.

Definition1.2.1.2: By a topological space, is meant a pair (X, τ) consisting of a set X and a topology τ defined on X.

A topological space is a pair made up of a set X and a topology defined on X, much like a metric space is a pair made up of a set X and a metric defined on X. Therefore, you need to specify a set X and a topology on X in order to specify a topological space. One and the same set can have various topologies attached to it, defining various distinct topological spaces. If there is no confusion, you may omit τ and refer to merely X as a topological space in the follow-up.

Definition1.2.1.3: The elements of the topologyτon$X$are called open sets

.

Example1.2.1.4 (The discrete topology): If X is a set, take τ_d to be the $P(X)$, power set of $X.\tau_d$ is clearly a topology on X, called the discrete topology. In the discrete topology, all subsets of X are open. It is the largest topology on X.

Example1.2.1.5 (The indiscrete topology): Let X be any non-empty set. The indiscrete topology on X is the family $\{0, X\}$.It is the smallest topology on X and (X, τ_t) is called the topological space of coalesced points. This is mainly of academic interest.

Example1.2.1.6 (Sierpinski topology): Let$X = \{a, b, c\}$. Many topologies on X can be defined. For example, you can define

$$\tau_s = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}\}$$

Then$\tau_s$is a topology on X called the Sierpinski topology.

Example 1.2.1.7 (Sierpinski space) The Sierpinski space$\mathbb{S}$ consists of two points $\{0, 1\}$ with the topology $\{\emptyset, \{1\}, \{0, 1\}\}$. The topology of the Sierpinski space is finer than the indiscrete topology $\{\emptyset, \{0, 1\}\}$ on $\{0, 1\}$ but coarser than the discrete topology $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ on $\{0, 1\}$.

Definition 1.2.1.8: Given two topologies τ_1 and τ_2 on the same set, we say that τ_1 is coarser than τ_2 if $\tau_1 \subseteq \tau_2$.
According to definition(3.4)you can observe that if τ is any topology on X, then

$$\tau_t \subset \tau \subset \tau_d$$

where$\tau_d$and $\tau_t$areas defined in examples(3.1)and(3.2).

Example1.2.1.9(Finite complement topology)

:Let$X$beaset,and let$\tau_f$be the collection of all subsets$U$of$X$such that $X\backslash U$ is finite or $U = \emptyset$,i.e., $\tau_f$is the collection of the form

$$\tau_f = \{U \subseteq X: X\backslash U is finite or X\backslash U = X\}$$

Then$\tau_f$is a topology of$X$called the finite complement topology (sometimes called the coffinite topology).

Example1.2.1.10: Let Xbeaset, and let $\tau_c$be the collection of subsets$U$of Xsuch that$X\backslash U$ is either countable or is$X$,i.e., $\tau_c$is a collection of the form

$$\tau_c = \{U \subseteq X: X\backslash U is at most countable or X\backslash U = X\}$$

Then $\tau_c$is a topology on X.

Theorem 1.2.1.11: The intersection $\tau = \bigcap_{\alpha\in\Delta} \tau_\alpha$ of topologies $\{\tau_\alpha\}_{\alpha\in\Delta}$ on X is itself a topology in X (where Δ is some indexing set).

Proof: You are required to verify the three (3)axioms of topology on a set$X$for

$$\tau = \bigcap_{\alpha\in\Delta} \tau_\alpha$$

Given that $\{\tau_\alpha\}_{\alpha\in\Delta}$ is family of topologies on X.

Therefore, we proceed as follows:

1. Since $\tau_\alpha$is a topology on$X$,for each $\alpha \in \Delta$, there exist$\emptyset$and Xineachτ_α, sothat

$$\emptyset, X \in \bigcap_{\alpha\in\Delta} \tau_\alpha =: \tau$$

2. Let$\{U_i\}_{i\in I}$bea collection of elements ofτ,where$I$is some index ingset.Let

$$U = \bigcup_{i\in I} U_i$$

You have to show that $U \in \tau$.

But you already have that for each $i \in I$, $U_i \in \tau$ implies that $U_i \in \tau_\alpha$ for fixed $\alpha \in \Delta$. Since τ_α is

a topology on X, $U = \bigcup_{i\in I} U_i \in \tau_\alpha$ for $\alpha \in \Delta$. Therefore, by taking intersections over $\alpha \in \Delta$, we have

$$U = \bigcup_{i\in I} U_i \in \bigcap_{\alpha\in\Delta} \tau_\alpha =: \tau$$

ie.,$U \in \tau$.

3. To verify axiom (3), it is enough to do it for two sets U_1 and U_2in τ. The results follow by induction on n. Therefore, take two sets U_1 and U_2in τ and let

$$U = U_1 \cap U_2$$

You have to show that $U \in \tau$. But$U_1, U_2 \in \tau$implies that $U_1, U_2 \in \tau_\alpha$ for each $\alpha \in \Delta$. Thus $U = U_1 \cap U_2 \in \tau_\alpha$ since eachτ_α, $\alpha \in \Delta$is a topology on X. Hence,

$$U = U_1 \cap U_2 \in \bigcap_{\alpha\in\Delta} \tau_\alpha =: \tau$$

i.e, $U \in \tau$. Therefore, the proof is over.■

1.2.2 Basis for Topology

For each example in the preceding section, you were able to specify the topology by describing the entire collection τ of open sets. This is usually difficult in general. In most cases, you will need to specify instead a smaller collection of subsets of X and then define the topology in terms of this collection.

Definition 1.2.2.1 (Basis): Let X be a set. A basis for a topology on X is a collection B of subsets of X (called basis elements) such that
1. For each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, or equivalently $X = \bigcup_{B \in \mathcal{B}} B$.
2. If $x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Definition 1.2.2.2(Topology generated by a Basis): If B satisfies the above two conditions, then we define the topology τ generated by B as follows:

A subset U of X is in τ (i.e.,U is open) if for each $x \in U$, there exists a basis element $B \in B$ such that$x \in B \subset U$. That is to say that τ is a collection of the form

$$\tau := \{U \in X : U = \emptyset \; or \; if \; x \in U, \text{there exists } B \in \mathcal{B} \text{ such that } x \in B \subset U\}$$

You can easily verify that τ is a topology on X. Note that each basis element is open.

Example 1.2.2.3: Let $B = \{(a,b): a,b \in \mathbb{R}, a < b\}$. Then B is a basis for a topology on$\mathbb{R}$ called the standard or Euclidean topology on $\mathbb{R}$.

Example 1.2.2.4: Let$B^0 = \{[a,b): a,b \in \mathbb{R}, a < b\}$. Then B^0 is a basis for a topology on $\mathbb{R}$ called the lower limit topology on $\mathbb{R}$.

Example 1.2.2.5: Let $B = \{\{x\}: x \in X\}$. Then B is a basis for the discrete topology on X.

Proposition 1.1.2.6: Let X be a set, and let B be a basis for a topology τ on X. Then τ equals the collection of all unions of elements of B.

Proof: Let $\{B_i\}_{i \in I}$be a collection of elements of B. Then for each$i \in I$, $B_i \in \tau$ (because each B_i is open). Since τ is a topology,

$$\bigcup_{i \in I} B_i \in \tau$$

Conversely, let $U \in \tau$, and let $x \in U$. B is a basis for τ implies there exist $B_x \in B$ such that $x \in B_x \subset U$. This implies that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B_x \subset U$$

Thus, $\bigcup_{x \in U} B_x$, so that U is a union of elements of B.■

Example 1.2.2.7: Let $X = \{a, b, c, d, e, f\}$ and $\tau^0 := \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then $B = \{\{a\}, \{c, d\}, \{b, c, d, e, f\}\}$ is a basis for τ^0 as$B \subset \tau^0$ and every element of τ^0 can be expressed as a union of elements of B.
Note thatτ^0itself is also a basis for τ^0.

So far, you have seen that when you are given a basis, you can define a topology. But the following example tells you that you have to be very careful when you have an arbitrary collection of subsets of a set X.

Example 1.2.2.8 Let $X = \{a, b, c\}$ and $B = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}\}$. Then B is not a basis for any topology on X. To see this, suppose that B is a basis for some topology τ. Then τ consists of all unions of sets in B. That is,

$$\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

However, τ is not a topology since $\{a, b\} \cap \{b, c\} = \{b\}$, $b \in \tau$. So τ does not have property (3) of Definition 3.1. This is a contradiction, and so your supposition is false. Thus B is not a basis for any topology on X.
In view of the above example, the question of interest now is; under what condition is a collection C of subsets of X a basis for a topology on X? The answer to this question is provided by the next proposition.

Proposition 1.2.2.9: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open subsets of X such that for each open set U of X and each $x \in U$, there exists $C \in \mathcal{C}$ such that

$$x \in C \subset U$$

Then $\mathcal{C}$ is a basis for a topology of X.
When topologies are given by basis, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than the other. One such criterion is the following proposition.

Proposition 1.2.2.10: Let B and B^0 0 be basis for the topologies τ and τ^0, respectively on X. Then the following are equivalent:
1. τ^0 is finer that τ.
2. For each $x \in X$ and each basis element $B \in B$ containing x, there exists a basis element B^0 such that $x \in B^0 \subset B$.

Proof:(1) $\Longrightarrow$ (2). Let $x \in X$ and $x \in X$ such that $x \in$. You know that $B \in \tau$ by definition and that $\tau \subset \tau^0$ by condition (1). Therefore, $B \in \tau^0$.

Since τ^0 is generated by B^0, then there exists an element $B^0 \in B^0$ such that $x \in B^0 \subset B$.

(2) $\Longrightarrow$ (1). Given an element $U \in \tau$. Our goal is to show that $U \in \tau^0$. So let $x \in U$. Since B generatesτ, there is an element $B \in B$ such that $x \in B \subset U$. By condition (2) there exists $B^0 \in B^0$ such that $x \in B^0 \subset B$. Then $x \in B^0 \subset U$. So $U \in \tau^0$, by definition.

Definition 1.2.2.11 The Metric Topology

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on a set. Topologies given in this way lie at the heart of modern analysis, for example. In this section, you shall be introduced with the metric topology and some of its examples.

Definition 1.2.2.12: A metric space is an ordered pair (X, d) where X is a set and d is a metric on X. i.e., a function

$$d: X \text{ x } X \rightarrow \mathbb{R}$$

Such that for any$x, y, z \in X$, the following holds:

i. $d(x, y) \geq 0$ $\forall$ x, y $\in$ X (Positivity)

ii. $d(x, y) = 0$ if and only if $x = y$ (By definiteness)

iii. $d(x, y) = d(y, x)$ $\forall\, x, y \in X$ (Symmetry)

iv. $d(x, y) \leq d(x, z) + d(z, y)$ $\forall x, y, z \in X$(Triangular inequality)

Given a metric d on a set X, (X, d) is a metric space and the number $d(x, y)$ is called the distance between x and y in the metric d.

Example 1.2.2.13: The most important example is the set $\mathbb{R}$ of real numbers with the metric $d(x, y) := \left| x - y \right|$. Recall the absolute value of a real number

$$|x| = \begin{cases} x, & if x > 0 \\ -x, & if x < 0 \end{cases}$$

Observe that

$x \leq |x|$ and $-x \leq |x|$ for $x \in \mathbb{R}$.$(*)$

It is easy to see that d satisfies the first two conditions of the metric. The triangle inequality follows form the triangle inequality of the absolute value:

$|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}(**)$

Let us quickly review a proof assuming the order relation on $\mathbb{R}$:

Case 1: Let $|x + y| = x + y$. Then $|x + y| = x + y \leq |x| + |y|$ by $(*)$.

Case 2: Let $|x+y| = -(x+y)$. We have $|x+y| = -x-y \leq |x| + |y|$ by $(*)$.

We have completed the proof of the triangle inequality $(**)$ for the absolute value. Also, note that the equality occurs in $(**)$ if x and y are both nonnegative or both non-positive. Assume that the equality occurs in the triangle inequality. Let us further assume that Case 2 occurs.

Then$|x+y| = -x-y = |x| + |y|$ holds so that $(|x| + x) + (|x| + y) = 0$. The terms on the left side of this equation are nonnegative so we conclude that$|x| = -x$ and $|y| = -y$. Hence both x and y are nonpositive. Similar analysis of Case 1 yields that both x and y are nonnegative. It is now an easy matter to derive the triangle inequality for d:

$$d(x,z) = |x-z|$$
$$= |(x-y) + (y-z)|$$
$\leq |x-y| + |y-z|$ (by triangle inequality for | |)
$$= d(x,y) + d(y,z)$$

We refer to d as the absolute value metric.

Definition 1.2.2.14: Let (X,d) be a metric space. Let $x \in X$and $r > 0$. The subsets
$B_d(x,r) := \{ y \in X : d(x,y) < r\}$ and $B_d[x,r] := y \in X: d(x,y) \leq r$
are respectively called the open and closed balls centered at x with radius r with respect to the metric d. We use this notation only when we want to emphasize that the metric under consideration is d. Otherwise, we denote $B_d(x,r)$ by $B(x,r)$ when there is no source for confusion. Similarly $B_d[x,r]$ will denote $B[x,r]$.

Example 1.2.2.15: Let $\mathbb{R}$ be with the standard metric. Then we claim that $B(x,r) = (x - r, x + r)$. For, if $y \in B(x,r)$ iff $d(x,y) < r$ iff $|x-y| < r$ iff $y \in (x - r, x + r)$.

Definition 1.2.2.16: If d is a metric on X then the collection of all $\varepsilon - balls B_d(x,\varepsilon)$ for $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the metric topology induced by d.

Lemma 1.2.2.17: Let $B_d(x,\varepsilon)$ be a $\varepsilon - ball$ in a topological space with the metric topology and metric d. Let $y \in B_d(x,\varepsilon)$. Then there is $\delta > 0$ such that $B_d(x,\delta) \subset B_d(x,\varepsilon)$.

Proof: Define $\delta = \varepsilon - d(x,y)$. Then for $z \in B_d(y,\delta)$ we have $d(y,z) < \delta = \varepsilon - d(x,y)$ and so, by the Triangle Inequality,

$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon$. So $y \in B_d(x,\varepsilon)$ and $B_d(y,\delta) \subset B_d(x,\varepsilon)$.

Remark 1.2.2.18: We must verify that the collection of sets in the previous definition really satisfies the definition of basis of a topology.

i. Firstly, every element $x \in X$ is in a basis element, say $B(x,1)$.
ii. Secondly, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. Then from Lemma 3.1, there are $\delta_1 > 0$ and $\delta_2 > 0$ with $B(y,\delta_1) \subset B_1$ and $B(y,\delta_2) \subset B_2$. With $\delta = min\{\delta_1,\delta_2\}$ we then have $B(y,\delta) \subset B_1 \cap B_2$. Since $B(y,\delta)$ is a basis element then the second part of the definition of basis is satisfied.

Example 1.2.2.19. Given a nonempty set X, define metric

$$d(x,y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

The topology reduces the discrete topology on X (It is in fact a metric).
Note: It is easy to check that d is a metric on X. The topology induced by this metric is the discrete topology; the basis element for example consists of the points x alone.

Example 1.2.2.20: The standard metric on the real numbers $\mathbb{R}$ is defined by $d(x,y) = |x-y|$. It is easy to check that d is a metric.

Definition 1.2.2.21 Product Topology

The product topology will be covered briefly here, but a more in-depth examination of this type of topology will be done in later units. Assume that X and Y are topological spaces. A topology on the cartesian product X x Y can be defined in a standard way. We now consider this topology and investigate some of its properties.

Lemma 1.2.2.22: Let X and Y be two topological spaces. Let B be the collection of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y. i.e.,

$$B := \{U \times V : U\ is\ open\ in\ X\ and\ V\ is\ open\ in\ Y\}$$

Then B is basis for a topology on $X \times Y$.

Proof: The first condition is trivial, since $X \times Y$ is itself a basis element. The second condition is almost easy, since the intersection of any two basis element $U_1 \times V_1$ and $U_2 \times V_2$ is another basis element. For $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, and the later set is a basis element because$U_1 \cap U_2$ and $(V_1 \cap V_2)$ are open in X and Y, respectively. ■

Definition 1.2.2.23: Let X and Y be topological spaces. The product topology on set $X \times Y$ is the topology having as basis the collection B of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

It is easy to check that B is not a topology itself on $X \times Y$.
You may now ask, what if the topologies on X and Y are given by basis? The answer to this question is in what follows.

Theorem 1.2.2.24: If $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ is the basis for the topology on Y, then the collection

$$D = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology on $X \times Y$.

Proof: You can use proposition 3.2. Given an open set W of $X \times Y$ and a point $(x, y) \in X \times Y$ of W, by definition of the product topology, there exists a basis element $U \times V$ such that $(x, y) \in U \times V \subset W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, you can choose an element $B \in \mathcal{B}$such that $x \in B \subset U$ and an element $C \in \mathcal{C}$ such that $y \in C \subset V$. So $(x, y) \in B \times C \subset U \times V \subset W$. Thus the collection D meets the criterion of proposition 3.2. so D is a basis of $X \times Y$. ∎

Example 1.2.2.25: You have the standard topology of $\mathbb{R}$. The product topology of this topology with itself is called the Product topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of open sets of $\mathbb{R}$, but the theorem we just proved tells us that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$ will also serve as a basis for the topology of $\mathbb{R}^2$. Each such set can be pictured as the interior of a rectangle in $\mathbb{R}^2$. It is sometimes useful to express the product topology in terms of subasis. To do this, we just define certain functions called projections.

Let $\pi_1 : X \times Y \to X$ be defined (pointwise) by the equation $\pi_1((x, y)) = x$. Let $\pi_2 : X \times Y \to Y$ be defined by the equation$\pi_2((x, y)) = y$. The maps π_1 and π_2 are the projections of $X \times Y$ onto its first and second factors, respectively.

The word *onto* is used because they are surjective (unless one of the spaces X *or* Y happens to be empty, in which case $X \times Y$ is empty and your whole discussion is empty as well).

If U is an open subset of X, then $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = \mathrm{X} \times \mathrm{V}$, which is also open in $X \times Y$. The intersection of these two sets in the set $U \times V$. This fact leads to the following theorem.

Theorem 1.2.2.26: The set

$$S = \{\pi_1^{-1}(V): U \; is \; open \; in \; X\} \cup \{\pi_2^{-1}(V): V \; is \; open \; in \; Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof: Let τ denote the product topology on $X \times Y$, let τ_0 be the topology generated by S. Since $S \subset \tau$ then arbitrary unions of finite intersections of elements of S stay in τ. Thus $\tau_0 \subset \tau$. On the other hand, every basis element $U \times V$ for the topology τ is a finite intersection of elements of S, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

Therefore $U \times V \in \tau_0$, so $\tau \subset \tau_0$ as well.■

1.2.3 The Subspace Topology

Definition 1.2.3.1: Let X be a topological space with topology τ. If Y is a subset of X, the collection

$$\tau_Y = \{Y \cap U : U \in \tau\}$$

is a topology on Y, called the subspace topology. With this topology. Y is called a subspace of X; it's open sets consists of all intersection of open sets of X with Y.

Lemma 1.2.3.2: If $\mathcal{B}$ is a basis for the topology on X, the collection

$$\mathcal{B}_Y = \{B \cap Y: B \in \mathcal{B}\}$$

is a basis for the subspace topology in Y.

Proof: Let U be an open set of X and $y \in U \cap Y$. By definition of basis, there exists $B \in \mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from proposition 3.2 that $B_Y$is a basis for the subspace topology on Y.■

When dealing with a space X and a subspace Y of X, you need to be careful when you use the term open set. The question is do you mean an element of the topology of Y or an element of the topology on X? The following definition is useful. If Y is a subspace of X, the set U is open in Y (or open relative to Y) if it belongs to the topology of Y: this implies in particular it is a subspace of Y. There is a special situation in which every open set in Y is also open in X.

Lemma 1.2.3.3: Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.

Proof: Since U is open in Y, $U = V \cap Y$ for some V open in X. Since Y and V are both open in X, so is $V \cap Y$. ■

Proposition 3.3.4: Let A be a subspace of X and B a subspace of Y. Then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

1.2.4 Closed Sets and Limit Points

Now that you have a few examples at hand, you can proceed to see some of the basic concepts associated with topological space. In this section, you shall be introduced to the notion of closed sets, interior, closure and limit point of a set.

Definition 1.2.4.1 (Closed Sets): A subset A of a topological space$X$ is said to be closed if $X\backslash A$, the complement of A in X is open.

Example 1.2.4.2: The subset $[a,b]$ of $\mathbb{R}$ is closed because its complement $R\backslash[a,b] = (-\infty,a) \cup (b,\infty)$is open. Similarly $[a,+\infty)$ is closed.

Example 1.2.4.3: Consider the following subset of the real line: $Y = [0,1] \cup (2,3)$, in the subspace topology. In this space, the set $[0,1]$ is open, since it is the intersection of the open set $-\frac{1}{2},\frac{3}{2}$ of $\mathbb{R}$ with Y. Similarly, $(2,3)$ is open as subset of Y. Since $[0,1]$ and $(2,3)$ are complement in Y of each other, you can conclude that both are closed as subset of Y.

The collection of closed subsets of a space X has properties similar to those satisfied by the collection of open subsets of X.

Theorem 1.2.4.4: Let X be a topological space. Then the following conditions hold:

1. $\emptyset$and X are closed.
2. Arbitrary intersection of closed sets is closed.
3. Finite unions of closed sets are closed.

Proof: Applying De Morgan's laws:

$$X\backslash\bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I}(X\backslash A_\alpha)$$

$$X\backslash\bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I}(X\backslash A_\alpha)$$

■

When dealing with subspaces, you need to be very careful in using the term open set. The following theorem is very important.

Theorem 1.2.4.5: Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof: Assume that $A = C \cap Y$, where C is closed in X, then $X \backslash C$ is open in X, so that $(X \backslash C) \cap Y$ is open in Y, by definition of the subspace topology. But $(X \backslash C) \cap Y = Y \backslash A$. Hence $Y \backslash A$ is open in Y, so that A is closed in Y. Conversely, assume that A is closed in Y. The set $X \backslash U$ is closed in X, and $A = Y \cap (X \backslash U)$, so that A equals the intersection of a closed set of X and Y, as desired.

Remark 1.2.4.6: A set that is closed in the subspace Y may not be closed in X. So the question now is, when is a closed set in a subspace Y closed in the space X? The next theorem provides an answer to this question.

Theorem 1.2.4.7: Let Y be a subspace of X. If A is closed in Y, and Y is closed in X, then A is closed in X.

Definition 1.2.4.8 (Closure and Interior of a Set): Suppose X is a topological space and$A \subset X$. The interior of A is the set given by

$$intA = \cup \{U \in X : U \subset A \text{ and } U \text{ is open}\}$$

That is, the $int(A)$ is the union of all open sets contained in A.

The closure of A denoted by $cl(A)$ or $\overline{A}$ is defined as the intersection of closed sets containing A. Clearly, the interior of A is an open set and the closure of A is a closed set; furthermore,

$$A^{\circ} \subset A \subset \overline{A}$$

If A is open, then $A = A^{\circ}$; on the other hand, if A is closed, then $A = \overline{A}$.

Proposition 1.2.4.9: Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ denote the clusure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$. Another useful way of describing the closure of a set is given in the following theorem.

Theorem 1.2.4.10: Let A be a subset of the topological space X.

a. Then$x \in \overline{A}$ if and only if every open set U containing x intersects A.

b. Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof: Consider the statement (a). It is a statement of the form $P \Leftrightarrow Q$. Transforming each statement to is contrapositive, gives you the logical equivalence $(not\ P) \Leftrightarrow (not\ Q)$. Explicitly, $x \in \overline{A}$ if and only if there exists an open set U containing x that does not intersect A.

In terms of this assertion, the theorem is easy to prove. If x is not in $\overline{A}$, the set $X\backslash A$ is open and contains x and does not intersect A as desired. Conversely, if there exists an open set U containing x which does not intersect A, then $X\backslash A$ is a closed set containing A. By definition of the closure $\overline{A}$, the set $X\backslash U$ must contain $\overline{A}$; therefore $x \in \overline{A}$.
Part (b) follows from the definition of basis.■

Definition 1.2.4.11: Let X be a topological space. Let $x \in X$ and V be a subset of X containing x. V is said to be a neighbourhood of x if there exist an open set U of X such that $x \in X \subset V$. The collection of all neighbourhoods of x is denoted by $N(x)$.

Proposition 1.2.4.12: Let X be a topological space and $x \in X$. Then
1. $N(x)$ is nonempty;
2. If $V \in N$ and $V \subset A$ then $A \in N(x)$;
3. A finite intersection of neighbourhoods of x is a neighbourhood of x.

Proposition 1.2.4.13: Let X be a topological space. Let U be a subset of X. Then U is open if and only in $U \in N(x)$ for every $x \in U$.

Lemma 1.2.4.14: If A is a subset of a topological space X, then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A. i.e., $x \in A$ if and only if for all $V \in N(x)$, $V \cap A = \emptyset$.

Proof:($\Longrightarrow$) Let $x \in \overline{A}$, and let $V \in N(x)$. Since $V \in N(x)$, there exist U open such that $x \in U \subset V$. It is enough for you to show that $U \cap A = \emptyset$. Suppose $U \cap A = \emptyset$, it implies that $A \subset U^c$ and U^c is closed since U is open, thus, $\overline{A} \subset U^c$, which implies that $x \in U^c$, which is a contradiction. Hence, $U \cap A = \emptyset$.

($\Longleftarrow$) Assume that for every neighbourhood V of x, $V \cap A = \emptyset$. You have to show that $x \in \overline{A}$. Suppose $x \in \overline{A}$, this implies that $x \in \overline{A}$ which is open (because $\overline{A}$ is closed) and so $\overline{A}^c \in N(x)$, and by hypothesis, $\overline{A}^c \cap A = \emptyset$. This is a contradiction, hence $x \in A$. ■

Example 1.2.4.15: Let X be the real line $\mathbb{R}$. If $A = (0,1]$, then $\overline{A} = [0,1]$, $\overline{B} = \{\frac{1}{n} : n \geq 1\}$ then $\overline{B} = B \cup \{0\}. If\ C = \{0\} \cup (1,2)$ then $\overline{C} = \{0\} \cup [1,2]$, $\overline{Q} = \mathbb{R}$.

Example 1.2.4.16 Consider the subspace $Y = (0,1]$ of the real line $\mathbb{R}$. The set $A = (0,\frac{1}{2})$ is a subset of Y. Its closure in R is the set $[0,1]$ and its closure in Y is the set $\overline{A} = [0,\frac{1}{2}] \cap Y = (0,\frac{1}{2}]$.

Definition 1.2.4.17(Limit Points): Let A be subset of a topological set X and let $x \in X$. x is said to be a limitpoint (or cluster point or point of accumulation) of A if every neighbourhood of x intersects A in some point other than that x itself.

$x \in X$ is a limit point of A if for all $V \in N(x)$, $V \cap (A\, r \,\{x\}) = \emptyset$ or x is a limit point of A if x belongs to the closure of $A \backslash \{x\}$. The point x may lie in A or not.

Theorem 1.2.4.18: Let A be a subset of the topological space X. Let A° be the set of all limit points of A.
Then

$$\overline{A} = A \cup A^{\circ}.$$

Proof: Clearly, $A \cup A^{\circ} \subset \overline{A}$. To prove the reverse inclusion, let $x \in \overline{A}$. If x happens to be in A, it is trivial that $x \in A \cup A^{\circ}$. Suppose that $x \in A$. Since $x \in A^{\circ}$, this implies that every neighbourhood U of x intersects A. Because $x \in A$, the set U intersects A in a point different from x. Then $\in A^{\circ}$, so that $x \in A \cup A^{\circ}$ as desired. ■

Corollary 1.2.4.19: A subset of a topological space is closed if and only if it contains all its limit points.

Proof: The set A is closed if and only if $A = \overline{A}$, and the later holds if and only if $A^{\circ} \subset A$.■

1.3 Self – Assessment Exercise(s)

1.In the following, answer true or false.

(a) The collection

$$\tau_{\infty} = \{U : X \backslash U is infinite or empty or all X\}$$

Is a topology in X.

(b) The union $\bigcup \tau_{\alpha}$ of a family $\{\tau_{\alpha}\}$ of topology on X is a topology on X.

(c) The countable collection $B = \{(a, b) : a < b, a, b \in \mathbb{Q}\}$ is a basis for a topology on $\mathbb{R}$.

(d) If A is a subset of a topological space X, and suppose that for each $x \in A$, there exists an open set U such that $x \in U \subset A$, then A is an open set in X.

2. Let $\mathbb{R}$ be with the standard topology and let $A \subset \mathbb{R}$. Then A is open in $\mathbb{R}$ if there exist an interval I such that $I \subset A$. For $a, b \in \mathbb{R}$, which of the following forms is the interval I

(a) $I = (a, b)$
(b) $I = (a, b]$
(c) $I = [a, b)$
(d) $I = [a, b]$

3. If τ is a topology on a set X, which of the following is not true about τ?

(a) Finite union of elements of τ is in τ.
(b) Finite intersection of elements of τ are in τ.
(c) The empty set $\emptyset$ and the whole set X are in τ.
(d) Arbitrary intersection of elements of τ are in τ.

4. Answer true or false. The collection
$B = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is
(a) a topology on the product space $X \times Y$.
(b) a basis for a topology on the product space $X \times Y$.

5. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. The collection

$$S = \{\pi_1^{-1}(V): U \text{ is open in } X\} \cup \{\pi_2^{-1}(V): V \text{ is open in } Y\}$$

Is______________________________ for the product topology on $X \times Y$.
(a) a collection of open sets
(b) a basis
(c) a sub – basis
(d) a topology

6. Let $\mathbb{R}$ be endowed with the standard topology. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y?
A = $x : \frac{1}{2} < |x| < 1$
B = $x : \frac{1}{2} < |x| \leq 1$
C = $x : \frac{1}{2} \leq |x| < 1$
D = $x : \frac{1}{2} \leq |x| \leq 1$
(a) A, B and C only
(b) A only
(c) B and C only
(d) D only.

7. With the standard topology on $\mathbb{R}$. Which of the sets in question **6** above are open in $\mathbb{R}$?
(a) A, B and C only
(b) A only
(c) B, C and D only
(d) D only.

8. Let $\mathbb{R}$ be endowed with the standard topology. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are closed in Y?
A = $x : \frac{1}{2} < |x| < 1$
B = $x : \frac{1}{2} < |x| \leq 1$
C = $x : \frac{1}{2} \leq |x| < 1$
D = $x : \frac{1}{2} \leq |x| \leq 1$
(a) A, B, C and D
(b) B and C only
(c) B, C and D only
(d) D only.

9. With the standard topology of $\mathbb{R}$. Which of the sets in question **8** above are closed in $\mathbb{R}$?
(a) A, B and C only
(b) B, C and D only
(c) B and C only
(d) D only.

10. For $A \subset X$, a topological space, and a boundary of A denoted by ∂A, defined by:
$\partial A = \overline{A} \cap \overline{X \backslash A}$.
The following are true;
1. A° and ∂A are disjoint, and $\overline{A} = A^\circ \cup \partial A$.
2. $\partial A = A$ if and only if A is both open and closed.
3. U is open if and only if $\partial U = \overline{U} \backslash U$.
Justify.

11. Hence or otherwise compute the boundary and interior of each of the following subsets of $\mathbb{R}^2$
(a) $A = \{(x, y) : y = 0\}$
(b) $B = \{(x, y) : x > 0 \text{ and } y = 0\}$
(c) $C = A \cup B$
(d) $D = \{(x, x) : x \text{ is rational}\}$

12. If $\mathbb{R}$, the real line is endowed with the indiscrete topology. Let $A = [0,1)$. What is $\overline{A}$?
(a) $[0,1]$
(b) $\mathbb{R}$
(c) $[0,1)$
(d) $\emptyset$

13. If $\mathbb{R}$, the real line is endowed with the usual metric topology, and let $A = (0,1)$. What is ∂A?
(a) $\mathbb{R}$
(b) $[0,1]$
(c) $\{0,1\}$
(d) $(0,1]$

1.4 Conclusion

The definition, examples, and basic concepts of topological spaces, such as the basis for a topology, closed sets, open sets, the interior, closure, neighborhood, and limit point of a set, have all been covered in this unit. You have seen some examples and proved some results.

1.5 Summary

Having gone through this unit, you now know that;

(i) a topology defined on a set X is a collection τ of subsets of X satisfying
(a) Theset$X$itselfandtheemptyset$\emptyset$arein τ
(b) The set X is in τ
(c) Arbitrary unions and finite intersections of elements of τ are in τ
(ii) a topological space is a pair (X,τ) consisting of a set X and a topology τ defined on it.
(iii) the elements of a topology on X are called open sets.
(iv) if τ_1 and τ_2 are topologies defined on X, then τ_1 is said to be finer that τ_2 if $\tau_2 \subset \tau_1$. In other words, we say that τ_2 is coarser than τ_1.
(iv) an arbitrary intersection of topologies is also a topology.
(v) a basis for a topology τ on X is a collection B of subsets of X (i.e., basis elements) such that
(a) for each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, or equivalently $X = \bigcup_{B\in\mathcal{B}} B$.

(b) if $x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

(vii) the topology generated by a basis B is given by
$\tau := \{U \in X : U = \emptyset \; or \; if \; x \in U,$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U\}$

(viii) The collection

$$B := \{U \times V : U \; is \; open \; in \; X \; and \; V \; is \; open \; in \; Y\}$$

Then B is basis for a topology on $X \times Y$.

(ix) The collection
$S = \{\pi_1^{-1}(V) : U \; is \; open \; in \; X\} \cup \{\pi_2^{-1}(V) : V \; is \; open \; in \; Y\}$
is a subbasis for the product topology on $X \times Y$, where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection maps defined (pointwise) on $X \times Y$ by $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = y$.

(x) if Y is a subset of a topological space (X, τ), the collection

$$\tau_Y = \{Y \cap U : U \in \tau\}$$

is a topology on Y, called the subspace topology. Y is called a subspace of X; it's open sets consisting of all intersection of open sets of X with Y.

(xi) A subset A of a topological space X is said to be closed in X if $X \backslash A$ (the complement of A in) is open.

(xii) if X is a topological space, then

(a) $\emptyset$ and X are closed.

(b) an arbitrary intersection of closed sets is closed.

(c) a finite union of closed sets is closed.

(xiii) if Y is a subspace of X, then a set A is closed in Y if and only if it equals the intersection of a closed set in X with Y.

(xiv) if A is a subset of a topological space X, then the interior of A, denoted by A° is the union of all open sets contained in A, while the closure of A denoted by $\overline{A}$ is the intersection of all closed sets contained in A.

(xv) if V is a subset of a topological space X and $x \in X$ such that $x \in V$, then V is called a neighbourhood of x if there exists an open set U of X such that of X such that $x \in X \subset V$.

(xvi) $N(x)$ denotes the collection of all neighbourhoods of x.

(xvii) if A is a subset of a topological space X, an element x of X is called a limit point of A if for all
$V \in N(x), V \cap (A \, r \, \{x\}) = \emptyset$.

(xviii) a subset of a topological space is closed if and only if it contains all its limit point.

1.6 References/Further Reading

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Discussion Forum

Discuss the topologies on a set X that contains three elements.

Feedback

There are 29 topologies on a set X that contains three elements. These topologies shall be discussed in class.

UNIT 2 SEPARATION AXIOMS

Unit structure

2.1 Introduction

Your understanding of the notions of closed and open sets as well as limit points in the real line or arbitrary metric space can be misleading when you carry such understanding to topological space. For example, in the space $\mathbb{R}$ and $\mathbb{R}^2$, each one-point set is closed. But this fact is not true for an arbitrary topological space. For if you consider the three-points set $X = \{a, b, c\}$, endowed with the sierpinski topology $\tau_s = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. In this space, the point set $\{b\}$ is not closed, because its complement $\{a, c\}$ is not open. Similarly, the understanding we have about convergence of a sequence in the real line can be misleading when you consider an arbitrary topological space. For example, on the real line, the limit of a sequence if it exists is unique, but this is not true in an arbitrary topological space. In this unit, you shall be introduced to the separation axioms, a natural restriction on the topological structure making the structure closer to that of a metric space (i.e., closer to being metrizable). A lot of separation axioms are known. Here you shall study five most important of them. They are numerated, and denoted by $\tau_0, \tau_1, \tau_2, \tau_3and\tau_4$, respectively.

2.1 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to;

- define a Hausdorff space and state some of its properties.
- prove that in a Hausdorff space, every point set is closed.
- define a convergent sequence and show that in a Hausdorff space, the limit is unique.
- prove that every metric topology is Hausdorff.
- know five separation axioms and their properties.

2.3 Axioms of Separation

Definition 2.2.1.1 Hausdorff Space ($\boldsymbol{\tau_2 - space}$)

The most celebrated of all the axioms of separation is the second axiom of separation τ_2 . It was suggested by the mathematician Felix Hausdorff, and so mathematicians have come to call it by his name. Therefore, Topological spaces that satisfy the second separation axiom will be called *Hausdorff space.*

A topological space is called a **Hausdorff space**, if for each x, y of distinct points of X, there exist neighbourhoods U_x and U_y of x and y respectively, that are disjoint. More formally X is Hausdorff if $\forall\, x, y \in X$ with $x = y$, there exist $U_x \in N(x)$, $U_y \in N(y)$: $U_x \cap U_y = \emptyset$.

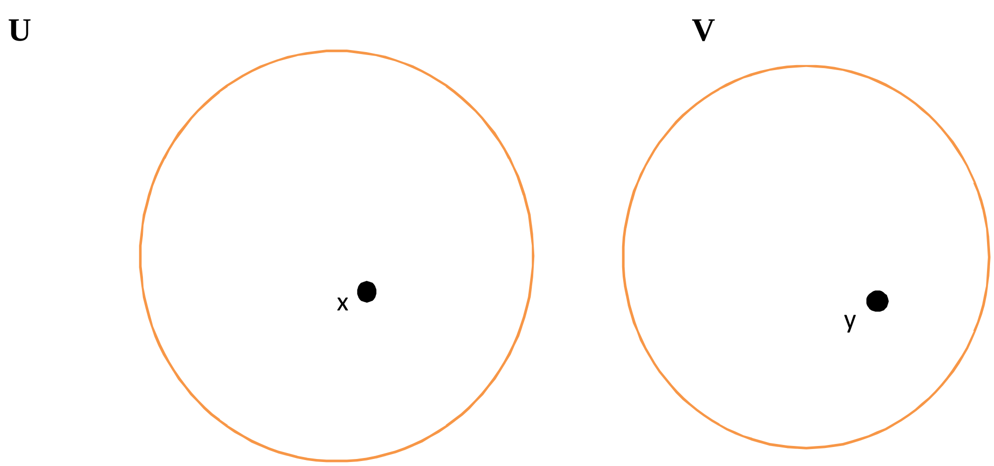

Figure 3.1.1 Hausdorff Space

As earlier remarked, Hausdorff space are τ_2. For example, consider the real line $\mathbb{R}$, with the standard topology, that is the topological spaces whose open sets are of the form (a, b), $a, b \in \mathbb{R}$ with $a < b$ (the open intervals). Take for instance the points$\frac{2}{2}, \frac{1}{4} \in \mathbb{R}$, the open intervals $(\frac{1}{6}, \frac{1}{3})$ and $(\frac{5}{12}, \frac{7}{12})$ are neighbourhoods of $\frac{1}{4}$and $\frac{1}{2}$ respectively and $\left(\frac{1}{6}, \frac{1}{3}\right) \cap \left(\frac{5}{12}, \frac{7}{12}\right) = \emptyset$. In fact, you know that the standard topology of$\mathbb{R}$is induced by the metric d defined by

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$. And for each $X \in \mathbb{R}$, the $open - ball$ centered at x with radius $r > 0$ is given by

$$B(x, r) = \{y \in \mathbb{R}: d(x, y) = |x - y| < r\} = (x - r, x + r)$$

Thus for each $x, y \in \mathbb{R}$, with $x = y$, just choose $x = \frac{1}{3}$,$d(x, y) > 0$ then $x \in (x - r, x + r) = B(x, r)$ and $y \in (y - r, y + r) = B(y, r)$ and $B(x, r) \cap B(y, r) = \emptyset$.

The above exercise can be done in an arbitrary space with the metric topology. and this gives you the first example of Hausdorff spaces.

Example 2.2.1.2: Every metric topology is Hausdorff.

Example 2.2.1.3: Every discrete space is Hausdorff.
To see this, Let X be a discrete topological space, and let $x, y \in X$ with $x = y$. Take $U_x = \{x\}$, and $U_y = \{y\}$, then U_x and U_y are open sets in the discrete topology, and $U_x \cap U_y = \emptyset$.

Exercise 2.2.1.4: Let $\mathbb{Q}$ be the set of rational numbers with the standard topology of $\mathbb{R}$, and let $\mathbb{Q}^0$ denote the set of all irrational numbers also with the standard topology of $\mathbb{R}$. Is $\mathbb{Q}$ and $\mathbb{Q}^0$ Hausdorff?
The following are some spaces that are not Hausdorff.

Example 2.2.1.5: The real line $\mathbb{R}$ with the finite complement topology is not Hausdorff.

To see this, recall first that the finite complement topology is defined by

$$\tau_f = \{U \subset X : X \backslash U \text{is either finite or the whole set} X\}$$

Now suppose $\mathbb{R}$ with the finite complement topology is Hausdorff, then for every $x, y \in \mathbb{R}$ there exists open neighbourhoodsU_x, U_y of x and y such that

$$(\mathbb{R} \backslash U_x) \cup (\mathbb{R} \backslash U_x) = \mathbb{R}$$

Which means that $\mathbb{R}$ is finite as a union of two finite sets, otherwise, the sets U_x and U_y would be empty sets and thus are no longer neighbourhoods of x and y respectively, this is a contradiction. Hence $\mathbb{R}$ with the finite complement topology is not Hausdorff.

Example 2.2.1.6: Let $X = \{a, b, c\}$ endowed with the topology

$$\tau_s = \{\emptyset, X, \{b\}, \{b, c\}, \{b, a\}\}$$

This is easy to see, because a and c are distinct points in X and there are no neighbourhoods of a and c with empty intersection.
The following important results makes Hausdorff space interesting.

Theorem 2.2.1.7: Let X be a Hausdorff space, then for all $x \in X$, the singleton set $\{x\}$is closed.

Proof: Let $x \in X$ be arbitrary and set $A = \{x\}$. It is enough to show that $A = \overline{A}$. You know that $A \subset \overline{A}$, so it is left for you to show that $\overline{A} \subset A$. You can do this by contraposition (i.e., you know that if $A \subset B$, then for every $y \in A, y \in B$; the contraposition is that if $y \notin B$ then $y \notin A$). Now, suppose that $y \notin A$, i.e., $y = x$, since X is Hausdorff, there exist $U_x \in N(x)$, $U_y \in N(y)$ such that $U_x \cap U_y = \emptyset$. This implies that $U_y \cap A = \emptyset$, i.e., $y \notin \overline{A}$. Hence, $\overline{A} \subset A$. Therefore, both inclusions $A \subset \overline{A}$ and $\overline{A} \subset A$gives you that $\overline{A} = A$ i.e., $A = \{x\}$is closed. ■

2.3.1 Sequences

In your course of elementary analysis, you can recall that a sequence $\{x_n\}$ of elements of $\mathbb{R}$ is said to converge to $x \in R$ if given any $\epsilon > 0$, there exist $N := N(\epsilon)$N such that for all $n \geq N$,

$$|x_n - x_0| < \epsilon(1)$$

The inequality (1) is equivalent to say that for all $n \geq N,\ x_n \in (x_0 - \epsilon, x_0 + \epsilon)$. Also you know that if X is a metric space, with a metric d, then a sequence $\{x_n\}$ in X converges to $x_0 \in X$ if given any $\epsilon > 0$, there exists $N := N(\epsilon)$ such that for all $n \geq N$,
$d(x_n, x_0) < \epsilon$ (2)

That is to say that for every $n \geq N, x_n \in B_d(x_0, \epsilon)$.

Suppose, now that you set $U = (x_0 - \epsilon, x_0 + \epsilon)$, or $U = B_d(x_0, \epsilon)$ accordingly, as you refer to the real line $\mathbb{R}$ or the metric space X, you will have that U is a neighbourhood of x_0 and depends on $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, then U is also arbitrary. This is now of great help to us to define convergent sequence in an arbitrary topological space since absolute value or distance does not make sense in an arbitrary topological space, but the concept of neighbourhood is meaningful in any topological space. Thus in an arbitrary topological space you have the following definition.

Definition 2.2.2.1 (Convergent sequence): Let X be a topological space, let $\{x_n\}$ be a sequence of elements of X. Then $\{x_n\}$ is said to converge to $x \in X$ if for all neighbourhoods $U$of x, there exists $N \in N$ such that for all $n \geq N, x_n \in U$. That is $x_n \longrightarrow x \in X$ as $n \to \infty$ if for all $U \in N(x)$, there exists $N \in N$ such that for all $n \geq N, x_n \in U$.

Recall that in the study of real line $\mathbb{R}$, and in a metric space X, we proved that the limit of a convergent sequence $\{x_n\}$ is unique. This is not true in an arbitrary topological space as shown in the following example.

Example 2.2.2.2: Let $\mathbb{R}$ the reals be endowed with the finite complement topology, and let $\{x_n\}$ be a sequence of elements of $\mathbb{R}$ defined by, $x_n = \frac{1}{n}$, for $n \geq 1$. If this sequence converges, every element of $\mathbb{R}$ is a limit of this sequence.

To see this, Let $x \in \mathbb{R}$, and suppose $x_n \to x$, then by definition, let U be a neighbourhood of x, there exists $N \in N$ such that for all $n \geq N, \frac{1}{n} \in U$ otherwise, $\frac{1}{n} \in U^c$ for all $n \geq N$ (i.e., $\left\{\frac{1}{n}\right\}$ does not converge to x). This would mean that infinitely many points of the sequence are contained in a finite set (since U belongs to the finite complement topology means that U^c is a finite set while it is assumed that U^c is not the whole $\mathbb{R}$ itself which would mean that $U = \emptyset$and thus would not be a neighbourhood of x). This is impossible, thus x must be the limit of the sequence $\left\{\frac{1}{n}\right\}$ and since x is arbitrary, $\left\{\frac{1}{n}\right\}$ converges to every element of $\mathbb{R}$. But you know vividly well that in the real line $\mathbb{R}$, the limit of the sequence $\left\{\frac{1}{2}\right\}$is 0. So you see that convergence of a sequence actually depends on the type of topology imposed on the space.

The next result tells us more about a sequence in a Hausdorff space. It says that in a Hausdorff space, the limit of a convergent sequence is unique. that is why you have terms like uniqueness of limits on the real line with the standard topology and in an arbitrary metric space, because they are Hausdorff.

Theorem 2.2.2.3: Let X be a Hausdorff space, then a sequence of points of X converges to at most one point of X. (i.e., if a sequence $\{x_n\}$ in X, a Hausdorff space, converges, the limit is unique.)

Proof: Let X be a Hausdorff space, and let $\{x_n\}$ be a convergent sequence of elements of X. Assume that x_n converges to x and y, you have to prove that $x = y$. Suppose for $x = y$, since X is Hausdorff, there exist$U_x \in N(x)$, and $U_y \in N(y)$such that $U_x \cap U_y = \emptyset$.$U_x \in N(x)$ and $x_n \to x$implies that there exists $N_1 \in N$ such that$x_n \in U_x$ for all$n \geq N_1$. Also $U_y \in N(y)$ and $x_n \to y$ implies that there exists $N_2 \in N$ such that $x_n \in U_y$ for all $n \geq N_2$. Now, choose $N := max\{N_1$, $N_2\}$ then $x_N \in U_x \cap U_y = \emptyset$(a contradiction). Hence $x = y$. ■

Having proven some of the basic results of Hausdorff spaces (i.e., $\tau_2 - space$), you will now be introduced to all the other axioms of separation.

2.3.2 The First Separation Axiom ($\boldsymbol{\tau_1 - space}$)

Definition 1.2.3.1($\tau_1 - space$): A topological space X satisfies the first separation axiom τ_1 if each one of any two points of X has a neighborhood that does not contain the other point. Thus X is called a $\tau_1 - space$. That is X is τ_1 if for all $x, y \in X$ with $x = y$, there exist $U_x \in N(x)$ such that $y \notin U_x$. Another name for a $\tau_1 - space$ is a Fréchet space.

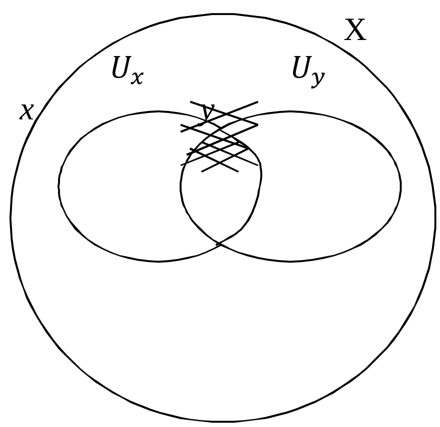

Figure 3.2.1: ($\boldsymbol{\tau_1 - axiom}$)

Theorem 2.2.3.2: A topological space X satisfies the first separation axiom

(i) if and only if all one-point set in X is closed.
(ii) if and only if every finite set in X is closed.

Proof:(i) ($\Longrightarrow$) Suppose X is τ_1 , and let $x \in X$. By the τ_1 axiom, for all $y \in X, x = y$, i.e. $y \in X\backslash\{x\}$, there exist an open set $U_y \in N(y)$ such that $x \notin U_y$. This implies that$U_y \subset X\backslash\{x\}$. $X\backslash\{x\}$ contains an open set

U_y, that tells us that it is open, and so its complement $(X\backslash\{x\})^c =$ {x}must be closed.
($\Longrightarrow$)Suppose X is a topological space in which all singletons are closed and let $x, y \in X$ such that $x = y$, then $X\backslash\{x\}$is open and contains y and $x \notin (X\backslash\{x\})$. This implies that X is τ_1.

(ii) ($\Longrightarrow$) Suppose X is τ_1, then every singleton $\{x\}$is closed. So also is a finite set, because it is a finite union of singletons which are closed sets.
(iii) ($\Longleftarrow$) Suppose that X is such that finite sets are closed, and let $x, y \in X, x = y$ then $\{x\}$ is a finite set, $(X\backslash\{x\})$ is an open neighbourhood of y and does not contain x. Hence, X is τ_1.

Example 2.2.3.3: Every Hausdorff space is τ_1. But the converse is not true.

Clearly, If you consider a set $X = \mathbb{R}$, the real line with the finite complement topology, then X is a $\tau_1 - space$. Since if $x, y \in X, U_x = X\backslash\{y\}$is an open set containing x that does not contain y, also, $U_y = X\backslash\{x\}$is an open set containing y that does not contain x. You have also seen in example 3.3 that $\mathbb{R}$ with this topology is not Hausdorff. Hence, we have given an example of a $\tau_1 - space$ that is not Hausdorff.

2.3.3 The zeroth Separation Axiom ($\boldsymbol{\tau_0 - space}$)

The zeroth separation axiom appears as a weakened first sepatation axiom. It states as follows:

Definition 2.2.4.1($\boldsymbol{\tau_0 - space}$): A topological space X satisfies the Kolmogorov axiom or the zeroth separation axiom τ_0 if at least one of any two distinct points of X has a neighborhood that does not contain the other point.

Spaces that satisfy the zeroth separation axiom or the Kolmogorov axiom τ_0 are regarded as $\tau_0 - space$. That is; X is τ_0 if for all $x, y \in X$ with $x = y$, there exist an open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. In other words, a topological space X is called a $\tau_0 - space$ if and only if for any two distinct points x, y of X ($\forall\, x, y \epsilon X$) there is an open subsets of U which contains one but not the other.

Example 2.2.4.2: Every τ_1 space is τ_0 so also is every τ_2 space. But the converse is not true in each case. **Example 2.2.4.3:** Let $X = \{a, b\}$ be endowed with the topology $\tau = \{X, \emptyset, \{a\}\}$. Then X is τ_0 but not τ_1.

Proposition 2.2.4.4: Let X be a topological space. The following properties of X are equivalent:

(a) X is τ_0;
(b) any two different points of X has different closures.

2.3.4 Third Separation Axiom. $\boldsymbol{\tau_3 - space}$

Definition 2.2.5.1($\boldsymbol{\tau_3 - spaces}$): A topological space X satisfies the third separation axiom if every closed set in X and every point of its complement have disjoint neighborhoods. τ_3 spaces are topological spaces that satisfy the third separation axiom. That is, X isτ_3 if for every closed set $F \subset X$ and every $x \in X$ such that $x \notin F$ there exists open sets $U_F, U_x \subset X$ with $F \subset U_F, x \in U_x$ such that $U_F \cap U_x = \emptyset$.

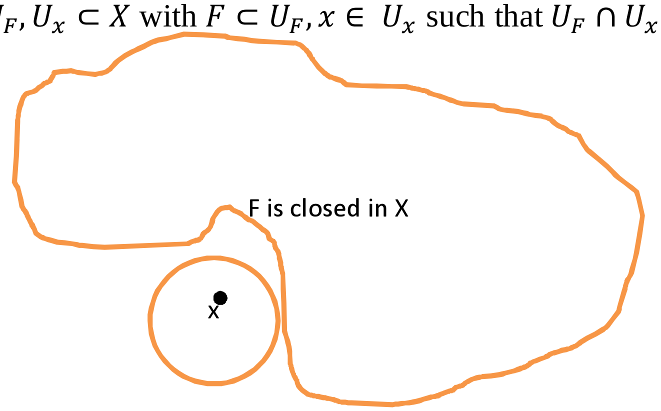

Figure 2.2.5.1:$\boldsymbol{\tau_3 - axiom}$

2.3.5 Regular Space

Definition 2.2.6.1 (Regular space): A topological space X is said to be a regular space if for any closed set F of X and any point $x \in X\backslash F$, there exists open sets U_F, $U_x \subset X$ such that $x \in U_x$, $F \subset U_F$ and $U_x \cap U_F = \emptyset$. If a topological space X is regular and is a $\tau_1$space, then X is a τ_3 space. On the other hand, if X is a τ_3 space and a $\tau_1$space, then X is regular.

Example 2.2.6.2:Any metric space is regular.

Example 2.2.6.3: Examples of regular spaces are $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$ and $\mathbb{R}^2$.

Example 2.2.6.4: Every regular τ_1 space X is τ_2 (Hausdorff).

2.2.7 Fourth Separation Axiom($\tau_4 - \boldsymbol{space}$)

Definition 2.2.7.1($\boldsymbol{\tau_4 - space}$): A topological space X satisfies the fourth separation axiom if any two disjoint closed sets in X have disjoint neighborhoods. Topological spaces that satisfy the fourth separation axiom are called τ_4 spaces. Thus X is a τ_4 if for any two closed sets E,

$F \subset X$ with $E \cap F = \emptyset$there exists open sets $U_E, U_F \subset X$ such that $E \subset U_E$, $F \subset U_F$ and $U_E \cap U_F = \emptyset$.

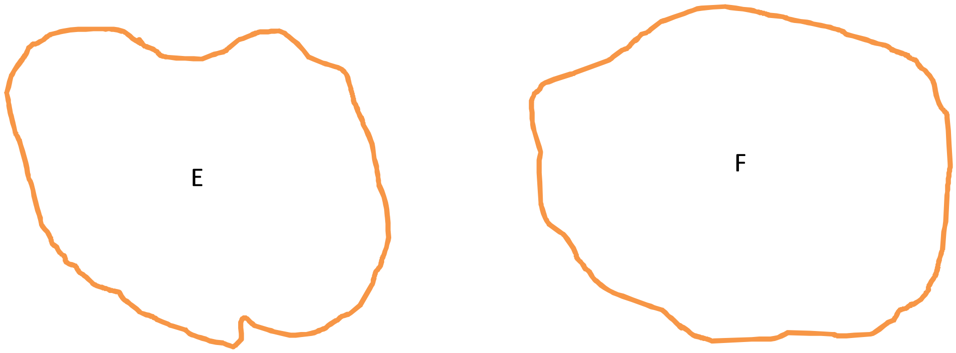

Figure 3.7.1: $\tau_4 - axiom$

Example 2.2.7.2: Any indiscrete topological space satisfies the fourth separation axiom. This is also an example of a τ_4 space that is not τ_2.

Definition 2.2.7.3(Normal Space): A topological space X is normal if it satisfies the first and the fourth separation axioms.

Example 2.2.7.4: Every metric space is normal.

2.3.7 Continuous Functions

Definition 2.2.8.1 (Continuous Function): Let X and Y be topological spaces. A function $f: X \longrightarrow Y$ is said to be continuous if for each open subset U_Y of Y, the set $f^{-1}(U_Y)$ is an open subset of X, where

$$f^{-1}(U_Y) = \{x \in X: f\ (x) \in U_Y\}$$

Continuity of a function depends not only on the function alone, but also on the topologies specified for its domain and range.

Theorem 2.2.8.2: If the topology on the range Y is given by a basis $\mathcal{B}$, then f is continuous if and only if any basis element $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.

Proof:($\Longrightarrow$) Let the topology Y be given by basis $\mathcal{B}$, and suppose that f is continous, then for all $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X since each $B \in \mathcal{B}$ is open.

($\Longleftarrow$) Suppose that each $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X, you have to show that f is continuous. So take an open set $V \in Y$, then you can write V as a union of basis elements, i.e.,

$$V = \bigcup_{i \in I} B_i$$

Therefore,

$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(B_i)$$

So that $f^{-1}(V)$ is open as a union of the sets $f^{-1}(B_i)$, $i \in I$, which are open by assumption. ■

Example 2.2.8.3: Any constant function is continuous.

Example 2.2.8.4: Consider a real valued function of real variable $f: \mathbb{R} \longrightarrow \mathbb{R}$. In analysis one defines continuity via $\epsilon - \delta$ definition. As you would see, the $\epsilon - \delta$ definition and your their equivalent.

Theorem 2.2.8.5: Let X and Y be topological spaces, let $f: X \longrightarrow Y$. Then the following are equivalent:

(1) f is continuous.
(2) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.
(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
(4) For each $x \in X$ and each neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subset V$.

If the condition in (4) holds for the point x, we say that f is continuous at x.

Proof: We show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) and that (1) $\Longrightarrow$ (4) $\Longrightarrow$ (1).
(1) $\Longrightarrow$ (2).Assume that f is continuous. Let A be a subset of X. We show that if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$. Let $x \in \overline{A}$ and let Vbe an open neighbourhood of $f(x)$. Then $f^{-1}(V)$ is an open subset X containing x. So $f^{-1}(V) \cap A = \emptyset$because $x \in \overline{A}$. Let $y \in f^{-1}(V) \cap A$, then $f(y) \in V \cap A$, thus $f(x) \in \overline{f(A)}$, as desired.

(2) $\Longrightarrow$ (3). Let B be a closed subset of Y and $A = f^{-1}(B)$. We wish to show that A is closed in X. We show that $\overline{A} = A$. By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$. Therefore, if $x \in \overline{A}$, then

$$f\,(x) \in f\,(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$$

so that $f(x) \in B$, thus $x \in f^{-1}(B) = A$, as desired.

(3) $\Longrightarrow$ (1). Let V be an open subset of Y. Set $B = Y \backslash V$. Then $f^{-1}(B) = X \backslash f^{-1}(V)$. Now B is closed set of Y, then $f^{-1}(B)$ is closed in X by hypothesis, so that $f^{-1}(V)$ is open in X, as desired.
(1) $\Longrightarrow$ (4). Let $x \in X$ and let V be an open neighbourhood of $f(x)$. Then the set $U = f^{-1}(V)$ is an open neighbourhood of x such that $f(U) \subset V$.
(4) $\Longrightarrow$ (1). Let V be an open set of Y. Let $x \in f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is an open neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open. ■

2.3.8 Homeomorphism

You are familiar with the following definitions about functions.

Definition 2.2.9.1: Let X and Y be sets, the map $f: X \longrightarrow Y$ is a surjective map or just a surjection if every element of Y is the image of at least one element of X. That is, f is a surjection if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

A map $f: X \longrightarrow Y$ is an injective map, injection or one-to-one map if every element of Y is the image of at most one element of X. That is f is an injection if for all $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$.

A map is a bijective map, bijection or invertible map if it is both surjective and injective.

Definition 2.2.9.2: Let X and Y be topological spaces; Let $f: X \longrightarrow Y$ be a bijection. If both f and its inverse $f^{-1}: Y \longrightarrow X$ are continuous, then f is called a homeomorphism.

Definition 2.2.9.3 (Equivalence Relation): Let X be a set and $\mathcal{R}$ be a relation on X. Then $\mathcal{R}$ is called an equivalence relation if $\mathcal{R}$ is

(a) Symmetric: $x\mathcal{R}x$ for all $x \in X$
(b) Reflective: If $x\mathcal{R}y$ then $y\mathcal{R}x$ for all $x, y \in X$.
(c) Transitive: If $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$ for all $x, y, z \in X$.

Definition 2.2.9.4: Two topological spaces X and Y are homeomorphic if there exists a homeomorphism $f: X \longrightarrow Y$ between the spaces.

Theorem 2.2.9.5: Being homeomorphic is an equivalence relation.
Suppose that $f: X \longrightarrow Y$ is an injective continuous map, where X and Y are topological spaces, Let Z be the image set $f(X)$, considered as a

subspace of Y; then the function $f^0: X \longrightarrow Z$ obtained by restricting the range of f is bijective.

Definition 2.2.9.6: If $f^0: X \longrightarrow Z$ is an homeomorphism, we say that the map $f: X \longrightarrow Y$is a topological imbedding or simply an imbedding of X in Y.

Example 2.2.9.7: The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 3x + 1$ is a homeomorphism.

Example 2.2.9.8: The function $F: (-1, 1) \longrightarrow \mathbb{R}$ given by

$$F(x) = \frac{x}{1 - x^2}$$

is a homeomorphism.

Example 2.2.9.9: The identity map $g: \mathbb{R}_1 \longrightarrow \mathbb{R}$ is bijective and continuous, but it is not a homeomorphism. **Example 2.2.9.10**: Let S^1 denote the unit circle,

$$S^1 = \{(x, y) : \ x^2 + y^2 = 1\}$$

considered as a subspace of the plane $\mathbb{R}^2$, and let $F: [0, 1] \longrightarrow S^1$ be a map defined by $f(t) \ = \ (cos\ 2\pi t, sin\ 2\pi t)$. The map F is bijective and continuous, but F^{-1} is not continuous.

Theorem 2.2.9.11: Let X, Y and Z be topological spaces. If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are continuous, then the map $g \circ f: X \longrightarrow Z$is continuous.

Proof: Let W be an open set in Z,

$$(g \circ f)^{-1}(W) = \ f^{-1} \circ g^{-1}(W) = f^{-1}(g^{-1}(W))$$

Since f and g are continuous, $g^{-1}(W)$ is open in Y implies that $f^{-1}(g^{-1}(W))$ are open in X. Thus, $g \circ f$ is continuous on X.

Theorem 2.2.9.12 (Restricting the domain): If $f: X \longrightarrow Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A: A \longrightarrow Y$ is continuous.

Proof: You have to show that ${f|_A}^{-1}(W)$is open in the subspace topology τ_A on A induced by the topology τ on X for any open set W in Y. So let W be an open set in Y. By the continuity of f on X, $f^{-1}(W)$ is open in X and

$$\begin{aligned} {f|_A}^{-1}(W) &= \{x \in A: f|_A(x) \in W\} \\ &= \{x \in A: f(x) \in W\} \\ &= A \cap \{x \in X: f(x) \in W\} \\ &= A \cap f^{-1}(W) \end{aligned}$$

which implies that${f|_A}^{-1}(W)$open in the subspace topology τ_A.

Theorem 2.2.9.13 (Restricting or expanding the range): Let $f: X \longrightarrow Y$ be continuous.

1. If Z is a subspace of Y containing the image set $f(X)$, then the map $g: X \longrightarrow Z$ obtained by restricting the range of f is continuous.
2. If Z is a space having Y as a subspace, then the function $h : X \longrightarrow Z$ obtained by expanding the range of f is continuous.

Proof: 1. You know that since Z is a subspace of Y, the subspace topology on τ_Z induced on Z by the topology τ on Y is given by

$$\tau_Z = \{V \cap Z : V \in \tau\}$$

Now, let V be open in Y (meaning that $Z \cap V$ is open in), you have to show that $g^{-1}(Z \cap V)$ is open in X. You can compute as follows:

$$\begin{aligned} g^{-1}(Z \cap V) &= \{x \in X : g(x) = f(x) \in Z \cap V\} = \{x \in X : f(x) \in V\} \\ &= f^{-1}(V) \end{aligned}$$

2. Using similar argument on the subspace topology as in (1) above, let W be open in Z, then $Y \cap W$ is open in Y (because Y is a subspace of Z) and

$$\begin{aligned} h^{-1}(W) &= \{x \in X : h(x) \in W\} \\ &= \{x \in X : f(x) \in W\} \\ &= \{x \in X : f(x) \in Y \cap W\} \\ &= f^{-1}(Y \cap W) \end{aligned}$$

is open in X because f is continuous and $f^{-1}(Y \cap W)$ is open in X. Hence h is continuous.■

Theorem 2.2.9.14 (The pasting lemma): Let $X = A \cup B$, where A and B are closed in X. Let $f: A \longrightarrow Y$ and $g: B \longrightarrow Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the function $h: A \longrightarrow Y$ defined by

$$h(x) = \begin{cases} f(x), & if \quad x \in A \\ g(x), & if \quad x \in B \end{cases}$$

is continuous.

Proof: Let f be a closed set in Y.

$$\begin{aligned} h^{-1}(F) &= \{x \in X : h(x) \in f\} \\ &= \{x \in A : f(x) \in F\} \cup \{x \in B : g(x) \in F\} \\ &= f^{-1}(F) \cup g^{-1}(F) \end{aligned}$$

$f^{-1}(F)$is closed in X because it is closed in A and A is closed in X, also $g^{-1}(F)$ is closed in X since it is closed in B and B is closed in X. Hence $h^{-1}(F)$ is closed in X as a finite union of closed sets in X. Hence, h is continuous.

Example 2.2.9.15: Let $h: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$h(x) = \begin{cases} \frac{x}{2}, & if \quad x \geq 0 \\ x, \quad if & x \leq 0 \end{cases}$$

then h is continuous.

To see this, let $A = [0, +\infty)$ and $f: A \longrightarrow \mathbb{R}$, defined by $f(x) = \frac{x}{2}$, also let $B = (-\infty, 0]$ and $g: \longrightarrow \mathbb{R}$, defined by $g(x) = x$. Observe that A and B are closed sets in $\mathbb{R}$ and $\mathbb{R} = A \cup B. f$ and g continuous functions, $A \cup B = \{0\}$ and $f(0) = g(0) = 0$. Hence by pasting lemma, h is continuous.

Theorem 2.2.9.16 (Maps in products): Let $f: Z \longrightarrow X \times Y$ be given by

$$f(z) = (f_1(x), f_2(y))$$

Then f is continuous if and only if the functions
$f_1: Z \longrightarrow X$ and $f_2: Z \longrightarrow Y$
are continuous.
The maps f_1 and f_2 are called coordinate functions of f.

Proof: Let $\pi_1: X \times Y \longrightarrow X$ and$\pi_2: X \times Y \longrightarrow Y$ be projections maps.

These maps are continuous.
Note that for each $z \in Z$,

$$f_1(z) = \pi_1(f(z)) \text{ and} f_2(z) = \pi_2(f(z))$$

If f is continuous then f_1 and f_2 are continuous as composites of continuous functions.

Conversely, suppose that f_1 and f_2 are continuous. Let $U \times V$ be a basis element of for the product topology in $X \times Y$. A point z is in $f^{-1}(U \times V)$ if and only if $f(z) \in U \times V$, that is, if and only if $f_1(z) \in U$ and $f_2(z) \in V$. Therefore

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

Since both of the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection.

2.3.7 More on Separation Axioms

Theorem 2.2.10.1: Let X be a topological space and Y a Hausdorff space. Let $f: X \longrightarrow Y$ be a map. If f is continuous, then the Graph of f, $Graph(f) = \{(x, f(x) : x \in X\}$.
is a closed subset of $X \times Y$.

Proof: Suppose f is continuous, you have to show that the graph of f is closed. It is enough for you to show that the complement of the graph of f is open in $X \times Y$. So let $U = (Graph(t))^c$, and let $(x_0, y_0) \in U$. This implies that $y_0 = f(x_0)$. Since $Y$is Hausdorff, there exist open sets W_{y_0} and $W_{f(x_0)}$in Y containing y_0 and $f(x_0)$ respectively such that

$$W_{y_0} \cap W_{f(x_0)} = \emptyset$$

Since f is continuous at x_0 (because f is continuous) and x_0 and $W_{f(x_0)} \in \mathrm{N}(f(\mathrm{x}_0))$, there exists $U_{x_0} \in N(x)$ such that $f(U_{x_0}) \subset W_{f(x_0)}$. Take

$$B = U_{x_0} \times W_{y_0}$$

B is a basis element for the product topology on $X \times Y$ and for $(x, y) \in B$, you have that $x \in U_{x_0}$ and $y \in W_{y_0}$. Also $x \in U_{x_0}$ implies that $f\ (x) \in W_{f(x_0)}$and so $y = f(x)$, thus $(x, y) \notin Graph(f)$, which implies that $(x, y) \notin U$. Thus $B \subset U$, and so U is open. Hence $Graph(f)$ is closed.

Theorem 2.2.10.2 (Urysohn's Lemma): Let A and B be two disjoint closed subsets of a normal space X. Then there exists a continuous function $f: X \longrightarrow I$ such that $f(A) = 0$ and $f(B) = 1$.

2.4 Self-Assessment Exercise(s)

1. Which of the following spaces is Hausdorff?

(a) The discrete space.
(b) The indiscrete space.
(c) $\mathbb{R}$ with the finite complement topology.
(d) $X = \{a, b\}$ endowed with the topology $\tau = \{\emptyset, X, \{a\}\}$.

2. Which of the following spaces is not Hausdorff?
(a) $\mathbb{R}$ with the standard topology.
(b) $\mathbb{R}$ with the lower limit topology.
(c) $\mathbb{R}$ with the metric topology.
(d) $\mathbb{R}$ with the finite complement topology.

3. If $\{x_n\}$ be a sequence in $\mathbb{R}$ endowed with the finite complement topology. If $\{x_n\}$ converges in $\mathbb{R}$ then
(a) the limit is unique.
(b) $\{x_n\}$ converges to only two points.
(c) $\{x_n\}$ converges to one point in R and one point outside $\mathbb{R}$.
(d) $\{x_n\}$ converges to every element of $\mathbb{R}$.

4. In the finite complement topology of $\mathbb{R}$, let the sequence $\{x_n\}$ be defined by $x_n = n$, for $n \in N$. If the limit of the sequence is x, then x must be
(a) ∞
(b) 0
(c) a unique constant
(d) arbitrary in $\mathbb{R}$

5. Which of the following spaces is not metrizable?
(a) Any discrete space
(b) X with the countable complement topology.
(c) $\mathbb{R}$ with the standard topology.
(d) $\mathbb{R}^2$ with the standard topology.

6. Which of the following is not true about τ_1 spaces?
(a) Every singleton is closed
(b) Every finite set is closed
(c) Every Hausdorff space is τ_1.
(d) Every τ_1 space is Hausdorff.

7. Let X be a topological space that satisfies the Kolmogorov axiom (τ_0). Which of the following is not true about X?
(a) Any two different points of X has different closures.
(b) X contains no indiscrete subspace consisting of two points.
(c) X contains no indiscrete subspace consisting of more than one point.
(d) X has an indiscrete subspace consisting of two points only.

8. Let X be a topological space. Then X is regular if
(a) X is both τ_1 and τ_2.
(b) X is τ_3 only
(c) X is both τ_2 and τ_3
(d) X is τ_2 only.

9. Which of the following spaces is not regular?
(a) $\mathbb{R}$
(b) $\mathbb{Q}$
(c) $\mathbb{Z}$
(d) Every Hausdorff space X
(Where $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Q}$ are with the standard topology on $\mathbb{R}$.)

10. In what follows, answer true of false. (Justify your claims).
(a) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x - 2, & for \quad x \leq 0 \\ x + 2, & for \quad x \geq 0 \end{cases}$$

then f is continuous.

(b) The identity map

$$id:(X,\Omega_1) \longrightarrow (X,\Omega_2)$$

if and only if $\Omega_1 \subset \Omega_2$, where Ω_1 and Ω_2 are topological structures on X.

(c) The function $f:\mathbb{R}_1 \longrightarrow \mathbb{R}$ defined by $f(x) = x$ is continuous, where $\mathbb{R}_1$ denotes the lower limit topology on $\mathbb{R}$ and $\mathbb{R}$ is endowed with the standard topology.

(d) Let $f:\mathbb{R} \longrightarrow \mathbb{R}_1$ be as defined in (c) above, with $\mathbb{R}_1$ and $\mathbb{R}$ are as in (c). Then f is continuous.

(e) If X is τ_4, then it must be τ_2.

(f) Every normal space is both regular and Hausdorff.

(g) Every open and bounded interval (a,b) of $\mathbb{R}$, $a < b$ is homeomorphic to $\mathbb{R}$.

(h) The closed and bounded interval $[a,b]$ of $\mathbb{R}$, is homeomorphic to $[0,1]$

(i) X is Hausdorff if and only if the diagonal $\Delta = \{(x,x): x \in X\}$ is closed in $X \times X$.

11. Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} x, & if\ x \leq 1 \\ x+2, & if\ x > 1 \end{cases}$$

Is f continuous?

12.Consider the map $f:[0,2] \longrightarrow [0,2]$

$$f(x) = \begin{cases} x, & if\ x \in [0,1) \\ 3-x, & if\ x \in [1,2] \end{cases}$$

Is it continuous (with respect to the topology induced from the real line)?

13**.** Let X be the subspace of $\mathbb{R}$ given by $X = [0,1] \cup [2,4]$. Define $f: X \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & if\ x \in [0,1] \\ 2, & if\ x \in [2,4] \end{cases}$$

prove that f is continuous.

2.5 Conclusion

You learned about Hausdorff, regular, and normal spaces as well as the five separation axioms in this unit. The ideas of continuity and homeomorphism were also studied. You also proved some important results which you have often used in your courses in analysis.

2.6 Summary

In this unit you now know that

(i) If X is a topological space, then X is

τ_0: If for all $x, y \in X$ with $x = y$, there exist an open set U such that either $x \in U$ and $y \notin U$or $y \in U$ and $x \notin U$.

τ_1: If for all $x, y \in X$ with $x = y$, there exist $U_x \in N(x)$ such that $y \notin U_x$. Or there exists $U_y \in N(y)$ such that $x \notin U_y$.

τ_2: If for all $x, y \in X$ with $x = y$, there exist $U_x \in N(x)$, $U_y \in N(y)$ such that $U_x \cap U_y = \emptyset$. $\tau_2$spaces are called Hausdorff spaces.

τ_3: If for every closed set $F \subset X$ and every $x \in X$ such that $x \notin F$ there exists open sets U_F, $U_x \subset X$ with $F \subset U_F$, $x \in U_x$ such that $U_x \cap U_F = \emptyset$.

τ_4: If for any two closed sets $E, F \subset X$ with $E \cap F = \emptyset$,there exists open sets U_E, $U_F \subset X$ such that $E \subset U_E$, $F \subset U_F$ and $U_E \cap U_F = \emptyset$.

(ii) X is a regular space if it is both τ_1 and τ_3.

(iii) X is a normal space if it is both τ_1 and τ_4. Also X is normal if and only if it is both Hausdorff (τ_2) and τ_4.

(iv) A function $f: X \longrightarrow Y$ between topological spaces X and Y is continuous if for every open set V of Y, the preimage

$$f^{-1}(V) = \{x \in X: f(x) \in V\}$$

is open in X.

(v) $f: X \longrightarrow Y$ is a homeomorphism if f is bijective and f and $f^{-1}: Y \longrightarrow X$ are continuous.

(vi) Topological spaces X and Y are homeomorphic if there exist a homeomorphism $f: X \longrightarrow Y$ between them.

(vii) A sequence $\{x_n\}$ in a topological space is convergent to $x \in X$ if given any neighbourhood V of x, we can find an integer $N \in N$ such that for all $n \geq N, x_n \in V$.

(viii) In a Hausdorff space, every singleton is closed.

(ix) In a Hausdorff space, the limit of a convergent sequence is unique.

(ix) Urysohn's lemma: If A and B be two disjoint closed subsets of a normal space X. Then there exists a continuous function $f: X \longrightarrow I$ such that $f(A) = 0$ and $f(B) = 1$.

(x) A topological space X is metrizable if its topological structure is generated by a certain metric.

(xi) Every metrizable space is Hausdorff.

2.7 References/Further Reading

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MODULE 2 SEPARABILITY, COMPACTNESS AND CONNECTEDNESS

Module Introduction

Separability is one of the requirements in topology to limit the size of the object that is under consideration. The actual definition is quite simple: a topological space X is separable if it has some subset Y which is both dense and countable. The concepts of separability is explained in this module A space is compact if every open cover of the space has a finite subcover. An *open cover* is a collection of open sets that covers a space. An example would be the set of all open intervals, which covers the real number line. The collection of all open intervals in the number line contains a lot of intervals. Compactness asks if there is a way to reduce that collection to a finite number of intervals and still cover the entire number line. That is, could we find a finite number of open intervals so that every point on the number line is in at least one of them? This question will be addressed in this module.

Connectedness is a topological property and is a powerful tool in proofs of well-known results. A connected topological space is one that is "in one piece". The way we will define this is by giving a very concrete notion of what it means for a space to be "in two or more pieces", and then say a space is connected when this is not the case. A topological space is said to be *connected* if it is not the union of two disjoint nonempty open sets. A set is open if it contains no point lying on its boundary; thus, the fact that a space can be partitioned into disjoint open sets suggests that the boundary between the two sets is not part of the space, and thus splits it into two separate pieces.

UNIT 1 CATEGORY AND SEPARABILITY

Unit structure

1.1 Introduction

In this unit, you shall be introduced to the notion of category, separability and axioms of countability. You shall be introduced with dense sets, and see some sets of the first and second categories.

1.2 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to:

- identify dense sets and nowhere dense sets.
- identify sets of first and second categories.
- identify separable spaces.
- state the first and second countability axioms.
- identify first and second countable space.
- state and prove the sequence lemma and its converse.

1.3 Dense Sets

Definition 1.2.1.1 (Dense Sets): Let X be a topological space and let A and B be two subsets of X. A is dense in B if $B \in \overline{A}$. A is dense in X or everywhere dense in X if $\overline{A} = X$.

Example 1.2.1.2: $\mathbb{Q}$ the set of rational numbers is a dense subset of $\mathbb{R}$ because $\overline{\mathbb{Q}} = \mathbb{R}$.

Proof: Suppose $\overline{\mathbb{Q}} = \mathbb{R}$. Then there exists an $x \in \mathbb{R}\backslash\overline{\mathbb{Q}}$. As $\mathbb{R}\backslash\overline{\mathbb{Q}}$ is open in $\mathbb{R}$, there exist a, b with $a < b$ such that $x \in (a, b) \subset \mathbb{R}\backslash\overline{\mathbb{Q}}$. But in every interval (a, b) there is a rational number q; that is$q \in (a, b)$. So,$q \in \mathbb{R}\backslash\overline{\mathbb{Q}}$ which implies $q \in \mathbb{R}\backslash\mathbb{Q}$. This is a contradiction, as $q \in \mathbb{Q}$. Hence $\overline{\mathbb{Q}} = \mathbb{R}$. ■

Example 1.2.1.3: Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. It is easy to see that $\overline{\{b\}} = \{b, e\}$, $\overline{\{a, c\}} = X$, and $\overline{\{b, d\}} = \{b, c, d, e\}$. Thus the set $\{a, c\}$ is dense in X.

Example 1.2.1.4: Let (X, τ) be a discrete space. Then every subset of X is closed (since its complement is open). Therefore, the only dense subset of X is X itself, since each subset of X is its own closure.

Theorem 1.2.1.5: Let (X, τ) be a topological space, and let A be a subset of X. A is dense in X if and only if every nonempty open subset U of X, $A \cap U = \emptyset$.

Proof: Assume that for all open sets$U$ of X, $U \cap A = \emptyset$. If $A = X$, then clearly A is dense in X. If A = X, let $x \in X \backslash A$. If $U \in \tau$ and $x \in U$ then $U \cap A = \emptyset$. So x is a limit point of$A$. As x is an arbitrary point in $X \backslash A$, every point of $X \backslash A$ is a limit point of A. So $X \backslash A \subset A^0$, and then by theorem 3.8 of unit 1, $\overline{A} = A^0 \cup A = X$; that is, A is dense in X.

Conversely, assume A is dense in X. Let U be a nonempty open subset of X. Suppose $U \cap A = \emptyset$. Then if $x \in U$, $x \notin A$ and x is not a limit point of A, since U is an open set containing x which does not contains any element of A. That is a contradiction since, as A is dense in X, every element of $X \backslash A$ is a limit point of A. So the supposition is false and $U \cap A = \emptyset$, as required. ■

Definition 1.2.1.6: A set is nowhere dense if the set $\overline{A}$ has empty interior.

Definition 1.2.1.7: Let A be a subset of a topological space (X, τ). Let $p \in X$. The point p is an isolated point of the set A if $p \in A$ and there exist $U_p \in N(p)$ such that $(A \backslash \{p\}) \cap U_p = \emptyset$.

1.3.1 Baire Spaces

Definition 1.2.2.1: Let Y be a subset of a topological space (X, τ). If Y is a union of a countable number of nowhere dense subsets of X, then Y is said to be a set of the first category or meager. If $Y$is not first category, it is said to be a set of the second category.

Definition 1.2.2.2: A topological space (X, τ) is said to be a *Baire Space* if for every sequence $\{X_n\}$ of open dense subsets of X, the set $\{X_n\}_{n=1}^{\infty}$is also dense in X.
Example 1.2.2.3: Every complete metric space is a *Baire space*.

1.3.2 The Axioms of Countability

In this section, you shall be introduced to three restriction on the topological structure. These are first and second countability axioms and the separability. Before proceeding to state these axioms, you have the following important definition and results.

Definition 1.2.3.1 (Equal Cardinality): Two sets A and B have equal cardinality if there exists a bijection between them.

Definition 1.2.3.2 (Countable Sets): A set A is said to be a countable set if it has the same cardinality as a subset of the set $\mathbb{N}$ of positive integers. While A is said to be at most countable if it has the same cardinality as the set $\mathbb{N}$ of positive integers.

Results: The following results will be stated without proof, because that is not the major interest here. You can find the proves in any good textbook on topology or analysis.

1. A set X is countable if and only if there exists an injection $\phi : X \longrightarrow \mathbb{N}$ (or, more generally, an injection of X into another countable set).
2. Any subset of a countable set is countable.
3. The image of a countable set under any map is countable.
4. $\mathbb{N}$ is countable.
5. The set $\mathbb{N}^2 = \{(k, n) : k, n \in \mathbb{N}\}$ is countable.
6. The union of a countable family of countable sets is countable.
7. $\mathbb{Q}$ is countable.
8. $\mathbb{R}$ is not countable.

1.3.3 Second Countability Axiom

First of all, you shall be introduced to the second countability axiom and separability.

Definition 1.2.4.1 (Second Countability axiom): A topological space X satisfies the second axiom of countability or is second countable if X has a countable basis.

Example 1.2.4.2 :$\mathbb{R}$ endowed with the standard topology is second countable. The basis

$$B = \{(a, b), a < b, a, b \in \mathbb{Q}\} = \mathbb{Q} \times \mathbb{Q}.$$

Hence is countable. Also

$$B = \left\{r - \frac{1}{n}, r + \frac{1}{n}, r \in \mathbb{Q}, n \geq 1\right\}$$

is a countable basis of $\mathbb{R}$.

Example 1**.2.4.3 :**$\mathbb{R}$ endowed with the lower limit topology is not second countable.

Example 1**.2.4.4 :** The discrete topology of any uncountable set is not second countable.

Example 1**.2.4.5 :** Not all metric spaces are second countable. For instance,$\mathbb{R}$ with the discrete metric. i.e.,

$$\rho_0(x,y) = \begin{cases} 1 & if x = y \\ 0 & if x \neq y \end{cases}$$

is not second countable.

1**.3.4 Separability and Separable Spaces**

Definition 1**.2.5.1 (Separability):** A topological space X is separable if it contains a countable dense subset.

Example 1**.2.5.2:**$\mathbb{R}$ endowed with the standard topology is separable because $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

Example 1**.2.5.3:** Any infinite set X endowed with the finite complement topology is separable since any infinite set is dense in X.

Example 1**.2.5.4:** The set of all points $x = (x_1, x_2, x_3 \ldots, x_n)$ with rational coordinates is a countable dense subset in the metric space $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is separable.

Example 1**.2.5.5:** The set of all points $x = (x_1, x_2, x_3 \ldots, x_k, \ldots)$ with only finitely many nonzero rational coordinates, is countably dense in the space

$$\ell_2 = \left\{ x = (x_1, x_2, x_3 \ldots, x_k, \ldots): \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}$$

Hence, ℓ_2 is separable.

Example 1**.2.5.6:** The set of all polynomials with rational coefficients is countably dense in the space $C[a,b]$ of continuous real valued function. Hence $C[a,b]$ is separable.

Theorem 1**.2.5.7:** Any second countable topological space X is separable.

Proof: Suppose X is second countable, then X contains a countable basis $\mathcal{B} = \{B_n, n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ choose $d_n \in B_n$ and define $D = \{d_n, n \geq 1\}$ then D is dense in X. ■

Remark 1**.2.5.8:** The converse of this theorem is not true in general. Notwithstanding in a metric space, second countability and separability are equivalent.

Theorem 1**.2.5.9:** Let (X, d) be a separable metric space then X is second countable.

Proof: Since X is separable, $D = \{d_n, n \in \mathbb{N}\}$ is a countable dense subset of X. Take $\mathcal{B} = \{B(d_n, \frac{1}{n}), n \geq 1, m \geq 1\}$. Then $\mathcal{B}$ is a countable basis for (X, d)

1**.3.5 Sequence Lemma**

Definition 1.2.6.1: A topological space (X, τ) is metrizable if there exists a metric d on the set X such that the topology τ on X is induced by d.

Theorem 1.2.6.2 (Sequence Lemma):

1. Let X be a topological space, and A be a subset of X. If there exists a sequence $\{x_n\}$ of elements of A converging to $x \in X$, then $x \in \overline{A}$. The converse holds if X is metrizable.
2. Let X and Y be topological spaces, and $f : X \longrightarrow Y$ be a function. If the function f is continuous, then for every sequence $\{x_n\}$ is in X such that $\{x_n\}$ converges to $x \in X$, The sequence $\{f(x_n)\}$ converges to $f(x)$ in Y. The converse is true if X is metrizable.

Proof: 1. Let $x \in X$. Suppose that there exists a sequence $\{x_n\}$ in A such that $x_n \longrightarrow x$. You have to show that $x \in \overline{A}$. Let U be a neighbourhood of x, $x_n \longrightarrow \infty$ as $n \longrightarrow \infty$ implies that there exist $N \in N$ such that for all $n \geq N$, $x_n \in U$. In particular, $x_N \in U$. But $x_N \in A$ implies that $U \cap A = \emptyset$. which implies that $x \in \overline{A}$.

Conversely, suppose that X is metrizable and $x \in \overline{A}$. Let d be a metric for the topology of X. For each $n \geq 1$, the neighbourhood

$B\left(x, \frac{1}{n}\right) \cap A = \emptyset$.

Choose $x_n \in B\left(x, \frac{1}{n}\right) \cap A$ for $n \geq 1$. Then, $\{x_n\}$ is a sequence of points of A and

$0 \leq d(x_n, n) < \frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$

which implies that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

2. Assume that f is continuous. Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. You have to show that $f(x_n) \longrightarrow f(x)$. Let V be a neighbourhood of $f(x)$. Then $f^{-1}(V)$ is a neighbourhood of x, and so there exists $N \geq 1$ such that $x_n \in f^{-1}(V)$ for $n \geq N$. Then $f(x_n) \in Vn \geq N$, which implies that $f(x_n) \longrightarrow f(x)$ as $n \longrightarrow \infty$ as desired.

Conversely, assume that the convergence condition is satisfied. Let A be a subset of X. You have to show that f is continuous, it suffices to show that $f(\overline{A}) \subset \overline{f(A)}$. If $x_n \in \overline{A}$, there exists a sequence $\{x_n\}$ of points of A converging to x (by seqence lemma). By assumption the sequence $\{f(x_n)\}$ converges to $f(x)$. Since $f(x_n) \in f(A)$, the sequence lemma implies that $f(x) \in \overline{f(A)}$, as desired. ■

1.3.6 Neighbourhood Basis

Definition 1.2.7.1 (Neighbourhood basis): Let (X, τ) be a topological space and let $x \in X$. The collection W is called a neighbourhood basis of the point x if the following conditions are satisfied;

(i) $\mathbb{W}$ is a subcollection of neighbourhoods of $x\big(\mathbb{W} \subset N(x)\big)$. i.e., for all $W \in \mathbb{W}, W \in N(x)$.

(ii) (ii) For all $V \in N(x)$, there exist $W \in \mathbb{W}$ such that $W \subset V$.

Example 1.2.7.2: Let $\mathbb{R}$ be endowed with the standard topology. Then for all $x \in \mathbb{R}$,

$$\mathbb{W} = \{(x - r, x + r), \qquad r > 0\}$$

is a neighbourhood basis of x.

Proof:

(i) Let $x \in X$. Clearly, for all $r > 0$, $(x - r, x + r)$ is a neighbourhood of x and so $W \subset N(x)$.

(ii) Let $V \in N(x)$ then there exist an open set U such that $x \in U \subset V$.

This implies that there exists $r > 0$ such that $(x - r, x + r\ ) \subset U \subset V$.

Example 1.2.7.3: Let (X, d) be a metric space, let $x \in X$, then

$$\mathbb{W} = \{B_d(x, r), r > 0\}$$

is a neighbourhood basis in the metric topology.

Example 1.2.7.4: Let $\mathbb{R}_1$ denote the real line endowed with the lower limit topology. Let $x \in X$, then

$$\mathbb{W} = \{[x, x + r), r > 0\}$$

is a neighbourhood basis for the lower limit topology on the real line.

Example 1.2.7.5: Let (X, τ) be a discrete topololgical space. Then for all $x \in X$,

$$\mathbb{W} = \{\{x\}, x \in X\}$$

is a neighbourhood basis of x in the discrete topology.

1.3.7 First Countability Axiom

Definition 1.2.8.1(First Countability Axiom): A topological space X satisfies the first countability axiom or is said to be first countable if any point $x \in X$ has a countable neighbourhood basis.

Example 1.2.8.2 : Let $\mathbb{R}$ be endowed with the standard topology. For all $x \in \mathbb{R}$ define

$$\mathbb{W} = \left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) : n \geq 1\right\}$$

Or

$$\mathbb{W} = \{(x - r, x + r) : r > 0, r \in \mathbb{Q}\}$$

In each case, $\mathbb{W}$ is a countable neighbourhood basis of x. Thus $\mathbb{R}$ is first countable.

Example 1.2.8.3: Let $\mathbb{R}$ be endowed with the lower limit topology. For all $x \in \mathbb{R}$, define

$$\mathbb{W} = \{(n = x, x + 1) : n \geq 1\}$$

Then $\mathbb{W}$ is a countable neighbourhood basis for x. Hence, $\mathbb{R}$ with the lower limit topology is first countable.

Example 1.2.8.4: Let (X, d) be a metric space. For every $x \in X$, define

$$W = B\left(x, \frac{1}{n}\right) : n \geq 1$$

Or

$$W = \{B(x, r) : r > 0, r \in \mathbb{Q}\}$$

Then in each case, W is a countable neighbourhood basis of x. Thus, every metric space is first countable.

Theorem 1.2.8.5: Let (X, τ) be a topological space. If X is second countable, then X is first countable.

Proof: Assume that X is second countable, then X has a countable basis $\mathcal{B} = \{b_n, n \in N\}$. Let $x \in X$, and define

$$\mathbb{W} = \{B_x, x \in B_n\}$$

then $W \subset \mathcal{B}$, so that $\mathbb{W}$ is countable.

1. For all $B_n \in \mathbb{W}$, $B_n \in N(x)$.
2. Let $V \in N(x)$, this implies that there exists an open set U such that $x \in U \subset V$. This implies that there exists $B_{n_x} \in \mathbb{W}$ such that $x \in B_{n_x} \subset$

$U \subset V$, so that $B_{n_x} \subset V$. Thus $\mathbb{W}$ is a countable neighbourhood of x. Hence X is first countable.

■

1.3.8 Sequence Lemma Revisited

Recall that in the sequence lemma which we proved above, It says that if A is subset of a topological space X and there exists a sequence$\{x_n\}$ of points of A such that $x_n \longrightarrow x$ in X as $n \longrightarrow \infty$, then $x \in \overline{A}$. And we proved the converse in a metrizable space. This tells us that the implication

$(\Longrightarrow)$ if $\{x_n\}$ is a sequence in A such that $x_n \longrightarrow x$ in X, then $x \in \overline{A}$ is true in any topological space. But the converse, if $x \in \overline{A}$ then there exists a sequence $\{x_n\}$ of A such that $x_n \longrightarrow x$ is only true if X is a metrizable space.

Similarly, for the continuous function $f: X \longrightarrow Y$, sequential continuity holds for topological spaces X and Y, i.e.,f is continuous, implies for all sequence $\{x_n\}$ of X such that $x_n \longrightarrow x$ in X, $f(x_n) \longrightarrow f(x)$ in Y.

The converse, i.e., for a sequence $\{x_n\}$ of X such tha if $x_n \longrightarrow x$ implies that $f(x_n) \longrightarrow f(x)$ then f is continuous; holds if and only if X is metrizable.

In what follows, you shall discover that if X is a first countable space the X also recorvers the converse of the sequence lemma. i.e., the converse of the sequential closure and the sequential continuity. Before you proceed, the following lemma will be useful.

Lemma 1.2.9.1: Let X be a topological space and let $x \in X$. Suppose X is first countable, then there exist a countable basis of x, say, $\mathbb{W} = \{W_n, n \geq 1\}$ such that $W_{n+1} \subset W_n$.

Proof: Let $x \in X$. Since X is first countable then there exists a countable neighbourhood basis $V = \{V_n, n \geq 1\}$ of x. Define for each $n \geq 1$,

$$W_n = \bigcap_{n=1}^{n} V_i$$

and let $\mathbb{W} = \{W_n, n \geq 1\}$. Then

(i) $\mathbb{W}$ is countable.

(ii) $W_n \in N(x)$, for each $n \geq 1$, because finite intersection of neighbourhoods of a point x is also a neighbourhood of x.

(iii) Let $V \in N(x)$, there exists $N \in N$ such that $V_N \in V$ and $x \in V_N \subset V$. But

$$W_N = \bigcap_{n=1}^{N} V_i \subset V_N \subset V$$

Thus, for every $V \in N(x)$ there exist N such that $W_N \in \mathbb{W}$ and $W_N \subset V$.

(iv) $W_{n+1} = \bigcap_{n=1}^{n+1} V_i = V_{n+1} \cap \bigcap_{n=1}^{n} V_i \subset \bigcap_{n=1}^{n} V_i = W_n$.That is $W_{n+1} \subset W_n$.Thus $\mathbb{W}$ is a countable neighbourhood basis of x that satisifies $W_{n+1} \subset W_n$, $n \geq 1$. ■

Theorem 1.2.9.2: Let X be a first countable topological space and A be a subset of X. Then if $x \in A$, there exists a sequence $\{x_n\}$ of A such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Proof: Since X is first countable, from lemma 3.1, there exists a countable neighbourhood $\mathbb{W} = \{W_n, n \geq 1\}$ such that $W_{n+1} \subset W_n$. Now let $x \in \overline{A}$. This implies that for all $n \geq 1$, $W_n \cap A = \emptyset$. Let $x_n \in W_n \cap A$. Then $\{x_n\}$ is a sequence of points of A.

Claim: $x_n \longrightarrow x$ as $n \longrightarrow \infty$

Proof of Claim: Let $V \in N(x)$. Then there exists $N \in N$ such that $x_n \in W_N \subset V$ and for all $n \geq N$,

$$x_n \in W_n \subset W_N \subset V$$

This implies that for all $n \geq N$, $x_n \in V$. Hence $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and the proof is complete.

Theorem 1.2.9.3: Let X and Y be two topological spaces and let $f: X \longrightarrow Y$ be a function. Suppose X is first countable. If for every sequence $\{x_n\}$ of X such that $x_n \longrightarrow x$ in X as $n \longrightarrow \infty$, one has that $f(x_n) \longrightarrow f(x)$ in Y then f is continuous.

Proof: It suffices to prove that if F is closed subset of Y, then the preimage $f^{-1}(F)$ is closed in X, i.e., $\overline{f^{-1}(F)} = f^{-1}(F)$. But you have already that $f^{-1}(F) \subset \overline{f^{-1}(F)}$, so it is left for you to show that $\overline{f^{-1}(F)} \subset f^{-1}(F)$. So let $x \in \overline{f^{-1}(F)}$, Since X is first countable, you have by seqeunce lemma that there exist a sequence $\{x_n\}$ of points of $f^{-1}(F)$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. This implies that $f(x_n)$ is a sequence of elements of F, and by assumption, $f(x_n) \longrightarrow f(x)$ in Y.

Since F is closed, $F = \overline{F}$ and so $f(x) \in F$, that is $x_n \in f^{-1}(F)$. Thus $\overline{f^{-1}(F)} \subset f^{-1}(F)$ as required. Therefore, $f^{-1}(F)$ is closed in X. Hence f is continuous. Type equation here.

1.4 Self – Assessment Exercise(s)

1. Given $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Let $A = \{a, c\}$. Then the set A^0 of limit points of A is given by
   (a) $A^0 = \{b, c, e\}$
   (b) $A^0 = \{b, d, e\}$
   (c) $A^0 = \{b, e\}$
   (d) $A^0 = X$.
2. Let $\mathbb{R}$ the real line be endowed with the discrete topology. Which of the following subsets of $\mathbb{R}$ is dense in $\mathbb{R}$?
   (a) $\mathbb{Q}$
   (b) $\mathbb{R}$ itself
   (c) $\mathbb{Q}^c$.
   (d) All singletons.
3. Let $A = (0, 1] \cup \{2\}$ be a subset of $\mathbb{R}$. Then the isolated points of A in $\mathbb{R}$ are
   (a) 0 and 1
   (b) 0 and 2
   (c) 1 and 2
   (d) 2 only 4.
4. For the set A in question 3, Which of the following are the limit points of A?
   (a) 0 and 1
   (b) 1 and 2
   (c) 0 only
   (d) 2 only
5. In $\mathbb{R}$ with the standard topology, which of the following sets is nowhere dense?
   (a) $\mathbb{Q}^c$
   (b) $\{\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$
   (c) $(0, 1)$
   (d) $[0, 1)$
6. The minimal neighbourhood basis of a point x in the discrete topology contains
   (a) the whole set X and the empty set $\emptyset$only.
   (b) Only the singletons.
   (c) All open sets of X only.
   (d) The whole set X only.
7. The minimal neighbourhood of a point x in the indiscrete topology contains
   (a) the whole set X and the empty set $\emptyset$ only.
   (b) Only the singletons.

(c) The empty set only.
(d) The whole set X only.

8. Which of the following spaces is second countable?
(a) $\mathbb{R}$ with the finite complement topology.
(b) $\mathbb{R}$ with the countable complement topology.
(c) $\mathbb{R}$ with the lower limit topology.
(d) $\mathbb{N}$ with the discrete topology.

9. Which of the following spaces is not first countable?
(a) $\mathbb{R}$ endowed with the lower limit topology.
(b) $\mathbb{R}$ endowed with the finite complement topology.
(c) $\mathbb{R}$ endowed with the discrete topology.
(d) $\mathbb{Q}$ endowed with the indiscrete topology.

1.5 Conclusion

In this unit, you were introduced to dense sets, sets of first and second category, and Baire spaces. You also studied the axioms of countability and separability and saw some examples of spaces that satisfy some of the axioms. You are able to prove that a first countable space satisfies the converse of the sequence lemma.

1.6 Summary

Having gone through this unit, you now know that;

(i) A subset A of a topological X is dense in $B \subset X$ if $B \subset \overline{A}$. A is everywhere dense in X if $\overline{A} = X$, while A is nowhere dense in X if $int(\overline{A}) = \emptyset$.
(ii) A subset Y of a topological space X is of the first category if Y is a countable union of sets of nowhere dense subsets of X. Otherwise Y is of the second category.
(iii) A set is countable if it has the same cardinality with at least a subset of a countable set.
(iv) A point $p \in X$ is called an isolated point of a subset A of a topological space X if there exists a neighbourhood U of p such that $(A\backslash\{p\}) \cap U = \emptyset$.
(v) $\mathbb{W}$ is a neighbourhood basis of a point $x \in X$ if
(a) for all $W \in \mathbb{W}$, $W \in N(x)$.
(b) $V \in N(x)$ then there exists $W \in \mathbb{W}$ such that $W \subset V$.
(vi) A topological space is first countable if it contains a countable neighbourhood basis.
(vii) A topological space is second countable if it contains a countable basis.

(viii) A topological space is separable if it contains a countable dense subset.
(ix) Every second countable space is first countable.
(x) Every second countable space if separable. The converse is true if the space is metrizable.
(ix) A topological space X is metrizable if its topological structures can be generated by a metric.
(x) Sequence Lemma
 (a) If there exists a sequence $\{x_n\}$ of elements of a subset A of a topological space X, such that $x_n \longrightarrow x \in X$, then $x \in A$.
 (b) If $f: X \longrightarrow Y$ is continuous, then for all sequence $\{x_n\}$ of elements of X, such that $x_n \longrightarrow x \in X$ then $f(x_n) \longrightarrow f(x)$ in Y. The converse of the sequence lemma is true if X is either first countable or metrizable.

1.7 References/Further Reading

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UNIT 2 COMPACT SETS AND SPACES

Unit structure

2.1 Introduction

In this unit, you shall be introduced to a topological property playing a very special and important role in topology and its application. It is a sort of topological counterpart for the property of being finite in the context of set theory.

2.2 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to;

- Give the definition of Covers and subcorvers.
- Define compact sets, subsets and compact spaces.
- Give the sequential characterization of compactness.
- Identify sequentially, countably and locally compact sets.

2.3 Definition and Examples

Definition 2.2.1.1 (Covering and Open Cover): A collection A of subsets of X is said to be a covering of X, if the union of the elements of A is X. i.e.,

$$X = \bigcup_{i \in I} O_i$$

where $U_i \in A$ for all $i \in I$, (I is an index set).

A is called open covering if its elements are open subsets of X.

Definition 2.2.1.2 (Subcover): If A is a covering of X and $O \subset A$ is also a covering of X, then O is a subcover or subcovering of A.

Definition 2.2.1.3 (Compact Set): A topological space X is compact if every open covering of X is reducible to a finite subcovering. That is, a topological space X is compact if for every open covering $\{O_i\}_{i \in I}$, there exists a finite subfamily $O_{i_1}, O_{i_2}, O_{i_{3,}} \dots, O_{i_n}$ such that such that

$$X = \bigcup_{k=1}^{n} O_{i_k}$$

Definition 2.2.1.4: Let A be a subset of a topological space X. Then A is said to be compact if for every family of open sets $\{O_i\}_{i \in I}$ such that

$$A \subseteq \bigcup_{i \in I} O_i$$

there exists a finite subfamily $O_{i_1}, O_{i_2}, O_{i_{3,}} \dots, O_{i_n}$ such that

$$A \subseteq \bigcup_{k=1}^{n} O_{i_k}$$

Example 2.2.1.5: Let X be endowed with the indiscrete topology. Then X is compact.

Proof: In the indiscrete topology, the only open covering of X is the $\emptyset$ and X itself. Hence, X is compact.■

Example 2.2.1.6: The real line $\mathbb{R}$ endowed with the standard topology is not compact.

Proof: It suffices to produce an open covering of $\mathbb{R}$ which cannot be reducible to a finite subcovering. Now

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$$

If there exist a finite open subcover, then there exists $n_1, n_2, n_3, \dots, n_m$ such that

$$\mathbb{R} = \bigcup_{n=1}^{m} (-n_i, n_i) = (-N, N)$$

Where $N = \max_{1 \le i \le m} n_i$ which is impossible. Hence $\mathbb{R}$ is not compact. ■

Example 2.2.1.7: Let $A = (0,1]$. Then A is not compact in $\mathbb{R}$.
Proof: In $(0,1]$ we have the trace topology (i.e., su24bspace topology) $\left(\frac{1}{n}, 2\right), n \in N$is an open covering of

$$(0,1] = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right)$$

Suppose that $(0,1]$ is compact, then there exists $n_1, n_2, n_3, \dots, n_m$ such that

$$(0,1] = \bigcup_{n=1}^{\infty} \left(\frac{1}{n_i}, 2\right) = \left(\frac{1}{N}, 2\right)$$

Where $N = \max_{1 \le i \le m} n_i$, which is a contradiction. Hence (0,1] is not compact.

Example 2.2.1.8:$\mathbb{R}_+^* = (0, +\infty)$ is not compact.

Proof: Suppose that $\mathbb{R}_+^*$ is compact, $\left\{\left(\frac{1}{n}, n\right), n \in N\right\}$is an open covering of $\mathbb{R}_+^*$ such that

$$\mathbb{R}_+^* = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, n\right)$$

So there exist $n_1, n_2, n_3, \dots, n_m$ such that

$$\mathbb{R}_+^* = \bigcup_{i=1}^{m} \left(\frac{1}{n_i}, n_i\right) = \left(\frac{1}{N}, N\right)$$

Where $N = \max_{1 \le i \le m} n_i$. This is impossible.■

Example 2.2.1.9: Let (X, τ) be a topological space and let $\{x_n\}$ be a sequence of points of X such that $x_n \longrightarrow x \in X$ in X, then $\{x_n, n \ge 1\} \cup \{x\}$ is compact.

Example 2.2.1.10: Any finite set of a topological set (X, τ) is compact.

Proof: Let $A \subset X$ be a finite set, the elements of A can be listed, i.e., $A = \{x_1, x_2, x_3, \dots, x_n\}$. Let $\{O_i\}_{i \in I}$ be an open covering for A, i.e.,

$$A \subseteq \bigcup_{i \in I} O_i$$

Then for each $x_j \in A$, choose an open set O_{i_j} such that $x_j \in O_{i_j}$. Thus

$$A \subseteq \bigcup_{j=1}^{n} O_{i_j} \blacksquare$$

Remark 2.2.1.11: So you see from Example 3.1.6 that every finite set (in a topological space) is compact. Indeed, as earlier mentioned in the beginning of this unit, “compactness” can be thought as a topological generalization of “finiteness”.

Example 2.2.1.12: A subset A of a discrete space is compact if and only if it is finite.

Proof: If A is finite then Example 3.1.6 shows that it is compact.

Conversely, let A be compact. Then the family of singleton sets $O_x = \{x\}$, $x \in A$ is such that each O_x is open and

$$A \subseteq \bigcup_{x \in A} O_x$$

Since A is compact, there exist $x_1, x_2, x_3, \ldots, x_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} O_{x_i};$$

That is $A \subseteq \{x_1, x_2, x_3, \ldots, x_n\}$. $\blacksquare$

Theorem 2.2.1.13: Any closed and bounded interval in $\mathbb{R}$ is compact.

Proof: Let $[a, b]$, $a < b$ be a closed and bounded interval of $\mathbb{R}$. Let $\{O_i\}_{i \in I}$ a family of open sets of $\mathbb{R}$ such that

$$[a, b] \subseteq \bigcup_{i \in I} O_i$$

Step 1: Suppose $a \leq x < b$. Then there exists $y > x$ such that $[x, y]$ can be covered by at most two $O^0 s$. For this end, if x has an immediate successor y, then the interval $[x, y]$ has only two elements. So it can be covered by at most two $U^0 s$. If x does not have an immediate successor, find U_i containing x. Pick $z > x$ such that $[x, z) \subset U_i$; this is possible because U_i is open. Since x does not have an immediate successor, there is $y$such that $x < y < z$. Then $[x, y] \subset U_i$. $\blacksquare$

Step 2: Now

let$A = \{y \in (a, b]: [a, y] \text{ can be covered by finitely many } U_i\}$

By step 1, there exists an element $y > a$ such that $[a, y]$ can be covered at most by two $U_i^0 s$. Therefore,A is nonempty and bounded above. Let $c = \sup A$.

Step 3: Claim: $c \in A$. Let i such that c$\in U_i$. Since U_i is open and $c > a$, there exists an interval $(d, c] \in U_i$. Since d cannot be an upper bound for A, there is an element of A larger than$d$. Let z such that $d < z < c$. Then $[a, c^0]$ can be covered by finitely many $U_i^0 s$ and $[c^0, c] \subset U_i$. Therefore $[a, c] = [a, c^0] \cup [c^0, c]$ can be covered by finitely many $U_i^0 s$. Hence $c \in A$.

Step 4: Claim: $c = b$. Suppose $c < b$. By step 1, there exist $y > c$ such that $[c, y]$ can be covered by at most two $U_i^0 s$. Since $c \in A$, $[a, c]$ can be covered by finitely many $U_i^0 s$. So $[a, y] = [a, c] \cup [c, y]$ can be covered by finitely many $U_i^0 s$ and therefore $y \in A$. This contradicts the fact that $c = \sup$. Hence $c = b$.■

Theorem 2.2.1.14: A closed subset A of a compact topological space (X, τ) is compact.

Proof: Let $\{O_i\}_{i \in I}$ be a family of open subsets of X such that

$$A \subseteq \bigcup_{i \in I} O_i$$

Now

$$X = A \cup A^c = \bigcup_{x \in A} O_x \cup A^c$$

Since X is compact, there exists $i_1, i_2, i_3, \dots, i_m$ such that

$$X = \bigcup_{j=1}^{m} O_{i_j} \cup A^c$$

This implies that

$$A \subseteq \bigcup_{j=1}^{m} O_{i_j}$$

Hence, A is compact.

Theorem 2.2.1.15: If A is a compact subset of a Hausdorff topological space (X, τ), then A is closed.

Proof: Suppose A is a compact subset of X and let $x \in A^c$. Then for all $y \in A$, $x = y$. Since X is Hausdorff, there exist U_y open in X and contains x, V_y open in X and contains y such that $U_y \cap V_y = \emptyset$. So

$$A \subseteq \bigcup_{y \in A} V_y$$

Since A is compact, there exists $y_1, y_2, y_3, \dots, y_m$ such that

$$A \subseteq \bigcup_{y \in A}^{m} V_{y_i}$$

Let

$$U = \bigcap_{i=1}^{m} U_{y_i}$$

and

$$A \subseteq \bigcup_{y \in A}^{m} V_{y_i}$$

Then U is open and contains x, V is open and contains A, and $U \cap V = \emptyset$. This implies that $U \cap A = \emptyset$, that is $U \subset A^c$. Thus A^c is open. Hence A is closed. ■

In the course of the proof of theorem 3.3, you proved the following result.

Theorem 2.2.1.16: Let A be a compact subset of a Hausdorff topological space X and let $x \in X$. Then there exists open sets U and V with $A \subset V$ and $x \in U$ such that $V \cap U = \emptyset$. This result is the third separation axiom τ_3.

Theorem 2.2.1.17: Let A and A be compact subsets of a Hausdorff topological space X such that $A \cap B = \emptyset$. Then there exists open sets U and V with $A \subset U$ and $B \subset V$ such that $U \cap V = \emptyset$.

2.3.1 Compactness in Product Spaces

Theorem 2.2.2.1 (Tube Lemma): Let $X \times Y$ be the product topology. Suppose that Y is compact. If W is an open subset of $X \times Y$ containing $\{x\} \times Y$ for some $x \in X$, then it contains some tube $U \times Y$ around $\{x\} \times Y$. Where U is an open set containing x.

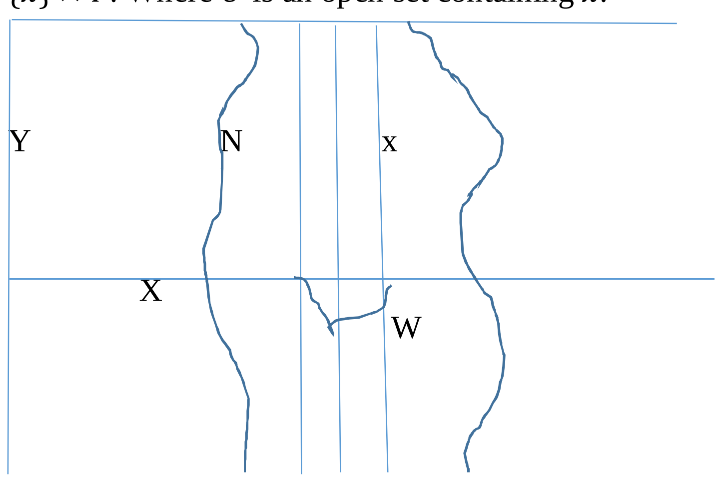

Figure 2.2.2.2: Tube Lemma

Proof: Observe that $\{x\} \times Y = Y$, and since Y is compact, $\{x\} \times Y$ is compact. Now for each $y \in Y$, you have $(x, y) \in \{x\} \times Y \subset W$. Therefore, there exists open sets U_y containing$x$, V_y containing y such that $(x, y) \in U_y \times V_y \subset W$. Thus $\{U_y \times V_y, y \in Y\}$ is an open cover of $\{x\} \times Y$. Since $\{x\} \times Y$ is compact, there exists $y_1, y_2, y_3, \dots, y_n$ such that

$$\{x\} \times Y \subset \bigcup_{i=1}^{m} U_{y_i} \times V_{y_i}$$

Take

$$U = \bigcap_{i=1}^{n} U_{y_i}$$

Then U is open, it contains x and $\{x\} \times Y \subset U \times Y \subset W$. For if $(z, y) \in U \times Y$, you have that $z \in U$ and $y \in Y$. $y \in Y$ implies that there exists i_0 such that $y \in V_{y_{i_0}}$. This implies that $z \in U_{y_{i_0}}$and $(z, y) \in U_{y_{i_0}} \times V_{y_{i_0}} \subset W$.

Theorem 2.2.2.3: A finite product

$$\prod_{i=1}^{n} X_i$$

of compact spaces $\{X_i\}_{i=1}^{n}$ is compact.

This theorem is called the ***Tychonoff product theorem***. The converse of the Tychonoff product theorem is also true.

Proof: You can prove this for a product $X \times Y$ of two compact spaces X and Y. The generalization follows by induction. So let $\{W_i\}_{i \in I}$ be a family of open sets of the product topology, such that

$$X \times Y \subseteq \bigcup_{i \in I} W_i$$

Let $x \in X$ be fixed. You have that

$$\{x\} \times Y \subseteq X \times Y \subseteq \bigcup_{i \in I} W_i$$

$\{x\} \times Y$ is compact since Y is also compact, and so there exists $i_1, i_2, i_3, \dots, i_m$ such that

$$\{x\} \times Y \subseteq \bigcup_{j=1}^{m} W_{i_j} = W_x$$

By tube lemma, there exists an open set U_x containing x such that$\{x\} \times Y \subseteq U_x \times Y \subseteq W_x$

And so

$$X \subseteq \bigcup_{x \in X} U_x$$

Since X is compact, there exists $x_1, x_2, x_3, \dots, x_n$ such that

$$X \subseteq \bigcup_{i=1}^{m} U_{x_i}$$

Therefore,

$$X \times Y \subseteq \bigcup_{i=1}^{n} (U_{x_i} \times Y) \subseteq \bigcup_{i=1}^{n} W_{x_i} \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} W_{i_j}$$

Hence, $X \times Y$ is compact.■

2.3.2 Heine-Borel Theorem

Theorem 2.2.3.1: A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof:($\Longrightarrow$) Let $\mathbb{R}^n$ be endowed with the Euclidean metric

$$d(x, y) = \sum_{i=1}^{n} [(x_i - y_i)^2]^{\frac{1}{2}}$$

Assume A is compact then$A$ is closed since $\mathbb{R}^n$ is Hausdorff. Also $\{B(0, n), n \in N\}$is a family of open sets of $\mathbb{R}^n$ and

$$A \subseteq \bigcup_{n=1}^{\infty} B(0, n)$$

Where $B(0, n) = \{y \in \mathbb{R}^n : \ d(y, 0) < n\}$is the open ball with center 0 and radius n. By the compactness of A, there exists $n_1, n_2, n_3, \dots, n_k$ such that

$$A \subseteq \bigcup_{i=1}^{k} B(0, n_i) \subseteq B(0, N)$$

where $N = \max_{1 \leq i \leq k} n_i$. Hence A is bounded.

($\Longrightarrow$)Suppose A is closed and bounded in $\mathbb{R}^n$, and show that A is compact. It suffices to show that A is a subset of a compact set. But A is bounded implies that there exist $R > 0$ such that

$$A \subseteq B(0, n_i) \prod_{i=1}^{n} [-\mathbb{R}, \mathbb{R}]$$

each $[-\mathbb{R}, \mathbb{R}]$ is compact in $\mathbb{R}$ and so

$$\prod_{i=1}^{n} [-\mathbb{R}, \mathbb{R}]$$

is compact as a finite product of compact sets. And so A is a closed subset of a compact set, therefore, A is compact.

Remark 1.2.3.1: Note that the above theorem was proved in $\mathbb{R}^n$. In an arbitrary metric space, what you have is that any compact space is closed and bounded but the converse is not true.

2.3.3 Finite Intersection Property (FIP)

Definition 2.2.4.1 (Finite intersection Property (FIP): Let X be a topological space. A collection C of subsets of X satisfies the Finite Intersection Property (FIP) if any intersection of a finite subcollection of C is nonempty.

$C = \{A_i, i \in I\}$ satisfy FIP if for any $J \in P_f(I)$,

$$\bigcap_{i \in J} A_i = \emptyset$$

Where $P_f(I)$ (finite part of) denotes a the set of all finite indexes of I.

Theorem 2.2.4.2: A topological space X is compact if and only if collection $C = \{C_i, i \in I\}$ of closed sets having the FIP, one has that

$$\bigcap_{i \in I} C_i = \emptyset$$

Proof:($\Longrightarrow$) Let X be a compact set and $C = \{C_i, i \in I\}$ be a collection of closed sets of X having the finite intersection property, i.e., for all $J \in P_f(I)$ such that

$$\bigcap_{i \in J} C_i = \emptyset$$

You have to show that

$$\bigcap_{i \in I} C_i = \emptyset$$

Suppose

$$\bigcap_{i \in I} C_i = \emptyset$$

Then

$$X = \bigcup_{i \in I} (X \backslash C_i)$$

Each $X \backslash C_i$ is open since C_i is closed Thus, $\{X \backslash C_i, i \in I\}$ is an open covering for X and since X is compact, there exists $J \in P_f(I)$ such that

$$X = \bigcup_{i \in j} (X \backslash C_i)$$

This implies that

$$\bigcap_{i \in J} C_i = \emptyset$$

contradicting the assumption that C satisfies FIP. Hence our supposition was wrong. Therefore

$$\bigcap_{i \in J} C_i = \emptyset \qquad \blacksquare$$

Corollary 2.2.4.3: Let X be a compact space and let $\{C_n, n \geq 1\}$ be a collection of nonempty closed sets such that $C_{n+1} \subset C_n$. Then

$$\bigcap_{n \geq 1} C_i = \emptyset.$$

Proof: Let $n_1, n_2, n_3, \dots, n_p \in N$, since $C_{n+1} \subset C_n$, and each C_n is nonempty, then

$$\bigcap_{i=1}^{p} C_{n_i} = C_N = \emptyset$$

Where $N = \max_{1 \leq i \leq p} n_i$. This implies that $\{C_n, n \geq 1\}$ satisfies the FIP. So by the last theorem,

$$\bigcap_{n \geq 1} C_n = \emptyset \qquad \blacksquare$$

Theorem 2.2.4.4: If X is a compact Hausdorff space having no isolated points, then X is uncountable.

Proof:

Step 1: First show that given any nonempty open set of X and any point x of X, there exists a nonempty set V contained in U such that $x \notin V$.

Choose a point $y \in U$ different from x, this is possible if x in U because x is not an isolated point of X and it is also possible if x in not U simply because U is nonempty. Now choose disjoint neighbourhood W_1 and W_2 of x and y respectively. Then take $V = U \cap W_2$.

Step 2: Let $f: N \longrightarrow X$. Then show that f is not injective.

Let $x_n = f(n)$. Apply step 1 to the nonempty open set $U = X$to choose a nonempty open set V_1such that $x \in V_1$. In general, given V_{n-1}, a nonempty open set, choose V_n to be a nonempty open set such that $V_n \subset V_{n-1}$ and $c_n \notin V_n$. Consider the nested sequence $\{V_1\}$ of nonempty closed sets of X. Since X is compact, there exists a point $x \in V_n$. Now if f is surjective, then there exists n such that $f(n) = x_n = x$, which implies that $x_n \in V_n$. Contradiction. $\blacksquare$

Corollary 2.2.4.5: Every closed and bounded interval of $\mathbb{R}$ is uncountable.

2.3.4 Compactness and Continuous function

Theorem 2.2.5.1: Let X and Y be topological spaces, and let $f: X \longrightarrow Y$ be a function. If X is compact and f is continuous, then $f(X)$ is compact.

Proof: Let $\{V_i\}_{i\in I}$ be a family of open sets of Y such that

$$f(X) \subseteq \bigcup_{i\in I} V_i$$

This implies that

$$X \subset f^{-1}(f(X)) \subseteq \bigcup_{i\in I} f^{-1}(V_i)$$

By the continuity of f, $\{f^{-1}(V_i)\}_{i\in I}$is a family of open sets of X, and since X is compact, there exists $i_1, i_2, i_3, \dots, i_m$ such that

$$X \subseteq \bigcup_{j=1}^{m} f^{-1}\left(V_{i_j}\right)$$

which implies that

$$f(X) \subseteq \bigcup_{j=1}^{m} f^{-1}\left(V_{i_j}\right) \subseteq \bigcup_{j=1}^{m} f\left(f^{-1}\left(V_{i_j}\right)\right) = \bigcup_{j=1}^{m} V_{i_j}$$

That is,

$$f(X) \subseteq \bigcup_{j=1}^{m} V_i$$

Hence, $f(X)$ is compact. ■

This theorem says that the continuous image of a compact set is compact.

Theorem 2.2.5.2: Let $f: X \longrightarrow Y$ be a continuous bijective function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof: Let F be a closed subset of X. Since X is compact, you have by theorem 3.3.1 that F is compact. Also by the continuity of f, and theorem 3.3.1, you have that $f(F)$ is compact. Since Y is Hausdorff, theorem 3.2.4gives you that $f(F)$ is closed in Y. And since f is a bijection, f^{-1} exists and is continuous. ■

2.3.5 The Extremum Value Theorem

Theorem 2.2.6.1 (The Extremum Value Theorem): Let $f: X \longrightarrow Y$ be continuous, where Y is an ordered set in the order topology. If K is a compact subset of X, then there exists points $\overline{c}$ and $\underline{c}$ in K such that

$$f(\underline{c}) = min f(x) \; and \; f(\overline{c}) = \max_{x\in K} f(x)$$

Proof: Since f is continuous, and K is compact, the set $A = f(K)$ is compact. So you can show that A has a largest element M and a smallest element m. Then since m and M belongs to A, you have to show that $m = f(\underline{c})$ and $M = f(\overline{c})$ for some points$\underline{c}$ and $\overline{c}$ in K.

By contradiction, assume that A has no largest element, then the collection

$$\{(-\infty, a) : a \in A\}$$

forms an open cover of A. Since A is compact, some finite subcover $(-\infty, a_1)$, $(-\infty, a_2)$, $\ldots, (-\infty, a_n)$ covers A. If a_{i_0} is the largest of the elements, $a_1, a_2, \ldots, a_n$ then a_{i_0} belongs to none of these sets, contrary to the fact that they cover A (because $a_{i_0} \in A$). A similar argument shows that A has a smallest element.■

Definition 2.2.6.2: (Lebesgue Number): Let $\mathbb{A}$ be an open cover of X. δ is a *Lebesgue number* on $\mathbb{A}$ if for all subsets A of X such that the diameter of A is less than δ, there exists $U \in \mathbb{A}$ such that $A \subseteq U$.

Theorem 2.2.6.3: Let (X, d) be a metric space. Let $\mathbb{A} = \{U_i, i \in I\}$ be an open cover of X. If X is compact, then there exists $\delta > 0$ such that any subset of X, having diameter less than δ is contained in one of the $U_i^0 s$.

Proof: Let $\mathbb{A} = \{U_i, i \in I\}$ be an open cover of X such that

$$X = \bigcup_{i \in I} U_i$$

If $X \in \mathbb{A}$, then any positive number is a lebesgue number of $\mathbb{A}$. So you can assume that $U_i \subset X$.
Take $C_i = X \backslash U_i$ and define $f : X \longrightarrow \mathbb{R}$ by

$$f\ (x) = \sum_{i=1}^{n} d_n(x, C_i)$$

Now for any $x \in X$, there exist $i_0 \in I$ such that $x \in U_{i_0}$. Since U_{i_0} is open, then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_{i_0}$. If $y \in C_{i_0}$ then $y \notin U_{i_0}$, i.e., $y \notin B(x, \epsilon)$ which implies that $d(x, y) \geq \epsilon$and so $d(x, C_{i_0}) \geq \delta$, thus $f(x) \geq \frac{1}{n}$.

Since f is continuous on X (which is compact), then f has a minumum value $\delta > 0$. You now have to show that δ is the Lebesgue number. For this, let A be a subset on X of diameter less than δ. Choose $x_0 \in B$, then $A \subset B(x_0, \delta)$. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

where $d(x_0, C_m)$ is the largest of the number $d(x_0, C_i)$. Then $B(x_0, \delta) \subset U_m$, as desired. ■

2.2.6.4: Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: (X, d_X) \longrightarrow (Y, d_Y)$ is said to be uniformly continuous if given any $\epsilon > 0$ there exists a $\delta > 0$ such that for every pair of points x_1, x_2 of X,

$$d_X(x_1, x_2) < \delta$$

implies that

$$d_Y(f(x_1), f(x_2)) < \epsilon$$

Theorem 2.2.6.5: Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \longrightarrow Y$ be continuous. If X is compact then f is uniformly continuous.

Proof: Let $\epsilon > 0$ be given. $\{B_Y(y, \frac{\epsilon}{2}), y \in Y\}$ is an open covering of Y. So that $\{f^{-1}(B_Y\left(y, \frac{\epsilon}{2}\right)), y \in Y\}$ is an open covering of X, and has a Lebegue number δ since X is compact. Let x_1, x_2 be points of X such that $d(x_1, x_2) < \delta$. This implies that diameter $(\{x_1, x_2\}) < \delta$. Thus $\{x_1, x_2\} \subseteq f^{-1}(B\left(y, \frac{\epsilon}{2}\right))$ and so $f(x_1), f(x_2) \in B(y_0, \frac{\epsilon}{2})$. Therefore,

$$d(f(x_1), f(x_2) \leq d(f(x_1), y_0) + d(f(x_2), y_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

i.e., $d(f(x_1), f(x_2))$ as desired. ■

2.4 Limit Point and Sequential Compactness

2.4.1 Limit Point Compactness

Definition 2.2.7.1.1: A space X is said to be limit point compact if every infinite subset of X has a limit point.

Theorem 2.2.7.1.2: Any compact space is limit point compact, but not conversely.

Proof: Let X be a compact space. Given a subset A of X, the goal is to prove that if A is infinite, then A has a limit point. The proof is done by contraposition. That is, if A has no limit point then A must be finite.

Suppose that A has no limit point. Then A is closed. Since X is compact. Furthermore, for each $a \in A$, you can choose an open neighbourhood U_a of a such that U_a intersects A in the point a alone. The subspace A is covered by the open cover $\{U_a: a \in A\}$; being compact, it can be covered by finitely many of these sets. Each U_a contains only one point of A, the set A must be finite.

The next is to show that for metrizable spaces, these two versions of compactness coincides. That is (X, ρ) is compact if and only if (X, ρ) is limit point compact. To this end, you shall be introduced to another version of compactness called sequential compactness.■

2.4.2 Sequential Compactness

Definition 2.2.8.1: A topological space X is said to be sequentially compact if every sequence of points of X has a convergence subsequence.

Theorem 2.2.8.2: Let X be a metrizable space. Then the following are equivalent.

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

Proof: You have already shown that $(1) \Longrightarrow (2)$ in theorem 3.14. To prove that $(2) \Longrightarrow (3)$, assume that X is limit point compact. Given a sequence (x_n) of points of X, consider the set $A = \{x_n : n \geq 1\}$. If the set A is finite, then there is a point x such that $x_n = x$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges. On the other hand, if A is infinite, then A has a limit point x. Define a subsequence of (x_n) converging to x as follows. First choose n_1 so that

$$x_{n_1} \in B(x, 1)$$

Then suppose the positive integer $n_{i-1}$is given. Because the ball $B\left(x, \frac{1}{i}\right)$ intersects A in infinitely many points, you can choose an index $n_i > n_{i-1}$ such that $x_{n_i} \in B\left(x, \frac{1}{i}\right)$. Then the subsequence $\left(x_{n_k}\right)$ converges to x.

Finally, you have to show that $(3) \Longrightarrow (1)$. This is the hardest part of the proof. First, show that if X is sequentially compact, then the Lebesgue number holds for X (This would form compactness, and compactness is what you want to prove.) Let A be an open cover of X. Assume that there exist no $\delta > 0$ such that each set of diameter less that δ has an element of A containing it. Your assumption implies in particular that for each positive integer n, there exists a set of diameter less than $\frac{1}{n}$ that is not contained in any element of A. Let C_n be such set. Choose a point $x_n \in C_n$ for each n. By hypothesis, some subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ converges, say to a point a. Now a is in some element U of the open cover A. Because U is open, you may choose $\epsilon > 0$ such that $B(a,\epsilon) \subset U$. Let k be sufficiently large such that $\frac{1}{n_k} < \frac{\epsilon}{2}$ and $d(x_{n_k}, a) < \frac{\epsilon}{2}$, then there exists $C_{n_k} \subset B(a,\epsilon)$. Contradiction.

Secondly, you have to show that if X is sequentially compact, then given ϵ, there exists a finite cover of $X\epsilon -$ balls. Once again, proceed by contradiction. Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many $\epsilon -$ balls. Construct a sequence of points x_n as follows: First, choose x_1 to be any point of X. Noting that the ball $B(x_1,\epsilon) = X$ (otherwise X could be covered by a single $\epsilon -$ balls) choose x_2 to be a point of X not in $B(x_1,\epsilon)$. In general, given $x_1, x_2, \ldots, x_n$, choose x_{n+1} to be a point of X not in the union

$$B(x_1,\epsilon) \cup B(x_2,\epsilon), \cup \cdots \cup B(x_n,\epsilon)$$

using the fact that these ball do not cover X. By construction $d(x_n + 1, x_i) \geq \epsilon$for $i = 1, \ldots, n$. Therefore, the sequence (x_n) can have no convergent subsequence. In fact, any ball of radius $\frac{\epsilon}{2}$ can contain x_n for at most one value of n.

Finally, show that if X is sequentially compact, then X is compact. Let A be an open cover of X. Because X is sequentially compact, then the open cover A has a Lebesgue number δ. Let $\epsilon = \frac{\delta}{3}$; using sequentially compact of X to find a finite cover of X by $\epsilon -$ balls. Each of these balls has diameter at most $\frac{2\delta}{3}$, so it lies in an element of A. Choosing one such element of A for each of these $\epsilon -$ balls, you obtain a finite subcollection of A that covers X. ■

2.4.3 Locally Compactness and One-point Compactification

2.4.4 Local Compactness

Definition 2.2.9.1.1: A topological space X is locally compact if each point of X has a neighbourhood with compact closure.

Example 2.2.9.1.2:$\mathbb{R}$ the real line endowed with the standard topology is locally compact because for all $x \in \mathbb{R}, (x-1, x+1)$ is a neighbourhood of x whose closure is the closed and bounded interval $[x-1, x+1]$ of $\mathbb{R}$, which is compact by theorem 2.1.

Example 2.2.9.1.3: The sets $\mathbb{Z}$, and $\mathbb{N}$ are locally compact sets in $\mathbb{R}$ but are not compact.

Example 2.2.9.1.4: In $\mathbb{R}$, $\mathbb{Q}$ the set of rational numbers is not locally compact.

Theorem 2.2.9.1.5: Every compact space is locally compact.

Proof: Let $x \in X$ and U be a neighbourhood of x. Suppose X is compact, then U is a closed subset of a compact space, and hence is compact. ■

2.4.5 One-Point Compactification

Let (X, Ω) be a Hausdorff topological space. Let X^* be the set obtained by adding a point x_* to X (of course, x_* does not belong to X). Let Ω^* be the collection of subsets of X^* consisting of

i. sets open in X and
ii. sets of the form $X^*\backslash C$, where $C \subset X$ is a compact set. i.e.,
$\Omega^* = \Omega \cup \{X^*\backslash C: C\backslash X \text{is a compact set}\}$.

Then
1. Ω^* is a topological structure on X^*.
2. (X^*, Ω^*) is compact.
3. The inclusion $(X, \Omega) \longrightarrow (X^*, \Omega^*)$ is a topological embedding.
4. If X is locally compact, then the space (X^*, Ω^*) is Hausdorff.

Definition 2.2.10.1: A topological embedding of a space X into a compact space Y is a compactification of X if the image of X is dense in Y. In this situation, Y is also called a compactification of X.

If X is a locally compact Hausdorff space, and Y is a compactification of X with one-point $Y\backslash X$, then there exists a homeomorphism $Y \longrightarrow X^*$ which is the identity on X.

Definition 2.2.10.2: Any space Y that satisfy the above condition is called a one-point compactification or Alexandrov compactification of X.

2.5 Self-Assessment Exercise(s)

1. Which of the following spaces is not compact?
(a) Every discrete space.
(b) Every indiscrete space.
(c) Any finite space.
(d) A finite discrete space.
2. Which of the following statements is false?
(a) Any closed subset of a compact space is compact.
(b) Any compact subset of a Hausdorff space is compact.

(c) Any finite set is compact.
(d) Any closed and bounded set of a metric space is compact.

3. Which of the following sets is compact in $\mathbb{R}$?
(a) $[0,1] \cap \mathbb{Q}$
(b) $[0,1] \cap \mathbb{Q}^c$
(c) $[0,1)$
(d) $[0,1]$

4. Let $f: [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then $f([a,b])$ is
(a) closed but not bounded.
(b) bounded but not closed.
(c) neither closed nor bounded.
(d) closed and bounded.

5. Which of the following sets is not compact?
(a) $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
(b) $S^n = \{(x_1, x_2, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\}$
(c) $\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0\}$
(d) $A = \{x = (x_1, x_2, \dots, x_n) : x_i = 0, \ i = 1, 2, \dots, n\}$

6. Let $X = [0,1) \cup [2,3]$ be a subspace of the standard topology on $\mathbb{R}$. The subset $A = [0,1)$ of X is
(a) closed, bounded and compact in X.
(b) closed, bounded and not compact in X.
(c) closed and compact in X.
(d) bounded and compact in X.

7. In an arbitray metric space (X, ρ)
(a) every closed and bounded set is compact.
(b) every compact set is closed and bounded.
(c) every bounded set is compact.
(d) every closed set is compact.

8. Let A_0 be the closed and bounded interval $[0,1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its middle third $\frac{1}{3}, \frac{2}{3}$. Let A_2 be the set obtained from A_1 by deleting its middle thirds $\frac{1}{9}, \frac{2}{9}$, and $\frac{7}{9}, \frac{8}{9}$. In general, define A_n by the equation

$$A_n = A_{n-1} \backslash \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

The intersection

$$K = \bigcap_{n \in N} A_n$$

is called the Cantor set. It is a subset of $[0, 1]$. Which of the following is not true about K?
(a) K is compact.
(b) K has no isolated points.
(c) K is countable.
(d) K is uncountable.

9. Which of the following sets is not locally compact?
(a) $\mathbb{R}$
(b) $\mathbb{Q}$
(c) $\mathbb{R}^n$
(d) a discrete space.

2.6 Conclusion

In this unit you have studied compactness; covers, compact sets and subsets of compact spaces and proved some important results as regards to compactness, some of them you have always used in its special case in your studies in Analysis and calculus. You were also introduced to the notions of limit point, sequentially and locally compactness and one-point compactification.

2.7 Summary

Having gone through this unit, you now know that;

(i) A collection $A = \{U_i, i \in I\}$ of open subsets of a topological space X is an open covering of X if

$$X = \bigcup_{i \in I} U_i$$

(ii) A topological space X is compact if every open covering of X can be reducible to a finite subcovering.
(iii) Every finite set is compact.
(iv) The real line R is not compact.
(v) Any closed and bounded interval of $\mathbb{R}$ is compact.
(vi) Any closed subset of a compact space is compact.
(vii) Any compact subset of a Hausdorff space is closed.
(viii) A finite product of compact spaces is compact.
(ix) Any compact set of a metric space is closed and bounded.

(x) In the metric space $\mathbb{R}^n$ compactness and closed and bounded are equivalent. This is the Heine Borel theorem.

(xi) A collection C of subsets of a topological space X satisfies the Finite Intersectiion Property (FIP) if any intersection of a finite subcollection of C is nonempty.

(xii) A topological space X is compact if and only if any collection X of closed sets of X satisfying the FIP, one has that the arbitrary intersection is nonempty.

(xiii) The continuous image of a compact set is compact.

(xiv) If K is a compact subset of a topological space X and f is a continuous function from X to an ordered space Y then f attains its maximum and minimum on K. This result is called the Extreme Value Theorem.

(xv) δ is a *Lebesgue number* on $\mathbb{A}$ if for all subsets A of X such that the diameter of A is less than δ, there exists $U \in \mathbb{A}$ such that $A \subseteq U$.

(xvi) A continuous function f from a compact metric space X to another metric space Y is uniformly continuous.

(xvii) A space X is called limit point compact if every infinite subset of X has a limit point.

(xviii) A topological space is sequentially compact if every sequence of points of X has a convergent subsequence.

(xix) A topological space X is locally compact if each point of X has a neighbourhood with compact closure.

(xx) A topological embedding of a space X into a space Y is a compactification of X if the image of X is dense in Y. In such situation, Y is also called a compactification of X.

(xxi) A space Y is called one-point compactification of X if X is a locally compact Hausdorff space, and Y is a compactification of X with one-point $Y \backslash X$, such that there exists a homeomorphism $Y \longrightarrow X^*$ which is identity on X.

2.8 References/Further Reading

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UNIT 3 CONNECTEDNESS

Unit structure

3.1 Introduction

In your study of calculus, you must have come across this all important results called the intermediate value theorem which states that if $f: I \longrightarrow \mathbb{R}$ is continuous, and r is a real number between $f(a)$ and $f(b)$ then there exists $c \in I$ such that $f(c) = r$, where I denotes an interval of $\mathbb{R}$. Although this theorem refers to continuous functions, notwithstanding it also depends on the topological property of the interval I. In fact we can restate the intermediate value theorem as follows; The continuous image of an interval I of $\mathbb{R}$ is also an interval. This topological notion property of the interval I on which the intermediate value theorem depends is called connectedness.

In this unit, you will be introduced to a generalization of the intermediate theorem, and some other related theorems which you have proved in particular cases of the real line.

3.2 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to;

- Differentiate between connected sets and separated spaces.
- Define connected spaces.
- Understand the connectedness to the real line.
- Identify the connected components of a given space.

- Identify locally connected spaces.
- Know and use of the concept of path connectedness.

3.3 Separated and Connected Sets

3.3.1 Definitions and Examples

Definition 3.2.2.1: Let X be a topological space. A Separation of X is a pair U, V of disjoint open sets of X, whose uninon is X.

Definition 3.2.2.2: A topological space X is connected if it has no separation.

Example 3.2.2.3: In $\mathbb{R}$, Let $X = [-1, 0) \cup (0, 1]$. $[-1, 0)$and $(0, 1]$ are open in X. They are nonempty and disjoint. And so is a separation of X. Therefore,X is not connected.

Example 3.2.2.4: Let $X = \{a, b\}$. If X is endowed with the indiscrete topology, then$X$ has no separation and thus is connected.

Another way of formulating the definition of connectedness is the following:

Theorem 3.2.2.5: A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Proof: If A is a nonempty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X \backslash A$ constitute a separation of X, for they are open, disjoint and nonempty, and their union is X.

Conversely, if U and V form a separation of X, then U is nonempty and different from X and it is both open and closed in X.■

Example 3.2.2.6: If X is any discrete space with more than one element, then X is not connected as each singleton set is both open and closed.

Example 3.2.2.7: If X is any indiscrete space, then it is connected as the only sets that are both closed and open are X and $\emptyset$.

3.3.2 Connected Sets

If you refer to a set Y as connected, you mean that Y lies in some topological space (which should be clear from the context) and, equipped with the subspace topology, thereby making Y a connected space. So Y is connected in a topological space X if Y is connected in the subspace topology induced by the topology on X.

Theorem 3.2.3.1: Let Y be a subspace of a topological space X. A separation of Y is a pair A, B of nonempty disjoints sets whose union is Y and neither of which contains a limit point of the other (i.e., $A \cap B^0 = \emptyset$ and $B \cap A^0 = \emptyset$).

Proof: Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\overline{A} \cap Y$, which implies that $\overline{A} \cap Y = \emptyset$. Since $\overline{A}$ is the union of A and its limit points, B contains no limit points of A. A similar argument shows that A contains no limit points of A.

Conversely, Suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\overline{A} \cap B = \emptyset$and $A \cap \overline{B} = \emptyset$. Therefore, we conclude that $\overline{A} = A \cap Y$ and $\overline{B} = B \cap Y$. Thus A and B are closed in Y, and since $A = Y \backslash B$, and $B = Y \backslash A$, they are open in Y, as desired. ■

Example 3.2.3.2: Let $X = [0, 1] \cup (1, 2] = A \cup B$. Then$A, B$ is not a separation of$X$ since $1 \in B^0 \cap A = \emptyset$.

Example 3.2.3.3:$\mathbb{Q}$ the set of all rational numbers is not a connected set. Indeed, the only connected subspace of $\mathbb{Q}$ are the one point sets. If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number a lying between p and q.

Having seen some examples of sets that are not connected, what follows are result that will help you determine how to construct connected sets from existing ones.

Lemma 3.2.3.4: If the sets A and B forms a separation of X, and Y is a connected subspace of X, then either Y lies entirely in either A or B.

Proof: Since A and B are both open in X, the set $A \cap Y$ and $B \cap Y$ are open in Y, and $Y = (A \cap Y) \cup (B \cap Y)$. If both of them are nonempty, then they constitute a separation, of Y. But since Y is connected, either $A \cap Y = \emptyset$or $B \cap Y = \emptyset$. So that Y either lies in A or B as required. ■

Theorem 3.2.3.5: The Union of a collection of connected subspaces of X that have one point in common is connected.

Proof: Let $\{C_i\}_{i \in I}$ be a collection of connected spaces on X; let p be a point of $\bigcap_{i \in I} C_i$. You have to prove that the space $Y = \bigcup_{i \in I} C_i$ is connected. Suppose that $Y = A \cup B$ is a separation of Y. The point p is in one of the sets A or B; suppose $p \in A$. Since C_i is connected, it must lie entirely in either A or B, and it cannot lie in B because it contains the point p of A. Hence, $C_i \subset A$ for every i, so $\bigcup_{i\in I} C_i \subset A$, contradiction the fact that B is nonempty. ■

Theorem 3.2.3.6: Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$ then B is connected and in particular A.

Proof: Let A be a connected subspace of X and let $A \subset B \subset \overline{A}$. Suppose $B = C \cup D$ is a separation of B, then by lemma 3.1, the set A lies entirely in C or in D. Suppose $A \subset C$, then $\overline{A} \subset \overline{C}$; since $\overline{C} \subset D \ = \emptyset$, B cannot intersect D, this contradicts the fact that D is a nonempty subset of B. ■

Theorem 3.2.3.7: The image of a connected space under a continuous function is connected.

Proof: Let $f: X \longrightarrow Y$ be a continuous map, let X be connected. You have to show that the space $Z = f(X)$ is connected. Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map

$$g: X \longrightarrow Z$$

Suppose $Z = A \cup B$ is a separation of Z into the disjoint nonempty open sets. Then $g^{-1}(A)$ and $g^{-1}(B)$ form a separation of X, contradicting the assumption that X is connected. ■

Theorem 3.2.3.8: A finite Cartesian product of connected spaces is connected.

Proof: You can prove this theorem for the product of two connected spaces X and Y. Choose a point (a, b) in $X \times Y$. Note that the horizontal slice $X \times \{b\}$is connected, being homeomorphic with X, and each vertical slice $\{x\} \times Y$ is connected being homeomorphic with Y. As a result each $T - shaped$ space

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected, being the union of two connected spaces that the point $\{x, b\}$is common. Now form the union $\bigcup_{x\in X} T_x$ of all this $T - shaped$ spaces. The union is connected because it is the union of collection of connected spaces that have the point (a, b) in common. Since this union equals $X \times Y$, the space $X \times Y$ is connected.■

The proof for any finite product of connected spaces follows by induction.

3.3.3 Connected Subspaces of the Real Line

Here you shall show that the real line is connected. So also is the intervals of $\mathbb{R}$ or the rays, i.e., sets of the form (a, ∞). You are also going to prove a generalization of the intermediate value theorem of calculus.

Definition 3.2.4.1: A simply ordered set L having more than one element is called linear continuum if the following hold:

1. L has the least upper bound property.
2. if $x < y$, there exists z such that $x < z < y$.

Theorem 3.2.4.2: If L is a linear continuum in the order topology, then L is connected, and so are the intervals and rays in L.

Proof: Recall that a subspace Y of L is said to be convex if for each points a, b of Y with $a < b$, one has the interval $[a, b]$ lies in Y. You have to prove that if Y is a convex subspace of L, then Y is connected.

Suppose that $Y = A \cup B$ is a separation of Y. Choose $a \in A$ and $b \in B$, suppose that $a < b$. The interval $[a, b]$ of points of L is the union of the disjoint sets

$$A_0 = A \cap [a, b] \text{ and} B_0 = B \cap [a, b]$$

each is open in $[a, b]$ in the subspace topology, which is the same as the order topology. The sets A_0 and B_0 are nonempty because $a \in A_0$ and $b \in B_0$. Thus A_0 and B_0 constitute a separation of $[a, b]$. Let $= \sup A_0$. You have to show that c belongs to A_0 or to B_0, which would contradict the fact that $[a, b]$ is the union of A_0 and B_0.

Case 1: Suppose that $c \in B_0$. Then $c = a$, so either $c = b$ or $a < c < b$. In either case, it follows from the fact that B_0 is open in $[a, b]$ that there exist some interval of the form $(d, c]$ contained in B_0. If $c = b$, you have a contradiction at once, for d is a smaller upper bound in A_0 than c. If $c < b$, observe that $(c, b]$ does not intersect A_0 (because c is an upper bound on A_0). Then

$$(d, b] = (d, c] \cup (c, b]$$

does not intersect A_0. Again, d is a smaller upper bound on A_0 than c, contrary to construction.

Case 2: Suppose that $c \in A_0$ then $c = b$, so either $c = a$ or $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. Because of the order property(2) of the linear continuum L, you can choose a point $z \in L$ such that $c < z < e$. Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0 . ■

Corollary 3.2.4.3: The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$. As an application, the intermediate value theorem of calculus is suitably generalized.

Theorem 3.2.4.4 (Intermediate Value Theorem): Let $f: X \longrightarrow Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c in X such that $f(c) = r$.

Proof: Assume by hypothesis of the theorem that the sets

$$A = f(X) \cap (-\infty, r) \text{ and } B = f(X) \cap (r, +\infty)$$

are disjoint, nonempty because one contains $f(a)$ and the other contains $f(b)$. Each is open in $f(X)$. If there is no point $c \in X$ such that $f(c) = r$, the A and B form a separation of $f(X)$which is connected. This is a contradiction.■

3.3.4 Path Connectedness

Definition 3.2.5.1: Given points x and y of the topological space X, a path in X from x to y is a continuous map $f: [a, b] \longrightarrow X$ of some closed interval in the interval in the real line to the space X, such that $f(a) = x$ and $f(b) = y$.

Definition 3.2.5.2 (Path Connectedness): A topological space X is said to be path connected if every pair of points of X can be joined by a path in X.

Theorem 3.2.5.3: If X is a path connected space then X is connected.

Proof: Suppose $X = A \cup B$ is a separation of X. Let $x \in A$ and $y \in B$.

Choose a path $f: [a, b] \longrightarrow X$ joining x and y. The subspace $f([a, b])$ of X is connected as a continuous image of a connected space. Therefore, it lies entirely in either A or B which contradicts the fact that A and B are disjoint. ■

Example 3.2.5.4: Define the unit ball $\mathcal{B}^n$in $\mathbb{R}^n$ by

$$\mathcal{B} = \{x \in \mathbb{R}^n : k \times k \leq 1\}$$

where

$$k \times k = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

The unit ball $\mathcal{B}^n$ is path connected, given any two points x, y in $\mathcal{B}^n$, the straight line path $f: [0, 1] \longrightarrow \mathbb{R}^n$ defined by

$$f(t) = (1 - t)x + ty$$

lies in $\mathcal{B}^n$.

3.3.5 Components and Local Connectedness

3.3.6 Connected Components

Definition 3.2.6.1.1(Connected Components): Given a topological space X, define anequivalence relation $\sim$ by $x \sim y$ if and only if there exists a connected subspace of X containing x and y.

Claim:$\sim$is an equivalence relation.

1. $x \sim x$ because $\{x\}$is connected (so $\sim$is reflexive).
2. $\sim$is symmetric by definition.
3. $\sim$is transitive, because $x \sim y$ and $y \sim z$ implies that there exists connected subspaces C_1 and C_2 of X such that $x, y \in C_1$ and $y, z \in C_2$. Let $C = C_1 \cup C_2$, then C is connected since $y \in C_1 \cap C_2$ and $x, z \in C$. Hence $x \sim y$.

A connected component or a component is all equivalence classes for this equivalence relation.

Theorem 3.2.6.1.2: The connected components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof: Being equivalence classes, the components of X are disjoint and their union is X. Each connected subspace A of X intersects only one of them. For if A intersects the components C_1 and C_2 of X, say in the points x_1 and x_2 respectively, then $x_1 \sim x_2$ by definition, this cannot happen unless $C_1 = C_2$. To show that the component C is connected, choose a point x_0 of C. For each point x of C, we know that $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x. By the result just proved, $A_x \subset C$. Therefore

$$C = \bigcup_{x \in C} A_x$$

since the subspaces A_x are connected and have the point x_0 in common, their union is connected.■

3.3.6.1 Locally Connectedness

Definition 3.2.6.2.1: A topological space (X, τ) is said to be locally connected if it has a basis $\mathcal{B}$ consisting of connected open sets.

Example 3.2.6.2.2: Z the set of integers is a locally connected space which is not connected.

Example 3.2.6.2.3: $\mathbb{R}^n$ is locally connected for all $n \geq 1$.

Example 3.2.6.2.4: Let (X, τ) be the subspace of $\mathbb{R}^2$ consisting of the points in the line segments joining $(0, 1)$ to $(0, 0)$ and to all the points $(\frac{1}{n}, 0), n = 1, 2, 3, \ldots$. Then the space (X, τ) is connected but not locally connected.

Proposition 3.4.7: Every open subset of a locally connected space is locally connected.

Proposition 3.4.8: A finite product of locally connected spaces is locally connected.

3.4 Self-Assessment Exercise(s)

1 Let X be a discrete topological space. If X is connected, then
(a) X is infinite
(b) X is countable
(c) X is finite with more than one element.
(d) X is a singleton.

2. Let $X = \{a, b, c, d, e\}$. Suppose X is connected when endowed with the topology τ, which of the following could be τ?
   (a) $\tau = \mathcal{P}(X)$, the power set of X.
   (b) $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}\}$
   (c) $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$
   (d) $\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, e\}, \{a, b, e\}\{b, c, d, e\}\}$

3. Let $X = \{a, b\}$. Which of the following topologies will make X disconnected?
   (a) $\tau = \{X, \emptyset, \{a\}, \{b\}\}$
   (b) $\tau = \{X, \emptyset, \{a\}\}$
   (c) $\tau = \{X, \emptyset, \{b\}\}$
   (d) $\tau = \{X, \emptyset\}$

4. In which of the following spaces is the subset $\{0, 1\}$ of real numbers connected?
   (a) $\mathbb{R}$ with the standard topology

(b) $\mathbb{R}$ with the finite complement topology

(c) $\mathbb{R} = [0, \infty)$ with the topology $\Omega = \{\emptyset, X, (a, +\infty)\}$

(d) $\mathbb{R}$ with the discrete topology

5. If $\mathbb{R}$ is endowed with the finite complement topology, then the following sets are connected except
   (a) the empty set
   (b) singleton sets
   (c) infinite sets
   (d) $\mathbb{N}$

6. Every connected space is path connected. (TRUE/FALSE)

7. Every connected space is locally connected. (TRUE/FALSE)

8. Every locally connected space is connected. (TRUE/FALSE)

9. Let A be a subset of a space X. Then the pair U, V is a separation such that $A = U \cup B$ if and only if

(a) $U^0 \subset V$ or$V^0 \subset U$
(b) $V^0 \cap U = \emptyset$ and$U^0 \cap V = \emptyset$
(c) $U \cap V = \emptyset$ and $U \cap V = \emptyset$
(d) V $V^0 \cap U = \emptyset$ and$U^0 \cap V = \emptyset$or$V^0 \cap U = \emptyset$ and$U^0 \cap V = \emptyset$

10. If C_1 and C_2 are connected components and A is a connected set, then

(a) either $C_1 \cap C_2 = \emptyset$ or$C_1 = C_1$, and A intersects both C_1 and C_2
(b) $C_1 \cap C_2 = \emptyset$ and$C_1 = C_1$, and A intersects both C_1 and C_2
(c) either $C_1 \cap C_2 = \emptyset$ and$C_1 = C_1$, and A intersects eitherC_1 orC_2
(d) $C_1 \cap C_2 = \emptyset$ and$C_1 = C_1$, and A intersects eitherC_1 orC_2.

11. A topological space is totally separated if all its components are singletons. Which of the following spaces is not totally separated?
   (a) Any discrete space
   (b) The space $\mathbb{Q}$ endowed the topology induced from standard topology of $\mathbb{R}$
   (c) The cantor set K
   (d) $\mathbb{R}$ with the standard topology.

12. If X is a connected space and $f: X \longrightarrow \mathbb{R}$ is a continuous function.

Then $f(X)$ is an interval I of $\mathbb{R}$. Which of the following is not correct about this assertion?

(a) $f(X)$ is connected
(b) The interval of $\mathbb{R}$ is connected

(c) $\mathbb{R}$ is connected
(d) The interval I is a continuous image of the connected space X.

3.5 Conclusion

In this unit, you were introduced to a topological property called connectedness. You studied connected and separated spaces with examples and the connectedness of the real line. You also studied the connected components of a given space, locally connected spaces and path connectedness. You also proved some important results such as the intermediate value theorem.

3.6 Summary

Having gone through this unit, you now know that;

(i) A separation of a topological space X is a pair U, V of disjoint open sets of X, whose union is X.
(ii) A topological space X is connected if it has no separation. Or X is connected if and only if the only closed and open sets in X is $\emptyset$ and X itself.
(iii) A set is connected if it is connected in the subspace topology induced by the topology in the topological space.
(iv) A union of a collection of connected subspaces of X that have one point in common is connected.
(v) The continuous image of a connected space is connected.
(vi) A finite Cartesian product of connected spaces is connected.
(vii) The real line is connected. So also is the intervals and rays.
(viii) A simply ordered set L having more than one element is called linear continuum if L has the least upper bound property and if $x < y$, then there exists z such that $x < z < y$. (
(ix) A linear continuum in the order topology is connected.
(x) If $f: X \longrightarrow Y$ is a continuous map from the connected space X to the ordered space Y in the order topology, a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c in X such that $f(c) = r$. This is the intermediate value theorem
(xi) A path from a point x to y in the topological space X is a continuous map $f: [a, b] \longrightarrow X$ of some closed interval in the real line to the space X, such that $f(a) = x$ and $f(b) = y$. X is called path connected if every pair of points of X can be joined by a path in X. If X is a path connected space then X is connected.

(xii) A connected component is an equivalence class for the equivalence relation;$x \sim y$ if and only if there exists a connected subspace X containing x and y. The connected components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

(xiii) A topological space X is said to be locally connected if it has a basis $\mathcal{B}$ consisting of connected open sets.

3.7 References/Further Reading

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MODULE 3 HOMOTOPY RELATIONS

Module Introduction

Two continuous functions from one topological space to another are called homotopic if one can be "continuously deformed" into the other, and such a deformation is called a homotopy between the two functions. Homotopy compares spaces by shape rather than topological structures. While homeomorphism preserves the lattice of open sets, homotopy equivalence preserves the shape of space, that is, characteristics of the space that is preserved up to deformation. Two topological spaces X and Y are homotopy equivalent if they can be transformed into one another or made homeomorphic by bending, shrinking and expanding operations.A connected open set $E \subset \mathbb{C}$ is simply connected if every closed path in E is homotopic to a point (can be any point in E).In this module, concepts of homotopy are discussed.

Unit 1 Homotopy of paths

Unit structure

1.1 Introduction

Paths and loops are central subjects of study in the branch of algebraic topology called homotopy theory. A homotopy of paths makes precise the notion of continuously deforming a path while keeping its endpoints fixed.

1.2 Intended Learning Outcomes (ILOs)

At the end of this unit, readers should be able to:

- Understand the concepts of homotopic paths.
- distinguish between paths and loops.

1.3 Homotopy of paths and Equivalence Relations

Definition 1.2.1.1 (Homotopic). Let X and Y be topological spaces. Let f, g : X $\longrightarrow$ Y be continuous. We say f is homotopic to g, denoted by f $\simeq$ g, if there exists a continuous function F : $X \times [0, 1] \longrightarrow Y$ such that F(x, 0) = f(x) and F(x, 1) = g(x) for all $x \in X$. In other words, we can continuously move the image f(x) to the image g(x). i.e., γt(x) = F(x, t): $X \longrightarrow Y$ for $0 \leq t \leq 1$ is a family of continuous functions, continuously deforming from f(x) to g(x). Now let's consider the special case where f and g are paths in X. Recall if γ : [0, 1] $\longrightarrow X$ is continuous, $\gamma(0) = x_0$ and $\gamma(1) = x_1$, then γ is called a path in X from x_0 to x_1.

Definition 1.2.1.2 (Path homotopic). Two paths γ and γ' in X from x_0 to x_1 are path homotopic, denoted by$\gamma \simeq_p \gamma'$, if there exists a continuous function F : [0, 1]×[0, 1] $\longrightarrow$ X such that F(s, 0) = γ(s) and F(s, 1) = γ'(s) (so homotopic) and F(0, t) = x_0 and F(1, t) = x_1 for all $0 \leq t \leq 1$(so at every t it is still a path fromx_0 to x_1.

Lemma 1.2.1.3. The homotopy relation $\simeq$ and the path homotopy relation $\simeq_p$ are equivalence relations on

$A = \{f : X \longrightarrow Y \text{ continuous}\}$
and
$A(x_0, x_1) = \{\gamma : [0,1] \longrightarrow X \text{ continuous}, \gamma(0) = x_0, \gamma(1) = x_0\}$
respectively.

Remark 1.2.1.4. If γ is a path, we denote its path homotopy equivalence class by [γ].

Proof: We will show $\simeq$ is an equivalence relation; path homotopy is very similar. Reflexivity is obvious: f $\simeq$ f by F(x, t) = f(x) for all x ∈ X and all t ∈ [0, 1]. For symmetry, suppose f $\simeq$g, say by a continuous F(x, t) such that F(x, 0) = f(x) and F(x, 1) = g(x). Take G(x, t) = F(x, 1 − t). Then G(x, 0) = F(x, 1) = g(x) and G(x, 1) = F(x, 0) = f(x). Hence g $\simeq$ f.

Finally, transitivity: Suppose f ≃ g and g ≃ h, say the first one by F(x, t) and the second by G(x, t). Take

$$H(x, t) = \begin{cases} F(x, 2t), & if\ 0 \le t \le \frac{1}{2}, \\ G(x, 2t-1), if\ \frac{1}{2} \le t \le 1. \end{cases}$$

Then H(x, 0) = F(x, 0) = f(x) and H(x, 1) = G(x, 1) = h(x). Moreover, H is continuous (only t = $\frac{1}{2}$ needs to be checked); it is made up of two continuous functions which agree on a closed set. ■

Example 1.2.1.5. Let γ_1 and γ_2 be two paths from x_0 to x_1 in $\mathbb{R}^2$. Then $\gamma_1 \simeq_p \gamma_2$. For instance, by taking the convex combination of the two paths,

F(s, t) = (1 − t)γ_1(s) + tγ_2(s).
This argument works in slightly more generally.

Remark 1.2.1.6. Let γ_1, γ_2 be two paths from x_0 to x_1 in a convex space X. Then $\gamma_1 \simeq_p \gamma_2$.(Since in a convex space the line segment connecting the two at a fixed time is still in the space because of convexity.)

1.3.1 Fundamental group and Changing Base Point

Let γ_0 be a path in X from x_0 to x_1 and let γ_1 be a path in X from x_1 to x_2. Define $\gamma_0 * \gamma_1$ to be the path from x_0 to x_2 given by

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s), \text{ if } 0 \le s \le \frac{1}{2}, \\ \gamma_1(2s-1), \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

This induces an operation on the path homotopy classes: $[\gamma_0] * [\gamma_1] := [\gamma_0 * \gamma_1]$.

Proposition 1.2.2.1.

(i) The operation $*$ is associative. In other words, let γ_0 be a path from x_0 to x_1, γ_1 be a path from x_1 to x_2, and γ_2 a path from x_2 to x_3. Then $([\gamma_0] * [\gamma_1]) * [\gamma_2] = [\gamma_0] * ([\gamma_1] * [\gamma_2])$.

(ii) The operation $*$ has identities. Given $x \in X$, $e_x\colon [0,1] \longrightarrow X$, $e_x(s)$ = x be the constant path. Let γ be a path from x_0 to x_1. Then $[e_{x_0}] * [\gamma] = [\gamma] = [\gamma] * [e_{x_1}]$.

(iii) The operation $*$ has inverses. Let γ be a path from x_0 to x_1. Let $\bar{\gamma}(s) := \gamma(1-s)$.

Then $[\gamma] * [\bar{\gamma}] = e_{x_0}$ and $[\bar{\gamma}] * [\gamma] = [e_{x_1}]$.

Remark 1.2.2.2. This means $*$ is a groupoid operation, but not a group operation, since the left and right identities are not necessarily equal.
On the other hand, this means:

Remark 1.2.2.3. If we consider A$(x_0, x_0) = \gamma : [0, 1] \to X$ continuous, $\gamma(0) = \gamma(1) = x_0$ then $*$ is a group operation on the path homotopy classes.

Definition 1.2.2.4. (i) Let X be a space and let $x_0 \in X$. A path in X that begins and ends at x_0 is called a loop at x_0. (ii) The set of pathhomotopy classes of loops based at x_0 with the operation $*$ is called the fundamental group of X relative to the base point x_0, denoted by $\pi_1(X, x_0)$.

Example 1.2.2.5. For any $x_0 \in \mathbb{R}^2$, $\pi_1(\mathbb{R}^2,\ x_0)$ = {e} = 0, the trivial group, since all paths in $\mathbb{R}^2$ are path homotopic by Example 3.1.5. In general, if X is convex, then π_1(X, x_0) = 0 for all $x_0 \in$ X. In particular, $\pi_1(\mathbb{R}^n, x_0) = 0$.

Remark 1.2.2.6 Let $x_0 \in X$ be a fixed base point. We refer to the pair (X, x_0) as a based space.

Definition 1.2.2.7. A path in X from x_0 to x_0 is called a loop in X based at x_0, or a loop in $(X,\ x_0)$. Let $\pi_1(X, x_0) = \{[f] | f\ is\ a\ loop\ in\ (X, x_0)\}$ be the set of path homotopy classes of loops in X based at x_0. $\pi_1(X, x_0)$ is the same set as $\pi_1(X, x_0, x_0)$. Note that the composition $f * g$ of two loops in $(X,\ x_0)$ is again a loop in $(X,\ x_0)$.

Lemma 1.2.2.8. The composition of path homotopy classes specializes to a pairing $\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)([f], [g]) \to [f] * [g] = [f * g]$ where f and g are loops in X based at x_0.

Theorem 1.2.2.9. The set $\pi_1(X, x_0)$ with the composition operation $*$ is a group, with neutral element $e = [cx_0]$ and group inverse $[f]^{-1} = [\bar{f}]$ for each loop f in (X, x_0).

Proof. The composition operation defines a group structure if it is (1) associative, (2) has a left and right unit, and (3) each element has a left and right inverse. All three conditions follow by specializing the previous theorem to the case where all paths are loops in X based at x_0. ■

Definition 1.2.2.10. $\pi_1(X, x_0)$, with this group structure, is called the fundamental group of X based at x_0.

Example 1.2.2.11. If $X \subset \mathbb{R}^n$ is convex, and $x \in X$, then $\pi_1(X, x_0) = \{e\}$ is the trivial group.

Theorem 1.2.2.12. If X is path connected and x_0, $x_1 \in X$ then$\pi_1(X, x_0) \sim = \pi_1(X, x_0)$.

Remark 1.2.2.13(Changing base point)

Let $x_0, x_1 \in X$and letα be a path fromx_0 to x_1. Then α induces a group homomorphism$\hat{\alpha}$:π_1(X, x_0)⟶ $\pi_1(X,\ x_1)$ given by $\hat{\alpha}([\gamma])$ = $[\bar{\alpha}] * [\gamma] * [\alpha]$ = $[\bar{\alpha} * \gamma * \alpha]$. (Recall $\bar{\alpha}(s) = \alpha(1 - s)$) is α in reverse.)

Theorem 1.2.2.14.$\hat{\alpha}$ is a group isomorphism.

Proof. Let $\beta(s) = \bar{\alpha}(s)$. This is a path from$x_1$ to x_0. Then$\hat{\beta}$:$\pi_1(X,\ x_1)$ → $\pi_1(X,\ x_0)$ is a group homomorphism, and $\hat{\alpha}$ and $\hat{\beta}$ are each other's inverse.■

1.4 Self-Assessment Exercise(s)

1. Check that $*$ on the path homotopy classes is well-defined (i.e., does not depend on the choice of representatives γ_0 and γ_1).
2. Prove Proposition 3.2.1.
3. Verify that $\bar{\alpha}$ is a group homomorphism.

1.5 Conclusion

Homotopy groups are used in algebraic topology to classify topological spaces. The first and simplest homotopy group is the fundamental group, denoted $\pi_1(X)$.

1.6 Summary

In this unit, you have learnt the principle of path homotopy, its equivalence relations and fundamental groups.

1.7 References/Further Readings

Lecture Notes in General Topology | Download book (freebookcentre.net)

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UNIT 2 COVERING SPACES

Unit structure

2.1 Introduction

2.2 Intended Learning Outcomes (ILOs)

At the end of this unit, readers should be able to:

- understand when a topological space is simply connected
- distinguish between a covering map and a covering space

2.3 Simply connected space

Definition 2.2.1.1 A path connected space X is said to be simply connected if$\pi_1(X, x_0)$ is the trivial group for some, hence any, base point$x_0 \in X$. A path connected space X is simply connected if and only if any two paths f and f_0 in X fromx_0 to x_1 are path homotopic.

Corollary 2.2.1.2 If X is path connected, then for any $x_0, x_1 \in X$, we have$\pi_1(X, x_0) \sim = \pi_1(X, x_1)$

Remark 2.2.1.3 This isomorphism depends on the pathα fromx_0 to x_1. Two different paths may induce different isomorphisms.

Definition 2.2.1.4 (Simply connected). A space X is simply connected if X is path connected and$\pi_1(X, x_0) = 0$. Note that since the space is path connected, the fundamental group$\pi_1(X, x_0)$ does not depend on the choice of $x_0 \in X$in the first place.

Definition 2.2.1.5 (Induced homomorphism) Let $h: X \to Y$be a continuous map such that $h(x_0) = y_0$. Write $h: (X, x_0) \to (Y, y_0)$. Then$h$ induces a homomorphism$h *: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by$h * ([\gamma]) = [h \circ \gamma]$.

Theorem 2.2.1.6 Let$h: (X, x_0) \to (Y, y_0)$and $k: (Y, y_0) \to (Z, z_0)$ be continuous.
Then $(k \circ h) * = k * \circ h *$.

Corollary 2.2.1.7 If $h: (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $h *: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Hence the fundamental group π_1 is a topological invariant.

2.3.1 Covering Map and Covering Space

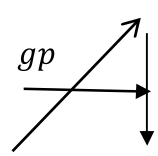
Definition 2.2.2 1 (Covering space). (i) Let $p: E \to X$ be a continuous surjective map. An open set$U \subset X$ is evenly covered by p if $p^{-1}(U)$ is a union of disjoint open subsets$V_\alpha \subset E$ such that $\backslash V_\alpha$: $V_\alpha \to U$ is a homeomorphism for all α. (ii) Let$p: E \to X$be a continuous surjective map. If each $x \in X$ has a neighbourhood U that is evenly covered by p, then p is called a covering map, and E is called a covering space of X.

Example 2.2.2.2 Consider $p: \mathbb{R} \to S^1$defined by$p(t) = e^{2\pi it}$. This p is a covering map.

Definition 2.2.2.3 (Fibre). Let$p: E \to X$ be a covering map. Let$x \in X$. Then$p^{-1}(x)$ is called the fibre over x.

Remark 2.2.2.4 The fibre$p^{-1}(x)$has the discrete topology, and for each$x \in X$there is an open neighbourhood U such that$p^{-1}(U)$ is homeomorphic to$p^{-1}(x) \times U$.

Definition 2.2.2.5 Let $p: E \to X$be a covering map. Let $f: Y \to X$be a continuous map. A continuous map$g: Y \to E$ is callled a lift of$f$ if $p \circ g = f$. In other words, a lift is a map making the diagram
E



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Definition 2.2.2.6 (Homotopy type). Let X and Y be topological spaces. (i) A map$f: X \to Y$ is a homotopy equivalence if there exists a map$g: Y \to X$ such that$f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$. The map g is called a homotopy inverse of f. (ii) The spaces X and Y are homotopy equivalent (or have the same homotopy type) if there is a homotopy equivalence between X and .

Theorem 2.2.2.7 Let $f: (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Then the induced map $f*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$is an isomorphism.

Proof. Let$g$ be a homotopy inverse of f, and $x_1 = g(y_0)$, i.e.g: $(Y, y_0) \to (X, x_1)$.

So,

$$\begin{tikzcd} (X, x_0) \arrow[r, "f"] \arrow[rr, bend right, "g \circ f \simeq Id_X"'] & (Y, y_0) \arrow[r, "g"] & (X, x_1) \end{tikzcd}$$

Then$(g \circ f)* = g * \circ f *: \pi_1(X, x_0) \to \pi_1(Y, y_0) \to \pi_1(X, x_1)$,
but$(Id_X)* = Id_{\pi_1}(X, x_0)$.

Since$g * \circ f * = Id_X$ so $f*$ is one-to-one and$g*$ is onto. Similarly, consider$f \circ g$, saying $f*$ is onto and $g*$ is one-to-one.

Remark 2.2.2.8 If X and Y are homeomorphic, then X and Y have the same homotopy type. The converse is not true: for instance, a single pointx_0 is homotopy equivalent to $\mathbb{R}$, but they are certainly not homeomorphic.

Definition 2.2.2.9 (Contractible). A space X is contractible if X is homotopy equivalent to a single-point space $Y = y_0$.

Corollary 2.2.2.6. A contractible space is simply connected.

2.4 Self-Assessment Exercise(s)

1. Verify that $h*$ is a group homomorphism
2. Let X be a simply connected topological space. Let $x_0, x_1 \in X$. Show that any two paths fromx_0 tox_1 are homotopic.
3. Prove Theorem 3.1.6.
4. Let X be a path-connected space and let $x_0, x_1 \in X$. Show that$\pi_1(X, x_0)$ is abelian if and only if for any pathsα, β fromx_0 tox_1, we have $\hat{\alpha} = \hat{\beta}$.
5. Let X be a topological space and let$x_0 \in X$. Suppose that there is a continuous map$F: X \times [0,1] \to X$ such that $F(x, 0) = x_0, x \in X, F(x, 1) = x$, F(x, 1) = x,$x \in X, F(x_0, t) = x_0$, $0 \le t \le 1$. (a) Show that X is path connected. (b) Show that X is simply connected.

6. A subset A of $\mathbb{R}^n$ is called star convex with respect to $a_0 \in A$ if all the line segments joining a_0 to any other points of A lie in A. (a) Find a star convex set that is not convex. (b) Show that a star convex set is simply connected.
7. Prove that if X is contractible and Y is path connected, then any two maps from X to Y are homotopic.