COURSE GUIDE

MTH103
ELEMENTARY MATHEMATICS III

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Introduction

MTH103 – Elementary Mathematics III is a 3-credit unit. This course is a compulsory course in first semester. It will take you 15 weeks to complete the course. You are to spend 65 hours of study for a period of 13 weeks while the first week is for orientation and last week is for end of semester examination.

You will receive the course material which you can read online or download and read off-line. The online course material is integrated in the Learning Management System (LMS). All activities in this course will be held in the LMS. All you need to know in this course is presented in the following sub-headings.

Course Competencies

By the end of this course, you will gain competency in mathematical skills on the followings

- Vectors
- Straight line
- Circle
- Conics section

Course Objectives

The course objectives are to:

- Apply vectors in computations of coordinates system
- Use information about points, gradient and intercept of a cartesian, to form different equations of straight lines
- Use information about a circle such as radius and centre to derive various equations of the circle.
- Obtain different types of conic section by making cuts at particular angles through a cone and also use the value of eccentricity to determine each of the conic section.

Working Through this Course

The course is divided into modules and units. The modules are derived from the course competences and objectives. The competencies will guide you on the skills you will gain at the end of this course. So, as you work through the course, reflect on the competencies to ensure mastery. The units are components of the modules. Each unit is sub-divided into introduction, intended, learning outcome(s), main content, self-assessment exercise(s), conclusion, summary, and further readings. The introduction introduces you to the unit topic. The intended learning outcome(s) is the central point which helps to measure your achievement or success in the course. Therefore, study the intended learning outcome(s) before going to the main content and at the end of the unit, revisit the intended learning outcome(s) to check if you have achieved the learning outcomes. Work through the unit again if you have not attained the stated learning outcomes.

The main content is the body of knowledge in the unit. Self-assessment exercises are embedded in the content which helps you to evaluate your mastery of the competencies. The conclusion gives you the takeaway while the summary is a brief of the knowledge presented in the unit. The final partis the further readings. This takes you to where you can read more on the knowledge or topic presented in the unit. The modules and units are presented as follows:

Module 1: Vectors

Unit 1: Vectors and Scalars
Unit 2: Addition of Vectors
Unit 3: Component of a vector in two dimensions
Unit 4: Component of a vector in three dimensions

Module 2: The straight line

Unit 1: Distance between two points
Unit 2: Gradients of lines
Unit 3: Equation of a line

Module 3: The geometry of a circle

Unit 1: Circle centred at the origin.
Unit 2: General equation of a circle
Unit 3: The equation of a tangent to a circle at given point
Unit 4: The equation of a normal to a circle at given point

Module 4: Conic sections

Unit 1: Parabola
Unit 2: Ellipse
Unit 3: Hyperbola

There are seventeen units in this course. All the units are for 13 weeks of study.

References and Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition
K. A. Stroud. *EngineeringMathematics*.8th Edition
Larson Edwards *Calculus: An Applied Approach*. 7th Edition
https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-conic section -2009-1.p

Presentation Schedule

The weekly activities are presented in Table 1 while the required hours of study and the activities are presented in Table 2. This will guide your study
time. You may spend more time in completing each module or unit.

Table 1: Weekly Activities

Week	Activities
1	Orientation and course guide
2	Module 1 unit 1 and 2
3	Module 1 unit 3 and 4
4	Module 1 unit 5 and 6
5	Module 2 unit1 and 2
6	Module 2 unit3
7	Module 3 unit1
8	Module 3 unit2
9	Module 3 unit 3
10	Module 3 unit 4
11	Module 4 unit 1
12	Module 4 unit 2
13	Module 4 unit 3
14	Revision
15	Examination

The activities in Table I include facilitation hours (synchronous and asynchronous), assignments, mini projects, and laboratory practical. How do you know the hours to spend on each? A guide is presented in Table 2.

Table 2: Required Minimum Hour of Study

S/N	Activity	Hour per Week	Hour per Semester
1	Synchronous Facilitation (Video Conferencing)	1	13
2	Asynchronous Facilitation (Read and respond to posts including facilitator's comment, self-study)	3	39
3	Assignments, mini-project	1	13
	Total	5	65

Assessment

Table 3 presents the mode you will be assessed.

Table 3: Assessment

S/N	Method of Assessment	Score (%)
1	Tutor Mark Assignment (TMAs)	30
2	Final Examination	70
	Total	100

Assignments

Take the assignment and click on the submission button to submit. The assignment will be scored, and you will receive feedback.

Examination

Finally, the examination will help to test the cognitive domain. The test items will be mostly application, and evaluation test items that will lead to creation of new knowledge/idea.

How to get the Most from the Course

To get the most in this course, you:

- Need a personal laptop. The use of mobile phone only may not give you the desirable environment to work.
- Need regular and stable internet.
- Must work through the course step by step starting with the programme orientation.

- Must do all the assessments following given instructions
- Must create time daily to attend to your study.

Online Facilitation

There will be two forms of facilitation – synchronous and asynchronous. The synchronous will be held through video conferencing according to weekly schedule. During the synchronous facilitation:

- There will be one hour of online real time contact per week making a total of 13 hours for thirteen weeks of study time.
- At the end of each video conferencing, the video will be uploaded for view at your pace.
- You are to read the course material and do other assignments as may be given before video conferencing time.
- The facilitator will concentrate on main themes.
- The facilitator will take you through the course guide in the first lecture at the start date of facilitation.

For the asynchronous facilitation, your facilitator will:

- Present the theme for the week.
- Direct and summarizes forum discussions.
- Coordinate activities in the platform.
- Score and grade activities when need be.
- Support you to learn. In this regard personal mails may be sent.
- Send you videos and audio lectures, and podcasts if need be.

Read all the comments and notes of your facilitator especially on your assignments, participate in forum discussions. This will give you opportunity to socialize with others in the course and build your skill for teamwork. You can raise any challenge encountered during your study. To gain the maximum benefit from course facilitation, prepare a list of questions before the synchronous session. You will learn a lot from participating actively in the discussions.

Learner Support

You will receive the following support:

- Technical Support: There will be contact number(s), email address and chat bot on the Learning Management System where you can chat or send message to get assistance and guidance any time during the course.

- 24/7 communication: You can send personal mail to your facilitator and the centre at any time of the day. You will receive answer to your mails within 24 hours. There is also opportunity for personal or group chats at any time of the day with those that are online.

- You will receive guidance and feedback on your assessments, academic progress, and receive help to resolve challenges facing your studies.

Module 1: Vectors

Module Introduction

This module introduces you to two physical quantities which are scalars and vectors. It then presents addition of vectors, multiplication of a vector by a scalar and position vectors, thereby showing how the position vectors are used in component of a vector in two and three dimensions.

Unit 1 Vectors and Scalars
Unit 2 Addition of Vectors
Unit 3 Component of a vector in two dimensions
Unit 4 Component of a vector in three dimensions

Unit 1: Vectors and Scalars

Unit Structure

1.0 Introduction
2.0 Intended Learning Outcomes (ILOs)
3.0 Main Content
  3.1 Physical Quantities
  3.2 Geometrical Representation of Vectors
  3.3 Types of Vectors
  3.4 Equality of Vectors
4.0 Self-Assessment Exercise(s)
5.0 Conclusion
6.0 Summary
7.0 References/Further Readings

1.0 Introduction

This unit will help you to understand the underlying meaning of physical quantities which in turn will help you to represent vectors with directions. Having understood different types of vectors with magnitude, you will be able to specify when two vectors are equal.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you will be able to:

- state the two physical quantities
- represent vectors graphically
- state types of vectors
- state when two vectors are equal

3.0 Main Content

3.1 Physical Quantities

Physical quantities can be classified into two main classes, namely:

(i) Scalar quantities: scalar quantities are defined by only magnitude (size). Examples are length, area, volume, mass, time, work, energy, distance, speed, temperature, density, etc.

(ii) Vector quantities: Vectors quantities have both magnitude and direction. Examples are displacement, velocity, acceleration, force, weight, momentum, etc.

3.2 Geometrical Representation of Vectors

Vector quantity can be represented graphically by lines, drawn such that

(i) the length of the line denotes the magnitude of the quantity, according to a stated vector scale.

(ii) the direction of the line represents the direction in which the vector quantity acts. The position of the direction is indicated by an arrow head.

Example 3.2.1 *A* vertical force of $20N$ acting downward would be indicated by a line as shown in **Fig 3.2.1**.

Fig 3.2.1: A force of $20N$

If the chosen vector scale were 1cm equivalent to 10N, the line would be 2.0cm long. Vector quantity AB is referred to as $\overline{AB}$ or $\boldsymbol{a}$.

The magnitude of the vector is simply written as $|\overline{AB}|$or $|\boldsymbol{a}|$. It is sometimes called the ***Modulus*** of the vectors

3.3 Types of Vector

There are different types of vector which are stated as follows:

(i) Zero vector: A vector with a magnitude of zero. The zero vector has no particular direction. It is represented by 0. The zero vector is sometimes called a ***null vector***

(ii) Position vector: For position vector $\overline{AB}$ to occurs, point A has to fixed.

(iii) A line vector: A line vector is one that can shift or slide along its line of action e.g. mechanical force acting on a body.

(iv) Free Vector: It is a vector which is not restricted in any way. It is defined by its magnitude and direction and can be drawn as any one of a set of equal length of parallel lines.

3.4 Equality of Vectors

Two vectors $\boldsymbol{a}$and $\boldsymbol{b}$are said to equal if they have same magnitude and move in the same direction. If $\boldsymbol{a} = \boldsymbol{b}$, then following conditions are satisfied.

(i) They have equal magnitude

(ii) $\boldsymbol{a}$moves in the same direction as $\boldsymbol{b}$

Equality of vectors $\boldsymbol{a}$and $\boldsymbol{b}$are illustrated in Fig 3.4.1.

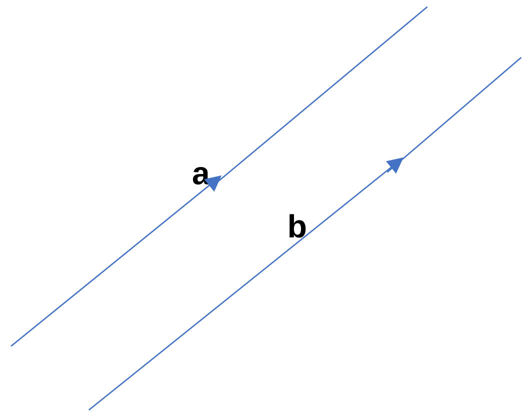


Fig 3.4.1: Equal of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

It is very important to note that $\overline{AB}$ would represent a vector quantity of the same magnitude but of opposite direction with $\overline{BA}$ as depicted in **Fig 3.4.2**.

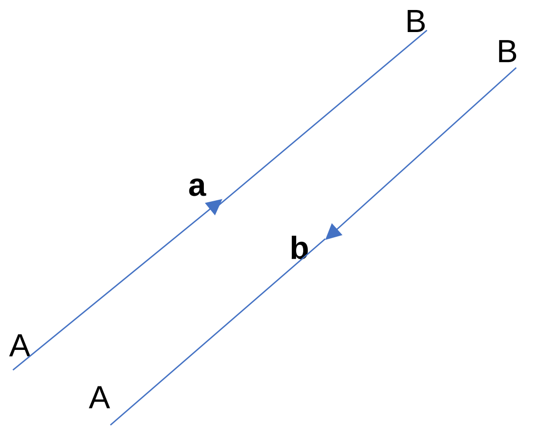


Fig 3.4.2: Negative a vector

$$\text{Given that } \overline{AB} = \boldsymbol{a} \text{ and } \overline{BA} = \boldsymbol{b}$$
$$\Longrightarrow \overline{AB} = -\overline{BA}$$
$$\therefore \quad \boldsymbol{a} = -\boldsymbol{b}$$

Thus, vector $\boldsymbol{b}$ is a negative vector of $\boldsymbol{a}$. It is because $\boldsymbol{b}$ vector has the same magnitude as $\boldsymbol{a}$but directions are opposite to each other.

4.0 Self-Assessment Exercise(s)

(1) State whether the following physical quantities are scalar or vectors:
- (i) Pressure
- (ii) Electric current
- (iii) Conductivity
- (iv) Frequency
- (v) Power
- (vi) Torque
- (vii) Latent heat
- (viii) Viscosity
- (ix) Surface
- (x) Reactance

(2) What is free and localized vector?

(3) Write short notes on vectors in opposite direction.

(4) What is a unit vector?

5.0 Conclusion

In this unit, you have studied the concept of physical quantities, geometrical representations of vector. You have also introduced to the different types of vector and able to distinguish when two vectors are equal or not.

6.0 Summary

In this unit, you have learned about scalars and vectors quantities with their examples. You have also learned about various representations and the forms of vectors. You have equipped yourself with the facts about equality of vectors.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition
K. A. Stroud. *Engineering Mathematics*.8th Edition
Larson Edwards *Calculus: An Applied Approach*. 7th Edition

Unit 2: Addition of Vectors

Unit Structure

1.0 Introduction

This unit introduces you to operations of vectors namely; addition and subtraction of vectors. You also learn the triangle and parallelogram laws of vector addition and subtraction. You would be able to sum two or more vectors with aid of geometrical presentation of vectors. Also, multiplication of a vector by a scale will be discussed with the illustrated examples.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you will be able to:

- state the triangle law of vector addition
- state the parallelogram of vector addition
- sum of two or more vectors

3.0 Main Content

3.1 Addition of Two Vectors

In **Fig 3.1**, the distance from point A to C can be covered in two ways:
(i) from point A to B and then from B to C.
(ii) directly from A to C

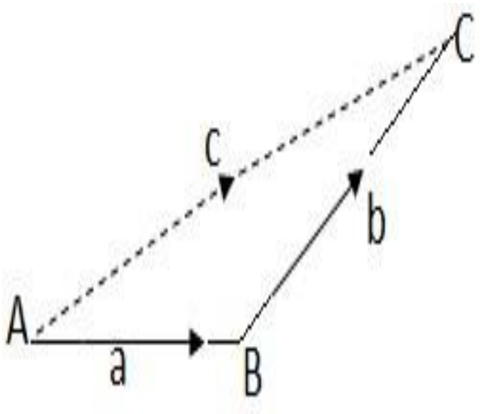


Fig 3.1.1: Distance from point A to B

The vector $\overrightarrow{AC}$ is the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. It can be written as

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

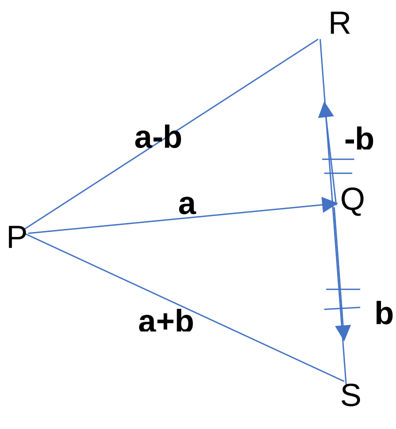
Thereby, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ is called the *triangle law of vector addition*

From **Fig 3.1**, $\overrightarrow{AC}$, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are represented by $\boldsymbol{c}$**,** $\boldsymbol{a}$ and $\boldsymbol{b}$. Then:

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$

$\boldsymbol{a} + \boldsymbol{b}$ is the *resultant* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

From the definition of a negative vector in unit 1, we can say that the subtraction of a vector is the same as the addition of its negative (see **Fig 3.2**).



In Fig 3.1. 2: Subtraction of vector

From Fig 3.1.2, $\overrightarrow{QS} = -\overrightarrow{QR}$

$$\begin{aligned}\overrightarrow{PQ} - \overrightarrow{QR} &= \overrightarrow{PQ} + \left(-\overrightarrow{QR}\right)\\ &= \overrightarrow{PQ} + \overrightarrow{QS}\\ &= \overrightarrow{PS}\end{aligned}$$

Fig 3.1.2, illustrates the sum and difference of vectors, thus

$$\overrightarrow{PQ} = \boldsymbol{a},\ \overrightarrow{QS} = \boldsymbol{b}, \overrightarrow{QR} = -\boldsymbol{b}$$

Therefore**,** $\overrightarrow{PS} = \boldsymbol{a} + \boldsymbol{b}$

$\overrightarrow{PR} = \boldsymbol{a} - \boldsymbol{b}$

Example 3.1.1

If ABC is a triangle, what is the sum of the vectors represented by $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$?

Solution 3.1.1

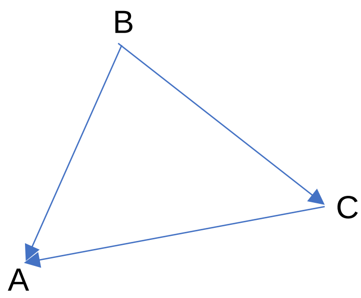


Fig 3.1.3: Triangle ABC

From **Fig 3.1.3,** $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{BA} + \overrightarrow{AB}$

$$= -\overrightarrow{AB} + \overrightarrow{AB}$$
$$= 0$$

Example 3.1.2

If the vector a and b are represented by the sides $\overrightarrow{AB}, \overrightarrow{AC}$ of a triangle ABC, what vectors are represented by:

(i) $\overrightarrow{BC}$

(ii) $\overrightarrow{CB}$

(iii) $\overrightarrow{AD}$

where D is the mid-point of $\overrightarrow{BC}$.

Solution 3.1.2

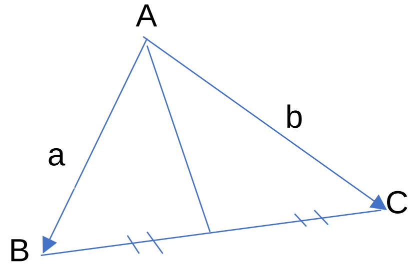


Fig 3.1.4: TriangleABC

From **Fig 3.1.4**,

(i) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$$= -\overrightarrow{AB} + \overrightarrow{AC}$$
$$= -a + b$$
$$= b - a$$

(ii) $\overrightarrow{CB} = -\overrightarrow{BC}$

$$= -(b - a)$$
$$= -b + a$$
$$= a - b$$

(iii) $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$$= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$$
$$= a + \frac{1}{2}(b - a)$$
$$= a + \frac{1}{2}b - \frac{1}{2}a$$
$$= \frac{1}{2}a + \frac{1}{2}b$$
$$= \frac{1}{2}(a + b)$$

3.2 Parallelogram Law of Vectors

The parallelogram law of vector addition states that resultant of two vectors at acting point is represented the diagonal of the parallelogram whose adjacent sides are the two vectors. With reference to **Fig 3.2.1**.

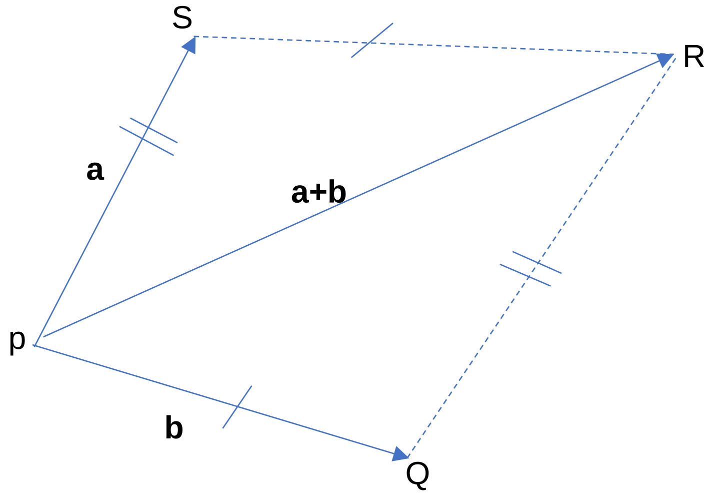


Fig 3.2.1: Parallelogram

$$\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR}$$
$$= \overrightarrow{PR}$$
$$\therefore \overrightarrow{PQ} + \overrightarrow{PS} = \boldsymbol{a} + \boldsymbol{b}$$

Note that the parallelogram law of vector addition is equivalent to the triangle law of vector of addition.

Example 3.2.1
Show that $a + b = b + a$,if a and b are the two vectors in **Fig 3.2.2**

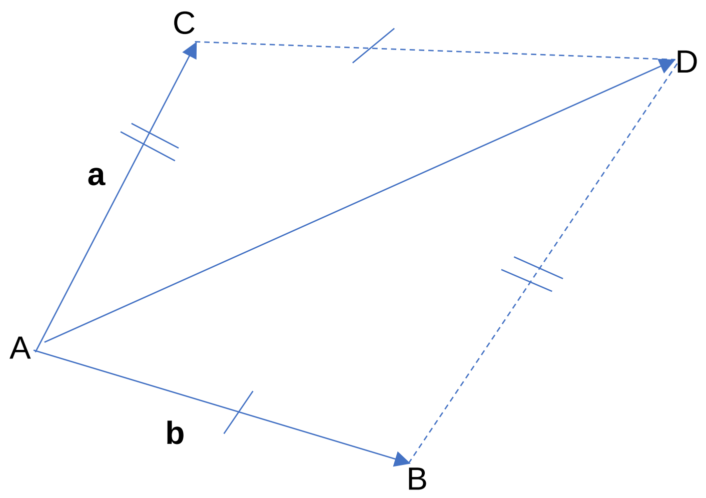


Fig 3.2.2: Parallelogram

From **Fig 3.2.2**, by applying parallelogram law, we have

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$$
$$= b + a$$

Also,

$$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$$
$$= a + b$$

$\therefore a + b = b + a$ (Satisfied commutative law of vector addition)

3.3: Addition of Several Vectors

Suppose there are more two vectors to be summed. The triangle law of vector addition can be used repeatedly to add more than two vectors as depicted in **Fig 3.3.1**.

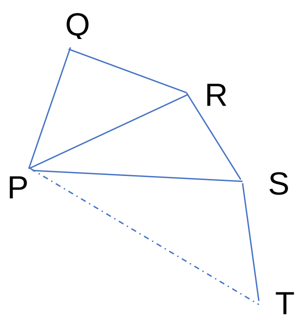


Fig 3.3.1: Polygon Vectors

From**Fig 3.3.1**, by using the triangle of vector addition repeatedly we have

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$
$$\overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$$
$$\overrightarrow{PS} + \overrightarrow{ST} = \overrightarrow{PT}$$

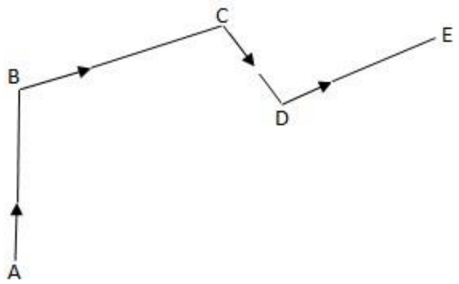
Hence,

$$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{PT}$$

This is called the *polygon of vector*.

Example 3.3.1

In Fig 3.3.2, what is the sum of the vectors represented by $\overrightarrow{AE}$?



Solution 3.3.1

From **Fig 3.3.2**, by using the triangle of vector addition repeatedly we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$
$$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$$
$$\overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE}$$

Hence,

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$$

3.4: Multiplication of a Vector by a Scalar

If $\boldsymbol{a}$ is a vector and k is a scalar, then $k\boldsymbol{a}$ is a vector with magnitude $k|\boldsymbol{a}|$. It means k times the magnitude of $\boldsymbol{a}$**,** whose direction is that of vector $\boldsymbol{a}$ or opposite to vector $\boldsymbol{a}$ according as k is positive or negative respectively. In particular, $\boldsymbol{a}$ and $-\boldsymbol{a}$ are opposite vectors.

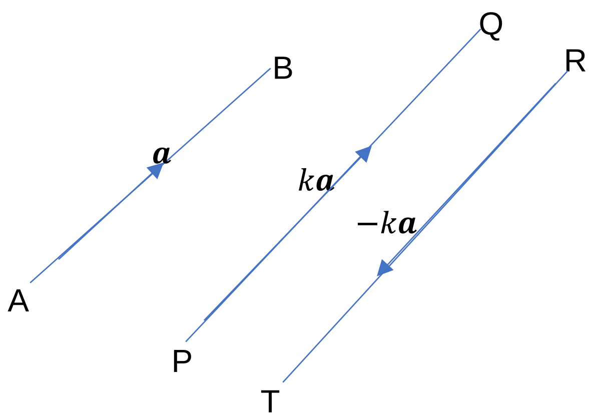


Fig 3.4.1: Scalar Multiplication of Vector

From **Fig 3.4.1**

Note that the direction may change, depending on the sign of the scalar k

- If $k > 0, k\boldsymbol{a}$ has the same direction as $\boldsymbol{a}$
- If $k < 0$, $k\boldsymbol{a}$ has the opposite direction as $\boldsymbol{a}$
- If $k = 0$, $k\boldsymbol{a}$is the zero vector

Example 3.4.1:

P, Q, R and S are the vertices of a quadrilateral such that A and B are mid -points of the $\overrightarrow{QR}$ and $\overrightarrow{PS}$ respectively. Show that:

$$\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{QP} + \overrightarrow{RS}\right)$$

Solution 3.4.1:

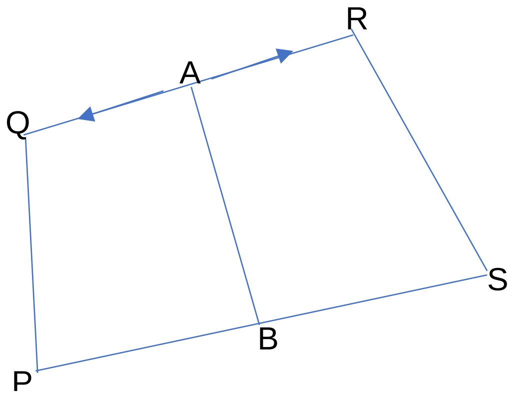


Fig 3.4.1: Quadrilateral

$$\overrightarrow{AB} = \overrightarrow{AQ} + \overrightarrow{QP} + \overrightarrow{PB}$$

Also, $\overrightarrow{AB} = \overrightarrow{AR} + \overrightarrow{RS} + \overrightarrow{SB}$

$$2\overrightarrow{AB} = \overrightarrow{AQ} + \overrightarrow{QP} + \overrightarrow{PB} + \overrightarrow{AR} + \overrightarrow{RS} + \overrightarrow{SB}$$

$$= \overrightarrow{AQ} + \overrightarrow{AR} + \overrightarrow{QP} + \overrightarrow{RS} + \overrightarrow{PB} + \overrightarrow{RS}$$

But,

$$\overrightarrow{AQ} + \overrightarrow{AR} = \overrightarrow{AQ} - \overrightarrow{AQ} = 0$$

Also,

$$\overrightarrow{PB} + \overrightarrow{RS} = \overrightarrow{PB} - \overrightarrow{PB} = 0$$

Therefore,

$$2\overrightarrow{AB} = \overrightarrow{QP} + \overrightarrow{RS}$$

$$\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{QP} + \overrightarrow{RS}\right)$$

4.0 Self-Assessment Exercise(s)

1. Find the sum of the following vectors

(i) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$

(ii) $\overrightarrow{AK} + \overrightarrow{KL} + \overrightarrow{LP} + \overrightarrow{PE}$

(iii) $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST}$

(iv) $\overrightarrow{BC} - \overrightarrow{DC} + \overrightarrow{DE} + \overrightarrow{FE}$

(v) $\overrightarrow{AC} + \overrightarrow{CL} - \overrightarrow{ML}$

(vi) $\overrightarrow{GH} + \overrightarrow{HJ} + \overrightarrow{JK} + \overrightarrow{KL} + \overrightarrow{LG}$

2. Find the resultant of the vectors $\overrightarrow{BC} - \overrightarrow{DC}$and $-\overrightarrow{FD}$

3. Find the resultant of the vectors $\overrightarrow{AC}, 3\overrightarrow{BC}, \overrightarrow{CD}, 3\overrightarrow{CD}$ and $\overrightarrow{DA}$
4. Prove that diagonals of a parallelogram bisect each other
5. If ABC is a triangle and M and N are the mid-points of AB and BC respectively, show that $MN = \frac{1}{2}AC$.
6. $ABCD$ is quadrilateral whose sides represent vectors, find a vector equivalent to $\overrightarrow{AD} + \overrightarrow{DC} + \vec{C}B$
7. Find the sum of the following vectors

(i) $\overrightarrow{AB}, -\overrightarrow{DC}, \overrightarrow{BC}$and $\overrightarrow{CE}$

(ii) $\overrightarrow{PR}, -\overrightarrow{SR}, \overrightarrow{ST}$ and $-\overrightarrow{QT}$

8. $ABCD$ is a rectangle in which $AB = 3a$ and $AD = 3b$. Find the magnitude of the vectors:

(i) $AB + AD$

(ii) $DC - BC$

(iii) $AB + BC$

5.0 Conclusion

The triangle and parallelogram of laws of vector addition were discussed. The two laws were applied in different examples to find sum of two and several vectors. Scalar multiplication and its applications are also expatiated with a complementary example The solutions of the addition of vectors are *resultant vectors*.

6.0 Summary

In this unit, you have learned about addition of two vector using the triangle and parallelogram laws of vectors addition. You have also learned addition of several

vectors using repeatedly triangle law of vector addition. More so, you have able to study multiplication of a vector by a scalar with examples.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A. Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards*Calculus:An Applied Approach*. 7th Edition

John Bird, *Engineering Mathematics*.4th Edition

Unit 3: Components of a Vector in Dimensions

Unit Structure

1.0 Introduction

Any vector may be expressed in Cartesian components, by using unit vectors in the directions of the coordinate axes. This unit describes the unit vectors in two dimensions and in three dimensions and shows how the units can used in calculations.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you will be able to:

- Identify the coordinate unit vectors in two dimensions
- Identify the coordinate unit vectors in three dimensions
- Calculate the modulus of a position vectors
- Calculate the angle between a position vector

3.0 Main Content

3.1 Components of a vector in two Dimensions

Consider the components of a vector in directions which are mutually perpendicular to each other. We shall refer to the rectangular cartesian coordinate system where the position of a point in the plane is completely specified x and y.

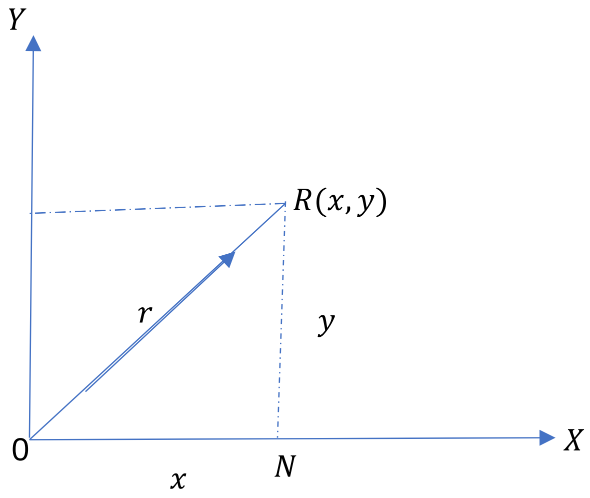


Fig 3.1.1: $x - y$ plane graph

From **Fig 3.1:**
Let i and j be unit vectors in the directions of $0X$ and $0Y$ respectively.
Then, $\overrightarrow{0N} = x\hat{\imath}$ and $\overrightarrow{NR} = y\hat{\jmath}$

$$\overrightarrow{0R} = r$$

$$\overrightarrow{0R} = \overrightarrow{0N} + \overrightarrow{NR}$$

Therefore,

$$r = x\hat{\imath} + y\hat{\jmath}$$

Note: Addition, subtraction and scalar multiplication of vectors are done component-wise. For instance,

if $r_1 = x_1\hat{\imath} + y_1\hat{\jmath}$ and $r_2 = x_2\hat{\imath} + y_2\hat{\jmath}$
(i) $r_1 + r_2 = (x_1 + x_2)\hat{\imath} + (y_1 + y_2)\hat{\jmath}$
(ii) $r_1 - r_2 = (x_1 - x_2)\hat{\imath} + (y_1 - y_2)\hat{\jmath}$
(iii) $kr_1 = kx_1\hat{\imath} + ky_1\hat{\jmath}$

3.1.1: Modulus of a vector in terms of its components

The modulus of a vector in terms of its components can be obtained from Fig 3.1.1.
i.e.$\Delta\ ONR$
$\left|\overrightarrow{OR}\right|^2 = \overrightarrow{0N}^2 + \overrightarrow{NR}^2$ (Pythagoras theorem)

$$|r|^2 = (x\hat{\imath})^2 + (y\hat{\jmath})^2$$

$|r|^2 = x^2 + y^2,\ (\hat{\imath}^2 = \hat{\jmath}^2 = 1)$
$\therefore |r| = \sqrt{x^2 + y^2}$B

$|r|$ is the modulus of vector r

Example 3.1.1:

Find the modulus of each of the following vectors:

(i) $3\hat{\imath} + 4\hat{\jmath}$

(ii) $\hat{\imath} + 3\hat{\jmath}$

(iii) $-2\hat{\imath} - 5\hat{\jmath}$

(iv) $\hat{\imath}\sin\theta - \hat{\jmath}\cos\theta$

Solution 3.1.1:

(i) Let $r_1 = 3\hat{\imath} + 4\hat{\jmath}$

$$|r_1| = \sqrt{(3\hat{\imath})^2 + (4\hat{\jmath})^2}$$
$$|r_1| = \sqrt{9 + 16}$$
$$|r_1| = \sqrt{25}$$
$$|r_1| = 5$$

(ii) Let $r_2 = \hat{\imath} + 3\hat{\jmath}$

$$|r_2| = \sqrt{(\hat{\imath})^2 + (3\hat{\jmath})^2}$$
$$|r_2| = \sqrt{1 + 9}$$
$$|r_2| = \sqrt{10}$$

(iii) Let $r_3 = -2\hat{\imath} - 5\hat{\jmath}$

$$|r_3| = \sqrt{(-2\hat{\imath})^2 + (-5\hat{\jmath})^2}$$
$$|r_3| = \sqrt{4 + 25}$$
$$|r_3| = \sqrt{29}$$

(iv) Let $r_4 = \hat{\imath}\sin\theta + \hat{\jmath}\sin\theta$

$$|r_4| = \sqrt{(\hat{\imath}\sin\theta)^2 + (\hat{\jmath}\cos\theta)^2}$$
$$|r_4| = \sqrt{\sin^2\theta + \cos^2\theta}$$
$$|r_4| = \sqrt{1}$$
$$|r_4| = 1$$

3.1.2: Direction cosines of vector in terms of its components

The direction of the vector r is specified by the angles which $\overrightarrow{0R}$ make with X and Y −axes.
Let represent the angles by α and β respectively (see **Fig 3.1.2**)

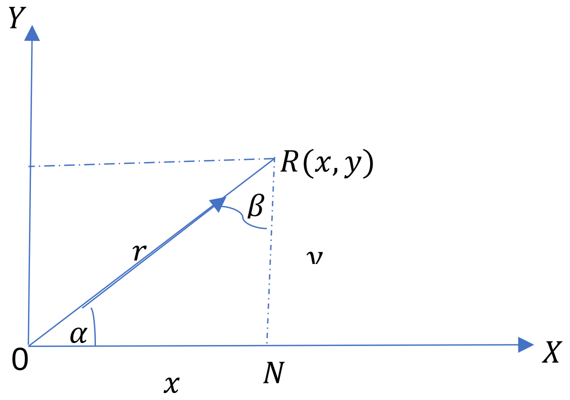


Fig 3.1.2: The graph of angles α and β along $0X$

From **Fig 3.1.2:**

$\cos\alpha = \frac{x}{|r|}$ and $\cos\beta = \frac{y}{|r|}$

Then, $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$ and $\cos\beta = \frac{y}{\sqrt{x^2+y^2}}$

$\cos\alpha$ and $\cos\beta$ are called the direction cosines of $\overrightarrow{0R}$.

Example 3.2.1:

If $r_1 = 7\hat{\imath} + 3\hat{\jmath},\ r_2 = 2\hat{\imath} - 5\hat{\jmath}$, find the modulus and direction cosines of

(i) $r_1 + r_2$

(ii) $r_1 - r_2$

Solution 3.2.1:

$$r_1 = 7\hat{\imath} + 3\hat{\jmath}$$
$$r_2 = 2\hat{\imath} - 5\hat{\jmath}$$

(i) $r_1 + r_2 = 7\hat{\imath} + 3\hat{\jmath} + 2\hat{\imath} - 5\hat{\jmath} = 9\hat{\imath} - 2\hat{\jmath}$

$$|r_1 + r_2| = \sqrt{(9\hat{\imath})^2 + (-2\hat{\jmath})^2}$$
$$= \sqrt{81 + 4}$$
$$= \sqrt{85}$$

Let $\cos\alpha_1$ and $\cos\beta_1$ be the direction cosines of $r_1 + r_2$

$\cos\alpha_1 = \frac{9}{\sqrt{85}}, \cos\beta_1 = \frac{-2}{\sqrt{85}}$

(ii) $r_1 - r_2 = (7\hat{\imath} + 3\hat{\jmath}) - (2\hat{\imath} - 5\hat{\jmath}) = -5\hat{\imath} - 8\hat{\jmath}$

$$|r_1 + r_2| = \sqrt{(-5\hat{\imath})^2 + (-8\hat{\jmath})^2}$$
$$= \sqrt{25 + 64}$$
$$= \sqrt{89}$$

Let $\cos\alpha_1$ and $\cos\beta_1$ be the directions cosines of $r_1 + r_2$

$\cos\alpha_2 = \frac{-5}{\sqrt{89}}, \cos\beta_2 = \frac{-8}{\sqrt{89}}$

3.2 Components of a vector in Three Dimensions

In **3.1**, we have considered vectors that lie in the $x-y$ plane. These vectors are described using the mutually perpendicular unit vectors $\hat{\imath}$ and $\hat{\jmath}$ that lie in the $x-$ and $y-$ directions respectively.
To consider vectors in space, we need a third direction with an axis that is perpendicular to both the x and y axes. This is provided by the z −axis (see **Fig 3.2**). The $x-, y-$ and z −axes define three mutually perpendicular directions for the axes of reference point 0.

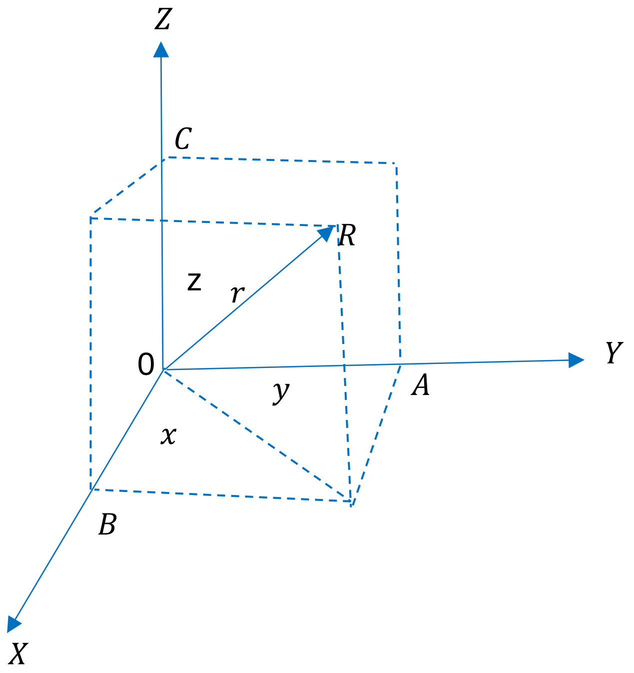


Fig. 3.2: Vector in space

Let $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ be unit vectors in the directions of $0X, 0Y$ and $0Z$ axes respectively.
Let $\overrightarrow{0A} = x\hat{\imath}$, $\overrightarrow{0C} = y\hat{\jmath}$ and $\overrightarrow{0B} = z\hat{k}$
Let $r = \overrightarrow{0R}$

From **Fig. 3.2:**

$$\overrightarrow{0R} = \overrightarrow{0A} + \overrightarrow{0C} + \overrightarrow{0B}$$
$$= \overrightarrow{0A} + \overrightarrow{OC} + \overrightarrow{0P}$$
$$r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

As in two dimensions, addition, subtraction and scalar multiplication of vectors are done component-wise. For example**:**

if $r_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $r_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

(i) $r_1 + r_2 = (x_1 + x_2)\hat{\imath} + (y_1 + y_2)\hat{\jmath} + (z_1 + z_2)\hat{k}$

(ii) $r_1 - r_2 = (x_1 - x_2)\hat{\imath} + (y_1 - y_2)\hat{\jmath} + (z_1 - z_2)\hat{k}$

(iii) $kr_1 = kx_1\hat{\imath} + ky_1\hat{\jmath} + kz_2\hat{k}$

3.2.1: Modulus of the vector of in terms of its components

The modulus of a vector in terms of its components can be obtained from **Fig 3.2.**

$$\left|\overrightarrow{0R}\right|^2 = \overrightarrow{0A}^2 + \overrightarrow{0C}^2 + \overrightarrow{0B}^2$$
$$= (x\hat{\imath})^2 + (y\hat{\jmath})^2 + (z\hat{k})^2$$
$$\left|\overrightarrow{0R}\right| = \sqrt{x^2 + y^2 + z^2}$$
$$\therefore |r| = \sqrt{x^2 + y^2 + z^2}$$

Example 3.2:

Find the modulus of each of the following vectors

(i) $2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

(ii) $\hat{\imath} - \hat{\jmath} + \hat{k}$

(iii) $3\hat{\imath} - 5\hat{\jmath} + \hat{k}$

(iv) $4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$

Solution 3.2:

(i) Let $r_1 = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

$$|r_1| = \sqrt{(2\hat{\imath})^2 + (3\hat{\jmath})^2 + (-4\hat{k})^2}$$
$$= \sqrt{4 + 9 + 16}$$
$$= \sqrt{29}$$

(ii) Let $r_2 = \hat{\imath} - \hat{\jmath} + \hat{k}$

$$|r_1| = \sqrt{(\hat{\imath})^2 + (-\hat{\jmath})^2 + (\hat{k})^2}$$
$$= \sqrt{1 + 1 + 1}$$
$$= \sqrt{3}$$

(iii) Let $r_2 = 3\hat{\imath} - 5\hat{\jmath} + \hat{k}$

$$|r_2| = \sqrt{(3\hat{\imath})^2 + (-5\hat{\jmath})^2 + (1\hat{k})^2}$$
$$= \sqrt{9 + 25 + 1}$$
$$= \sqrt{35}$$

(iv) Let $r_4 = 4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$

$$|r_4| = \sqrt{(4\hat{\imath})^2 + (3\hat{\jmath})^2 + (-2\hat{k})^2}$$
$$= \sqrt{16 + 9 + 4}$$
$$= \sqrt{29}$$

3.2.1: Direction cosines of vector in terms of its components

The direction of the vector r is specified by the angles which $\overrightarrow{0R}$ make with, Y and Z $-$axes (see **Fig 3.2.1**).

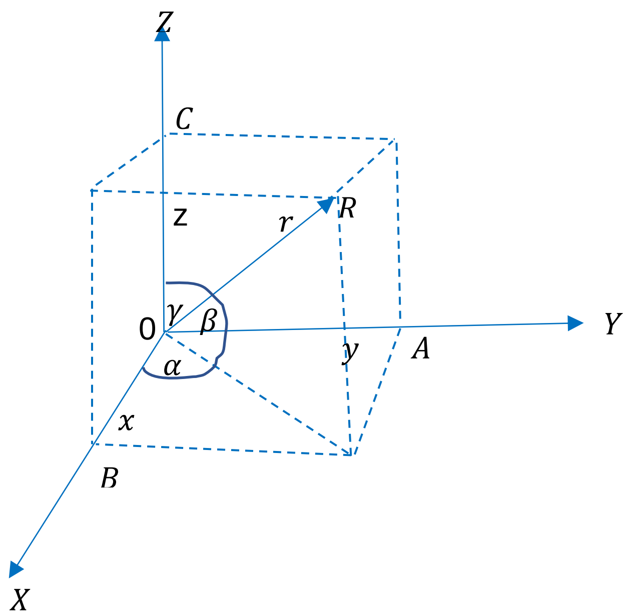


Fig 3.2.1: The graph of angles $\alpha\beta$ and γ along $0R$

Let α, β and γ be the angle which $\overrightarrow{0R}$ makes with $0X, 0Y$ and $0Z$ axes respectively, then

(i) $\cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2+y^2+z^2}}$

(ii) $\cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2+y^2+z^2}}$

(iii) $\cos\gamma = \frac{x}{|r|} = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ defined the direction cosines of the vector $\overrightarrow{0R}$

Example 3.2:

Find the direction cosines of each of the following vectors:

(i) $r_1 = 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$

(ii) $r_2 = -3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$

Solution 3.2:

Let α_1, β_1 and γ_1 be the angles which r_1 makes with $0X, 0Y$ and $0Z$ axes respectively, then,

(i) $r_1 = 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$

$$\cos\alpha_1 = \frac{x}{|r_{1|}}$$

$$= \frac{2}{\sqrt{(2\hat{\imath})^2 + (-5\hat{\jmath})^2 + (\hat{k})^2}}$$

$$= \frac{2}{\sqrt{4 + 25 + 1}}$$

$$= \frac{2}{\sqrt{30}}$$

$$\cos\beta_1 = \frac{y}{|r_{1|}}$$

$$= \frac{-5}{\sqrt{(2\hat{\imath})^2 + (-5\hat{\jmath})^2 + (\hat{k})^2}}$$

$$= \frac{-5}{\sqrt{4 + 25 + 1}}$$

$$= \frac{2}{\sqrt{30}}$$

$$\cos\gamma_1 = \frac{y}{|r_{1|}}$$

$$= \frac{1}{\sqrt{(2\hat{\imath})^2 + (-5\hat{\jmath})^2 + (\hat{k})^2}}$$

$$= \frac{1}{\sqrt{4 + 25 + 1}}$$

$$= \frac{2}{\sqrt{30}}$$

(ii) $r_2 = -3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$

$$\cos\alpha_2 = \frac{x}{|r_{2|}}$$

$$= \frac{-3}{\sqrt{(-3\hat{\imath})^2 + (2\hat{\jmath})^2 + (-6\hat{k})^2}}$$

$$= \frac{-3}{\sqrt{9 + 4 + 36}} = \frac{-3}{\sqrt{49}}$$

$$= \frac{2}{7}$$

$$\cos \beta_2 = \frac{y}{|r_{1|}}$$

$$= \frac{2}{\sqrt{(-3\hat{\imath})^2 + (2\hat{\jmath})^2 + (-6\hat{k})^2}}$$

$$= \frac{-3}{\sqrt{9+4+36}} = \frac{-3}{\sqrt{49}}$$

$$= \frac{2}{7}$$

$$\cos \gamma_2 = \frac{z}{|r_{1|}}$$

$$= \frac{z}{\sqrt{(-3\hat{\imath})^2 + (2\hat{\jmath})^2 + (-6\hat{k})^2}}$$

$$= \frac{-3}{\sqrt{9+4+36}} = \frac{-3}{\sqrt{49}}$$

$$= \frac{2}{7}$$

4.0 Self-Assessment Exercise(s)

1. Find the modulus of each of the following vectors

(i) $r_1 = 2\hat{\imath} - 5\hat{\jmath} + 7\hat{k}$

(ii) $r_2 = 3\hat{\imath} + 8\hat{\jmath} - \hat{k}$

(iii) $r_3 = 3\hat{\imath} - 3\hat{\jmath} + 5\hat{k}$

2. Find the unit vectors in the directions of the following vectors

(i) $r_1 = 2\hat{\imath} + 3\hat{\jmath}$

(ii) $r_2 = -3\hat{\imath} - 5\hat{\jmath}$

(iii) $r_3 = -3\hat{\imath} + \hat{\jmath}$

(iv) $r_4 = -2\hat{\imath} - 2\hat{\jmath}$

3. Find the unit vectors in the directions of the following vectors

(iv) $r_1 = 7\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

(v) $r_2 = 3\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$

4. The vectors a, b and c are given by:

$a = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

$b = 5\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$

$c = 6\hat{\imath} + 2\hat{\jmath} - \hat{k}$

Find:

(i) $2a + 3b - c$

(ii) $a - 2b + 4c$

(iii) $5a + 2b - c$

(iv) $a - b - 3c$

5.The vectors a, b and c are given by $a = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

Find:

(i) $2a + 3b - 7c$

(ii) $4a - 5b + 3c$

(iii) $a + 2b + 4c$

(iv) $3a + 2b - c$

6. The vectors a, b and c are given by $a = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ and $c = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Find:

(i) $4a - 2b + 3c$

(ii) $a - 7b + 5c$

(iii) $4a - 3b + c$

(iv) $a + 5b - 4c$

7. Three points A, B and C have the coordinate $(4, 0), (6, 2)$ and $(-2,1)$ respectively.

(i) Express in component from $\overrightarrow{AB}$,$\overrightarrow{BC}$ and $\overrightarrow{AC}$

(ii) Calculate the unit vector along AC

8.Given that

$$a = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$$
$$b = 4\hat{\imath} + 3\hat{\jmath} + \hat{k}$$

Find a unit vector in the direction of:

(i) $a + b$

(ii) $2b + 3b$

(iii) $a - b$

(iv) $4a - 3b$

5.0 Conclusion

In this unit, you have been introduced to the coordinate unit vectors in two dimensions and in three dimensions. The unit vectors also expressed the position vector of a point in terms of the coordinates. The direction of cosines of vector in terms of its components for two and three dimensions are depicted with worked examples.

6.0 Summary

In this unit, you have learnt that the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are the components of vectors in two dimensions and $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the units in the three dimensions. The modulus of these vectors' units is also calculated. You also learnt the direction cosines of unit vectors in the two and three dimensions. Lastly, you have been able to work through the examples enumerated in this unit.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A. Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards *Calculus: An Applied Approach*. 7th Edition

John Bird. *Engineering Mathematics*.4th Edition

Unit 4: Products of Two Vectors

Unit Structure

1.0 Introduction
2.0 Intended Learning Outcomes (ILOs)
3.0 Main Content
    3.1 Scalar Product
    3.2 Vector Product
4.0 Self-Assessment Exercise(s)
5.0 Conclusion
6.0 Summary
7.0 References/Further Readings

1.0 Introduction

This unit introduces you to two types of products of two vectors which are scalar and vector products of two vectors. Also in this unit, you will learn about the projection of one vector over another vector which defined both the products of two vectors. The basic properties of the two products of vectors are also specified. The working of examples and application of the two products of two are depicted in detailed.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you will be able to:

(i) define the scalar or dot product of two vectors
(ii) solve problems on the dot product of two vectors
(iii) derive some basic results in geometry using vectors
(iv) define the vector or cross product of two vectors
(v) solve problems on the cross product of two vectors

3.0 Main Content

3.1 Scalar or dot product of two vectors

Scalar or dot product of two vectors is defined as the product of magnitude of vector A and magnitude of component of vector B in the direction of vector$A$ (vice-versa) as shown in **Fig 3.1**.

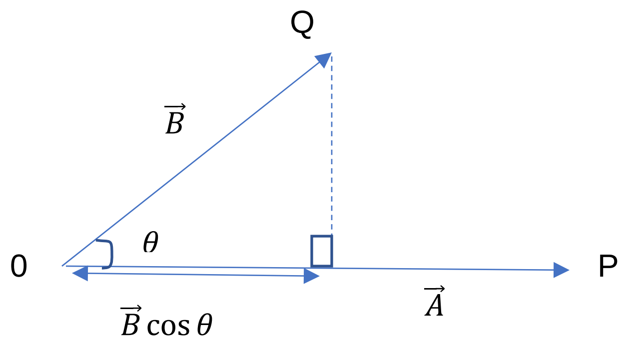


Fig 3.1: Projection of $\vec{B}$ over $\vec{A}$

Scalar or dot product of two vectors A and B may also be defined as the product of magnitudes of the two vectors and the cosine of the angle between them.

In **Fig 3.1** and from definition

$$\therefore A \cdot B = |A||B| \cos\theta$$

where θ is the angle between the vector A and B.

If $A = a_1\hat{\imath} + a_2\hat{\jmath}$ and $B = b_1\hat{\imath} + b_2\hat{\jmath}$, then

$$A \cdot B = (a_1\hat{\imath} + a_2\hat{\jmath}) \cdot (b_1\hat{\imath} + b_2\hat{\jmath})$$
$$= a_1b_1\hat{\imath}\cdot\hat{\imath} + a_1b_2\hat{\imath}\cdot\hat{\jmath} + a_2b_1\hat{\jmath}\cdot\hat{\imath} + a_2b_2\hat{\jmath}\cdot\hat{\jmath}$$

Now, let the two vectors expressed in terms of the unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$, then

$A = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and

$$B = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$
$$A \cdot B = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \cdot (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$
$$= a_1b_1\hat{\imath}\cdot\hat{\imath} + a_1b_2\hat{\imath}\cdot\hat{\jmath} + a_1b_3\hat{\imath}\cdot\hat{k} + a_2b_1\hat{\jmath}\cdot\hat{\imath} + a_2b_2\hat{\jmath}\cdot\hat{\jmath} + a_2b_3\hat{\jmath}\cdot\hat{k}$$
$$+ a_3b_1\hat{k}\cdot\hat{\imath} + a_3b_2\hat{k}\cdot\hat{\jmath} + a_3b_3\hat{k}\cdot\hat{k}$$

Recall that $\hat{\imath}$ and $\hat{\jmath}$are mutually perpendicular unit vectors, consequently,

$$\hat{\imath}\cdot\hat{\imath} = 1 \times 1 \cos 0^0 = 1$$
$$\hat{\imath}\cdot\hat{\jmath} = 1 \times 1 \cos 90^0 = 0$$
$$\hat{\imath}\cdot\hat{k} = 1 \times 1 \cos 90^0 = 0$$
$$\hat{\jmath}\cdot\hat{\imath} = 1 \times 1 \cos 90^0 = 0$$
$$\hat{\jmath}\cdot\hat{\jmath} = 1 \times 1 \cos 0^0 = 1$$
$$\hat{\jmath}\cdot\hat{k} = 1 \times 1 \cos 90^0 = 0$$
$$\hat{k}\cdot\hat{\imath} = 1 \times 1 \cos 90^0 = 0$$
$$\hat{k}\cdot\hat{\jmath} = 1 \times 1 \cos 90^0 = 0$$
$$\hat{k}\cdot\hat{k} = 1 \times 1 \cos 0^0 = 1$$

For the unit vector $\hat{\imath}$ and $\hat{\jmath}$,$A \cdot B = a_1b_1 + a_2b_2$.

And also,for the unit vector $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$,$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$

3.1.1 Angle between two vectors

From dot product of two vectors A and B formular, the angle between the two vectors can be defined as

$$\cos\theta = \frac{A \cdot B}{|A||B|}$$

Example 3.1.1: Find the scalar products of the following pairs of vectors:

(i) $A = 2\hat{\imath} + 3\hat{\jmath}$ and $B = -\hat{\imath} + 4\hat{\jmath}$
(ii) $A = 5\hat{\imath} + 2\hat{\jmath}$ and $B = 2\hat{\imath} - 3\hat{\jmath}$
(iii) $A = 2i + 3\hat{\jmath} + 5\hat{k}$ and $B = 4\hat{\imath} + \hat{\jmath} + 6\hat{k}$
(iv) $A = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $B = -2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

Solution 3.1.1:

(i) $A = 2\hat{\imath} + 3\hat{\jmath}$ and $B = -\hat{\imath} + 4\hat{\jmath}$

$$A \cdot B = a_1b_1 + a_2b_2$$
$$A \cdot B = 2(-1) + 3(4)$$
$$= -2 + 12$$
$$= 10$$

(ii) $A = 5\hat{\imath} + 2\hat{\jmath}$ and $B = 2\hat{\imath} - 3\hat{\jmath}$

$$A \cdot B = a_1b_1 + a_2b_2$$
$$A \cdot B = 5(2) + 2(-3)$$
$$= 10 - 6$$
$$= 4$$

(iii) $A = 2i + 3\hat{\jmath} + 5\hat{k}$ and $B = 4\hat{\imath} + \hat{\jmath} + 6\hat{k}$

$$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$$
$$A \cdot B = 2(4) + 3(1) + 5(6)$$
$$= 8 + 3 + 30$$
$$= 41$$

(iv) $A = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $B = -2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

$$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$$
$$A \cdot B = 3(2) + (-2)(3) + 1(-4)$$
$$= 6 - 6 - 4$$
$$= -4$$

Example 3.1.2:

Find the angles between $A = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $B = 6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

Solution 3.1.2:
(i) Let θ be the angle between the vectors A and B
$A = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $B = 6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

$$\cos\theta = \frac{A \cdot B}{|A||B|}$$

$$|A| = \sqrt{(2\hat{\imath})^2 + (2\hat{\jmath})^2 + (-\hat{k})^2} = \sqrt{4+4+1}$$
$$= \sqrt{9} = 3$$
$$|B| = \sqrt{(6\hat{\imath})^2 + (-3\hat{\jmath})^2 + (2\hat{k})^2} = \sqrt{36+9+4}$$
$$= \sqrt{49} = 7$$
$$A \cdot B = 2(6) + 2(-3) + (-1)(2) = 12 - 6 - 2$$
$$A \cdot B = 4$$

Then, $\cos\theta = \frac{A \cdot B}{|A||B|} = \frac{4}{3(7)} = 0.1905$

$$\theta = cos^{-1}\,(0.1905)$$

Therefore, $\theta = 79^o$

Example 3.1.3:

Determine the value of a so that $A = 2\hat{\imath} + a\hat{\jmath} + \hat{k}$ and $B = 4\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ are perpendicular.

Solution 3.1.2:

A and B are perpendicular if only if $A \cdot B = 0$.
Then, $A \cdot B = 2(4) + a(-2) + 1(-2) = 0$

$$= 8 - 2a - 2 = 0$$
$$-2a = -6$$
$$a = 3$$

3.2 Vector product of two Vectors

The vector or cross product of two vectors A and B is denoted by $A \times B$, and its resultant is perpendicular to the vectors a and b. It is used to determine the vector, which is perpendicular to the plane surface spanned by two vectors.

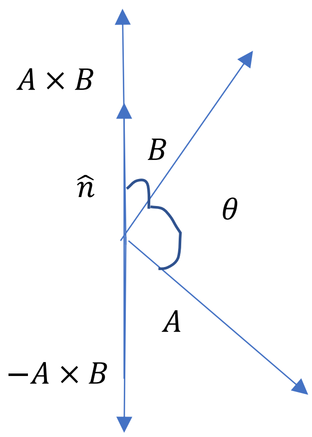


Fig 3.2: Cross Product

If θ is the angle between the vectors A and B, the cross product of the vectors is given by:

$$A \times B = |A||B| \sin\theta$$

or

$$A \times B = |A||B| \sin\theta \, \hat{n}$$

$|A|$ and $|B|$ are the magnitudes of the vectors A and B
$\hat{n}$ is the unit vector perpendicular to the plane containing the vectors, in the direction given by the right-hand ruler.

3.2.1: Cross product of two vector formular

Consider two vectors

$$A = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$
$$B = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

When know that standard basis vectors $\hat{\imath}, \hat{\jmath}and\hat{k}$ satisfy the below-given equalities

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \text{and } \hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath} \text{and } \hat{k} \times \hat{\jmath} = -\hat{\imath}, \quad \hat{k} \times \hat{\imath} = \hat{\jmath} \text{and} \hat{\imath} \times \hat{k} = \hat{\jmath}$$

Also, the anti-commutativity of the cross product and the distinct absence of linear independence of the vectors signifies that:

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

Now,

$$A \times B = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$A \times B = a_1b_1(\hat{\imath} \times \hat{\imath}) + a_1b_2(\hat{\imath} \times \hat{\jmath}) + a_1b_3(\hat{\imath} \times \hat{k}) + a_2b_1(\hat{\jmath} \times \hat{\imath}) + a_2b_2(\hat{\jmath} \times \hat{\jmath}) + a_2b_3(\hat{\jmath} \times \hat{k}) + a_3b_1(\hat{k} \times \hat{\imath}) + a_3b_2(\hat{k} \times \hat{\jmath}) + a_3b_3(\hat{k} \times \hat{k})$$

By applying the above-mentioned equalities

$$A \times B = a_1b_1(0) + a_1b_2(\hat{k}) + a_1b_3(-\hat{\jmath}) + a_2b_1(-\hat{k}) + a_2b_2(0) + a_2b_3(\hat{\imath}) + a_3b_1(\hat{\jmath}) + a_3b_2(-\hat{\imath}) + a_3b_3(0)$$

$$= (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_3b_2)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$

3.2.2 Cross product matrix

We can also derive the formula for the cross product of two vectors using the determinant of the matrix as given below.

$$A = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$
$$B = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

Thus,

$$A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$A \times B = (a_2b_3 - a_3b_2)\hat{\imath} - (a_3b_2 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$
$$= (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_3b_2)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}$$

3.2.3: Cross product properties

To find the cross product of two vectors, we can use the properties such as anti-commutative property, zero vector property etc. These properties play essential roles in finding the cross product of two vectors. Apart from these properties, some other properties include Jacobi property and distributive property. The properties of cross-product are given below**:**

(i) Anti-commutative property: $A \times B = -B \times A$
(ii) Distributive property: $A \times (B + C) = A \times B + A \times C$
(iii) Jacobi Property: $A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$
(iv) Zero Vector Property: $a \times b = 0$ if $a = 0$ or $b = 0$.

Example 3.2.3.1

Find the cross product of the given vectors

$$A = 5\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$$
$$B = \hat{\imath} + \hat{\jmath} + \hat{k}$$

Solution: 3.2.3.1:

Given:

$$A = 5\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$$
$$B = \hat{\imath} + \hat{\jmath} + \hat{k}$$

To find the cross product of two vectors, we have to write the given vectors in determinant form. Using the determinant form, we can find the cross product of two vectors as:

$$A \times B = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 6 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

By expanding

$$A \times B = (6-2)\hat{\imath} - (5-2)\hat{\jmath} + (5-6)\hat{k}$$

$$A \times B = 4\hat{\imath} - 3\hat{\jmath} - \hat{k}$$

Example 3.2.3.2:

The position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are $\hat{\imath} + 2\hat{\jmath} - 3\hat{k},\ 4\hat{\imath} - 5\hat{\jmath} + \hat{k}$ and $3\hat{\imath} + 2\hat{\jmath} - \hat{k}$ respectively, find the:

(i) Vector $\overline{AB}$
(ii) Vector $\overline{BA}$
(iii) Vector $\overline{BC}$
(iv) Vector $\overline{CB}$
(v) Cross product $\overline{AB} \times \overline{BC}$

Solution 3.2.3.2:

$$\overrightarrow{OA} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$$
$$\overrightarrow{OB} = 4\hat{\imath} - 5\hat{\jmath} + \hat{k}$$
$$\overrightarrow{OC} = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$$

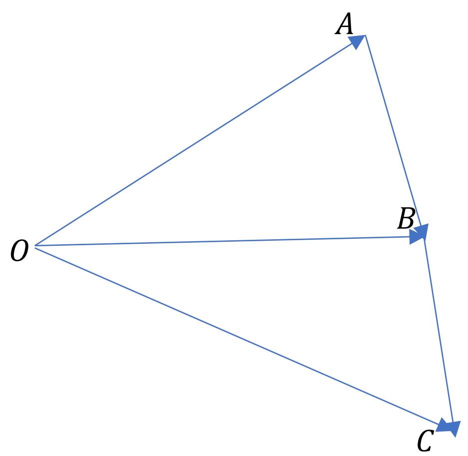


Fig .3.2.3.1: Position vector

(i) $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$

$$= \left(\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) - (4\hat{\imath} - 5\hat{\jmath} + \hat{k})$$
$$= \hat{\imath} + 2\hat{\jmath} - 3\hat{k} - 4\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$$
$$= -3\hat{\imath} + 7\hat{\jmath} - 2\hat{k}$$

(ii) $\overrightarrow{BA} = -\overrightarrow{AB}$

$$= -(-3\hat{\imath} + 7\hat{\jmath} - 2\hat{k})$$
$$= 3\hat{\imath} - 7\hat{\jmath} + 2\hat{k}$$

(iii) $\overrightarrow{BC} = \overrightarrow{OB} - \overrightarrow{OC}$

$$= \left(4\hat{\imath} - 5\hat{\jmath} + \hat{k}\right) - (3\hat{\imath} + 2\hat{\jmath} - \hat{k})$$
$$= 4\hat{\imath} - 5\hat{\jmath} + \hat{k} - 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$$
$$= \hat{\imath} - 7\hat{\jmath} + 2\hat{k}$$

(iv) $\overrightarrow{CB} = -\overrightarrow{BC}$

$$= -(\hat{\imath} - 7\hat{\jmath} + 2\hat{k})$$
$$= -\hat{\imath} + 7\hat{\jmath} - 2\hat{k}$$

(v) $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 7 & -2 \\ 1 & -7 & 2 \end{vmatrix}$

$$= \hat{\imath}(14 - 14) - \hat{\jmath}(-6 + 2) + \hat{k}(21 - 7)$$
$$= 4\hat{\jmath} + 14\hat{k}$$

4.0 Self-Assessment Exercise(s)

1. Calculate the scalar product of vectors a and b when the modulus of a is 9, modulus of b is 7 and the angle between the two vectors is 60°

2. If $A = 5\hat{\imath} + 4\hat{\jmath} + 2\hat{k}, B = 4\hat{\imath} - 5\hat{\jmath} + 3\hat{k}$ and $C = 2\hat{\imath} - \hat{\jmath} - 7\hat{k}$. Find the scalar product of:

(i) A and B
(ii) A and C
(iii) B and C

3. The position vectors of point P, Q and R in the $X - Y$ plane are $\overrightarrow{OP} = 3\hat{\imath} + 4\hat{\jmath}$, $\overrightarrow{OQ} = 5\hat{\imath} + 6\hat{\jmath}$ and $\overrightarrow{OR} = 7\hat{\imath} + 2\hat{\jmath}$ respectively, where k is a scalar. If the resultant of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ is perpendicular to OQ, find the value of k.

4. Find the angles between the following between the following pair vectors:

(i) $3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$
(ii) $5\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$ and $2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$
(iii) $-2\hat{\imath} - 4\hat{\jmath} + 3\hat{k}$ and $2\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

(iv) $\hat{\imath} - 2\hat{\jmath} - \hat{k}$ and $2\hat{\imath} + 3\hat{\jmath} - \hat{k}$

5. If $p = (2k+1)\hat{\imath} + 3\hat{\jmath}, q = -5\hat{\imath} + (k-4)\hat{\jmath}$ and $p \cdot q = 11$, where k is a constant, find (i) Value of k
(ii) cosine of the angle between p and q.

6. Given that $A = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}, B = 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$ and $C = 6\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$. Find the vector product of:
(i) A and B
(ii) A and C
(iii) B and C

7. If $a = 4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $b = 2\hat{\imath} - \hat{\jmath} - 5\hat{k}$. Show that $a \times b$ is perpendicular to both a and b.

8. Show that if θ is the acute angle between the vector a and b then:

$$\sin^2\theta = \frac{(a \times b)^2}{|a|^2|b|^2}$$

9. Show that $(a \times b)^2 = a^2b^2 - (a \cdot b)^2$

10. The adjacent sides of a parallelogram are $\overrightarrow{AB} = 4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\overrightarrow{AC} = \hat{\imath} + 3\hat{\jmath} - \hat{k}$. Find the area of the parallelogram.

5.0 Conclusion

This unit presents the two products of vectors which are scalar and vector products. The scalar is defined as the product of magnitude of vector A and magnitude of component of vector B in the direction of vector A (vice-versa). It is also itemized the results of scalar product unit vectors and calculation of angles between vectors. More so, the unit dealt with vector product and showed the application of determinant of matrices in solving the vector product.

6.0 Summary

In this unit, you have learnt the scalar and cross product of vectors. The mutual perpendicular of the unit vectors is used in calculating the scalar product and the angel between vectors. You have also studied the cross product of the unit vectors and the use of the result of the unit vectors in solving the cross product of vector. Importantly, you have learnt the application of the determinant of matrices in solving problems on the cross product.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A.. Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards *Calculus: An Applied Approach*. 7th Edition

John Bird. *Engineering Mathematics*.4th Edition

Module 2: The Straight Line

Module Introduction

In the module, you are to study Cartesian coordinates, distance between two points and the mid-point of a line segment. You will also learn gradient of points, angle of slope and angle between two lines. Equation of a straight line and general form of the equation of a straight line are inclusive in the module.

Unit 1 Distance between two points
Unit 2 Gradients of lines
Unit 3 Equation of a line

Unit 1: Distance between two points

Unit Structure

1.0 Introduction
2.0 Intended Learning Outcomes (ILOs)
3.0 Main Content
  3.1 Cartesian coordinates
  3.2 Distance between two points
  3.3 Mid-point of a line segment
4.0 Self-Assessment Exercise(s)
5.0 Conclusion
6.0 Summary
7.0 References/Further Readings

1.0 Introduction

This unit introduces you to Cartesian coordinates of points represented by (x, y). Thus, (x, y) coordinates are used in the formulation of distance between two points with aids of Pythagoras theorem. You will also study the proof and formula of mid-point of a line segment with examples.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should able to:

- Define the coordinates of points
- Determine the distance between two points
- Determine the mid-point of line segment

3.0 Main Content

3.1 Cartesian coordinates

The position of a point in a plane is specified by the distances from two perpendicular lines. The lines are called $x - axis$ and $y - axis$ and their point of intersection is called the origin.

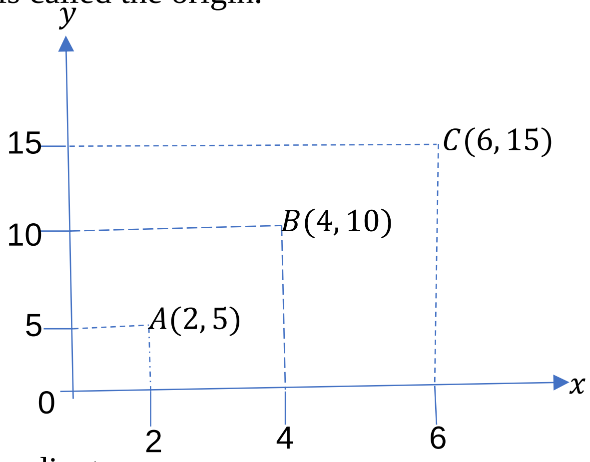


Fig 3.1: Cartesian coordinates

The perpendicular distance from the $x -$ axisis called $x -$co-ordinate or abscissa and the perpendicular distance from $y - axis$ is called $y -$co-ordinate or ordinate.

The co-ordinates form an ordered pair with the abscissa written first. For example, the coordinate of A are $A(2, 5)$ and the coordinates of B are $B(4, 10)$. The coordinates for a general point P are taken as (x, y) and specific points are taken as (x_1, y_1) and (x_2, y_2), etc.

3.2 Distance between two points

In Fig 3.2, P and Q have coordinates (x_1, y_1) and (x_2, y_2) respectively.

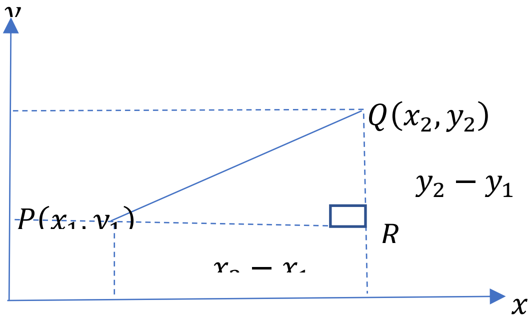


Fig 3.2: Graph of Distance between two points

From **Fig 3.2**,
$PQ^2 = PR^2 + RQ^2$ (Pythagoras theorem)

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 3.2.1

Find the distance between the following pairs of points
(i) $A(3, 2)$ and $B(4, 6)$
(ii) $C(-1, 3)$ and $D(2, -7)$
(iii) $P(5, -8)$ and $Q(-4, -2)$
(iv) $S(-3, -2)$ and $T\ (1, -4)$

Solution 3.2.1
(i) Let d_1 be the distance between A and B, then

$$d_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4 - 3)^2 + (6 - 2)^2}$$
$$= \sqrt{1 + 16}$$
$$= \sqrt{17}$$

(ii) Let d_2 be the distance between A and B, then

$$d_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(2 + 1)^2 + (-7 - 3)^2}$$
$$= \sqrt{9 + 100}$$
$$= \sqrt{109}$$

(iii) Let d_3 be the distance between A and B, then

$$d_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-4 - 5)^2 + (-2 + 8)^2}$$
$$= \sqrt{81 + 36}$$
$$= \sqrt{117}$$

(iv) Let d_4 be the distance between A and B, then

$$d_4 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(1 + 3)^2 + (-4 + 2)^2}$$
$$= \sqrt{16 + 4}$$
$$= \sqrt{20}$$

3.3 The Midpoint of a line segment

In fig 3.3, P and Q have coordinates (x_1, y_1) and (x_2, y_2) respectively. Let R with coordinates (x, y) be the midpoint of PQ.

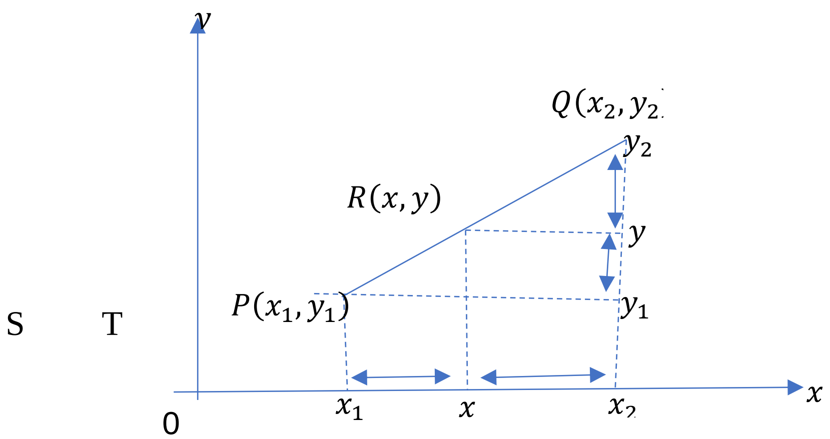


Fig 3.3: Midpoint of segment of line

As $\Delta s PRS$ and PQT are similar:

$$= \frac{PR}{RQ} = \frac{PS}{ST}$$

As $PR = RQ$

$$PS = ST$$
$$\therefore x - x_1 = x_2 - x$$
$$2x = x_2 + x_1$$
$$x = \frac{x_1 + x_2}{2}$$

Similarly, obtained

$$y = \frac{y_1 + y_2}{2}$$

Hence, the coordinate of the midpoint of the line joining (x_1, y_1) and (x_2, y_2) are:

$$\left\{\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right\}$$

Example 3. 3.1

Find the midpoints of the line joining the following pairs of points

(i) $A(3,5)$ and $B(1,3)$
(ii) $P(-1,7)$ and $Q(-3,-5)$
(iii) $C(3,5)$ and $D(1,3)$
(iv) $E(p, 0)$ and $F(0, q)$

Solution 3.3.1

(i) Midpoint of $AB = \left\{\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right\}$

$$= \left\{\frac{1}{2}(1+3), \quad \frac{1}{2}(5+3)\right\}$$
$$= (2, 4)$$

(ii) Midpoint of $PQ = \left\{\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right\}$

$$= \left\{\frac{1}{2}(-1 - 3), \frac{1}{2}(7 - 5)\right\}$$

$$= (-2, 1)$$

(iii) Midpoint of $CD = \left\{\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right\}$

$$= \left\{\frac{1}{2}(3 + 1), \frac{1}{2}(5 + 3)\right\}$$

$$= (2, 4)$$

(iv) Midpoint of $EF = \left\{\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right\}$

$$= \left\{\frac{1}{2}(p + 0), \frac{1}{2}(0 + q)\right\}$$

$$= (\frac{1}{2}p, \frac{1}{2}q)$$

3.4 Division of a line in a Given Ratio

Division of a line in given ration can be internal and external. The internal division of line can be depicted in fig 3.4.1.

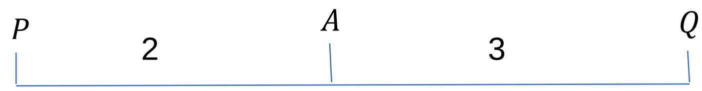


Fig: 3.4.1: Internal Division

We say A divides PQ internally in the ratio 2:3 (see Fig: 3.4.1).

$\frac{PA}{AQ} = \frac{2}{3}$

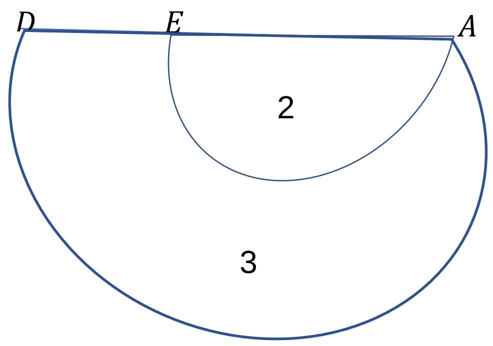


Fig: 3.4.2: External Division I

We say A divides DE externally in the ratio $3: -2$ (see **Fig: 3.4.2**).
$\frac{DA}{AE} = -\frac{3}{2}$

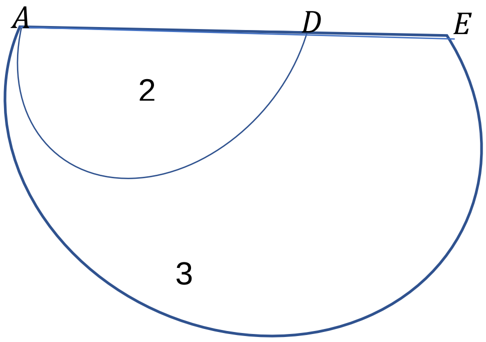


Fig: 3.4.3: External Division II

We say A divides DE externally in the ratio $-2: 3$ (see **Fig: 3.4.3**).
$\frac{DA}{AE} = -\frac{2}{3}$

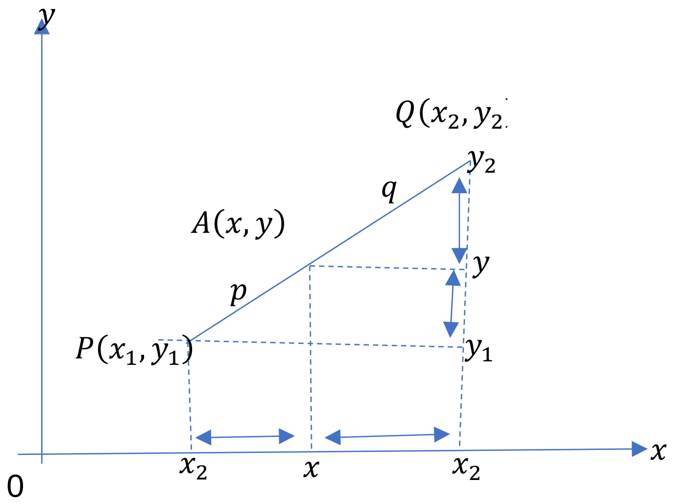


Fig 3.4.4: External division in the ratio $p: q$

In **Fig 3.4.4**, A divides PQ in the ratio $p: q$.
Therefore,

$$\frac{x - x_1}{x_2 - x} = \frac{p}{q}$$
$$qx - qx_1 = px_2 - x$$
$$px + qx = px_2 + qx_1$$
$$(p + q)x = px_2 + qx_1$$

$$\therefore x = \frac{px_2 + qx_1}{p + q}$$

Similarly, obtained

$$y = \frac{py_2 + qy_1}{p + q}$$

Hence, the coordinates of internal or external division ratio are given as

$$\left(x = \frac{px_2 + qx_1}{p + q}, y = \frac{py_2 + qy_1}{p + q}\right)$$

The result obtained is applied to external if p or q is taken as negative.

Example 3.4.1

R divides the line PQ, where the coordinates of P and Q are (5,2) and (3,7) respectively in the ratio 4:3, find the coordinates of R

Solution 3.4.1

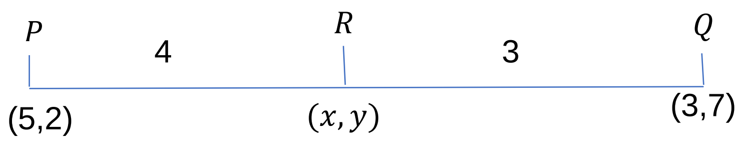


Fig 3.4.5: Coordinates of eternal division ratio

The coordinates of $R = \left(x = \frac{px_2+qx_1}{p+q}, y = \frac{py_2+qy_1}{p+q}\right)$
From **Fig 3.4.5**

$$x_1 = 5, x_2 = 3, y_1 = 2, y_2 = 7, p = 4 \text{ and } q = 3$$

$$x = \frac{px_2 + qx_1}{p + q} = \frac{4(3) + 3(5)}{4 + 3}$$
$$= \frac{12 + 15}{7} = \frac{27}{7}$$
$$y = \frac{py_2 + qy_1}{p + q} = \frac{4(7) + 3(2)}{4 + 3}$$
$$= \frac{28 + 6}{7} = \frac{34}{7}$$

$$R = \left(x = \frac{px_2 + qx_1}{p + q}, y = \frac{py_2 + qy_1}{p + q}\right) = \left(\frac{27}{7}, \frac{34}{7}\right)$$

Example 3.4.2

The point C divides the line AB where the coordinates of P and Q are $(3,2)$ and $(4,1)$ respectively in the ratio $3:-2$. Find coordinates of C.

Solution 3.4.2:

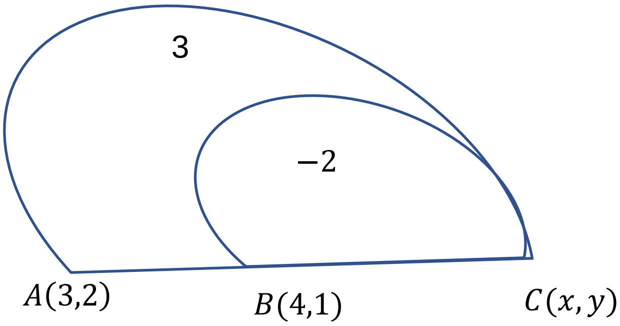


Fig 3.4.5: Coordinate of external division ratio

The coordinates of $C = \left(x = \frac{px_2+qx_1}{p+q}, y = \frac{py_2+qy_1}{p+q}\right)$
From **Fig 3.4.5**

$$x_1 = 3, x_2 = 4, y_1 = 2, y_2 = 1, p = 4 \text{ and } q = -2$$

$$x = \frac{px_2 + qx_1}{p+q} = \frac{3(4) + (-2)(3)}{3-2}$$
$$= \frac{12-6}{1} = 6$$
$$y = \frac{py_2 + qy_1}{p+q} = \frac{3(1) + (-2)(2)}{3-2}$$
$$= \frac{3-4}{1} = -1$$

$$C = \left(x = \frac{px_2 + qx_1}{p+q}, y = \frac{py_2 + qy_1}{p+q}\right) = (6, -1)$$

4.0 Self-Assessment Exercise(s)

1. Find the lengths of the lines joining the following pairs of points:

(a) $(3, 5)$ and $(-2, -3)$
(b) $(-1, -1)$ and $(7, -3)$
(c) $(-2,7)$ and $(3, -2)$
(d) $\left(5, 3\frac{1}{2}\right)$ and $\left(-1, -2\frac{1}{2}\right)$

(e) $(10, -1)$ and $(3, -2)$
(f) $(p, 2q)$ and $(2p, q)$

2. Find the coordinates of the midpoints of the lines joining the following pairs of points:
(a) $(3, -6)$ and $(5,8)$
(b) $(-4, -2)$ and $(1, -3)$
(c) $(-6, -7)$ and $(-5, -3)$
(d) $(11, 8)$ and $(9, 6)$
(e) $(3p, q)$ and $(q, 3p)$
(f) $(-3, 1)$ and $(5, -1)$

3. Find the coordinates of the points which divide the lines joining the following pairs of points in the given ratio:

Pairs of points Ratio
(a) $(\mathbf{5}, \mathbf{8})$ and $(-\mathbf{1}, \mathbf{3})\mathbf{2} : \mathbf{3}$
(b) $(\mathbf{4}, \mathbf{7})$ and $(\mathbf{3}, -\mathbf{2})\mathbf{4} : \mathbf{5}$
(c) $(-\mathbf{8}, \mathbf{5})$ and $(\mathbf{4}, -\mathbf{7}) - \mathbf{3} : \mathbf{4}$
(d) $(\mathbf{10}, -\mathbf{2})$ and $(-\mathbf{6}, -\mathbf{4})\mathbf{3} : -\mathbf{2}$

5.0 Conclusion

You have been introduced to the coordinates of points (x, y) in a plane that represent the perpendicular distance from x and y −axies. The distance between two points is formulated by using Pythagoras theorem of triangular solution. The coordinates of the midpoint of line joining points (x_1, y_1) and (x_2, y_2) are also considered. The internal and external divisions of line ratio are considered for the coordinates of the division of the line.

6.0 Summary

You have learnt the coordinates of point and the formular of the distance two points by using Pythagoras theorem. Also, you have learnt the coordinates of midpoint of a line segment and division of a line in given ratio. The solved examples of the distance two points, midpoint coordinates, internal and external division ratio coordinates are also studied.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th.Edition
K.A Stroud. *Engineering Mathematics*.8th Edition
Larson Edwards, *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-strtlines-2009-1.

Unit 2: The Gradient of a Straight-Line Segment

Unit Structure

1.0 Introduction
2.0 Intended Learning Outcomes (ILOs)
3.0 Main Content
  3.1 The gradient of line
  3.2 Angle of slope
  3.3 Angle between two lines
    3.3.1 Conditions for parallelism
    3.3.2 Condition for perpendicularity
4.0 Self-Assessment Exercise(s)
5.0 Conclusion
6.0 Summary
7.0 References/Further Readings

1.0 Introduction

In this unit, you find the gradient of a straight-line segment, angle of slope and the relationships between the gradients and the angle of slope of the parallel and perpendicular lines.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able:

- Calculate the gradient of a line through two points;
- Calculate the angle of slope
- Use gradient to determine whether two lines are parallel;
- Use gradient to determine whether two lines are perpendicular.

3.0 Main Content

3.1 The gradient of a line

The gradient of a line is a measure of how steep the line is. A line with large gradient will be steep; a line with a small gradient will be relatively shallow; and a line with zero gradient will be horizontal.

Consider Fig 3.1.1 which shows three-line segments. The line segment AD is steeper than the line segment AC. In turn, this is steeper that AB which is horizontal. We can quantify this steepness mathematically by measuring the relative changes in x and y as we move from the beginning to the end of the line segment.

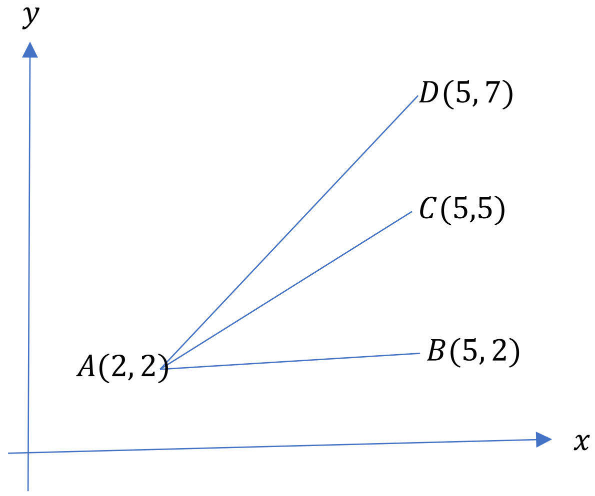


Fig 3.1.1: Three-line segments

From Fig 3.1.1,

On the segment AD, y changes from 2 to 7 as x changes from 2 to 5, so that the change in y is 5 and the change in x is 3. The relative change is

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{7-2}{5-2} = \frac{5}{3}$$

On the segment AC, y changes from 2 to 5 as x changes from 2 to 5,so that in y is 3 and the change in x is 3. The relative change is

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{5-2}{5-2} = \frac{3}{3} = 1$$

On the segment AB, y changes from 2 to 2 as x changes from 2 to 5, so that the change in y is 0 and the change in x is 3. The relative change is

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{2-2}{5-2} = \frac{0}{3} = 0$$

The relative change, $\frac{\text{Change in } y}{\text{Change in } x}$ is the gradient of the line segment. By defining the gradient in this way, we see that it is consistent with our ideas of steepness-lines in which the steeper have a larger gradient than lines which are less steep. Lines which are horizontal have zero gradient.

In the general case, if we take two points $A(x_1, y_1)$ and $B(x_2, y_2)$ (see **Fig 3.1.2**), then we can work out the gradient using the same method as above, by finding the point C and then finding the length AC and BC.

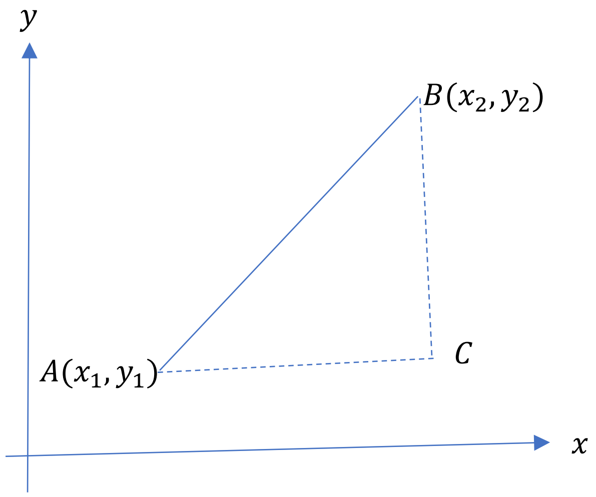


Fig 3.1.2: Gradient of a line

From **Fig 3.1.2,** the point C is given by (x_2, y_1). The length AC is $x_2 - x_1$ and the length BC is $y_2 - y_1$. Therefore, the gradient of AB is

$$\frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

It is a common convention that the gradient of a line is written as m, then it can be written as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 3.1.1

Find the gradient of the line joining points $A(3,4)$ and $B(8,14)$.

Solution 3.1.1

Given points $A(3, 4)$and $B(8, 14)$

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 4}{8 - 3}$$
$$= \frac{10}{5} = 2$$

3.2 Angle of slope

In **Fig 3.2.1**, the line AB makes an angle θ with the positive x-axis. The angle θ is called the angle of slope of the line.

Recall that in the right-angled triangle in **Fig 3.2.**1, $\tan\theta$ equals the opposite over the adjacent, or the change in y over the change in x as we move from the beginning to the end of the hypotenuse.

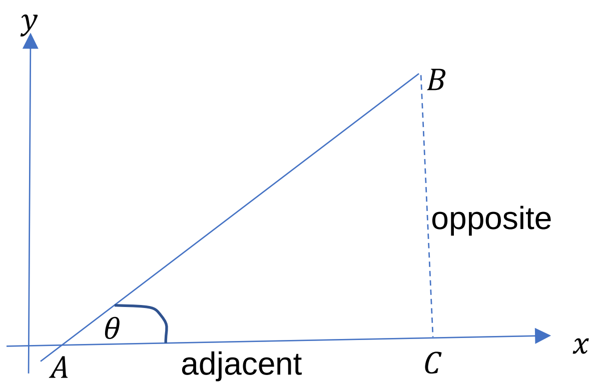


Fig 3.2.1: Angle of slope

$$\tan\theta = \frac{AB}{AC}$$

Therefore, the tangent of the angle of slope is equal to the gradient of the line.

$$\tan\theta = m$$

Example 3.2.1

Find the gradient of the line joining $(-3, -5)$ and $(4, -1)$ and the angle of slope of the line.

Solution 3.2.1

The points given are $(-3, -5)$ and $(4, -1)$

$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-5)}{4-(-3)}$$

$$= \frac{4}{7}$$

The angle of slope of the line, then

$$\tan\theta = m$$
$$\tan\theta = \frac{4}{7}$$
$$\tan\theta = 0.5714$$
$$\theta = \tan^{-1}(0.5714)$$
$$\theta = 29.74^0$$

3.3 Angles between two lines

In **Fig 3.3.1**, the line AB and CD make angles θ_1 and θ_2 respectively with x-axis. The acute angle between the lines is θ.

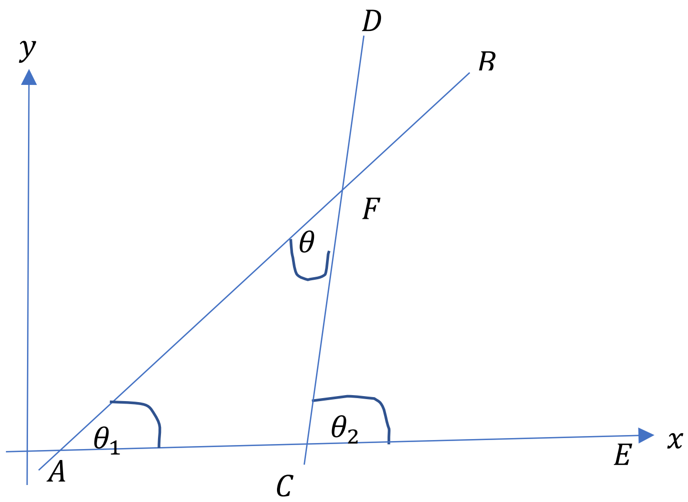


Fig 3.3.1: Angles between two lines

$$\tan\theta_1 = \frac{AB}{AE} = m_1$$

$$\tan\theta_2 = \frac{CD}{CE} = m_2$$

The exterior angle of $\Delta AFC\theta_2 = \theta + \theta_1$

$$\therefore \theta = \theta_2 - \theta_1$$

$\tan\theta = \tan(\theta_2 - \theta_1)$ (Take tangent of the both sides)

$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2 \tan\theta_{1}{}_2}$ (Trigonometric identity)

$$\therefore \tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

The acute angle θ between the two lines is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

3.3.1 Condition for parallelism

In **Fig 3.3.1**, l_1 and l_2 are two parallel lines that make the angles $\theta_1$and θ_2 with the x-axis.

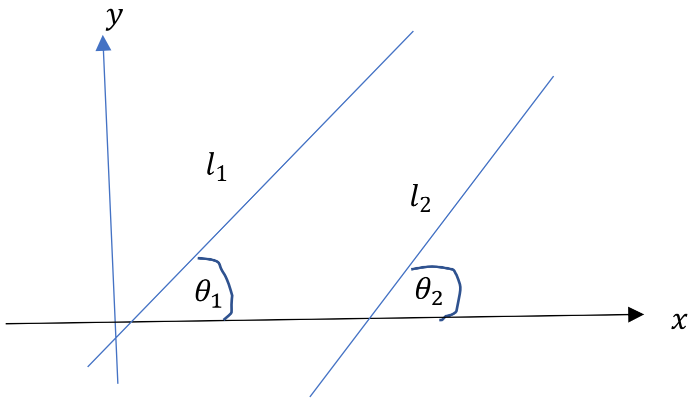


Fig 3.3.1: Parallel lines

From **Fig 3.3.1**,
$l_1 \,//\, l_2$ (The two parallel lines)
Then, $\theta_1 = \theta_2$.

$$\tan\theta_1 = \tan\theta_2$$
$$\therefore m_1 = m_2$$

Note: If the lines l_1 and l_2 are have gradients m_1 and m_2 are parallel,$m_1 = m_2$. This relationship between m_1 and $m_2$are used to decide whether or not two lines are parallel.

3.3.2 Condition for perpendicularity

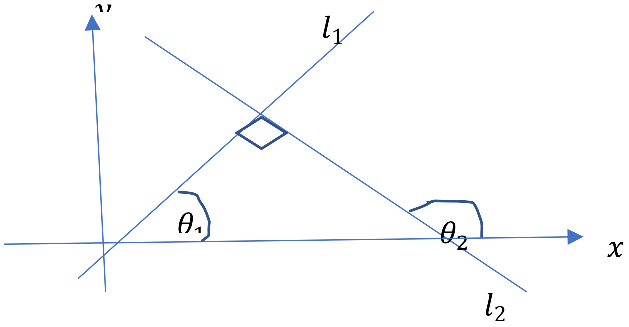


Fig 3.3.2: Perpendicular lines

From **Fig 3.3.2**,

$l_1 \perp l_2$ (The two perpendicular lines)

$\theta_2 = 90^0 + \theta_1$ (The exterior angle of Δ)

$$\tan\theta_2 = \tan(90^0 + \theta_1)$$

$\tan\theta_2 = -\cot\theta_1$ (Trigonometric identity)

$\tan\theta_2 = \frac{-1}{\tan\theta_1}$ (Trigonometric identity)

$$m_2 = \frac{-1}{m_1}$$

$$\therefore m_1 m_2 = -1$$

Note: If the lines l_1 and l_2 are have gradients m_1 and m_2 are perpendicular, $m_1 m_2 = -1$. This relationship between m_1 and $m_2$are used to decide whether or not two lines are perpendicular.

Example 3.3.1

Determine if AB is parallel or perpendicular to PQ in each of the following:

(a) $A(3,1),\ B(4,3),\ P(4,6), Q(5,8)$

(b) $A(5,-1),\ B(3,2),\ P(2\ ,4), Q(5,6)$

(c) $A(4,\ 7),\ B(6,8),\ P(3,\ 5), Q(5,6)$

(d) $A(-1,-2),\ B(2,-3),\ P(5,4), Q(6,7)$

Solution 3.3.1:

(a) $A(3,1),\ B(4,3),\ P(4,6), Q(5,8)$

Let $m_1$be the gradient of the line AB and let $m_2$be the gradient of the line PQ.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{3-1}{4-3} = \frac{2}{1}$$

$$m_1 = 2$$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_2 = \frac{8-6}{5-4} = \frac{2}{1}$$

$$m_2 = 2$$

Since,

$$m_1 = m_2, \quad AB//PQ$$

(b) $A(5,-1),\ B(3,2),\ P(2,4), Q(5,6)$

Let $m_1$be the gradient of the line AB and let $m_2$be the gradient of the line PQ.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{2-(-1)}{3-5} = -\frac{3}{2}$$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_2 = \frac{6-4}{5-2} = \frac{2}{3}$$

$$m_1 m_2 = -\frac{3}{2} \times \frac{2}{3} = -1$$

Since,

$$m_1 m_2 = -1, \quad AB \perp PQ$$

(c) $A(4,7),\ B(6,8),\ P(3,6), Q(5,6)$

Let $m_1$be the gradient of the line AB and let $m_2$be the gradient of the line PQ.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{8-7}{6-4} = \frac{1}{2}$$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_2 = \frac{6-5}{5-3} = \frac{1}{2}$$

Since,

$$m_1 = m_2, \quad AB//PQ$$

(d) $A(-1,-2),\ B(2,-3),\ P(5,4), Q(6,7)$

Let $m_1$be the gradient of the line AB and let $m_2$be the gradient of the line PQ.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{-3-(-2)}{2-(-1)} = \frac{-3+2}{3} = \frac{-1}{3}$$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m_2 = \frac{7-4}{6-5} = 3$$
$$m_1 m_2 = -\frac{1}{3} \times 3 = -1$$

Since,

$$m_1 m_2 == -1, \quad AB \perp PQ$$

4.0 Self-Assessment Exercise(s)

1. Find the gradient of the lines joining the following pairs of points:
(a) $(5,-4)$ and $(3,-2)$
(b) $(4,3)$ and $(5,-2)$
(c) $(13,-4)$ and $(11,8)$
(d) $(4,-7)$ and $(-3,\ 8)$
(e) $(-2,-3)$ and $(-6,-7)$
(f) $(0,\ p)$ and $(q,0)$

2. Determine if AB is parallel or perpendicular to CD in each of the following:
(a) $A(1,1),\ B(4,2),\ C(1,-2), D(-2,-3)$
(b) $A(1,\ 2),\ B(3,4),\ C(1,2), D(0,2)$

3. Show that the line segment joining the points $(1,4)$ and $(3,10)$ is parallel to the line segment joining the points $(-5,-10)$ and $(-2,-)$.

4. The line segment joining the point $(4,\ 1)$ and $(8,\ 17)$ is parallel to the line segment joining the points $(-2,\ 5)$ and $(0,\ a)$. What is the value of a?

5. The point A, B and Chave coordinates $(4,1), (6,-3)$and $(8,-2)$ respectively. Show that triangle ABC has a right angle at B.

6. The line segment joining the points $(1,5)$ and $(4,\ 20)$ is perpendicular to the line segment $(0,3)$ and $(k,0)$. What is the value of k?

5.0 Conclusion

This unit teaches you the gradient of a line as a measure of how steep the line is. It further defined the gradient as change in y over change in x, which is noted by m. It is also expounded that the tangent of angle of the slope is equal to the gradient of the line. This explained the relationship between the angle of slope and the gradient of the lines for the conditions for parallel and perpendicular lines.

6.0 Summary

In this unit, you have learnt the gradient of a line segment between two points. It is also taught that the angle of slope of a line is equal the gradient of the line. The angle between two lines which is the rudiment for the establishment of the condition for parallel and perpendicular lines was studied. You have also scrutinized the worked examples of gradients of lines, angles of slopes, parallel and perpendicular lines.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards, *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-gradstlnseg-2009-1.pd

Unit 3: Equations of Straight Lines

Unit Structure

1.0 Introduction
2.0 Intended Learning Outcomes (ILOs)
3.0 Main Content
  3.1 The equation of a line through the origin with a given gradient
  3.2 The y −intercept of a line
  3.3 The equation of straight line with a given gradient, passing through a given point
  3.4 The equation of straight line through two given points
  3.5 The general equation of a straight line
4.0 Self-Assessment Exercise(s)
5.0 Conclusion
6.0 Summary
7.0 References/Further Readings

1.0 Introduction

In this unit, you find the equation of a straight line, when you are given some information about the line. The information could be the value of its gradient, together with coordinate of a point on the line. Alternatively, the information might be the coordinates of two different points on the line. There are different ways of expressing the equation of straight line.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to:

(i) find the equation of a straight line, given its gradient and its intercept on the y-axis;
(ii) find the equation of a straight line, given its gradient and one point lying on it;
(iii) find the equation of a straight line given two points lying on it;
(iv) give the equation of a straight line either of the form $y = mx + c$ or $ax + by + c = 0$.

3.0 Main Content

3.1 The equation of a line through the origin with a given gradient

Let (x, y) are variable points on a line with equation $y = x$. That is, for every point on the line, the y coordinate is equal to the$x$ coordinate. The line contains the following list:

x	y
0	0
1	1
2	2
3	3
4	4

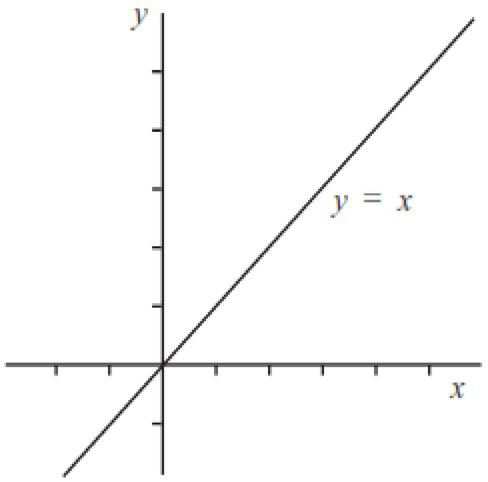


Fig 3.1.1: Graph of $y = x$

We can find the gradient of the line using the formula for gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and substitute in the first two sets of values from the tables. We obtain

$$m = \frac{1-0}{1-0} = 1$$

Therefore, the gradient of the line is 1.

What about the equation $y = 2x$? This also represents a straight line and for all the point on the line each y value is twice the corresponding x value. The line contains points in the following list.

x	y
0	0
1	2
2	4
3	6

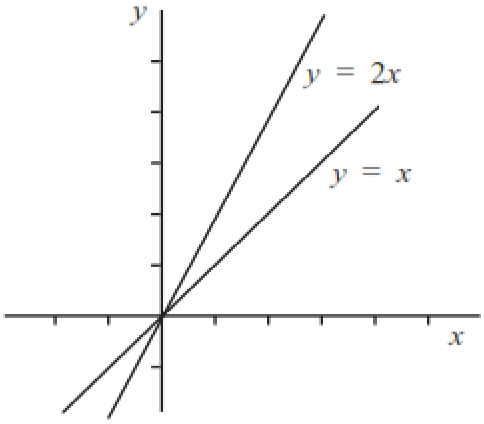


Fig 3.1.2: Graph of $y = 2x$

If we find the gradient of the line $y = 2x$, using the first two sets of values from the tables. We obtain

$$m = \frac{2-0}{1-0} = 2$$

Therefore, the gradient of the line is 2.

Now take the equation $y = 3x$. This also represents a straight line, and for all the points on the line each y value is three times the corresponding x value. The line contains points in the following list.

x	y
0	0
1	3
2	6
3	9

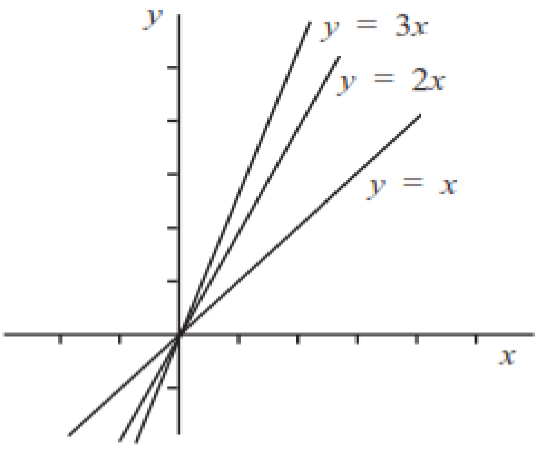


Fig 3.1.2: Graph of $y = 2x$

If we find the gradient of the line $y = 3x$, using the first two sets of values from the tables. We obtain

$$m = \frac{3-0}{1-0} = 3$$

Therefore, the gradient of the line is 3.

We can follow the pattern of the equations; we can see that y equals some number times x. In all the case, the line passes through the origin and the gradient of the line is given by the number that is multiplying x. For instance, if we have a line with equation $y = 13x$, then, the gradient of the line is 13. Similarly, if we have a line with equation $y = -2x$, then the gradient is -2.

In general, the equation $y = mx$ represents a straight line passing through the origin with gradient m.

3.2 The y-intercept of a line

Consider the straight line with equation $y = 2x + 1$. This equation is slightly different from the equations of straight lines we have seen before. The equation $y = 2x + 1$ can be sketched by calculating some values of x and y as follows:

x	y
0	1
1	3
2	5
3	7

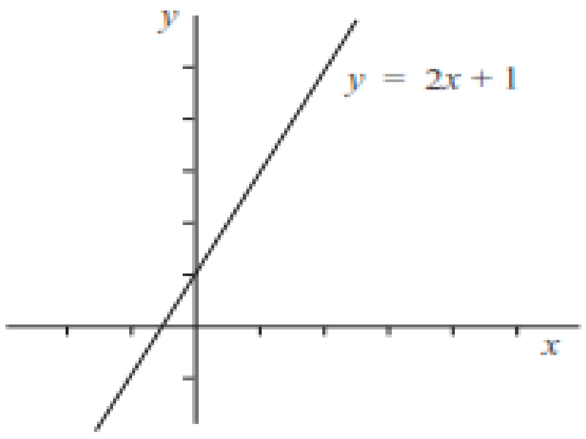


Fig 3.2.1: Graph of $y = 2x + 1$

Notice that when $x = 0$, the value of $y = 1$. The line cuts the y-axis at $y = 1$

What about the line $y = 2x + 4$? Again, we can calculate some values of x and y as follows:

x	y
-1	2
0	4
1	6
2	8

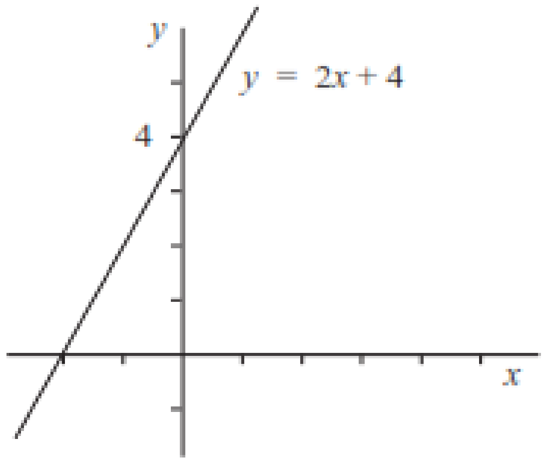


Fig 3.2.2: Graph of $y = 2x + 4$

The line cuts the y-axis at $y = 4$

What about the line $y = 2x - 1$, Again, we can calculate some values of x and y as follows:

x	y
-1	-3
0	-1
1	1
2	3

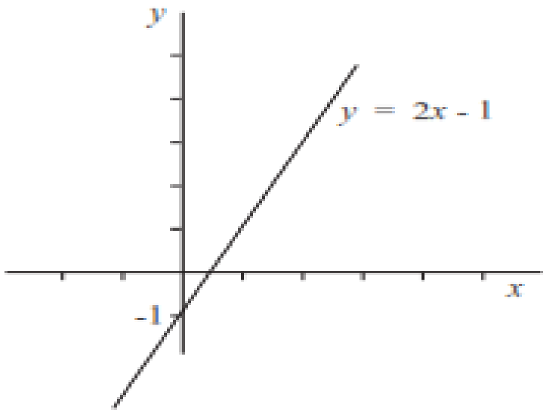


Fig 3.2.2: Graph of $y = 2x - 1$

The line cuts the y-axis at $y = -1$.

In general, the equation $y = mx + c$ represents a straight line with intercept on y-axis, where m is the gradient, $y = c$ is the value where the line cuts the y-axis. This number c is called the intercept on the y-axis.

Example 3.2.1

Determine the equations of the following lines in the gradient intercept form:

(a) gradient 3; intercept on y-axis: 2

(b) gradient -2; intercept on y-axis:3

(c) gradient $+4$; intercept on y-axis:-1

(d) gradient $-\frac{7}{2}$; intercept on y-axis:-2

Solution 3.2.1

(a) Let the equation of the line be $y = mx + c$, then

$$m = 3 \text{ and } c = 2$$

Therefore, the equation of the line is $y = 3x + 2$

(b) Let the equation of the line be $y = mx + c$, then

$$m = -2 \text{ and } c = 3$$

Therefore, the equation of the line is $y = -2x + 3$

(c) Let the equation of the line be $y = mx + c$, then

$$m = 4 \text{ and } c = -1$$

Therefore, the equation of the line is $y = 4x - 1$

(d) Let the equation of the line be $y = mx + c$, then

$$m = -\frac{7}{2} \text{ and } c = -2$$

Therefore, the equation of the line is $y = -\frac{7}{2}x - 2$

Example 3.2.2

Find the gradients and intercepts on the y-axis of the following lines:

(a) $y = 3x - 4$

(b) $y = -5x + 1$

(c) $y = -\frac{1}{2}x - 3$

(d) $y = \frac{4}{5}x - 3$

Solution 3.2.2

(a) Compare $y = 3x - 4$ with $y = mx + c$.
Then, $m = 3$ and $c = -4$

(b) Compare $y = -5x + 1$ with $y = mx + c$.
Then, $m = -5$ and $c = 1$

(c) Compare $y = -\frac{1}{2}x - 3$ with $y = mx + c$.
Then, $m = -\frac{1}{2}$ and $c = -3$

(d) Compare $y = \frac{4}{5}x + 7$ with $y = mx + c$.
Then, $m = \frac{4}{5}$ and $c = 7$

3.3 The equation of straight line with a given gradient, passing through a given point

In **Fig 3.3.1**, a line passing through $Q(x_1, y_1)$ with a gradient m, the equation of the line can be obtained by considering point $P(x, y)$ on the line.

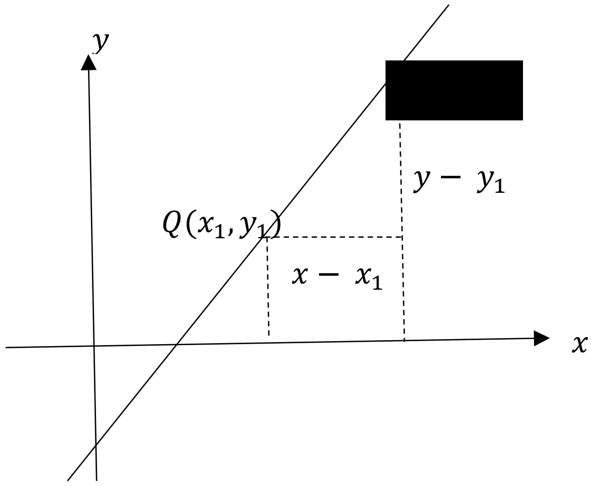


Fig 3.3.1: Gradient and one point

From Fig 3.3.1,
Gradient of $QP = \frac{y-y_1}{x-x_1}$

$$m = \frac{y - y_1}{x - x_1}$$
$$\therefore y - y_1 = m(x - x_1)$$

Note: It represents the equation of straight line with gradient m, passing through the point (x_1, y_1).

Example 3.3.1

Find the equation of a straight line of slope 2, if it passes through the point $(3, -2)$)

Solution 3.3.1

The equation of a straight line of gradient m, passing through the points (x_1, y_1) is

$y - y_1 = m(x - x_1)$.

$$m = 2, x_1 = 3, y_1 = -2$$

Then, $y - (-2) = 2(x - 3)$

$$y + 2 = 2x - 6$$
$$y = 2x - 6 - 2$$

Hence, $y = 2x - 8$

It is the equation of the straight line

3.4 The equation of a straight line through two given points

If a straight line passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$. The equation of the straight line can be obtained at point $P(x, y)$ on the line (see **Fig 3.3.1**).

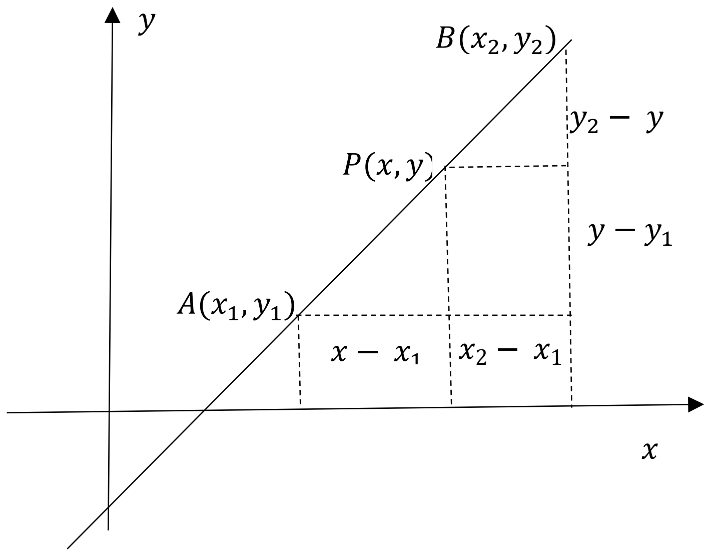


Fig 3.4.1: Two given points on a line

From **Fig 3.4.1,**

Gradient of $AP = \frac{y-y_1}{x-x_1}$

Gradient of $AB = \frac{y_2-y_1}{x_2-x_1}$

Since, the gradient of AP must be the same as the gradient of AB, as all three points on the same line.

Therefore, Gradient of AP =Gradient of AB

i.This implies $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$ or $y - y_1 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x - x_1)$

Note: It is the equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2).

Example 3.4.1

Find the equation of the straight line which passes through the point $P(2, -3)$ and $Q(-4,\ 2)$.

Solution 3.4.1

$$(x_1, y_1) = (2, -3)$$
$$(x$$

However, we observed that each equation is a linear or first-degree relationship between x and y. Therefore, an equation of the form

$$ax + by + c = 0$$

This is general form of the equation of a straight line.

The general form can be transformed into other forms. For example, there are special cases of the equation by setting either a or b equal to zero.

If $a = 0$ then we obtain lines with general equation $by + c = 0$, i.e. $y = -\frac{c}{b}$, these lines are horizontal, so that they are parallel to the x-axis.

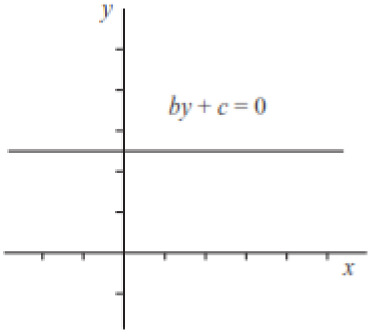


Fig 3.5.1: Horizontal line.

If $b = 0$ then we obtain lines with general equation $ax + c = 0$i.e.,$x = -\frac{c}{a}$. These lines are vertical, so that they are parallel to the y-axis.

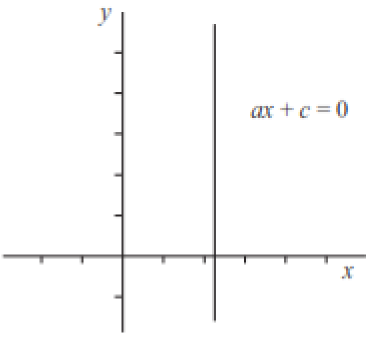


Fig 3.5.2: Vertical line

Note: The general equation of a striaght line may be rearranged as

$y = \frac{-ax}{b} - \frac{c}{b}$.

This is the gradient intercept form.

Example 3.5.1

Find the equation of the straight line which passes through the point $(-3, 5)$ and is perpendicular to the line $2x - 4y + 3 = 0$.

Solution 3.5.1

Let m_1 be gradient of the line

From the equation of the line $2x - 4y + 3 = 0$

$$-4y = -2x - 3$$
$$y = \frac{1}{2}x + \frac{3}{4}$$

Therefore, $m_1 = \frac{1}{2}$

Since, the line is perpendicular to the point $(-3, 5)$

$$m_1 m_2 = -1$$
$$m_2\left(\frac{1}{2}\right) = -1$$
$$m_2 = -2$$

The equation of the straight line is

$$y - y_1 = m(x - x_1)$$

$$(x_1, y_1) = (-3, 5)$$

Then, $y - 5 = -2(x + 3)$

$$y = -2x - 6 + 5$$

$$y = -2x - 1$$

$$\therefore y + 2x + 1 = 0$$

Example 3.5.2

Find the equation of the straight line which passes through $(2, 4)$ and parallel to $\therefore 3y - 5x + 2 = 0$

Solution 3.5.2

Let $m_1$be gradient of the line

From the equation of the line $3y - 5x + 2 = 0$

$$3y = 5x - 2$$
$$y = \frac{5}{3}x - \frac{2}{5}$$

Therefore, $m_1 = \frac{5}{3}$

Since, the line is parallel to the point $(2, 4)$

$$m_{1=}\, m_2 = \frac{5}{3}$$
$$m_2 = \frac{5}{3}$$

The equation of the straight line of given a point $(2, 4)$ is

$$y - y_1 = m(x - x_1)$$

$$(x_1, y_1) = (2, 4)$$

Then, $y - 4 = \frac{5}{3}(x - 2)$

$$y = \frac{5}{3}x - \frac{10}{3} + 4$$
$$y = \frac{5}{3}x + \frac{2}{3}$$
$$3y = 5x - 2$$
$$3y - 5x + 2 = 0$$

4.0 Self-Assessment Exercise(s)

1. Determine the gradient and y-intercept for each of the straight lines in table below.

Equation	Gradient	y-intercept
$y = 3x + 2$		
$y = 5x - 2$		
$y = -2x + 4$		
$y = 12x$		
$y = \frac{1}{2}x - \frac{2}{3}$		
$2y - 10x = 8$		
$x + y + 1 = 0$		

2. Find the equation of the lines described below.
(a) Gradient 5, y-intercept 3
(b) Gradient -2, y-intercept -1
(c) Gradient 3, passing through the origin
(d) Gradient $\frac{1}{3}$, passing through $(0, 1)$
(e) Gradient $-\frac{3}{4}$, y-intercept $\frac{1}{2}$

3. Find the equation of the following lines.
(a) Gradient 3, passing through $(1, 4)$
(b) Gradient -2 passing through $(2, 0)$
(c) Gradient $\frac{2}{5}$, passing through $(5,\ -1)$
(d) Gradient 0, passing through $(-1\ , 2)$
(e) Gradient -1, passing through $(1, -1)$

4. Determine the equation of the following lines:
(a) Passing through $(4,\ 6)$ and $(8,\ 26)$
(b) Passing through $(1,\ 1)$ and $(4, -8)$
(c) Passing through $(3,\ 4)$ and $(5,\ 4)$
(d) Passing through $(0,\ 2)$ and $(4,\ 0)$
(e) Passing through $(-2,\ 3)$ and $(2, -5)$

5. Find the equations of the following lines:
(a) Parallel to $6x + 5y = 1$ and passing through $(4, -2)$
(b) Perpendicular to $3x - 2y = 3$ and passing through $(3, -7)$
(c) Whose perpendicular distance is of length 3 units and 60^0 from the x-axis
(d) Which is the perpendicular bisector of the line joining $(4, 7)$ and $(5,\ 3)$.

6. Find the equation of the lines described below.
(a) The line through $(3, -2)$ and $(-3, 2)$;
(b) The vertical line passing through the point $\left(0, \frac{2}{3}\right)$.

7. Find the acute angles between the following pair of lines:
(a) $4y + 3x = 2$ and $x - 2y = 3$;
(b) $y = 5x - 2$ and $y - 3x + 1 = 0$;
(c) $3x - 2y + 5 = 0$ and $y - 4x + 3 = 0$
(d) $2x + 3y = 1$ and $4x - 7y = 5$.

8. Find the equation of the line which is parallel to the line $5x + 4y = 18$ and makes an intercept of 2 units on the x-axis

9. The $y = 2x - 1$ meets the y-axis at B and the line $y = 5$ at C:
(a) Calculate the coordinate of B and C;

(b) If A is the point $(-4, 1)$, show that AB is perpendicular to BC;
(c) Find the area of triangle ABC.

10. (a) A point P moves in the $x - y$ plane in such a way that its distance from the point $(1, -3)$ is equal to its distance from line $y - 4 = 0$. Find the equation of the locus of the point P.
(c) Find:
(i) the equation of the line through the points $A(2, -3)$ and $B(3, 2)$;
(ii) the equation of the line through the origin perpendicular to AB;
(iii) the coordinate of the point of intersection of lines in (i) and (ii).

5.0 Conclusion

In this unit, you have been introduced to different forms of equation of a straight line such as the equation of a line through the origin, the y-intercept, through a given point and through two given points. These forms are all linear relationship between x and y, which can be presented by general form of the equation $ax + by + c = 0$. There are unique characteristics of the equations of lines which are the gradient, interception on the y −axis and points which the line passing through. These characteristics determine the different ways of calculating the equations of straight lines.

6.0 Summary

In this unit, you have learnt the equation of a line through the origin with a given gradient. You are also studied the y-intercept of a line with the worked examples. The equations of lines with a given gradient passing through a point and two points are illustrated and solved. Also in this unit, you have learnt thatall the different forms of the equations of straight lines are represented by $ax + by + c = 0$ which is called general form of the equation of straight lines.

7.0 References/Further Readings

Blitzer, *Algebra and Trigonometry custom*. 6th Edition
K. A. Stroud. *Engineering Mathematics*. 8th Edition
Larson Edwards, *Calculus: An Applied Approach*. 7th Edition
https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-strtlines-2009-1.p

Module 3: The Geometry of a Circle

Module Introduction

In this module, you will learn the circle geometry and its properties that can be proved using the tradition Euclidean format. You will also learn how to work in coordinates and this requires you to know the standard equation of a circle, how to interpret the equation of the circle and how to find the equation of a tangent to a circle standard equation of a circle.

Unit 1 Circle centred at the origin.
Unit 2 General equation of a circle
Unit 3 The equation of a tangent to a circle at given point
Unit 4 The equation of a normal to a circle at given point

Unit 1: Circle Centered at the Origin

Unit Structure

1.0 Introduction
2.0 Intended Learning Outcomes (ILOs)
3.0 Main Content
 3.1 Circle
  3.1.1 Parts of a circle
  3.1.2 Properties of a circle
 3.2 Equation of a circle centred at the origin
4.0 Self-Assessment Exercise(s)
5.0 Conclusion
7.0 Summary
7.0 References/Further Readings

1.0 Introduction

In this unit, you will learn the definition of a circle, its specifications which are center and radius of the circle, parts and properties of the circle. Also, in this unit, you will study the circle centred at the origin and find the equation of the circle centered at the origin.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to:

- Define a circle
- Identify a circle centred at the origin
- Find the equation of a circle centred at the origin

3.0 Main Content

3.1 Circles

A circle is a two-dimensional figure formed by a set of points that are at a fixed distance from a fixed point on a plane. The fixed point is called the origin or centre of the circle and fixed distance of the points from the origin is called the radius.

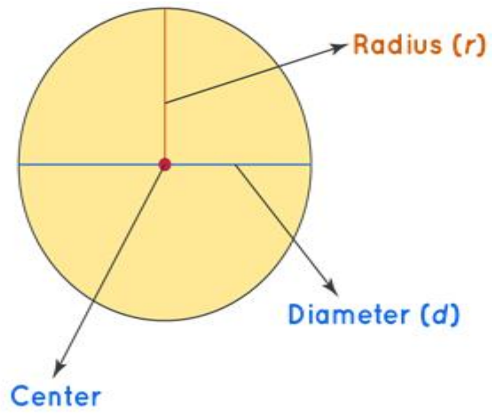


Fig 3.1: A Circle

3.1.1 Parts of a circle

There are many parts of a circle that we should know to understand its properties. A circle has the following parts:

(i) **Circumference:** It is also referred to as the perimeter of a circle and can be defined as the distance around the boundary of the circle.

(ii) **Radius of Circle:** Radius is the distance from the center of a circle to any point on its boundary. A circle has many radii as it is the distance from the center and touches the boundary of the circle at various points.

(iii) **Diameter:** A diameter is a straight line passing through the center that connects two points on the boundary of the circle. We should note that there can be multiple diameters in the circle, but they should:

a. pass through the center;
b. be straight lines.
c. touch the boundary of the circle at two distinct points which lie opposite to each other.

(iv) **Chord of a Circle:** A chord is any line segment touching the circle at two different points on its boundary. The longest chord in a circle is its diameter which passes through the center and divides it into two equal parts.

(v) **Tangent:** A tangent is a line that touches the circle at a unique point and lies outside the circle.

(vi) **Secant:** A line that intersects two points on an arc/circumference of a circle is called the secant.

(vii) **Arc of a circle:** An arc of a circle is referred to as a curve that is a part or portion of its circumference.

(viii) **Segment in a circle:** The area enclosed by the chord and the corresponding arc in a circle is called a segment. There are two types of segments - minor segment, and major segment.

(ix) **Sector of a circle:** The sector of a circle is defined as the area enclosed by two radii and the corresponding arc in a circle. There are two types of sectors - minor sector, and major sector.

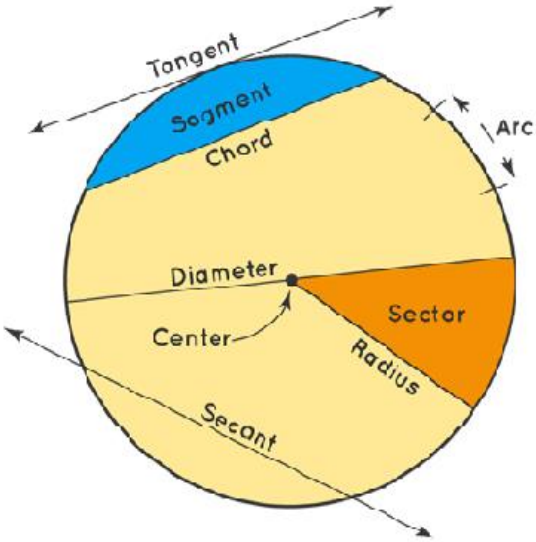


Fig 3.1.2: Part of a circle

3.1.2: Properties of a circle

Let us move ahead and learn about some interesting properties of circles that make them different from other geometric shapes. Here is a list of properties of a circle:

(i) A circle is a closed 2D shape that is not a polygon. It has one curved face.

(ii) Two circles can be called congruent if they have the same radius.

(iii) Equal chords are always equidistant from the center of the circle.
(iv) The perpendicular bisector of a chord passes through the center of the circle.
(v) When two circles intersect, the line connecting the intersecting points will be perpendicular to the line connecting their center points.
(vi) Tangents drawn at the endpoints of the diameter are parallel to each other.

3.2 Equation of a circle centred at the origin

The simplest form of equation of a circle is the equation of a circle whose centre is at the origin. In Fig **3.2.1**, $P(x, y)$ is a point on a circle centred at the origin$(0, 0)$ with radius r.

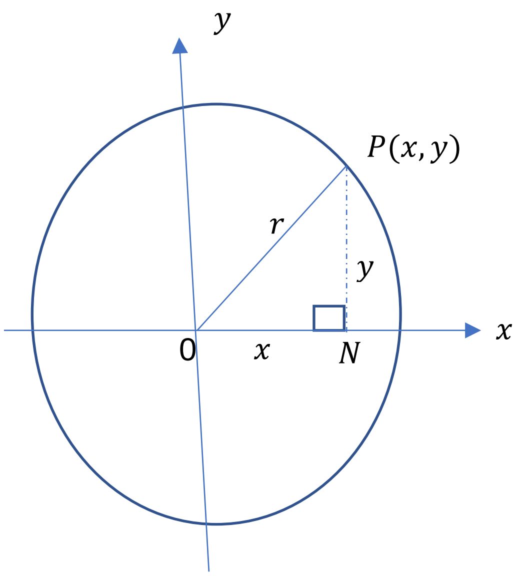


Fig 3.2.1: Circle centred at origin

From **Fig 3.2.1**, in right-angled triangle $0PN(\Delta 0PN)$,

$0N^2 + NP^2 = 0P^2$ (Pythagoras theorem)

$$\therefore \; x^2 + y^2 = r^2$$

Hence, this is the equation of a centre centred at the origin.

Example 3.2.1

What will be the equation of a circle centre on the origin with radius 5units?

Solution 3.2.1

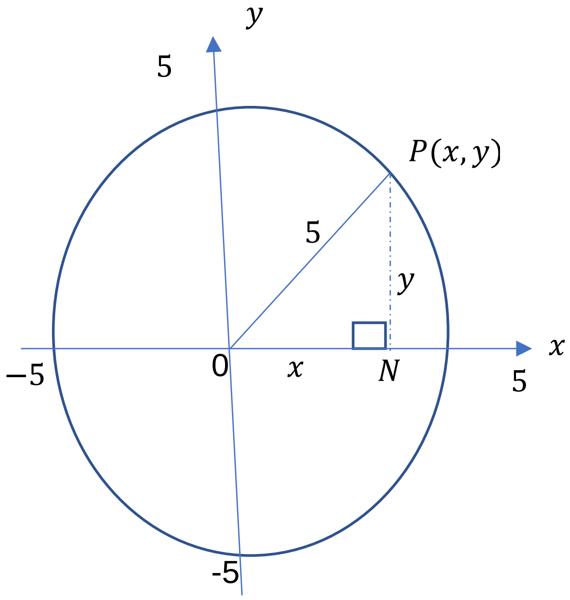


Fig 3.2.2: A circle centre on the origin with radius 5units

From **Fig 3.2.2**, in right-angled triangle $0PN(\Delta 0PN)$

$0N^2 + NP^2 = 0P^2$ (Pythagoras theorem)

$$x^2 + y^2 = 5^2$$

$$\therefore\ x^2 + y^2 = 25$$

4.0 Self-Assessment Exercise(s)

1. What is a circle
2. State 5 parts of a circle
3. Mention 5 properties of a circle
4. With aid of diagram, identify major components of a circle.
5. Find the equation of a circle centered at the origin with 3 units radius.
6. Find the equations of the following circles:
   (i) Centre $(0, 0)$, radius 7
   (ii) Centre $(0, 0)$, radius $\sqrt{5}$

5.0 Conclusion

This unit discuss a circle, parts and properties of the circle. The illustration of different components of circle depicted with aid of diagram. The circle centred at the origin with given a radius also discussed. The equation of a circle centred at the origin are derived with worked examples

6.0 Summary

In this unit, you have learnt the definition of a circle, its components and characteristics of the circle. Also, this unit provides you with derivation of the equation of a circle centred at origin with aids of diagrams. Then, you are able to solve to the equations of the circle at the origin with given radius.

7.0 References/Further Readings

Blitzer, *Algebra and Trigonometry custom*. 6th Edition

K. A. Stroud. *Engineering Mathematics*. 8th Edition

Larson Edwards, *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-circles-2009-1.p

Unit 2: General Equation of a Circle

Unit Structure

1.0 Introduction

In the last unit, you have learnt the equation of a circle centred at $(0, 0)$ with given radius r. Now in this unit, you will be discussing the equation of a circle (a, b) with given radius r. All the different forms of the equations of the circle are grouped together to establish the general equation of the circle.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to:

- find the equation of a circle centre (a, b) with given radius
- find the general equation of a circle.

3.0 Main Content

3.1 Equation of a circle centre (a, b) with given radius

Fig 3.1.1 shows a circle centre (a, b) with given radius r.

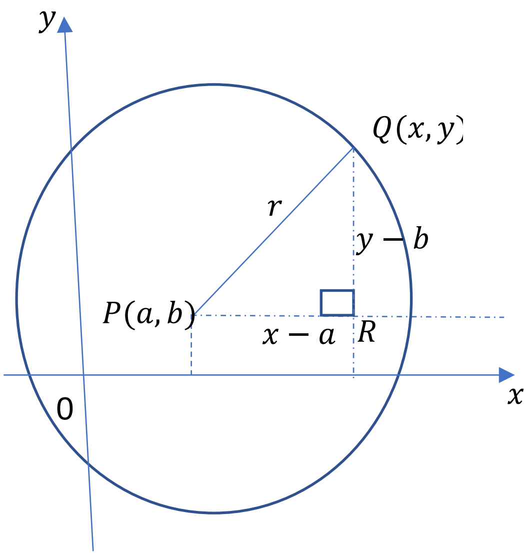


Fig 3.1.1: A circle centre (a, b)

Consider ΔPQR in **Fig 3.1.1**.

$$PR = x - a$$

$$RQ = y - b$$

Since, ΔPQR is a right-angled triangle, we have:

$PR^2 + RQ^2 = PQ^2$ (Pythagoras theorem)

$$\therefore (x - a)^2 + (y - b)^2 = r^2$$

This is the equation of a circle centre (a, b) with radius r.

Example 3.1.1

Find the equation of a circle centre $(3, -2)$ with radius 2 units.

Solution 3.1.1

The equation of a circle centre (a, b) with radius r is given as:

$$\therefore (x - a)^2 + (y - b)^2 = r^2$$

centre $(3, -2), r = 2$

Then, $(x-3)^2 + (y-(-2))^2 = 2^2$

$$(x-3)^2 + (y+2)^2 = 4$$
$$x^2 - 6x + 9 + y^2 + 4y + 4 = 4$$
$$x^2 + y^2 - 6x + 4y + 9 = 0$$

Example 3.1.2

Find the equation of a circle whose centre is $(5,-4)$ passing through point $(-3,\ 2)$.

Solution 3.1. 2

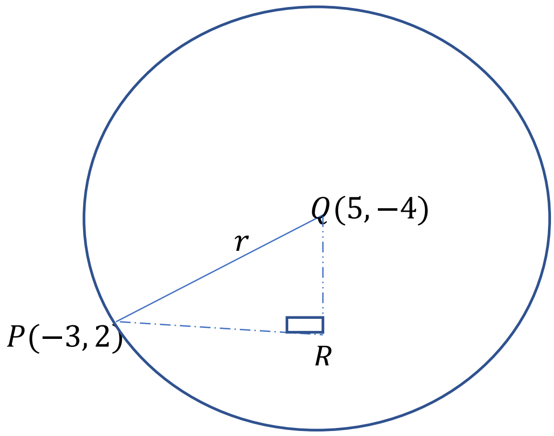


Fig 3.1.2: A circle centre $(5,-4)$ passing through point $(-3,\ 2)$.

Consider ΔPQR in Fig 3.1.2 and
let $PQ = r$ be the radius of the circle.

$PQ^2 = PR^2 + RQ^2$ (Pythagoras theorem)

$$r^2 = (-3-5)^2 + (-4-2)^2$$
$$r^2 = (8)^2 + (-6)^2$$
$$r^2 = 64 + 36$$
$$r^2 = 100$$
$$r = 10$$

Therefore, the equation of the circle centre $(5,-4)$ with radius 10 units is given as

$$\therefore (x-a)^2 + (y-b)^2 = r^2$$

The centre = $(5,-4), r = 10.$

Then, $(x-5)^2 + (y-(-4))^2 = 10^2$

$$(x-5)^2 + (y+4)^2 = 100$$
$$x^2 - 10x + 25 + y^2 + 8y + 16 = 100$$
$$x^2 + y^2 - 10x + 8y - 59 = 0$$

3.2 The general equation of a circle

Fig 3.2.1 depicts a circle of radius r, centred at the point $P(a, b)$ with given radius r.

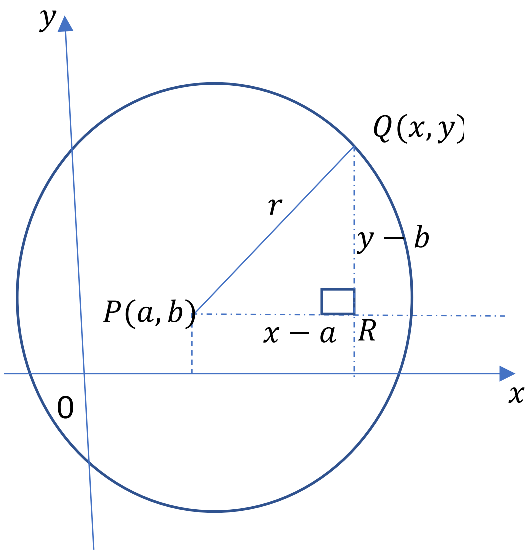


Fig 3.1.1: A circle centre (a, b)

Consider ΔPQR in **Fig 3.2.1,**.

$$PR = x - a$$

$$RQ = y - b$$

Since, ΔPQR is a right-angled triangle, we have:

$PR^2 + RQ^2 = PQ^2$ (Pythagoras theorem)

$$(x - a)^2 + (y - b)^2 = r^2$$

Then, expanding the brackets gives

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 - r^2 = 0$$

It is a convention, at this point, to replace $-a$ by g and $-b$ by f. This gives

$$x^2 + 2gx + y^2 + 2fy + g^2 + f^2 - r^2 = 0$$

Now look at the last three terms on the left-hand side, $g^2 + f^2 - r^2$. These do not involve x or y at all, so together they just represent a single number that we can call c. Substituting c into the equation, it finally gives us

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

This is the general equation of a circle, where the centre is given by $(-g, -f)$ and the radius is given by$r = \sqrt{g^2 + f^2 - c}$

Note: The following from the general equation of a circle.
(i) It is a quadratic expression in both x and y.
(ii) The coefficient of x^2 and y^2 are equal
(iii) It has no xy term.

Example 3.2.1

Find the centre and radius of the circle

$$x^2 + y^2 - 6x + 4y - 3 = 0$$

Solution 3.2.1

If we compare this $x^2 + y^2 - 6x + 4y - 3 = 0$ with the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2gx = -6x$,$2fy = 4y$ and $c = -3$

$$\therefore g = -3 \text{ and } f = 2$$

The centre is given by $(-g, -f) = (3, -2)$.

The radius is given by $r = \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + (-2)^2 - (-3)}$

$$= \sqrt{9 + 4 + 3}$$
$$= \sqrt{16}$$
$$= 4$$

Hence, the centre of circle is $(3, -2)$, radius is 4

Example 3.2.2

Find the centre and radius of a circle

$$2x^2 + 2y^2 - 8x + 12y - 24 = 0$$

Solution 3.2.1

Note that coefficient of x^2 and y^2 of this equation $2x^2 + 2y^2 - 8x + 12y - 24 = 0$ are equal but not 1. However, we can divide throughout by 2, we get

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

If we compare the equation with the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2gx = -4x$, $2fy = 6$ and $c = -12$

$$\therefore g = -2 \text{ and } f = 3$$

The centre is given by $(-g, -f) = (2, -3)$.

The radius is given by $r = \sqrt{g^2 + f^2 - c} = \sqrt{2^2 + (-3)^2 - (-12)}$

$$= \sqrt{4 + 9 + 12}$$
$$= \sqrt{25}$$
$$= 5$$

Hence, the centre of circle is $(2, -3)$, radius is 5.

4.0 Self-Assessment Exercise(s)

1. Find the equation of the following circles
   (a) centre $(2,-3)$,radius 4
   (b) centre $(-1,-4)$, radius 5
   (c) centre $(3,5)$, radius 3
   (d) centre $(-2,\ 3)$, radius 1
   (e) centre $(-1,-3)$, radius 2
   (f) centre $(2,-2)$, radius 5
   (g) centre $(0,\ 5)$, radius 4

2. Find the equation of the circle whose diameter has the end points $A(5,4)$ and $(7,\ 4)$.
3. The end points of a diameter of a circle have coordinates $(2,\ 3)$ and $(5,-6)$, find the equation of the corresponding circle.
4. Find the equation of the circle passing through $(3,-2)$, $(4,\ 5)$ and $(-6,-3)$.
5. Find the centre and radius of each of the following circles:
   (a) $(x-5)^2+(y-8)^2=36$
   (b) $x^2+(y-1)^2=16$
   (c) $(x-1)^2+(y-4)^2=9$
6. Determine which the following curves are circles
   (a) $x^2+y^2+2x+3y+1=0$
   (b) $2x^2+3y^2-x+2y+1=0$
   (c) $3x^2-3y^2+2xy+3y+3=0$
   (d) $x^2+y^2-4x+8y+5=0$
   (e) $2x^2+2y^2-3x+5y-6=0$
7. Find the centres and radii of the following circles
   (a) $x^2+y^2-6x+8y+5=0$
   (b) $5x^2+5y^2-3x+7y-1=0$
   (c) $3x^2-3y^2+4x-5y+2=0$
   (d) $2x^2+2y^2-x+y-1=0$
8. Find the equation of a circle of radius and concentric with $(x+1)^2+(y-2)^2=4$
9. Find the coordinates of the centre of the circle $2x^2+2y^2-14x+12y-7=0$. Find the length of its diameter
   10. Find the equation of the circle that passes through the point $(0,0)$, $(2,0)$ and $(3,-3)$.

5.0 Conclusion

This unit teaches you the equation of circle centre (a, b) with a given radius r. The equation was derived using Pythagoras theorem for right-angled triangle. The general equation of a circle is formed by expansion the equation of circle centre (a, b) and replaced the variables a and b by g and f respectively.

6.0 Summary

In this unit, you have learnt and able to solve problems on the equation of circle centre (a, b) with a given radius r. You are equally able to work out problems on the general equation of a circle which was derived from the equation of circle centre (a, b) with a given radius r.

7.0 References/Further Readings

Blitzer, *Algebra and Trigonometry custom*. 6th Edition

K. A. Stroud. *Engineering Mathematics*. 8th Edition

Larson Edwards, *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-circles-2009-1.p
Equation of a Tangent to the Circle: Definition, Properties - Embibe

Unit 3: The Equation of Tangent to a Circle

Unit Structure

1.0 Introduction

This unit introduces you to a tangent to a circle which is defined as a straight line that touches the circle at a unique point and lies outside the circle. To find the equation of a tangent, you need to know either two points or one point on the circle with its gradient. You will also need to know that the tangent to the circle is perpendicular to the radius at point of contact called the point of tangency.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to:

- define a tangle to a circle and a point of tangency
- state the properties of the tangent tangle to a circle
- write equation of a tangent to circle
- solve problems on the equation of tangent to circle

3.0 Main Content

3.1 Definition of tangent of a circle

3.1.1 A tangent is a line that passes through a point on a curve. **Fig 3.1.1** depicts a tangent P drawn to S on the curve.

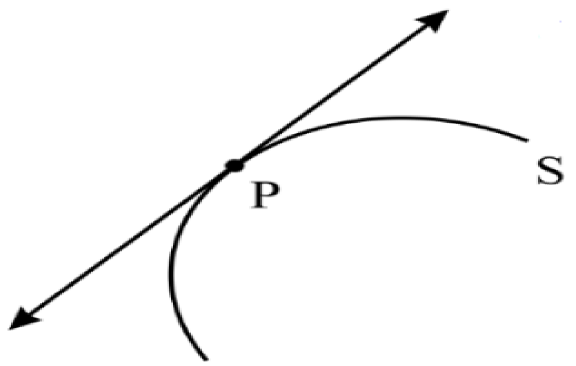


Fig 3.1.1: A tangent on a curve.

3.1.2 The point of tangency is where the line touches the curve at one point. In **Fig** above, P point represents the point of tangency.

3.1.3 A tangent to a circle is the line that touches the circle at one point only. A circle can have only one tangent at a point on the circle. The point where the tangent comes in contact with the circle is called the point of tangency.

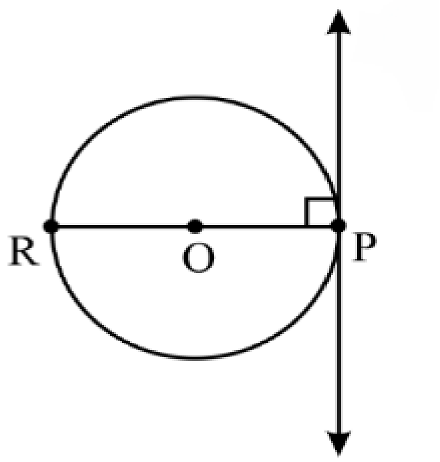
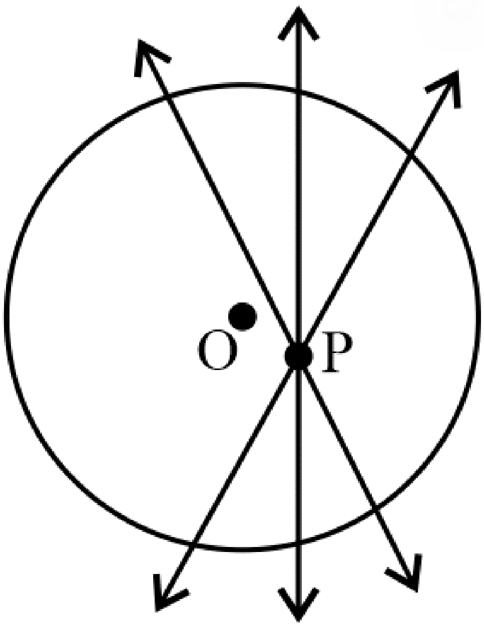


Fig 3.1.2: A circle with a point P.

Based on where the point of tangency lies with respect to the circle, we can define the conditions for tangent as:

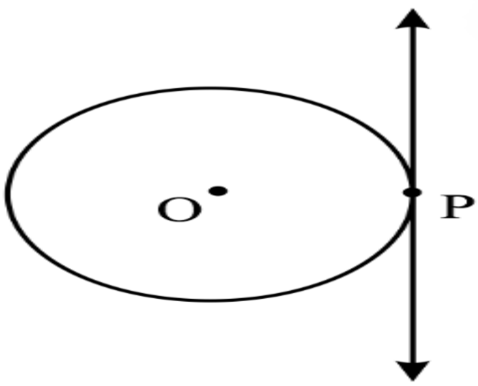
- When the point lies inside the circle
- When the point lies on the circle
- When the point lies outside the circle

3.1.4 When the point lies inside the circle



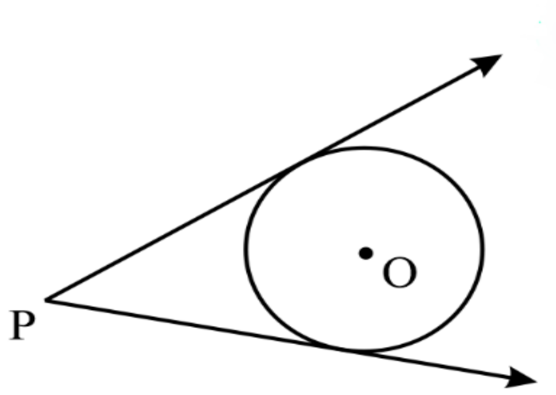
Observe that PP lies inside the circle as shown in the figure. All the lines through point *P* intersect the circle at two distinct points. So, no tangent can be drawn to a circle that passes through a point lying inside the circle.

3.1.5 When the Point Lies on the Circle



Only one tangent can be drawn to the circle at PP that lies on the circle. There is only one point of tangency.

3.1.5 When the Point Lies Outside the Circle

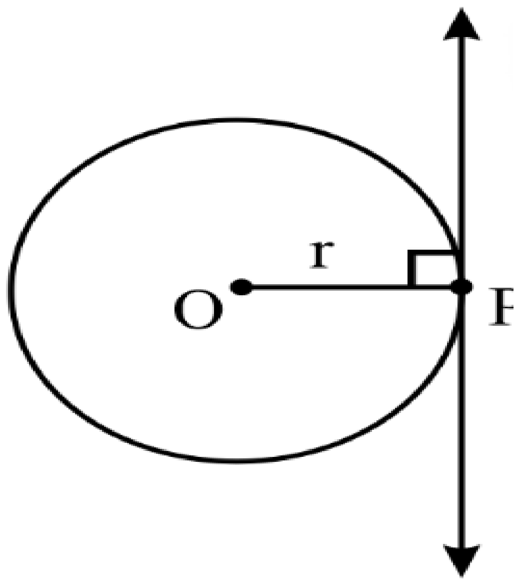


When *P* lies outside the circle, observe that two tangents can be drawn to the given circle. Hence, there are two points of tangency. Also, the length of tangents from the points of tangency to *P* are always equal.

3.1.6 Properties of Tangents

A tangent has the following properties:

- Tangent always touches the circle at one point only.
- Tangent is perpendicular to the radius (r) of the circle at the point of tangency.



- Tangent never intersects the circle at more than one point.
- The length of tangents drawn to a circle from an external point are equal

3.2 The equation of tangent to a circle

The equation of a tangent to a circle at the point of tangency $T\ (x_1, y_1)$ as shown in **Fig 3.21**.

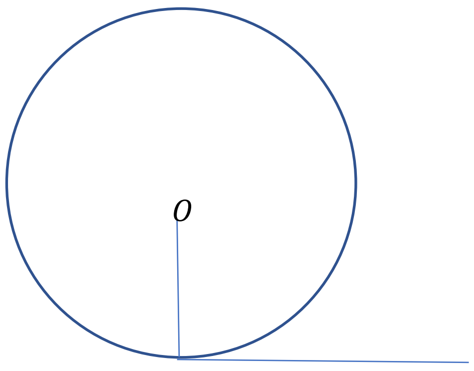


Fig 3.2.1 Point of tangency T

To derive equation of tangent to a circle, we have to find

(i) Point of tangency $P\ (x_1, y_1)$
(ii) Slope of tangent

Note: Slope of a circle is actually the slope of tangent at point $P\ (x_1, y_1)$

Recall the equation of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiate (2) with respect to x, we have

$$2x + 2y\frac{dy}{dx} + 2gx + 2f\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + g + \frac{dy}{dx} = 0$$

$$(y + f)\frac{dy}{dx} + (x + g) = 0$$

$$\frac{dy}{dx} = \frac{-(x + g)}{y + f}$$

Then, the slope of the circle is $\frac{dy}{dx} = \frac{-(x+g)}{y+f}$

Therefore, the slope of tangent at point $P\ (x_1, y_1)$ is

$$m = \left.\frac{dy}{dx}\right|_{x_1,y_1} = \frac{-(x_1 + g)}{y_1 + f}$$

Using point slope formula at point $P\ (x_1, y_1)$

$$y - y_1 = m(x - x_1)$$

Then, $y - y_1 = \frac{-(x_1+g)}{y_1+f}(x - x_1)$

$$(y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$$
$$yy_1 + xx_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

Add $gx_1 + fy_1 + c$ in both sides, we have

$$yy_1 + xx_1 + gx + fy + gx_1 + fy_1 + c$$
$$= x_1^2 + y_1^2 + gx_1 + fy_1 + gx_1 + fy_1 + c$$

$$yy_1 + xx_1 + g(x + x_1) + f(y+y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Therefore, $yy_1 + xx_1 + g(x + x_1) + f(y+y_1) + c = 0$

Is the equation of tangent to a circle.

Example 3.2.1

Show that the point $(2, 3)$ lies on the circle $x^2 + y^2 - 3x + 4y - 19 = 0$

Hence or otherwise, determine the equation of the tangent to the circle at the $(2, 3)$.

Solution

Substituting the coordinates $(2, 3)$into the equation

$x^2 + y^2 - 3x + 4y - 19 = 0$ (1)

$$L.H.S = 2^2 + 3^2 - 3(2) + 4(3) - 19$$
$$= 4 + 9 - 6 + 12 - 19$$

$$= 25 - 25$$
$$= 0$$
$$= R.H.S$$

Hence the point $(2, 3)$ lies on the circle Eq. (1)

Equation of the tangent at (x_1, y_1) is

$$yy_1 + xx_1 + g(x + x_1) + f(y + y_1) + c = 0 \qquad (2)$$

From, Eq. (1) $g = \frac{-3}{2}, f = 2, c = -19$ (3)

Substitute Eq (3) with point of tangency $(2, 3)$ in (2), we have the equation of the tangent as

$$3y + 2x - \frac{3}{2}(x + 2) + 2(y + 3) - 19 = 0 \qquad (4)$$

By simplification, Eq. (4) becomes

$$x + 10y - 32 = 0$$

3.3 Length of a tangent to a circle from an external point

Suppose we wish to find the length of the tangent to a circle (a, b) from external point (x_1, y_1) as shown in **Fig 3.3.1**.

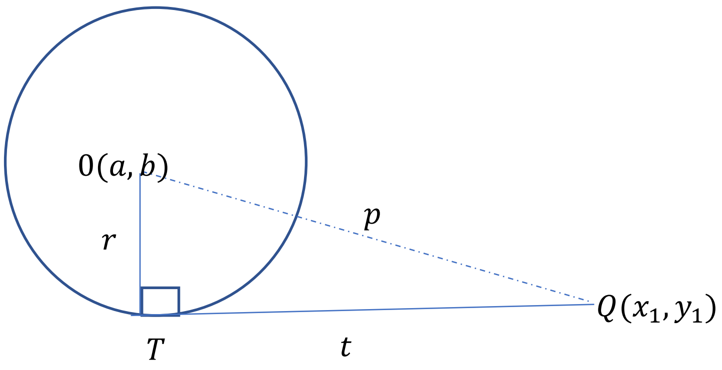


Fig 3.3.1 Length of a tangent from $Q(x_1, y_1)$

Consider Fig 3.3.1,

since $\Delta 0TQ$ is right angled triangle

$$OQ^2 = OT^2 + TQ^2$$

$$p^2 = r^2 + t^2$$

$$t^2 = p^2 - r^2$$

Also,

$$p^2 = (a - x_1)^2 + (b - y_1)^2$$

Example 3.3.1

Find the length of the tangent to the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ from the point $(8, 10)$.

Solution 3.3.1

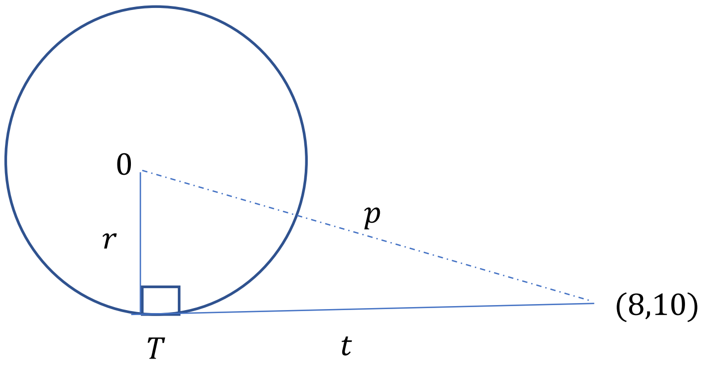


Fig 3.3.2: Length of a tangent from (8,10)

Given the equation of the circle
$x^2 + y^2 - 2x - 4y - 4 = 0$ (1)
From Eq. (1), we have

$$2gx = -2x, 2fy = -4y, \qquad c = -4$$
$$g = -1, \quad f = -2$$
$$\text{centre} = (-g, -f) = (1\ ,2)$$

$$\text{radius (r)} = \sqrt{g^2 + f^2 - c} = \sqrt{(-1)^2 + (2)^2 - (-4)}$$
$$= \sqrt{1 + 4 + 4}$$
$$= \sqrt{9}$$
$$r = 3$$
$$p^2 = (a - x_1)^2 + (b - y_1)^2$$

$$p^2 = (1-8)^2 + (2-10)^2$$
$$p^2 = 49 + 64$$

$$p^2 = 113$$

Then, $t^2 = p^2 - r^2$

$$t^2 = 113 - 9$$
$$t^2 = 104$$
$$t = \sqrt{104}$$

3.4 Equation of normal to a circle

A normal to a circle is a perpendicular line to the tangent at the point of contact of the tangent with the circle as shown in **Fig 3.4.1**.

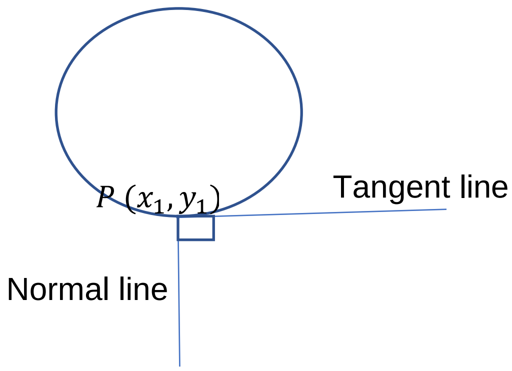


Fig 3.4.1: Normal to a circle

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

From Eq. (1), we obtained the gradient (m) of tangent at point $P\ (x_1, y_1)$ to the circle as

$$m = \frac{dy}{dx} = \frac{-(x_1+g)}{y_1+f} \qquad (2)$$

Since, the normal is the perpendicular line to the tangent at point$P\ (x_1, y_1)$, then, the condition of gradients (m_1) of the normal and gradient (m) of the tangent is given as

$$mm_1 = -1$$

$\therefore\ m_1 = \frac{-1}{m}$ (3)

Substitute Eq. (2) into Eq. (3), we have

$$m_1 = \frac{y_1 + f}{x_1 + g}$$

Using point slope formula at point $P\ (x_1, y_1)$

$$y - y_1 = m_1(x - x_1)$$

Then, $y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1)$

$$(x_1 + g)(y - y_1) = (y_1 + f)(x - x_1)$$

$$x_1y - x_1y_1 + gy - gy_1 = xy_1 - x_1y_1 + fx - fx_1$$

$$x_1y + gy - gy_1 = xy_1 + fx - fx_1$$

$$y(x_1 + g) - gy_1 = x(y_1 + f) - fx_1$$

$$y(x_1 + g) - x(y_1 + f) + fx_1 - gy_1 = 0$$

Hence, it is the equation of the normal to the circle.

Example 3.41:

Find the equation of the normal to the circle $x^2 + y^2 - 8x - 10y - 128 = 0$ at point $(-8, 10)$

Solution 3.4.1

$x^2 + y^2 - 8x - 10y - 128 = 0$ at point $(-8, 10)$

To the centre of the circle,

$$2gx = -8x, 2fy = -10$$

$$g = -4, f = -5$$

Then, centre= $(-g, -f) = (4, 5)$

Therefore, equation of the normal to the circle given as

$y(x_1 + g) - x(y_1 + f) + fx_1 - gy_1 = 0$ at point $(-8, 10)$, we have

$$y(-8-4) - x(10-5) + (-5)(-8) - (-4)(10) = 0$$
$$-12y - 5x + 40 + 40 = 0$$

$$-12y - 5x + 80 = 0$$

It is the equation of the normal to the circle

4.0 Self-Assessment Exercise(s)

1. (a) Define point of tangency
   (b) What is a tangent to a circle?

2. (a) Based on where the point of tangency lies with respect to the circle, state the three conditions for tangent
(c) Enumerate the four properties of a tangent to a circle

3. Find the equation of the tangent to the following circles at the given points on the circles:
   (a) $x^2 + y^2 + 2x - 3y + 13 = 0$ at $(1, 2)$
   (b) $2x^2 + 2y^2 - 3x + 4y - 32 = 0$ at $(2, 3)$
   (c) $x^2 + y^2 - 2x - y - 15 = 0$ at $(-1, -3)$
   (d) $3x^2 + 3y^2 - 8x - 6y - 61 = 0$ at $(4, 5)$

4. Find the length of the tangent of the following circles from the given points:
   (a) $x^2 + y^2 + 5x + 4y - 20 = 0$ at $(2, 3)$
   (b) $x^2 + y^2 - 4x + 2y + 10 = 0$ at $(-4, 1)$
   (c) $2x^2 + 2y^2 + x - 2y - 17 = 0$ at $(5, -3)$
   (d) $3x^2 + 3y^2 + 5x + 6y - 30 = 0$ at $(2, 2)$

5. $P(-3, 1)$ and $Q(9, -6)$ are the ends of diameter of a circle, find:
(a) the coordinates of the centre of the circle
(b) the equation of the tangent to the circle at P

6. Find the equation of the normal to the following circles from the given points:
(a) $x^2 + y^2 + 3x + 5y - 18 = 0$ at $(1, 2)$
(b) $x^2 + y^2 - 4x + 2y - 4 = 0$ at $(2, -4)$
(c) $4x^2 + 4y^2 - 5x - 6y - 19 = 0$ at $(3, 1)$
(d) $2x^2 + 2y^2 - x - 3y - 41 = 0$ at $(2, -5)$

7. Find the equation of the normal to the circle $x^2 + y^2 = 15$ at point $(-3, -4)$

8. Find the point of intersection of the line $y = 2x + 1$ and the circle $x^2 + y^2 - 2y + \ 4 = 0$. Show that line $y = 2x + 1$ is a diameter of the circle. Find the equation of the tangent to circle at one of the points of intersection.

5.0 Conclusion

In this unit, you have been introduced to tangent to a curve. The definitions of point of tangency, tangent to a circle, conditions and properties of the tangent were also stated. The derivation of the equation of tangent is clearly written and discussed with its application. The length of tangent to the circle with worked examples was discussed. Lastly, the normal to the circle was derived from the gradient of tangent at a point which is perpendicular to the point of tangency.

6.0 Summary

In this unit, you have leant the definitions of tangent, point of tangency and properties of tangent. You are able to find the equation and the length of tangent to the circle at a point. Also, you derived the equation of normal to the circle and solve the problem of the equation of normal to the circle.

7.0 References/Further Readings

Blitzer, *Algebra and Trigonometry custom*. 6th Edition

K. A. Stroud. *Engineering Mathematics*. 8th Edition

Larson Edwards, *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-circles-2009-1.p

Unit 4: Parametric Equation of a circle

Unit Structure

1.0 Introduction

In the unit, you will learn that a circle can be also defined as the locus of points that satisfy the equations $x = r\cos\theta$ and $y = r\sin\theta$, where $x,$ and y are the coordinates, r is the radius of the circle and θ is the parameter -the angle subtended by the point at the circle's centre. The equations are called parametric equations of a circle. You will also learn the parametric equations of tangent and normal to the to the circle at $(r\cos\theta,\ r\sin\theta)$.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to find:

- parametric equations of a circle
- the equations of tangent to the to the circle at $(r\cos\theta,\ r\sin\theta)$
- the equations of normal to the to the circle at $(r\cos\theta,\ r\sin\theta)$

3.0 Main Content

3.1 Parametric equations of a circle

Consider a circle, whose centre is at $(0, 0)$ and radius r (see **Fig 3.1**).

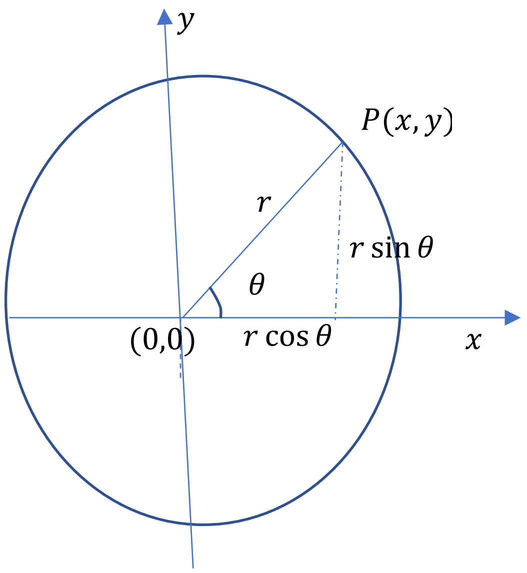


Fig 3.1: Circle centred at (0,0)

From **Fig 3.1,**

$P(x, y)$ be point on the circle such that OP makes an angle θ with the $x-$ axis.
Using trigonometry, we will get

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The equations are called parametric equations of a circle, where x, and y are the coordinates, r is the radius of the circle and θ is the parameter - the angle subtended by the point at the circle's centre.

In other words, for all values of θ, the point $(r\cos\theta,\ r\sin\theta)$ lies on the circle $x^2 + y^2 = r^2$.

Example 3.1.1

Find the parametric equation of the circle $x^2 + y^2 = 16$

Solution 3.1.1

Compare $x^2 + y^2 = 16$ with $x^2 + y^2 = r^2$

Then, $r^2 = 16, \Rightarrow r = 4,$

$x^2 + y^2 = r^2$ in parameter θ are $x = r\cos\theta\,,\ y = r\sin\theta.$

Therefore, the parametric equations of the given circle $x^2 + y^2 = 16$ are

$x = 4\cos\theta\,,\ y = 4\sin\theta$ and $0 \le \theta \le 2\pi.$

Example 3.1.2

Find the cartesian equation of the circle whose parametric equation of the whose the parametric equation are $x = 2\cos\theta\,,\ y = 2\sin\theta$ and $0 \le \theta \le 2\pi.$

Solution 3.1.2

To find the cartesian equation of the circle, eliminate parameter θ from the given equations $x = 2\cos\theta\,,\ y = 2\sin\theta$ and $0 \le \theta \le 2\pi.$

Then, $\cos\theta = \frac{x}{2};\ \sin\theta = \frac{y}{2}$

$\cos^2\theta + \sin^2\theta = 1$ (Trigonometric identity)

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$\therefore x^2 + y^2 = 4$$

3.2 Parametric equation for the circle centred at point $(-g, -f)$

Consider the general equation of the circle:

$x^2 + y^2 + 2gx + 2fy + c = r^2.$

This can be written as:

$$(x+g)^2 + (y+f)^2 = r^2$$

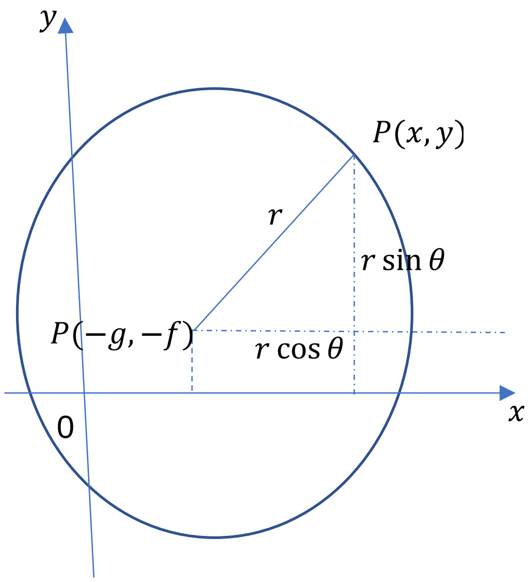


Fig 3.2.1: Circle with centre $(-g, -f)$

From **Fig 3.2.1**

$P(x, y)$ be point on the circle such that OP makes an angle θ with the $x-$ axis.
Using trigonometry, we will get

$$x + g = r\cos\theta$$
$$y + f = r\cos\theta$$

This gives

$$x = -g + r\cos\theta$$
$$y = -f + r\cos\theta$$

This is the parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Example 3.2.1

Find the parametric equation of the circle $(x-3)^2 + (y-6)^2 = 8^2$

Solution 3.2.1

Compare $(x-3)^2 + (y-6)^2 = 8^2$ with $(x+g)^2 + (y+f)^2 = r^2$

Then, $g = -3, f = -6, \ r = 8$

The parametric equations are

$x = -g + r\cos\theta$ and $y = -f + r\cos\theta$

Therefore, $x = 3 + 8\cos\theta$, $y = 6 + 8\cos\theta$ and $0 \leq \theta \leq 2\pi$

Example 3.2.2

Find the parametric equation of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$

Solution 3.2.2

Compare $x^2 + y^2 - 6x + 4y - 12 = 0$ with the general equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Then, $2gx = -6x, 2fy = 4y, c = -12$

$$g = -3, f = 2,$$
$$r = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{(-3)^2 + (2)^2 - (-12)}$$
$$= \sqrt{9 + 4 + 12}$$
$$\sqrt{25} = 5$$

The parametric equations are

$x = -g + r\cos\theta$ and $y = -f + r\cos\theta$

Therefore, $x = 3 + 5\cos\theta$, $y = -2 + 5\cos\theta$ and $0 \leq \theta \leq 2\pi$

3.3 Parametric equation of the tangent to the circle

Given the equation of a circle centred $(0, 0)$

$x^2 + y^2 = r^2$ (1)

Differentiating (1) implicitly, we have

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$\frac{dy}{dx} = \frac{-x}{y}$ (2)

At the point with coordinates $(r\cos\theta,\ r\sin\theta)$ is

$\frac{dy}{dx} = \frac{-\cos\theta}{\sin\theta}$ (3)

Equation of tangent at the point $(r\cos\theta,\ r\sin\theta)$

$$\frac{y - r\sin\theta}{x - r\cos\theta} = \frac{-\cos\theta}{\sin\theta}$$

$$\sin\theta\,(\mathrm{y} - \mathrm{r}\sin\theta) = -\cos\theta\,(x - r\cos\theta)$$
$$y\sin\theta - r\sin^2\theta = \mathrm{x}\cos\theta + r\cos^2\theta$$
$$y\sin\theta + x\cos\theta = r(\sin^2\theta + \cos^2\theta)$$

$y\sin\theta + x\cos\theta = r \qquad (\sin^2\theta + \cos^2\theta = 1)$

The equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point $(r\cos\theta,\ r\sin\theta)$ is $x\cos\theta + y\sin\theta = r$

Example 3.3.1

Find the parametric equation of tangent to the circle $x^2 + y^2 = 9$ at point $(r\cos\theta,\ r\sin\theta)$.

Solution 3.3.1

Compare the equation of the circle $x^2 + y^2 = 9$ with $x^2 + y^2 = r^2$

Then, $r^2 = 9, \Rightarrow r = 3$

Therefore, the parametric equation of the tangent to the circleis

$$x\cos\theta + y\sin\theta = 3$$

3.4 Parametric equation of the normal to a circle

Equation of the circle $x^2 + y^2 = r^2$ at point $(r\cos\theta,\ r\sin\theta)$.

Let m be the slope of of the tangent to the circle at the point $(r\cos\theta,\ r\sin\theta)$, then

$$m = \frac{-\cos\theta}{\sin\theta}$$

Let m_1 be the gradient of the normal then

$$m_1 = \frac{-1}{m}$$

$$m_1 = \frac{\sin\theta}{\cos\theta}$$

Equation of the normal at the point $(r\cos\theta,\ r\sin\theta)$ is

$$\frac{y - r\,sin\theta}{x - r\cos\theta} = \frac{\sin\theta}{\cos\theta}$$
$$\cos\theta(y - r\sin\theta) = \sin\theta(x - r\cos\theta)$$
$$y\cos\theta - r\sin\theta\cos\theta = x\sin\theta - r\sin\theta\cos\theta$$

$$y\cos\theta - x\sin\theta = 0$$

Hence, the equation of the normal to the circle at the point $(r\cos\theta,\ r\sin\theta)$ is $y\cos\theta - x\sin\theta = 0$.

4.0 Self-Assessment Exercise(s)

1. Find the cartesian equation of the circle whose parametric equations are $x = \frac{1}{4}\cos\theta\,,\ y = \frac{1}{4}\sin\theta$ and $0 \le \theta \le 2\pi$.

2. Find the parametric equation of the circle $4x^2 + 4y^2 = 9$

3. Find the parametric equation of the circle $(x-1)^2 + (y-1)^2 = 5^2$
4. Find the parametric equation of the circle $x^2 + y^2 - 10x + 8y - 59 = 0$

5. Find the parametric equation of tangent to the circle $x^2 + y^2 = 9$ at point.
6. Show that the parametric equation of normal to the circle $x^2 + y^2 = r^2$ is $y\cos\theta - x\sin\theta = 0$.

5.0 Conclusion

In this unit, you have been introduced to parametric equation of the circle centred at $(0, 0)$as well as the circle centredat point $(-g, -f)$. The detail worked examples are also depicted. You have studied the derivation of the parametric equation of the tangent and normal from the equation of the circle $x^2 + y^2 = r^2$.

6.0 Summary

In the unit, you have learnt that a circle as the locus of points that satisfy $x = r\cos\theta$ and $y = r\sin\theta$, which are parametric equations of the circle. You have alsolearnt the parametric equations of tangent and normal to the to the circle at $(r\cos\theta,\ r\sin\theta)$.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition

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Larson Edwards *Calculus: An Applied Approach*. 7th Edition

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Module 4: Conic Sections

Module Introduction

In this module, you will study the curves obtained by making sections or cut at particular angle through a cone. These curves are parabola, hyperbola and ellipse and were known to the ancient Greeks, who first explored their properties. These curves have exceeding modern technological applications, for examples, the television dish aerial, radio telescopes, solar radiation collectors, etc., all depend upon what is known as the reflective property of one of the conic sections.

In **Fig1**, you can see parts of a cone which comprises the axis and generator of the cone. The axis is the central line about which the cone is symmetric. A generator is a line which, when rotated about the axis, sweeps out the cone.

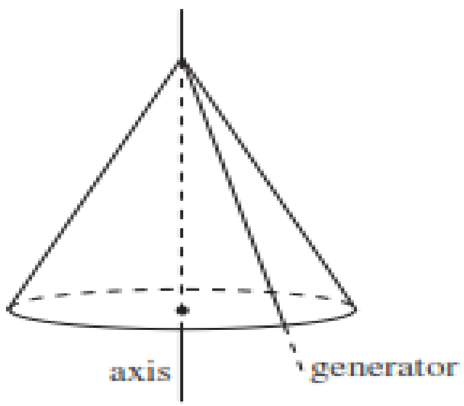


Fig1: Parts of a cone

Unit 1 Parabola
Unit 2 Ellipse
Unit 3 Hyperbola

Unit 1: Parabola

Unit Structure

1.0 Introduction

In this unit, you will learn the different curves of conic section. These curves are parabola, hyperbola and ellipse. You would be able to categorized the curves based on the values of the constant *e* called eccentricity. Then, you would be able to derive the equation of parabola and solve problems on the parabola using the directrix and the focus point.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should able to:

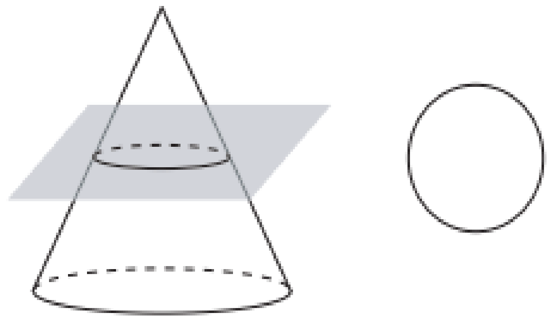
- state the four types of conic section
- state the values of eccentricity for the conic sections
- solve the problem on the parabola

3.0 Parabola

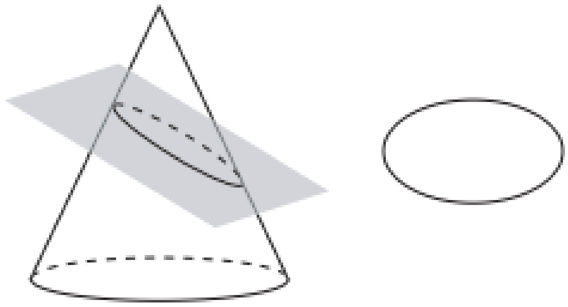
3.1 The sections of a cone

If we cut a cone at different angles, then we will obtain different types of conic section. There are four types we can obtain.

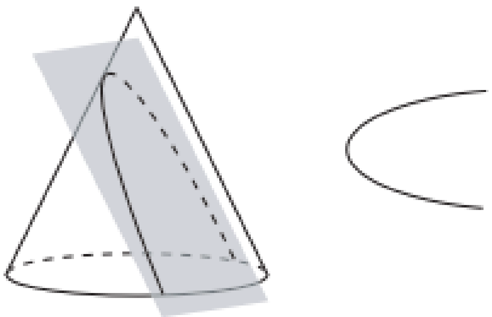
1. We can make the obvious cut or section perpendicular to the axis of the cone. This gives a circle.



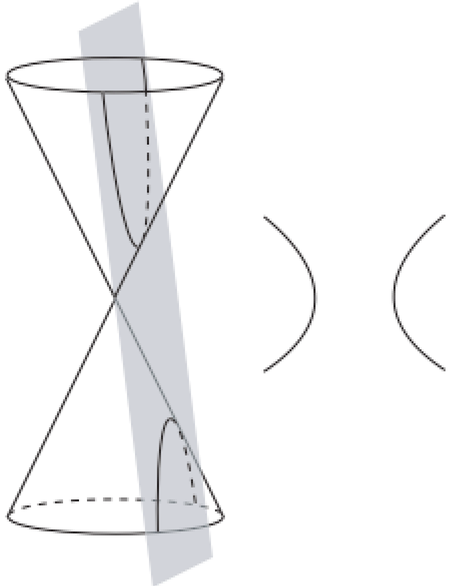
2. We can make the cut at an angle to the axis of the cone, so that we still get a closed curve which is no longer a circle. This curve is an ellipse.



3. If we now make the cut parallel to the generator of the cone, we obtain an open curve. This is called a parabola.



4. Finally, we make the cut at an even steeper angle. If we imagine that we have a double cone, that is, two cones vertex to vertex, then we obtain the branches of a hyper bola.



3.2 Eccentricity

In order to be able to obtain the equations of conics, we need definitions that are given in terms of algebra as well as geometry.

Suppose that we are given a fixed straight line called the *directrix* and a fixed point called the *focus*. If we have another point P, when we can consider the perpendicular distance of P from the line, and also the distance of P from the focus. What would
happen if one of these distances is a fixed multiple of the other?

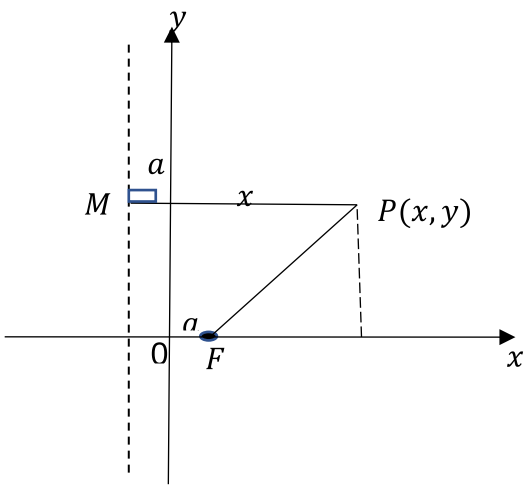


Fig 3.2.1: Eccentricity e

Suppose in **Fig3.2.1**, there is a constant e such that

$$PF = ePM$$

$PF = \sqrt{(x-a)^2 + y^2}$ (Pythagoras theorem)

$$PM = x + a$$

Therefore, $\sqrt{(x-a)^2 + y^2} = e(x+a)$

Hence, the curves of the sections of conic depend upon the value of the constant e:

(i) If $e = 0$, then the curve is a circle
(ii) If $e = 1$, then the curve is a parabola; and
(iii) If $e > 1$, then the curve is a hyperbola
(iv) If $0 < e < 1$, then the curve is an ellipse

The constant$e$ is called the eccentricity of the conic.

Example 3.2.1

In each case below, the given point on a conic with focus $(2, 0)$ and directrix $x = -2$
Find the value of its eccentricity and the type of conic.
(a) $(1, 0)$ (b) $(0, 0)$ (c) $(-1,0)$

Solution 3.2.1

The equation uses to determine the value of eccentricity is given as

$\sqrt{(x-a)^2 + y^2} = e(x+a)$,
Focus $= (2, 0)$,
Directrix $x = -2$
$a = 2$ from focus point

(a) At point $(1, 0)$, the equation becomes

$\sqrt{(1-2)^2 + (0)^2} = e(1+2)$,

$\sqrt{1} = 3e$,

$$\therefore e = \frac{1}{3}$$

Hence, the curve is ellipse

(b) At point $(0, 0)$, the equation becomes

$$\sqrt{(0-2)^2 + (0)^2} = e(0+2),$$

$$\sqrt{4} = 2e \Rightarrow 2e = 2,$$

$$\therefore e = 1$$

Hence, the curve is parabola.

(c) At point $(-1, 0)$, the equation becomes

$$\sqrt{(-1-2)^2 + (0)^2} = e(-1+2),$$

$$\sqrt{9} = e,$$

$$\therefore e = 3$$

Hence, the curve is hyperbola

3.3 The parabola

The equation of parabola can be obtained in Cartesian coordinates (see **Fig 3.3.1**).

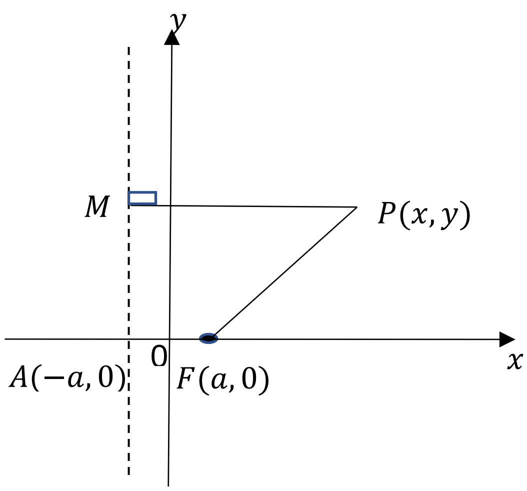


Fig 3.3.1: Derivation of Parabola

In **Fig 3.3.1,** $e = 1$, for the parabola so that $PF = PM$. Thus, each point on the curve is equidistant from the focus and the directrix, and so the curve will pass through the mid-point of AF. The origin 0 is the mid-point of AF, this gives the position of the y-axis.

In order to have a scale where we placing the curve, $0F$ is a distance $a > 0$, giving F as the point$(a, 0)$ and thus A is the point $(-a, 0)$.

Now as P moves aways from 0, we can see that PF and PM can be become as large as we wish, provided they equal in length. Furthermore, P can be below the x-axis, and as the value of PM is unaffected by this switch, the magnitude of PF is unchanged. So, we can sketch an open curve that is entirely to the right of the y −axis, and symmetrical about the$x$ −axis.

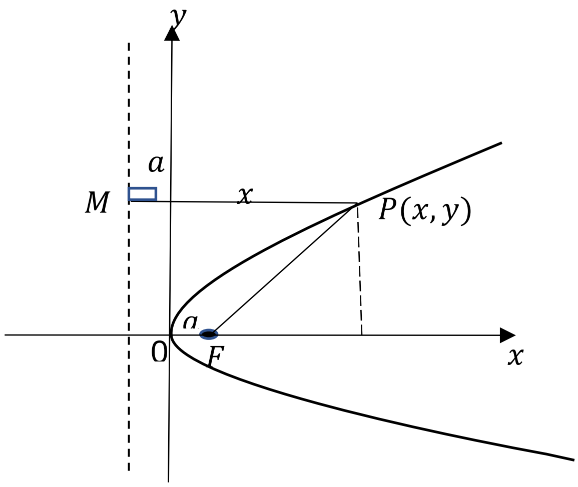


Fig 3.3.2: Parabola curve

Now, we can find the equation of the parabola. To do this, we need to write down some of the lengths in **Fig 3.3.2**. If P is the point (x, y), then the length PM is the distance a from the directrix to the y-axis, plus the x coordinate of P, so that

$$PM = x + a$$

$$PF = \sqrt{(x-a)^2 + y^2} \qquad \text{(Pythagoras' theorem)}$$

For parabola, we have $PF = PM$, so that

$$\sqrt{(x-a)^2 + y^2} = x + a$$

Squaring both sides, we obtain

$$(x-a)^2 + y^2 = (x+a)^2$$

and expanding the bracket, gives

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

Which simplifies to $y^2 = 4ax$.

This is the standard Cartesian equation of the parabola.

Example 3.3.1

The equation of a parabola is $y^2 = 24x$. Find the length of the latus rectum, focus and vertex.

Solution 3.3.1

To find the length of latus rectum, focus and vertex of the parabola, given the equation of the parabola $y^2 = 24x$.

Compare $y^2 = 24x$ with the standard equation of parabola $y^2 = 4ax$.

$$a = \frac{24x}{4x} = 6$$

Now, parabola formula for latus rectum is:

The length of latus rectum= $4a$

$$= 4(6) = 24$$

Focus= $(a, 0) = (6, 0)$

Vertex= $(0, 0) = (0, 0)$

4.0 Self-Assessment Exercise(s)

1. State the four types of conic section
2. In each case below, the given point on a conic with focus $(2, 0)$ and directrix $x = -2$. Find the value of its eccentricity and the type of conic.
   (a) $(8,\ 0)$ (b) $(2,\ 5)$ (c) $(5, -4)$ (d) $(2, 4)$ (e) $(-6,\ 6)$

3. The equation of a parabola is $2(y - 3)^2 + 24 = x$. Find the length of the latum, focus and vertex.

4. Which equation represent a parabola that has a focus of $(0, 0)$ and a diretrix $y = 4$

5. In each case below, the given point lies on the parabola $y^2 = 4ax$ for some value of a. Find a, and give the equation of the tangent to the parabola at that point.

5.0 Conclusion

In this unit, you have studied the cutting position of the circle, ellipse, parabola and hyperbola from cone and constant eccentricity value of each of the curves. Essentially, the directrix and focus which are two major parts of the geometry which determined the value of the eccentricity have been discussed. Lastly, the equation of parabola is derived from the eccentricity equation.

6.0 Summary

In this unit, you have learnt the four different types of the sections of a cone and the particular angles of cuts through the cone. You have also studied and calculated the values of eccentricity of parabola, hyperbola and ellipse. The parabolic equation is also derived from the value of constant eccentricity couple with application of Pythagoras' theorem. And, you have to study through the worked examples of problems on eccentricity and parabola.

7.0 References/Further Readings

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A. Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-conic sections-2009-1.p

Unit 2: Ellipse

Unit Structure

1.0 Introduction

This unit introduces you to ellipse which is an integral part of the conic section and is similar in properties to a circle. An ellipse is oval in shape and has an eccentricity less than one, and it represents the locus of points, the sum of whose distances from the two foci of the ellipse is a constant value. Examples of the ellipse in our daily life are shape of an egg in a two-dimensional form and the running track in sport stadium. In this unit, you will be studying the definition, derivation and different standard forms of the equation of the ellipse.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should able to:

- define and describe an ellipse
- identify the foci, vertices, axes and centre of an ellipse
- derive and solve equation of ellipses centred at the origin
- derive and solve equation of ellipses not centred at the origin

3.0 The Ellipse

3.1 Description of an ellipse

An ellipse is the locus of points in a plane such that the sum of their distances from two fixed points in the plane, is constant. The fixed points are known as the foci (singular focus), which are surrounded by the curve, The fixed line is directrix and constant ratio is eccentricity of ellipse. Eccentricity is a factor of the ellipse, which demonstrates the elongation of it and is denoted by e.

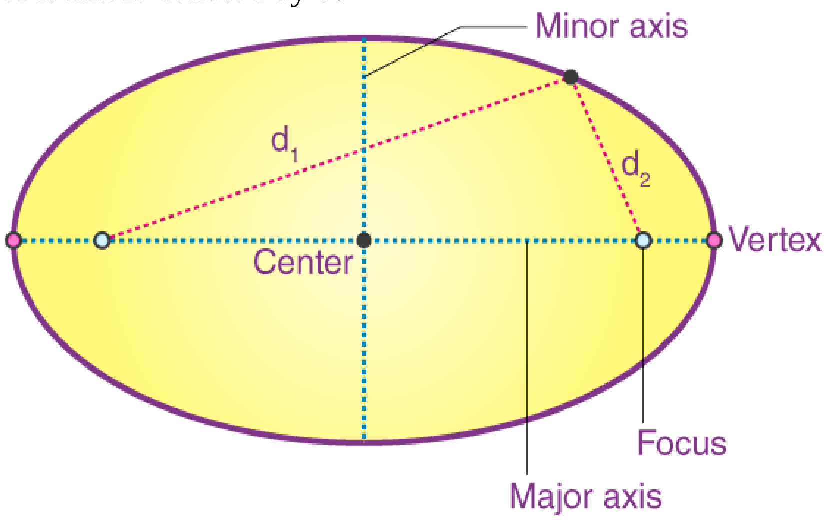


Fig 3.1.1: Ellipse

The shape of the ellipse is an oval shape and the area of an ellipse is defined by its major axis and minor axis, area of ellipse $= \pi ab$, where a and b are length of semi-major and semi-minor axis of an ellipse respectively. Ellipse is similar to other conic sections such as parabola and hyperbola, which are open in shape and unbounded (see **Fig 3.1.1**).

The major axis is the longest diameter of the ellipse, going through the centre from one end to the other, at the broad part of the ellipse. Whereas, the minor axis is the shortest diameter of ellipse, crossing through the centre from one end to the other (see **Fig 3.1.1**).

Half of major axis is called semi-major axis and half of minor axis is called semi-minor axis (see **Fig 3.1.1**).

3.1.1 Properties of ellipse

The followings are the properties of ellipse:

(i) Ellipse has two focal points, also called foci
(ii) The fixed distance is called a directrix.
(iii) The eccentricity of ellipse lies between 0 to 1 $(0 < e < 1)$.
(iv) The total sum of each distance from the locus of an ellipse to the two points is constant
(v) Ellipse has one major axis, minor axis and a centre

3.2. Equation of Ellipse at the origin

The ellipse is the set of all points $(x,\ y)$ such that the sum of the distances from $(x,\ y)$ to the foci is constant as shown Fig 3.2.1.

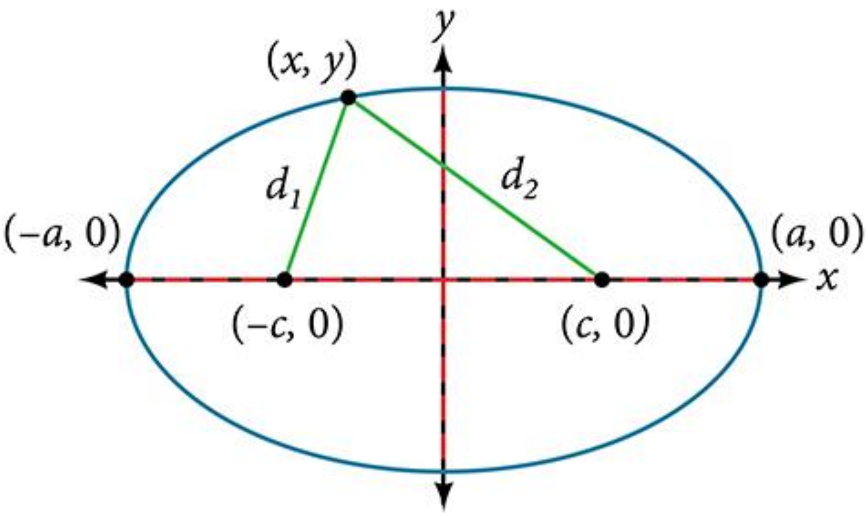


Fig 3.2.1: Ellipse

In **Fig 3.2.1**, $(a,\ 0)$ is a vertex of the ellipse, then, the distance from $(-c,\ 0)$ to $(a,\ 0)$ is $a-(-c)=a+c$ and the distance from $(c,\ 0)$ to $(a,\ 0)$ is $a-c$. The sum of the distances from the foci$(-c,\ 0)$ and and $(-c,\ 0)$ to the vertex $(a,\ 0)$ is

$$(a+c\ )+(a-c)=2a$$

(x, y) is point on the ellipse, then we can define the following variables

$d_1=$ the distance from $(-c,\ 0)$ to $(x,\ y)$
$d_2=$ the distance from $(c,\ 0)$ to $(x,\ y)$

By the definition of an ellipse, d_1+d_2 is constant for any point $(x,\ y)$ on the ellipse. We know that the sum of these distances is $2a$ for the vertex

$(a, 0)$. It follows that $d_1 + d_2 = 2a$ for any point on the ellipse. We will be the derivation by applying the distance formula. The rest of the derivation is algebraic

$d_1 + d_2 = \sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$ (Distance formula)

$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$ (Simplifying)

$(x+c)^2 + y^2 = \left[2a - \sqrt{(x-c)^2 + y^2}\right]^2$ (Squaring both sides)

$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$ (Expanding)

$2cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2cx$ (Combine like terms)

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$ (Divides through by 4)

$$[cx - a^2]^2 = a^2\left[\sqrt{(x-c)^2 + y^2}\right]^2$$

$c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2)$ (Expanding)

$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$ (Combine like terms)

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$ (Factor common terms)

$x^2b^2 + a^2y^2 = a^2b^2$ (Set $b^2 = a^2 - c^2$)

$\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$ (Divide both sides by a^2b^2)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This equation defines an ellipse centred at the origin. If $a > b$, the ellipse is stretched further in the horizontal direction, and if $b > a$, the ellipse is stretched further in the vertical direction.

The key features of the ellipse are its centre, vertices, co-vertices, foci, and lengths and positions of the major and minor axes. We can identify all of these features just by looking at the standard form of the equation.

There are four variations of the standard forms of the ellipse. These variations are categorized first by the location of the centre (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph.

Example 3.2.1

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

Solution 3.2.1

The foci are on the x-axis, so the major axis is the x-axis. Thus, the equation will have the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The vertices are $(\pm 8, 0)$, so $a = 8$ and $a^2 = 64$

The foci are $(\pm 5, 0)$, $c = 5$ and $c^2 = 25$

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$

Solving for b^2, $b^2 = a^2 - c^2$

$$b^2 = 64 - 25$$
$$b^2 = 39$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ in the standard equation form of the ellipse. Then, the equation of the ellipse is

$$\frac{x^2}{64} + \frac{y^2}{39} = 1$$

3.3 Equation of the ellipse centre at (h, k)

The standard form of the equation of an ellipse with centre (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$

- the coordinates of the foci are $(h \pm c,\ k)$, where $c^2 = a^2 - b^2$(see **Fig 3.3.1 a**).

The standard form of the equation of an ellipse with centre $(h,\ k)$ and major axis parallel to the y-axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h + b,\ k)$
- the coordinates of the foci are $(k, k \pm c)$, where $c^2 = a^2 - b^2$(see **Fig 3.3.1 b**)

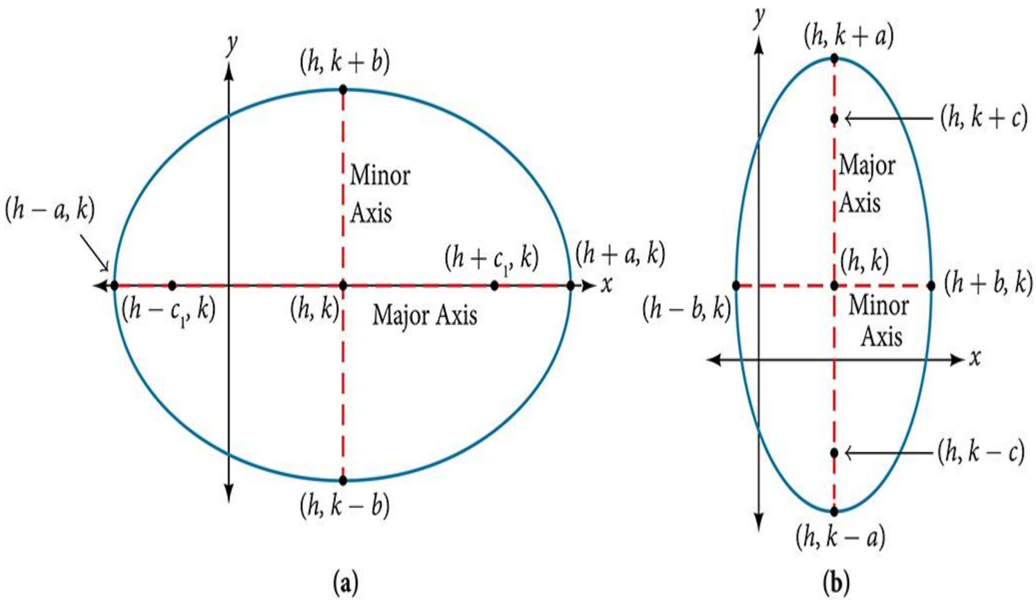


Fig 3.3.1 (a) Horizontal ellipse with center $(h,\ k)$**(b)** vertical ellipse with center $(h,\ k)$

Example 3.3.1

What is the standard form equation of the ellipse that has vertices $(-2, -8)$ and $(-2, 2)$ and foci $(-2, -7)$ and $(-2, 1)$?

Solution 3.3.1

The x-coordinates of the vertices and foci are the same, so the major axis is parallel to the y-axis. thus, the equation of the ellipse is of the form

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

First, we identify the center, $(h,\ k)$. The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have

$$(h,k)=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)$$

$$=(-2,\qquad -3)$$

Next, we find a^2. The length of the major axis, $2a$, is bounded by the vertices. We solve for a by finding the distance between the y-coordinates of the vertices.

$$2a=2-(-8)$$
$$2a=10$$
$$a=5$$

So $a^2=25$

Now, we find c^2. The foci are given by $(h,\ k\pm c)$. So, $(h,k-c)=(-2,-7)$ and $(h,k+c)=(-2,1)$. We have substituted $k=-3$. Using either of the points to solve for c.

$$k+c=1$$
$$-3+c=1$$
$$c=4$$

So $c^2=16$

Next, we solve for b^2 using the equation $c^2=a^2-b^2$.

$$c^2=a^2-b^2$$
$$16=25-b^2$$
$$b^2=25-16$$
$$b^2=9$$

Finally, we substitute the values found for $h,\ k,a^2$ and b^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9}+\frac{(y+3)^2}{25}=1$$

4.0 Self-Assessment Exercise(s)

1. (a) Define an ellipse.
   (b) State of the properties of an ellipse.
2. (a) Write the standard form of equation of an ellipse centre at (0.0).
   (b) Write the standard form of equation of an ellipse centre at $(h.k)$ where major axis parallel to the y-axis

3. What is the standard form equation of the ellipse that has vertices $(\pm 9,\ 0)$ and foci $(\pm 5,\ 0)$?

4. What is the standard form equation of the ellipse that has vertices $(0,\ \pm 4)$ and foci $(0,\ \pm\sqrt{15})$?

5. What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5,\ 3)$ and foci $(1-2\sqrt{3},\ 3)$ and $(1+2\sqrt{3}, 3)$?

6. Graph the ellipse given by the equation $\frac{x^2}{9}+\frac{y^2}{25}=1$. Identify and label the centre, vertices, co-vertices, and foci

7. Graph the ellipse given by the equation $4x^2+25y^2=100$. Rewrite the equation in standard form. Then, identify and label the centre, vertices, co-vertices and foci.

8. Graph the ellipse given by the equation $\frac{(x+2)^2}{4}+\frac{(y-5)^2}{9}=1$. Identify and label the centre, vertices, co-vertices, and foci

9. Graph the ellipse given by the equation $4x^2+9y^2-40x+100=0$. Identify and label the centre, vertices, co-vertices and foci.

10. Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices and foci of the ellipse.
$$4x^2+y^2-24x+2y+21=0$$

5.0 Conclusion

In this unit, you studied the definition, description and properties of an ellipse. The derivation of equation of the standard form of the equation of an ellipse centre at the origin was carried out. Also, the standard form of the equations of an ellipse with centre $(h,\ k)$ were stated and itemized based on the parallel to the major axis. More so, the worked examples are done in details.

6.0 Summary

In this unit, you have learnt the components of the ellipse such, its centre, foci, vertices, co-vertices, major and minor axes. Also, you have learnt that ellipse as the set of all points $(x,\ y)$ such that the sum of the distances from $(x,\ y)$ to the foci is constant which aids the derivation of the standard form of equation of the ellipse at the origin. Furthermore, the standard equationsof the ellipse with centre $(h,\ k)$ also specified with the major axis parallel to the$x$ and y-axis. Finally, you have able to work through all the examples enumerated in this unit.

7.0 References/Further Reading

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A. Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-conic section -2009-1.p
https://openstax.org/books/college-algebra-2e/pages/8-1-

Unit 3: Hyperbola

Unit Structure

1.0 Introduction

This unit teaches you to a hyperbola as a cone section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror of images each other. You will also learn the derivation of the equation of a hyperbola centred at the origin and the equation of a hyperbola with centre (h, k). Finally, you will study the worked examples of the both equations.

2.0 Intended Learning Outcomes (ILOs)

At the end of this unit, you should be able to:

- locate a hyperbola's vertices and foci
- write equations of hyperbolas in standard form
- solve applied problems involving hyperbolas

3.0 Main Content

3.1 Locate a hyperbola's vertices and foci

Hyperbola can be defined as set of points in the coordinate plane. It is the set of all point (x, y) in a plane such that the difference of the distance between (x, y) and the foci is a positive constant.

Every hyperbola has two axes of symmetry. The transverse axis is a line segment that passes through the centre of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The conjugate axis is perpendicular to the transverse axis and has the co-vertices as its endpoints. The centre of a hyperbola is the midpoint of both the transverse and conjugate axes, where they intersect.

Every hyperbola has also two asymptotes that pass through its centre. As a hyperbola recedes from the centre, its branches approach these asymptotes. The central rectangle of the hyperbola is centred at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle (see **Fig. 3.1.1**)

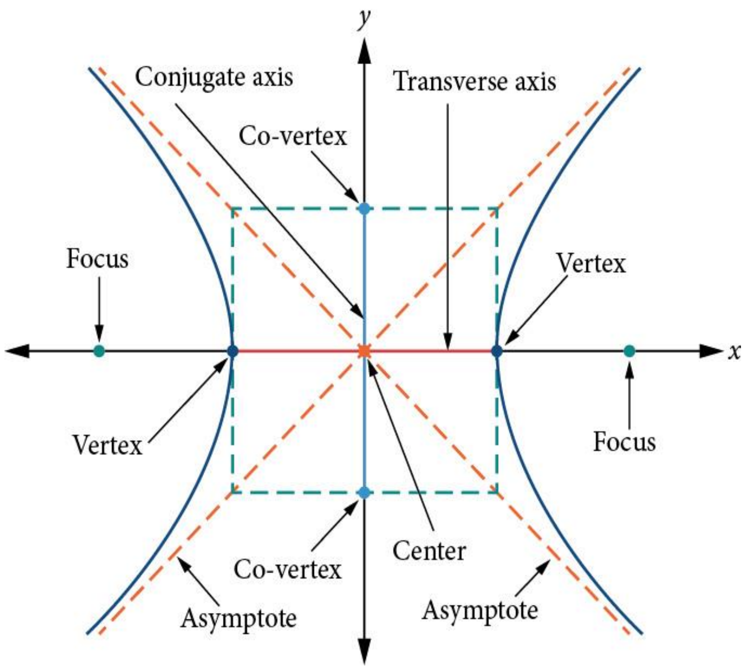


Fig 3.1.1: Key Features of the hyperbola

3.2 Deriving the Equation of a hyperbola centred at the origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centred at the origin. The hyperbola is the set of all point (x, y) such that the difference of the distances from (x, y) to the foci is constant (see **Fig 3.2.1**).

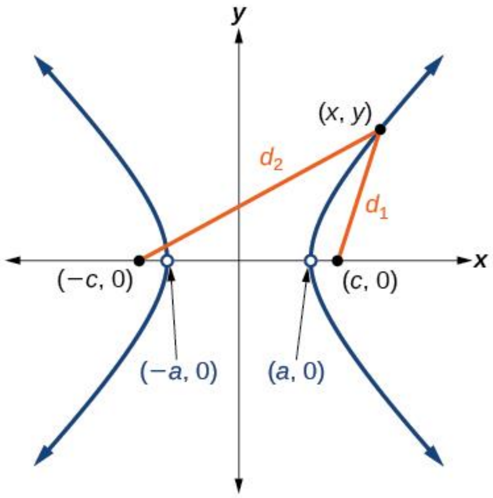


Fig 3.2.1 Hyperbola

If $(a,\ 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c)=a+c$. The distance from $(c, 0)$ ti $(a, 0)$ is $c-a$. The difference of the distances from the foci to the vertex is

$$(a+c)-(c-a)=2a$$

If $(x,\ y)$ is a point on the hyperbola, we can define the following variable

d_2 = the distance from $(-c, 0)$ to (x, y)
d_1 = the distance from $(c, 0)$ to (x, y)

By definition of a hyperbola d_2-d_1 is constant for any point (x, y) on the hyperbola.
We know that the differences is$2a$ forth vertex $2a$ for the vertex $(a, 0)$. It follows that $d_2-d_1=2a$ for any point on the hyperbola.

$$d_2-d_1=$$
$$d_1+d_2=\sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$
(Distance formula)

$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$ (Simplifying)

$(x+c)^2+y^2=\left[2a+\sqrt{(x-c)^2+y^2}\right]^2$ (Squaring both sides)

$x^2+2cx+c^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$ (Expanding)

$2cx = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} - 2cx$ (Combine like terms)

$$4cx - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$cx - a^2 = a\sqrt{(x-c)^2 + y^2}$ (Divides through by 4)

$$[cx - a^2]^2 = a^2\left[\sqrt{(x-c)^2 + y^2}\right]^2$$

$c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2)$ (Expanding)

$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$ (Combine like terms)

$$a^4 + c^2x^2 = a^2x^2a^4 + a^2c^2 + a^2y^2$$

$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$ (Factor common terms)

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$x^2b^2 - a^2y^2 = a^2b^2$ (Set $b^2 = c^2 - a^2$)

$\frac{x^2b^2}{a^2b^2} - \frac{y^2b^2}{a^2b^2} = 1$ (Divide both sides by a^2b^2)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This equation defines a hyperbola centred at the origin with vertices $(\pm a, 0)$ and co-vertices $(0, \pm b)$

3.2 Standard forms of the equation of a hyperbola with centre $(\mathbf{0}, \mathbf{0})$

The standard form of the equation of a hyperbola with centre $(0, 0)$ and transverse axis on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{a}{b}x$

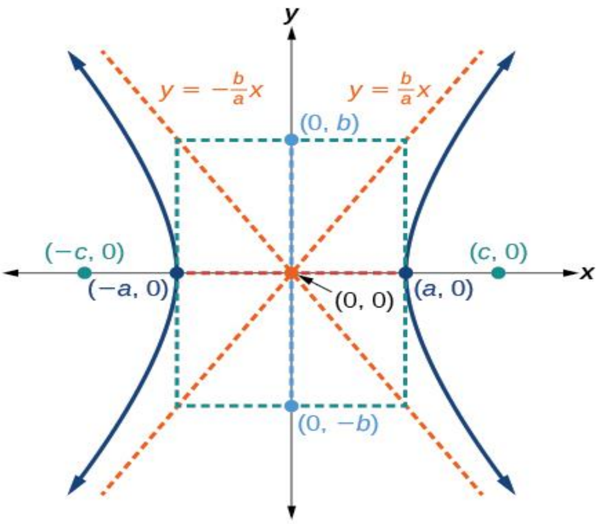


Fig 3.2.1 Horizontal hyperbola with centre $(0, 0)$

The standard form of the equation of a hyperbola with centre $(0, 0)$ and transverse axis on the y-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the major axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b}x$

Note that the vertices, co-vertices and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relation to identify its vertices and foci.

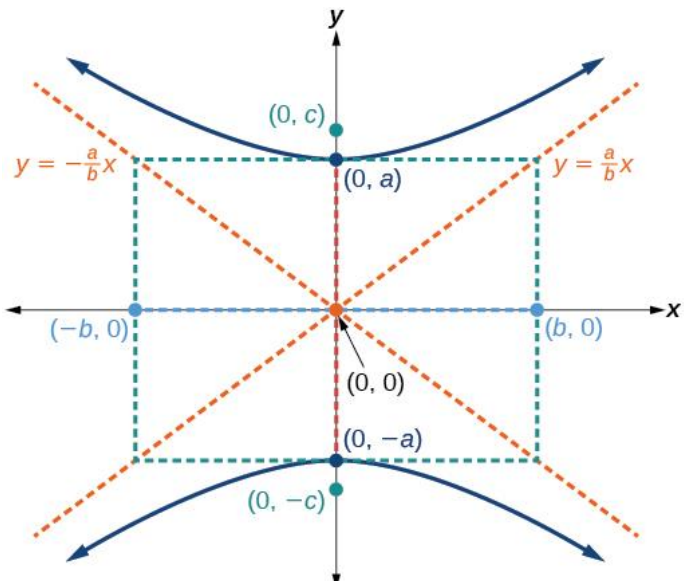


Fig 3.2.2 Vertical hyperbola with centre $(0, 0)$

Example 3.2.1

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$

Solution 3.2.1

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the y-axis. The hyperbola is centred at the origin, so the vertices serve as the y- intercept of the graph. To find the vertices, set $x = 0$, and solve for y.

$$\frac{y^2}{49} - \frac{x^2}{32} = 1$$
$$\frac{y^2}{49} - \frac{0^2}{32} = 1$$
$$\frac{y^2}{49} = 1$$
$$y^2 = 49$$
$$y = \pm 7$$

The foci are located at $(0, \pm c)$. Solving for c

$$c = \sqrt{a^2 + b^2} = \sqrt{49 + 32}$$
$$\sqrt{81} = 9$$

Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0, 9)$.

3.3 Standard forms of the equation of a hyperbola with centre $(\boldsymbol{h}, \boldsymbol{k})$

The standard form of the equation of a hyperbola with centre (h, k) and transverse axis on the x-axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

where

- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h,\ k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, 0)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h) + k$.

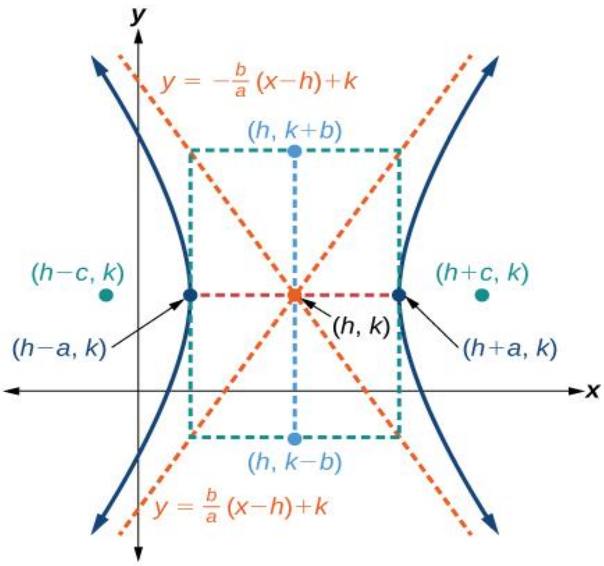


Fig 3.3.1: Horizontal hyperbola with centre (h, k)

The standard form of the equation of a hyperbola with centre (h, k) and transverse axis on the y-axis is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

where

- the length of the major axis is $2a$
- the coordinates of the vertices are $(h,\ k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h \pm b,\ k)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$

- the coordinates of the foci are $(h,\ k \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h)+k$.

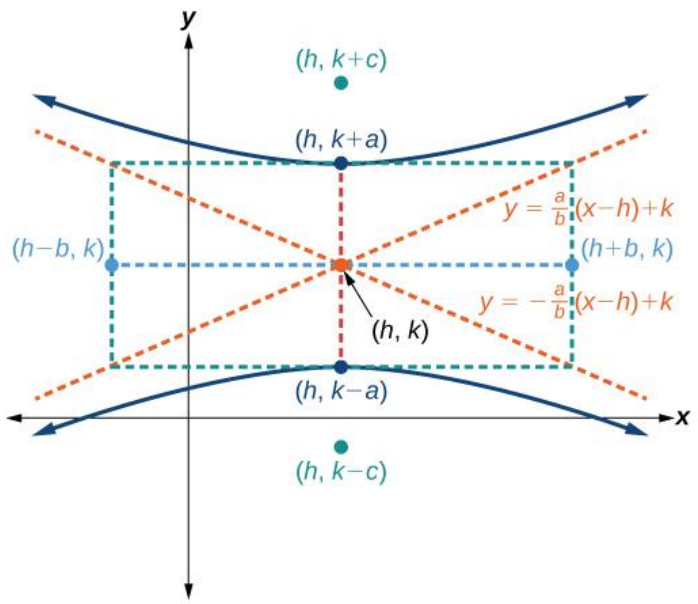


Fig 3.3.2: vertical hyperbola with centre (h,k)
Example 3.3.1

What is the standard form equation of the hyperbola that has vertical at $(0,-2)$ and $(6,-2)$ and foci $(-2,-2)$ and $(8,-2)$?

Solution 3.3.1

The y-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the x-axis. Thus, the equation of the hyperbola will take the form

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

First, we identify the centre, (h,k). The centre is halfway between the vertices $(0,-2)$ and $(6,-2)$. Applying the midpoint formula, we have

$$(h,k)=\left(\frac{0+6}{2},\frac{-2+(-2)}{2}\right)$$

$$=(3,2)$$

Next, we find a^2. The length of the transverse axis, $2a$, is bounded by the vertices. So, we can find a^2 by finding the distance between the x-coordinates of the vertices.

$$2a = |0 - 6|$$
$$2a = 6$$
$$a = 3$$
$$a^2 = 9$$

Now we need to find c^2. The coordinates of the foci are (h, k). So $(h - c, k) = (-2, -2)$ and $(h + c, k) = (8, -2)$. We can the x-coordinate from either of these points to solve for c. Using the point $(8, -2)$, and substitute $h = 3$,

$$h + c = 8$$
$$3 + c = 8$$
$$c = 5$$
$$c^2 = 25$$

Next, solve for b^2 using the equation $b^2 = c^2 - a^2$

$$b^2 = 25 - 9$$
$$b^2 = 16$$

Finally, substitute the value found for h, k, a^2 and b^2 into the standard form of the equation.

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

4.0 Self-Assessment Exercise(s)

1. (a) Define a hyperbola
   (b) State of the properties of a hyperbola

2. (a) Write the standard form of equation of an ellipse centre at (0.0), where major axis parallel to the y-axis,
   (b) Write the standard form of equation of an ellipse centre at $(h.k)$, where major axis parallel to the y-axis.

3. Find the vertices and foci of the hyperbola with equation $\frac{y^2}{9} - \frac{x^2}{25} = 1$

4. What is the standard form equation of the hyperbola that has vertices $(\pm 6,\ 0)$ and foci $(\pm 2\sqrt{10},\ 0)$?

5. What is the standard form equation of the hyperbola that has vertices $(0,\ \pm 2)$ and foci $(0, \pm 2\sqrt{5})$?

6. What is the standard form equation of the hyperbola that has vertical at $(1, -2)$ and $(1,\ 8)$ and foci $(1, -10)$ and $(1, 16)$?

7. Given the hyperbola equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$, find the vertices, co-vertices, foci, and asymptotes

8. Given the hyperbola equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$, find the vertices, co-vertices, foci, and asymptotes

9. Given the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$, find the centre, vertices, co-vertices, foci and asymptotes.

10. Given the equation of hyperbola $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$, find the center, vertices, co-vertices, foci and asymptotes.

5.0 Conclusion

In this unit, you have studied the definition, description and properties of a hyperbola. You have able to locate a hyperbola's vertices and foci, and derivation of equation of the standard form of the equation of a hyperbola centre at the origin. Also, you have able to state the two standard forms of the equations of hyperbola with centre $(h,\ k)$.

6.0 Summary

In this unit, you have learnt the components of a hyperbola such, its centre, foci, vertices, co-vertices, major and minor axes. Also, you have learnt that hyperbola as the set of all point (x, y) in a plane such that the difference of the distance between (x, y) and the foci is a positive constant which aids the derivation of the standard form of equation of the hyperbola at the origin. You have been introduced to different forms of the standard equations of the hyperbola either with the major axis parallel to both x and y-axis. Finally, you have been able to work through all the examples in this unit.

7.0 References/Further Reading

Blitzer. *Algebra and Trigonometry custom*.6th Edition

K.A. Stroud. *Engineering Mathematics*.8th Edition

Larson Edwards, *Calculus: An Applied Approach*. 7th Edition

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/mc-TY-conic section -2009-1.p
https://openstax.org/books/college-algebra-2e/pages/8-2-the-hyperbol