COURSE GUIDE

BUS 800
QUANTITATIVE ANALYSIS

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Introduction

BUS800: Quantitative Analysis is a two-credit course for students offering M.Sc. Business Administration in the Department of Business Administration, Faculty of Management Sciences.

The course will consist of fifteen (15) units, that is, three modules of five (5) units for each module. The material has been developed to suit postgraduate students in Business Administration at the National Open University of Nigeria (NOUN) by using an approach that treats Quantitative Analysis.

A student who successfully completes the course will surely be in a better position to manage operations of organizations in both private and public organizations.

The course guide tells you briefly what the course is about, what course materials you will be using and how you can work your way through these materials. It suggests some general guidelines for the amount of time you are likely to spend on each unit of the course in order to complete it successfully. It also gives you some guidance on your tutor-marked assignments. Detailed information on tutor-marked assignment is found in the separate assignment file which will be available in due course.

What You Will Learn in This Course

This course will introduce you to some fundamental aspects of Quantitative Analysis, set theory; basic concepts in probability; probability distributions; decision theory; forecasting models and techniques, introduction to operations research; network models, simulation, sequencing and scheduling, Game Theory and Inventory Control.

Course Material

The major component of the course, what you have to do and how you should allocate your time to each unit in order to complete the course successfully on time are listed follows:

1. Course guide
2. Study unit
3. Textbook
4. Assignment file
5. Presentation schedule

Study Unit

There are 15 units in this course which should be studied carefully and diligently.

Module 1

Module 2

Module 3

Each study unit will take at least two hours, and it include the introduction, objective, main content, self-assessment exercise, conclusion, summary and reference. Other areas border on the Tutor-Marked Assessment (TMA) questions. Some of the self-assessment exercise will necessitate discussion, brainstorming and argument with some of your colleges. You are advised to do so in order to understand and get acquainted with historical economic event as well as notable periods.

There are also textbooks under the reference and other (on-line and off-line) resources for further reading. They are meant to give you additional information if only you can lay your hands on any of them. You are required to study the materials; practice the self-assessment exercise and tutor-marked assignment (TMA) questions for greater and in-depth understanding of the course. By doing so, the stated learning objectives of the course would have been achieved.

Textbook and References

For further reading and more detailed information about the course, the following materials are recommended:

Ackoff, R., and Sisieni, M. (1991). Fundamentals of Operations Research, New York: John Wiley and Sons Inc.

Akingbade, F. (1995). Practical Operational Research for Developing Countries: A Process Framework approach, Lagos: Panaf Publishing Inc.

Adebayo, O.A., Ojo, O., and Obamire, J.K. (2006). Operations Research in Decision Analysis, Lagos: Pumark Nigeria Limited.

Beyer, W. H. CRC Standard Mathematical Tables, 28th ed. Boca Raton, FL: CRC Press, p. 532, 1987.

Churchman, C.W., Ackoff, R.L., and Arnoff, E.L.(1957). Introduction to Operations Research, New York: John Wiley and Sons Inc.

Denardo, E.V. (2002). The Schience of Decision making: A Problem-Based Approach Using Excel. New York: John Wiley.

Dixon – Ogbechi, B.N (2001). Decision Theory in Business, Lagos: Philglad Nig. Ltd.

Eiselt, H.A., and Sandblom, C. (2012). Operations Research: A Model Based Approach, 2nd ed., New York: Springer Heidelberg

Gupta, P.K and Hira, D.S. (2012). Operations Research, New – Delhi: S. Chand & Company.

Harper, W.M. (1975). Operational Research, London: Macdonald & Evans Ltd.

Howard, A. (2004). Speaking of Decisions: Precise Decision Language. Decision Analysis, Vol. 1 No. 2, June).

Ihemeje, J.C. (2002). Fundamentals of Business Decision Analysis, Ibadan: Sibon Books Limited.

Jawahar, S. (2006). Overview of System Analysis & Design- Lesson Note no. 1, Ashok Educational Foundation

Lucey, T. (1988). Quantitative Techniques: An Instructional Manual, London: DP Publications. Microsoft Encarta Premium, (2009).

Murthy, R.P. (2007). Operations Research 2nd ed. New Delhi: New Age International Publishers.

Papoulis, A. Probability, Random Variables, and Stochastic Processes, 2nd ed. New York: McGraw-Hill, 1984.

Reeb, J., and Leavengood , S. (1998). "*Operations Research, Performance Excellence in the Wood Products Industry",* October EM 8718

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable, Vol. 1)

Skemp, R.R. (1991). The Psychology of Learning Mathematics, Harmonds Worth: Penguin Book.

Assignment File

Assignment files and marking scheme will be made available to you. This file presents you with details of the work you must submit to your tutor for marking. The marks you obtain from these assignments shall form part of your final mark for this course. Additional information on assignments will be found in the assignment file and later in this Course Guide in the section on assessment.

There are four assignments in this course. The four course assignments will cover:

Assignment 1 - All TMAs' question in Units 1 – 5 (Module 1)
Assignment 2 - All TMAs' question in Units 6 – 10 (Module 2)
Assignment 3 - All TMAs' question in Units 11 – 15 (Module 3)

Presentation Schedule

The presentation schedule included in your course materials gives you the important dates for this year for the completion of tutor-marking assignments and attending tutorials. Remember, you are required to submit all your assignments by due date. You should guide against falling behind in your work.

Assessment

There are two types of the assessment of the course. First are the tutor-marked assignments; second, there is a written examination.

In attempting the assignments, you are expected to apply information, knowledge and techniques gathered during the course. The assignments must be submitted to your tutor for formal Assessment in accordance with the deadlines stated in the Presentation Schedule and the Assignments File. The work you submit to your tutor for assessment will count for 30 % of your total course mark.

At the end of the course, you will need to sit for a final written examination of three hours' duration. This examination will also count for 70% of your total course mark.

Tutor-Marked Assignments (TMAs)

There are four tutor-marked assignments in this course. You will submit all the assignments. You are encouraged to work all the questions thoroughly. The TMAs constitute 30% of the total score.

Assignment questions for the units in this course are contained in the Assignment File. You will be able to complete your assignments from the information and materials contained in your set books, reading and study units. However, it is desirable that you demonstrate that you have read and researched more widely than the required minimum. You should use other references to have a broad viewpoint of the subject and also to give you a deeper understanding of the subject.

When you have completed each assignment, send it, together with a TMA form, to your tutor. Make sure that each assignment reaches your tutor on or before the deadline given in the Presentation File. If for any reason, you cannot complete your work on time, contact your tutor before the assignment is due to discuss the possibility of an extension. Extensions will not be granted after the due date unless there are exceptional circumstances.

Final Examination and Grading

The final examination will be of three hours' duration and have a value of 70% of the total course grade. The examination will consist of questions which reflect the types of self-assessment practice exercises and tutor-marked problems you have previously encountered. All areas of the course will be assessed

Revise the entire course material using the time between finishing the last unit in the module and that of sitting for the final examination to. You might find it useful to review your self-assessment exercises, tutor-marked assignments and comments on them before the examination. The final examination covers information from all parts of the course.

Course Marking Scheme

The Table presented below indicates the total marks (100%) allocation.

Assignment	**Marks**
Assignments (Three assignments that is marked)	30%
Final Examination	70%
Total	**100%**

Course Overview

The Table presented below indicates the units, number of weeks and assignments to be taken by you to successfully complete the course, Quantitative Analysis (BUS800).

Units	**Title of Work**	**Week's Activities**	**Assessment (end of unit)**
	Course Guide		
Module 1			
1	Sets and Subset	Week 1	Assignment 1
2	Basic Set Operations	Week 2	Assignment 1
3	Set of Numbers	Week 3	Assignment 1
4	Probability Theory and Applications Contents	Week 4	Assignment 1
5	Elements of Decision Analysis	Week 5	Assignment 1
Module 2			
1	Types of Decision Situations	Week 6	Assignment 2
2	Decision Trees	Week 7	Assignment 2
3	Operations Research (OR)	Week 8	Assignment 2
4	Modelling in Operations Research	Week 9	Assignment 2
5	The Transportation Model	Week 10	Assignment 2
Module 3 Need for Organizational Structure			
1	Simulation	Week 11	Assignment 3
2	System Analysis	Week 12	Assignment 3
3	Sequencing and Scheduling	Week 13	Assignment 3
4	Games Theory	Week 14	Assignment 3
5	Inventory Control	Week 15	Assignment 3
	Total	**15 Weeks**	

How to Get the Most from This Course

In distance learning the study units replace the university lecturer. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace and at a time and place that suit you best.

Think of it as reading the lecture instead of listening to a lecturer. In the same way that a lecturer might set you some reading to do, the study units tell you when to read your books or other material, and when to embark on discussion with your colleagues. Just as a lecturer might give you an in-class exercise, your study units provides exercises for you to do at appropriate points.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit.

You should use these objectives to guide your study. When you have finished the unit you must go back and check whether you have achieved the objectives. If you make a habit of doing this you will significantly improve your chances of passing the course and getting the best grade.

The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a readings section. Some units require you to undertake practical overview of historical events. You will be directed when you need to embark on discussion and guided through the tasks you must do.

The purpose of the practical overview of some certain historical economic issues are in twofold. First, it will enhance your understanding of the material in the unit. Second, it will give you practical experience and skills to evaluate economic arguments, and understand the roles of history in guiding current economic policies and debates outside your studies. In any event, most of the critical thinking skills you will develop during studying are applicable in normal working practice, so it is important that you encounter them during your studies.

Self-assessments are interspersed throughout the units, and answers are given at the ends of the units. Working through these tests will help you to achieve the objectives of the unit and prepare you for the assignments and the examination. You should do each self-assessment exercises as you come to it in the study unit. Also, ensure to master some major historical dates and events during the course of studying the material.

The following is a practical strategy for working through the course. If you run into any trouble, consult your tutor. Remember that your tutor's job is to help you. When you need help, don't hesitate to call and ask your tutor to provide it.

1. Read this Course Guide thoroughly.
2. Organize a study schedule. Refer to the `Course overview' for more details. Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the semester is available from study centre. You need to gather together all this information in one place, such as your dairy or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates for working breach unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the `Overview' at the beginning of each unit. You will also need both the study unit you are working on and one of your set books on your desk at the same time.
6. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit you will be instructed to read sections from your set books or other articles. Use the unit to guide your reading.
7. Up-to-date course information will be continuously delivered to you at the study centre.
8. Work before the relevant due date (about 4 weeks before due dates), get the Assignment File for the next required assignment. Keep in mind that you will learn a lot by doing the assignments carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the exam. Submit all assignments no later than the due date.
9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study material or consult your tutor.
10. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.
11. When you have submitted an assignment to your tutor for marking do not wait for it return `before starting on the next units. Keep to

your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also written on the assignment. Consult your tutor as soon as possible if you have any questions or problems.

12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this Course Guide).

Tutors and Tutorials

There are some hours of tutorials (2-hours sessions) provided in support of this course. You will be notified of the dates, times and location of these tutorials. Together with the name and phone number of your tutor, as soon as you are allocated a tutorial group.

Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter, and provide assistance to you during the course. You must mail your tutor-marked assignments to your tutor well before the due date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail, or discussion board if you need help. The following might be circumstances in which you would find help necessary. Contact your tutor if.

- You do not understand any part of the study units or the assigned readings
- You have difficulty with the self-assessment exercises
- You have a question or problem with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should try your best to attend the tutorials. This is the only chance to have face to face contact with your tutor and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating in discussions actively.

Summary

On successful completion of the course, you would have developed critical thinking skills with the material necessary for efficient and effective discussion on macroeconomic issues: national income analysis, monetary issue, government expenditure and macroeconomics in open economy. However, to gain a lot from the course please try to apply anything you learn in the course to term papers writing in other economic development courses. We wish you success with the course and hope that you will find it fascinating and handy.

MAIN COURSE

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Unit 1 Sets and Subsets

Unit Structure

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    1.3.2 Types of Sets
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1.5 Axiomatic development of set theory
1.6 Summary
1.7 References/Further Readings/Web Resources
1.8 Possible Answers to Self-Assessment Exercise(s)

1.1 Introduction

This first unit provides an understanding of the Set Theory. It is the branch of mathematical logic that studies sets, which can be informally described as collections of objects. Although objects of any kind can be collected into a set, set theory, as a branch of mathematics, is mostly concerned with those that are relevant to mathematics as a whole.

The modern study of set theory was initiated by the German mathematicians Richard Dedekind and Georg Cantor in the 1870s. In particular, Georg Cantor is commonly considered the founder of set theory. The non-formalized systems investigated during this early stage go under the name of naive set theory. After the discovery of paradoxes within naive set theory (such as Russell's paradox, Cantor's paradox and the Burali-Forti paradox), various axiomatic systems were proposed in the early twentieth century, of which Zermelo–Fraenkel set theory (with or without the axiom of choice) is still the best-known and most studied.

Set theory is commonly employed as a foundational system for the whole of mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Besides its foundational role, set theory also provides the framework to develop a mathematical theory of infinity, and has various applications in computer science (such as in the theory of relational algebra), philosophy and formal semantics. Its foundational appeal, together with its paradoxes, its implications for the concept of infinity and its multiple applications, have made set theory an area of major interest for logicians and philosophers of mathematics. Contemporary research into set theory covers a vast array of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

The theory of sets lies at the foundation of mathematics. It is a concept that rears its head in almost all fields of mathematics; pure and applied. This unit aims at introducing basic concepts that would be explained further in subsequent units. There will be definition of terms and lots of examples and exercises to help you as you go along. You can click on the link below to watch video on set theory.
https://www.youtube.com/watch?v=tyDKR4FG3Yw

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Discuss what Set is all about
- Demonstrate Set Notation
- Evaluate Types of Sets

1.3 Definition of Set

Set theory begins with a fundamental binary relation between an object o and a set A. If o is a member (or element) of A, the notation o ∈ A is used. A set is described by listing elements separated by commas, or by a characterizing property of its elements, within braces { }. Since sets are objects, the membership relation can relate sets as well.

A derived binary relation between two sets is the subset relation, also called set inclusion. If all the members of set A are also members of set B, then A is a subset of B, denoted A ⊆ B. For example, {1, 2} is a subset of {1, 2, 3}, and so is {2} but {1, 4} is not. As implied by this definition, a set is a subset of itself. For cases where this possibility is unsuitable or would make sense to be rejected, the term proper subset is defined. A is called a proper subset of B if and only if A is a subset of B, but A is not equal to B. Also, 1, 2, and 3 are members (elements) of the set {1, 2, 3}, but are not subsets of it; and in turn, the subsets, such as {1}, are not members of the set {1, 2, 3}.

A set is a collection of objects. The objects in a set are called its elements or members. The elements in a set can be any types of objects, including sets! The members of a set do not even have to be of the same type. For example, although it may not have any meaningful application, a set can consist of numbers and names.

Well, simply put, it's a collection. First, we specify a common property among "things" (we define this word later) and then we gather up all the "things" that have this common property.

For example, the items you wear: hat, shirt, jacket, pants, and so on, as you can see in figure 1.

Figure 1: Wears

Or another example is **types of fingers**.
This set includes index, middle, ring, and pinky. This is indicated in figure 2 below.

Figure 2: Types of Fingers

So, it is just things grouped together with a certain property in common. We usually use capital letters such as A, B, C, S, and T to represent sets, and denote their generic elements by their corresponding lowercase letters a, b, c, s, and t, respectively. To indicate that b is an element of the set B, we adopt the notation $b \in B$, which means "b belongs to B" or "b is an element of B.

Self-Assessment Exercise 1

What are set of even numbers less than 8?

1.3.1 What is Set Notation?

Set notations are the basic symbols used to denote the various representations across set operations. Set notation is used to denote any working within and across the sets. All the symbols except the number elements can be easily considered as the notations for sets. The simplest set notation is the Curley brackets, which are used to enclose and represent the elements of the set. The elements of a set are written using flower brackets { }, or by using parenthesis ( ).

The elements of a set are written and separated by commas. For example, set A containing the five vowels of the English alphabets is written as A = {a, e, i, o, u}. The sets are denoted by capital letters and the elements of the set are denoted by small letters.

Set notation is further used to represent various sets and operations. Further, it is possible to represent the various relations and functions across sets, only with the help of set notation.

Broadly the set notations can be classified for set representations and for set operations. Let us now study in detail, the set notation for set representation, and the set formation for set operations.

Self-Assessment Exercise 2

If A={4,9} and B={n2:n=2 or n=3}

1.3.2 Types of Sets

We know that a set is a well–defined collection of objects. There are different types of sets depending on the objects and their characteristics. Some of these are explained below. Let us understand the various types of sets along with illustrations.

Types of sets are classified according to the number of elements they have. Sets are the collection of elements of the same type. For example, a set of prime numbers, natural numbers, etc. There are various types of sets such as unit sets, finite and infinite sets, null sets, equal and unequal sets, etc. Let us learn more about the various forms of sets in detail.

Self-Assessment Exercise 3

What are the basic types of sets?

i. Singleton Sets or Unit Sets

A set that has only one element is called a singleton set. It is also known as a unit set because it has only one element. Example1: Set A = { k | k is an integer between 5 and 7} which is A = {6}.

Example 2: Set A = {8} is a singleton set.

ii. Finite Sets

As the name implies, a set with a finite or exact countable number of elements is called a finite set. If the set is non-empty, it is called a non-empty finite set. Some examples of finite sets are: For example, Set B = {k | k is an even number less than 20}, which is B = {2,4,6,8,10,12,14,16,18}. Let us consider one more illustration, Set A = {x : x is a day in a week}; Set A will have 7 elements.

iii. Infinite Sets

A set with an infinite number of elements is called an infinite set. In other words, if a given set is not finite, then it will be an infinite set. For example, A = {x : x is a real number}; there are infinite real numbers. Hence, here A is an infinite set. Let us consider one more example, Set B = {z: z is the coordinate of a point on a straight line}; there are infinite points on a straight line. So, here B is an example of an infinite set. Another example could be Set C = {Multiples of 3}. Here we can have infinite multiples of 3. The illustration of both sets is as shown in figure 3.

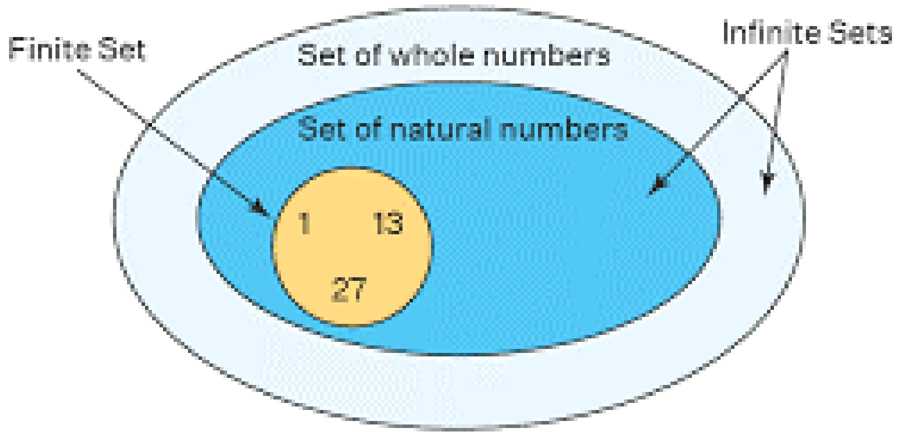

Figure 3: Finite and Infinite Sets
Source: https://www.cuemath.com/algebra/finite-and-infinite-sets/

iv. Empty or Null Sets

A set that does not contain any element is called an empty set or a null set. An empty set is denoted using the symbol '∅'. It is read as 'phi'. Example: Set X = {}.

v. Equal Sets

Equal sets are sets in set theory in which the number of elements is the same and all elements are equal. It is a concept of set equality. Before getting into the detail of the concept of equal sets, let us recall the meaning of sets. A set is a well-defined collection of objects such as letters, numbers, people, shapes, etc.

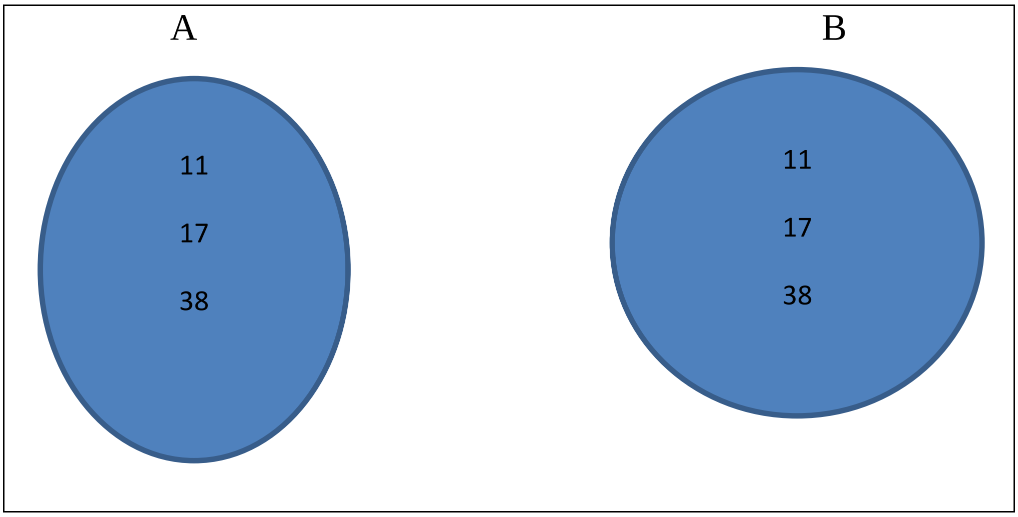

Figure 4: Example of Equal Sets

If two sets have the same elements in them, then they are called equal sets. Example: A = {11,17,38} and B = {11,17,38}. Here, set A and set B are equal sets. This can be represented as A = B.

Example: Check if the sets A = {a, e, i, o, u} and B = {e, i, a, o, u} are equal sets or unequal sets.
Solution: We know that the order of the elements does not impact the equality of the two sets.

Therefore, set B can be written as B = {a, e, i, o, u} after rearranging the elements of B.
Hence, A = {a, e, i, o, u} = B
Sets A and B are equal sets.

vi. Unequal Sets

If two sets have at least one element that is different, then they are unequal sets.Example: X = {4, 5, 6} and Y = {2,3,4}. Here, set X and set Y are unequal sets. This can be represented as X ≠ Y.

vii. Equivalent Sets

Two sets are said to be equivalent sets when they have the same number of elements, though the elements are different. Example: A = {7, 8, 9, 10} and B = {a,b,c,d}. Here, set A and set B are equivalent sets since n(A) = n(B)

Self-Assessment Exercise 4

Difference between Equal and Equivalent Sets

viii. Overlapping Sets

Two sets are said to be overlapping if at least one element from set A is present in set B. Example: A = {4,5,6} B = {4,9,10}. Here, element 4 is present in set A as well as in set B. Therefore, A and B are overlapping sets.

ix. Disjoint Sets
Two sets are disjoint sets if there are no common elements in both sets. Example: A = {1,2,3,4} B = {7,8,9,10}. Here, set A and set B are disjoint sets.

x. Subset and Superset
For two sets A and B, if every element in set A is present in set B, then set A is a subset of set B($A \subseteq B$) and B is the superset of set A($B \supseteq A$).
Example: A = {1,2,3} B = {1,2,3,4,5,6}
$A \subseteq B$, since all the elements in set A are present in set B.
$B \supseteq A$ denotes that set B is the superset of set A.

xi. Universal Set
A universal set is the collection of all the elements in regard to a particular subject. The set notation used to represent a universal set is the letter 'U'. Example: Let U = {The list of all road transport vehicles}. Here, a set of cars is a subset for this universal set, the set of cycles, trains are all subsets of this universal set.

xii. Power Sets
Power set is the set of all subsets that a set could contain. Example: Set A = {1,2,3}. Power set of A is = {{∅}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3}}.

Self-Assessment Exercise 5

What is a Power set? Define with example. A = {1, 3, 5}

xiii. Union of Sets
Union of sets is one of the set operations that is used in set theory. In addition to the union of sets, the other set operations are difference and intersection. All the set operations are represented by using a unique operator. The union of sets is analogous to arithmetic addition. The union of two given sets is the set that contains all the elements present in both sets. The symbol for the union of sets is "∪". For any two sets A and B, the union, $A \cup B$ (read as A union B) lists all the elements of set A as well as set B. Thus, for two given sets, Set A = {1,2,3,4,5} and Set B = {3,4,6,8}, $A \cup B$ = {1,2,3,4,5,6,8}

Notation of Union of Sets
We use a unique mathematical notation to represent each set operation. The mathematical notation that is used to represent the union between two sets is '∪'. This operator is called infix notation and it is surrounded by the operands.

Consider two sets P and Q, where P = {2,5,7,8} and Q = {1,4,5,7,9}. P ∪ Q = {1,2,4,5,7,8,9}.

Self-Assessment Exercise 6

Find equal set of A = {1, 2, 3, 4, 5} and B = {1, 2, 3, 4, 5}.

1.4 Venn vs Euler Diagram

What is a Venn diagram?
In the simplest terms, a Venn diagram illustrates the logical relationship between two or more sets of items. It visually represents the differences and similarities between the two concepts.

What is an Euler diagram?
An Euler diagram is another diagram that represents sets and their relationships. It's similar to a Venn diagram as both use circles to create the diagram. However, while a Venn diagram represents an entire set, an Euler diagram represents a part of a set. A Venn diagram shows an empty set by shading it out, whereas in an Euler diagram that area could simply be missing altogether.

What are the Differences between a Venn Diagram and Euler Diagram?
Both sets of diagrams are based on the set theory. A Venn diagram shows all possible logical relationships between a collection of sets. But an Euler diagram only shows relationships that exist in the real world.

While Venn diagrams and Euler diagrams are both tools used to represent sets and their relationships, there are some important differences between the two.

Table 1.1: Differences between a Venn Diagram and Euler Diagram

Feature	Venn	Euler
Overlap	The overlap between two or more sets is represented by a shared region in the diagram.	Can use overlapping or nested shapes to represent the relationships between sets, and may not have any shared regions at all.
Completeness	Designed to be complete, meaning that they show all possible relationships between sets.	Can be partial or incomplete, meaning that they may not show all possible relationships between sets.

Expressiveness	Limited in their expressiveness, as they can only represent set relationships in terms of union, intersection, and difference.	Can use overlapping and nested shapes to represent a wider range of relationships and dependencies between sets.
Complexity	Can become complex and difficult to read when representing relationships between more than three sets.	Can be designed to handle more complex relationships and can be easier to interpret in these cases.

In general, Venn diagrams are ideal to visualize simple set relationships and can be useful for teaching basic concepts in set theory. However, Euler diagrams are more flexible and versatile, therefore they can be used to visualize a wider range of relationships between sets. This makes them useful for more complex situations and applications.

Venn Diagrams vs Euler Diagrams Examples

Let us start with a very simple example. Let us consider Animals superset with mammals and birds as subsets. A Venn diagram shows an intersection between the two sets even though that possibility does not exist in the real world. Euler diagram, on the other hand, does not show an intersection. These examples are shown in the Figure 1.1 below.

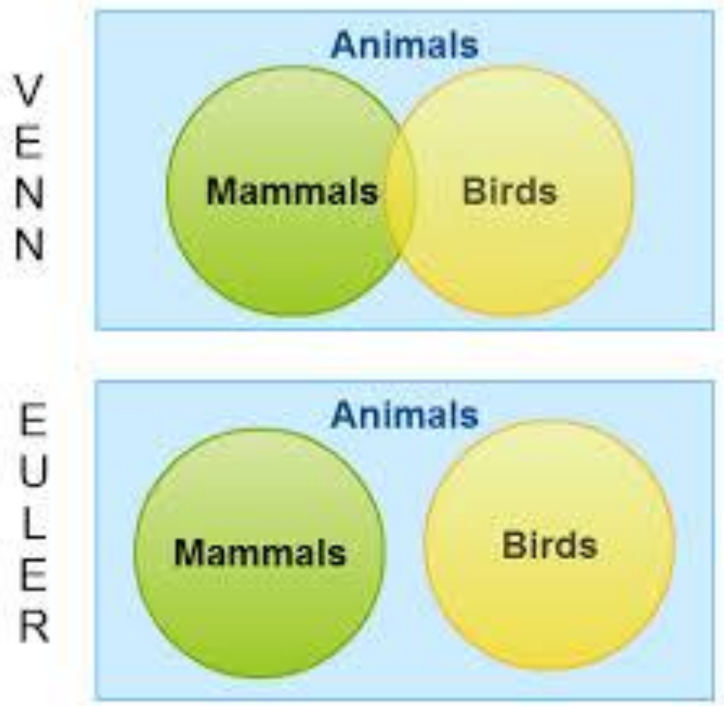

Figure 5: Venn versus Euler Diagram

Source:https://medium.com/thousand-words-by-creately/an-introduction-to-venn-diagrams-and-euler-diagrams-b24d569a2322

1.5 Axiomatic development of set theory

The axioms for set theory now most often studied and used, although put in their final form by Skolem, are called the Zermelo-Fraenkel set theory (ZF). Actually, this term usually excludes the axiom of choice, which was once more controversial than it is today. When this axiom is included, the resulting system is called ZFC.

An important feature of ZFC is that every object that it deals with is a set. In particular, every element of a set is itself a set. Other familiar mathematical objects, such as numbers, must be subsequently defined in terms of sets.

The ten axioms of ZFC are listed below. (Strictly speaking, the axioms of ZFC are just strings of logical symbols. What follows should therefore be viewed only as an attempt to express the intended meaning of these axioms in English. Moreover, the axiom of separation, along with the axiom of replacement, is actually an infinite schema of axioms, one for each formula.) Each axiom has further information in its own article.

i. Axiom of extensionality: Two sets are the same if and only if they have the same elements.
ii. Axiom of empty set: There is a set with no elements. We will use {} to denote this empty set.
iii. Axiom of pairing: If x, y are sets, then so is {x,y}, a set containing x and y as its only elements.
iv. Axiom of union: Every set has a union. That is, for any set x there is a set y whose elements are precisely the elements of the elements of x.
v. Axiom of infinity: There exists a set x such that {} is in x and whenever y is in x, so is the union y U {y}.
vi. Axiom of separation (or subset axiom): Given any set and any proposition P(x), there is a subset of the original set containing precisely those elements x for which P(x) holds.
vii. Axiom of replacement: Given any set and any mapping, formally defined as a proposition P(x,y) where P(x,y) and P(x,z) implies y = z, there is a set containing precisely the images of the original set's elements.
viii. Axiom of power set: Every set has a power set. That is, for any set x there exists a set y, such that the elements of y are precisely the subsets of x.
ix. Axiom of regularity (or axiom of foundation): Every non-empty set x contains some element y such that x and y are disjoint sets.
x. Axiom of choice: (Zermelo's version) Given a set x of mutually disjoint nonempty sets, there is a set y (a choice set for x) containing exactly one element from each member of x.

The axioms of choice and regularity are still controversial today among a minority of mathematicians. Other axiom systems for set theory are Von Neumann-Bernays-Gödel set theory (NBG), the Kripke-Platek set theory (KP), Kripke-Platek set theory with urelements (KPU) and Morse-Kelley set theory.

1.6 Summary

A summary of the basic concept of set theory is as follows:
A set is any well-defined list, collection, or class of objects.
Given a set A with elements 1,3,5,7 the tabular form of representing this set is $A = \{1, 3, 5, 7\}$.

The set-builder form of the same set is $A = \{x| \ x = 2n + 1, 0 \leq n \leq 3\}$
Given the set $N = \{2,4,6,8,....\}$ then N is said to be infinite, since the counting process of its elements will never come to an end, otherwise it is finite

Two sets of A and B are said to be equal if they both have the same elements, written $A = B$
The null set, $\emptyset$, contains no elements and is a subset of every set
The set A is a subset of another set B, written $A \subset B$, if every element of A is also an element of B, i.e. for every $x \in A$ then $x \in B$
If $B \subset A$ and $B \neq A$, then B is a proper subset of A
Two sets A and B are comparable if $A \subset B$ and $B \subset A$

1.7 References/Further Readings/Web Resources

Seymour, L.S.(1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.
Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable, Vol. 1).
https://www.youtube.com/watch?v=tyDKR4FG3Yw
https://medium.com/thousand-words-by-creately/an-introduction-to-venn-diagrams-and-euler-diagrams-b24d569a2322
https://www.cuemath.com/algebra/finite-and-infinite-sets/

1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1
8= {2,4,6}

Answer to SAE 2
A=B. A⊆B if and only if every element of A is also an element of B.

Answer to SAE 3

- Empty Sets
- Singleton Sets.
- Finite and Infinite Sets.
- Equal Sets.
- Subsets.
- Power Sets.
- Universal Sets.
- Disjoint Sets.

Answer to SAE 4

Equal Sets	Equivalent Sets
If all elements are equal in two or more sets, then they are equal.	If the number of elements is the same in two or more sets, then are equivalent.
Equal sets have the same cardinality	Equivalent sets have the same cardinality.
They have the same number of elements.	They have the same number of elements.
The symbol used to denote equal sets is '='	The symbol used to denote equivalent sets is ~ or ≡
All equal sets are equivalent sets.	Equivalent sets may or may not be equal.
Elements should be the same.	Elements need not be the same.

Answer to SAE 5
In set theory, the power set of a set A is defined as the set of all subsets of the Set A including the Set itself and the null or empty set.
Power set of A, P(A) = {{}, {1}, {3}, {5}, {1,3}, {3,5}, {1,5}, {1,3,5}}

Answer to SAE 6
Sets A and B are said to be equal sets as their elements are the same and they have the same cardinality.

Unit 2 Basic Set Operations

Unit Structure

1.1 Introduction

The set operations are performed on two or more sets to obtain a combination of elements as per the operation performed on them.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Discuss Set Operations
- Demonstrate Union of Sets (∪)?
- Evaluate Properties of Set Operations

1.3 Definitions of Set Operations

Set operations can be defined as the operations that are performed on two or more sets to obtain a single set containing a combination of elements from both all the sets being operated upon. There are basically three types of operation on sets in Mathematics; they are: The Union of Sets (∪) The Intersection of Sets (∩)

1.3.1 The Union of Sets (∪)?

In Operation set in math, the union of sets can be described as the set that contains all the elements of all the sets on which the union operation is applied. The union of sets can be denoted by the symbol ∪. The set formed by the union of P and Q will contain all the elements of set P and set Q together. The union of sets can be interpreted as:

$P \cup Q = n(P) + n(Q)$
Where n(P) represents, the cardinal number of set P and
n(Q) represents the cardinal number of set Q
Let's take an example:
Set P- {1, 2, 3, 4, 5} and Set Q- {7, 8, 9, 10}
Therefore, $P \cup Q = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$

Self-Assessment Exercise 1

If set A = {1,2,3,4} and B {6,7}

1.3.2 The Intersection of Sets (∩)

The intersection of sets is referred to as a set containing the elements that are common to all the sets being operated upon. It is denoted by the symbol ∩. The set that is formed by the intersection of both the sets will contain all the elements that are common in set P as well as Set Q.
Let's take an example:

Set P= {1, 2, 3, 4} and Set Q= {3, 4, 5, 6}
Then, $P \cap Q = \{3, 4$
Example
$A \cap B = \{x : x \in A \text{ and } x \in B\}$
Where x is the common element of both sets A and B.
The intersection of sets A and B, can also be interpreted as:
$A \cap B = n(A) + n(B) – n(A \cup B)$
Where,
n(A) = cardinal number of set A,
n(B) = cardinal number of set B,
$n(A \cup B)$ = cardinal number of union of set A and B.

Self-Assessment Exercise 2

Let A = {1,2,3} and B = {3,4,5}

1.3.3 The Difference between Sets (-)

The difference of two sets refers to a set that contains the elements of one set that are not present in the other set. It is denoted by the symbol -. Let us say there are two sets P and Q, then the difference between P and Q can be represented as P - Q.
Let's take an example:
Set P= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and Set Q= {8, 9, 10}
Therefore, P - Q= {1, 2, 3,4, 5, 6, 7}, here we can see that P - Q contains all those elements that are present in P but not in Q.

Self-Assessment Exercise 3

If A = {1,2,3,4,5,6,7} and B = {6,7} are two sets.

1.3.4 Complement of Set

If U is a universal set and X is any subset of U then the complement of X is the set of all elements of the set U apart from the elements of X.

$X' = \{a : a \in U \text{ and } a \notin A\}$

Self-Assessment Exercise 4

U = {1, 2, 3, 4, 5, 6, 7, 8}
A = {1, 2, 5, 6}

1.4 Properties of Set Operations

There are certain properties of set operations; these properties are used for set operations proofs. The properties are as follows:

i. Distributive Property states that:
If there are three sets P, Q and R then,

$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

$P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$

ii. The Commutative Property states that:
If there are two sets P and Q then,

$P \cup Q = Q \cup P$

$P \cap Q = Q \cap P$

iii. Associative Property states that:

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

1.5 Summary

The basic set operations are Union, Intersection, Difference and Complement defined as:

The Union of sets A and B, denoted by AUB, is the set of all elements, which belong to A or to B or to both.

The intersection of sets A and B, denoted by $A \cap B$, is the set of elements, which are common to A and B. If A and B are disjoint then their intersection is the Null set $\emptyset$

The difference of sets A and B, denoted by A –B, is the set of elements which belong to A but which do not belong to B.

The complement of a set A, denoted by A‘, is the set of elements, which do not belong to A, that is, the difference of the universal set U and A.

1.6 References/Further Readings/Web Resources

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp.
1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real – valued functions of a real variable), Vol. 1

1.7 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Union of sets, $A \cup B = \{1,2,3,4,6,7\}$

Answer to SAE 2

Then, $A \cap B = \{3\}$; because 3 is common to both the sets.

Answer to SAE 3

The difference of set A and set B is given by;

$A – B = \{1, 2, 3, 4, 5\}$

We can also say, that the difference of set A and set B is equal to the intersection of set A with the complement of set B. Hence,

$A-B=A\cap B'$

Answer to SAE 4

Complement of A will be; $A' = \{3, 4, 7, 8\}$

Unit 3 Set of Numbers

Unit Structure

1.1 Introduction
1.2 Learning Outcomes
1.3 Define Set of Numbers
 1.3.1 Natural Numbers and Whole Numbers
 1.3.2 Integers
 1.3.3 Rational, Irrational, and Real Numbers
1.4 Interval Notation and Set-builder Notation
1.5 Characteristics of Number Sets
1.6 Summary
1.7 References/Further Readings/Web Resources
1.8 Possible Answers to Self-Assessment Exercise(s)

1.1 Introduction

Although, thetheory of sets is verygeneral, important sets, which we meet in elementary mathematics, are sets of numbers. Of particular importance, especially in analysis, is the set of real numbers, which we denote by R In fact, we assume in this unit, unless otherwise stated, that the set of real numbers R is out universal set. We first review some elementary properties of real numbers before applying our elementary principles of set theory to sets of numbers. The set of real numbers and its properties is called the real number system.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Discuss Set of Numbers
- Demonstrate Natural Numbers and Whole Numbers
- Evaluate Characteristics of Number Sets

1.3 Title of the main section: Define Set of Numbers

One of the basic tools of higher mathematics is the concept of sets. A set of numbers is a collection of numbers, called elements. The set can be either a finite collection or an infinite collection of numbers. One way of denoting a set, called roster notation, is to use "{{"

and "}}", with the elements separated by commas; for instance, the set {2,31}{2,31} contains the elements 2 and 31.

In set notation, there is a symbol "∪" to represent "or," and we say we are taking the union of the two sets. For example, we can take the union of the sets{2,3,5}{2,3,5} and {4,5,6}{4,5,6}: {2,3,5}∪{4,6}= {2,3,4,5,6}{2,3,5}∪{4,6}={2,3,4,5,6}. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set.

1.3.1 Natural Numbers and Whole Numbers

When a set contains an infinite number of elements, it is impossible to list all of them, so we need ways to indicate which numbers are included. Some infinite sets are very well-known, and form the basis of our number system. These are the numbers we use to count objects in our world: 1,2,3,4, and so on. They are called the counting numbers, or natural numbers and they are so important that they are designated by the special symbol N. If we add zero to the counting numbers, we get the set of whole numbers.

- Counting Numbers: N={1,2,3,…}
- Whole Numbers: {0,1,2,3,…}

The notation "……" is called an ellipsis and means "and so on," or that the pattern continues endlessly. Note that all of the natural numbers are included in the set of whole numbers. We say that the natural numbers are a subset of the whole numbers.

1.3.2 Integers

A negative number is a number less than 0. The negative numbers are to the left of zero on the number line as shown in Figure 6.

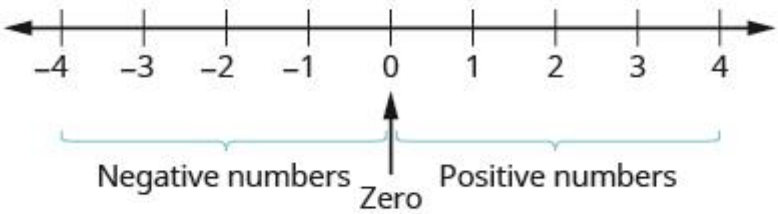

Figure 6: Location of Positive and Negative Numbers
Source: https://www.ipracticemath.com/learn/integer

The number line shows the location of positive and negative numbers. You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 22 and −2−2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 22 is −2−2, and the opposite of −2−2 is 22.

1.3.3 Rational, Irrational, and Real Numbers

We often see only the integers marked on the number line, which may cause us to forget (temporarily) that there are many numbers in between every pair of integers; in fact, there are infinite amounts of numbers in between every pair of integers! The next set we consider is the set of rational numbers, designated by Q. You have worked with rational numbers before, but we will give a careful definition of Q. (Using this definition, it can be seen that the set of integers is a subset of the rational numbers.)

A rational number is a number that can be expressed as a ratio of two integers (with the second integer not equal to zero). Hence, a rational number can be written as m/n for some integers m and n, where $n \neq 0$. The set of rational numbers is denoted as Q

Any decimal that terminates, or ends after a number of digits (such as 7.3 or −1.2684), can be written as a ratio of two integers, and thus is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction. For example, -1.2684 can be written as −12684/10000. Also, any rational number can be written in decimal form where the decimal terminates or begins to repeat its digits in the same pattern, infinitely. The decimal for 1/3 is the number 0.3. The bar over the 3 indicates that the number 3 repeats infinitely. The number(s) under the bar is called the repeating block and it repeats itself forever. .

Every rational number can be written both as a ratio of integers and as a decimal that terminates or begins to repeat.

It may seem strange to you, as it seemed not only strange, but impossible to early mathematicians 6,000 years or so ago; but it turns out that there are numbers on the number line that cannot be expressed as a ratio of two integers. These numbers are called irrational numbers. When we include the irrational numbers along with the rational numbers, we get the set of numbers called the real numbers, denoted R. Some famous irrational numbers that you may be familiar with are: π and $\sqrt{2}$

An irrational number is a real number that cannot be expressed as a ratio of two integers; i.e., is not rational.

The real numbers, denoted R, are the numbers corresponding to all the points on the number line. This is illustrated on Figure 7 below.

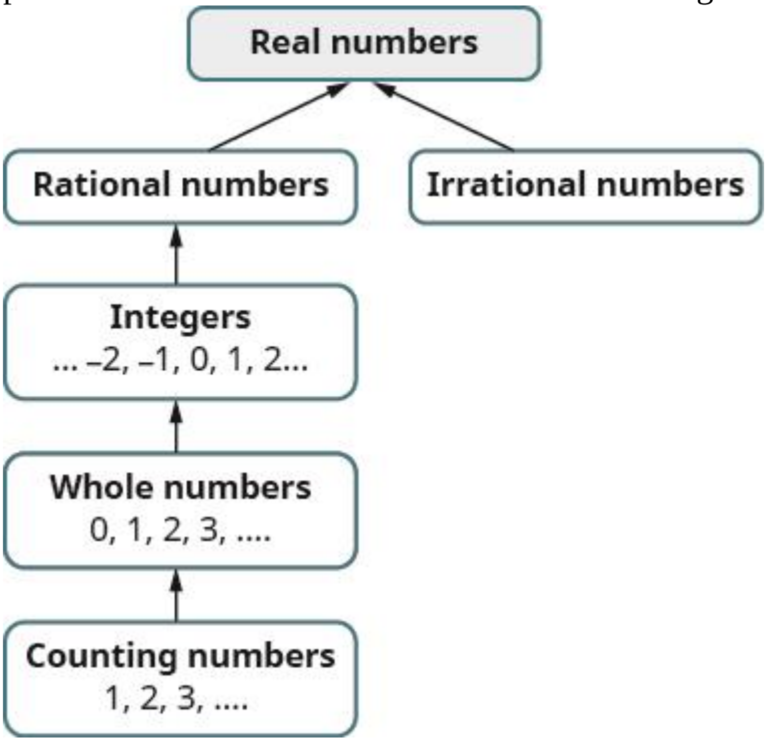

Figure 7: Number sets that make up the set of real numbers.
Source: https://www.kristakingmath.com/blog/real-number-sets

Self-Assessment Exercise 1

Given the set {−7, 14/5, 8, √5, 5.9, −√64, list the a) whole numbers b) integers c) rational numbers d) irrational numbers e) real numbers.

1.4 Interval Notation and Set-builder Notation

In calculus+, sets of real numbers that span a given interval on the number line are often important. A special notation called interval notation is often used, in which only the beginning number and end number of the interval are named, and it is understood that all numbers in between them are included in the set. The braces that are used for roster notation are not used in interval notation; rather, we use a square bracket "[" when the set includes the endpoint and a parenthesis "(" to indicate that the endpoint is either not included or the interval is unbounded. For example, we express the set of real numbers that are greater than 3 as $(3,\infty)$. There is no upper end to this set; the interval is unbounded. The symbol ∞ is read as "infinity." It is not an actual number.

Conventions for sets written using interval notation

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- All real numbers between the smallest and largest term are included in the set.
- Parentheses, "(" or ")," are used to signify that an endpoint is not included, called **exclusive**.
- Brackets, "[" or "]," are used to indicate that an endpoint is included, called inclusive.
- An interval that is unbounded at the upper end uses the symbol ∞ as its second term, and an interval that is unbounded at the lower end uses the symbol $-\infty$ as its first term.

1.5 Characteristics of Number Sets

Number sets are groups of numbers that have a series of structural properties. Outside of the main examples of number sets, they are arbitrarily named. For example, the set of all numbers that appear in the number Pi would be called exactly that.

If performing operations on two or more elements from defined set results in a number that belongs to the same set, the set is called a closed order set. An example of a well defined set is that of the first ten non-zero positive numbers of the integer numbers set: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. If addition is performed on elements 2 and 4, the result is 6, which belongs to the set. Thus, the set of the first ten positive non-zero numbers of the integer number set is a closed set.

The characterization of finite and infinite sets is defined by their order. The natural and integer number sets are examples of infinite sets. Thus, they are sets of infinite order. The set of the ten first numbers of the natural numbers, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, is a set of finite order. or more specifically, a set of order 10. This is because, unlike the whole set of natural numbers, the number of elements in it can be counted.

Number sets also have a commutative property. However, this is only true for operations were the different order of the elements yields the same result. The commutativity property is a+b = b +aa . b = b.a

Self-Assessment Exercise 2

Discuss Characteristics of Number Sets

1.6 Summary

In this unit, you have been introduced to the sets of numbers. The set of real numbers, Ᵹ, contains the set of integers, Z, Rational numbers, Q, Natural numbers, N, and Irrational numbers, Q‘. Intervals on the real line are open, closed, open-closed or closed-open depending on the nature of the endpoints.

1.7 References/Further Readings/Web Resources

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp.1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable), Vol. 1.

https://www.ipracticemath.com/learn/integer

https://www.kristakingmath.com/blog/real-number-sets

1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

a) Remember, the whole numbers are {0,1,2,3,…}so 8 is the only whole number given.

b) The integers are the whole numbers and their opposites (which includes 0). So the whole number 8 is an integer, and −7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. The integers are −7,8, and $-\sqrt{64}$.

c) Since all integers are rational, then −7, 8, and $-\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so 14/5 and 5.9 are rational. The list of rational numbers is −7, 14/5, 8, 5.9, and $-\sqrt{-64}$.

d) Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.

e) All the numbers listed are real numbers.

Answer to SAE 2

The foremost property of a set is that it can have elements, also called members. Two sets are equal when they have the same elements. More precisely, sets A and B are equal if every element of A is an element of B, and every element of B is an element of A; this property is called the extensionality of sets.

Unit 4 Probability Theory and Applications Contents

Unit Structure

1.1 Introduction

In this unit, we pay a special attention to the concept of probability theory, applications of simple probability experiments, principle of additivity, multinomial probability, conditional probability, applications of conditional probability, independence, bayes theorem, random variables, distributions, expectation, and variance. At the end of this lecture, students will be expected to be able to make effective decisions under uncertainties. The basic elements of probability theory are the outcomes of the process or phenomenon under study.

Each possible type of occurrence is referred to as an event. The collection of all the possible events is called the sample space. A compound or joint event is an event that has two or more characteristics. For example, the event of a student who is an economics major and B or above average‖ is a joint or compound event since the student must be an economics major and have a B or above average. The event black ace is also a compound event since the card must be both black and ace in order to qualify as a black ace.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Probability Theory
- Demonstrate Additivity Rule

1.3 What is Probability Theory?

Probability theory, a branch of mathematics concerned with the analysis of random phenomena. The outcome of a random event cannot be determined before it occurs, but it may be any one of several possible outcomes. The actual outcome is considered to be determined by chance.

The word probability has several meanings in ordinary conversation. Two of these are particularly important for the development and applications of the mathematical theory of probability. One is the interpretation of probabilities as relative frequencies, for which simple games involving coins, cards, dice, and roulette wheels provide examples. The distinctive feature of games of chance is that the outcome of a given trial cannot be predicted with certainty, although the collective results of a large number of trials display some regularity. For example, the statement that the probability of “heads” in tossing a coin equals one-half, according to the relative frequency interpretation, implies that in a large number of tosses the relative frequency with which “heads” actually occurs will be approximately one-half, although it contains no implication concerning the outcome of any given toss. There are many similar examples involving groups of people, molecules of a gas, genes, and so on. Actuarial statements about the life expectancy for persons of a certain age describe the collective experience of a large number of individuals but do not purport to say what will happen to any particular person. Similarly, predictions about the chance of a genetic disease occurring in a child of parents having a known genetic makeup are statements about relative frequencies of occurrence in a large number of cases but are not predictions about a given individual.

Self-Assessment Exercise 1

Define Probability Theory

1.3.1 Applications of simple probability experiments

The fundamental ingredient of probability theory is an experiment that can be repeated, at least hypothetically, under essentially identical conditions and that may lead to different outcomes on different trials. The set of all possible outcomes of an experiment is called a “sample space.” The experiment of tossing a coin once results in a sample space with two possible outcomes, “heads” and “tails.” Tossing two dice has a sample space with 36 possible outcomes, each of which can be identified with an ordered pair (i, j), where i and j assume one of the values 1, 2, 3, 4, 5, 6 and denote the faces showing on the individual dice. It is important to think of the dice as identifiable (say by a difference in colour), so that the outcome (1, 2) is different from (2, 1). An “event” is a well-defined subset of the sample space. For example, the event “the sum of the faces showing on the two dice equals six” consists of the five outcomes (1, 5), (2, 4), (3, 3), (4, 2), and (5, 1).

A third example is to draw n balls from an urn containing balls of various colours. A generic outcome to this experiment is an n-tuple, where the ith entry specifies the colour of the ball obtained on the ith draw (i = 1, 2,…, n). In spite of the simplicity of this experiment, a thorough understanding gives the theoretical basis for opinion polls and sample surveys. For example, individuals in a population favouring a particular candidate in an election may be identified with balls of a particular colour, those favouring a different candidate may be identified with a different colour, and so on. Probability theory provides the basis for learning about the contents of the urn from the sample of balls drawn from the urn; an application is to learn about the electoral preferences of a population on the basis of a sample drawn from that population.

1.3.2 Additivity Rule

The most simple method to determine the probability of a union of events is to count the sample points of the compound set (provided that the individual probabilities are equal). Sample points which belong to both sets of the compound event are counted only once. If we naively tried to calculate the probability of the union by simply adding the probabilities of the original sets A and B, we would find that the sum of the probabilities of the sets A and B is greater than the probability of the union of A and B. The difference is given by the probability of the intersection of A and B.

From these considerations the probability of the union of two events can be calculated as follows:

$P(A) + P(B) = P(A \cup B) + P(A \cap B)$,

So the probability of the union of two events A and B is given by:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example: We toss a die and define the event A as the set of even numbers, and the event B as the set of numbers less than 3. The probabilities of A, B and the intersection of A and B are: 1/2, 1/3, and 1/6, respectively. The probability of the union of A and B is then P(A ∪ B) = 1/2 + 1/3 - 1/6 = 2/3.

Mutually exclusive events:

Events A and B are mutually exclusive events, if A ∩ B contains no sample points, i.e. A and B have no sample points in common. In that case, the probability of the union of mutually exclusive events is just the sum of their probabilities.

P(A ∩ B) = P(A) + P(B)

Example: Two coins are tossed and we want to know the probability of getting at least one head. We can represent the event A (at least one head) as the union of the set B (exactly one head) and set C (exactly two heads). Since the events B and C are mutually exclusive, we can just add their probabilities to get the probability of event A:

P(A)=P(B ∪ C)=P(B)+P(C)

P(A) = 1/2 + 1/4 = ¾

Self-Assessment Exercise 2

There are 13 spades and 12 face cards, and of the 12 face cards, 3 are spades. Therefore, the number of cards that are either a spade or a face card or both is 13+9 = 22. That is, event E occurs when 1 of 22 cards is selected, the 22 cards being the 13 spade cards and the 9 face cards that are not spade. To find P(E), we use the Additive Rule of Probability.

1.3.3 Multinomial probability

A basic problem first solved by Jakob Bernoulli is to find the probability of obtaining exactly *i* red balls in the experiment of drawing *n* times at random with replacement from an urn containing *b* black and *r* red balls. To draw at random means that, on a single draw, each of the $r + b$ balls is equally likely to be drawn and, since each ball is replaced before the next draw, there are $(r + b) \times\cdots\times (r + b) = (r + b)^n$ possible outcomes to the experiment. Of these possible outcomes, the number that is favourable to obtaining *i* red balls and $n - i$ black balls in any one particular order is

$$\overbrace{r \times r \times \cdots \times r}^{i} \times \overbrace{b \times b \times \cdots \times b}^{n-i} = r^i \times b^{n-i}.$$

The number of possible orders in which *i* red balls and $n - i$ black balls can be drawn from the urn is the binomial coefficient

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}, \qquad (2)$$

where $k! = k \times (k - 1) \times\cdots\times 2 \times 1$ for positive integers *k*, and $0! = 1$. Hence, the probability in question, which equals the number of favourable outcomes divided by the number of possible outcomes, is given by the binomial distribution.

$$\binom{n}{i} \frac{r^i b^{n-i}}{(r+b)^n} = \binom{n}{i} p^i q^{n-i} \qquad (i = 0, 1, 2, \ldots, n), \qquad (3)$$

where $p = r/(r + b)$ and $q = b/(r + b) = 1 - p$.
For example, suppose $r = 2b$ and $n = 4$. According to equation (3), the probability of "exactly two red balls" is

$$\binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = 6 \times \frac{4}{81} = \frac{8}{27}.$$

In this case the

$$\binom{4}{2} = 6$$

possible outcomes are easily enumerated: (*rrbb*), (*rbrb*), (*brrb*), (*rbbr*), (*brbr*), (*bbrr*).

(For a derivation of equation (2), observe that in order to draw exactly i red balls in n draws one must either draw i red balls in the first $n - 1$ draws and a black ball on the nth draw or draw $i - 1$ red balls in the first $n - 1$ draws followed by the ith red ball on the nth draw. Hence,

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1},$$

from which equation (2) can be verified by induction on n.)
Two related examples are (i) drawing without replacement from an urn containing r red and b black balls and (ii) drawing with or without replacement from an urn containing balls of s different colours. If n balls are drawn without replacement from an urn containing r red and b black balls, the number of possible outcomes is

$$\binom{r+b}{n},$$

of which the number favourable to drawing i red and $n - i$ black balls is

$$\binom{r}{i}\binom{b}{n-1}.$$

Hence, the probability of drawing exactly i red balls in n draws is the ratio

$$\frac{\binom{r}{i}\binom{b}{n-i}}{\binom{r+b}{n}}.$$

If an urn contains balls of s different colours in the ratios $p_1:p_2:\ldots:p_s$, where $p_1 + \cdots + p_s = 1$ and if n balls are drawn with replacement, the probability of obtaining i_1 balls of the first colour, i_2 balls of the second colour, and so on is the multinomial probability

$$\frac{n!}{i_1!i_2!\cdots i_s!}p_1^{i_1}p_2^{i_2}\cdots p_s^{i_s}.$$

The evaluation of equation (3) with pencil and paper grows increasingly difficult with increasing n. It is even more difficult to evaluate related cumulative probabilities for example the probability of obtaining "at most j red balls" in the n draws, which can be expressed as the sum of equation (3) for $i = 0, 1, \ldots, j$.

Example: Three card players play a series of matches. The probability that player A will win any game is 20%, the probability that player B will win is 30%, and the probability player C will win is 50%. If they play 6 games, what is the probability that player A will win 1 game, player B will win 2 games, and player C will win 3?

$$\mathrm{P} = \frac{n!}{(n_1!)(n_2!)\ldots(n_x!)} P_1{}^{n1} P_2{}^{n2} \ldots P_x{}^{nx}$$

$$\Pr(A = 1, B = 2, C = 3) = \frac{6!}{1!2!3!}(0.2^1)(0.3^2)(0.5^3) = 0.135$$

1.3.4 Conditional probability

Suppose two balls are drawn sequentially without replacement from an urn containing r red and b black balls. The probability of getting a red ball on the first draw is $r/(r + b)$. If, however, one is told that a red ball was obtained on the first draw, the conditional probability of getting a red ball on the second draw is $(r - 1)/(r + b - 1)$, because for the second draw there are $r + b - 1$ balls in the urn, of which $r - 1$ are red. Similarly, if one is told that the first ball drawn is black, the conditional probability of getting red on the second draw is $r/(r + b - 1)$.

In a number of trials the relative frequency with which B occurs among those trials in which A occurs is just the frequency of occurrence of $A \cap B$ divided by the frequency of occurrence of A. This suggests that the conditional probability of B given A (denoted $P(B|A)$) should be defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}. \qquad (4)$$

If A denotes a red ball on the first draw and B a red ball on the second draw in the experiment of the preceding paragraph, then $P(A) = r/(r + b)$ and

$$P(A \cap B) = \frac{r(r-1)}{[(r+b)(r+b-1)]},$$

which is consistent with the “obvious” answer derived above.

Rewriting equation (4) as $P(A \cap B) = P(A)P(B|A)$ and adding to this expression the same expression with A replaced by A^c (“not A”) leads via equation (1) to the equality

$$\begin{aligned} P(B) &= P(A \cap B) + P(A^c \cap B) \\ &= P(A)P(B \mid A) + P(A^c)P(B \mid A^c). \end{aligned}$$

More generally, if $A_1, A_2, \ldots, A_n$ are mutually exclusive events and their union is the entire sample space, so that exactly one of the A_k must occur, essentially the same argument gives a fundamental relation, which is frequently called the law of total probability:

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n).$$

1.3.5 Applications of conditional probability

An application of the law of total probability to a problem originally posed by Christiaan Huygens is to find the probability of "gambler's ruin." Suppose two players, often called Peter and Paul, initially have x and $m - x$ dollars, respectively. A ball, which is red with probability p and black with probability $q = 1 - p$, is drawn from an urn. If a red ball is drawn, Paul must pay Peter one dollar, while Peter must pay Paul one dollar if the ball drawn is black. The ball is replaced, and the game continues until one of the players is ruined. It is quite difficult to determine the probability of Peter's ruin by a direct analysis of all possible cases. But let $Q(x)$ denote that probability as a function of Peter's initial fortune x and observe that after one draw the structure of the rest of the game is exactly as it was before the first draw, except that Peter's fortune is now either $x + 1$ or $x - 1$ according to the results of the first draw. The law of total probability with A = {red ball on first draw} and A^c = {black ball on first draw} shows that

$$Q(x) = pQ(x+1) + qQ(x-1). \qquad (5)$$

This equation holds for $x = 2, 3,\ldots, m - 2$. It also holds for $x = 1$ and $m - 1$ if one adds the boundary conditions $Q(0) = 1$ and $Q(m) = 0$, which say that if Peter has 0 dollars initially, his probability of ruin is 1, while if he has all m dollars, he is certain to win.

It can be verified by direct substitution that equation (5) together with the indicated boundary conditions are satisfied by

$$\begin{aligned} Q(x) &= \frac{\left(\frac{q}{p}\right)^x - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^m} \quad \left(p \neq \frac{1}{2}\right) \\ &= 1 - \frac{x}{m} \quad \left(p = \frac{1}{2}\right). \end{aligned} \qquad (6)$$

With some additional analysis it is possible to show that these give the only solutions and hence must be the desired probabilities.

Suppose $m = 10x$, so that Paul initially has nine times as much money as Peter. If $p = 1/2$, the probability of Peter's ruin is 0.9 regardless of the values of x and m. If $p = 0.51$, so that each trial slightly favours Peter, the situation is quite different. For $x = 1$ and $m = 10$, the probability of Peter's ruin is 0.88, only slightly less than before. However, for $x = 100$ and $m = 1,000$, Peter's slight advantage on each trial becomes so important that the probability of his ultimate ruin is now less than 0.02.

Generalizations of the problem of gambler's ruin play an important role in statistical sequential analysis, developed by the Hungarian-born American statistician Abraham Wald in response to the demand for more efficient methods of industrial quality control during World War II. They also enter into insurance risk theory, which is discussed in the section Stochastic processes: Insurance risk theory.

The following example shows that, even when it is given that A occurs, it is important in evaluating $P(B|A)$ to recognize that A^c might have occurred, and hence in principle it must be possible also to evaluate $P(B|A^c)$. By lot, two out of three prisoners Sam, Jean, and Chris are chosen to be executed. There are

$$\binom{3}{2} = 6$$

possible pairs of prisoners to be selected for execution, of which two contain Sam, so the probability that Sam is slated for execution is 2/3. Sam asks the guard which of the others is to be executed. Since at least one must be, it appears that the guard would give Sam no information by answering. After hearing that Jean is to be executed, Sam reasons that, since either he or Chris must be the other one, the conditional probability that he will be executed is 1/2. Thus, it appears that the guard has given Sam some information about his own fate. However, the experiment is incompletely defined, because it is not specified how the guard chooses whether to answer "Jean" or "Chris" in case both of them are to be executed. If the guard answers "Jean" with probability p, the conditional probability of the event "Sam will be executed" given "the guard says Jean will be executed" is

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{p}{3}} = \frac{1}{1+p}.$$

Only in the case $p = 1$ is Sam's reasoning correct. If $p = 1/2$, the guard in fact gives no information about Sam's fate.

1.3.6 Independence

One of the most important concepts in probability theory is that of "independence." The events A and B are said to be (stochastically) independent if $P(B|A) = P(B)$, or equivalently if

$$P(A \cap B) = P(A)P(B). \qquad (7)$$

The intuitive meaning of the definition in terms of conditional probabilities is that the probability of B is not changed by knowing that A has occurred. Equation (7) shows that the definition is symmetric in A and B.

It is intuitively clear that, in drawing two balls with replacement from an urn containing r red and b black balls, the event "red ball on the first draw" and the event "red ball on the second draw" are independent. (This statement presupposes that the balls are thoroughly mixed before each draw.) An analysis of the $(r + b)^2$ equally likely outcomes of the experiment shows that the formal definition is indeed satisfied.

In terms of the concept of independence, the experiment leading to the binomial distribution can be described as follows. On a single trial a particular event has probability *p*. An experiment consists of *n* independent repetitions of this trial. The probability that the particular event occurs exactly *i* times is given by equation (3).

Independence plays a central role in the law of large numbers, the central limit theorem, the Poisson distribution, and Brownian motion.

1.4 Bayes theorem

Consider now the defining relation for the conditional probability $P(A_n|B)$, where the A_i are mutually exclusive and their union is the entire sample space. Substitution of $P(A_n)P(B|A_n)$ in the numerator of equation (4) and substitution of the right-hand side of the law of total probability in the denominator yields a result known as Bayes's theorem (after the 18th-century English clergyman Thomas Bayes) or the law of inverse probability:

$$P(A_n \mid B) = \frac{P(A_n)P(B \mid A_n)}{\sum_i P(A_i)P(B \mid A_i)}.$$

As an example, suppose that two balls are drawn without replacement from an urn containing *r* red and *b* black balls. Let *A* be the event "red on the first draw" and *B* the event "red on the second draw." From the obvious relations $P(A) = r/(r + b) = 1 - P(A^c)$, $P(B|A) = (r - 1)/(r + b - 1)$, $P(B|A^c) = r/(r + b - 1)$, and Bayes's theorem, it follows that the probability of a red ball on the first draw given that the second one is known to be red equals $(r - 1)/(r + b - 1)$. A more interesting and important use of Bayes's theorem appears below in the discussion of subjective probabilities.

1.5 Random variables, distributions, expectation, and variance

Random variables

Usually it is more convenient to associate numerical values with the outcomes of an experiment than to work directly with a nonnumerical description such as "red ball on the first draw." For example, an outcome of the experiment of drawing *n* balls with replacement from an urn containing black and red balls is an *n*-tuple that tells us whether a red or a black ball was drawn on each of the draws. This *n*-tuple is conveniently represented by an *n*-tuple of ones and zeros, where the appearance of a one in the *k*th position indicates that a red ball was drawn on the *k*th draw. A quantity of particular interest is the number of red balls drawn, which is just the sum of the entries in this numerical description of the experimental outcome. Mathematically a rule that associates with every element of a given set a unique real number is called a "(real-valued) function." In the history of statistics and probability, real-valued functions defined on a sample space have traditionally been called "random variables." Thus, if a sample space S has the generic element e, the outcome of an experiment, then a random variable is a real-valued function $X = X(e)$. Customarily one omits the argument e in the notation for a random variable. For the experiment of drawing balls from an urn containing black and red balls, R, the number of red balls drawn, is a random variable. A particularly useful random variable is $1[A]$, the indicator variable of the event A, which equals 1 if A occurs and 0 otherwise. A "constant" is a trivial random variable that always takes the same value regardless of the outcome of the experiment.

Probability distribution

Suppose X is a random variable that can assume one of the values $x_1, x_2, \ldots, x_m$, according to the outcome of a random experiment, and consider the event $\{X = x_i\}$, which is a shorthand notation for the set of all experimental outcomes e such that $X(e) = x_i$. The probability of this event, $P\{X = x_i\}$, is itself a function of x_i, called the probability distribution function of X. Thus, the distribution of the random variable R defined in the preceding section is the function of $i = 0, 1, \ldots, n$ given in the binomial equation. Introducing the notation $f(x_i) = P\{X = x_i\}$, one sees from the basic properties of probabilities that

$$f(x_i) \geq 0 \quad \text{for all } i, \qquad \sum_i f(x_i) = 1,$$

And

$$P\{a < X \leq b\} = \sum_{a < x_i \leq b} f(x_i),$$

for any real numbers a and b. If Y is a second random variable defined on the same sample space as X and taking the values $y_1, y_2, \ldots, y_n$, the function of two variables $h(x_i, y_j) = P\{X = x_i, Y = y_j\}$ is called the joint distribution of X and Y. Since $\{X = x_i\} = \cup_j\{X = x_i, Y = y_j\}$, and this union consists of disjoint events in the sample space,

$$f(x_i) = \sum_j h(x_i, y_j), \quad \text{for all } i. \qquad (8)$$

Often f is called the marginal distribution of X to emphasize its relation to the joint distribution of X and Y. Similarly, $g(y_j) = \Sigma_i h(x_i, y_j)$ is the (marginal) distribution of Y. The random variables X and Y are defined to be independent if the events $\{X = x_i\}$ and $\{Y = y_j\}$ are independent for all i and j—i.e., if $h(x_i, y_j) = f(x_i)g(y_j)$ for all i and j. The joint distribution of an arbitrary number of random variables is defined similarly.

Suppose two dice are thrown. Let X denote the sum of the numbers appearing on the two dice, and let Y denote the number of even numbers appearing. The possible values of X are 2, 3,…, 12, while the possible values of Y are 0, 1, 2. Since there are 36 possible outcomes for the two dice, the accompanying table giving the joint distribution $h(i, j)$ ($i = 2, 3, \ldots, 12$; $j = 0, 1, 2$) and the marginal distributions $f(i)$ and $g(j)$ is easily computed by direct enumeration.

For more complex experiments, determination of a complete probability distribution usually requires a combination of theoretical analysis and empirical experimentation and is often very difficult. Consequently, it is desirable to describe a distribution insofar as possible by a small number of parameters that are comparatively easy to evaluate and interpret. The most important are the mean and the variance. These are both defined in terms of the "expected value" of a random variable.

Expected Value

Given a random variable X with distribution f, the expected value of X, denoted $E(X)$, is defined by $E(X) = \Sigma_i x_i f(x_i)$. In words, the expected value of X is the sum of each of the possible values of X multiplied by the probability of obtaining that value. The expected value of X is also called the mean of the distribution f. The basic property of E is that of linearity: if X and Y are random variables and if a and b are constants, then $E(aX + bY) = aE(X) + bE(Y)$. To see why this is true, note that $aX + bY$ is itself a random variable, which assumes the values $ax_i + by_j$ with the probabilities $h(x_i, y_j)$. Hence,

$$E(aX + bY) = \sum_{i,j}(ax_i + by_j)h(x_i, y_j)$$
$$= a\sum_{i,j} x_i h(x_i, y_j) + b\sum_{i,j} y_j h(x_i, y_j).$$

f the first sum on the right-hand side is summed over j while holding i fixed, by equation (8) the result is

$$\sum_i x_i f(x_i),$$

which by definition is $E(X)$. Similarly, the second sum equals $E(Y)$.

If 1[A] denotes the "indicator variable" of A—i.e., a random variable equal to 1 if A occurs and equal to 0 otherwise—then $E\{1[A]\} = 1 \times P(A) + 0 \times P(A^c) = P(A)$. This shows that the concept of expectation includes that of probability as a special case.

As an illustration, consider the number R of red balls in n draws with replacement from an urn containing a proportion p of red balls. From the definition and the binomial distribution of R,

$$E(R) = \sum_i i \binom{n}{i} p^i q^{n-i},$$

which can be evaluated by algebraic manipulation and found to equal np. It is easier to use the representation $R = 1[A_1] + \cdots + 1[A_n]$, where A_k denotes the event "the kth draw results in a red ball." Since $E\{1[A_k]\} = p$ for all k, by linearity $E(R) = E\{1[A_1]\} + \cdots + E\{1[A_n]\} = np$. This argument illustrates the principle that one can often compute the expected value of a random variable without first computing its distribution. For another example, suppose n balls are dropped at random into n boxes. The number of empty boxes, Y, has the representation $Y = 1[B_1] + \cdots + 1[B_n]$, where B_k is the event that "the kth box is empty." Since the kth box is empty if and only if each of the n balls went into one of the other $n - 1$ boxes, $P(B_k) = [(n - 1)/n]^n$ for all k, and consequently $E(Y) = n(1 - 1/n)^n$. The exact distribution of Y is very complicated, especially if n is large.

Many probability distributions have small values of $f(x_i)$ associated with extreme (large or small) values of x_i and larger values of $f(x_i)$ for intermediate x_i. For example, both marginal distributions in the table are symmetrical about a midpoint that has relatively high probability, and the probability of other values decreases as one moves away from the midpoint. Insofar as a distribution $f(x_i)$ follows this kind of pattern, one can interpret the mean of f as a rough measure of location of the bulk of the probability distribution, because in the defining sum the values x_i associated with large values of $f(x_i)$ more or less define the centre of the distribution. In the extreme case, the expected value of a constant random variable is just that constant.

It is also of interest to know how closely packed about its mean value a distribution is. The most important measure of concentration is the variance, denoted by $\mathrm{Var}(X)$ and defined by $\mathrm{Var}(X) = E\{[X - E(X)]^2\}$. By linearity of expectations, one has equivalently $\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2$. The standard deviation of X is the square root of its variance. It has a more direct interpretation than the variance because it is in the same units as X. The variance of a constant random variable is 0. Also, if c is a constant, $\mathrm{Var}(cX) = c^2\mathrm{Var}(X)$.

There is no general formula for the expectation of a product of random variables. If the random variables X and Y are independent, $E(XY) = E(X)E(Y)$. This can be used to show that, if $X_1,\dots, X_n$ are independent random variables, the variance of the sum $X_1 +\cdots+ X_n$ is just the sum of the individual variances, $\mathrm{Var}(X_1) +\cdots+ \mathrm{Var}(X_n)$. If the Xs have the same distribution and are independent, the variance of the average $(X_1 +\cdots+ X_n)/n$ is $\mathrm{Var}(X_1)/n$. Equivalently, the standard deviation of $(X_1 +\cdots+ X_n)/n$ is the standard deviation of X_1 divided by Square root of$\sqrt{n}$. This quantifies the intuitive notion that the average of repeated observations is less variable than the individual observations. More precisely, it says that the variability of the average is inversely proportional to the square root of the number of observations. This result is tremendously important in problems of statistical inference. (*See* the section The law of large numbers, the central limit theorem, and the Poisson approximation.)

1.6 Summary

Probability is a concept that most people understand naturally, since such words as a chance, likelihood, possibilities and proportion are used as part of everyday speech. It is a term used in making decisions involving uncertainty. Though the concept is often viewed as very abstract and difficult to relate to real world activities, it remains the best tool for solving uncertainties problems.

To remove some of the abstract nature of probabilities, this unit has provided you with the simplest approach to understanding and calculating, as well as applying the probability concept.

1.7 References/Further Readings/Web Resources

Beyer, W. H. CRC Standard Mathematical Tables, 28th ed. Boca Raton, FL: CRC Press, p. 532, 1987.

Papoulis, A. Probability, Random Variables, and Stochastic Processes, 2nd ed. New York: McGraw-Hill, 1984.

1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Probability theory or probability calculus is the branch of mathematics concerned with probability. Although there are several different probability interpretations, probability theory treats the concept in a rigorous mathematical manner by expressing it through a set of axioms. Typically these axioms formalise probability in terms of a probability space, which assigns a measure taking values between 0 and 1, termed the probability measure, to a set of outcomes called the sample space. Any specified subset of the sample space is called an event.

Answer to SAE 2

$P(A \cup B) = P(A) + P(B) – P(A \cap B)$

$13/52 + 12/52 – 3/52$

$0.250 + 0.231 – 0.058$

0.423

42.3%

Unit 5 Elements of Decision Analysis

Unit Structure

1.1 Introduction

Business Decision Analysis takes its roots from Operations Research (OR). Operation Research as we will learn later is the application of scientific method by interdisciplinary teams to problems solving and the control of organized (Man Machine) systems so as to provide solution which best serve the purpose of the organization as a whole (Ackoff and Sisieni 1991). In other words, Operations Research makes use of scientific methods and tools to provide optimum or best solutions to problems in the organization. Organisations are usually faced with the problem of deciding what to do; how to do it, where to do it, for whom to do it etc. But before any action can be taken, it is important to properly analyse a situation with a view to finding out the various alternative courses of action that are available to an organization. Operations Research helps the organisation with the job of critically analysing a situation and finding out the various alternatives available to choose from. OR also helps the organization to identify the best alternative available there by enabling the enterprise to make the most rational decision after having identified and analysed all available alternatives.

In the light of the above, it could be said that Operations Research provides the scientific process, tools, techniques, and procedure for optimum decision in business analysis. In this chapter, we shall concern ourselves with those critical elements and tools that organisations utilise to make sound decisions.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Decision
- Discuss who is a Decision Maker
- Demonstrate Components of Decision Making

1.3 What is a Decision?

Decision originally comes from the Latin *decidere* ("determine"). You make decisions every day: what to wear, what to eat, how to spend your money, who to vote for, what movie to go to. A court judge makes a decision in a trial (and actually "hands down" or "announces" that decision). If judges make the decision in a boxing match, the winner wins "by decision." More loosely, *decision* can also refer to the outcome of any game or contest.

A decision can be defined as an action to be selected according to some pre-specified rule or strategy, out of several available alternatives, to facilitate a future course of action. This definition suggests that there are several alternative courses of action available, which cannot be pursued at the same time. Therefore, it is imperative to choose the best alternative base on some specified rule or strategy. Decision making is the process of selecting the best out of several alternatives.

Self-Assessment Exercise 1

Discuss what decision is all about

1.3.1 Who is a Decision Maker?

A decision maker is one who takes decision. It could be an individual or a group of individuals. It is expected that a good decision maker should be skilled in art of making decisions. He is also expected to apply all necessary procedure, rules and techniques in arriving at the best alternative which will either maximize wellbeing or profit, or minimize losses, or help to achieve any other objective or goal that has been set.

Self-Assessment Exercise 2

Who is a decision maker?

1.3.2 Decision Analysis

Decision Analysis (DA) is a systematic, quantitative, and visual approach to addressing and making optimal decisions under uncertain conditions. DA incorporates different elements, including the decision maker's values and judgments, uncertainty, trade-offs, and risk tolerance.

It aids in breaking down complex decisions into comprehensible components, enabling the decision-maker to understand the decision problem better.

The origins of decision analysis can be traced back to the mid-20th century when Ronald A. Howard, a pioneer in the field, began formalizing the methodology.

Initially employed in operations research and management science, DA quickly found its application in various fields.

Over the years, decision analysis evolved, incorporating advancements in computational techniques and decision theory. Today, it's a vital tool in many disciplines, including finance, where it's used to navigate complex investment decisions and strategic planning.

1.3.3 Components of Decision Making

The five distinct parts of consumer decision making presented are input, information processing, a decision process, decision process variables, and external influences.

1. Input: Input includes all kinds of stimuli from our contact with the world around us, such as our experiences, marketer-controlled stimuli (e.g., advertising, store display, demonstrations), other stimuli (e.g., personal recollections, conversations with friends) and external search.

2. Information processing: Stimuli are processed into meaningful information and this task includes the stages of exposure, attention, comprehension, yielding and retention.

a. Exposure: Exposure refers to a kind of stimulation closer to consumer physical, and one of our five senses may be activated by the kind of stimulation. And then, the information that is encoded will be sent to the brain through the nerve conduction. Subsequently, the consuming consciousness may be excited by these stimuli and the consumer decision making begins.

b. Attention: Attention refers to the activity or ability which makes one's thoughts on the matter. That is to say, the basic factor of attention is focus. Generally speaking, a product which has more relevant information and content can attract more people's attention.

c. Comprehension: When the product information intended to attract consumers, the information will be further classified, and stored in memory. Because a consumer is attentive toward a stimulus does not mean that the stimulus can be comprehended as intended. Enterprises expect the information can be comprehended exactly.
d. Yielding: After consumers comprehend this product information, they either accept this product or refuse it. Product information is aimed at changing or rectifying the image of products in people's minds. Making consumers accept the information is the purpose of enterprises.
e. Retention: Retention refers to the memory process that determines which of the many stimuli that have gone through the initial four stages of consumer information processing will be remembered. This stage prepares for future purchases.
In order to attract attention, the product information is often overloading, everyone will face a host of advertising. So, not all stimuli that have been comprehended will be stored in memory. In fact, the percentage of stimuli remaining in memory is generally quite small, particularly after a period of weeks has passed since exposure.

3. Decision process: According to the processed information, consumers will make consuming decision. Decision process is triggered at any time during information processing. It consists of five steps. They are problem recognition, search, alternative evaluation, choice and outcomes (post-purchase evaluation and behavior). Consumer demand is the starting point of any purchase decision.
a. Problem recognition: When consumers realize that they need something, problem recognition is the first step in the decision-making process. It is the psychological process used to determine the difference between the consumer's actual benefits state (where you are) and the desired benefits state (where you want to be). The essence of problem recognition is the perception gap between the reality and consumer desire. According to the gap for the importance of consumers and the variability of the gap, consumer will take appropriate purchase decision. Problem recognition is influenced by situation, consumer and marketing.
b. Search: Information collected by consumers is the basis for evaluation and choice behavior. It is important for marketers to know why consumers are searching for information, where will they look, what information consumers seek and how extensively they are willing to search. The breadth and width of the collected information is decided by personality, social class, income, purchases, past experience, the understanding of previous brand, customer satisfaction and other factors.
c. Alternative evaluation: Consumers will choose the product from the brand group which has a number of options to choose. When consumers choose the product brand, the attribute of product will be considered. For instance, the photo resolution or the price of camera will be considered when consumers want to buy a camera. Most consumers will consider a

few attributes, but the importance of these attributes should have different weight.

d. Choice: In this part, consumers choose one of many retailers firstly. And then, consumers choose product which can meet their demands in the shop. These behaviors are influenced by salespeople, product display and advertisement. The best salespeople will introduce all the good features of the product and attempt to maintain the shop image as much as possible for attract customers to patronize once again.

e. Outcomes: After purchasing the product, consumers will produce the sense of satisfaction and dissatisfaction. The level of satisfaction or dissatisfaction we experience depends upon how well the product's performance meets our expectations. Objective performance is product-related and depends on whether the product meets all functional expectations Affective performance is consumer-related and depends on whether the purchase meets the emotional expectations of the consumer. Marketers must understand consumer expectations and the extent to which purchases satisfy them. Post-purchase behavior is as important as understanding what causes consumers to buy. It has an impact on future sales. The information can be used to improve products and services, undertake better targeted promotions, and design more effective strategies to keep actual customers and attract new ones.

4. Decision Process variables: Those individual qualities what make consumers unique. Decision process variables include motives, beliefs, attitudes, lifestyles, intentions, evaluative criteria, normative compliance and informational influence and other aspects of self. Motives refer to a tendency for people to behave in a general way in order to satisfy a need. Motive is an inherent power which can promote individual activities for attain consumer purpose. In fact, buying motives indicates that consumer behavior is generally purposive or goal-directed. Closely related to the concept of motives are consumers' evaluative criteria. Evaluative criteria refer to the dimensions or performance characteristics desired from a product or service. Evaluative criteria are developed from a consumer's past experiences, personality traits and the influences of other people, and so, are more than just manifestations of consumer motives. Beliefs are foundation of will. It combines the individual goal and the overall goal. Belief is a psychological momentum, the role of its behavior aims to arouse people's potential energy, physical, intellectual and other capabilities for achieving a certain purpose. Attitudes refer to people will assess things based on morality and values. Lifestyle refers to a pattern of consumption reflecting a person's choices of how he or she spends time and money. Intentions are subjective judgment about future activities of people. Purchase intentions refer to what people think they will buy. One of these variables can influence consumers' behavior.

5. External influences: Such influences are called "Circles of Social Influence." They are: culture, sub-culture (co-culture), social class, reference groups, and family or household influences. Culture refers to non-individual, as members of society, exchanges and understandings each other. Culture also is the symbol of values, ideas and other meaning. Culture is described as the foundation of human activities what determine the interaction between social activities and production activities. Social class refers to relatively permanent and identical part in society. In the same social class, depending on the same or similar values, lifestyles, similar interests, wealth, status, education, economic status, behavior and other factors, individuals or families are classified into different separate parts. Family or household influences are also important to consumers' behavior.

Self-Assessment Exercise 3

Discuss Components of Decision Making

1.4 Phases of Decision Analysis

Deterministic Analysis Phase: This phase accounts for certainties rather than uncertainties. Here, graphical and diagrammatic models like influence diagrams and flow charts can be translated into mathematical models. Necessary tools are used for predicting consequences of alternatives and for evaluating decision alternatives.

Probabilistic Analysis: Probabilistic analyses cater for uncertainties in the decision making process. We can use the decision tree as a tool for probabilistic analysis.

Evaluation Phase: At the phase, the alternative strategies are evaluated to enable one identify the decision outcomes that correspond to sequence of decisions and events.

Choice Activity Phase: This is the final phase of the decision analysis cycle. It is the judgemental stage where the decision maker decides on the best strategy to adopt having carefully analyses all other options.

1.5 Errors that can occur in Decision Making

1. Holding out for the perfect decision

Striving for perfection in our decisions adds unnecessary pressure and often leads to "analysis paralysis." No one likes to be wrong, but we must shake our fear and accept that decision-making means taking risks: sometimes we'll get it right, other times we won't. Mistakes are a part of learning.

"I've failed over and over and over again in my life. And that is why I succeed," boasts Michael Jordan, arguably the best basketball player of all time.

2. Failing to face reality

We tend to see things as we would like them to be, confusing wishful thinking with reality. For example, 75 percent of drivers think they are above average behind the wheel, which is statistically impossible.

Faced with a situation, we tend to take a stance and may fail to see beyond it, ignoring what might be better options out there. Furthermore, we tend to magnify the positive aspects of our stance and minimize the negative ones.

One good way to avoid this bias is to try and distinguish facts (objective) from opinions (subjective).

3. Falling for self-deceptions

The way we are presented with a situation, and the way we present it to ourselves, affects our final decisions. For example, when some cancer patients were told that the survival rate one year after a surgery was 68 percent, a significant percentage opted for that surgery. Meanwhile, when others were told that 32 percent of patients die within a year of the operation, no one elected to undergo it. The same information was given, just presented in a different way.

To avoid falling prey to self-deceptions, it is important to seek alternatives and consider them from different angles. Finally, sleep on it before making the decision.

4. Going with the flow

There is something worse than being wrong: being the only one who is wrong. Doing what everyone else does is easier and, more importantly, may save us from embarrassment. Hence our tendency to follow the herd, even if it is heading to a precipice.

We saw this with the dotcom bubble, for example. Everyone wanted to invest in tech companies when the bubble was inflating, even when most investors knew little about them.

The problem with imitation (and not thinking before deciding) is that we eliminate the possibility of finding wiser alternatives than what is fashionable.

5. Rushing and risking too much

Before deciding hastily, we should consider whether a decision is truly urgent. We tend to rush into things, crossing things off our list to feel accomplished. But all we're really doing in a rush is taking unnecessary risks.

For example, the Chernobyl disaster was caused by an unnecessary test that simulated a power-failure at the nuclear power plant. By going through the motions for security testing, the outcome was exactly what they sought to avoid: the reactor exploded. There was no urgent need to run that test, but it happened anyway, risking far too much.

6. Relying too heavily on intuition

Intuition can be an asset, but when we allow it to outweigh analytical thinking, it leads to mistakes. What's more, testing our hunches with low-cost experiments is important.

The authors offer Samsung chairman Lee Kun-hee as a cautionary tale. In the 1990s, he reportedly decided to get into automobile manufacturing because he "sensed" the market would take off in Asia. The project resulted in a loss of $2 billion and 50,000 layoffs.

7. Being married to our own ideas

It's hard for us to change a prior decision, even if keeping the status quo is clearly inefficient or harmful.

The year 2003 saw the grounding of the Concorde, a supersonic jet airliner that was never profitable. But it took a fatal accident, with over 100 fatalities, to put it into permanent retirement. Economically speaking, the right decision should have been made long before then, but that meant acknowledging a failure. And no one likes doing that.

8. Paying little heed to consequences

Sometimes we don't consider the consequences of a decision. Or we only consider the most direct and immediate ones, ignoring the side effects. And that can cause even bigger problems than the ones we were trying to solve in the first place.

That's what happened to those in charge of the Titanic, who wanted to arrive at their destination 24 hours ahead of schedule in order to silence critics who claimed their large ship would be slow. They ignored warnings about icebergs, warnings that should have slowed them down for safety's sake.

9. Overvaluing consensus

We often think group decisions are more effective, but that's not always true. Reaching consensus also has its drawbacks: it may take longer, accountability tends to be diluted, and people may not say what they really think due to peer pressure or the desire to be accepted.

The latter may have occurred with the Kennedy administration's botched Bay of Pigs invasion. What was supposed to be a surprise attack on Cuba turned out to be an open secret, the authors relate. Since no one wanted to appear to be a "dissident," no one dared question the orders — even when the most sensible thing to do was abort the mission.

To avoid such a situation, remember to consult with people who hold different views and are willing to question our arguments.

10. Not following through

The decision-making process does not end with the decision: implementation and the monitoring of it are essential, too. However, some resolutions are not put into practice due to our own personal limitations (e.g., lack of will power, commitment or time) or external factors (e.g., lack of authority or support).

For example, a multinational decided to establish a corporate headquarters for southern Europe. But the idea was eventually discarded in order to avoid upsetting any of the managing directors of the three potential host countries ultimately causing damage to the whole enterprise.

It is vital to consider the implementation of a decision. And that means assessing our own ability to commit to a particular course of action and accepting that others will have their own interests and needs.

If we consider these common mistakes, our decision-making abilities will improve substantially.

1.6 Summary

In this unit, the elements of decision analysis were discussed. It began with defining a decision, and who a decision maker is. Further, it considers the components of decision making, structure of a decision problem and finally errors that can occur in decision making. This unit provides us with concepts that will help us in understanding the subsequent units and modules.

1.7 References/Further Readings/Web Resources

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

A decision can be defined as an action to be selected according to some pre-specified rule or strategy, out of several available alternatives, to facilitate a future course of action. This definition suggests that there are several alternative courses of action available, which cannot be pursued at the same time. Therefore, it is imperative to choose the best alternative base on some specified rule or strategy. Decision making is the process of selecting the best out of several alternatives.

Answer to SAE 2

Decision-makers are people within a company who have the power to make strategic decisions like acquisitions, expansion, or investment.

A decision maker is the person or group of individuals who is responsible for making strategically important decisions based on a number of variables, including time constraints, resources available, the amount and type of information available and the number of stakeholders involved.

Answer to SAE 3

Identify the Problem-Then Simplify It

As soon as a potential problem comes to light, some teams tend to spring immediately into problem-solving mode without knowing at the most basic level what the problem actually is. What is the underlying issue at hand? What needs to be fixed or changed? Where did the problem originate? What implications does it have? Effective leaders don't overcomplicate or add layers to a problem; they identify it, then simplify it to its most basic form. This allows them to create an effective path to determine the most appropriate solution to a problem and make the best decision even with imperfect information.

Embrace the Pre-Mortem

Effective leaders take time to consider the potential negative outcomes before deciding what action to pursue. They ask what failure would look like if the resulting outcome went down a specific path. Many teams and leaders wait until the dust settles after the decision has run its course and then conduct a post-mortem. Effective teams take a run at a pre-mortem to explore the reasons why something could potentially fail and what failure would look like. This reframing will keep the team flexible and open to adapting plans as needed. We have many times observed teams that are too focused on why and how an idea could lead to success and neglect to ask what could go wrong.

Welcome the Pivot

A change in organizational direction, or pivot, is inevitability in today's business world. An organization with no change mechanisms in place will quickly become stagnant and regress. The same principles apply to the individuals who are making decisions as leaders of a team. Every leader must be prepared to change course when necessary. Becoming locked into one approach and refusing to alter it, even when the key indicators are pointing in a different direction, often leads to bad outcomes. Successful teams realize that the best path forward almost always requires a detour if not many.

Think Differently

The most successful teams in our simulations consistently consider alternative viewpoints. Many times, success is possible only when it disrupts the status quo. Teams that ensure their decision-making process is open to thinking outside the proverbial box typically succeed more than teams that follow the same recipe in every simulated quarter. Effective leaders consider each problem individually, from multiple angles, often playing devil's advocate against themselves to make sure they've covered all the bases.

MODULE 2

Unit 1 Types of Decision Situations
Unit 2 Decision Trees
Unit 3 Operations Research (OR)
Unit 4 Modelling in Operations Research
Unit 5 The Transportation Model

Unit 1 Types of Decision Situations

Unit Structure

1.1 Introduction
1.2 Learning Outcomes
1.3 What is a Decision Situation?
    1.3.1 Elements of Decision Situation
    1.3.2 Types of Decision Situations
1.4 Steps in Decision Theory Approach
1.5 Decision Making Criteria
1.6 Summary
1.7 References/Further Readings/Web Resources
1.8 Possible Answers to Self-Assessment Exercise(s)

1.1 Introduction

In this unit, we shall delve fully into considering these situations and learn how we can use different techniques in analysing problems in certain decision situations i.e Certainty, Uncertainty, Risk, and Conflict situations.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Decision Situations
- Demonstrate Steps in Decision Theory Approach
- Discuss types of Decision Situations

1.3 What is a Decision Situation?

Decision situation, or decision-demanding situation, refers to a situation when decision making is inevitable. In the decision situation, the need for a decision is apparent and the decision maker must begin to define the problem and get involved in the decision making process." (Intezari & Pauleen, 2019 , p. 11).

Self-Assessment Exercise 1

What is Decision Situations?

1.3.1 Elements of Decision Situation

These elements are discussed in the below:

1. The Decision markers: Decision markers the first elements of the decision situation. According 10 Ernest Dale weak decision makers usual have one of four orientations

a. Receptive
b. Exploitative
c. Hoarding
d. Marketing

a. Receptive: Decision markers who have a receptive orientation believe that the source of all good is outside themselves and therefore they rely heavily on suggestions from other organization members.

b. Exploitative: Decision makers With an exploitative orientation also believe that the source of all good is outside themselves and they are willing to steal ideas as necessary in order to make good decisions.

c. Hoarding: the hoarding orientation is characterized by the desire to preserve the unchangeable as much as possible
d. Marketing: Marketing oriented decision takers look upon themselves as commodities that are only as valuable as the decisions they make.

2. Goals to be served: The goals that decision makers seek to attain are another element of the decision situation. In the case of managers, these goals should often be organizational objectives.

3. Relevant Alternatives: Relevant alternatives are those alternatives that are considered feasible for solving an existing problem and for implementation.

4. Ordering of Alternative: The decision situation requires a process or mechanism for ranking alternatives most desirable to least desirable. The process can be subjective, objective or some combination of the two.

5. Choice of Alternatives: The last element of the decision situation is the actual choice between available alternatives. This choice establishes the decision.

Typically managers choose the alternative that maximizes long-term return for the organization.

1.3.2 Types of Decision Situations

According to Gupta and Hira (2012), there are four types of environments under which decisions can be made. These differ according to degree of certainty. The degree of certainty may vary from complete certainty to complete uncertainty. The region that lies between corresponds to decision making under risk.

1. Decision Making Under Condition of Certainty

In this environment, only one state of nature exits for each alternative. Under this decision situation, the decision maker has complete and accurate information about future outcomes. In other words, the decision maker knows with certainty the consequence of every alternative course of action. It is easy to analyse the situation and make good decisions. Since the decision maker has perfect knowledge about the future outcomes, he simply chooses the alternative with the optimum payoff. The approach to analysing such decision problem is deterministic. Decision techniques used here include simple arithmetic for simple problem, and for complex decision problems, methods used include cost-volume analysis when information about them is precisely known, linear programming, transportation and assignment models, deterministic inventory models, deterministic quelling models and network model. We shall discuss these models later.

2. Decision Making Under Conditions of Uncertainty

Here, more than one state of nature exists, but the decision maker lacks sufficient knowledge to allow him assign probabilities to the various state of nature. However, the decision maker knows the states of nature that may possibly occur but does not have information which will enable him to determine which of these states will actually occur. Techniques that can be used to analyse problem under this condition include the Maximax criterion, equally likely or Laplace's criterion, and Hurwicz criterion or Criterion of Realism. These techniques have earlier been discussed. We shall consider a more difficult problem for further illustration.

Self-Assessment Exercise 2

Discuss types of decision situation

Example 1

A farmer is considering his activity in the next farming season. He has a choice of three crops to select from for the next planting season – Groundnuts, Maize, and Wheat. Whatever is his choice of crop; there are four weather conditions that could prevail: heaving rain, moderate rain, light rain, and no rain. In the event that the farmer plants Ground nuts and there is heavy rain, he expects to earn a proceed of N650,000 at the end of the farming season, if there is moderate rain- N1,000,000, high rain – N450,000 and if there is no rain – (-N1,000). If the farmer plants Maize, the following will be his proceeds after the harvest considering the weather condition: heavy rain – N1,200,000, moderate rain –N1,500,000, Light rain –N600,000 and no rain N2000. And if the farmer decides to plant wheat, he expects to make the following: heavy rain – N1,150,000, moderate rain –N1,300,000, Light rain - N800,000 and No rain – N200 -000. The farmer has contact you, an expert in OR to help him decide on what to do.

Question: Construct a payoff matrix for the above situation, analyse completely and advise the farmer on the course of action to adopt. Assume = 0.6. Solution

First, construct a contingency matrix from the above problem.

Contingency Matrix 1a

Alterative Crops	Weather conditions			
	Heavy Rain (S_1) ₦	**Moderate Rain (S_2) ₦**	**Light Rain (S_3) ₦**	**No Rain (S_4) ₦**
Groundnut(d1)	750,000	1,000.000	450,000	-1,000
Maize (d2)	1,200,000	1,500,000	600,000	2000
Wheat (d3)	1,150,000	1,300,000	800,000	-200,000

Figure 1a: Pay off table

Contingency Matrix 1b

Alternative Crops	Weather conditions				Max col	Min Col
	S_1 (₦"000)	**S_2 S_3 S_4 (₦"000)**	**(₦"000)**	**(₦"000)**		
d1	750	1,000	450	-1	1,000	-1
d2	1,200	1,500	600	2	1,500	2
d3	1,150	1,300	800	-200	1,300	-200

Fig. 1b: Pay- off Table

Regret Matrix 1

Alternative Crops	**Weather conditions**				**Max col**	**Min Col**
	S1 (₦"000)	**S2 S3 (₦"000)**	**(₦"000)**	**(S4) (₦"000)**		
d1	1200 – 750 450	1500-1000 500	800-450 350	2-(1) 3	500	
d2	1200 – 1200 0	1500-1500 0	800-600 200	2-2 0	200	200
d3	1200-1150 50	1500-1300 200	800-800 0	2-(-200) 202	202	
Col max	1200	1500	800	2		

Fig. 2: Regret Matrix1

1. Maximax Criterion

Alternative	**Maximum Coloum**
d1	1,000
d2	**1,500**
d3	1,300

Recommendation: Using the maximax criterion, the farmer should select alternative d2 and plant maize worth ₦1,500

2. Maximin Criterion

Alternative	**Minimum Coloum**
d1	-1
d2	**2**
d3	-200

Recommendation: Using the maximum criterion, the farmer should select alternative d2 and plant maize worth ₦2,000.

3. Minimax Regret Criterion

Choice of crops	**Weather conditions**				**Max Col**	**Min Col**
	S1 S2 S3	**(S4)**				
d1	450	500	350	3	500	
d2	0	0	200	0	200	200
d3	50	200	0	202	202	

Fig. 3: Pay- off Table

Recommendation: Using the minimax Regret Criterion, the decision maker should select alternative d2 and plant maize to minimize loss worth ₦200

4. Laplace Criterion

$$d1 = \frac{750 + 1000 + 450 - 1}{4} = 549.75$$

$$d2 = \frac{1200 + 1500 + 600 + 2}{4} = \mathbf{825.50}$$

$$d3 = \frac{1150 + 1300 + 800 - 200}{4} = 762.50$$

Recommendation: Using the equally likely or Savage Criterion, the farmer should select alternative d2 to plant maize worth ₦825.50

5. Hurwicz Criterion

$\alpha = 0.6$, $1 - \alpha = 0.4$

CRi = (max in row) + (1-α) (min in row)

$CR_1 = 0.6 (1000) + (0.4) (-1) = 600 + (-0.4) = 599.6$

$CR_2 = 0.6 (1500) + (0.4) (2) = 900 + 0.8 = \underline{\mathbf{900.8}}$

$CR_3 = 0.6 (1300) + (0.4) (-200) = 780 + (-80) = 700$

Recommendation: Using the Hurwicz criterion the farmer should select alternative d2 and cultivate maize worth ₦900.8

1.4 Steps in Decision Theory Approach

Decision theory approach generally involves four steps. Gupta and Hira (2012) present the following four steps.

Step 1: List all the viable alternatives

The first action the decision maker must take is to list all viable alternatives that can be considered in the decision. Let us assume that the decision maker has three alternative courses of action available to him a, b, c.

Step 2: Identify the expected future event

The second step is for the decision maker to identify and list all future occurrences. Often, it is possible for the decision maker to identify the future states of nature; the difficulty is to identify which one will occur. Recall, that these future states of nature or occurrences are not under the control of the decision maker. Let us assume that the decision maker has identified four of these states of nature: i, ii, iii, iv.

Step 3: Construct a payoff table

After the alternatives and the states of nature have been identified, the next task is for the decision maker to construct a payoff table for each possible combination of alternative courses of action and states of nature. The payoff table can also be called contingency table.

Step 4: Select optimum decision criterion
Finally, the decision maker will choose a criterion which will result in the largest payoff or which will maximize his wellbeing or meet his objective.

Self-Assessment Exercise 3

What are the steps in decision theory approach?

1.5 Decision Making Criteria

There are five criteria with which a decision maker can choose among alternatives given different states of nature. Gupta and Hira (2012) are of the view that choice of a criterion is determined by the company's policy and attitude of the decision maker. They are:
1. Maximax Criterion or Criterion of Optimism
2. Maximin Criterion or Criterion of Pessimism (Wald Criterion)
3. Minimax Regret Criterion (Savage Criterion)
4. Laplace Criterion or Equally likely criterion or criterion of Rationality (Bayes' Criterion)
5. Hurwicz Criterion or Criterion of Realism
Now let us see how we can solve problems using the above criteria.

Example 2.1: Consider the contingency matrix given below
Contingency table 2

Alternative Products	**Market Demand**		
	High (₦)	**Moderate (₦)**	**Low (₦)**
Body Cream	500	250	-75
Hair Cream	700	300	-60
Hand Lotion	400	200	-50

Pay-off table
The matrix above shows the payoffs of an investor who has the choice either investing in the production of Body Cream, or Hair cream, or hand lotion. Whichever of the three products he decides to produce; he will encounter three types of market demand. It may turn out that the market demand for any of the product is high, or moderate of low. In other words, the production of body cream, or hair cream, or hand lotion represent the alternative courses of action or strategies available to the investor, while the occurrence of either high demand, or moderate demand, or low demand represent the states of nature for which the investor has no control

over. Now, how would the investor arrive at the choice of product to manufacture? We are going to analyse the decision problem using the five criteria earlier listed below.

1. Maximax Criterion (Criterion of Optimism)
The maximax criterion is an optimistic criterion. Here, the decision maker aims to maximize profit or his outcome. It involves an optimistic view of future outcomes. This is done by selecting the largest among maximum payoffs. However, the disadvantage of this criterion is that it does not make use of all available information in getting the quantitative values. This is not often the case on real life situations. The criterion has also been criticizes for being too optimistic and assumes that the future will always be rosy.(Adebayo et al, 2006)

Contingency table 3

Alternative Products	**Market Demand High (₦)**	**(₦) Moderate**	**Low (₦)**	**Max Column (₦)**	**Maxi max (₦)**
Body Cream	500	250	-75	500	
Hair Cream	700	300	-60	700	700
Hand Lotion	400	200	-50	400	

Let us now try to solve the decision problem in the matrix above using the maximax criterion.

Step 1: Create and additional column to the right hand side of the matrix and call it max column as shown below.

Step 2: Identify the maximum pay-off in each alternative course of action (i.e. Either the role for Body Cream, or Hair Cream, or Hand Lotion) and place it in the corresponding cell on the maximum column.

Step 3: Identify and select the pay-off with the highest value on the maximum column. This value becomes your optimal value using the maximax criterion.

Step 4: Make recommendations.
As we can see from Contingency table 3 above, the maximax value is ₦ 700.

Recommendation: Using the maximax decision criterion, the decision maker should manufacture hair cream to maximize worth N700.

2. Maximin Criterion (Criterion of Pessimism)

Under the maximin Criterion, the decision maker is assumed to be pessimistic. The objective here is to maximize the minimum possible outcome. It is a decision situation where the decision maker tries to make the most of bad situations and avoids taking risks and incurring huge losses. According to Adebayor et al (2006), the weakness of this criterion is that the result may not always be unique. It has also been criticized for being an unduly careful. However, it has the advantage of helping one to be in the best possible condition in case the worst happens.

In analysing a decision situation using this criterion, we use the following steps.

Step 1: Create an additional column to the rights hand side of your pay-off matrix-minimum column.

Step 2: Select the minimum pay-off from each alternative and place on the corresponding call in the minimum column.

Step 3: Identify and select the maximum pay-off in the minimum column

Step 4: Make recommendation

Using data in contingency matrix 2

Minimum Coloum (~~N~~)	Maximin (~~N~~)
75	
60	
50	-50

Payoff table- Minimum and Maximum Columns.

Recommendation: Using the maximin decision criterion, the decision maker should manufacture hand lotion with a pay-off of - N50.

3. Minimax Regret Criterion (Savage Criterion)

This decision criterion was developed by Savage. He pointed out that the decision maker might experience regret after the decision has been made and the states of nature i.e. events have occurred. Thus the decision maker should attempt to minimize regret before actually selecting a particular alternative (strategy) (Gupta and Hira, 2011). The criterion is aimed at minimizing opportunity loss.

The following steps are used to solve problems using this criterion. Step 1: For each column, identify the highest payoff

Step 2: Subtract the value from itself every other pay-off in the column to obtain the regret matrix.

Step 3: Create an additional column to the right of your regret matrix and call it maximum column.

Step 4: Identify and select the maximum value from each alternative strategy
Step 5: Find the minimum value in the maximum column created.
Step 6: Make Recommendations

Contingency table 4

Alternative Products	**Market Demand**		
	High (₦)	**Moderate (₦)**	**Low (₦)** -
Body Cream	500	250	-75
Hair Cream	**700**	**300**	-60
Hand Lotion	400	200	**-50**

Regret Matrix 1

Alternative Products	**Market Demand**			**Max Column**	**Mini max**
	High (₦)	**Moderate (₦)**	**Low (₦)**		
Body Cream	200	50	25	200	
Hair Cream	0	0	10	10	10
Hand Lotion	300	100	0	300	

Recommendation: Using the minimax regrets criterion, the decision maker should manufacture hair cream to minimize loss worth N10.

4. Equally Likely of Laplace Criterion (Bayes' or Criterion of Rationality)

This criterion is based upon what is known as the principle of insufficient reasons. Since the probabilities associated with the occurrence of various events are unknown, there is not enough information to conclude that these probabilities will be different. This criterion assigns equal probabilities to all the events of each alternative decision and selects the alternative associated with the maximum expected payoff. Symbolically, if "n" denotes the number of events and "s" denotes the pay-offs, then expected value for strategy, say s_1 is

$$1/N[P_1 + P_2 + \ldots. + P_n]$$

or simply put

$$\frac{P_1 + P_2 + \ldots.. + P_n}{n}$$

The steps to follow are:

Step 1: Compute the average for each alternative using the above formula.
Step 2: Select the maximum outcome from the calculation in step 1 above
Step 3: Make recommendations

Example

Contingency table 5

Alternative Products	Market Demand High (N)	Moderate (N)	Low (N)	Average Column	Max Col.
Body Cream	500	250	-75	500 +250 + 75 = 675 = 225 3 3	
Hair Cream	700	300	-60	700 + 300 – 60 = 940 =313.3 3 3	313.3
Hand Lotion	400	200	-50	400 + ~~200 – 5 =~~ 550 =183.3 30 3	

Payoff table.

Recommendation: Using the equally likely criterion, the decision should manufacture Hair Cream worth N313.3

5. Hurwicz Criterion (Criterion of Realism)

This criterion is also called weighted average criterion. It is a compromise between the maximax (optimistic) and maximin (Pessimistic) decision criteria. This concept allows the decision maker to take into account both maximum and minimum for each alternative and assign them weights according to his degree of optimism or pessimism. The alternative which maximises the sum of these weighted pay-offs is then selected. (Gupta and Hira, 2012)

The Hurwicz Criterion Comprises the following steps:

Step 1: Choose an appropriate degree of optimism (lies between zero and one (0 <<1)), so that (1-) represents the degree of pessimism is called coefficient or index of optimism.

Step 2: Determine the maximum as well as minimum value of each alternative course of action.

Step 3: Determine the criterion of realism using the following formula CRi = (Max in Row$_1$) + (1-) (Min in Row$_1$)

Step 4: Select the maximum outcome in step 3 above

Step 5: Make Recommendation

Example: Using the contingency above,

Maximum	Min in Row
500	-75
700	-60
400	-50

For alternative Body Cream (b)
$(R_b =$ (Maxim Row_b) + (1 -) (min in Row_b)
Let us assume = 0.5
CR_{bc} = 0.5 (500) + (1 – 0.5) (-75)
0.5 (500) + 0.5 (-75)
250 – 37.5 = 212.5
For alternative Hair Cream CR_{hc} = 0.5 (700) + (0.5) (-60)
350 + (- 30)
350 – 30 = 320
Therefore
For alternative Hand Lotion
CR_{hl} = 0.5 (400) + (0.5) (-50)
200 – 25 = 177
Therefore
CR_{bc}=N212.5
CR_{hc} **= N320**
CR_{hl} = N175

Recommendation: Using the Hurwicz Criterion, the decision maker should manufacture Hair Cream worth N320.

We have seen how interesting and simple it is to use the five criteria in analysing decision problems. However, the above analysis can only be used under a situation of uncertainty where the decision maker neither knows the future states of nature nor have the probability of occurrence of the states of nature. This will be discussed in greater detail in the next unit.

1.6 Summary

Every individual, group of individuals, as well as organizations are faced with decision problems every day. An individual or a small group of people faced with simple decision may apply common sense in solving their problems. However, this is not the case with big corporate organizations which are faced with very complex decision problems. An application of common sense in such complex situations will not be appropriate as it will lead mostly to wrong decisions. Complex decision problems demand the use of specialized tools and techniques for analysis of problem and eventual arrival at the best alternative.

1.7 References/Further Readings/Web Resources

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Is an organizational situation that demands for a managerial decision-making process to avoid organizational damages, to improve organizational performance, or to keep the organizational state.

Answer to SAE 2

- Types of Decision-Making in Management
- Routine and Basic Decision-making.
- Personal and Organizational Decision-making.
- Individual and Group Decision-making.
- Programmed and Non-Programmed Decision-making.
- Policy and Operating Decision-making.
- Tactical and Strategic Decision-making.
- Planned and Unplanned Decision-making.

Answer to SAE 3

Step 1: Identify the decision that needs to be made.
Step 2: Gather relevant information.
Step 3: Identify alternative solutions. ...
Step 4: Weigh the evidence. ...
Step 5: Choose among the alternatives. ...
Step 6: Take action. ...
Step 7: Review your decision and its impact (both good and bad)

Unit 2 Decision Trees

Unit Structure

1.1 Introduction

So far, we have been discussing the techniques used for decision analysis. We have demonstrated how to solve decision problems by presenting them in a tabular form.

However, if decision problems can be presented on a table, we can also represent the problem graphically in what is known as a decision tree. Also the decision problems discussed so far dealt with only single stage decision problem.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Decision Trees
- Demonstrate Components of the decision tree
- Discuss Benefits of using decision tree

1.3 What is a Decision Trees?

A decision tree is a graphical representation of the decision process indicating decision alternatives, states of nature, probabilities attached to the states of nature and conditional benefits and losses (Gupta & Hira 2012). A decision tree is a pictorial method of showing a sequence of inter-related decisions and outcomes. All the possible choices are shown on the tree as branches and the possible outcomes as subsidiary branches. In summary, a decision tree shows: the decision points, the outcomes (usually dependent on probabilities and the outcomes values) (Lucey, 2001).

Self-Assessment Exercise 1

Define Decision Trees

1.3.1 Benefits of using decision tree

Dixon-Ogbechi (2001) presents the following advantages of using the decision tree

- They assist in the clarification of complex decisions making situations that involve risk.
- Decision trees help in the quantification of situations.
- Better basis for rational decision making are provided by decision trees.
- They simplify the decision making process.

1.3.2 Disadvantage of the decision tree

The disadvantage of the decision tree is that it becomes time consuming, cumbersome and difficult to use/draw when decision options/states of nature are many.

1.3.3 Components of the Decision Tree

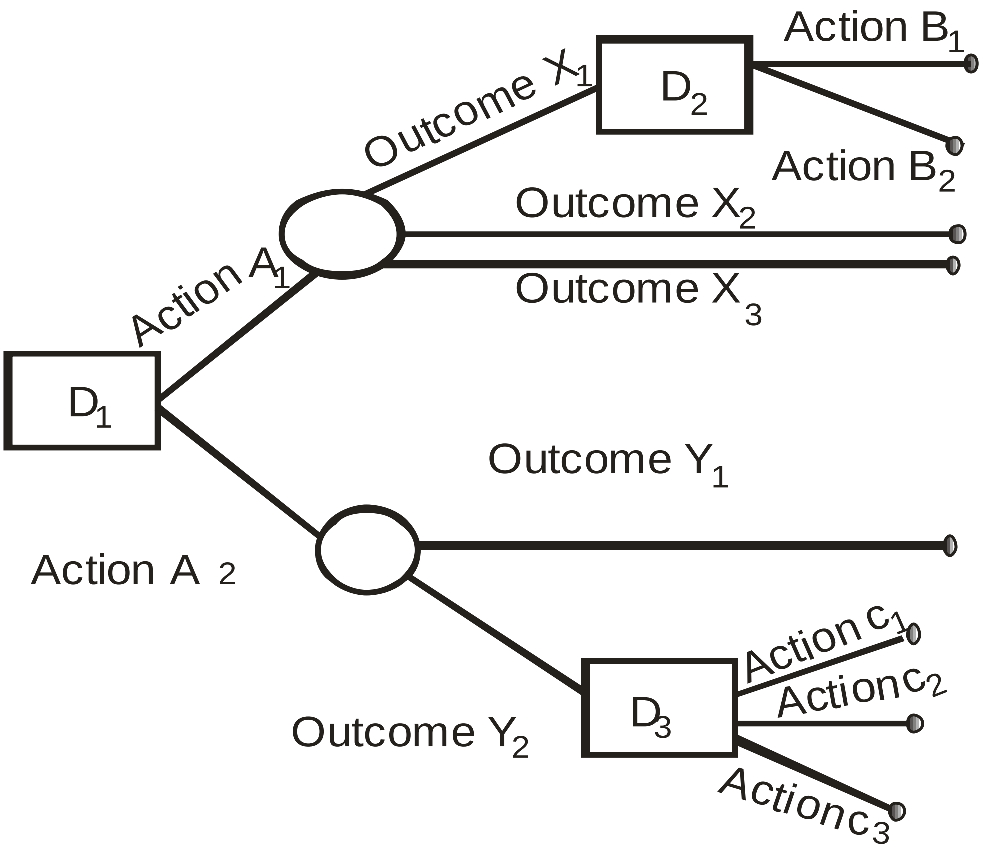
It is important to note the following components of the structure of a decision problem

The choice or Decision Node: Basically, decision trees begin with choice or decision nodes. The decision nodes are depicted by square. It is a point in the decision tree were decisions would have to be made. Decision nodes are immediately by alternative courses of action in what can be referred to as the decision fork. The decision fork is depicted by a square with arrows or lines emanating from the right side of the square.The number of lines emanating from the box depend on the number of alternatives available.

Change Node: The chance node can also be referred to as state of nature node or event node. Each node describes a situation in which an element of uncertainty is resolved. Each way in this uncertainty can be resolved is represented by an arc that leads rightward from its chance node, either to another node or to an end-point. The probability on each such arc is a conditional probability, the condition being that one is at the chance node to its left. These conditional probabilities sum to 1 (0ne), as they do in probability tree (Denardo, 2002).

1.3.4 Structure of a Decision Tree

The structure and the typical components of a decision tree are shown in the diagram below.



Adapted from Lucey, T (2001), Quantitative Techniques, 5th London: Continuum.

The above is a typical construction of a decision tree. The decision tree begins with a decision node D_1 signifying that the decision maker is first of all presented with a decision to make. Immediately after the decision node, there are two courses of Action A_1 and A_2.If the decision maker chooses A1, there are three possible outcomes – X_1 X_2, X_3. And if chooses A_2, there will be two possible outcomes Y_1 and Y_2 and so on.

Self-Assessment Exercise 2

What are Components of the Decision Tree?

1.4 How to Analyse a Decision Tree

The decision tree is a graphical representation of a decision problem. It is multi-state in nature. As a result, a sequence of decisions are made repeatedly over a period of time and such decisions depend on previous decisions and may lead to a set of probabilistic outcomes. The decision tree analysis process is a form of probabilistic dynamic programming (Dixon-Ogbechi, 2001).

Analysing a decision tree involves two states:

Backward Pass: This involves the following steps

- Starting from the right hand side of the decision tree, identify the nearest terminal. If it is a chance event, calculate the EMV (Expected Monetary Value). And it is a decision node, select the alternative that satisfies your objective.
- Repeat the same operation in each of the terminals until you get to the end of the left hand side of the decision tree.

Forward Pass: The forward pass analysis involves the following operation.

- Start from the beginning of the tree at the right hand side, at each point, select the alternative with the largest value in the case of a minimization problem or profit payoff, and the least payoff in the case of a minimization problem or cost payoff.
- Trace forward the optimal contingency strategy by drawing another tree only with the desired strategy.

These steps are illustrate below

Example

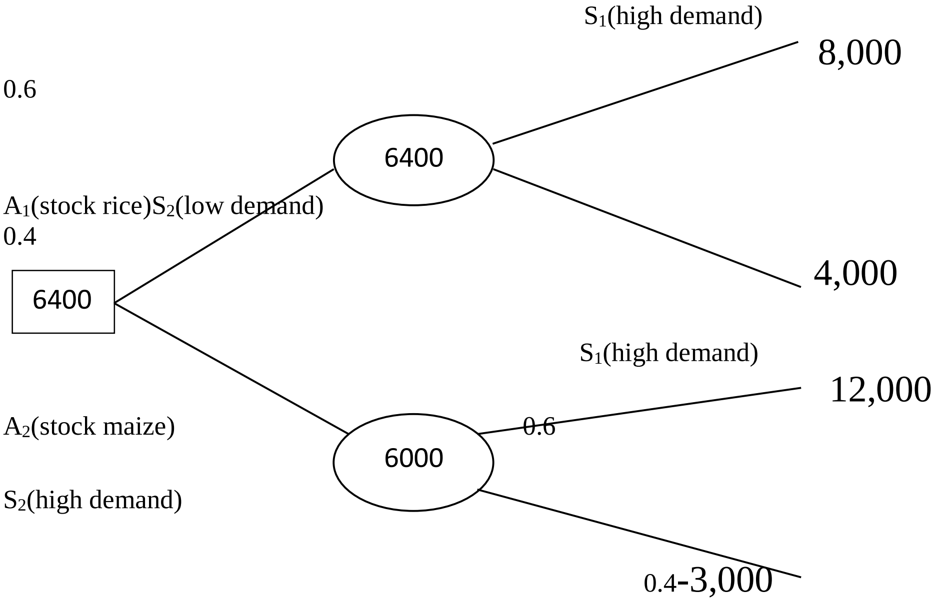
Contingency Matrix 1

States of Nature	Alternatives		Probability
	Stock Rice (A_1)	Stock Maize (A_2)	
High demand (S_1) (₦)	8,000	12,000	0.6
Low demand (S_2) (₦)	4,000	-3,000	0.4

Pay-off Matrix

Question: Represent the above payoff matrix on a decision tree and find the optimum contingency strategy.

We can represent the above problem on a decision tree thus:



A Decision Tree.

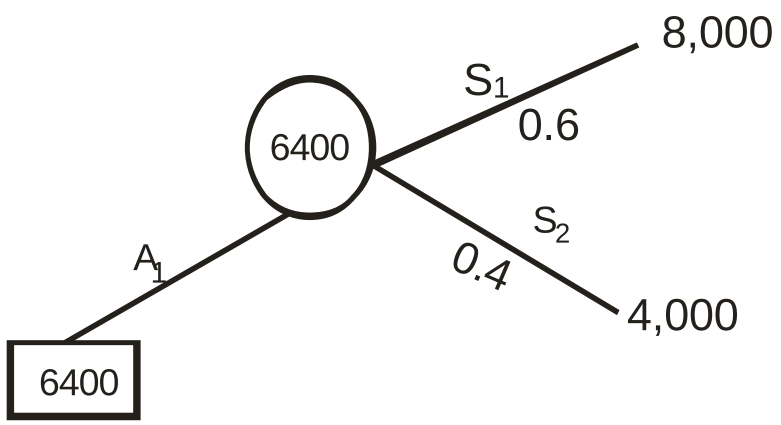
Next, we compute the EMY for alternatives A_1 and A_2.

EMV_{A1} = 8,000 x 0.6 + 4,000 x 0.4 = 6400
= 4800 x 1,600

EMV_{A2} = 12,000 x 0.6 + (-3,000) x 0.4
= 7,200 – 1,200 = <u>₦6,000</u>

EMV_{A1} gives the highest payoff

We can now draw our optimal contingency strategy thus:



Optimal Contingency Strategy

The above decision tree problem is in its simplest form. They also could be word problem to be represented on a decision tree diagram unlike the above problem that has already been put in tabular form. Let us try one of such problems.

Example
A client has contracted JELFAD, a real estate firm to help him sell three properties A,B,C that he owns in Banana Island. The client has agreed to pay JELFAD 5% commission on each sale. The agent has specified the following conditions: JELFAD must sell property A first, and this he must do within 60days. If and when A is sold, JELFAD receives 5% commission on the sale, JELFAD can then decide to back out on further sale or go ahead and try to sell the remaining two property B and C within 60 days. If they do not succeed in selling the property within 60days, the contract is terminated at this stage. The following table summarises the prices, selling Costs (incurred by JELFAD whenever a sale is made) and the probabilities of making sales

Property	**Prices of property**	**Selling Cost**	**Probability**
A	12,000	400	0.7
B	25,000	225	0.6
C	50,000	450	0.5

Pay-off Matrix

(i) Draw an appropriate decision tree representing the problem for JELFAD
(ii) What is JELFAD best strategy under the EMV approach?

(Question Adapted from Gupta and Hira (2012))

Solution
Hint: Note that the probabilities provided in the table are probabilities of sale. Therefore, to get the probability of no sale, we subtract the prob. Of sale from 1
Prob. of no Sales = 1 – prob. of sales

JELFAD gets 5% Commission if they sell the properties and satisfy the specified conditions.

The amount they will receive as commission on sale of property A, B and C are as follows
Commission on A = 5/100 x 12,000 = ₦6000
Commission on B = 5/100 x 25,000 = ₦1250
Commission on C = 5/100 x 50,000 = ₦2500

The commission calculated above are conditional profits to JELFAD. To obtain the actual profit accrued to JELFAD from the sale of the properties, we subtract the selling cost given in the table above from the commission.

JELFAD Actual profit
A = ₦600 – ₦400 = ₦200
B = ₦1250 - ₦225 = ₦1025
C = ₦2500 – ₦450 = ₦2050

We now construct our decision tree

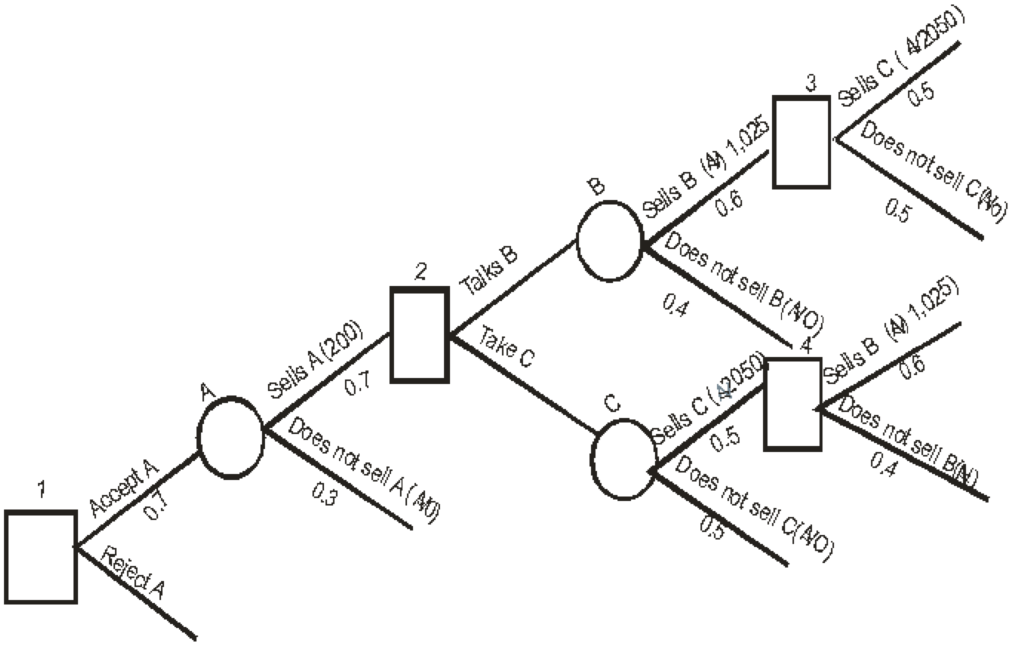
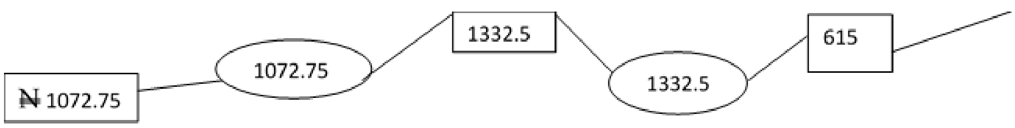


Fig. 4.4: A Decision Tree

BACKWARD PASS ANALYSIS
EMV of Node 3 = ₦ (0.5 2050 + 0.5 x0) = ₦1025
EMV of Node 4 = ₦ (0.6 x 1,025 + 0.4 x 0) =₦615
EMV of Node B = [0.6 (1025 + 1025) + 0.4 x 0] = 1230
Note: 0.6 (EMV of node 3 + profit from sales of B at node B)
EMV of Node C = [0.5 (2050 + 615) + 0.5 x 0] = ₦1332.50
Note: same as EMV of B above
EMV of Node 2 =₦1332.50 (Higher EMV at B and C)
EMV of Node A = ₦[0.7 (200 + 1,332.50) + 0.3 x 0] = ₦1072.75
EMV of Node 1 =₦1072.75

Optimal contingency strategy



The optimal contingency strategy path is revealed above. Thus the optimum strategy for JELFAD is to sell A, if they sell A, then try sell C and if they sell C, then try sell B to get an optimum expected amount of ₦1072.75.

Let us try another example as adapted from Dixon – Ogbechi (2001).

1.5 The Secretary Problem

The secretary problem was developed to analyse decision problems that are complex and repetitive in nature. This type of decision tree is a modification upon general decision tree in that it collapses the branches of the general tree and once an option is jettisoned, it cannot be recalled.

1.5.1 Advantages of the Secretary Problem Over the General Decision Tree

In addition to the advantages of the general decision try the secretary problem has the following added advantages
(1) It is easy to draw and analyse.
(2) It saves time.

1.5.2 Analysis of the Secretary Problem

The analysis of a secretary decision tree problem is similar to that of the general decision tree. The only difference is that since the multi stage decision problem could be cumbersome to formulate when the branches become too many, the secretary problem collapses the different states of nature into one. This will be demonstrated in the example below.

1.6 Summary

Now let us cast our minds back to what we have learnt so far in this unit. We learnt that the decision tree is mostly used for analysing a multi-stage decision problem. That is, when there is a sequence of decisions to be made with each decision having influence on the next. A decision tree is a pictorial method of showing a sequence of inter-related decisions and outcomes. It is a graphical representation that outlines the different states of nature, alternatives courses of actions with their corresponding probabilities. The branches of a decision tree are made up of the decision nodes at which point a decision is to be made, and the chance node at which point the EMV is to be computed.

1.7 References/Further Readings/Web Resources

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Gupta, P.K., & Hira, D.S. (2012). Operations Research, New – Delhi: S. Chand & Company.

1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

A decision tree is a decision support hierarchical model that uses a tree-like model of decisions and their possible consequences, including chance event outcomes, resource costs, and utility. It is one way to display an algorithm that only contains conditional control statements.

Answer to SAE 2

Decision trees are composed of three main parts decision nodes (denoting choice), chance nodes (denoting probability), and end nodes (denoting outcomes).

Unit 3 Operations Research (OR)

Unit Structure

1.1 Introduction

We mentioned in Unit 1, module 1, that the subject Business Decision Analysis takes its root from the discipline Operations Research or Operational Research (OR). This unit is devoted to giving us background knowledge of OR. It is however, not going to be by any way exhaustive as substantial literature been developed about quantitative approaches to decision making. The root of this literature are centuries old, but much of it emerged only during the past half century in tandem with the digital computer (Denardo, 2002). The above assertion relates only to the development of the digital computer for use in solving OR problems.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Operations Research?

- Demonstrate Characteristics of OR
- Explain Scope of Operations Research

1.3 What is Operations Research?

Many definitions of OR have been suggested by writers and experts in the field of operations Research. We shall consider a few of them.

1. Operations Research is the applications of scientific methods by inter disciplinary teams to problems involving the control of organized (Man-Machine) Systems so as to provide solutions which best serve the purpose of the organization as a whole (Ackoff & Sasieni, 1991).

2. Operations Research is applied decision theory. It uses any scientific, Mathematical or Logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems (Miller and Starr, 1973).

3. Operations research is a scientific approach to problem solving for executive management (Wagner, 1973).

4. Operations Research is the art of giving bad answers to problems, to which, otherwise, worse answers are given (Saaty, 1959).

Self-Assessment Exercise 1

Define Operations Research?

1.3.1 Development of Operations Research

Gupta and Hira (2012) traced the development of Operations Research (OR) thus:

i The Period Before World War II

The roots of OR are as old as science and society. Though the roots of OR extend to even early 1800s, it was in 1885 when Ferderick, W. Taylor emphasized the application of scientific analysis to methods of production, that the real start took place. Taylor conducted experiments in connection with a simple shovel. His aim was to find that weight load of Ore moved by shovel would result in the maximum amount of ore move with minimum fatigue. After many experiments with varying weights, he obtained the optimum weight load, which though much lighter than that commonly used, provided maximum movement of ore during a day. For a “first-class man” the proper load turned out to be 20 pounds. Since the density of Ore differs greatly, a shovel was designed for each density of Ore so as to assume the proper weight when the shovel was correctly filled. Productivity rose substantially after this change.

Henry L. Gantt, also of the scientific management era, developed job sequencing and scheduling methods by mapping out each job from machine to machine, in order to minimize delay. Now, with the Gantt procedure, it is possible to plan machine loading months in advance and still quote delivery dates accurately.

However, the first industrial Revolution was the main contributing factor towards the development of OR. Before this period, most of the industries were small scale, employing only a handful of men. The advent of machine tools – the replacement of man by machine as a source of power and improved means of transportation and communication resulted in fast flourishing industries. If became increasingly difficult for a single man to perform all the managerial functions (Planning, sales, purchasing production, etc). Consequently, a division of management functions took place. Managers of production marketing, finance, personal, research and development etc. began to appear. For example, production department was sub-divided into sections like maintenance, quality control, procurement, production planning etc.

ii World War II

During War II, the military management in England called on a team of scientists to study the strategic and tactical problems of air and land defence. This team was under the leadership of Professor P. M. S. Blackett of University of Manchester and a former Naval Officer. “Blackett’s circus”, as the group was called, included three Physiologists, two Mathematical Physicists, one Astrophysicist, one Army officer, one Surveyor, one general physicist and two Mathematicians. The objective of this team was to find out the most effective allocation of limited military resources to the various military operations and to activities within each operation. The application included effective use of newly invented radar, allocation of British Air Force Planes to missions and the determination best patters for searching submarines. This group of scientist formed the first OR team..

iii Post World War II

Immediately after the war, the success of military teams attracted the attention of industrial mangers who were seeking solutions to their problems. Industrial operations research in U.K and USA developed along different lines, and in UK the critical economic efficiency and creation of new markets. Nationalisation of new key industries further increased the potential field for OR. Consequently OR soon spread from military to government, industrial, social and economic planning.

In the USA, the situation was different impressed by is dramatic success in UK, defence operations research in USA was increased. Most of the war-experienced OR workers remained in the military services. Industrial executives did not call for much help because they were returning to peace and many of them believed that it was merely a new applicati0on of an old technique. Operations research has been known by a variety of names in that country such as Operational Analysis, Operations Evaluation, Systems Analysis, Systems Evaluation, Systems Research, Decision Analysis, Quantitative Analysis, Decision Science, and Management Science.

1.3.2 Characteristics of OR

Ihemeje (2002) presents four vital characteristics of OR.

1 The OR approach is to develop a scientific model of the system under investigation with which to compare the probable outcomes of alternative management decision or strategies.

2. OR is essentially an aid to decision making. The result of an operation study should have a direct effect on managerial action, management decision based on the finding of an OR model are likely to be more scientific and better informed.

3. It is based on the scientific method. It involves the use of carefully constructed models based on some measurable variables. It is, in essence, a quantitative and logical approach rather than a qualitative one. The dominant techniques of OR are mathematical and statistical.

Other characteristics of OR are:

4. It is system (Executive) Oriented
5. It makes use of interdisciplinary teams
6. Application of scientific method
7. Uncovering of new problems
8. Improvement in quality of decision

1.3.3 Scientific Method in Operations Research

Of these three phases, the research phase is the longest. The other two phases are equally important as they provide the basis of the research phase. We now consider each phase briefly as presented by Gupta & Hira (2012).

1.3.3.1 The Judgement Phase

The judgement phases of the scientific method of OR consists of the following:

A. **Determination of the Operation:** An operation is the combination of different actions dealing with resources (e.g men and machines) which form a structure from which an action with regards to broader objectives is maintained. For example an act of assembling an engine is an operation.

B. Determination of Objectives and Values Associated with the operation: In the judgement phase, due care must be given to define correctly the frame of references of operations. Efforts should be made to find the type of situation, e.g manufacturing, engineering, tactical, strategic etc.

C. Determination of Effectiveness Measures: The measure of effectiveness implies the measure of success of a model in representing a problem and providing a solution. It is the connecting link between the objectives and the analysis required for corrective action.

1.3.3.2 The Research Phase

The research phase of OR includes the following:

A. Observation and Data Collection: This enhances better understanding of the problem.

B. Formulation of Relevant Hypothesis: Tentative explanations, when formulated as propositions are called hypothesis. The formulation of a good hypothesis depends upon the sound knowledge of the subject–matter. A hypothesis must provide an answer to the problem in question.

C. Analysis of Available Information and Verification of Hypothesis: Quantitative as well as qualitative methods may be used to analyse available data.

D. Prediction and Generalisation of Results and Consideration of Alternative Methods: Once a model has been verified, a theory is developed from the model to obtain a complete description of the problem. This is done by studying the effect of changes in the parameters of the model. The theory so developed may be used to extrapolate into the future.

1.3.3.3 The Action Phase

The action phase consists of making recommendations for remedial action to those who first posed the problem and who control the operations directly. These recommendations consists of a clear statement of the assumptions made, scope and limitations of the information presented about the situation, alternative courses of action, effects of each alternative as well as the preferred course of action.

A primary function of OR group is to provide an administrator with better understanding of the implications of the decisions he makes. The scientific method supplements his ideas and experiences and helps him to attain his goals fully.

1.3.4 Necessity of Operations Research in Industry

Having studied the scientific methods of operations research, we now focus on why OR is important or necessary in industries. OR came into existence in connection with war operations, to decide the strategy by which enemies could be harmed to the maximum possible extent with the help of the available equipment. War situations required reliable decision making. But the need for reliable decision techniques is also needed by industries for the following reasons.

A. Complexity: Today, industrial undertakings have become large and complex. This is because the scope of operations has increased. Many factors interact with each other in a complex way. There is therefore a great uncertainty about the outcome of interaction of factors like technological, environmental, competitive etc. For instance, a factory production schedule will take the following factors into account:

i. Customer demand.
ii. Raw material requirement.
iii. Equipment Capacity and possibility of equipment failure.
iv. Restrictions on manufacturing processes.

It could be seen that, it is not easy to prepare a schedule which is both economical and realistic. This needs mathematical models, which in addition to optimization, help to analyse the complex situation. With such models, complex problems can be split into simpler parts, each part can be analysed separately and then the results can be synthesized to give insights into the problem.

B. Scattered Responsibility and Authority: In a big industry, responsibility and authority of decision-making is scattered throughout the organization and thus the organization, if it is not conscious, may be following inconsistent goals. Mathematical quantification of OR overcomes this difficulty to a great extent.

C. Uncertainty: There is a lot of uncertainty about economic growth. This makes each decision costlier and time consuming. OR is essential from the point of view of reliability.

1.4 Scope of Operations Research

We now turn our attention towards learning about the areas that OR covers. OR as a discipline is very broad and is relevant in the following areas.

A. In Industry: OR is relevant in the field or industrial management where there is a chain of problems or decisions starting from the purchase of raw materials to the dispatch of finished goods. The management is interested in having an overall view of the method of optimizing profits.

In order to take scientific decisions, an OR team will have to consider various alternative methods of producing the goods and the returns in each case. OR should point out the possible changes in overall structure like installation of a new machine, introduction of more automation etc.

Also, OR has been successfully applied in production, blending, product mix, inventory control, demand forecast, sales and purchases, transportation, repair and maintenance, scheduling and sequencing, planning, product control, etc.

B. In Defence: OR has wide application in defence operations. In modern warfare, the defence operations are carried out by a number of different agencies, namely – Air force, Army, and Navy. The activities perfumed by each of these agencies can further be divided into sub-activities viz: operations, intelligence, administration, training, etc. There is thus a need to coordinate the various activities involved in order to arrive at an optimum strategy and to achieve consistent goals.

C. In Management: Operations Research is a problem-solving and decision-making science. It is a tool kit for scientific and programmable rules providing the management a qualitative basis for decision making regarding the operations under its control. Some of the area of management where OR techniques have been successfully applied are as follows:

A. **Allocation and Distribution**

i. Optimal allocation of limited resources such as men, machines, materials, time, and money.

ii. Location and size of warehouses, distribution centres retail depots etc

iii. Distribution policy

B. Production and Facility Planning

i. Selection, location and design of production plants, distribution centre and retail outlets.

ii. Project scheduling and allocation of resources.

iii. Preparation of forecast for the various inventory items and computing economic order quantities and reorder levels.

iv. Determination of number and size of the items to be produced.

C. Procurement

i. What, how, and when to purchase at minimum purchase cost.

ii. Bidding and replacement policies.

iii. Transportation planning and vendor analysis.

D. Marketing

i. Product selection, timing, and competitive actions.

ii. Selection of advertisement media.
iii. Demand forecast and stock levels.

E. Finance
i. Capital requirement, cash-flow analysis.
ii. Credit policy, credit risks etc.
iii. Profit plan for the organisation.

F. Personnel
i. Selection of personnel, determination of retirement age and skills.
ii. Recruitment policies and assignment of jobs.
iii. Wages/salaries administration.

G. Research and Development
i. Determination of areas for research and development.
ii. Reliability and control of development projects.
iii. Selection of projects and preparation of budgets.

Self-Assessment Exercise 2

Discuss Scope of Operations Research

1.5 Scope of Operations Research in Financial Management

The scope of OR in financial management covers the following areas
i. **Cash Management:** Linear programming techniques are helpful in determining the allocation of funds to each section.
ii. Inventory Control: Inventory control techniques of OR can help management to develop better inventory policies and bring down the investment in inventories. These techniques help to achieve optimum balance between inventory carrying costs and shortage cost.
iii. Simulation Technique: Simulation considers various factors that affect and present and projected cost of borrowing money from commercial banks, and tax rates etc. and provides an optimum combination of finance (debt, equity, retained earnings) for the desired amount of capital.
iv. Capital Budgeting
It involves evaluation of various investment proposals (viz, market introduction of new products and replacement of equipment with a new one).

1.6 Summary

This has provided us with background information on the area of Operations Research. As stated in the opening unit of this study material, the discipline Business Decision Analysis or Analysis for Business

Decisions takes its root from operations research. The History and Development of operations research which is as old as science and society. Though the roots of or extend to even early 1800s, it was in 1885 when Ferderick, w. Taylor emphasized the application of scientific analysis to methods of production, that the real start took place. Taylor conducted experiments in connection with a simple shovel. However, Operations Research as we know it today can be traced to the period during War II, when the military management in England called on a team of scientists to study the strategic and tactical problems of air and land defence.

1.7 References/Further Readings/Web Resources

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Operations research (British English: operational research) (U.S. Air Force Specialty Code: Operations Analysis), often shortened to the initialism OR, is a discipline that deals with the development and application of analytical methods to improve decision-making. The term management science is occasionally used as a synonym.

Employing techniques from other mathematical sciences, such as modeling, statistics, and optimization, operations research arrives at optimal or near-optimal solutions to decision-making problems. Because of its emphasis on practical applications, operations research has overlapped with many other disciplines, notably industrial engineering. Operations research is often concerned with determining the extreme values of some real-world objective: the maximum (of profit, performance, or yield) or minimum (of loss, risk, or cost). Originating in military efforts before World War II, its techniques have grown to concern problems in a variety of industries.

Answer to SAE 2

Operation Research is today recognized as an Applied Science concerned with a large number of diverse human activities. To be precise an operation uses some valuable resources like men money machines time effort etc. the outcome of the operation has also some value. An Operation Research worker is required: (i) to minimize the input value for a specific output, or/and (ii) to maximize the output value for a specific input or/and (iii) maximize some function of these values for example the profit function (difference between output and input values) or return on investment function (ratio of output and input values).

Unit 4 Modelling in Operations Research

Unit Structure

1.1 Introduction

The construction and use of models is at the core of operations research. Operations research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources. Modelling is a scientific activity that aims to make a particular part or feature of the world easier to understand, define, quantify, visualize, or simulate. Models are typically used when it is either impossible or impractical to create experimental conditions in which scientists can directly measure outcomes. Direct measurement of outcomes under controlled conditions will always be more reliable than modelled estimates of outcomes.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Model**?**
- Demonstrate Classification of Models
- Explain Characteristics of Good models

1.3 Title of the main section: Definition of Model

Scientific modelling is an activity the aim of which is to make a particular part or feature of the world easier to understand, define, quantify, visualize, or simulate. It requires selecting and identifying relevant aspects of a situation in the real world and then using different types of

models for different aims, such as conceptual models to better understand, operational models to operationalize, mathematical models to quantify, and graphical models to visualize the subject (http://en.wikipedia.org/wiki/Scientific modelling)
Adebayo et al (2010) define Modelling as a process whereby a complex life problem situation is converted into simple representation of the problem situation. They further described a model as a simplified representation of complex reality. Thus, the basic objective of any model is to use simple inexpensive objects to represent complex and uncertain situations. Models are developed in such a way that they concentrate on exploring the key aspects or properties of the real object and ignore the other objects considered as being insignificant. Models are useful not only in science and technology but also in business decision making by focusing on the key aspects of the business decisions (Adebayo et al, 2010).

Self-Assessment Exercise 1

Differentiate between model and modelling.

1.3.1 Classification of Models

The following are the various schemes by which models can be classified:

i. By degree of abstraction
ii. By function
iii. By structure
iv. By nature of the environment

Let us now briefly discuss the above classifications of models as presented by Gupta and Hira (2012)

i. By Degree of Abstraction.

Mathematical models such as Linear Programming formulation of the blending problem, or transportation problem are among the most abstract types of models since they require not only mathematical knowledge, but also great concentration to the real idea of the real-life situation they represent.

Language models such as languages used in cricket or hockey match commentaries are also abstract models.
Concrete models such as models of the earth, dam, building, or plane are the least abstract models since they instantaneously suggest the shape or characteristics of the modelled entity.

ii. By Function

The types of models involved here include

Descriptive models which explain the various operations in non-mathematical language and try to define the functional relationships and interactions between various operations. They simply describe some aspects of the system on the basis of observation, survey, questionnaire, etc. but do not predict its behaviour. Organisational charts, pie charts, and layout plan describe the features of their respective systems.

Predictive models explain or predict the behaviour of the system. Exponential smoothing forecast model, for instance, predict the future demand

Normative or prescriptive models develop decision rules or criteria for optimal solutions. They are applicable to repetitive problems, the solution process of which can be programmed without managerial involvement. Liner programming is also a prescriptive or normative model as it prescribes what the managers must follow.

iii. By Structure

- **Iconic or physical models**

In iconic or physical models, properties of real systems are represented by the properties themselves. Iconic models look like the real objects but could be scaled downward or upward, or could employ change in materials of real object. Thus, iconic models resemble the system they represent but differ in size, they are images. They thus could be full replicas or scaled models like architectural building, model plane, model train, car, etc.

- **Analogue or Schematic Models**

Analogue models can represent dynamic situations and are used more often than iconic models since they are analogous to the characteristics of the system being studied. They use a set of properties which the system under study possesses. They are physical models but unlike iconic models, they may or may not look like the reality of interest. They explain specific few characteristics of an idea and ignore other details of the object. Examples of analogue models are flow diagrams , maps, circuit diagrams, organisational chart etc.

- **Symbolic or mathematical models**

Symbolic models employ a set of mathematical symbols (letters, numbers etc.) to represent the decision variables of the system under study. These variables are related together by mathematical equations/in-equations which describe the properties of the system. A solution from the model is, then, obtained by applying well developed mathematical techniques. The relationship between velocity, acceleration, and distance is an example of a mathematical model. Similarly, cost-volume-profit relationship is a mathematical model used in investment analysis.

iv. By Nature of Environment

- **Deterministic models**

In deterministic models, variables are completely defined and the outcomes are certain. Certainty is the state of nature assumed in these models. They represent completely closed systems and the parameters of the systems have a single value that does not change with time. For any given set of input variables, the same output variables always result. E.O.Q model is deterministic because the effect of changes in batch size on total cost is known. Similarly, linear programming, transportation, and assignment models are deterministic models.

- **Probabilistic Models**

These are the products of the environment of risk and uncertainty. The input and/or output variables take the form of probability distributions. They are semi-closed models and represent the likelihood of occurrence of an event. Thus, they represent to an extent the complexity of the real world and uncertainty prevailing in it. As a example, the exponential smoothing method for forecasting demand a probabilistic model.

Self-Assessment Exercise 2

List the different classifications of models

1.3.2 Characteristics of Good models

The following are characteristics of good models as presented by Gupta and Hira (2012)

1. The number of simplifying assumptions should be as few as possible.
2. The number of relevant variables should be as few as possible. This means the model should be simple yet close to reality.
3. It should assimilate the system environmental changes without change in its framework.

Self-Assessment Exercise 3

Outline five characteristics of a good model.

1.3.3 Advantages of Models

1 It provides a logical and systematic approach to the problem.
2. It indicates the scope as well as limitation of the problem.
3. It helps in finding avenues for new research and improvement in a system.
4. It makes the overall structure of the problem more comprehensible and helps in dealing with the problem in its entirety.

1.3.4 Limitations of Models

1. Models are more idealised representations of reality and should not be regarded as absolute in any case.

2. The reality of a model for a particular situation can be ascertained only by conducting experiments on it.

1.4 Model Construction

Formulating a problem requires an analysis of the system under study. This analysis shows the various phases of the system and the way it can be controlled. Problem formulation is the first stage in constructing a model. The next step involves the definition of the measure of effectiveness that is, constructing a model in which the effectiveness of the system is expressed as a function of the variables defining the system. The general Operations Research form is

$$E = f(x_i, y_i),$$

Where E = effectiveness of the system,
x_i = controllable variables,
y_i = uncontrollable variables but do affect E.

Deriving a solution from such a model consists of determining those values of control variables xi, for which the measure of effectiveness is measure of effectiveness is optimised. Optimised includes both maximisation (in case of profit, production capacity, etc.) and minimisation (in case of losses, cost of production, etc.).

The following steps are involved in the construction of a model

1. Selecting components of the system
2. Pertinence of components
3. Combining the components
4. Substituting symbols

1.5 Types of mathematical Models

The following are the types of mathematical models available:

1. Mathematical techniques
2. Statistical techniques
3. Inventory models
4. Allocation models
5. Sequencing models

1.6 Summary

This unit introduced us to the concept of models. We have learnt about the importance of models to operations research. The unit opened with a consideration of various definitions of models. Among the definitions is that by Adebayo et al (2010) who defined modelling as a process whereby a complex life problem situation is converted into simple representation of the problem situation. A model as used in Operations Research is defined as an idealised representation of real life situation. It represents one of the few aspects of reality.

1.7 References/Further Readings/Web Resources

Adebayo, O.A. et al (2006). Operations Research in Decision and Production Management.

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Models are abstractions of reality, and modeling is the process of creating these abstractions of reality (Wallace 1994). Models take a variety of forms based upon their function, structure, and degree of quantification (Tersine and Grasso 1979).

Answer to SAE 2

There are two different flavors of classification models:

Binary classification models, where the output variable has a Bernoulli distribution conditional on the inputs;

Multinomial classification models, where the output has a Multinoulli distribution conditional on the inputs.

Remember that a Bernoulli random variable can take only two values, either 1 or 0. So, a binary model is used when the output can take only two values.

The Multinoulli distribution is more general. It can be used to model outputs that can take two or more values. If the output variable can take J different values, then it is represented as a $1 imes J$ Multinoulli random vector, that is, a random vector whose realizations have all entries equal to 0, except for the entry corresponding to the realized output value, which is equal to 1$.$

Answer to SAE 3

Characteristics of a Good Model A model does not always have the characteristic of being a yard stick it can be explanatory rather than merely descriptive. Following are the chief characteristics that a good OR model should have.

1. A good number should be capable of taking into account new formulations without having any significant change in its frame.

2. The number of assumptions made should be as small as possible.

3. It should be simple and coherent, i.e. number of variables used should be small.

4. It should be open to a parametric type of treatment. Such situations are often foced when response to an advertising campaign or the customer acceptance of a new product is studied.

5. A model should not take much time in its construction for any problem.

6. It should express the relations and interrelations of action and reaction of cause and effect in operational situations.

Unit 5 The Transportation Model

Unit Structure

1.1 Introduction

We now go on to another mathematical programming problem which is a special case of linear programming model. Unlike the linear programming method, treated in the last chapter of this book, which focuses on techniques of minimizing cost of production or maximizing profit, this special linear programming model deals with techniques of evolving the lowest cost plan for transporting product or services from multiple origins which serve as suppliers to multiple destinations that demand for the goods or services.

As an example, suppose cows are to be transported in a day from 5 towns in the northern part of Nigeria to 4 towns in the south. Each of the five northern towns has the maximum they can supply in a day, while each of the town in the southern part also has the specified quantities they demand for. If the unit transportation cost from a source to each of the destinations is known it is possible to compute the quantity of cows to be transported from each of the northern towns to the southern towns in order to minimize the total transportation cost.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Demonstrate four assumptions of the transportation model.

- Explain Vogel’s Approximation Method
- Describe the nature of a transportation problem.

1.3 Assumptions Made in the Use of the Transportation Model

The transportation model deals with a special class of linear programming problem in which the objective is to transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost (Murthy, 2007)

In using transportation model the following major assumptions are made.

1. The Homogeneity of materials to be transported. The materials or items to be transported by road, sea, air or land must be the same regardless of their specific source or specified locations.
2. Equality of transportation cost per unit. The transportation cost per unit is the same irrespective of which of the materials is to be transported.
3. Uniqueness of route or mode of transportation between each source and destination.

In using the transportation model it is essential that the following information are made available.

• The list of each source and its respective capacity for supplying the materials

• The list of each destination and its respective demand for that period.

• The unit cost of each item from each source to each destination.

Self-Assessment Exercise 1

Describe the nature of a transportation problem.

1.3.1 Theoretical Consideration

Suppose we have a transportation problem involving movement of items from m sources (or origins) to n location (destination) at minimum cost. Let c_{ij} be the unit cost of transporting an item from source i to location j; a_1 be the quantity of items available at source i and b_j the quantity of item demanded at location j. Also, let x_{ij} be the quantity transported from i^{th} source to j^{th} location then total supply = $\sum a_{ij}$, while total demand = $\sum b_{ij}$. This problem can be put in tabular form as shown below:

	1	2	3	…. n	Supply
1	c_{11}	c_{12}	c_{13}	…. n	a_1
2	c_{21}	c_{22}	c_{23}	…. n	a_2
3	c_{31}	c_{32}	c_{33}	…. n	a_3
:	:	:	:	:	:
m	c_{m1}			c_{mn}	a_m
Demand	b_1	b_2	b_3	…. b_m	

$when \sum_{i=1}^{m} a_{ij} = \sum_{j=1}^{n} b_{ij}\ then\ we\ have\ the\ balanced\ case.$ The linear

Programming model can be formulated as follows: $\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}$
Subject to the constraints-

$$\sum_{i=1}^{n} a_1 =\ldots \ldots i = 1,2,\ldots..$$

$$\sum_{i=1}^{m} = b_{1=\ldots\ \ldots\ldots\ i=1\ \ldots\ldots} n$$
$$X_{ij} \geq 0$$

1.3.2 General Procedure for Setting up a Transportation Model

Convert statement of the problem into tabular form showing the total supply and total demand for each of the sources and destinations.
• Check that total number of supply equals the total number of demand to know whether the transportation model is of the balanced or unbalanced type.

• Allocate values into the necessary cells using the appropriation techniques for the method of allocation of quantities that you have selected. We expect the number of allocated cells to be m+n-l where m is the number of rows and n is the number of columns otherwise degenerating occurs.
. Compute the total cost of transportation.
Solution of transportation problem comes up in two phases namely:

- The initial feasible solution
- The optimum solution to the transportation problem

1.3.3 Developing an Initial Solution

When developing an initial basic feasible solution there are different methods that can be used. We shall discuss three methods used namely;
1. The North West Corner Method
2. The Least Cost Method
3. Vogel's Approximation Method
It is assumed that the least cost method is an improvement on the North West Corner method, while the Vogel's approximation method is an improvement of the least cost method.

The North West Corner Method
This is the simplest and most straight forward format of the method of developing an initial basic feasible solution. The initial solution resulting from this method usually Operations Research in Decision Analysis and Production Management results in the largest total transportation cost among the three methods to be discussed. To explain how to use this method, we present an illustrative data of a transportation problem in the example below:

Table 1 Supply and demand of cows

Sources	**Lagos**	**Akure**	**Awka**	**Supply**
Sokoto	90	85	70	600
Kano	175	110	95	1400
Maiduguri	205	190	130	1000
Demand	1600	1050	350	

Example 1
Suppose the table above gives us the supply of cows from three sources in the north and the demands by three locations in the southern part of Nigeria. The quantities inside the cell represent the unit costs, in naira. of transporting one cow from one source to one location. Use the North West Corner method to allocate the cows in such a way as to minimise the cost of transportation and find the minimum cost.

Solution
We observe that the total demand = 1600 + 1050 + 350 =3U00 and total supply = 600 + 1400 + 1000 = 3000. Since demand supply we have a balanced transportation problem
To use the North West Corner method to allocate all the cows supplied to the cells where they are demanded, we follow this procedure:

a. Starting from the North West Corner of the table allocate as many cows as possible to cell (1,1). In this case it is 600. This exhausts the supply from Sokoto leaving a demand of 1000 cows for Lagos.

Sources	**Lagos**	**Akure**	**Awka**	**Supply**
Sokoto	90 600	85 -	70 -	 ~~600~~
Kano	175	110	95	1400
Maiduguri	205	190	130	~~350~~ 100
Demand	~~1600~~ 1000	1050	350	

b. Allocate 1000 cows to cell (2,1) to meet Lagos demand leaving a supply of 400 cows in Kano. Cross out the 1000 in column 1 where the demand has been met

Sources	**Lagos**	**Akure**	**Awka**	**Supply**
Sokoto	90 600	85 -	70 -	 ~~600~~
Kano	175 1000	110 -	95	~~1400~~ 400
Maiduguri	205	190	130	100
Demand	~~1600~~ ~~1000~~	1050	350	

1. Allocate 400 cows to cell (2, 2) to exhaust the supply from Kano leaving a demand of 650 in Akure. Cross out the 1400 in row 2 which has been satisfied.

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 600	85 -	70 -	 ~~600~~
Kano	175 1000	110 400	95 -	~~400~~ ~~1400~~
Maiduguri	205 -	190 650	130 350	1000
Demand	~~1600~~ ~~1000~~	~~1050~~ 650	350	

1.0 Allocate 650 cows to cell (3,2) to satisfy the demand in Akure

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 600	85 -	70 -	 600
Kano	175 1000	110 400	95 -	400 ~~1400~~
Maiduguri	205 -	190 650	130 350	350 ~~1000~~
Demand	~~1600~~ ~~1000~~	~~1050~~ 650	350	

1. Allocate 350 cows to cell (3, 3) to satisfy the demand in Awka and exhaust the supply in Maiduguri. Cross out the 350.

Location

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 600	85 -	70 -	~~600~~
Kano	175 1000	110 400	95 -	~~400~~ ~~1400~~
Maiduguri	205 -	190 650	130 350	~~350~~ ~~1000~~
Demand	~~1600~~ ~~1000~~	~~1050~~ 650	~~350~~	

This completes the allocation. We observe that out of the 9 cells only 5 cells have been allocated. If m is the number of rows. n is the number of columns the total number of allocated cells in this case is m + n — 1 i.e 3+ 3 — 1 = 5.
The transportation cost is found by multiplying unit cost for each cells by its unit allocation and summing it up.
i.e. C =∑ (unit cost x cell allocation)
= (600 x 90) + (l000x 175) + (110 x 400) + (190 x 650) + (130 x 350)
= 54000 + 175000 x 44000 + 123500 + 45500
= N442000
This can be summarised in tabular form as follows

Cell	Quantity	Unit Cost	Cost
(1, 1)	600	90	54000
(2, 1)	1000	175	17500
(2, 2)	400	110	4400
(3, 2)	650	190	123500
(3, 3)	130	350	45500
			442000

The Least Cost Method
This method is also known as the minimum cost method. Allocation commences with the cell that has the least unit cost and other subsequent method of allocation is similar to the North West Corner method

Example 2
Solve example 2 using the Least Cost method
Solution
We observe that the total demand = 1600 + 1050 + 350 = 3000 and total supply = 600 + 1400 + 1000 3000. Since demand = supply we have a balanced transportation problem.
We note that the least cost per unit in this problem is N70 in cell (1, 3). We do the allocation as follows:

Step 1: Allocate 350 to cell (1, 3) to satisfy the demand at Awka and leaving a supply of 250 cows at Sokoto. Cross out column3 that has been satisfied.

Table 2

Location

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 -	85 250	70 350	250 ~~600~~
Kano	175	110	95 -	1400
Maiduguri	205	190 -	130 -	1100
Demand	1600	1050	~~350~~	

Step 2:

Allocate 250 cows to cell (1, 2) that has the next smallest unit cost of N85 to complete the supply from Sokoto leaving us with demands of 800 cows at Akure. Cross out exhausted row one.

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 -	85 250	70 350	~~250~~ ~~600~~
Kano	175	110	95 -	~~600~~ 1400
Maiduguri	205	190	130	1100
Demand	1600	~~1050~~ 800	~~350~~	

Step 3: Allocate 800 cows to cell (2, 2) with the next least cost leaving 110 to satisfy demand at Akure, leaving us with supply of 600 ram at Kano cross out satisfied column 2.

Location

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 -	85 250	70 350	~~250~~ ~~600~~
Kano	175 600	110 800	95 -	600 ~~1400~~
Maiduguri	205 1000	190 -	130 -	1100
Demand	1600	~~1050~~ ~~800~~	~~350~~	

Step 4: Allocate 600 cows to cell (2, 1) which has the next least cost of 175 thereby exhausting supply from Kano, leaving 1000 cows demand in Lagos. Cross out exhausted row 2.

Location

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 -	85 250	70 350	~~250~~ ~~600~~
Kano	175 600	110 800	95 -	~~600~~ ~~1400~~
Maiduguri	205	190 -	130 -	 1100
Demand	~~1600~~ 1000	~~1050~~ ~~800~~	~~350~~	

Step 5: Allocate the remaining 1000 cows to cell (3, 1) to exhaust the supply from Maiduguri. This completes the allocation.

Location

Sources	Lagos	Akure	Awka	Supply
Sokoto	90 -	85 250	70 350	~~250~~ ~~600~~
Kano	175 600	110 800	95 -	~~600~~ ~~1400~~
Maiduguri	205 1000	190	130 -	 ~~1100~~
Demand	~~1600~~ ~~1000~~	~~1050~~ ~~800~~	~~350~~	

The minimum cost of transportation is given by:
(85 x 250) + (70 x 350) + (175 x 360) + (110 x 800) + (205 x 1,000) =2 1,250 + 24,500 + 105,000 + 88,000 + 205,000 = 443,750
We can represent it in a tabular form as follow

Cells	Quantity	Unit Cost	Cost
(1, 2)	250	80	21250
(1, 3)	350	70	24500
(2, 1)	600	175	105000
(2, 2)	800	110	88000
(3, 1)	1000	205	205000
			443750

Example 3

In the table below, items supplied from origins A, B, C and D and those demanded in locations 1, 2, 3 and 4 are shown. If the figures in the boxes are the unit cost of moving an item from an origin to a destination, use the least cost method to allocate the material in order to minimize cost of transportation.

Destination

Origin	**1**	**2**	**3**	**4**	**Supply**
A	29	41	25	46	1250
B	50	27	45	33	2000
C	43	54	49	40	500

D	60	38	48	31	2750
Demand	3250	250	1750	1250	

Solution:
We check the total demand and supply. In this case both totals are equal to 6500. It is balanced transportation problem. We set up the allocation as follows;

Step 1
Look for the least cost. It is 25 in cell (1, 3) Allocate 1250 to cell (1, 3) and thus exhaust the supply by A while leaving a demand of 500 in column three. Cross out the 1250 in A and the other cells on row 1.

Step 2
Look for the least cost among the remaining empty cells. This is 27 in cell (2,2) Allocate all the demand of 250 to that cell. Cross out the 250 in column 2 and the 2000 supply in row 2 becomes 1750. Cross out the empty cells in column 2.

Step 3
Once again identify the cell with the least cost out of the remaining empty cells. This is 31 in cell (4, 4). Allocate all the 1250 in column 4 to this cell to satisfy the demand by column 4. This leaves a supply of 1500 for row D. Cross out all the other cells in column 4. Step 4.
Identify the cell having the least cost among the remaining cell. This is 43 in cell (3, 1). Allocate all the 500 supply to this cell and cross out the 500 in satisfied row 3 as well as any empty cell in that row. Also cross out the 3250 row and replace with 3250 — 500 = 2750.

Step 5
Identify the cell with the least cost among the empty cell. This is 45 in cell (2,3). Since column 3 needs to exhaust 500 we allocate this to cell (2, 3). Cross out the 500 and any empty cell in that column. Row 2 needs to exhaust 1250 quantity.

Step 6
Examine the remaining empty cells for the cell with the least cost. This is cell (2, 1) with N50. Allocate all the 1250 into this cell. Cross out 2750 in column 1 and write the balance of 1500.

Step 7
The last cell remaining is (4, 1). Allocate the remaining 1500 to this cell thus satisfying the remaining demands of row 4 and the remaining supply of column 1.

Destination

Origin	**1**	**2**	**3**	**4**	**Supply**
A	29 -	41 -	25 1250	46 -	~~1250~~
B	50 1250	27 250	45 500	33	~~7501250~~ ~~2000~~
C	43	54	49	40	~~500~~

	500	-	-	-	
D	60 1500	38 -	48 -	31 1250	~~27501500~~
Demand	~~3250~~ ~~2750~~	~~250~~	~~1750~~ ~~500~~	~~1250~~	

We then calculate the cost as follows

Cells	Quantity	Unit Cost	Cost
(2, 1)	1250	50	62500
(3, 1)	500	43	21500
(4, 1)	1500	60	90000
(2, 2)	250	27	6750
(1, 3)	1250	25	31250
(2, 3)	500	45	22500
(4, 4)	1250	31	38750
			273,250

Note that total number of cell allocations m + n -1 where m = 4 and n = number of columns = 4

Vogel's Approximation Method (VAM)

This technique of finding an initial solution of the transportation is an improvement on both the least cost and North West corner methods. It involves minimization of the penalty or opportunity cost. This penalty cost is the cost due to failure to select the best alternatives. This technique can thus be regarded as the penalty or regret method.

The steps for using the VAM method can be presented as follows.

• Check the row and column totals to ensure they are equal.

• Compute the row and column parallels for the unit costs. This is done by finding the difference between the smallest cell cost and the next smallest cell cost for each row and column.

. Identify the row or column with the highest penalty cost.

• Allocate to the cell with the least cost in the identified cell in step 3 the highest possible allocation it can take.

• Cross out all the redundant cell.

• Re compute the penalty cost and proceed to allocate as done in the previous steps until all the cells have been allocated.

• Check for the m + n — 1 requirement.

• Compute the total cost.

Example 4

Solve example 4 using the Vogel's Approximation Method

Solution

(i) The first step is to check the row and column totals and since both totals equal 6500, it is a balanced transportation problem.

(ii) Next we compute the row and column penalty costs denoted by d_1! and d_1 respectively and obtain the following table:

	1	2	3	4	Supply	d_1!
A	29	41	25	46	1250	4
B	50	27	45	33	2000	6
C	43	54	49	40	500	3
D	60	38	48	31	2750	7
Demand	3250	2500	1750	1250	6500	
d_1	14	11	20	2		

(iii) The highest penalty cost is 20; we allocate to the least unit cost in that column the highest it can take. The least is 25 and is allocated 1250 as shown below.
(iv) The next step is to re-compute the penalty costs, d_2! and d_2 for the unbalanced cells in both rows and column. The results obtained are as follows.

	1	2	3	4	Supply	d_1!	d_2!
A	29	41	25 1250	46 -	~~1250~~	4	-
B	50	27 250	45	33	~~2000~~ 1750	6	6
C	43	54 -	49	40	500	3	3
D	60	38 -	48	31	2750	7	7
Demand	3250	~~250~~	~~1750~~ 500	1250	6500		
d_1	14	11	20	2			
d_2	7	11	3	2			

Since all cells in the row 1 have all been allocated d2 0 for that row.
(v) The highest penalty cost is 11. We allocate the maximum allocation for cell (2,1). which has the least cost of 27 which is 250. Row 2 has a balance of 1750 to be exhausted while column 2 is satisfied.
(vi) Next we compute penalty costs d_3! and d_3 for the unallocated cells and obtain the following.

	1	2	3	4	Supply	d_1!	d_2!	d_3!
A	29	41	25 1250	46 -	~~1250~~	4	-	-
B	50	27 250	45	33	~~2000~~ 1750	6	6	12
C	43	54 -	49	40	500	3	3	3
D	60	38 -	48	31 1250	~~2750~~ 1500	7	7	17
Demand	3250	250	~~1750~~ 500	1250	6500			

d_1	14	11	20	2				
d_2	7	11	3	2				
d_3	7	-	3	2				

The highest penalty cost is 17 and the unit cost is 31. We give the cell with unit cost of 31 its maximum allocation of 1250 thereby exhausting the demand in column 3 and leaving a balance of 1500 in row 4.
(vii) We re-compute the penalty cost $d_4!$ and d_4 and then fill up all the other cells

	1	2	3	4	Supply	$d_1!$	$d_2!$	$d_3!$	$d_4!$
A	29 -	41 -	25 1250	46 -	1250	4	-	-	
B	50 1750	27 250	45	33	2000 1750	6	6	12	5
C	43 500	54 -	49	40	500	3	3	3	6
D	60 1000	38 -	48 500	31 1250	2750 1500	7	7	17	12
Demand	3250	250	1750 500	1250	2750				
d_1	14	11	20	2					
d_2	7	11	3	2					
d_3	7	-	3	2					
d_4	7	-	3	-					

The highest penalty costs is 12, we allocate cell (4, 3) having the least unit cost of 48 maximally with 500 to exhaust row 3. All the remaining cells in column 3 are given 0 allocations since column 3 has now been exhausted.
(viii) We then allocate the remaining empty cells as follows: cell (2, 1) is given the balance of 1750 to exhaust the supply of row 2. Cell (3, 1) is given supply of 500 to exhaust the supply of row 3 while cell (4, 1) is allocated to the balance of 1000.
(ix) A total of 7 cells have been allocated satisfying m + n — 1 criterion. We then compute the minimum cost of allocation in the transportation model and obtain the following;

Cells	Quantity	Unit Cost	Cost
(1, 3)	25	1250	62500
(2, 1)	50	1750	21500
(2, 2)	27	250	90000
(3, 1)	43	500	6750
(4, 1)	60	1000	31250
(4, 3)	48	500	22500
(4, 4)	31	1250	38750
Total			**269750**

We observed that this value is an improvement on the value of N273, 250 obtained by the Least Cost Method.

1.3.4 The Unbalanced Case

Suppose the total number of items supplied is not equal to the total number of items demanded. When this happens then we have an unbalanced transportation problem. To solve this type of problem we adjust the transportation table by creating a dummy cell for source or demand column or row to balance the number. The dummy cells created are allocated zero transportation unit cost and the problem is solved using appropriate method as before. We have two cases, namely (1) the case when supply is greater than demand (SS>DD) (2) the case when the demand is greater than the supply (DD >SS). The next two examples will show us how the dummy is created and how the problem is solved.

Example 5

The table below shows us how some items are transported from five locations A,B,C,D to four location P,Q,R,S with the unit cost of transportation in them being shown in the box. Determine the initial feasible solution by finding minimum cost of transportation using the North West Corner method.

	P	Q	R	S	Supply
A	150	120	135	105	2000
B	90	140	130	140	8000
C	120	100	120	150	7000
D	180	140	200	162	3000
E	110	130	100	160	2500
Demand	1000	4000	8500	4500	

The total from the supply is 2000 + 8000 + 7000 + 3000 + 2500 = 22500. The total quantity demanded is 1000 + 4000 + 8500 + 4500 = 18,000. Since the supply is more than the demand. We then create out a new dummy variable, with column T to take care of the demand with value of 22500 - 18000 = 4500. We now have a table with five rows and time columns.

	P	Q	R	S	T	Supply
A	150	120	135	105	0	2000
B	90	140	130	140	0	8000
C	120	100	120	150	0	7000
D	180	140	200	162	0	3000
E	110	130	100	160	0	2500
Demand	1000	4000	8500	4500	4500	

We then carry out the allocation using the usual method to get the table below. So the table becomes.

	P	Q	R	S	T	Supply
A	150 1000	120 1000	135	105	0	~~1000~~ ~~2000~~
B	90	140 3000	130 5000	140	0	~~5000~~ ~~8000~~
C	120	100	120 3500	150 3500	0	~~3500~~ ~~7000~~
D	180	140	200	162 1000	0 2000	~~2000~~ ~~3000~~
E	110	130	100	160	0 2500	~~2500~~
Demand	~~1000~~	~~4000~~ ~~3000~~	~~8500~~ ~~3500~~	~~4500~~ ~~1000~~	~~4500~~ ~~2000~~	

The allocations are shown above. The cost can be computed as follows

Cells	Quantity	Unit Cost	Cost
(1, 1)	1000	180	180000
(1, 2)	1000	120	120000
(2, 2)	3000	140	420000
(2, 3)	5000	130	650000
(3, 3)	3500	120	420000
(3, 4)	3500	150	525000
(4, 4)	1000	162	162000
(4, 5)	2000	0	0
(5, 5)	2500	0	0
Total			2477000

Example 6

Find the minimum cost of this transportation problem using the North West Corner method.

	1	2	3	Supply
A	10	8	12	150
B	16	14	17	200
C	19	20	13	300
D	0	0	0	250
Demand	300	200	400	900

Solution

Total for demand = 300 + 200 + 400 = 900 Total for supply is 150 + 200 + 300 = 650

Here Demand is greater than Supply. According to Lee (1983) one way of resolving this is to create a dummy variable to make up for the 900 — 600 = 250 difference in the supply and to assign a value of 0 to this

imaginary dummy variable. We then end up with 4 x 3 table as shown below:
The cells are now allocated using the principles of North West Corner method

	1	2	3	Supply
A	10 150	8 -	12 -	150
B	16 150	14 50	17 -	~~200~~ ~~150~~
C	19 -	20 - 150	13 - 150	~~300~~
D	0 -	0 -	0 250	~~250~~
Demand	~~300~~ ~~150~~	~~200~~ ~~150~~	~~400~~ ~~150~~	900

The cost can be computed as follows

Cells	Quantity	Unit Cost	Cost
(1, 1)	150	10	1500
(2, 1)	150	16	2400
(2, 2)	50	14	700
(3, 2)	150	20	3000
(3, 3)	150	13	1950
(4, 3)	250	0	0
Total			**9550**

1.3.5 Formulating Linear Programming Model for the Transportation Problem

The linear programming model can also be used for solving the transportation problem. The method involves formulating a linear programming model for the problem using the unit costs and the quantities of items to be transported. In case the decision variables are the quantities to be transported, we may represent the decision variable for cell 1 column 1 as x_{11}, cell 2 column 2 as x_{22} e.t.c. Constraints are created for both rows (supply) and column (demand). There is no need of constraints for the total. For the balanced case we use equality for the supply and demand constants. For the unbalanced case we use equality for the lesser quantity between supply and demand while the greater of the two will use the symbol “less than or equal to (≤)“. Dummy variables will be created to balance the requirements for demand and supply.

Example 7

Formulate a linear programming model for the transportation problem

	Ofada	Ewokoro	Abeokuta	Supply
Ikeja	5	8	2	250
Yaba	4	3	7	100
Agege	9	6	5	450
Lagos	3	4	6	300
Demand	600	200	300	

Solution

Total demand 600 + 200 + 300 = 1100 while

Total supply 250 + 100 + 450 + 300 = 1100. This is a balanced transportation problem.

We therefore use equality signs for the supply and demand constraints.

Let X_{11}, X_{12}, X_{13} be the quantities for row 1. The other quantities for the remaining rows are similarly defined.

The objective function consists of all the cell costs as follows

Minimize $5X_{11} + 8X_{12} + 2X_{13} + 4X_{21} + 3X_{22} + 7X_{23} + 9X_{31} + 6X_{32} + 5X_{33} + 3X_{41} + 4X_{42} + 6X_{43}$

The constraints are

Supply (Row) $X_{11} + X_{12} + X_{13} = 250$

$X_{21}+X_{22}+X_{23} = 100$

$X_{31} +X_{32}+X_{33} = 450$

$X_{41} +X_{42}+X_{43} = 300$

Demands (column) $X_{11} + X_{11} + X_{11} = 600$

$X_{21}+X_{22}+X_{23}=200$

$X_{31} + X_{32} + X_{33} = 300$

1.3.6 Improving the Initial Feasible Solution Through Optimization

After the feasible solutions have been found using the North West Corner Method, the Least Cost Method and the Vogel's Approximation Method (VAM) we move on to the next and final stage of finding the minimum transportation cost using optimisation technique on the obtain feasible solution. Various methods have been proffered for finding this optimum solution among which are the following:

1) The Stepping Stone Method

2) The Modified Distribution Method (MOD1) which is an improvement on the stepping stone method and is more widely accepted.

1.3.7 Determination of the Optimal Transportation Cost Using the Stepping Stone Method

The optimal solution is found due to need to improve the result obtained by the North West Corner Method, the Least Cost Method and the Vogel's Approximation Method.

The stepping stone method is used to improve the empty or unallocated cells by carefully stepping on the other allocated cells. The method was pioneered by Charnes. A and Cooper W. W, and is based on the idea of the Japanese garden which has at the center stepping stones carefully laid across the path which enables one to cross the path by stepping carefully on the stones.

The criterion of m + n - 1 number of occupied cells must be satisfied to avoid degeneracy. The stepping stone method is similar to the simplex method in the sense that occupied cells are the basic variables of the simplex method while the empty cells are the non-basic variables. To find the optimum solution we assess a stepping stone path by stepping on allocated cells in order to evaluate an empty cell. The set of allocated cells that must be stepped on in order to evaluate an unallocated cell is known as the stepping stone path. It is identical to the positive or negative variables on a non-basic column of the simplex tableau. The critical thing to do is to find the stepping stone path in order to find out the net change in transportation by re-allocation of cells. In re-allocating cells it is very important that the total supply and total demand is kept constant.

The following steps are essential in using the stepping stone method:

• Identify the stepping stone path for all the unallocated cells.

• Trace the stepping stone paths to identify if transportation of one unit will incur a difference in total transportation cost. One may need to skip an empty cell or even an occupied cell when tracing the path. We usually represent increase with a positive sign and a decrease with negative sign.
• Using the traced stepping stone paths analyze the unit transportation cost in each cell. Compute the Cost Improvement Index (CII) for each empty cell.
• Select the cell with the largest negative CII for allocation, bearing in mind the need to ensure that the demand and supply are both kept constant and calculate the cost of transportation. Re-compute the CII for the new table. if all the CIls are positive then we have reached the optimum allocation otherwise the procedure is iterated until we get positive values for all the CIIs in the transportation table.

The following points should be noted when using the stepping stone method:

• It will be observed that if iteration is necessary the transportation cost in cacti of the subsequent table will reduce until we obtain the optimum solution.
• Only sources transport goods to destinations. Re-allocation is done using horizontal movements for rows and vertical movement columns.
• Every empty cell has a unique stepping stone path.
• The stepping stone path consists of allocated cells.

Example 8
You are given the following transportation table. Find (a) the initial basic feasible solution using the Least Cost Method (b) the optimum solution using the Stepping Stone.
Method

	Abuja	**Bauchi**	**Calabar**	**Supply**
Ibadan	6	7	9	70000
Jos	5	8	7	10000
Kano	7	9	6	150000
Demand	130000	90000	110000	

Solution
This is a case of unbalanced transportation problem since the total demand is 330,000 while the total supply is 230,000. We therefore create a dummy row of 100,000 to balance up. The result obtained by the Least Cost Method is shown in the table below;

	Abuja	**Bauchi**	**Calabar**	**Supply**
Ibadan	6 70000	7	9	70000
Jos	5 10000	8	7	10000
Kano	7 50000	9	6 100000	150000
Dummy	0	0 90000	0 10000	100000
Demand	130000	90000	110000	330000

Minimum cost is given as follows

Cell	**Quantity**	**Unit cost**	**Cost**
(1,1)	70000	6	420000
(2,1)	10000	5	50000
(3,1)	50000	7	350000
(3,3)	100000	6	600000
(4,2)	90000	0	0
(4,3)	10000	0	0
			1420000

(b) We now find the optimum solution using the result obtained by the Least Cost Method. We first identify the empty cells in the table of initial feasible solution. The cells are **Cell (1,2), Cell (1, 3), Cell (2,2), Cell (2,3) and Cell (4,1)**
Next we evaluate empty cells to obtain the stepping stone path as well as the Cost Improvement indices (CII) as follows:
Cell (1,2): The Stepping Stone Path for this cell is
+ (1,2) - (4,2) + (4,3) - (1,1)+(3,l) - (3,3)

	Abuja	Bauchi	Calabar	Supply
Ibadan	(-) 6	(+) 7	9	

		70000		70000
Jos	5 10000	8	7	 10000
Kano	(+) 7	9	(-) 6	

120000 30000
150000

Dummy 0 (-) 0 (+) 0
20000 80000
100000

Demand 130000 90000 110000
330000

The CII for cell (1,2) is +7 – 0 + 0 – 6 + 7 = +2

Cell (1,3) The Stepping Stone Path for this cell is +(1,3) — (1,1) + (3,1) — (3.3). The allocation matrix for this cell is shown below:

	Abuja	Bauchi	Calabar	Supply
Ibadan	(-) 6	7 70000	(+)9 70000	 70000
Jos	5 10000	8	7	 10000
Kano	(+) 7 120000	9	(-)6 30000	 150000

Dummy 0 0 0
90000 10000
100000

Demand 130000 90000 110000
330000

CII is given as +9 – 6 + 7 – 6 = +4

Cell (2,2) the stepping stone path for the cell is
+(2,2) – (4,2) + (4,3) – (3,3) + (3,1) – (2,1) the obtained matrix is as follows

	Abuja	Bauchi	Calabar	Supply
Ibadan	6 70000	7	9	 70000
Jos	(-5)5	(+)8	7	 10000
Kano	(+) 7 60000	9	(-)6 90000	 150000

120000 30000
150000

Dummy 0 (-) 0 (+) 0
8000 20000
100000

Demand 130000 90000 110000
330000

The cell CII for cell (2, 2) is given as +8 – 0 + 0 – 6 + 7 – 5 = +4
Cell (2, 3) the stepping stone path is + (2, 3) – (3, 3) + (3,1) – (2,1)
The allocation matrix for the cell is as follows

	Abuja	Bauchi	Calabar	Supply
Ibadan	6 70000	7	9	70000
Jos	(-)5	8	(+)7 10000	10000
Kano	(+) 7 60000	9	(-) 6 90000	150000
Dummy	0	(-)0 20000	0 10000	100000
Demand	130000	90000	110000	330000

The CII for cell (2, 3) is given by +7 - 6 + 7 – 5 = +3
Cell (3, 2) the stepping stone Path is
+(3,2) – (4,2) + (4,3) – (3,3).
The matrix of the allocation is shown below;

	Abuja	Bauchi	Calabar	Supply
Ibadan	6 70000	7	9	70000
Jos	5 10000	8	7	10000
Kano	(+) 7 50000	(+)9 10000	(-) 6 90000	150000
Dummy	0	(-)0 80000	0 20000	100000
Demand	130000	90000	110000	330000

The CII for cell (3, 2) is given by +9–0 + 0 – 6 = +3
Cell (4, 1) the stepping stone Path is
+(4,1) – (3,1) + (3,3) – (4,3).

The allocation matrix is given below

	Abuja	Bauchi	Calabar	Supply
Ibadan	6 70000	7	9	70000
Jos	5 10000	8	7	10000
Kano	(-) 7 40000	(+)9	(-) 6 110000	150000
Dummy	(+)0 10000	0 90000	(-)0 100000	
Demand	130000	90000	110000	330000

The CII for cell (4,1) is given as +0-7+6-0=-1
Since cell (4,1) has negative CII the optimum solution has not been reached. We need to compute the CII for the un allocated cells in the table for cell (4,1) shown above.
The empty cells are **Cell (1,2); Cell (1,3); Cell (2,2); Cell (2,3); Cell (3,2) and Cell (4,3)**
For Cell (1,2) The Stepping Stone Path is +(1,2) - (4,1) + (4,2) - (1,1).
The CIl =+7 – 0 + 0 – 6 =+1
For Cell(1,3)The Stepping Stone Path is +(1,3)-(1,1)+(3,1) - (3,3)
The CII =+9 – 6 + 7 - 6 =+4
For Cell (2,2) The Stepping Stone Path is + (2,2) - (4,2) + (4,1) - (2,1)
The CIl =+8 – 0 + 0 - 7= +3
For Cell (2,3) The Stepping Stone Path is +(2,3) - (3,3) + (3,1) - (2,1)
The CIl = +7 – 6 +7 – 5 = +3
For Cell (3,2) The Stepping Stone Path is +(3,2) — (4,2) + (4,1) — (3,1)
TheCil
For Cell (4,3) The Stepping Stone Path is + (4,3)— (3,3) + (3,1) — (4,1)
The CIl =+0—6+7-0=1
Since the CIIs are all positive an optimal solution has been found in the last table

	Abuja	Bauchi	Calabar	Supply
Ibadan	6 70000	7	9	70000
Jos	5 10000	8	7	10000
Kano	7 40000	9	6 110000	150000
Dummy	0 10000	0 90000	0	100000
Demand	10000	90000	110000	330000

Ibadan to Abuja 70,000 @ N6 = 70,000 x 6= 420,000
From Jos to Abuja 1,000 @ N5 = 10,000 x 5 = 50,000
From Kano to Abuja 40,000 @ N7 = 40,000 x 7 = 280,000

From Kano to Calabar 110,000 @ N6 = 110,000 x 6= 660,000
Dummy to Abuja 10,000 @ NO = 10,000 x 0=0
Dummy to Bauchi 90,000 @ NO = 90,000 x 0=0

1,410,000

We observe that the minimum cost of 1, 420, 000 obtained by the Least Cost Method has been reduced by the stepping stone method to give s the optimum transportation cost of 1,410, 000.

1.4 The Modified Distribution Method

This method is usually applied to the initial feasible solution obtained by the North West Corner method and the Least Cost method since the initial feasible solution obtained by the Vogel's Approximation Method, is deemed to be more accurate than these two. To use this method we take the following steps:

Step 1: Using the obtained feasible solution, compute the row dispatch unit cost r_1 and the column reception unit cost c_{ij} at location j for every cell with allocation using

$C_{ij} = r_i+c_j$

Conventionally, $r_i = 0$

Note that r_i is the shadow cost of dispatching a unit item from source to cell k_{ij} while c_j is the shadow cost of receiving a unit of the item from location j to cell k_{ij} and c_{ij} is the cost of transporting a unit of the item from source i to location j in the corresponding cell k_{ij}.

If we have a 3 x 3 cell we obtain r_1, r_2, r_3, c_1,c_2, and c_3 respectively.

Step 2

Compute the unit shadow costs for each of the empty unallocated cells using the various obtained c_i and r_i

Step 3

Obtain the differences in unit costs for the unallocated cells using

$C^1_{ij} =c_{ij-} (r_i+c_j)$

If these differences are all positive for the empty cells the minimum optimum solution has been obtained. If we have one or more records of any negative difference then it implies that an improved solution can still be obtained and so we proceed to step 4.

Step 4

We select the cell with the highest negative value of C_{ij}. If more than one of them have the same negative C_{ij} (i.e. the unit shadow cost is greater than the actual cost), that is a tie occurs we select any one of them arbitrarily for transfer of units.

Step 5

Transfer to the empty cells the minimum value possible from an allocated cell, taking care that the values of the demand and supply are unaffected by the transfer and that no other empty cell is given allocation.

Step 6

Develop a new solution and test if it is the optimum solution

Step 7
If it is not, repeat the procedures by starting from step 1 until the optimum solution is obtained.

Example 9
In the transportation table given below:
(a) Find the initial feasible solution using (i) the least cost method (ii) the Vogel's approximation method.
Use the modified Distribution method to find the optimum solution using the initial feasible solution obtained by the Least Cost Method.

	1	2	3	Supply
X	9	11	15	400
Y	15	7	17	500
Z	11	5	7	600
Demand	500	450	550	

Solution
We use the least cost method to obtain this table

	1	**2**	**3**	**Supply**
X	9	11	15	
	400	-	-	400
Y	15	7	17	
	100	-	400	500
Z	11	5	7	
	-	450	150	600
Demand	500	450	550	

The Least Cost value is (400 x 9) + (100 x 15) + (450 x 5) + (400 x 17) + (550 x 7)
= 3600+1500+2250+6800+1050= 15200
Minimum cost by least cost = 15200
ii. Using the Vogel approximation method,

	X	Y	Z	Supply	d_1	d_2	d_3
A	9 400	11 -	15 -	400	2	6	6*
B	15 50	7 450	17 -	500	8*	2	2
C	11 50	5 -	7 550	600	2	4	-
Demand	150	100	130				
d_{11}	2	2	8				
d_{21}	2	-	8*				
d_{31}	6	-	2				

The cost by the Vogel Approximation method is (400 x 9) + (50 x 15) + (50 x 11) + (550 x7)
=3600+750÷550+ 3150+3850=11900
(b) Using the Modified Distribution Method on the Least Cost Risk
(c) We now use the Modified Distribution method on the initial solution obtained by the Least Cost Method We follow the steps allowed as shown below

Step 1

Reproduce the obtained feasible solution by least cost method

	1	**2**	**3**	**Supply**
X	9	11	15	
	400	-	-	400
Y	15	7	17	
	100	-	400	500
Z	11	5	7	
	-	450	150	600
Demand	500	450	550	

Minimum cost by least cost = 15200
We then compute the unit shown costs for each of the allocated cells as follows
By convention r_1 =0
In cell (1,1) $r_1+c_1=9$ ….. $c_1=9$
In cell (2,1) $r_2+c_1=15$:.$r_2 = 15 – 9 =6$
In cell (2,3) $r_2+c_3= 17$:.$c_3 = 17 – 6 =11$
In cell (3,3) $r_3+c_3 =7$:.$r_3 = 7 – 11 =-4$
In cell (3,2) $r_3 + c_2 = 5$:.$c_2 = 5 - (-4) = 9$
We summarise as follows
$r_1 = 0$ $c_1=9$
$r_2 = 6$ $c_2=9$
$r_3 = -4$ $c_3=11$

Steps 2 and 3

We compute the difference in unit cost for the unoccupied cells as follows
For cell (1,2) $c_{12} = 11 - (r_1 + c_2) = 11 - 9 = 2$
For cell (1,3) $c_{13} = 15 - (r_1+c_3) = 15 - 11= 4$
For cell (2,2) $c_{22} = 7 - (r_2+c) = 7 – 15 = -8$*
For cell (3,1) $c_{31} = 11 - (r_3+c_1) = 11 – 5 = 6$

Step 5

The negative value in asterisk implies we have to do some transfer to cell (2,2) while ensuring that the supply and demand quantities are kept constant and no other empty cell expect (2,2) is given allocation. We must also ensure that the m + n — 1 criterion is maintained to avoid degeneracy. We obtain the table below.

	1	**2**	**3**	**Supply**
X	9	11	15	
	400	-	-	400
Y	15	7	17	

	100	-	400	500
Z	11	5	7	
	-	450	150	600
Demand	500	450	550	

Cost =(400x9)+(100x15)+(400x7)+(50x5)+(550x7)
= 3600 + 1500 + 2800 + 250 + 3850 = 12000
Which is less than 15200. However, we need to check if this is an optimum value

Step 6

This is done by computing the c^i_{ij} and c^i_{ij} or the new table. If none of the c_{ij} is negative
then it Is the optimum value.
As before we get
For allocated cells

$r_1=0$, $1+c_1=9$ $:.c_1=9$
$r_2+c_1=15$ $:.r_2=15 – 9 = 6$
$r_2+c_2=7$ $:,c_2=7-6 =1$
$r_3+c_2=5$ $:.r_3=5-1=4$
$r_3+c_3=7$ $:.c_3=7-4=3$

We summarise and get the following
$r_1=0$ $c_1=9$
$r_2=6$ $c_2=1$
$r_3=4$ $c_3=3$

For unallocated cells we have the following
For cell (1,2) we have $c_{12} = 11 - (r_1 + c_2) = 11 - 1 =$ l0 for cell (1,3) we have $c_{13} = 15 -\ r_1 \div c_3) = 15 - 13 = 2$
For cell (1,3) we have $c^i_{13} = 15 - (r_1 + c_3) = 15 - 13 = 2$
For cell (2,3) we have $c^i_{23} = 17 - (r_2 + c_3) = 17 - (6+3) = 17 - 9 = 8$
For cell (3,1) we have $c^i_{31} = 11 - (r_3 + c_1) = 11(4 + 9) = 11 - 13 = -2^*$
Since (3,1)has negative c^i_{31} value of -2 we do some transfer to (3,1) in the usual member together.

	1	**2**	**3**	**Supply**
X	9 400	11 -	15 -	400
Y	15 50	7 450	17	500
Z	11 50	5	7 550	600
Demand	500	450	550	

Cost= (400x9)+(50x15)+(50x11)+(450x7)+(550x7)
=3600+70+550+3150+3850= 11,900
We check if this is the optimum solution by computing the differences in the unit costs an unit shadow costs c_{ij} in the usual way.

For allocated cells

$r_1=0$, $1+c_1 = 9$.: $c_1 = 9$

$r_2 + c_1 = 15$.: $r_2 = 15 – 9 = 6$

$r_2 + c_2 = 7$.: $c_2 = 7 – 6 = 1$

$r_3+c_1= l1$.: $r_3 = 11 – 9 = 2$

$r_3+c_3 = 7$.: $c_3 = 7 – r_3 = 7 – 2 = 5$

We summarize and get

$r_1= 0$ $c_1=9$

$r_2=6$ $c_2=1$

$r_3=2$ $c_3=5$

For unallocated cells

(1,2) we have $c_{12} = 11 - (0+1) = 10$

for(1,3)we have $c_{13} =15 - (0+5)=10$

for (2,3) we have $c_{23}=17 - (6+5)=6$

for cell (3,2) we have $c_{32} = 5 - (2 + 1) = 2$

Since all these values are positive then the last table is the optimum assignment.

The optimum assignment is thus

	1	**2**	**3**	**Supply**
X	400	-	-	400
Y	50	450	-	500
Z	50	-	550	600
Demand	500	450	550	

The optimum cost is N11,900

We observe that the value obtained is the same as that of Vogel's Approximation method so Vogel is the best of all the methods.

1.5 Degeneracy

This condition arises when the number of allocated cell does not satisfy the m + n — 1 criterion Degeneracy prevents us from utilizing the optimisation technique to get the minimization cost of the transportation model.

Exanp1e 10

Find the initial feasible solution of the transportation problem below using the Vogel s Approximation Method. Comment on your result.

	1	2	3	Supply
X	8	7	6	40
Y	16	10	9	120
Z	19	18	12	90
Demand	130	15	25	

Solution

	1	2	3	Supply	d_1	d_2	d_3
X	8 40	7 -	6 -	 400	 1	 -	 -
Y	16 -	10 95	9 25	25 120	 1	 1	 1
Z	19 90	18 -	12 -	 90	 6	 6	 7
Demand	130	15	25				
d_1	8*	3	3				
d_2	3	8*	3				
d_3	3	-	3				

Correct: Only 4 cells are allocated. So degeneracy occurs.
The feasible solution can be obtained as follows
40 x 8 = 320
95 x 10 = 950
90 x 19 = 1710
25 x 9 = 225
N3,205
Let us compute the shadow costs
By convention $r_1 = 0$
$r_1+c_1 = 8$.: $r_1 = 8$
$r_2+c_2 = 10$
$r_2+c_3 = 9$
$r_3+c_1=19$:$r_3=19 – 8 =11$
Due to degeneracy we cannot get enough information to enable us calculate r+, c+ and c_3. This implies that we cannot determine the needed row and column values for the unallocated cells.
The way out is to create a dummy allocated cell which is assigned a value of 0. Others advocate adding a small value to an empty cell and then proceeding with using MODI to obtain the optimum solution in the usual way.

1.6 Summary

The transportation model deals with a special class of linear programming problem in which the objective is to transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost. In this unit, we discussed the following issues- assumptions of the transportation model which include the Homogeneity of materials to be transported, equality of transportation cost per unit, uniqueness of route or mode of transportation between each source and destination. It also discussed extensively, various methods of solving the transportation problem viz: the North West corner method, the least cost method, Vogel's approximation method (VAM), the unbalanced case, formulating linear programming model for the transportation problem, and determination of the optimal transportation cost using the stepping stone method.

1.7 References/Further Readings/Web Resources

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

The transportation problem is a distribution-type problem, the main goal of which is to decide how to transfer goods from various sending locations (also known as origins) to various receiving locations (also known as destinations) with minimal costs or maximum profit.

MODULE 3

Unit 1 Simulation

Unit Structure

1.1 Introduction

Simulation is primarily concerned with experimentally predicting the behaviour of a real system for the purpose of designing the system or modifying behaviour (Budnick et al., 1988). The main reason for a researcher to resort to simulation is twofold. First of all, simulation is probably the most flexible tool imaginable. Take queuing as an example. While it is very difficult to incorporate reneging, jumping queues and other types of customer behaviour in the usual analytical models this presents no problem for simulation. A system may have to run for a very long time to reach a steady state. As a result, a modeller may be more interested in transient states, which are easily available in a simulation.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define Simulation
- Demonstrate four assumptions of the transportation model.

- Explain Vogel’s Approximation Method
- Describe the nature of a transportation problem.

1.3 Definition of Simulation

According Budnick et al (1988), Simulation is primarily concerned with experimentally predicting the behaviour of a real system for the purpose of designing the system or modifying behaviour. In other words, simulation is a tool that builds a model of a real operation that is to be investigated, and then feeds the system with externally generated data. We generally distinguish between deterministic and stochastic simulation. The difference is that the data that are fed into the system are either deterministic or stochastic. This chapter will deal only with stochastic simulation, which is sometimes also referred to as Monte Carlo simulation in reference to the Monte Carlo Casinos and the (hopefully) random outcome of their games of chance.

According to Gupta and Hira (2012), simulation is an imitation of reality. "They further stated that simulation is the representation of reality through the use of models or other device which will react in the same manner as reality under given set of conditions. Simulation has also been defined the use of a system model that has the designed characteristics of reality in order to produce the essence of actual operation. According to Donald G. Malcom, a simulated model may be defined as one which depicts the working of a large scale system of men, machine, materials, and information operating over a period of time in a simulated environment of the actual real world condition.

1.3.1 Advantages of Simulation Technique

When the simulation technique is compared with the mathematical programming and slandered probability analysis, offers a number of advantages over these techniques. Some of the advantages are:

1. Simulation offers solution by allowing experimentation with models of a system without interfering with the real system. Simulation is therefore a bypass for complex mathematical analysis.
2. Through simulation, management can foresee the difficulties and bottlenecks which may come up due to the introduction of new machines, equipment or process. It therefore eliminates the need for costly trial and error method of trying out the new concept on real methods and equipment.
3. Simulation is relatively free from mathematics, and thus, can be easily understood by the operating personnel and non-technical managers. This helps in getting the proposed plan accepted and implemented.

1.3.2 Application of Simulation

Simulation is quite versatile and commonly applied technique for solving decision problems. It has been applied successfully to a wide range of problems of science and technology as given below:

1. In the field of basic sciences, it has been used to evaluate the area under a curve, to estimate the value of π, in matrix inversion and study of particle diffusion.
2. In industrial problems including shop floor management, design of computer systems, design of queuing systems, inventory control, communication networks, chemical processes, nuclear reactors, and scheduling of production processes.
3. In business and economic problems, including customer behaviour, price determination, economic forecasting, portfolio selection, and capital budgeting.
4. In social problems, including population growth, effect of environment on health and group behaviour.

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1.4 Limitations of Simulation Technique

Despite the many advantages of simulation, it might suffer from some deficiencies in large and complex problems. Some of these limitations are given as follows:

i. Simulation does not produce optimum results when the model deals with uncertainties, the results of simulation only reliable approximations subject to statistical errors.

ii. Quantification of variables is difficult in a number of situations; it is not possible to quantify all the variables that affect the behaviour of the system.

iii. In very large and complex problems, the large number of variables and the interrelationship between them make the problem very unwieldy and hard to program.

iv. Simulation is by no means, a cheap method of analysis.

v. Simulation has too much tendency to rely on simulation models. This results in application of the technique to some simple problems which can more appropriately be handled by other techniques of mathematical programming.

1.5 Monte Carlo Simulation

The Monte Carlo method of simulation was developed by two mathematicians Jon Von Neumann and Stainslaw Ulam, during World War II, to study how far neurone would travel through different materials. The technique provides an approximate but quite workable solution to the problem. With the remarkable success of this technique on the neutron problem, it soon became popular and found many applications in business and industry, and at present, forms a very important tool of operation researcher's tool kit.

The technique employs random number and is used to solve problems that involve probability and where physical experimentation is impracticable, and formulation of mathematical model is impossible. It is a method of simulation by sampling technique. The following are steps involved in carrying out Monte Carlo simulation.

1. Select the measure of effectiveness (objective function) of the problem. It is either to be minimised or maximised.
2. Identify the variables that affect the measure of effectiveness significantly. For example, a number of service facilities in a queuing problem or demand, lead time and safety stock in inventory problem.
3. Determine the cumulative probability distribution of each variable selected in step 2. Plot these distributions with the values of the variables along the x-axis and cumulative probability values along the y-axis.
4. Get a set of random numbers.

Self-Assessment Exercise 1

Customers arrive at a service facility to get required service. The interval and service times are constant and are 1.8minutes and minutes respectively. Simulate the system for 14minutes. Determine the average waiting time of a customer and the idle time of the service facility.

1.6 Summary

This unit provides for us an overview of simulation. It takes us through various conceptualisations on the definition of simulation. Simulation has been defined as the representation of reality through the use of models or other device which will react in the same manner as reality under given set of conditions. A good example of simulation is a children amusement or a cyclical park where children enjoy themselves in a simulated environment like Amusement Parks, Disney Land, Planetarium shows where boats, train rides, etc. are done to simulate actual experience. It is quite versatile and commonly applied technique for solving decision problems such as basic sciences, in industrial problems including shop floor management, in business and economic problems etc.

1.7 References/Further Readings/Web Resources

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

The arrival times of customers at the service station within 14 minutes will be:

Customer :	1	2	3	4	5	6	7	8
Arrival time : *(minutes)*	0	1.8	3.6	5.4	7.2	9.0	10.8	12.6

The time at which the service station begins and ends within time period of 14 minutes is shown below. Waiting time of customers and idle time of service facility are also calculated

Customer	*Service Begins*	*Service ends*	*Waiting time of customer*	*Idle time of service facility*
1	0	4	0	0
2	4	8	4-1.8 = 2.2	0
3	8	12	8-3.6 = 4.4	0
4	12	16	12-5.4 = 6.6	0

The waiting time of the first four customers is calculated above. For the remaining, it is calculated below.

Customer	:	5	6	7	8
Waiting time (min)	:	14 – 7.2 = 6.8	5.0	3.2	1.4

Therefore, average waiting time of a customer

$$= \frac{0 + 2.2 + 4.4 + 6.6 + 6.8 + 5 + 3.2 + 1.4}{8} = \frac{29.6}{8} = 3.7 \text{ minutes}$$

Idle time of facility = nil.

Unit 2 Systems Analysis

Unit Structure

1.1 Introduction

The word system has a long history which can be traced back to Plato (Philebus), Aristotle (Politics) and Euclid (Elements). It had meant "total", "crowd" or "union" in even more ancient times, as it derives from the verb sunìstemi, uniting, putting together.

"System" means "something to look at". You must have a very high visual gradient to have systematization. In philosophy, before Descartes, there was no "system". Plato had no "system". Aristotle had no "system"(McLuhan. 1967)

In the 19th century the first to develop the concept of a "system" in the natural sciences was the French physicist Nicolas Léonard Sadi Carnot who studied thermodynamics. In 1824 he studied the system which he called the working substance, i.e. typically a body of water vapour, in steam engines, in regards to the system's ability to do work when heat is applied to it. The working substance could be put in contact with either a boiler, a cold reservoir (a stream of cold water), or a piston (to which the working body could do work by pushing on it). In 1850, the German physicist Rudolf Clausius generalized this picture to include the concept of the surroundings and began to use the term "working body" when referring to the system.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Define a system
- Identify and describe the types of systems
- Highlight the different forms of systems we have
- Describe how a system is analysed
- Discuss the concept of entropy

1.3 Definition of Systems Analysis

The term system is derived from the Greek word *systema*, which means an organized relationship among functioning units or components. A system exists because it is designed to achieve one or more objectives. We come into daily contact with the transportation system, the telephone system, the accounting system, the production system, and, for over two decades, the computer system. Similarly, we talk of the business system and of the organization as a system consisting of interrelated departments (subsystems) such as production, sales, personnel, and an information system. None of these subsystems is of much use as a single, independent unit. When they are properly coordinated, however, the firm can function effectively and profitably.

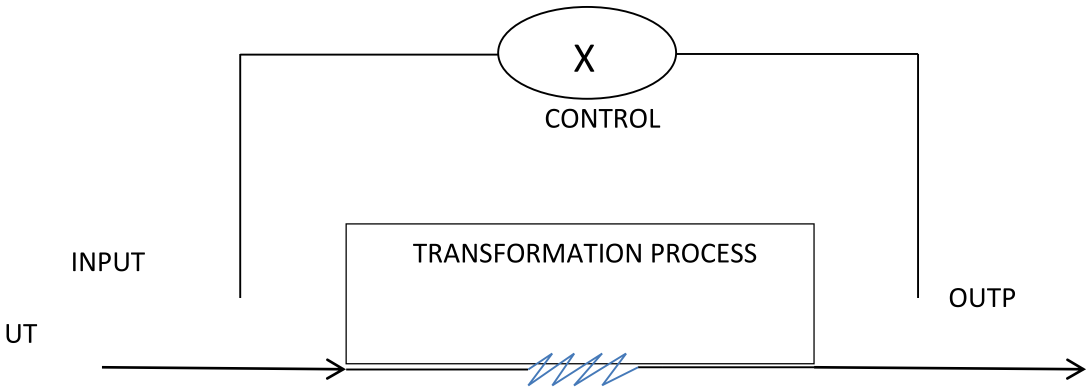
There are more than a hundred definitions of the word system, but most seem to have a common thread that suggests that a system is an orderly grouping of interdependent components linked together according to a plan to achieve a specific objective. The word component may refer to physical parts (engines, wings of aircraft, car), managerial steps (planning, organizing and controlling), or a system in a multi-level structure. The component may be simple or complex, basic or advanced. They may be single computer with a keyboard, memory, and printer or a series of intelligent terminals linked to a mainframe. In either case, each component is part of the total system and has to do its share of work for the system to achieve the intended goal. This orientation requires an orderly grouping of the components for the design of a successful system.

Self-Assessment Exercise 1

What do you understand term system?

1.3.1 The Systems Theory

The general systems theory states that a system is composed of inputs, a process, outputs, and control. A general graphic representation of such a system is shown below.



An Operational System
Adapted from Ihemeje, (2019), Fundamentals of Business Decision Analysis, Lagos- Sibon Books Limited.

The input usually consists of people, material or objectives. The process consists of plant, equipment and personnel. While the output usually consists of finished goods, semi-finished goods, policies, new products, ideas, etc.

The purpose of a system is to transform inputs into outputs. The system theory is relevant in the areas of systems design, systems operation and system control. The systems approach helps in resolving organisational problems by looking at the organisation as a whole, integrating its numerous complex operations, environment, technologies, human and material resources. The need to look at the organisation in totality is premised on the fact that the objective if the different units of the organisation when pursued in isolation conflict with one another. For instance, the operation of a manufacturing department favours long and uninterrupted production runs with a view to minimising unit cost of production, including set-up costs. However, this will result in large inventories, and leading to high inventory costs. The finance department seeks to minimise costs as well as capital tied down in inventories. Thus, there is a desire for rapid inventory turnover resulting in lower inventory levels. The marketing department seeks favourable customer service and as a result, will not support any policy that encourages stock outs or back ordering. Back ordering is a method of producing later to satisfy a previously unfulfilled order. Consequently, marketing favours the maintenance of high inventory levels in a wide variety of easily accessible locations which in effect means some type of capital investment in warehouse or sales outlets. Finally, personnel department aims at stabilizing labour, minimizing the cost of firing and hiring as well as employee discontentment. Hence, it is desirable from the point of view of personnel to maintain high inventory level of producing even during periods of fall in demand.

1.3.2 Elements of a System

Following are considered as the elements of a system in terms of Information systems:

- Input
- Output
- Processor
- Control
- Feedback

- INPUT: Input involves capturing and assembling elements that enter the system to be processed. The inputs are said to be fed to the systems in order to get the output. For example, input of a 'computer system' is input unit consisting of various input devices like keyboard, mouse, joystick etc.

- OUTPUT: The element that exists in the system due to the processing of the inputs is known as output. A major objective of a system is to produce output that has value to its user. The output of the system maybe in the form of cash, information, knowledge, reports, documents etc. The system is defined as output is required from it. It is the anticipatory recognition of output that helps in defining the input of the system. For example, output of a 'computer system' is output unit consisting of various output devices like screen and printer etc.

- PROCESSOR(S): The processor is the element of a system that involves the actual transformation of input into output. It is the operational component of a system. For example, processor of a 'computer system' is central processing unit that further consists of arithmetic and logic unit (ALU), control unit and memory unit etc.

- CONTROL: The control element guides the system. It is the decision-making sub-system that controls the pattern of activities governing input, processing and output. It also keeps the system within the boundary set. For example, control in a 'computer system' is maintained by the control unit that controls and coordinates various units by means of passing different signals through wires.

- FEEDBACK: Control in a dynamic system is achieved by feedback. Feedback measures output against a standard in some form of cybernetic procedure that includes communication and control. The feedback may generally be of three types viz., positive, negative and informational. The positive feedback motivates the persons in the system. The negative indicates need of an action, while the information. The feedback is a reactive form of control. Outputs from the process of the system are fed back to the control mechanism. The control mechanism then adjusts the control signals to the process on the basis of the data it receives. Feed forward is a protective form of control. For example, in a 'computer system' when logical decisions are taken, the logic unit concludes by comparing the calculated results and the required results.

Self-Assessment Exercise 2

List Elements of a System

1.3.3 Types of Systems

Systems are classified in different ways:

1. Physical or abstract systems.
2. Open or closed systems.
3. 'Man-made' information systems.
4. Formal information systems.
5. Informal information systems.
6. Computer-based information systems.
7. Real-time system.

Physical systems are tangible entities that may be static or dynamic in operation.

An open system has many interfaces with its environment. i.e. system that interacts freely with its environment, taking input and returning output. It permits interaction across its boundary; it receives inputs from and delivers outputs to the outside. A closed system does not interact with the environment; changes in the environment and adaptability are not issues for closed system.

1.4 Forms of Systems

A system can be conceptual, mechanical or social. A system can also be deterministic or probabilistic. A system can be closed or open.

Conceptual system

A system is conceptual when it contains abstracts that are linked to communicate ideas. An example of a conceptual system is a language system as in English language, which contains words, and how they are linked to communicate ideas. The elements of a conceptual system are words.

Mechanical system

A system is mechanical when it consists of many parts working together to do a work. An example of a social system is a typewriter or a computer, which consists of many parts working together to type words and symbols. The elements of the mechanical system are objects.

Social system

A system is social when it comprises policies, institutions and people. An example of a social system is a football team comprising 11 players, or an educational system consisting of policies, schools and teachers. The elements of a social system are subjects or people.

Deterministic system

A system is deterministic when it operates according to a predetermined set of rules. Its future behaviour can therefore be predicted exactly if it's present state and operating characteristics are accurately known. Example s of deterministic systems are computer programmes and a planet in orbit. Business systems are not deterministic owing to the fact that they interfere with a number of in determinant factors, such as customer and supplier behaviour, national and international situations, and climatic and political conditions.

Probabilistic system

A system is probabilistic when the system is controlled by chance events and so its future behaviour is a matter of probability rather than certainty. This is true of all social systems, particularly business enterprises. Information systems are deterministic enterprises in the sense that a pre-known type and content of information emerges as a result of the input of a given set of data.

Closed system

A system is closed when it does not interface with its environment i.e. it has no input or output. This concept is more relevant to scientific systems that to social systems. The nearest we can get to a closed social system would be a completely self-contained community that provides all its own food, materials and power, and does not trade, communicate or come into contact with other communities.

Open system

A system is open when it has many interfaces with its environment, and so needs to be capable of adopting their behaviour in order to continue to exist in changing environments. An information system falls into this category since it needs to adapt to the changing demands for information. Similarly, a business system must be capable of reorganizing itself to meet the conditions of its environment, as detected from its input; it will more rapidly tend towards a state of disorganization. (Ihemeje, 2002).

Self-Assessment Exercise 3

What are forms of system?

1.5 The Concept of Entropy in a System

The term entropy is used as a measure of disorganisation. Thus, we can regard open systems as tending to increase their entropy unless they receive negative entropy in the form of information from their environment. In the above example, if increased cost of cost of materials were ignored, the product will become unprofitable and as a result, the organisation may become insolvent, that is, a state of disorganisation.

1.6 Summary

This unit discusses the concept of systems analysis. The origin of system analysis has been traced to the Greek word *systema*, which means an organized relationship among functioning units or components. A system exists because it is designed to achieve one or more objectives. It can be defined is a collection of elements or components or units that are organized for a common purpose. The general systems theory states that a system is composed of inputs, a process, outputs, and control. The input usually consists of people, material or objectives. The process consists of plant, equipment and personnel. While the output usually consists of finished goods, semi-finished goods, policies, new products, ideas, etc. A system consists of the following element: input, output, processor, control, feedback, boundary and interface, and environment. Depending on the usage, a system has the following are types of systems: Physical or abstract systems, Open or closed systems, Man-made information systems, Formal information systems, Informal information systems, Computer-based information systems and Real-time system. A system can be conceptual, mechanical or social. A system can also exist in the following forms- it can be deterministic or probabilistic, closed or open, mechanical, social, and conceptual.
It has been quite an exciting journey through the world of systems analysis.

1.7 References/Further Readings/Web Resources

Ihemeje, J.C. (2002). Fundamentals of Business Decision Analysis, Lagos- Sibon Books.

Jawahar, S. (2006). Overview of System Analysis & Design- Lesson Note no. 1, Ashok Educational Foundation.

1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

The term system is derived from the Greek word *systema*, which means an organized relationship among functioning units or components. A system exists because it is designed to achieve one or more objectives. We come into daily contact with the transportation system, the telephone system, the accounting system, the production system, and, for over two decades, the computer system.

Answer to SAE 2

- Input
- Output
- Processor
- Control
- Feedback

Answer to SAE 3

A system can be conceptual, mechanical or social. A system can also be deterministic or probabilistic. A system can be closed or open.

Unit 3 Sequencing and Scheduling

Unit Structure

1.1 Introduction

Sequencing problems involves the determination of an optimal order or sequence of performing a series jobs by number of facilities (that are arranged in specific order) so as to optimize the total time or cost. Sequencing problems can be classified into two groups:

The group involves *n* different jobs to be performed, and these jobs require processing on some or all of m different types of machines. The order in which these machines are to be used for processing each job (for example, each job is to be processed first on machine A, then B, and thereafter on C i.e., in the order ABC) is given. Also, the expected actual processing time of each job on each machine is known. We can also determine the effectiveness for any given sequence of jobs on each of the machines and we wish to select from the $(n!)^m$ theoretically feasible alternatives, the one which is both technologically feasible and optimises the effectiveness measure.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Demonstrate Sequencing and Scheduling
- Explain Assumptions Made in Sequencing Problems
- Discuss nature of Scheduling

1.3 Definition of Sequencing and Scheduling

Sequencing is the order of tasks to be done in chain. Hence the next task is started once the previous one is completed.

Scheduling, on the other hand is the process in which people are assigned to time to accomplish different tasks.

Scheduling refers to establishing the timing of the use of equipment, facilities and human activities in an organization, that is, it deals with the timing of operations. Scheduling occurs in every organization, regardless of the nature of its operation. For example, manufacturing organizations, hospitals, colleges, airlines e.t.c. schedule their activities to achieve greater efficiency. Effective Scheduling helps companies to use assets more efficiently, which leads to cost savings and increase in productivity. The flexibility in operation provides faster delivery and therefore, better customer service. In general, the objectives of scheduling are to achieve trade-offs among conflicting goals, which include efficient utilization of staff, equipment and facilities and minimization of customer waiting tune, inventories and process times (Adebayo et al, 2006).

Self-Assessment Exercise 1

Define Sequencing

1.3.1 Assumptions Made in Sequencing Problems

Principal assumptions made for convenience in solving the sequencing problems are as follows:

1. The processing times A_i and B_i etc. are exactly known to us and they are independent of order of processing the job on the machine. That is whether job is done first on the machine, last on the machine, the time taken to process the job will not vary it remains constant.

2. The time taken by the job from one machine to other after processing on the previous machine is negligible. (Or we assume that the processing time given also includes the transfer time and setup time).

3. Only one operation can be carried out on a machine at a particular time.
4. Each job once started on the machine, we should not stop the processing in the middle. It is to be processed completely before loading the next job.
5. The job starts on the machine as soon as the job and the machine both become idle (vacant). This is written as job is next to the machine and the machine is next to the job. (This is exactly the meaning of transfer time is negligible).

1.3.2 Nature of Scheduling

Scheduling technique depends on the volume of system output, the nature of operations and the overall complexity of jobs. Flow shop systems require approaches substantially different from those required by job shops. The complexity of operations varies under these two situations.

1. **Flow Shop**
Flow shop is a high-volume system, which is characterized by a continuous flow of jobs to produce standardized products. Also, flow shop uses standardized equipment (i.e. special purposed machines) and activities that provide mass production. The goal is to obtain a smooth rate of flow of goods or customer through the system in order to get high utilization of labour and equipment. Examples are refineries, production of detergents etc.

2. **Job Shop**
This is a low volume system, which periodically shift from one job to another. The production is according to customer's specifications and orders or jobs usually in small lots. General-purpose machines characterize Job shop. For example, in designer shop, a customer can place order for different design.

Job-shop processing gives rise to two basic issues for schedulers: how to distribute the workload among work centre and what job processing sequence to use.

Self-Assessment Exercises 2

Explain the following concepts (a) Flow shop (b) Job shop

1.3.3 Loading Jobs in Work Centres

Loading refers to the assignment of jobs to work centres. The operation managers are confronted with the decision of assigning jobs to work centres to minimize costs, idle time or completion time.
The two main methods that can be used to assign jobs to work centres or to allocate resources are:
1. Gantt chart
2. Assignment method of linear programming

Gantt Charts

Gantt charts are bar charts that show the relationship of activities over some time periods. Gantt charts are named after Henry Gantt, the pioneer

who used charts for industrial scheduling in the early 1900s. A typical Gantt chart presents time scale horizontally, and resources to be scheduled are listed vertically, the use and idle times of resources are reflected in the chart.
The two most commonly used Gantt charts are the schedule chart and the load chart.

Assignment Method

Assignment Model (AM) is concerned specifically with the problem of job allocation in a multiple facility production configuration. That is, it is useful in situations that call for assigning tasks or jobs to resources. Typical examples include assigning jobs to machines or workers, territories to sales people e.t.c. One important characteristic of assignment problems is that only one job (or worker) is assigned to one machine (or project). The idea is to obtain an optimum matching of tasks and resources. A chapter in this book has treated the assignment method.

1.3.4 Priority Rules for Job Sequencing

Priority rules provide means for selecting the order in which jobs should be done (processed). In using these rules, it is assumed that job set up cost and time are independent of processing sequence. The main objective of priority rules is to minimize completion time, number of jobs in the system, and job lateness, while maximizing facility utilization. The most popular priority rules are:
I. First Come, First Serve (FCFS): Job is worked or processed in the order of arrivals at the work centre.
2. Shortest Processing Time (SPT): Here, jobs are processed based on the length of processing time. The job with the least processing time is done first.
3. Earliest Due Date (EDD): This rule sequences jobs according to their due dates, that is, the job with the earliest due date is processed first.
4. Longest Processing Time (LPT): The job with the longest processing time is started first.
5. Critical Ratio: Jobs are processed according to smallest ratio of time remaining until due date to processing time remaining.
The effectiveness of the priority rules is frequently measured in the light of one or more performance measures namely; average number of jobs, job flow time, job lateness, make span, facility utilisation etc.

1.4 Applicability

The sequencing problem is very much common in Job workshops and Batch production shops. There will be number of jobs which are to be processed on a series of machine in a specified order depending on the

physical changes required on the job. We can find the same situation in computer centre where
number of problems waiting for a solution. We can also see the same situation when number of critical patients waiting for treatment in a clinic and in Xerox centres, where number of jobs is in queue, which are to be processed on the Xerox machines. Like this we may find number of situations in real world.

1.5 Types of Sequencing Problems

There are various types of sequencing problems arise in real world. All sequencing problems cannot be solved. Though mathematicians and Operations Research scholars are working hard on the problem satisfactory method of solving problem is available for few cases only. The problems, which can be
solved, are:

(a) '*n*' jobs are to be processed on two machines say machine *A* and machine *B* in the order *AB*. This means that the job is to be processed first on machine *A* and then on machine *B*.
(b) '*n*' jobs are to be processed on three machines *A*,*B* and *C* in the order *ABC i.e.* first on machine *A*, second on machine B and third on machine *C*.
(c) '*n*' jobs are to be processed on '*m*' machines in the given order.
d) Two jobs are to be processed on '*m*' machines in the given order. (Murthy, 2007)

- **Single Machine Scheduling Models**

The models in this section deal with the simplest of scheduling problems: there is only a single machine on which tasks are to be processed. Before investigating the solutions that result from the use of the three criteria presented in the introduction

- **'N' Jobs and Two Machines**

If the problem given has two machines and two or three jobs, then it can be solved by using the Gantt chart. But if the numbers of jobs are more, then this method becomes less practical. (For understanding about the Gantt chart, the students are advised to refer to a book on Production and Operations Management (chapter on Scheduling). Gantt chart consists of *X* -axis on which the time is noted and *Y*-axis on which jobs or machines are shown. For each machine a horizontal bar is drawn. On these bars the processing of jobs in given sequence is marked. Let us take a small example and see how Gantt chart can be used to solve the same.

EXAMPLE 1

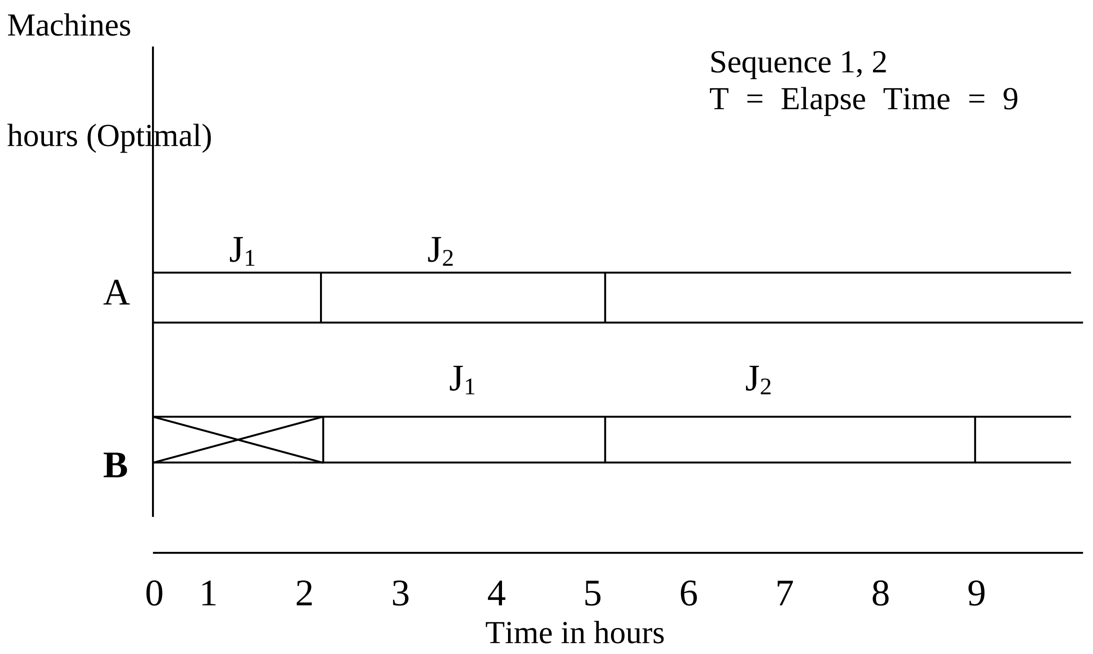
There are two jobs job 1 and job 2. They are to be processed on two machines, machine *A* and Machine *B* in the order *AB*. Job 1 takes 2 hours on machine *A* and 3 hours on machine *B*. Job 2 takes 3 hours on machine *A* and 4 hours on machine *B*. Find the optimal sequence which minimizes the total elapsed time by using Gantt chart.

Solution

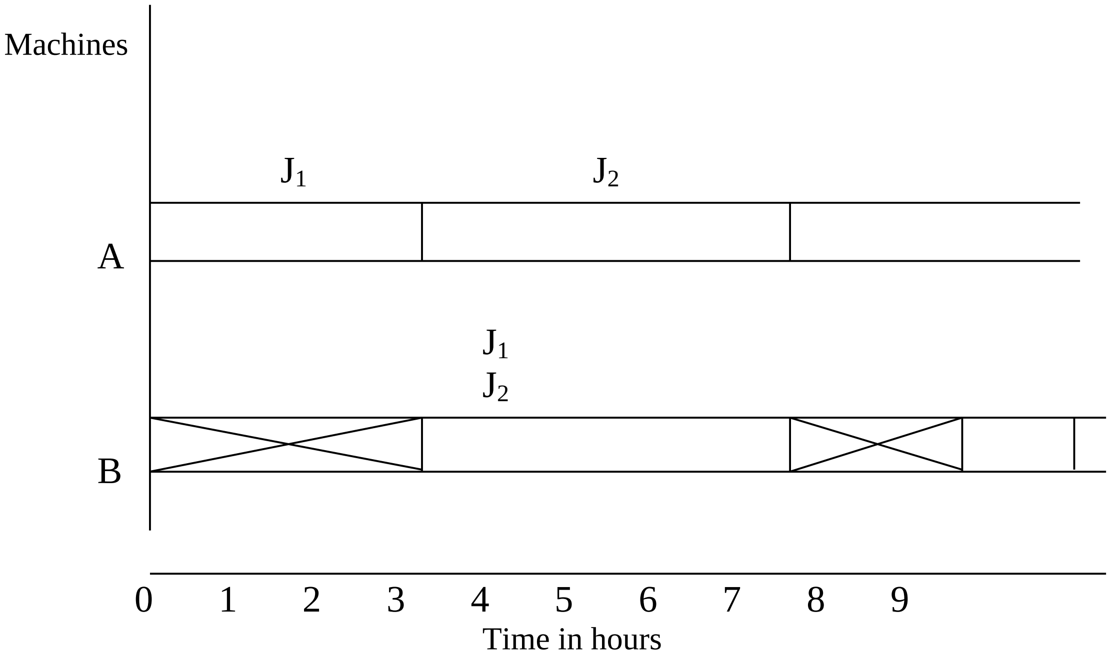
Jobs.	*Machines (Time in hours)*	
	A	*B*
1	*2*	*3*
2	*3*	*4*

(*a*) Total elapsed time for sequence 1,2*i.e.* first job 1 is processed on machine *A* and then on second machine and so on.

Draw *X* - axis and Y- axis, represent the time on *X* - axis and two machines by two bars on *Y axis*. Then mark the times on the bars to show processing of each job on that machine.



Sequence 1, 2
T = Elapse Time = 9 hours (Optimal sequence)



Gantt chart.

Source: Murthy, R. P. (2007), Operations Research, 2nd ed., New Delhi: New Age International (P) Limited Publisher

Both the sequences shows the elapsed time = 9 hour
The drawback of this method is for all the sequences, we have to write the Gantt chart and find the total elapsed times and then identify the optimal solution. This is laborious and time consuming. If we have more jobs and more machines, then it is tedious work to draw the chart for all sequences. Hence we have to go for analytical methods to find the optimal solution without drawing charts.

Analytical Method
A method has been developed by **Johnson and Bellman** for simple problems to determine a sequence of jobs, which minimizes the total elapsed time. The method:

1. '*n*' jobs are to be processed on two machines *A* and *B* in the order *AB* (*i.e*. each job is to be processed first on *A* and then on *B*) and passing is not allowed. That is whichever job is processed first on machine *A* is to be first processed on machine *B* also, whichever job is processed second on machine *A* is to be processed second on machine *B* also and so on. That means each job will first go to machine *A* get processed and then go to machine *B* and get processed. *This rule is known as no passing rule.*

2. Johnson and Bellman method concentrates on minimizing the idle time of machines. Johnson and Bellman have proved that optimal sequence of '*n*' jobs which are to be processed on two machines *A* and *B* in the order *AB* necessarily involves the same ordering of jobs on each machine. This result also holds for three machines but does not necessarily hold for more than three machines. Thus total elapsed time is minimum when the sequence of jobs is same for both the machines.

3. Let the number of jobs be 1,2,3,…………*n*

The processing time of jobs on machine *A* be*A*1, *A*2, *A*3 ………… *An*

The processing time of jobs on machine *B* be*B*1, *B*2, *B*3 …………..*Bn*

Jobs	*Machine Time in Hours*		
	Machine A	*Machine B*	*Order of Processing is AB*
1	A_1	B_1	
2	A_2	B_2	
3	A_3	B_3	
I	A_I	B_I	
S	A_S	B_S	
T	A_T	B_T	
N	A_N	B_N	

4. Johnson and Bellman algorithm for optimal sequence states *that identify the smallest element in the given matrix. If the smallest element falls under column 1 i.e under machine I then do that job first.* As the job after processing on machine 1 goes to machine2, it reduces the idle time or waiting time of machine 2. *If the smallest element falls under column 2 i.e under machine 2 then do that job last*. This reduces the idle time of machine1. *i.e*. if r the job is having smallest element in first column, then do the r^{th} job first. If *s* the job has the smallest element, which falls under second column, then do the *s* the job last. Hence the basis for Johnson and Bellman method is to keep the idle time of machines as low as possible. Continue the above process until all the jobs are over.

1	*2*	*3………..*		*n-1*	*n*
r					*s*

If there are '*n*' jobs, first write '*n*' number of rectangles as shown. Whenever the smallest elements falls in column 1 then enter the job number in first rectangle. If it falls in second column, then write the job number in the last rectangle. Once the job number is entered, the second rectangle will become first rectangle and last but one rectangle will be the last rectangle.

5. Now calculate the total elapsed time as discussed. Write the table as shown. Let us assume that the first job starts at Zero th time. Then add the processing time of job (first in the optimal sequence) and write in out column under machine 1. This is the time when the first job in the optimal sequence leaves machine 1 and enters the machine 2. Now add processing time of job on machine 2. This is the time by which the processing of the job on two machines over. Next consider the job, which is in second place in optimal sequence. This job enters the machine 1 as soon the machine becomes vacant, i.e first job leaves to second machine. Hence enter the time in-out column for first job under machine 1 as the starting time of job two on machine 1. Continue until all the jobs are over. Be careful to see that whether the machines are vacant before loading. Total elapsed time may be worked out by drawing Gantt chart for the optimal sequence.

Example 2

There are seven jobs, each of which has to be processed on machine *A* and then on Machine *B*(order of machining is *AB*). Processing time is given in hour. Find the optimal sequence in which the jobs are to be processed so as to minimize the total time elapsed.

JOB:	1	2	3	4	5	6	7
MACHINE: A (TIME IN HOURS).	3	12	15	6	10	11	9
MACHINE: B (TIME IN HOURS).	8	10	10	6	12	1	3

Solution

By Johnson and Bellman method the optimal sequence is:

1	4	5	3	2	7	6

Optimal Sequence	Machine:A		Machine:B		Machine idle time		Job idle time	Remarks.
	In	out	In	out	A	B		
1	0	3	3	11		3	-	
4	3	9	11	17			2	Job finished early
5	9	19	19	31		2		Machine A take more time
3	19	34	34	44		3		Machine A takes more time.
2	34	46	46	56		2		- do-
7	46	55	56	59			1	Job finished early.
6	55	66	66	67	1	7		Machine A takes more time. Last is finished

								on machine A at 66 th hour.
	Total		**Elapsed**		**Time = 67 houN**			

Example 3

Assuming eight jobs are waiting to be processed. The processing time and due dates for the jobs are given below: Determine the sequence processing according to (a) FCFS (b) SPT (c) EDD and (d) LPT in the light of the following criteria:

(i) Average flow time,
(ii) Average number of jobs in the system,
(iii) Average job lateness,
(iv) Utilization of the workers

JOB	PROCESSING TIME	DUE DATE (DAYS)
A	4	9
B	10	18
C	6	6
D	12	19
E	7	17
F	14	20
G	9	24
H	18	28

Solution:

(a) **To determine the sequence processing according to FCFS**

The FCFS sequence is simply A-B-C-D-E-F-G-H- as shown below

Job	Processing Time	Flow time	Job due date	Job lateness (0 of negative)
A	4	4	9	0
B	10	14	18	0
C	6	20	6	14
D	12	32	19	13
E	7	39	17	22
F	14	53	20	33
G	9	62	24	38
H	18	80	28	52
	80	304		172

The first come, first served rule results is the following measures of effectiveness:

1. Average flow time = $\dfrac{\text{Sum of total flow time}}{\text{Number of jobs}}$

$$= \frac{304\text{days}}{8} = 38\text{jobs}$$

2. Average number of jobs in the system = $\frac{\text{Sum of total flow time}}{\text{Total processing time}}$

$= \frac{304\text{days}}{80} = 3.8\text{jobs}$

3. Average job lateness = $\frac{\text{Total late days}}{\text{Number of days}} = \frac{172}{8}$ x 21.5 = 22days

4. Utilization = $\frac{\text{Total processing time}}{\text{Sum of total flow time}} = \frac{80}{304} = 0.2631579$

0.2631579 x 100% = 26.31579 = 26.32%

(b) To determine the sequence processing according to SPT

SPT processes jobs based on their processing times with the highest priority given to the job with shortest time as shown below:

Job	Processing Time	Flow time	Job due date	Job lateness (0 of negative)
A	4	4	9	0
B	6	10	6	4
C	7	17	17	0
D	9	26	24	2
E	10	36	18	18
F	12	48	19	29
G	14	62	20	42
H	18	80	28	52
	80	283		147

The measure of effectiveness are:

1. Average flow time = $\frac{\text{Sum of total flow time}}{\text{Number of jobs}} = \frac{283}{8}$

$= 35.375\text{days} = 35.38$ days

2. Average number of jobs in the system = $\frac{\text{Sum of total flow time}}{\text{Total processing time}}$

$= \frac{283\text{days}}{80} = 3.54\text{jobs}$

3. Average job lateness = $\frac{\text{Total late days}}{\text{Number of days}} = \frac{147}{8}$

= 18.375days
=18.38days

4. Utilization = $\frac{\text{Total processing time}}{\text{Sum of total flow time}} = \frac{80}{283}$

0.2826855 x 100%

= 28.27%

(c) To determine the sequence processing according to EDD

Using EDD, you are processing based on their due dates as shown below:

Job	Processing Time	Flow time	Job due date	Job lateness (0 of negative)
A	6	6	6	0
B	4	10	9	1
C	7	17	17	0
D	9	27	18	9
E	10	39	19	20
F	12	53	20	33
G	14	62	24	38
H	18	80	28	52
	80	294		153

The measure of effectiveness are:

1. Average flow time = $\frac{294}{8}$ = 36.75days

2. Average number of jobs in the system = $\frac{294}{80}$
3.675 = 3.68days

3. Average job lateness = $\frac{153}{8}$ = 19.125

= 19.13days
=18.38days

4. Utilization = $\frac{80}{294}$ = 0.272108843

0.282108843 x 100
= 27.21%

(d) **To Determine the Sequence Processing According to LPT**
LPT selects the longer, bigger jobs first as presented below:

Job	Processing Time	Flow time	Job due date	Job lateness (0 of negative)
A	18	18	28	0
B	14	32	20	12
C	12	44	19	25
D	10	54	18	36
E	9	63	24	39
F	7	70	17	53
G	6	76	6	70
H	4	80	9	71
	80	437		306

The measure of effectiveness are:

1. Average flow time = $\frac{437}{8}$ = 54.625days
   = 54.63days

2. Average number of jobs in the system = $\frac{437}{80}$ = 5.4625
   5.46days

3. Average job lateness = $\frac{306}{8}$ = 38.25days

4. Utilization = $\frac{80}{437}$ = 0.183066361
   0.183066361 x 100%
   = 18.31%

The summary of the rules are shown in the table below:

	Average flow time (days)	**Average number of jobs in the system**	**Average job lateness job**	**Utilization%**
FCFS	38	3.8	21.5	26.32
SPT	35.38	3.54	18.38	28.27
EDD	36.75	3.68	19.13	27.21
LPT	54.63	5.46	38.25	18.31

As it can be seen from the table, SPT rule is the best of the four measures and is also the most superior in utilization of the system. On the other hand, LPT is the least effective measure of the three,

Sequencing Jobs in Two Machines

Johnson's rule is used to sequence two or more jobs in two different machines or work centres in the same order. Managers use Johnson rule method to minimize total timer for sequencing jobs through two facilities. In the process, machine total idle time is minimised. The rule does not use job priorities.

Johnson's rule involves the following procedures

1) List the jobs and their respective time requirement on a machine.
2) Choose the job with the shortest time. if the shortest time falls with the first machine, schedule that job first; if the time is at the second machine, schedule the job last. Select arbitrary any job if tie activity time occur.
3) Eliminate the scheduled job and its time
4) Repeat steps 2 and 3 to the remaining jobs, working toward the centre of the sequence until all the jobs are properly scheduled.

Example 4

You arc given the operation times in Hours for 6 jobs in two machines as follow:

Job	**Machine 1 Time (Hours)**	**Machines 2 Time (Hours)**
P	20	20
Q	16	12
R	33	36
S	8	28
T	25	33
U	48	60

(a) Determine the sequence that will minimize idle times on the two machines
(b) The time machine I will complete its jobs
(c) The total completion time for all the jobs
(d) The total idle time

Solution

Using the steps outlined earlier for optimum sequencing of jobs, we obtained

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{t7h}
S	T	R	U	E	Q

We then use tabular method to solve the remaining questions

Job sequence	**1 Machine 1 Duration**	**II Machine 1 in**	**III Machine I Out**	**IV Machine 2 Duration**	**V Machine 2 In**	**VI Machines 2 Out**	**VII Idle Time**
S	8	0	8	28	8	36	8
T	25	8	33	33	36	69	0
R	33	33	66	36	69	105	0
U	48	66	114	60	114	174	9
P	20	114	134	20	174	194	0
Q	16	134	150	12	194	206	0

(a) Machine 1 will complete his job in150 hours
(b) Total completion time is 206 hours
(c) Total idle time is 17 hours

Note that machine 2 will wait 8 hours for its first job and also wait 9 hours after completing job R.
In general, idle time can occur either at the beginning of job or at the end of sequence of jobs. In manufacturing organizations, idle times can be used to do other jobs like maintenance, dismantling or setting up of other equipment.

1.6 Summary

Scheduling, which occurs in every organisation, refers to establishing the timing of the use of equipment, facilities and human activities in an organization and so it deals with the timing of operations. Scheduling technique depends on the volume of system output, the nature of operations and the overall complexity of jobs. The complexity of operation varies under two situations, namely, Flow Shop system and Job Shop system. Flow Shop is a high volume system while Job Shop is a low volume system. Lading refers to assignment of jobs to work centres. The two main methods that can be used to assign jobs to work centres are used of Gant chart and Assignment Method. Job sequencing refers to the order in which jobs should be processed at each work station.

1.7 References/Further Readings/Web Resources

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1.8 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Sequencing is the order of processing a set of tasks over available resources.

Scheduling involves sequencing" task of allocating as well as the determination of process commencement and completion times i.e., time-tabling. Sequencing problems occur whenever there is a choice to the order in which a group of tasks can be performed. The shop supervisor or scheduler can deal with sequencing problems in a variety of ways. The simplest approach is to ignore the problem and accomplish the tasks in any random order.

The most frequently used approach is to schedule heuristically according to predetermined "rules of thumb". In certain cases, scientifically derived scheduling procedures can be used to optimize the scheduling objectives.

Answer to SAE 2

1. **Flow Shop**

Flow shop is a high-volume system, which is characterized by a continuous flow of jobs to produce standardized products. Also, flow shop uses standardized equipment (i.e. special purposed machines) and activities that provide mass production. The goal is to obtain a smooth rate of flow of goods or customer through the system in order to get high utilization of labour and equipment. Examples are refineries, production of detergents etc.

2. **Job Shop**

This is a low volume system, which periodically shift from one job to another. The production is according to customer's specifications and orders or jobs usually in small lots. General-purpose machines characterize Job shop. For example, in designer shop, a customer can place order for different design.

Job-shop processing gives rise to two basic issues for schedulers: how to distribute the workload among work centre and what job processing sequence to use.

Unit 4 Games Theory

Unit Structure

1.1 Introduction

The theory of games (or game theory or competitive strategies) is a mathematical theory that deals with the general features of competitive situations. This theory is helpful when two or more individuals or organisations with conflicting objectives try to make decisions. In such a situation, a decision made by on person affects the decision made by one or more of the remaining decision makers, and the final outcome depend depends upon the decision of all the parties. (Gupta and Hira, 2012)

According to Adebayo et al (2006), Game theory is a branch of mathematical analysis used for decision making in conflict situations. it is very useful for selecting an optimal strategy or sequence of decision in the face of an intelligent opponent who has his own strategy. Since more than one person is usually involved in playing of games, games theory can be described as the theory of multiplayer decision problem. The Competitive strategy is a system for describing games and using mathematical techniques to convert practical problems into games that need to be solved. Game theory can be described as a distinct and interdisciplinary approach to the study of human behaviour and such disciplines include mathematics, economics, psychology and other social and behavioural sciences. If properly understood it is a good law for studying decision- making in conflict situations and it also provides mathematical techniques for selecting optimum strategy and most rational solution by a player in the face of an opponent who already has his own strategy.

1.2 Learning Outcomes

By the end of this unit, you will be able to:

- Demonstrate assumptions of games theory
- Define the concept of a game
- Describe the two-person zero-sum games

1.3 Description of a Game

In our day-to-day life we see many games like Chess, Poker, Football, Base ball etc. All the games are pleasure-giving games, which have the character of a competition and are played according to well- structured rules and regulations and end in a *victory* of one or the other team or group or a player. But we refer to the word game in this unit the competition between two business organizations, which has more earning competitive situations. In this chapter game is described as:
A competitive situation is called a game if it has the following characteristics (Assumption made to define a game):
1. There is finite number of competitors called *Players*. This is to say that the game is played by two or more number of business houses. The game may be for creating new market, or to increase the market share or to increase the competitiveness of the product.

2. A play is played when each player chooses one of his courses of actions. The choices are made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action. But in real world, a player makes the choices after the opponent has announced his course of action.
Algebraic Sum of Gains and Losses*:* A game in which the gains of one player are the losses of other player or the algebraic sum of gains of both players is equal to zero, the game is known as Zero sum game (ZSG). In a zero sum game the algebraic sum of the gains of all players after play is bound to be zero. i.e. If g_i as the pay of to a player in an-person game, then the game will be a zero sum game if sum of all g_i is equal to zero.

In game theory, the resulting gains can easily be represented in the form of a matrix called pay–off matrix or gain matrix as discussed in 3 above. A pay- off matrix is a table, which shows how payments should be made at end of a play or the game. Zero sum game is also known as constant sum game. Conversely, if the sum of gains and losses does not equal to zero, the game is a non zero-sum game. A game where two persons are playing the game and the sum of gains and losses is equal to zero, the game is known as Two-Person Zero-Sum Game (TPZSG). A good example of two- person game is the game of chess. A good example of n- person game is the situation when several companies are engaged in an intensive advertising campaign to capture a larger share of the market (Murthy, 2007)

Self-Assessment Exercise 1

What is Games Theory?

1.3.1 Some Important Definitions in Games Theory

Adebayo et al (2010) provide the following important definitions in game theory.

- **Player**: A player is an active participant in a game. The games can have two persons(Two-person game) or more than two persons (Multi person or n-person game)
- **Moves**: A move could be a decision by player or the result of a chance event.
- **Game**: A game is a sequence of moves that are defined by a set of rules that governs the players' moves. The sequence of moves may be simultaneous.
- **Decision maker**: A decision-maker is a person or group of people in a committee who makes the final choice among the alternatives. A decision-maker is then a player in the game.
- **Objective:** An objective is what a decision-maker aims at accomplishing by means of his decision. The decision-maker may end up with more than one objective.
- **Behaviour**: This could be any sequence of states in a system. The behaviours of a system are overt while state trajectories are covert.
- **Decision**: The forceful imposition of a constraint on a set of initially possible alternatives.
- **Conflict**: A condition in which two or more parties claim possession of something they cannot all have simultaneously. It could also be described as a state in which two or more decision-makers who have different objectives, act in the same system or share the same resources. Examples are value conflicts, territorial conflict, conflicts of interests etc.
- **Strategy**: it is the predetermined rule by which a player decides his course of action from a list of courses of action during the game. To decide a particular strategy, the player needs to know the other's strategy.
- **Perfect information**. A game is said to have perfect information if at every move in the game all players know the move that have already been made. This includes any random outcomes.
- **Payoffs**. This is the numerical return received by a player at the end of a game and this return is associated with each combination

of action taken by the player. We talk of "expected payoff' if its move has a random outcome.

- **Zero-sum Game**. A game is said to be zero sum if the sum of player's payoff is zero. The zero value is obtained by treating losses as negatives and adding up the wins and the losses in the game. Common examples are baseball and poker games.

1.3.2 Assumptions Made in Games Theory

The following are assumptions made in games theory.

- Each player (Decision-maker) has available to him two or more clearly specified choices or sequence of choices (plays).
- A game usually leads to a well-defined end-state that terminates the game. The end state could be a win, a loss or a draw.
- Simultaneous decisions by players are assumed in all games.
- A specified payoff for each player is associated with an end state (eg sum of payoffs for zero sum-games is zero in every end-state).
- Repetition is assumed. A series of repetitive decisions or plays results in a game
- Each decision-maker (player) has perfect knowledge of the game and of his opposition i.e. he knows the rules of the game in details and also the payoffs of all other players. The cost of collecting or knowing this information is not considered in game theory.
- All decision-makers are rational and will therefore always select among alternatives, the alternative that gives him the greater payoff.

The last two assumptions are obviously not always practicable in real life situation. These assumptions have revealed that game theory is a general theory of rational behaviour involving two or more decision makers who have a limit number of courses of action of plays, each leading to a well-defined outcome or ending with games and losses that can be expressed as payoffs associated with each courses of action and for each decision maker. The players have perfect knowledge of the opponent's moves and are rational in taking decision that optimises their individual gain.

The various conflicts can be represented by a matrix of payoffs. Game theory also proposes several solutions to the game. Two of the proposed solutions are:

1. **Minimax or pure Strategy**: In a minimax strategy each player selects a strategy that minimises the maximum loss his opponent can impose upon him.
2. **Mixed Strategy**: A mixed strategy which involves probability choices.

Lot of experiments have been performed on games with results showing conditions for (i) Cooperation (ii) Defection and (iii) Persistence of conflict,

1.4 Types of Games

A. Two-Person Zero-Sum Game

This game involves two players in which losses are treated as negatives and wins as positives and the sum of the wins and losses for each set of strategies in the game is zero. Whatever player one wins player two loses and vice versa. Each player seeks to select a strategy that will maximise his payoffs although he does not know what his intelligence opponent will do. A two-person zero-sum game with one move for each player is called a rectangular game.

Formally, a two-person zero-sum game can be represented as a triple (A, B, y) where A [al, a2.. . .a_{mj} and B [b 1, b2 bn] and are payoff functions, e_{ij} such that y [ai bj = eij. This game can be represented as an m x n matrix of payoffs from player 2 to player 1 as follows:

$$\begin{bmatrix} [\gamma[a_1, b_1] \; \gamma[aj, b_2] \; \ldots\ldots.. \; \gamma[a_i, b_n] \\ \gamma(a_m, b_1] \; y \; [a_m \; b_2] \; \ldots\ldots.. \; \gamma \; [a_m, b_n] \end{bmatrix}$$

The two-person zero-sum games can also be represented as follows: Suppose the choices or alternatives that are available for player 1 can be represented as l,2,3...m. While the options for player two can be represented as 1,2,3.,.n. If player 1 selects alternative i and player 2 selects alternative j then the payoff can be written as a. The table of payoffs is as follows:

		Alternatives for player 1				
		1	2	3	…	n
	1	a_{11}	a_{12}	a_{13}	...	a_{1n}
Alternative for player 2	2	a_{21}	a_{22}	a_{23}	...	a_{2n}
	3	a_{31}	a_{32}	a_{33}	...	a_{3n}
	.					
	.					
	m	a_{m1}	a_{m2}	a_{m3}	…	a_{mn}

A saddle point solution is obtained if the maximum of the minimum of rows equals the minimum of the maximum of columns i.e maximin = minimax
i.e max(min a9) = min(max a9)

Example 1
Investigate if a saddle point solution exists in this matrix

$$\begin{bmatrix} 2 & 1 & -4 \\ -3 & 6 & 2 \end{bmatrix}$$

Solution

$$\begin{array}{l|c} & \text{min} \\ \begin{bmatrix} 2 & 1 & 1 \\ -3 & 6 & 2 \end{bmatrix} & \begin{matrix} -4 \\ -3 \end{matrix} \\ \text{Max} \quad \begin{matrix} 2 & 6 & 3 \end{matrix} & \end{array}$$

$\max_i (\min_{ij}) = \max(-4, -3) = -3$
$\min_j (\max_j a_{ij}) = \min(2,6,3) = 2$
$\max_i (\min_j a_{ij}) \neq \min_j (\max_i a_{ij})$
So a saddle point solution does not exist.

Example 2
We shall consider a game called the "matching penny" game which is usually played by children. In this game two players agree that one will be even and the other odd. Each one then shows a penny. The pennies are shown simultaneously and each child shows a head or tail. If both show the same side "even" wins the penny from odd and if they show different sides odd wins from even. Draw the matrix of payoffs

Solution
The pay-off table is as follows:

		Odd (Player 2) Head	Tail
	Head	(1,-1)	(-1, 1)
Even (Player 1)	Tail	(-1, 1)	(1,-1)

The sum in each cell is zero, hence it is a zero sum game. Now A (H, T), B (H, T) and y (H,H)= y(T,T) 1 while y (H,T)=(T,H)=-1, In matrix form, if row is for even and column is for odd we have the matrix of payoffs given to player I by players 2 as

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Solution of Two-Person Zero-Sum Games

Every two-person zero-sum game has a solution given by the value of the game together with the optimal strategies employed by each of the two players in the game. The strategies employed in a two person zero sum game could be

i. Pure Strategies

ii. Dominating Strategies

iii. Mixed Strategies

Example 3

Find the solutions of this matrix game

$$\begin{pmatrix} -200 & -100 & -40 \\ 400 & 0 & 300 \\ & & 300 \end{pmatrix} \quad -20 \quad 400$$

Solution

We check if max (min a_{ij}) min (max aij) in order to know whether it has a saddle point solution. We first find the minimum of rows and miximum of columns as follows.

$$\begin{pmatrix} -200 & -100 & -40 \\ 400 \quad 0 & 300 & 0 \\ 300 & -20 & 400 \end{pmatrix} \begin{matrix} -200 \\ \\ -20 \end{matrix}$$

$$\text{Max} \quad 400 \qquad 0 \quad 400$$

So $\max_i (\min_j a_{ij})$ = max (-200, 0, -20) = 0

min(max a_{ij}) = mm (400, 0, 400). So a saddle point solution exists at (row2, column2)

i.e (r_2 c_2) The value of the game is 0.

B. Dominating Strategies

In a pay-off matrix row dominance of i oven occurs if a_i>a_j, while column dominance of I over occurs if b_1 b_1. If dominance occurs, column j is not considered and we reduce the matrix by dominance until we are left with 1 x I matrix whose saddle point, solution can be easily found. We consider the matrix

$$\begin{pmatrix} 3 & 4 & 5 & 3 \\ 3 & 1 & 2 & 3 \\ 1 & 3 & 4 & 4 \end{pmatrix}$$

Observation shows that every element in column 1 is less than or equal to that of column 4 and we may remove column 4 the dominating column. Similarly b_3 dominates b_2 and we remove the dominating column b_3. The game is reduced to

$$\begin{pmatrix} 3 & 4 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$$

In row dominance, we eliminate the dominated rows a, (where a.> a,) while in column dominance we eliminate the dominating column b_j (where$_i$≤b_j) since player 2 desired to concede the least payoff to the row player and thus minimise his losses.

This procedure is iterated using row dominance. Since a1 dominates a2 and also dominates a3 we remove the dominated rows a2 and a3. This is due to the fact that player 1, the row player, wishes to maximise his payoffs. We then have a 1 x 1 reduced game [3 4] which has a saddle point solution. Generally if a dominated strategy is reduced for a game, the solution of the reduced game is the solution of the original game.

C. Mixed Strategies

Suppose the matrix of a game is given by

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 3 & -2 \end{pmatrix}$$

Inspection shows that *i* column dominance cannot be used to obtain a saddle point solution. If no saddle point solution exists we randomise the strategies. Random choice of strategies is the main idea behind a mixed strategy. Generally a mixed strategy for player is defined as a pro6a6iffty distribution on the set of pure strategies. The minimax theorem put forward by von Neumann enables one to find the optimal strategies and value of a game that has no saddle point solution and he was able to show that every two-person zero-sum game has a solution in mixed if not in pure strategy.

D. Optimal Strategies in 2 X 2 Matrix Game

Linear optimisation in linear programming enables one to calculate the value and optimal actions especially when the elements of A are more than 2. We now demonstrate how to solve the matching pennies matrix with a simple method applicable when A has two elements and B is finite. Here the value is given as maximin ($\theta\varphi\ [a_1,b_1] + (1\text{-}\theta)\ \varphi\ (a_2\ b_2), \theta\varphi\ ,(a_1,b_2) + (1\text{-}\theta)\ \varphi\ (a_2, b_2)$)

The matric is odd

$$\begin{array}{cc} & \begin{array}{cc} \theta_1 & 1\theta_1 \end{array} \\ \theta 1 & \begin{bmatrix} 1 & \\ 1\text{-}\theta\text{-}1 & 1 \\ & \end{bmatrix} \end{array}$$

We note here that the maximin criterion cannot hold since max (mm of row) max (-1, -1)-1 while min (max of columns) = min (1,1) 1 and no saddle point solution exists.

Let "even" choose randomised action (θ, 1 -θ) i.e θ = a (a_1) and (1-θ) = θ (a_2). Using formula above, we have max mm (θ - 1 + θ, - θ + 1 - θ)
θ + -l (1 - θ) = -θ+ 1 (1 —θ) using principle of equalising expectations.
This gives 2θ -1, 1 -2θ
4θ =2. And $\theta = {}^1/_2$

Similarly if optimal randomised action by player 2= θ_1,
then we get $\theta_1 + (1\text{-}\theta_1)\text{-}1, \theta,+1 -\theta_1)$

$\theta 1(1) + \text{-}1(1\text{-}\ \theta_1) = \theta_1\ (\text{-}1) + (1 - \theta_1)$. Simplify both sides of the equation to get $2\theta_1\text{-}1 = 1 - 2\theta_1\theta_1={}^1/_2$ and so randomised action by player 1 is (${}^1/_2$, ${}^1/_2$) and also (${}^1/_2$, ${}^1/_2$) by player 1

The value can be obtained by substituting = ${}^1/_2$ into 2θ - 1 or I - 2θ or by substituting $\theta_1 = {}^1/_2$ into $2\theta_1$ - I or 1 -$2\theta_1$. If we do this we get a value of zero. So the solution is as follows:
Optimal strategies of (${}^1/_2$,${}^1/_2$) for player 1 and (${}^1/_2$, ${}^1/_2$) for player 2 and the value of the game is 0.

It is Obvious that there is no optimal mixed strategy that is independent of the opponent.

Example 4
Two competing telecommunication companies MTN and Airtel both have objective of maintaining large share in the telecommunication industry. They wish to take a decision concerning investment in a new promotional campaign. Airtel wishes to consider the following options:

r_1: advertise on the Internet
r_2: advertise in all mass media

MTN wishes to consider these alternatives
c_1: advertise in newspapers only
c_2: run a big promo

If Airtel advertise on the Internet and MTN advertises in newspapers, MTN will increase its market share by 3% at the expense of V-Mobile. If MTN runs a big promo and Airtel advertises on the Internet, Airtel will lose 2% of the market share. If Airtel advertises in mass media only and MTh advertises in newspapers, Airtel will lose 4%. However, if Airtel advertises in mass media only and MTN runs a big promo, Airtel will gain 5% of the market share.
a) Arrange this information on a payoff table
b) What is the best policy that each of the two companies should take?

Solution
a) The matrix of payoff is as follows

MTN
c_1 c_2
Airtelr_1

$$\begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix}$$

We first cheek if a saddle point solution exists. We use the minimax criterion to do this. Now for the rows,
Minimax (3,5) = 3 while for the columns
Maximin = Max (-4, -2) = -2.

Since minimax is not equal to maximin, no saddle point solution exists. We then randomise and use the mixed strategy.
Let (θ, 1 - θ) be the mixed strategies adopted by Airtel while (8, 1-8) be the strategies adopted by MTN
Then for Airtel. $\theta (3)+-4(1 - \theta)-2\theta+5(1 -\theta)$
$3\theta - 4+4\theta \ =-2\theta +5 - 5\theta$
$7\theta - 4 -7\theta +5.$
Solving we obtain
$\theta = {}^{9}/_{14}$ and $1 - \theta^{5}/_{14}$
The randomised strategies by V-Mobile will be $({}^{9}/_{4})$
For MTN, $3\theta -2(1-\theta)=-4\theta_1 i+5(1-\theta_1)$
$3\theta+2\theta_{1} {}_{-} 2=-4\theta+5-5\theta_1$
$5\theta_1 -2= -9\theta_1+5$. Solving, we obtain $\theta_1= {}^{1}/_{2}$and $1-\theta_1={}^{1}/_{2}$

The value of the game can be found by substituting 9/14 into 78-4 or – 79+5; or V2 into 5θ -2 or -9θ+5. When we do this we obtain the value ${}^{1}/_{2}$. So Airtel should advertise on the Internet ${}^{9}/_{14}$ of the time and advertise on the mass media ${}^{5}/_{14}$ of the time. On the other hand, MTN should advertise in the newspapers only 50% ($^1/_2$) of the time and run a big promo $^1/_2$ of the time. The expected gain of Airtel is $^1/_2$ of the market share.

F. Equilibrium Pairs

In mixed strategies, a pair of optimal strategies a* and b* is in equilibrium if for any other a and b, E(a,b*) <E(a*,b*) <E(a*, b)

A pair of strategies (a*, b*) in a two person zero sum game is in equilibrium if and only if {(a*, b*), E(a*, b*)} is a solution to the game. Nash Theory states that any two person game (whether zero-sum or non-zero-sum) with a finite number of pure strategies has at least one equilibrium pair. No player can do better by changing strategies, given that the other players continue to follow the equilibrium strategy.

G. Optimal Strategies in 2 X N Matrix Game

Suppose we have a matrix game of

$$\begin{pmatrix} 5 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Now

$\max_i (\min_j a_{ij})= \max(2,3)=3$ while = min(max) 4.

The two players now have to look for ways of assuring themselves of the largest possible shares of the difference

$\max_i (\min_j a_{ij}) - \min_i (\max_j a_{ij}) \geq 0$

They will therefore need to select strategies randomly to confuse each other. When a player chooses any two or more strategies at random according to specific probabilities this device is known as a mixed strategy.

There are various method employed in solving 2x2, 2xn, mx2 and m x n game matrix and hence finding optimal strategies as we shall discuss in this and the next few sections. Suppose the matrix of game is m x n. If player one is allowed to select strategy I. with probability pi and player two strategy II with probability q. then we can say player 1 uses strategy

$P=(P_1,P_2\ldots P_m)$

While player 2 selects strategy

$q=(q_1,q_2,...q_n)$.

The expected payoffs for player 1 by player two can be explained in

$$E\sigma\ \sigma^* \sum_{i=1}^{m} \sum_{j=1}^{n} \text{pi}\varphi\ (\text{pi})q$$

In this game the row player has strategy q = (q1, q2.. .q). The max-mm reasoning is used to find the optimal strategies to be employed by both player. We demonstrate with a practical example:

Example 5

Let the matrix game be

$$\begin{pmatrix} 5 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Solution

Inspection shows that this does not have a saddle point solution. The optimal strategy p" for the row player is the one that will give him the maximum pay-off. Since p = (p p2). Let the expected value of the row be represented by E_1player. If player 2 plays column 1 is =

5p+3(l .p) 2p+3p

If player 2 plays column 2 we have

$E_{2(p)}$ = 2P+ 4 (l-P) = -2P+4

and if player 2 plays colunm 3 we have

E_3(p) $4_{(p)}$+5(1-p) =p+5.So, $E_{1(p)}$ = 2p +3; $E_{2(p)} =$ 2p+4 and $E_{3(p)}$ = p+5 are the payoffs for player 1 against the three part strategies of player 2, we give arbitrary values for p to check which of these strategies by player2 will yield the largest payoff for

player 1.

Let p $^3/_4$ ……. E_1=-2x$^3/_4$+3=4$^1/_2$

$E_{2(p)}$ = 2x¼+42$^1/_2$ $E_{2(p)}$ =-$^3/_4$+5=4$^1/_4$.

So the two largest are $E_{1(p)},$ $E_{3(p)}$ and we equate them to get

2p = 3 = p+5

so 3p= 2, p=$^2/_3$

$E_{j(p)}$ = (2 x $^2/_3$) + 3 4$^1/_3$

E = -$2_{(p)}$ -2 x 2$^2/_3$ + 4 = 2$^2/_3$ and $E_{3(p)}$ = $^2/_3$ + 5 = 74$^1/_3$

So($^2/_3$, $^1/_3$) is optimal for player 1. To get the optimal strategy for player 2, we observe that it is advisable for player 2 to play column 2 in other to ensure that the payoff to row player is minimal. So the game is reduced to

$$\begin{pmatrix} 5 & 4 \\ 3 & 5 \end{pmatrix}$$

Let (q, l-q) be the strategy for player 2 in a required game.

So 5q + 4 (1-q) 3q +5(l-q)

5q+4-4q=3q+5-5q

q +45-2q
3q=l q=1/3
So it is optimal for player 2 to play mixed strategy with probability $q(^1/_3,O,^2/_3)$.If we substitute $q = {^1/_3}$ into q+4 or 5-2q, we obtain $4^1/_3$ as before. This is the value of the game.

1.5 Summary

Making decision is an integral and continuous aspect of human life. For child or adult, man or woman, government official or business executive, worker or supervisor, participation in the process of decision- making is a common feature of everyday life. A competitive situation is called a game if it has the following characteristics- there is finite number of competitors called Players. A list of finite or infinite number of possible courses of action is available to each player; a list of finite or infinite number of possible courses of action is available to each player; a play is played when each player chooses one of his courses of actions; all players act rationally and intelligently. Each player is interested in maximizing his gains or minimizing his losses; each player makes individual decisions without direct communication between the players; it is assumed that each player knows complete relevant information.

1.6 References/Further Readings/Web Resources

Adebayo, O.A., Ojo, O., & Obamire, J. K. (2006). Operations Research in Decision Analysis, Lagos: Pumark Nigeria Limited.

Murthy, R. P. (2007). Operations Research, 2nd ed., New Delhi: New Age International (P) Limited Publishers

1.7 Possible Answers to Self-Assessment Exercise(s)

Answer to SAE 1

Game theory is the study of mathematical models of strategic interactions among rational agents. It has applications in many fields of social science, used extensively in economics as well as in logic, systems science and computer science.

Answer to SAE 2

- Cooperative and non-cooperative games.
- Normal form and extensive form games.
- Simultaneous move and sequential move games.
- Constant sum, zero-sum and non-zero-sum games.
- Symmetric and asymmetric games.

Unit 5 Inventory Control

Unit Structure

1.1 Introduction

One of the basic functions of management is to employ capital efficiently so as to yield the maximum returns. This can be done in either of two ways or by both, *i.e*. (a) By maximizing the margin of profit; or (b) By maximizing the production with a given amount of capital, *i.e.* to increase the productivity of capital. This means that the management should try to make its capital work hard as possible. However, this is all too often neglected and much time and ingenuity are devoted to make only labour work harder. In the process, the capital turnover and hence the productivity of capital is often totally neglected.

Inventory management or Inventory Control is one of the techniques of Materials Management which helps the management to improve the productivity of capital by reducing the material costs, preventing the large amounts of capital being locked up for long periods, and improving the capital - turnover ratio. The techniques of inventory control were evolved and developed during and after the Second World War and have helped the more industrially developed countries to make spectacular progress in improving their productivity.

1.2 Learning Outcomes

- Define inventory control

- Explain the basic concepts in inventory control
- Identify the issues that necessitate maintaining inventory
- Identify causes of poor inventory control systems
- Discuss the various classifications of inventories.

1.3 Definition of Inventory and Inventory Control

The word *inventory* means a physical stock of material or goods or commodities or other economic resources that are stored or reserved or kept in stock or in hand for smooth and efficient running of future affairs of an organization at the minimum cost of funds or capital blocked in the form of materials or goods (Inventories). The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finished goods in an orderly manner to meet the objectives of maximum customer service with minimum investment and efficient (low cost) plant operation is termed as *inventory control.* (Murthy, 2007)

Gupta and Hira (2012) defined an inventory as consisting of usable but idle resources such as men, machines, materials, or money. When the resources involved are material, the inventory is called stock. An inventory problem is said to exist if either the resources are subject to control or if there is at least one such cost that decrease as inventory increases. The objective is to minimise total (actual or expected) cost. However, in situations where inventory affects demand, the objective may also be to minimise profit.

1.3.1 Basic Concepts in Inventory Planning

For many organizations, inventories represent a major capital cost, in some cases the dominant cost, so that the management of this capital becomes of the utmost importance. When considering the inventories, we need to distinguish different classes of items that are kept in stock. In practice, it turns out that about 10% of the items that are kept in stock usually account for something in the order of 60% of the value of all inventories. Such items are therefore of prime concern to the company, and the stock of these items will need close attention. These most important items are usually referred to as “A items” in the ABC classification system developed by the General Electric Company in the 1950s. The items next in line are the B items, which are of intermediate importance. They typically represent 30% of the items, corresponding to about 30% of the total inventory value. Clearly, B items do require some attention, but obviously less than A items. Finally, the bottom 60% of the items are the C items. They usually represent maybe 10% of the monetary value of the total inventory. The control of C items in inventory planning is less crucial than that of the A and B items. The models in this chapter are mostly aimed at A items.

1.3.2 Necessity for Maintaining Inventory

Though inventory of materials is an idle resource (since materials lie idle and are not to be used immediately), almost every organisation. Without it, no business activity can be performed, whether it is service organisation like a hospital or a bank or it a manufacturing or trading organisation. Gupta and Hira (2012) present the following reasons for maintain inventories in organisations.

1. It helps in the smooth and efficient of an enterprise.
2. It helps in providing service to the customer at short notice.
3. In the absence of inventory, the enterprise may have to pay high prices due to piecemeal purchasing.
4. It reduces product cost since there is an added advantage of batching and long, uninterrupted production runs.
5. It acts as a buffer stock when raw materials are received late and shop rejection is too many.

1.3.3 Causes of Poor Inventory Control Systems

a. Overbuying without regard to the forecast or proper estimate of demand to take advantages of favourable market.
b. Overproduction or production of goods much before the customer requires them
c. Overstocking may also result from the desire to provide better service to the custom.
d. Cancellation of orders and minimum quantity stipulations by the suppliers may also give rise to large inventories.
(Gupta and Hira, 2012)

1.3.4 Classification of Inventories

Inventories may be classified as those which play direct role during manufacture or which can be identified on the product and the second one are those which are required for manufacturing but not as a part of production or cannot be identified on the product. The first type is labelled as ***direct inventories*** and the second are labelled as ***indirect inventories***. Further classification of direct and indirect inventories is as follows:

A. Direct inventories

(*i*) *Raw material inventories or Production Inventories*: The inventory of raw materials is the materials used in the manufacture of product and

can be identified on the product. In inventory control manager can concentrate on the
(*a*) Bulk purchase of materials to save the investment,
(*b*) To meet the changes in production rate,
(*c*) To plan for buffer stock or safety stock to serve against the delay in delivery of inventory against orders placed and also against seasonal fluctuations. Direct inventories include the following:

- **Production Inventories-** items such as raw materials, components and subassemblies used to produce the final products.
- **Work-in-progress Inventory-** items in semi-finished form or products at different stages of production.
- **Miscellaneous Inventory-** all other items such as scrap, obsolete and unsaleable products, stationary and other items used in office, factory and sales department, etc.

(*ii*) ***Work-in -process inventories or in process inventories:*** These inventories are of semi-finished type, which are accumulated between operations or facilities.As far as possible, holding of materials between operations to be minimized if not avoided. This is because; as we process the materials the economic value (added labour cost) and use value are added to the raw material, which is drawn from stores. Hence if we hold these semi-finished material for a
long time the inventory carrying cost goes on increasing, which is not advisable in inventory control. These inventories serve the following purposes:
(*a*) Provide economical lot production,
(*b*) Cater to the variety of products,
(*c*) Replacement of wastages,
(*d*) To maintain uniform production even if sales varies.

(*iii*) ***Finished goods inventories:*** After finishing the production process and packing, the finished products are stocked in stock room. These are known as finished goods inventory. These are maintained to:
(*a*) To ensure the adequate supply to the customers,
(*b*) To allow stabilization of the production level and
(*c*) To help sales promotion programme.

(*iv*) ***MRO Inventory or Spare parts inventories:*** Maintenance, Repair, and Operation items such as spare parts and consumable stores that do not go into final products but are consumed during the production process. Any product sold to the customer, will be subjected to wear and tear due to usage and the customer has to replace the worn-out part. Hence the manufacturers always calculate the life of the various components of his product and try to supply the spare components to the market to help after sales service. The use of such spare parts inventory is:
(*a*) To provide after sales service to the customer,
(*b*) To utilize the product fully and economically by the customer.

(*iv*) *Scrap or waste inventory or Miscellaneous Inventory:* While processing the materials, we may come across certain wastages and certain bad components (scrap), which are of no use. These may be used by some other industries as raw material. These are to be collected and kept in a place away from main stores and are disposed periodically by auctioning.

B. Indirect Inventories

Inventories or materials like oils, grease, lubricants, cotton waste and such other materials are required during the production process. But we cannot identify them on the product. These are known as indirect inventories. In our discussion of inventories, in this chapter, we only discuss about the direct inventories. Inventories may also be classified depending on their nature of use. They are:

(*i*) *Fluctuation Inventories:* These inventories are carried out to safeguard the fluctuation in demand, non-delivery of material in time due to extended lead-time. These are sometimes called as Safety stock or reserves. In real world inventory situations, the material may not be received in time as expected due to trouble in transport system or some times, the demand for a certain material may increase unexpectedly. To safeguard such situations, safety stocks are maintained. The level of this stock will fluctuate depending on the demand and lead-time etc.

(*ii*) *Anticipation inventory:* When there is an indication that the demand for company’s product is going to be increased in the coming season, a large stock of material is stored in anticipation. Some times in anticipation of raising prices, the material is stocked. Such inventories, which are stocked in anticipation of raising demand or raising rises, are known as anticipation inventories.

(*iii*) *Lot size inventory or Cycle inventories:* This situation happens in batch production system. In this system products are produced in economic batch quantities. It sometime happens that the materials are procured in quantities larger than the economic quantities to meet the fluctuation in demand. In such cases the excess materials are stocked, which are known as lot size or cycle inventories.

1.3.5 Costs Associated with Inventory

While maintaining the inventories, we will come across certain costs associated with inventory, which are known as *economic parameters*. Most important of them are discussed below:

A. Inventory Carrying Charges, or Inventory Carrying Cost or Holding Cost or Storage Cost (C_1) or (i%)

This cost arises due to holding of stock of material in stock. This cost includes the cost of maintaining the inventory and is proportional to the quantity of material held in stock and the time for which the material is maintained in stock. The components of inventory carrying cost are:

i. Rent for the building in which the stock is maintained if it is a rented building. In case it is own building, depreciation cost of the building is taken into consideration. Sometimes for own buildings, the nominal rent is calculated depending on the local rate of rent and is taken into consideration.

ii. It includes the cost of equipment if any and cost of racks and any special facilities used in the stores.

iii. Interest on the money locked in the form of inventory or on the money invested in purchasing the inventory.

iv. The cost of stationery used for maintaining the inventory.

v. The wages of personnel working in the stores.

vi. Cost of depreciation, insurance.

B. Shortage cost or Stock - out - cost- (C_2)

Sometimes it so happens that the material may not be available when needed or when the demand arises. In such cases the production has to be stopped until the procurement of the material, which may lead to miss the delivery dates or delayed production. When the organization could not meet the delivery promises, it has to pay penalty to the customer. If the situation of stock out will occur very often, then the customer may not come to the organization to place orders that is the organization is losing the customers In other words, the organization is losing the goodwill of the customers The cost of good will cannot be estimated. In some cases it will be very heavy to such extent that the organization has to forego its business. Here to avoid the stock out situation, if the organization stocks more material, inventory carrying cost increases and to take care of inventory cost, if the organization purchases just sufficient or less quantity, then the stock out position may arise. Hence the inventory manager must have sound knowledge of various factors that are related to inventory carrying cost andstock out cost and estimate the quantity of material to be purchased or else he must have effective strategies to face grave situations. The cost is generally represented as so many naira and is represented by C_2.

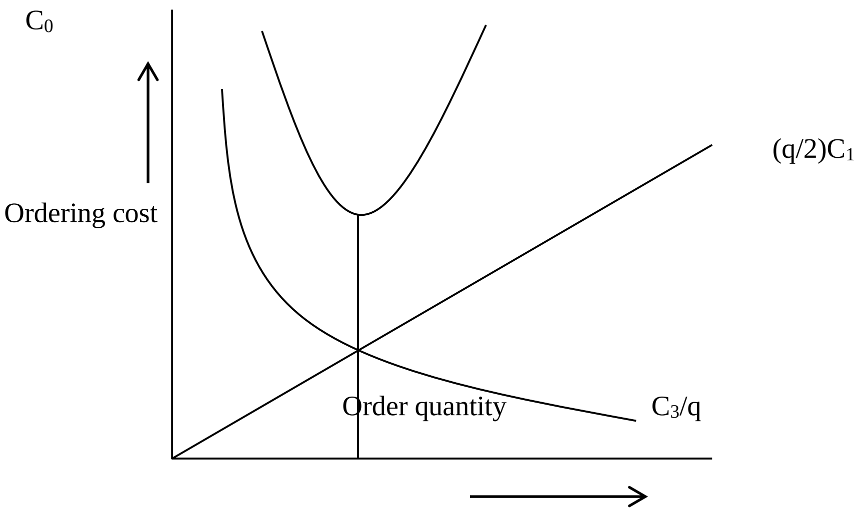
C. Set up cost or Ordering cost or Replenishment Cost (C_3)

For purchase models, the cost is termed as ordering cost or procurement cost and for manufacturing cost it is termed as set up cost and is represented by C_3.

(*i*) *Set up cost:* The term set up cost is used for production or manufacturing models. Whenever a job is to be produced, the machine is to set to produce the job. That is the tool is to be set and the material is to be fixed in the jobholder. This consumes some time. During this time the machine will be idle and the labour is working. The cost of idle machine and cost of labour charges are to be added to the cost of production. If we produce only one job in one set up, the entire set up cost is to be charged to one job only. In case we produce '*n*' number of jobs in one set up, the set up cost is shared by '*n*' jobs. In case of certain machines like N.C machines, or Jig boarding machine, the set up time may be 15 to 20 hours. The idle cost of the machine and labour charges may work out to few thousands of naira. Once the machine set up is over, the entire production can be completed in few hours if we produce more number of products in one set up the set up cost is allocated to all the jobs equally. This reduces the production cost of the product. For example let us assume that the set up cost is N 1000/-. If we produce 10 jobs in one set up, each job is charged with ₦ 100/- towards the set up cost. In case, if we produce 100 jobs, the set up cost per job will be ₦10/-. If we produce, 1000 jobs in one set up, the set up cost per job will be Re. 1/- only. This can be shown by means of a graph as shown in figure 15.1.

(*ii*) *Ordering Cost or Replenishment Cost:* The term Ordering cost or Replenishment cost is used in purchase models. Whenever any material is to be procured by an organization, it has to place an order with the supplier. The cost of stationary used for placing the order, the cost of salary of officials involved in preparing the order and the postal expenses and after placing the order enquiry charges all put together, is known as ordering cost. In Small Scale Units, this may be around ₦ 25/- to ₦ 30/- per order. In Larger Scale Industries, it will be around ₦ 150 to N 200 /- per order. In Government organizations, it may work out to ₦ 500/- and above per order. If the organization purchases more items per order, all the items share the ordering cost. Hence the materials manager must decide how much to purchase per order so as to keep the ordering cost per item at minimum. One point we have to remember here, to reduce the ordering cost per item, if we purchase more items, the inventory carrying cost increases. To keep inventory carrying cost under control, if we purchase less quantity, the ordering cost increase. Hence one must be careful enough to decide how much to purchase? The nature of ordering cost can also be shown by a graph as shown in figure 8.1. If the ordering cost is C_3 per order (can be equally applied to set up cost) and the quantity ordered / produced is '*q*' then the ordering cost or set up cost per unit will

be $C3/q$ is inversely proportional to the quantity ordered, *i.e*. decreased with the increase
in '*q*' as shown in the graph below.



Ordering Cost
Source : Murthy, P. R. (2007) Operations Research 2nd ed. New Delhi: New Age International Publishers

(*iii*) *Procurement Cost:* These costs are very much similar to the ordering cost / set up cost. This cost includes cost of inspection of materials, cost of returning the low quality materials, transportation cost from the source of material to the purchaser's site. This is proportional to the quantity of materials involved. This cost is generally represented by '*b*' and is expressed as so many naira per unit of material. For convenience, it always taken as a part of ordering cost and many a time it is included in the ordering cost / set up cost.

D. Purchase price or direct production cost

This is the actual purchase price of the material or the direct production cost of the product. It is represented by '*p*'. *i.e*. the cost of material is ₦ '*p*' per unit. This may be constant or variable. Say for example the cost of an item is N 10/- item if we purchase 1 to 10 units. In case we purchase more than 10 units, 10 percent discount is allowed. *i.e*. the cost of item will be ₦9/- per unit. The purchase manager can take advantage of discount allowed by purchasing more. But this will increase the inventory carrying charges. As we are purchasing more per order, ordering cost is reduced and because of discount, material cost is reduced. Materials manager has to take into consideration these cost – quantity relationship and decide how much to purchase to keep the inventory cost at low level.

1.3.6 Purpose of Maintaining Inventory or Objective of Inventory cost Control

The purpose of maintaining the inventory or controlling the cost of inventory is to use the available capital optimally (efficiently) so that inventory cost per item of material will be as small as possible. For this the materials manager has to strike a balance between the interrelated inventory costs. In the process of balancing the interrelated costs *i.e.* Inventory carrying cost, ordering cost or set up cost, stock out cost and the actual material cost. Hence we can say that *the objective of controlling the inventories is to enable the materials manager to place and order at right time with the right source at right price to purchase right quantity.* The benefits derived from efficient inventory control are:

i. It ensures adequate supply of goods to the customer or adequate of quantity of raw materials to the manufacturing department so that the situation of stock out may be reduced or avoided.

ii. By proper inventory cost control, the available capital may be used efficiently or optimally, by avoiding the unnecessary expenditure on inventory.

iii. In production models, while estimating the cost of the product the material cost is to be added. The manager has to decide whether he has to take the actual purchase price of the material or the current market price of the material. The current market price may be less than or greater than the purchase price of the material which has been purchased some period back. Proper inventory control reduces such risks.

iv. It ensures smooth and efficient running of an organization and provides safety against late delivery times to the customer due to uncontrollable factors

v. A careful materials manager may take advantage of price discounts and make bulk purchase at the same time he can keep the inventory cost at minimum.

1.3.7 Other Factors to be considered in Inventory Control

There are many factors, which have influence on the inventory, which draws the attention of an inventory manager, they are:

(*i*) Demand

The demand for raw material or components for production or demand of goods to satisfy the needs of the customer, can be assessed from the past consumption/supply pattern of material or goods. We find that the demand may be deterministic in nature *i.e.*, we can specify that the demand for the item is so many units for example say '*q*' units per unit of time. Also the demand may be static, *i.e*. it means constant for each time period (uniform over equal period of times).

The supply of inventory to the stock may deterministic or probabilistic (stochastic) in nature and many a times it is uncontrollable, because, the rate of production depends on the production, which is once again depends on so many factors which are uncontrollable / controllable factors Similarly supply of inventory depends on the type of supplier, mode of supply, mode of transformation etc.

(*iii*) Lead time or Delivery Lags or Procurement time
Lead-time is the time between placing the order and receipt of material to the stock. In production models, it is the time between the decisions made to take up the order and starting of production. This time in purchase models depends on many uncontrollable factors like transport mode, transport route, agitations etc. It may vary from few days to few months depending on the nature of delay.

(*iv*) Type of goods
The inventory items may be discrete or continuous. Sometimes the discrete items are to be considered as continuous items for the sake of convenience.

(*v*) Time horizon
The time period for which the optimal policy is to be formulated or the inventory cost is to be optimized is generally termed as the Inventory planning period of Time horizon. This time is represented on X - axis while drawing graphs. This time may be finite or infinite.

In any inventory model, we try to seek answers for the following questions:

(*a*) When should the inventory be purchased for replenishment? For example, the inventory should be replenished after a period't' or when the level of the inventory is *qo*.

(*b*) How much quantity must be purchased or ordered or produced at the time of replenishment so as to minimize the inventory costs? For example, the inventory must be purchased with the supplier who is supplying at a cost of N*p*/- per unit. In addition to the above depending on the data available, we can also decide from which source we have to purchase and what price we have to purchase? But in general time and quantity are the two variables, we can control separately or in combination.

1.4 Inventory Control Problem

The inventory control problem consists of the determination of three basic factors:

1. When to order (produce or purchase)?
2. How much to order?
3. How much safety stock to be kept?

When to order: This is related to lead time (also called delivery lag) of an item. Lead time may interval between the placement of an order for an item and its receipt in stock. It may be replenishment order on an outside or within the firm. There should be enough stock for each item so that customers' orders can be reasonably met from this stock until replenishment.
How much to order: Each order has an associated ordering cost or cost of acquisition. To keep this cost low, the number of orders has to be as reduced as possible. To achieve limited number of orders, the order size has to be increased. But large order size would imply high inventory cost.

How much should the safety stock be. This is important to avoid overstocking while ensuring that no stock out takes place.

The inventory control policy of an organisation depends upon the demand characteristics. The demand for an item may be dependent or independent. For instance, the demand for the different models of television sets manufactured by a company does not depend upon the demand for any other item, while the demand for its components will depend upon the demand for the television sets.

1.5 The Classical EOQ Model (Demand Rate Uniform, Replenishment Rate Infinite)

According Gupta and Hira 2012, the EOQ model is one of the simplest inventory models we have. A store keeper has an order to supply goods to customers at a uniform rate R per unit. Hence, the demand is fixed and known. Not shortages are allowed, consequently, the cost of shortage $C_2$is infinity. The store keeper places an order with a manufacturer every *t* time units, where *t* is fixed; and the ordering cost per order is C_3. Replenishment time is negligible, that is, replenishment rate is infinite so that the replacement is instantaneous (lead time is zero). The holding cost is assumed to be proportional to the amount of inventory as well as the time inventory is held. Hence the time of holding inventory I for time T is C_1IT, where C_1, C_2 and C_3 are assumed to be constants. The store keeper's problem is therefore to the following

i. How frequently should he place the order?
ii. How many units should he order in each order placed?

This model is represented schematically below.
If orders are placed at intervals *t*, a quantity *q* = R*t*must be ordered in each order. Since the stock in small time *dt* is R*tdt* the stock in time period *t* will be

$$\int_0^t Rt.dt = \frac{1}{2}Rt^2 = \frac{1}{2}qt$$

$$= Area\ of\ inventory\ triangle\ OAP.$$

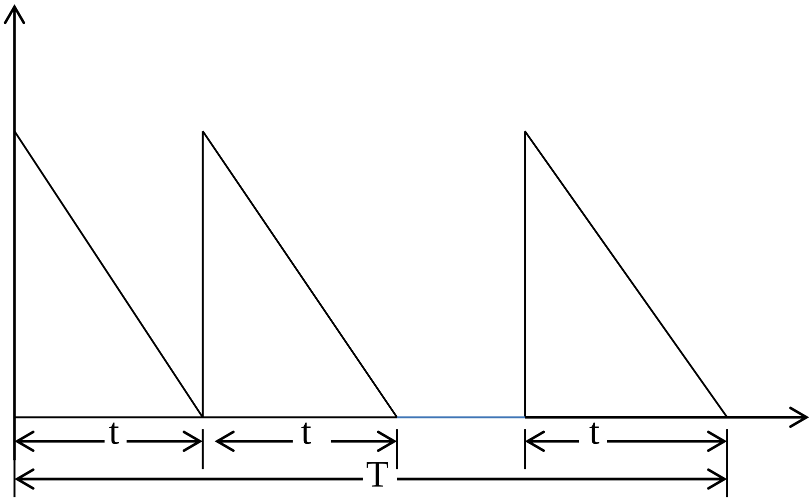


Fig. Inventory situation for EOQ model

$\therefore$ Cost of holding inventory during time t = $\frac{1}{2} C_1Rt^2$.

Order cost to place an order = C_3.

$\therefore$ Total cost during time $t = \frac{1}{2} C_1Rt^2 + C_3$.

$\therefore$Average total cost per unit, $C(t) = \frac{1}{2} C_1Rt + \frac{C_3}{t}$……………. *(1)*

C will be minimum if $\frac{dC(t)}{dt} = 0$ and $\frac{d_2C(t)}{dt^2}$ is positive.

Differentiating equation (1) w.r.t ‘*t*’

$$\frac{d^2C(t)}{dt} = \frac{1}{2} C_1R - \frac{C_3}{t^2} = 0, \text{ which gives } t = \sqrt{\frac{2C_3}{C_1R}}$$

Differentiating w.r.t. ‘*t*’

$\frac{d^2C(t)}{dt^2} = \frac{2C_3}{t^3}$, which is positive for value of *t* given by the above equation.

Thus C*(t)* is minimum for optimal time interval,

$$t_o = \sqrt{\frac{2C_3}{C_1R}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textit{(2)}$$

Optimum quantity q_0 to be ordered during each order,

$$q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textit{(3)}$$

This is known as the optimal lot size (or economic order quantity) formula by r. H. Wilson. It is also called Wilson's or square root formula or Harris lot size formula.

Any other order quantity will result in a higher cost.
The resulting minimum average cost per unit time,

$$C0(q) = \frac{1}{2}C_1R.\sqrt{\frac{2C_3}{C1R}} + C_3\sqrt{\frac{C_1R}{2C_3}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{C_1C_3R} + \frac{1}{\sqrt{2}}\sqrt{C_1C_3R} = \sqrt{2C_1C_3R} \ldots\ldots \textit{(4)}$$

Also, the total minimum cost per unit time, including the cost of the item

$$= \sqrt{2C1C3R} + CR, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textit{(5)}$$

Where C is cost/unit of the item
Equation *(1)* can be written in an alternative form by replacing *t* by *q*/R as

$$C(q) = \frac{1}{2}C_1q + \frac{C_3R}{q}$$

The average inventory is $\frac{q_{0+}\,0}{2} = \frac{q_0}{2}$ and it is time dependent.

It may be realised that some of the assumptions made are not satisfied in actual practice. For instance, in real life, customer demand is usually not known exactly and replenishment time is usually not negligible.

Corollary 1. In the above model, if the order cost is $C_3 + bq$ instead of being fixed, where b is the cost of order per unit of item, we can prove that there no change in the optimum order quantity due to changed order cost.

Proof. The average cost per unit of time, $C(q) = \frac{1}{2}C_1 q + \frac{R}{q}(C_3 + bq)$.
From equation *(5),*

$\frac{dC(q)}{dq} = 0\ and\ \frac{d^2\ C(q)}{dq^2}$is positive

That is, $\frac{1}{2}C_1 - \frac{RC_3}{q^2} = 0\ or\ q = \sqrt{\frac{2RC_3}{C_1}}$,
$and\ \frac{d^2\ C(q)}{dq^2} = \frac{2RC_3}{q^3}$, which is necessarily positive for above value of q.

$q_0 = \sqrt{\frac{2C_3R}{C_1}}$, which is the same as equation *(3)*

Hence, there is no change in the optimum order quantity as a result of the change in the cost of order.

Corollary 2. In the model in figure …… discussed above, the lead time has been assumed to be zero. However, most real life problems have positive lead time L from the order for the item was placed until it is actually delivered. The ordering policy of the above model therefore, must satisfy the reorder point.

If L is the lead time in days, and R is the inventory consumption rate in units per day, the total inventory requirements during the lead time = LR. Thus we should place an order q as soon as the stock level becomes LR. This is called reorder point p = LR.

In practice, this is equivalent to continuously observing the level of inventory until the reorder point is obtained. That is why economic lot size model is also called continuous review model.

If the buffer stock B is to maintained, reorder level will be

P = B + LR ……………………………………….. *(6)*

Furthermore, if D days are required for reviewing the system,

$$p = B + LR = \frac{RD}{2} = B + R[L + \frac{D}{2}] \qquad \text{……………….. (7)}$$

Assumptions in the EOQ Formula

The following assumptions have been made while deriving the EOQ formula:

1. Demand is known and uniform (constant)
2. Shortages are not permitted; as soon as the stock level becomes zero, it is instantaneously replenished.
3. Replenishment stock is instantaneous or replenishment rate is infinite.
4. Lead time is zero. The moment the order is placed, the quantity ordered is automatically received.
5. Inventory carrying cost and ordering cost per order remain constant over time. The former has a linear relationship with the quantity ordered and the latter with the number of order.
6. Cost of the item remains constant over time. There are no price-breaks or quantity discounts.
7. The item is purchased and replenished in lots or batches.
8. The inventory system relates to a single item.

Limitations of the EOQ Model

The EOQ formula has a number of limitations. It has been highly controversial since a number of objections have been raised regarding its validity. Some of these objections are:

1. In practice, the demand neither known with certainty nor it is uniform. If the fluctuations are mild, the formula can be applicable but for large fluctuations, it loses its validity. Dynamic EOQ models, instead, may have to be applied.
2. The ordering cost is difficult to measure. Also it may not be linearly related to the number of orders as assumed in the derivation of the model. The inventory carrying rate is still more difficult to measure and even to define precisely.
3. It is difficult to predict the demand. Present demand may be quite different from the past history. Hardly any prediction is possible for a new product to be introduced in the market.
4. The EOQ model assumes instantaneous replenishment of the entire quantity ordered. The practice, the total quantity may be supplied in parts. EOQ model is not applicable in such a situation.

5. Lead time may not be zero unless the supplier is next-door and has sufficient stock of the item, which is rarely so.
6. Price variations, quantity discounts and shortages may further invalidate the use of the EOQ formula.

However, the flatness of the total cost curve around the minimum is an answer to the many objections. Even if we deviate from EOQ within reasonable limits, there is no substantial change in cost. For example, if because of inaccuracies and errors, we have selected an order quantity 20%more (or less) than $q_0$the increase in total cost will be less than 20%.

EXAMPLE 1

A stock keeper has to supply 12000 units of a product per year to his customer. The demand is fixed and known and the shortage cost is assumed to be infinite. The inventory holding cost is ₦ 0.20k per unit per month, and the ordering cost per order is N350. Determine

i. The optimum lot size q_0
ii. Optimum scheduling period t_0
iii. Minimum total variable yearly cost.

Solution

Supply rate R = $\frac{12,000}{12} = 1,000\ unit/month,$

C1 = N 0.20K per unit per month, C3 = ₦350 per order.

i. $q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2X350X1000}{0.20}} = 1870 units/order$

ii. $t_0 = \sqrt{\frac{2C_3}{C_1R}} = \sqrt{\frac{2X350}{0.20X1000}} = 1.87\ months = 8.1 weeks\ between\ orders$

iii. $C_0 = \sqrt{2C_1C_3R} = \sqrt{2X0.2X12X350X(1000X12)} =$ ₦$4490\ per\ year$

EXAMPLE 2

A particular item has a demand of 9000 unit/year. The cost of a single procurement is ₦100 and the holding cost per unit is ₦ 2.40k per year. The replacement is instantaneous and no shortages are allowed. Determine

i. The economic lot size,
ii. The number of orders per year,
iii. The time between orders
iv. The total cost per if the cost of one unit is ₦1

Solution

R = 9000 units/year

C3 = N100/procurement, C1 = N2.40/unit/year

i. $q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\ X\ 100\ X\ 9000}{2.40}} = 866\ units/procurement$

ii. $n_0 = \frac{1}{t_0} = \sqrt{\frac{2.40\ X\ 9000}{2\ X\ 100}} = \sqrt{108} = 10.4\ orders/year$

iii. $t_0 = \frac{1}{n_0} = \frac{1}{10.4} = 0.0962 years = 1.15\ months\ between\ procurement$

iv. $C_0 = 900\ X\ 1 + \sqrt{2C_1C_3R} = 9000 + \sqrt{2\ X\ 2.40\ X 100\ X 9000}$
= 9000 + 2080 = ₦11080/year

EXAMPLE 3

A stockist has to supply 400 units of a product every Monday to his customer. He gets the product at ₦ 50 per unit from the manufacturer. The cost of ordering and transportation from the manufacturer is ₦75 per order. The cost of carrying the inventory is 7.5% per year of the cost of the product. Find

i. The economic lot size
ii. The total optimal cost (including the capital cost)
iii. The total weekly profit if the item is sold for ₦ 55 per unit

Solution

R = 400 units/week
C3 = N75per order
C1 = 7.5% per year of the cost of the product

$$= N\left(\frac{7.5}{100} X 50\right) per\ unit\ per\ year$$

$$= \left(\frac{7.5}{100} X \frac{50}{2}\right) per\ unit\ per\ week$$

$$= N\frac{3.75}{52}\ \ per\ unit\ per\ week$$

i. $q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\ x\ 75\ x\ 400\ x\ 52}{52}}\ = 912\ unita\ per\ order$

ii. $C_0 = 400\ x\ 50 + \sqrt{2C_1C_3R}$

$$= 20000 + \sqrt{\frac{2\ x\ 3.75}{52} x\ 75\ x\ 400} = 20000 + 65.8$$
$$= N20065.80\ per\ week$$

iii. $Profit\ P = 55\ x\ 400 - C_0 = 22000.80 = N1934.20\ per\ week$

Self-Assessment Exercises

1. What do you understand by the term inventory control?
2. Identify and discuss the different classifications of inventories.
3. Give six limitations of the EOQ model.
4. Outline the assumptions of the EOQ formula

1.6 Summary

It has been an interesting journey through the subject of inventory control systems. This unit has provided us with vital information about the inventory control model. An inventory control model has been defined an inventory as consisting of usable but idle resources such as men, machines, materials, or money. When the resources involved are material, the inventory is called stock. Though inventory of materials is an idle resource (since materials lie idle and are not to be used immediately), almost every organisation. It helps in the smooth and efficient of an enterprise. It helps in providing service to the customer at short notice. In the absence of inventory, the enterprise may have to pay high prices due to piecemeal purchasing. It reduces product cost since there is an added advantage of batching and long, uninterrupted production runs. It acts as a buffer stock when raw materials are received late and shop rejection is too many.

1.7 References/Further Readings/Web Resources

Eiselt, H.A. & Sandblom, C.L. (2012). Operations Research: A Model Based Approach, 2nd ed., New York: Springer Heidelberg

Gupta, P.K., & Hira, D.S. (2012). Operations Research, New – Delhi: S. Chand & Company.

1.8 Possible Answers to Self-Assessment Exercise(s)