COURSE GUIDE

ECO 256
MATHEMATICS FOR ECONOMISTS II

Course Team Wilson Friday Ohioze (Course Developer/Writer) - NOUN
Dr. Sunday F. Ayadi (Course Editor) - UNILAG

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
University Village
Plot 91, Cadastral Zone,
Nnamdi Azikwe Express way
Jabi-Abuja

Lagos Office
14/16 Ahmadu Bello Way
Victoria Island, Lagos

e-mail: centralinfo@nou.edu.ng
URL: www.nou.edu.ng

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INTRODUCTION

ECO 256: Mathematics for Economists II is a two-credit unit and one-semester undergraduate course for Economics students. The course is made up of nineteen units spread across fifteen lecture weeks. This Course Guide tells you how important research is to students of economics, and how statistical tools can be applied in solving some basic economic problems. It tells you about the course materials and how you can work your way through these materials. It suggests some general guidelines for the amount of time required of you on each unit in order to achieve the course aims and objectives successfully. It will also provide answers to your tutor-marked assignments (TMAs) to assist you work on the materials and examine yourself unaided.

COURSE CONTENTS

This course is a continuation of Mathematics for Economists I (ECO 255). The topics covered include derivatives, integration, economic applications of derivatives and integration, introduction to optimisation, functions of several variables, optimisation with constraints, differentials, matrix, matrix operations, matrix inversion and economic applications of matrix. As it is, the course will take you through derivatives to matrix and their applications.

COURSE AIM

The aim of this course is to give you in-depth understanding of mathematical applications in economics in the following areas:

- Basic concepts in mathematics for economists at year two undergraduate studies.
- To familiarise you with the basic topics in the course.
- To stimulate your understanding in the applications of mathematics in decision-making.
- To expose you to the task ahead in their study of higher level mathematics for economists as they progress in the course.

COURSE OBJECTIVES

To achieve the aims of this course, there are overall objectives which the course is out to achieve. In addition, there are set objectives for each unit. The unit objectives are included at the beginning of each unit. You should endeavour to read them before you start working through the unit. You may want to refer to them during your study of the units to check on your progress through the units. You should always look at each unit objectives after completing the unit to ascertain the level of achievements. This is to assist you in accomplishing the tasks involved in this course. In this way, you can be sure you have done what was required of you by the units. The objectives serve as study guide, such you would know if you are able to grab an understanding of each unit through the sets objectives. At the end of the course period, you are expected to be able to:

- Define and explain the concept of derivative.
- State the rules of differentiation
- Apply it in economics.
- Define and understand integration
- Differentiate between integration and differentiation.
- Learn the rules of integration, and apply it accordingly
- Clearly state the difference between function of single variable and multi-variables models.
- Define and explain linear/matrix algebra.
- State and apply the laws of matrix and, its operations.
- Apply matrix to solve complex systems of equations, and a lot more.

WORKING THROUGH THE COURSE

To successfully complete this course, you are required to read the study units, referenced books and other supplementary materials on the course. Each unit contains self-assessment exercises called Student Assessment Exercises (SAE). At some points in the course, you will be required to submit assignments for assessment purposes. At the end of the course there is a final examination. This course should take about 15weeks to complete and some components of the course are outlined under the course material subsection.

COURSE MATERIALS

The major components of the course, what you have to do and how you should allocate your time to each unit in order to complete the course successfully on time are listed below:

1. Course Guide
2. Study Unit
3. Textbook
4. Assignment File
5. Presentation Schedule

STUDY UNITS

There are 12 units in this course which should be studied carefully and thoroughly.

Module 1 Calculus

Unit 1 Derivatives I
Unit 2 Derivatives II
Unit 3 Integration
Unit 4 Economic Applications of Derivatives and Integration

Module 2 Optimisation

Unit 1 Introduction to Optimisation
Unit 2 Functions of Several Variables
Unit 3 Optimisation with Constraints
Unit 4 Differentials

Module 3 Linear Algebra

Unit 1 Matrix
Unit 2 Matrix Operations
Unit 3 Matrix Inversion
Unit 4 Economics Applications of Matrix

Each study unit will take at least two hours, and it includes the introduction, objectives, main content, self-assessment exercise, conclusion, summary and reference. Other areas bordered on the Tutor-Marked Assignment (TMA) questions. Some of the self-assessment exercise will necessitate discussion, brainstorming and argument with some of your colleagues. You are advised to do so in order to understand and get acquainted with historical economic events as well as notable periods.

There are also textbooks under the references and other (on-line and off-line) resources for further reading. They are meant to give you additional information if only you can lay your hands on any of them. You are required to study the materials; practice the self-assessment exercises and tutor-marked assignment (TMA) questions for greater and in-depth understanding of the course. By doing so, the stated learning objectives of the course would have been achieved.

TEXTBOOK AND REFERENCES

For further reading and more detailed information about the course, the following materials are recommended:

Adams, R.A. (2006). *Calculus: A Complete Course*. (6th ed.). Toronto, Ontario, Canada: Pearson Education Inc.

Chinang, A.C. & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics*. (4thed.). New York, USA: McGraw-Hill/Irwin.

Dowling, E. T. (2001). *Introduction to Mathematical Economics*. (3rd ed.). USA: McGraw-Hill: Schaum's Outline Series.

Ekanem, O.T. (2000). *Mathematics for Economics and Business*. (2nd ed.). Benin City: Uniben Press.

Lial, M.L., Greenwill, R.N, & Ritchey, N.P. (2005).*Calculus with Applications*. (8thed.): Boston, Massachusetts: Pearson Education, Inc., USA.

Sydsaeter, K., & Hammond, P. (2002). *Essential Mathematics for Economic Analysis.* Edinburgh Gate: Pearson Education Ltd, England.

ASSIGNMENT FILE

Assignment files and marking scheme will be made available to you. This file presents you with details of the work you must submit to your tutor for marking. The marks you obtain from these assignments shall form part of your final mark for this course. Additional information on assignments will be found in the assignment file and later in this Course Guide in the section on assessment.

There are three assignments in this course. The three assignments will cover:

Assignment 1 - All TMAs question in Units 1 – 4 (Modules 1)
Assignment 2 - All TMAs question in Units 5 – 8 (Module 2)
Assignment 3 - All TMAs question in Units 9 – 12 (Module 3)

PRESENTATION SCHEDULE

The presentation schedule included in your course materials gives you the important dates in this year for the completion of tutor-marking assignments and attending tutorials. Remember, you are required to submit all your assignments by the due dates. You are to guide against falling behind in your work.

ASSESSMENT

There are two types of assessment for the course. First are the tutor-marked assignments and second, is a written examination.

In attempting the assignments, you are expected to apply information, knowledge and techniques gathered during the course. The assignments must be submitted to your tutor for formal assessment in accordance with the deadlines stated in the Presentation Schedule and the Assignment File. The work you submit to your tutor for assessment will count as 30 % of your total course mark.

At the end of the course, you will need to sit for a final written examination of three hours' duration. This examination will also count for 70% of your total course mark.

TUTOR-MARKED ASSIGNMENTS (TMAs)

There are four tutor-marked assignments in this course. You will submit all the assignments. You are encouraged to work all the questions thoroughly. The TMAs constitute 30% of the total score.

Assignment questions for the units in this course are contained in the Assignment File. You will be able to complete your assignments from the information and materials contained in your set books, reading and study units. However, it is desirable that you demonstrate that you have read and researched more widely than the required minimum. You should use other references to have a broad viewpoint of the subject and also to give you a deeper understanding of the subject.

When you have completed each assignment, send it, together with a TMA form, to your tutor. Make sure that each assignment reaches your tutor on or before the deadline given in the Presentation File. If for any reason, you cannot complete your work on time, contact your tutor before the assignment is due to discuss the possibility of an extension. Extensions will not be granted after the due date unless there are exceptional circumstances.

FINAL EXAMINATION AND GRADING

The final examination will be of three hours' duration and have a score of 70% of the total course grade (100%). The examination will consist of questions which reflect the types of self-assessment practice exercises and tutor-marked problems you have previously encountered. All areas of the course are possible candidates for assessment and students are encouraged to read wide.

Revise the entire course material using the time between finishing the last unit in the module and that of sitting for the final examination to do a thorough revision of the course material. You might find it useful to review your self-assessment exercises, tutor-marked assignments and comments on them before the examination. The final examination covers information from all parts of the course.

COURSE MARKING SCHEME

The table below indicates the total marks (100%) allocated for the course.

Table 1: Course Marking Scheme

Assignment	**Marks**
Assignments (Best three assignments out of four that is marked)	30%
Final Examination	70%
Total	**100%**

COURSE OVERVIEW

The table presented below indicates the units, number of weeks and assignments to be taken by you to successfully complete the course - Mathematics for Economists II (ECO 256).

Table 2: Course Organiser

Units	**Title of Work**	**Week's Activities**	**Assessment (end of unit)**
	Course Guide		
Module 1 Calculus			
1	Derivative I	Week 1	Assignment 2

2	Derivative II	Week 1	Assignment 2
3	Integration	Week 2	Assignment 3
4	Differentiation and Integration in Economics	Week 2	Assignment 2
Module 2 Optimisation			
1	Introduction to Optimisation	Week 3	Assignment 2
2	Function of Several Variables	Week 4	Assignment 3
3	Optimisation with Constrains	Week 5	Assignment 3
4.	Differentials	Week 6	Assignment 3
Module 3 Linear Algebra			
1	Matrix	Week 7	Assignment 3
2	Matrix Operations	Week 8	Assignment 2
3	Matrix Inversion	Week 9	Assignment 3
4	Economic applications of matrix	Week 10	Assignment 2

HOW TO GET THE MOST FROM THIS COURSE

In distance learning, the study units replace the university lecturer. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace and at a time and place that suits you best. Think of it as reading the lecture instead of listening to a lecturer. In the same way that a lecturer might give you some reading to do, the study units tell you when to read your books or other material, and when to embark on discussion with your colleagues. Just as a lecturer might give you an in-class exercise, your study units provides exercises for you to do at appropriate points in time.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit.

You should use these objectives to guide your study. When you have finished the unit you must go back and check whether you have achieved the objectives. If you make a habit of doing this, you will significantly improve your chances of passing the course and getting the best grade.

The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a reading section. Some units require you to undertake practical overview of historical events. You will be directed when you need to embark on discussion and guided through the tasks you must do.

The purpose of the practical overview of some certain historical economic issues are in twofold. First, it will enhance your understanding of the material in the unit. Second, it will give you practical experience and skills to evaluate economic arguments, and understand the roles of history in guiding current economic policies and debates outside your studies. In any event, most of the critical thinking skills you will develop during studying are applicable in normal working practice, so it is important that you encounter them during your studies.

Self-assessments are interspersed throughout the units, and answers are given at the ends of the units. Working through these tests will help you to achieve the objectives of the unit and prepare you for the assignments and the examination. You should do each self-assessment exercises as you come to it in the study unit. Also, ensure to master some major historical dates and events during the course of studying the material.

The following is a practical strategy for working through the course. If you run into any trouble, consult your tutor. Remember that your tutor's job is to help you. When you need help, don't hesitate to call and ask your tutor to provide the assistance.

1. Read this Course Guide thoroughly.
2. Organise a study schedule. Refer to the `Course Overview' for more details. Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the semester is available from study centre. You need to gather together all these information in one place, such as your dairy or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates for working each unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the `Overview' at the beginning of each unit. You will also need both the study unit you are working on and one of your set books on your desk at the same time.
6. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit you will be instructed to read sections from your set books or other articles. Use the unit to guide your reading.

7. Up-to-date course information will be continuously delivered to you at the study centre.
8. Work before the relevant due date (about 4 weeks before due dates), get the Assignment File for the next required assignment. Keep in mind that you will learn a lot by doing the assignments carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the exam. Submit all assignments no later than the due date.
9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study material or consult your tutor.
10. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.
11. When you have submitted an assignment to your tutor for marking do not wait for its return `before starting on the next units. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and on the assignment. Consult your tutor as soon as possible if you have any questions or problems.
12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this Course Guide).

TUTORS AND TUTORIALS

There are some hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials. Together with the name and phone number of your tutor, as soon as you are allocated a tutorial group.

Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter, and provide assistance to you during the course. You must mail your tutor-marked assignments to your tutor well before the due date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail, or discussion board if you need help. The following might be circumstances in which you would find help necessary. Contact your tutor if.

- You do not understand any part of the study units or the assigned readings
- You have difficulty with the self-assessment exercises
- You have a question or problem with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should try your best to attend the tutorials. This is the only chance to have face to face contact with your tutor and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating in discussions actively.

SUMMARY

The course, Mathematics for Economists II (ECO 256), exposes you to the basic concept of derivative, integration, derivative and integration in economics, introduction to optimisation, functions of several variables, optimisation with constraints, differential, matrix, matrix operations, matrix inversion and matrix in economics. As it is, the course will take you through derivative to matrix in economics. Thereafter it shall enlighten you about decision making as regard using mathematics for economics.

On successful completion of the course, you would have developed critical analytical skills with the material necessary for efficient and effective discussion of basic areas in economics. However, to gain a lot from the course, please try to apply whatever you learn in the course to term papers writing in other aspect of economics courses. We wish you success with the course and hope that you will find it fascinating.

MAIN COURSE

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MODULE 1 CALCULUS

UNIT 1 DERIVATIVES I

CONTENTS

1.0 INTRODUCTION

Mathematic for Economists that is, ECO 255 is the foundation course upon which this course (Mathematics for Economists II) will be built. In fact, it is a prerequisite course for ECO 256. This means as a student of ECO 256 you are required to have registered for the ECO 255, read it, understand it and passed it before you are allowed to register for this advanced version, ECO 256. Mathematics for economists I would have introduced you to what you are expected to study in this advanced version at the rounding-off of the course (ECO 255). Meanwhile, in the prerequisite course, that is, Mathematics for Economists I, you were expected to learn number system, exponents and roots, equations, logarithms, and a lot more. In this aspect (Mathematics for Economists II), you will be studying mathematical economics at a level higher than what you have studied in the foundation class. In this advanced version, you will be discussing basically calculus and its periphery. We shall be treating topics like derivatives, integration, optimisation with constraint, functions of variables, linear algebra, and etcetera. In the very first unit of this module, we shall be looking at the concept of derivatives, notations of derivatives, differentiation rules, and many others and its application to economics. It is important to make it clear to students that, mathematical economics is not pure mathematics but an application of mathematical concepts in explaining economic issues. All these we shall be discussing in this version.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define and explain the concept of derivatives
- discuss the notations for derivatives
- state the rules of differentiation.

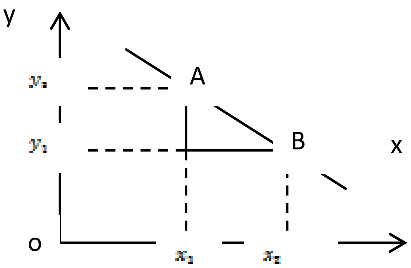
3.0 MAIN CONTENT

3.1 The Concept of Derivatives

The concept of derivative is a significant aspect of comparative statics, because it is the most basic part of mathematics known as differential calculus, which is concerned about the notion of rate of change. Comparative static has to do with the comparison of various states of equilibrium that are related to kind of sets of values of parameters and exogenous variables. For instance, in a of demand function model, such an initial equilibrium will be denoted by a determinate price P^* and quantity Q^*. If there disequilibrium occurs in the demand function or model in a way that there will be changes in the exogenous variables (they are variables in which the values are determined outside the model. In order words, are variables on the right-hand of an equation), there will definitely be an upset in the initial equilibrium which must lead to certain adjustments in the endogenous variable (variable in which the value is determined within the model. In fact, it is always on the left-hand of an equation). In the scenario just given, the issue is about the response of the endogenous variables to the changes that occur in the exogenous variables, which is the 'rate of change.' The rate of change of the endogenous variables with respect to the change in a certain exogenous variable is what the concept of derivatives deals with in mathematics.

Slope of a line

To fully understand the concept of derivatives using a curve, it is imperative for the leaner to first understand the concept using a linear graph (straight line). One important feature of a linear graph is its slope, a number which signifies the steepness of the line. See diagram below. The slope of a linear graph is often referred to as "slope of a line." The slope of linear graph is the vertical change over the horizontal change as a body moves along the linear path.

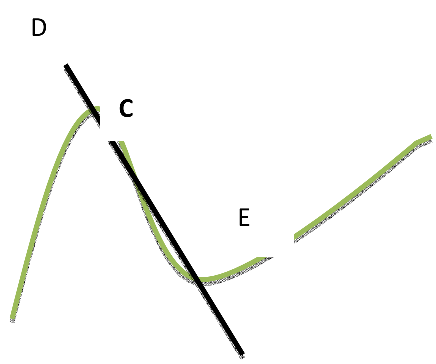


From the figure above, the slope of the linear graph $\overline{AB}$ can be interpreted to mean the average amount of change in the vertical axis represented by 'y', per unit change in the horizontal axis represented by 'x', fondly written as $\Delta y/\Delta x$.

Assuming we have a function $y = f(x) = a + bx$, the slope of this function is *(b)*, the coefficient of *x*. If we have (x_1, y_1) and (x_2, y_2) as coordinate points on a linear graph, the slope which is as *b*, is defined thus:

$$b = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Derivative is all about slope. In fact, the derivate at a point of a function of a single variable is the slope of the tangent line to the graph of the function at that very point. For example, assuming a curve is drawn which is either maximum or minimum in shape, to find the gradient or slope of the curve, a straight line will be drawn that will be tangential or meet the curve at any point. At this point (see point *c* in the figure below), the slope of the curve can be determined.



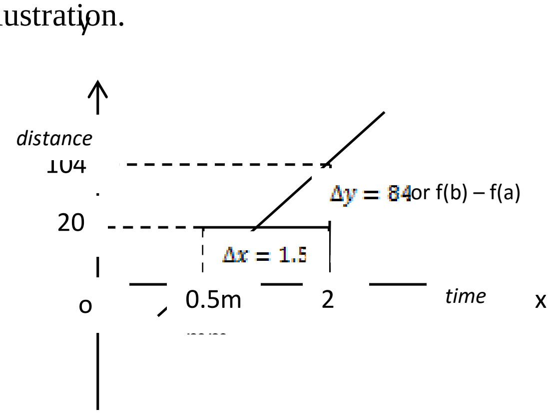
The graph of a function drawn in green colour above has a straight line *DE* in black colour drawn to be tangential to the function. At that point, as seen above, the slope of the tangential line is equal to the derivative of the function at the point where the graph and the line coincide. The slope of the tangential line is equal to the derivative of the function at the point where the curve intersects with the line, and the derivative measures the change in one variable as result of the changes in some other variables. In order words, the slope of a curve at a point on the curve is the same as the average rate of change. By definition, the rate of change determines how one parameter changes in relation to the other. From this definition, we can infer one fundamental issue, that is, two points are involved here (points A and B), see figure above. For instance, a car travelling from Lagos to Benin, started-off at a point 'A' in Lagos, and stopped at a final point 'B' in Benin with the hours or minutes traveled recorded against the distance made. In this case, we can find the average rate of change between the two points.

The average rate of change of function of *xf(x)* with respect to *x* for a function *f* as *x* changes from point A to point B is

$$\frac{f(b)-f(a)}{b-a}$$

Note, the formula as stated is the same as the formula for the slope of a line through (*a*, *f(a)*) and (*b*, *f(b)*). Where *f(a)* and *f(b)* are functional values of the variables (*a* and *b*).

Going back to our earlier example on a car travelling from Lagos to Benin, assuming we have the following points where (x, y) =*a* = (.5, 20),*b* =(1, 48),*c* = (1.5, 80) and *d*= (2, 104) each representing time taken in hour and distance in kilometer made. To determine the average speed or average rate of change, we will apply the formula above. See diagram below for further illustration.



Meanwhile, recall that

Average speed = $\frac{Distance\ traveled}{Time\ taken}$

Therefore, to calculate the rate of change or speed over the time interval from *t* = 0.5 to *t* = 2.

To be able to calculate the rate of change or average speed over time, an understanding of the concept of delta is very important. Delta is denoted thus (Δ); it is synonymous to the concept of limit. Delta is the rate of change in one variable to the change in another variable. For instance if we have $\Delta y/\Delta x$, it means that as *y* moves from one stationary position to another stationary point, *x* changes too. To determine the extent of the change called rate of change (slope of the function), we apply the delta notation as given above. To calculate the rate of change using points *a* and *f*[(.5, 20) and (2, 104)]as indicated above,

Rate of change = $\frac{change\ in\ y}{Change\ in\ x}$ = $\frac{\Delta y}{\Delta x}$ =$\frac{y_2 - y_1}{x_2 - x_1}$, note, let *b* represent rate of change.

$$b = \frac{f(2) - f(0.5)}{2 - 0.5} = \frac{\Delta y}{\Delta x} = \frac{104 - 20}{2 - 0.5} = \frac{84}{1.5}.$$

The slope of the linear graph indicates how fast *y* changes for each unit of change in *x.*

Let us consider a scenario where the formula can be applicable to resolving practical issue(s). Assuming an organisation determines to ascertain the cost in naira [C(x)] of producing *x* numbers of school bags using the mathematical equation $C(x) = 150 + 10x - x^2$ where, *x* is between is greater than zero, but less than eight ($0 < x < 8$). Find the average rate of change or slope of cost per bag for producing between 1 and 6 bags.

Solution

Apply the formula for average rate of change. The cost of producing 1 bag is $C(1) = 150 + 10(1) - 1^2 = 159$ or ₦159. Also, the cost of producing 6 bags is $C(6) = 150 + 10(6) - 6^2 = 174$ or ₦174.

Therefore, the average rate of change in the cost of production of the bags is

$$\frac{C(6) - C(1)}{6 - 1} = \frac{174 - 159}{5} = 15.$$

On the average, the cost of production increases at the rate of ₦15 per bag as production increases from 1 to 6 bags.

Given a function y = *f(x)*, the derivative of the function *f* at *x*, written $f'(x)$ or d_y/d_x, is define as

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ if the limit exists, or

$$f'(x) = \lim_{h\to 0} \frac{f(x_1 + h) - f(x)}{h}$$

Where *f'(x)* is read "the derivative of *f* with respect to *x* or *f* prime of *x*". The derivative of a function *f'(x)*, or simply *f'*, is itself a function which that measures both the slope and the instantaneous rate of change of the initial function *f(x)* at a point. Meanwhile, a function is differentiable at a point if the derivative exists at a given point. Two conditions must be met for a function to be differentiable: a) the function must be continuous, and b) has a tangent at a given point.

For simplicity, a function is continuous when a curve is drawn without any break. That is, a situation whereby a pen or pencil is used to draw a curve on a paper without lifting the pen. A function *f* is continuous at a point *x = a* only if: i) *f(x)* is defined, that is *x = a*ii) limit of *f(x)* exists and iii) the limit of *f(x)* = the limit of *f(a).*All polynomial and rational functions are good examples of a continuous function. However, where a function is undefined, that is, the denominator is equal to zero.

3.2 Derivative Notations

Notations are generally used in mathematics and mathematical subjects alike. It is define as a system of symbolic representations of expressions and ideas in mathematics. Notations are used in mathematics, physical sciences, engineering, and economics. They are simple symbolic representations, such as the numbers 0, 1 and 2, function symbols, signs, conceptual symbols, such as lim, dy/dx, equations and variables.
In calculus, the notations of derivative of a function can be represented in many forms. For instance, if *y = f(x)*, the notations of the derivative of this function can be written as follows

$$f'(x) \quad y' \quad \frac{dy}{dx} \; \frac{df}{dx} \; \frac{d}{dx}[f(x)] \quad \text{or} \quad D_x[f(x)]$$

If we equate *f* to ϕ and we rewrite the function, you will have y = ϕ *(j)*, the derivative notations can be represented thus:

$$\phi'(j) \quad y' \quad \frac{dy}{dj} \; \frac{d\phi}{dj} \; \frac{d}{dj}[\phi(j)] \quad \text{or} \quad D_j[\phi(j)]$$

If *y = f(x)* and is estimated at *x = z*, the right derivative notation will be $\frac{dy}{dx}|z$ *and f' (z).*
Let us use one or two examples worked based on all we have discussed concerning derivative.

Example 1. If $y = 2x^2 + 7x - 18$, write the derivative of the equation

$$y' \quad \frac{dy}{dx} \quad \frac{d}{dx}\left(2x^2 + 7x - 18\right) \quad \text{or} \quad D_x\ (2x^2 + 7x - 18)$$

Example 2. If $p = \sqrt{6z - 12}$, the derivative of the equation will be written thus

$$p' \quad \frac{dp}{dz} \quad \frac{d}{dp}\left(\sqrt{6z-12}\right) \quad \text{or} \quad D_p\left(\sqrt{6z-12}\right)$$

SELF-ASSESSMENT EXERCISE

Derivative is mainly concerned about the rate of change, discuss.

3.3 Rules of Differentiation

Recall that we have discussed two conditions that must be met before a function is differentiable. When a function or a mathematical model is differentiable, we can observe from the model or function how the endogenous variables change with respect to change in the exogenous variables. All these will be discussed in this section. Differentiation is the process of determining the derivative of a function or a mathematical model. It is about applying some basic rules to a given function or mathematical model. In discussing the rules of differentiation, authors or writers also use functions such as *g(x)* and *h(x)*, where *g* and *h* are both unspecified functions of *x*.

The variables *f, g* and *h* are used in any mathematical model to indicate the functional relationship between left hand variable(s) and right hand variable(s). That is, any change that is observed in the left hand variable(s) is dependent on the behaviours of the variable(s) on the right hand of the model. For instance in the function $y = f(x)$, the variable *y* is dependent on the variable *x* which is on the right hand side of the model or function. The dependability of the *y* on *x* can be determined using differential calculus called derivative as already explained. To do this effectively, you differentiate the function base on certain guarding rules. These rules are regard as rules for a function of one variable.

Rule 1: The derivative of a constant

The derivative of a constant function like $p = k$, *or* $f(t) = k$ is zero. Note that *k* is constant because it is a numerate standing alone without any variable attached to it, or multiplying it.

If $p = f(t) = k$, and k is a constant, the derivative is

$$\frac{dp}{dt} = \frac{dk}{dt} = 0 \quad \text{or} \qquad f'(t) = 0$$

Example 3 : Find the derivative of the followings if:

i) $p = f(t) = 12$
ii) $q = 4^2$
iii) $f(x) = \pi$

Solution

i) $\frac{dp}{dt} = 0$ or $f'(t) = 0$
ii) $\frac{dq}{dk} = 0$
iii) $D_x [f(x)] = 0$ or $f'(x) = 0$

Rule 2: The linear function rule

A linear function is a function that is to the power of one. If we have a model or a function $g(x) = mx + d$ which is a linear function, the derivative will be equal to m (the coefficient of x). Note, the derivative of any variable to the power of one is all the time equal to the coefficient of that variable, knowing full well that the derivative of a constant is always zero.

If $g = mx + d$, the derivative will be dg/dx or $g'(x)$ equal to m.

Example 4 Find the derivative of the following functions

i) $f(x) = 5x - 3$
ii) $f(x) = 10 + \frac{1}{4}x$
iii) $g(t)= 13t$

Solution

i) $f'(x) = 5$
ii) $f'(x) = \frac{1}{4}$
iii) $g'(t) = 12$

Rule 3: The power function rule

The derivative of any power function is determined by multiplying the coefficient of the function by the exponential and the alphabet variable to the power of the exponential minus one. Let assume we have $y = Mx^n$, where M is a constant, x an alphabetical variable, and n is any real number which is in power. Now given that $y = Mx^n$, to differentiate this function (that is, the derivative) $\frac{dy}{dx} = m*n*x^{n-1}$ or $f'(x) = mnx^{n-1}$.

Example 5 Differentiate the followings using power function rule.

i) $y = x^4$
ii) $y = \frac{1}{x^4}$
iii) $D_x (x^{2/4})$

Solution

i) $\frac{dy}{dx}$ or $D_x y = 4x^{4-1} = 4x^3$

ii) In this case, you apply one of the laws of indices to get a negative exponent, and rewrite the function thus:

$y = x^{-4}$; then $f'(x) = -4x^{-4-1} = -4x^{-5}$ or $\frac{-4}{x^5}$.

iii) $D_x(x^{2/4}) = \frac{2}{4}x^{2/4-1} = \frac{2}{4}x^{-2/4} = \frac{1}{2}x^{-1/2}$

Rule 4: The product of a constant and a function

Considering this rule, to differentiate any function or any economic model that has the combination of a constant term and a function, is the product of the constant term and the derivate of the function.

Given that k is a real number. If $y = h(x)$ exist, then

$$\frac{dy}{dx} = kh'(x) \quad 0r D_x[kh(x)] = kh'(x).$$

Example 6 Given that y is equal to the following functions below, find their respective derivatives.

i) $7x^3$
ii) $\frac{8}{x}$
iii) $(-6x)$

Solution

i) $\frac{dy}{dx} = 7(3x^{3-1}) = 7(3x^2) = 21x^2$

ii) Applying one of the laws of indices, $\frac{8}{x}$ will now be $8x^{-1}$ then

$f'(x) = 8(-1x^{-1-1}) = 8(-1x^{-2}) = -8x^{-2}$ or $\frac{-8}{x^2}$

iii) Open the bracket, and you will have -6x, then
$f'(x) = -6(1x^{1-1}) = -6x^0 = -6(1) = -6.$

Recall that any number other than Zero raised to the power of Zero is equal to one.

Example 7: if z is a function of the function given below, determine the derivative.

$(2k^4 + 5)(3k^5 - 8).$

Solution

$\frac{dz}{dk} = (2k^4 + 5)(15k^4) + (3k^5 - 8)(8k^3)$ – product rule (see unit 2, rule 2)

$= 30k^8 + 75k^4 + 24k^8 - 64k^3$
$= 54k^8 + 75k^4 - 64k^3.$

SELF-ASSESSMENT EXERCISE

i. Define and explain differentiation the way you will understand it.
ii. Write and explain in your own way, the rules of differentiation you have studied thus far.

4.0 CONCLUSION

In this unit, you have learnt about derivative and some rules of differentiation. Differentiation is vital instance in the discussion of derivative. In fact, we have identified that differentiation is the process of determining the derivative of a function or a mathematical model. One basic use of differentiation in the determination of derivative is that, it helps to smooth a function (curve) as well as making it to continue. Most functions used in economics are specific, and have the property that they can be differentiated at every point on the curve. Therefore, we can conclude that is vital to the understanding of economics.

5.0 SUMMARY

Thus far, you have learnt about the concept of derivative as basic part of comparative - static analysis. Recall that in the begin, we have said that the concept of derivative is a significant aspect of comparative statics because it is the most basic part of mathematics known as differential calculus, which is concerned about the rate of change. Concerning the notion of rate of change, you have learnt that a moving object has both starting and ending points. In-between these two points, there is either increase or decrease in speed as time changes. This led us to comparison of various equilibrium states that are related with sets of variables. For instance, distance and time are good examples of the said sets of variable. In economics, the understanding of this concept will quicken the understanding of certain aspect. For example, the study of an isolated market model in economics will be made simpler with a good understanding of the concept of derivative.

6.0 TUTOR - MARKED ASSIGNMENT

Determine the derivatives of these functions using any of the rules already studied.

1. $y = \frac{9}{z}$, $f'(z) = 12\sqrt{z}$ and $q = 8p^3 + 11p^{-5}$.
2. $p = 7q^3(5q - 150)$, $R = (t^4 + 12)(t^6 + 8)$.
3. $f(j) = 20j - 10$, $f(q) = -9q^{-6}$ and $y = 150x$.

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UNIT 2 DERIVATIVES II

CONTENTS

1.0 INTRODUCTION

In the last unit, you have seen the usefulness of the concept of derivative in comparative static analysis. The concept of derivative is directly concerned with the notion of rate of change in mathematical sciences which is applicable to other fields of study such as Economics. Also, we have explained that, in the determination of derivative, differentiation is used. In applying this, some fundamental guidelines were given. These guidelines are referred to as “Rules of Differentiation.”

In this unit, we shall be starting off by continuing our discussion on the rules of differentiation. Thereafter, we shall go further in our study by looking at higher order derivatives.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- discuss other rules of differentiation
- apply these rules to solve problems
- explain higher order derivatives.

3.0 MAIN CONTENT

3.1 Rules of Differentiation (Cont.)

Recall that we have stated that the rules of differentiation are like guiding principles which help in the determination of derivatives of certain functions. You have learnt some earlier in the preceding unit. The ones you are to study now are considered higher rules.

Rule 1: The sums or differences rules

Now we want to consider a situation where two or more components of a function are differentiated separately, and their products either added or subtracted. This is a case of sums and differences rules. The derivative of functions in this category is equal to the sum or difference of the individual components.

Suppose we have a situation of $f(x) = u(x) \pm g(x)$, and if the derivatives of the components are $u'(x)$ and $g'(x)$, then $f'(x) = u'(x) \pm g'(x)$.

Example 1: solve the following using the rules of sums and differences in derivatives.

i) $y = 4x^2 + 6x^3$
ii) $p = 15q^3 - 3q^4$
iii) $R = 10t^3 - 3t^2 + 8$

Solution

i) In this problem, we have two components of a function these are $4x^2$ and $6x^3$, and it is case of summation. So, let $u(x) = 4x^2$ and $g(x) = 6x^3$; then $y = u(x) + g(x)$. Now, $u'(x) = 8x$ and $g'(x) = 18x^2$,

$$\frac{dy}{dx} = 8x + 18x^2 \text{ or } 18x^2 + 8x.$$

ii) This is a case of subtraction. So if $p = u(q) - g(q)$, we have $p' = u'(q) - g'(q)$ as the individual differentiable value in difference form. It therefore follows that,

$$\frac{dp}{dq} = 45q^2 - 12q^3.$$

iii) In (iii), it is a situation of more than two functions, and both addition and subtraction signs are observed. So, $R' = u'(t) - g'(t) + h'(t)$ which is the difference and sum of the individual differentiated functions. Appling the rule, we will have

$$\frac{dR}{dt} = 30t^2 - 6t.\text{(Since } \frac{dR}{dt} \text{ of a constant (8) = 0)}$$

Rule 2: The product rule

This rule is about two or more components of a function that are being considered wholly, neither added nor subtracted, but multiplied. Note that, whatever is product in mathematics is considered multiplication. Assuming we have two components of a function, the derivative will be

the first component times the derivative of the second component plus the second component times the derivative of the first component.
If *f(x) = g(x),.h(x),*where *g(x)* and *h(x)*are functions that can be differentiated. The derivative will be:

$$\frac{dy}{dx} = g(x)\,.\,h'(x) + h(x)\,.\,g'(x)$$

Example 2: if given the function $y = (3x + 2)(4x^3)$, determine the derivative using product rule.

Solution

let $3x + 2 = g(x)$ and $4x^3 = h(x)$ so, $g'(x) = 3$ and $h'(x) = 12x^2$.

$$\therefore \frac{d}{dx}[(3x + 2)(4x^3)] = (3x + 2)(12x^2) + (4x^3)(3) = 48x^3 + 24x^2.$$

Note: In performing mathematical operations as regards the use of the product rule, students should remember that the derivative of a product of two functions is not mere multiplication of two individual derivatives. Rather, it is a weighted sum of the functions as demonstrated above.

Rule 3: The quotient rule

Merely looking at the quotient rule, we can observe that this rule is the direct opposite of the product rule just discussed. In this case two functions are involved; one is a numerator and the other a denominator. The derivative in this situation is the product of the denominator and the differentiated numerator, minus the product of the numerator and the differentiated denominator, all over the square of the denominator.

Supposing we have $y = g(x)/h(x)$ where $g(x)$ is the numerator, and $h(x)$ is the denominator, then

$$\frac{dy}{dx} = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}$$

Example 3: let $y = \frac{4x-2}{x+2}$. Find the derivative y'

Solution

let $4x\text{-}2 = u$ and $x+2 = v$, therefore, $u' = 4$ and $v' = 1$. Appling the rule,

$$f'(x) = \frac{v.u' - u.v'}{v^2}$$

$$= \frac{(x+2)4-(4x+2)1}{(x-2)^2}$$

$$= \frac{4x+8-4x-2}{(x-2)^2}$$

$$= \frac{6}{(x-2)^2}.$$

Rule 4: An inverse function rule

Before now, we have been dealing with functions in which one variable is directly dependent on the other. In a situation where we have an inverse of a function, it indicates the direct oppose of the formal.
Recall that, $y = f(x)$ is a function which represents one on one mapping. That is, any change in variable $y(\Delta y)$ is a result of change in x (Δx). In this instance, the relationship shows in this function is a direct one. However, where $x = f(y)$ or $y = f^{-1}(x)$, this is an inverse function of the just considered function. The derivative of an inverse function as already given is the reciprocal of the derivative of the direct function given.

That is, $\frac{dy}{dx} = \frac{1}{dy/dx}$.

Example 4: $q = 5p + 45$, find the derivative of q^{-1}

Solution

$f'(p) = 5$. But $q^{-1} = \frac{1}{f'(p)}$

$\therefore \quad q\ \text{-}1 = \frac{1}{5}$.

Also, find the derivative of the inverse function of q, if $q = p^5 + p$.
Therefore, $\frac{dq}{dp}$ = $5p^4 + 1 = \frac{1}{5p^4 - 1}$.

Rule 4: The chain rule

This rule comes up under the discussion of the derivative of a function of a function. It is also known as the composite rule. In this, there is more than one function, and the functions are dependent on one another. This is why it is referred to as a function of a function. A good instance is where y is function of u and u in turn is a function of z so,

$y = f(u)$ and $u = h(z)$, then
$y = f[h(z)]$.

The derivate of *y* with respect to *z* is equal to the differentiation of the 1st function with respect to *u* multiply by the differentiation of the 2nd with respect to *z*.

$$\therefore \qquad \frac{dy}{dz} = \frac{dy}{du}.\frac{du}{dz} = f'(y)h'(z)$$

Example 5: If $y = (x^2+5x)^2$ where $y = u^2$ and $u = x^2+5x$. Differentiate the function using chain rule.

Solution

From the problem given, we can see that *y* is a function of *x*, and *x* is in turn a function of *u*. using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{du} = 2u = 2(x^2+5x) = (2x^2+10x)$$

$$\frac{du}{dx} = (2x+5)$$

$$\therefore \frac{dy}{dx} = (2x^2+10x)(2x+5)$$

Also, if $y = (x^3+4x-6)^{19}$, determine the dy/dx using chain's rule. Note, without the use of chain's rule, solving the problem will be very cumbersome. With chain's rule, solving the problem becomes easier. To solve this, let the variables in bracket equal *z* to create two components' effects in a single function.

$$z = x^3+4x-6$$

Therefore, $y = z^{19}$. It then follows that $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$

$$= 19z^{18}(3x^2+4) = 19(x^3+4x-6)^{18}(3x^2+4).$$

Rule 5: The implicit functions rule

This form of differentiation is somehow cumbersome. But if given the required attention, problems that arise from this form of differentials calculus are always easy to solve. Meanwhile, implicit functions are the direct opposite of explicit functions. Thus far, we have been looking at explicit functions where *y* is expressed as a function of *x*. There are times when *y* cannot be solved explicitly, so, we use the rule of implicit differentiation to solve the problem. For instance $y = -2x + 4$ is an explicit equation. Here, *y* is seen explicitly in the function, say *f* of *x*. Bring in the principle of Economics to further explain the function, *y* can be seen as an output that is determined by an input *x*. on the other hand, the function could take the form $4x + 2y = 8$. In this equation, the variable *y* is not being seen as an explicit function of *x* rather, it is being displayed as an implicit function of *x*. so, solving for *y* in the equation will make it an explicit of *x*.

$$2y = -4x + 8 \rightarrow y = -2x + 4.$$

However, there are times some functions become complicated to determine the explicit function. In spite of the difficulties involved in solving it, it is surprisingly easy to find the derivative of the equation implicitly without needing to solve the function itself.

Three basic steps are considered in determining the derivatives of an implicit function of *x*. these steps are:

i) Consider *y* as an unknown function of *x*
ii) Determine the d*y*/d*x* in the function by differentiating the implicit function term by term.
iii) Solve the equation that resulted from the derivative.

Example 6: assuming we have the function $y^3-2x^2y^2+x^4 = 0$. Find *dy/dx*

$y^3-2x^2y^2+x^4 = 0$

Solution

$\frac{d(y^3)}{dx} - \frac{d(2x^2y^2)}{dx} + \frac{d(x^4)}{dx} = 0$(Note: $\frac{d(2x^2y^2)}{dx}$ was solved using the product rule)

$3y^2 d/dx - 4xy^2 - 4yx^2 d/dx + 4x^3 = 0$
$3y^2 d/dx - 4x^2 y d/dx = 4xy^2 - 4x^3$

Factorize both sides of the equation, and we have

$\frac{d}{dx}(y)(3y-4x^2) = 4x(y^2-x^2)$
$= \frac{dy}{dx}(3y-4x^2) = 4x(y^2-x^2)$

Make *dy/dx* the subject of the equation, we will have

$$dy/dx = \frac{4x(y^2 - x^2)}{y(3y - 4x^2)}.$$

Also, consider this example: $4x^3 - 5y^3 - 20 = 0$

Solution

$$\frac{d}{dx}(4x^3 - 5y^3 - 20) = \frac{d}{dx}(0)$$

$$= \frac{d}{dx}(4x^3) - \frac{d}{dx}(5y^3) - \frac{d}{dx}(20) = \frac{d}{dx}(0)$$

$$= 12x^2 - \frac{d}{dx}(5y^3) - 0 = 0$$

But, note that $\frac{d}{dx}(y) = \frac{dy}{dx} \rightarrow \frac{d}{dx}(5y^3) = 12x^2$

$$= 5y^2 \frac{dy}{dx} = 12x^2$$

Make *dy/dx* the subject of the equation, we will have

$$\frac{dy}{dx} = \frac{12x^2}{15y^2}.$$

SELF-ASSESSMENT EXERCISE

i. How can you differentiate a product rule from a quotient rule?

ii. Differentiate the following functions: i) $y = \frac{3z^2}{z^3}$, ii) $y = (1/5x^2 + 5)4x^3$.

3.2 Higher-Order Derivatives

Thus far all we have discussed concerning derivatives of functions can be called ‘first derivatives.’ In this part, we shall be considering a higher version of derivatives called ‘second derivatives.’ It is circumstance where the first derivative of the original function is itself a differentiable function. We can then take the derivative of the first derivative as we did to the initial function. If the function is further differentiable, a third derivative exists and on like that. All these are referred to as higher-order derivatives. Higher-order derivatives are estimated by the successive application of the guiding rules of differentiation to the derivative of the preceding order.

As we have discussed in unit one concerning derivative notations, the same applies to higher-order derivatives but with little difference. Suppose we have an equation of the form

$q = 4p^3 + p^2 + 20,$

$\frac{dq}{dp} = q' = 8p^2 + 2p + 0$ → 1st differentiation (1st derivative)

$\frac{d^2q}{dp^2} = q'' = 16p + 2$ → 2nd differentiation (2nd derivative)

$\frac{d^3q}{dp^3} = q''' = 16$ → 3rd differentiation (3rd derivative)

$\frac{d^4q}{dp^4} = q'''' = 0$ → 4th differentiation (4th derivative).

Peradventure, we have a situation where you are required to find the nth derivative of a function, we have

$$\frac{d^nq}{dp^n} = q^{(n)} \text{or } y^{(n)} \text{or } f^{(n)}(x).$$

Example 7: find the 5th derivative of the function y = $4x^4 + 7x^3 + 2x^2$.

Solution

This is a case of higher-order derivatives. The way to go in this case is to start the differentiation with the primitive or initial equation already given. Thus we have the followings,

$D_x[f(x)] = 16x^3 + 21x^2 + 4x$
$D^2y = 48x^2 + 42x + 4$
$D^3y = 96x + 42$
$D^4y = 96$
$D^5y = 0.$

Note that the notation used in this solution is not amongst the ones given above. Meanwhile, it is one of the derivative notations used in calculus (see unit one).

Also, within the purview of higher-order derivatives, there is what is commonly referred to as 'higher-order partial derivatives.' It is higher-order derivatives too, but with slight differences. In calculus, the second derivative of any function would bring to fur the relative extrema (maxima and minima) of that function. In this instance, the second higher-order partial derivatives, that is, the partial differentiation of the initially partial differentiated function, are used in a similar way for functions of two or more variables. See unit two of module two for detailed account on partial differentiation.

SELF-ASSESSMENT EXERCISE

i. Briefly explain what you know by the term primitive function in higher-order derivative.
ii. Which order of derivative is '*f*' in differential calculus?

4.0 CONCLUSION

In this part, we have discussed other rules of differentiation and their applicability to solving mathematical problems. You have equally learnt that there are higher version of derivatives refer to as "higher-order derivatives." It allows for the differentiation of a particular function for as many times as possible, if the need arises. Higher-order derivative is vital in the discussion optimisation in calculus. See discussion on optimisation in module two.

5.0 SUMMARY

In this unit, you have leant about the following:

- Other rules of differentiations such as product rule, quotient rule, a function of a function rule, and many more that were perceived more complicated than the ones discussed in unit one.
- Higher-order derivatives. In this topic, you were exposed to the fact that an equation can be differentiated more than one if circumstance surrounding it demands it.

6.0 TUTOR-MARKED ASSIGNMENT

1. Compute $\frac{d}{dx}(3x^5 + \frac{x^{60}}{60})$
2. Compute $\frac{d}{dx}(\frac{C(x)}{x})$ using quotient rule.
3. Find the derivative of the function $q = \sqrt{p^3 + 2}$ using the chain rule.
4. If $y = Ak^a$, compute $\frac{d^3y}{dk^3}$.

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UNIT 3 INTEGRATION

CONTENTS

1.0 INTRODUCTION

We have seen in units one and two what differential calculus is all about. Recall, we have mentioned that, it (differential calculus) measures the rate of change of a function or an equation. This led us to the discussion on derivative of a function, which is an integral part of differential calculus. To determine the derivative of a particular function, the function (i.e. the original) would have to be differentiated. Often in economics, the rate of change of a set of functions is given, and the onus is always on economists to determine the initial function. Working from the rate of change to determine the original equation amounts to revising the process of differentiation, this mathematical method is called integration, antiderivative or antidifferentiation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define and explain integration
- explain the rules, and apply them accordingly
- explain the difference between integration and differentiation.

3.0 MAIN CONTENT

3.1 Indefinite Integrals

In a simple understanding, integration is the direct opposite of differentiation. Here, *F(x)* is the primitive function, and it is termed integral of *F'(x)*. That is, *F'(x)* is a function that has been differentiated, and *F(x)* is the integral of the differentiated form which serves as the answer in this circumstance. To further understand the difference between integration and differentiation, we consider this simple analogy. When descendants or group of people find their ways back to their origin, what these people have done has can be likened to *integration* and *differentiation*. It is integration when the descendants were able to locate their origin (which is the primitive function). However, it is differentiation when a lot of descendants came out from a single parent/origin. In a clearer form, integration is *movement backward* while, the *forward movement* is differentiation.

Basically, the concept of integration is about the study of an area under a curve. When a function is specified say, $y = f(x) = x^4$ this represents an area under a curve and is between two distinct points *b* and *a*; which can be determined by integrating the function. Let us assume we do not know the primitive function whose derivative *f'(x)* is given above, and we want to compute the primitive function, we can do this through the process of integration. For instance, ifx^6 is differentiated, and it resulted to $6x^5$, it then means that the derivative of$\frac{1}{5}x^5$will bex^4.$\frac{1}{5}x^5$in this scenario, is the primitive function of the derivative function x^4. So, if we have the primitive function added to a constant i.e.$\frac{1}{5}x^5 + c$ in this case, the *c* is a constant that will be equal to zero when differentiated. It therefore means that,

$f'(x) = x^4$(derivative function) $\Rightarrow$ $f(x) = \frac{1}{5}x^5 + c$.

3.2 Notation of Integration

As we have seen in unit one, we stated some vital notations to represent differential calculus such as *f(x), dy/dx*, and a lot more. Also in integral calculus, there is a notation that symbolizes integration. That is, if you see and able recognise the sign or notation, it then means that the problem or mathematical issue is about integration. For indefinite integral, we use $\int$. The symbol as stated means integral notation, and it is an elongated *S* which as some elements of summation in it. We have said it is an 'integral sign', whereas *f(x)* which is always part of it is termed 'integrand', and the *dx* aspect is the differentiation operator. Let see a typical integral sign based on the explanation done so far.

$$\int f(x)dx$$

From the notation given above, three parts can be explained here. i) the elongated *S*, ii) the *f(x)* and iii) the *dx*. If we decide to make two parts out of the three to be one (i.e. *f(x)dx*), you will have an entity called differential of original function *F(x)* which mean $d\,F(x) = f(x)\,dx$. So if we have

$$\int f(x)dx = F(x) + c$$

The variable *c* is an arbitrary constant integration that vanishes with differentiation. The notation can be interpreted thus, 'the indefinite integral of *f(x)* with respect to *x* is *F(x)* added to a constant. Also, the integral symbol shows an instruction of what to do. That is, to reverse the own process of differentiation.

SELF-ASSESSMENT EXERCISE

i. In your own understanding, distinguished between integration and differentiation.
ii. Explain the inclusion of an arbitrary constant in integration notation.

3.3 Rules of Integration

As we have rules in differential calculus that guide its operations, so it is integral calculus. Recall we said in simple term that integration is the opposite of differentiation. That is, if we work from the primitive function to get the derivative in the case of differential calculus, for integration, we work from the derivatives to arrive at the initial function (the primitive one). In the same vein, the rules guiding the operations of integration are in away a reversal of the basic rules of differentiation. Note that, the rules of integration are dependent on the rules of derivatives. Let us consider again the power function rule in derivative, we have,

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n (n \neq -1)$$

Primitive function derivative function

We can see from the expression that $x^{n+1}/(n + 1)$ is the initial or primitive function for the x^n which is the derivative function. If we then substitute these in expression we had earlier above, what we will have is integration rule.

Rule 1: The power rule

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \; (n \neq -1)$$

The expression as stated above is the power rule of integration. Let apply this to solve some mathematical problems.

Example 1: use the power rule of integration to solve the following problems.

i) $\int x^6 \, dx$.

ii) $\int \sqrt{x^3} \, dx$.

iii) $\int x dx$.

Solutions

i) x^6 is a derivative function, and to determine the primitive function which is the integration itself, we shall apply the power rule already stated. Remember that our $n = 6$, therefore

$$\int x^6 \, dx = \frac{1}{7} x^7 + C$$

ii) Note that $\sqrt{x^3}$ is same as $x^{3/2}$, where n in this case is $3/2$ and n+1 = 5/2, therefore

$$\int \sqrt{x^3} \, dx = \frac{x^{5/2}}{\frac{5}{2}} + C = \frac{2}{5}\sqrt{x^5} + C$$ (Note that 1/(5/2) = 2/5)

iii) The derivative function in this instance is x, our n is equal to 1. If we apply the power rule (see rule 3 under rules of differentiation), we have:

$$\int x dx = \frac{1}{2} x^2 + C$$

Note that the derivative of the integral (primitive function) must be equal to the integrand also known as the derivative function. This is a way of checking results in integration.

Rule 2: The constant rule

The integral of a constant say k is

$$\int k\,dx = kx + C$$

Example 2: compute the integral of $\int 10\,dx$ and $\int -3\,dx$ using the constant rule.

Solution

10 and -3 are constant values and are expressed in the form of derivatives. Reversing them to get the original functions use rule 2, we have

i) $\int 10\,dx = 10x + C$, and

ii) $\int -3\,dx = -3x + C$.

Rule 3: The integral of a constant and a function rule

$$\int kf(x)dx = k\int f(x)\,dx$$

In this case, outcome is constant value multiplied by the integral of the function.

Example 3: determine the initial functions of the followings:

i. $\int 3x^2 dx$

ii $\int 2a^8\,da$

Solution

In applying rule 3 to solve this problem, we have to first identify the constant value and the function. $3$and x^2 are the constant value and functional variable respectively. Therefore:

$\int 3x^2\,dx = 3\int x^2\,dx$, we now apply rule 1 at this

point and determine the initial function

$= 3\left(\frac{1}{3}x^3 + C\right)$

$= x^3 + C.$

iii. Applying the method in solving above (i), we will have:

$$\int 2a^8 da = 2\int a^8 da.$$

Therefore, we have $9\left(\frac{1}{9}a^9 + C\right) = a^9 + C$.

Rule 4: The integral of x^{-1} rule

This is also known as logarithmic rule of integration. The integrand being considered is *1/x* which is same as x^{-1}. You will recall that under the power rule of integration i.e. rule 1, n = -1 is not accepted. However, under this rule, n = -1is welcomed based on the law of indices.

$$\int \frac{1}{x} dx = \ln x + C \qquad (x > 0)$$

The rule stated is applicable where *x* is positive that is $x > 0$. Where it is otherwise, that is *x* is not equal to zero $(x \neq 0)$, which mean that *x* can take a negative value, we use

$$\int \frac{1}{x} dx = \ln|x| + C \; (x \neq 0)$$

Note that as a matter of notation, the integral of $\int \frac{1}{x} dx$ can sometimes be stated thus $\int \frac{dx}{x}$. Also, under this same rule, we have

$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$,where *f(x)* is positive or

$= \ln|f(x)| + C$, in this case, *f(x)* not positive. (where

| *f(x)*| means the absolute value of the *f(x)*).

Example 4: integrate the function $\int \frac{6}{x} dx$.

Solution

Note that the function is same as $\int 6\frac{1}{x} dx$. Therefore, applying rule 3 and 4, we will have

$$\int \frac{6}{x} dx = 6\int \frac{1}{x} dx = 6 \ln x + c.$$

Rule 5: The exponential rule

To find the integral of an exponential function using exponential rule, it is advisable to understand derivative of an exponential function. Without much ado, the derivative of exponential e^x is the e^x itself. Thus,

$\int e^x dx = e^x + C$. More generally, (where e = exponential)
$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$. Where $a$is a positive.

Rule 6: The sum or difference of two functions rule

The integral of sum or difference of two or more functions is the sum or difference of the individual integral.

$$\int [f(x) + h(x)]dx = \int f(x)dx + \int h(x)dx$$ Or
$$\int [h(x) - g(x)]dx = \int h(x)dx - \int g(x)dx$$

Example 6: compute the $\int (3x^3 - x + 2)dx$.

Solution

To resolve this problem, we need the combined application of the rules we have discussed thus far. In this case, we will combine rules 3 and 6, therefore,

$$\int (3x^3 - x + 2)dx = 3\int x^3 dx - \int xdx + \int 2dx \quad \rightarrow \quad \text{rule 6}$$

We shall then apply rule 1and have,

$= 3(\frac{1}{4}x^4) - \frac{1}{2}x^2 + x + c$, note that integral of dx is $x + c$

$$= \frac{3}{4}x^4 - \frac{1}{2}x^2 + 2x + c$$

3.3.1 Integration by Substitution

This is another way of resolving some difficult problems in integral calculus. Integration by substitution is useful when the integral function is becoming too large and difficult to handle. Three main steps are involved in this method, these are: state another variable say u to represent the function *f(x),* replace *dx* with *du* in the function and you arrive at answer by putting the value of u in terms of x with the derived integral.

Example 7: Compute the integral $\int (x+2)^3 dx$.

Solution

To be able to tackle this problem, the method of integration by substitution will be applied. In doing this, the three basic steps stated above shall be considered.

a) Equate u to $x + 2$ and $(x + 2)^3$ equals u^3

b) Now that $u = x+2$, $\frac{du}{dx} = 1$ and $du = dx$

Then $\int (x+2)^3 dx = \int u^3 du$. We shall now use rule 1 (i.e. power rule) to evaluate this stage, then you have:

$$= \frac{u^4}{4} + c$$

c) We shall now put the value of u in the integral.
Doing that we therefore have,

$$\frac{(x+2)^4}{4} + c \text{ or } \frac{1}{4}(x+2)^4 + c .$$

3.3.2 Integration by Parts

Outside integration by substitution just discussed, one other method of solving cumbersome mathematical problems is by using integration by parts. Integration by parts is the process of reversing the processing of differentiation using product rule. It is basically used when an integrand is a product or quotient of differentiable function and cannot be stated as a constant multiple.

Assuming we have the following derivative function using product rule,

$$\frac{d}{dx}[g(x)h(x)] = g(x)h'(x) + h(x)g'(x) ,$$

Now taking the integral, we shall have,

$$g(x)h(x) = \int g(x)h'(x)dx + \int h(x)g'(x)dx$$

Then, we can solve the integrals algebraically. For instance, the second integral on the right-hand side can be solved thus:

$$\int h(x)g'(x)dx = h(x)g(x) - \int g(x)h'(x)dx$$

Example 8: Integrate the following function $\int x(x+3)^{2/3}dx$.

Solution

Unlike the problem in example 7, this very one is not solvable using the method of substitution as earlier applied. This is because the problem is in the form ($\int vdu$), can only be solved using the method of integration by parts. Therefore, let *v* = *x*, and *dv* = *dx*. Also, let *u* $= \frac{3}{4}(x+3)^{4/3}$, since $du = (x+3)^{2/3}\, dx$. We can now determine the integral thus:

$$\int x(x+3)^{2/3}dx = \int h(x)g'(x)dx = h(x)g(x) - \int g(x)h'(x)dx$$

$$= \frac{3}{4}(x+3)^{4/3}x - \int \frac{3}{4}(x+3)^{4/3}dx$$

$$= \frac{3}{4}(x+3)^{4/3}x - \frac{9}{28}(x+3)^{7/3} + c.$$

It is important to note that arriving at the required answer, combination of integral rules is very essential. And it depends on the student's full grasp of the studied area or topic (e.g. integration and its rules). There are complicated functions in integration. These are solved using integration table. These can be found in many available mathematical books and tables.

SELF-ASSESSMENT EXERCISE

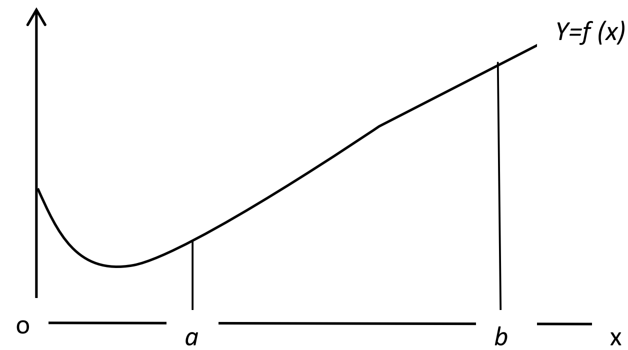
Determine the following integrals applying the rules discussed so far.

i. $\int 7.5dx$, $\int 6x^3dx$, $\int 3x^{-1}dx$, $\int (5x^4 + 3x^3 + 4x)dx$ and $\int \frac{dx}{\sqrt[4]{x}}$

ii. $\int 7xe^7x^2 + 4dx$, $\int \frac{dx}{4x-2}$ and $\int \frac{8x}{(x-3)^3}dx$.

3.4 Definite Integral

All we have discussed thus far in this section (integration) is about indefinite integral. It is so, because the functions were dependent on a single variable and had no precise numerical value. Now if *f* is a continuous function defined between the *(p, q)*, and *F* is continuous, and has a derivative with *F'(x) = f'(x)* for every *x* in *(p, q)*. In this instance, *F(P) – F(q)* is referred to as "definite integral" of *f* over*(p, q)*. In this instance, *p* is known as the lower limit of the integration and *q* as the upper limit.



As such, the already familiar integral sign is somehow modified to the form $\int_p^q$ .Therefore, the definite integral of *f* over *(p, q)* or *(a, b)* is number that hinges only on the function *f* and the numbers *p* and *q*.we then have:

$$\int_p^q f(x)dx \text{ Or } \int_a^b f(x)dx \,.$$

The evaluation of a definite integral founded on the expression above is symbolized as follows

$$\int_a^b f(x)dx = \Big|_a^b F(x) = F(b) - F(a) \,.$$

The notation $\Big|_a^b$ is a command that b and a should be substituted for x in the outcome of the integration to determine F (b) and F(a), and subtract accordingly as indicated in the symbol above.

Let see one or two examples of this method.

Example 9: Evaluate $\int_2^7 4x^3 dx$.

Solution

In the final outcome for this problem, we shall have a precise value as answer, unlike the indefinite. For instance, the indefinite integral for this very function is $x^4 + c$. the definite integral will then be

$$\int_2^7 4x^3 dx = x^4\Big]_2^7 = (7)^4 - (2)^4 = 2401 - 16 = 2385$$

Example 10: Evaluate $\int_0^2 (\frac{1}{1+x} + 2x)dx$.

Solution

Note that the indefinite integral is $\ln|1+x| + x^2 + c$, therefore the definite integral will be

$$\int_0^2 (\frac{1}{1+x} + 2x)dx =$$

$\ln|1+x| + x^2\Big]_0^2 = (\ln 3 + 4) - (\ln 1 + 0) = \ln 3 + 4$ (note that ln1is zero).

3.4.1 Properties of Definite Integrals

Definite integrals have the following properties for any function that is continuous:

1) $\int_p^q f(x)dx = -\int_q^p f(x)dx$
2) $\int_p^p f(x)dx = 0$
3) $\int_p^q \alpha f(x)dx = \alpha \int_p^q f(x)dx$. (where α is any arbitrary number)
4) $\int_p^q f(x)dx = \int_p^r f(x)dx + \int_r^q f(x)dx$
5) $\int_p^q - f(x)dx = -\int_p^q f(x)dx$
6) $\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx$. (Where a < b < c < d).

These properties of definite integration are vital to solving definite integral problems. Students are advice to read more about them.

SELF-ASSESSMENT EXERCISE

Evaluate the following integrals:

$\int_1^3 xdx$, $\int_{-2}^3 (\frac{1}{2}x^2 - \frac{1}{3}x^3)dx$ and $\int_2^4 (\frac{1}{z-1} + z)dz$.

4.0 CONCLUSION

In this section, we have studied another aspect of calculus called integration. It is a topic that is related to economic dynamics. Dynamics is a term which has to do with the study of the specific time paths of variables. It is to ascertain if these variables will tend to converge towards a particular value given enough time. What integration as topic has done in the body of mathematical science, is to fill the gap left by differential calculus, which is the 'time paths.' Conclusively, if differentiation is about looking ahead to a destination that is from point A to another point B or C, integration on the other hand is about reversing the trend, which is tracing the journey back to where it started using point B or C as the base.

5.0 SUMMARY

This unit is fundamentally about integral calculus. We have said earlier that integral calculus is the direct opposite of differential calculus. All you have learnt in this unit is about the reversal of you learnt in the previous units. Basically, what we did here was to study some vital areas in integration. These are: indefinite integration, rules of integration, definite integration, and properties of integration. All these we have fully dealt with.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the following:

   $\int (4cx+d)(cx^2+ax)^5 dx$, $\int 3e^{-2x} dx$, and $\int \frac{x}{4x^2+7} dx$

2. Evaluate the followings:

   $\int_0^5 (x+x^3) dx$, $\int_1^2 \frac{x^2+x+\sqrt{x+2}}{x+1} dx$ and $\int_4^4 (3x+2) dx$

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UNIT 4 ECONOMIC APPLICATIONS OF DERIVATIVES AND INTEGRATION

CONTENTS

1.0 INTRODUCTION

From unit one to unit three of this module, what we have done is to do a mathematical study of both differential and integral calculus. That is, we looked at it the way people in the field of mathematical sciences will do. Recall that we have stated that, differentiation is the process of ascertaining the derivative of a function. That, derivative is about the rate of change between two or more variables. You will also recall that, we stated in simple understanding that integral calculus is the opposite of differential calculus. That the other name of integration is antiderivative or antidifferentiation. A good instance is, if derivative is movement from 'G to *g*,' the reverse process i.e. going from '*g* to G' is integration or antiderivative/antidifferentiation.

Therefore, the understanding of all these mathematics, and it's applications in economics by Economists make the subject interesting and grey areas in it are easy to resolve. So, this unit is about the application of differential and integral calculus to Economics.

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- express the need for mathematics in economics
- apply mathematical tools in solving economic problems.

3.0 MAIN CONTENT

3.1 Marginal Concept

The concept of marginal in economics is divided into two. These are the concept of marginal revenue and the concept of marginal cost. These concepts are best discussed under profit maximisation or cost minimisation in either the principle or intermediate economics. In any business organisation (that is privately owned), the primary aim of setting it up is to make gain/profit. Therefore, the onus is on the management team to pursue this noble objective. To achieve this, the ideal of profit maximisation or cost minimization comes to mind. All this is about the concepts of total revenue and total cost. The manipulations of the two (revenue and cost) mathematically in conjunction with some economic principles bring about profit or loss in an organisation. This is impossible without a good understanding of the concept of marginal in economics.

Marginal revenue is defined as the change in total revenue as a result of change in the sale of an additional unit of a particular product. In the same vein, marginal cost is the change in total cost owing to the production of an additional unit of any product. Now, if we are to construct a function or model in which marginal cost and marginal revenue can be mathematically derived differentially respectively, total concepts and production outputs will play a major role. So if *TC* is total cost and *TR* is total revenue are both linearly related positively to the production output *Q*, then their function will be expressed thus:

If $TR = f(Q)$, then $MR = \frac{dTR}{dQ}$

Also, if $TC = f(Q)$, then $MC = \frac{dTC}{dQ}$

In a nutshell, the marginal concepts of any function in economics are determined by differentiating the total function.

Example 1: if the total cost function $TC = 5Q^2 + 7Q + 20$, find the marginal function and evaluate it at $Q = 6$ and $Q = 9$.

Solution

Given the total cost function, the marginal cost function is an expression of the derivative of the total function, thus we have:

$$MC = \frac{dTC}{dQ} = 10Q + 7 .$$

To evaluate the marginal function based on the outputs given, substitute the output one after the other and you will arrive at the results. That is

At $Q = 6$, $MC = 10(6) + 7 = 67$
At Q= 9, $MC = 10(9) + 7 = 97$.

Also, determine the marginal revenue function if the total revenue function is $TR = 15Q - Q^2$. Estimate the marginal revenue *(MR)* function base on the outputs level given above.

Solution

As we have seen it done in example one, the marginal revenue (*MR*) function is gotten by differentiating the total revenue (*TR*) function. Therefore:

$$MR = \frac{dTR}{dQ} = 15 - 2Q$$

Now, to estimate the marginal revenue function based on the level of outputs given, substitute the output one after the other and you will get the marginal revenue at the various levels of production. That is,

At $Q = 6$, $MR = 15 - 2(6) = 3$
Also at $Q = 9$, $MR = 15 - 2(9) = -3$.

You will recall that the primary aim of any businessman is to make profit. Meanwhile, all other cost incurred would have been adjusted for to arrive at the expected profit. That gain is what is referred to as profit. In economics, the gain/profit that accrued to the business owner is sales made at the prevailing market price (total revenue *TR*) less all expenses incurred (total cost *TC*). We can from this, formulate a function called profit function. The profit function is a combination of the total functions (i.e. total revenue and total cost).Note that, the notation for profit in economics is represented by the sign π, and the function is expressed thus

$$\pi = \pi(Q) = TR - TC \text{ or } PQ - CQ$$

What we have seen in the profit function is that, the amount any businessman would gain or make as profit is dependent on the volume of sales or volume of production *(Q)*. Having known the total profit, we may decide to probe further mathematically the profit made per unit item as the production progress. This brings us to the issue of additional or marginal gain/profit. This (marginal profit) is an expression of the derivative of the profit function. Note that, *P* and *C* in the model $PQ - CQ$ are per unit price and per unit cost respectively.

Example 2: Determine and evaluate the marginal gain function of the profit function $\pi = Q^2 - 16Q + 50$ at *Q = 4 and Q = 6*.

Solution

Given the profit function, the marginal gain function is an expression of the derivative of the total profit/profit function, thus we have,

$M\pi = \frac{d\pi}{dQ} = 2Q - 16$. Note that the notation *Mπ* as used in the expression denotes marginal profit/gain. To evaluate the function as required based on the outputs given, substitute the output one after the other and you will arrive at the answers. That is

At *Q = 4, Mπ = 2(4) - 16 = -8*
Also at *Q = 6, Mπ = 2(6) - 16 = -4*.

What we have seen with marginal concept is to estimate the additional cost or gain per unit of any product produced in any firm. However, there is the average concept. This concept estimate the total function to get either the cost of producing a unit of product or the revenue per a unit of good sold. To determine the average function, divide the total cost or total revenue function including the constant term with *Q*. That is, if we are to determine for instance the average revenue (*AR*), we will have:

$$AR = \frac{TR}{Q}$$

After you have determined the function, you can proceed to estimate the average function by substituting for *Q* in the function.

SELF-ASSESSMENT EXERCISE

i. Determine and evaluate the marginal expenditure of the function $P = Q^3 + 4Q + 3$ at *Q = 4 and Q = 7*. Hint: note that the *TE = P*Q*.
ii. If the total consumption function is $C = 1000 + 0.88Y$, evaluate the marginal propensity to consume (*MPC*).

3.2 The Production Functions

The theory of production is about input and output analysis in the production of goods and services in any firm. It explains the combination of certain items or materials called input to produce something valuable or special items tagged output. The basic inputs considered in production theory are always factors of production that has been narrowed down to Capital *(K)* and Labour *(L)*, while the output is represented by (Q). By definition, production is the process of combining raw materials input in certain required proportion, and transform them into a different but useful form (output).

The transformation of these inputs into useful item is through a process called production technique. This cannot be divulged from a mathematical representation known as production function. Production function by any means is a statement relating how inputs can be combined to achieve various possible levels of output. Algebraically, production function (*FP*) can be represented mathematically thus:

$$Q = f(K, L);$$

Where:

Q is output per time period,

K is the amount of capital employed, and

L is the number of labour employed.

From the information given above, we can determine the marginal and the average products if the total production function is known.

3.3 Integration Applied

Integrals are vital to analysing issues in Economics. These are done in several ways.

a) If net investment I is the rate of change in capital stock formation k over time $t(k(t))$.Therefore, we determine net investment over time $I(t)$ by the process of differentiation. That is, $I(t) = dK(t)/dt = K't$. Now that we know the net investment, the level of capital stock formation can be computed. This we can do by integrating the capital stock with respect to time of net investment:

$$K_t = \int I(t)dt = K(t) + c = K(t) + K_0 .$$

C in the expression is equal to K_o, and it is the initial capital stock.

In the same vein, the total cost of a product can be determined from the marginal cost of that product through the process of integration. If marginal cost is computed using total cost parameter overtime as the output change, that is $MC = dTC/dQ$, then the TC will be

$$\int MCdQ = VC + c = VC + FC \,.$$

Again in the expression c (small lettered) is equal to fixed or initial cost. Let see one or two worked instances where integration is applied.

Example 1: Assuming that marginal cost (MC) is $50 + 60Q - 18Q^2$, if fixed cost is *75*; determine the total cost (TC).

Solution

The secret in this question for the students to discover is that, we want to find the total cost given the marginal cost. The best way to do that is apply integration. Thus:

$$TC = \int MCdQ = \int (50 + 60Q - 18Q^2)dQ$$
$$= 50Q + 30Q^2 - 6Q^3 + c$$

Our fixed cost (FC) is *75*, and is also equal to c, because c denotes constant term which in this case is equal to *75*. Therefore,

$$TC = 50Q + 30Q^2 - 6Q^3 + 75.$$

Assuming, the question requires you to find the cost per unit (i.e. AC) and variable cost (VC), the way out is simple. The cost per unit (AC) will be:

$$AC = \frac{TC}{Q} = 50 + 30Q - 6Q^2 + \frac{75}{Q}.$$

However, $VC = TC - FC = 50Q + 30Q^2 - 6Q^3$.

SELF-ASSESSMENT EXERCISE

i. Find the consumption function *(C)* if marginal propensity to consume (MPC) is *0.6*, and consumption is *70* with income*(Y)* equal to zero.
ii. Determine the capital function (K) if the rate of net investment (I) is $20t^{3/5}$, and stock of capital at *t equal* to zero is *50*.

4.0 CONCLUSION

In this unit, what we have done is to practically apply the principles of differentiation and integration to issues as they relate to economics. So, the module, confirms that, these two principles are important in economic analysis of certain issues.

5.0 SUMMARY

This unit is fundamentally about the application of differentiation and integration into economics. You will recall that, in unit three of this module we stated that integral calculus is the direct opposite of differential calculus (derivative). All you have learnt in this unit is basically about the application of differentiation and integration in resolving some basic economic problems. We looked at marginal concept. It was discovered that, given total cost/revenue function, the marginal cost/revenue can be ascertained applying the principle of derivative. However, in a reverse manner, the total cost/revenue can be determined from the marginal cost/revenue using integration.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the total revenue function and the per unit price given that marginal revenue (MR) is $40 - 4Q - Q^{3.}$
2. If marginal cost (MC) is $24e^{0.5Q}$, and the fixed cost is 50, ascertain the total cost function.
3. Find the marginal revenue (MR) function of the demand function $Q = 72 - 4p$.
4. Assuming we have a consumption function (*C*) which is $600 + 0.4Yd$, where *Yd* is $Y - T$, and *T* is *200*, find marginal propensity to consume (*MPC*).

   Hints: note that the demand function is same as the price function.

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MODULE 2 OPTIMISATION

UNIT 1 INTRODUCTION TO OPTIMISATION

CONTENTS

1.0 INTRODUCTION

The just concluded module (that is module one) to be exact, in the first two units (units one and two) where we treated basically differential calculus, the study on derivative was limited to functions of a single independent variable such as $q = f(p)$. However, a lot of economic models or functions involve more than one independent variable. For example, we could have a situation where $Q = f(L, K)$. In this case, we can define the model as a function of two independent variables where Q is the endogenous variable; L and K are the exogenous variables.

In this part, we are basically going to be discussing optimisation which is about equilibrium analysis (which is called goal equilibrium). By definition, equilibrium is the state of optimum position for a given economic unit such as household, business entity, or the entire economy, in which they strive to attain that equilibrium. Optimisation by understanding is the quest for the best. In a quest to have the best as mentioned, economic agents try to understand the effect of one exogenous variable on the endogenous variable which is basically measured using partial derivative and others that are discussed below.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain optimisation problem
- discuss what partial differentiation is all about
- apply optimisation to solve economic problems.

3.0 MAIN CONTENT

3.1 Optimisation of Functions

Recall we stated that optimisation is about the quest for the best. However, by definition optimisation is the process of finding the relative maximum or minimum of a function or model in mathematical sciences, business studies and economics. Without the aid of a graph, this is done with the methods of relative extreme and inflection points and a lot more. All these are discussed below.

An economic model or function of the nature $g(x)$ is said to be rising or falling at $x = a$. If in the immediate vicinity of the point $[a, g(x)]$ the graph of the function rises or falls as it goes from left to right. Recall that in unit one of module one, we stated that derivative (that is the differentiation of the initial/original function) measures the rate of change and slope of a function. A positive initial derivative at $x = a$ shows that the function is rising and a negative one means it is falling. See figure below:

Slope > 0
Rising function at x = a
(a)

Slope < 0
Falling function at x = a
(b)

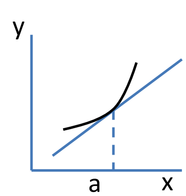
$g'(x) > 0$: rising function at $x = a$

$g'(x) < 0$: falling function at $x = a$

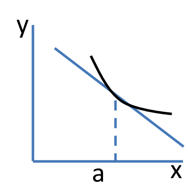
When an economic model or function is rising or falling over its entire area is referred to as amonotonic function.

3.1.1 A Case of Concave and Convex Functions

Still discussing optimisation, a function $g(x)$ is concave at $x = a$ if in some small region close to the point the $[a, g(x)]$ graph of the function lies completely beneath its tangential line. On the other hand, a function is convex at $x = a$ if in an area very close to $[a, g(x)]$ the graph of the function is wholly above the tangential line. Meanwhile, a positive second differentiation of the function or model at $x = a$ signifies that the function or model is convex at $x = a$(that is, $g''(x) > 0$); also, a negative second derivative of the function or model at $x = a$ denotes the function or model is concave at $x = a$ (that is $g''(x) < 0$). It is important to note that, if $g''(x) > 0$ for all x in the region, $g(x)$is convex. Also, if $g''(x) < 0$ for all x in the region, $g(x)$is concave. However, the sign of the first differentiation does not matter for concavity. Again, see figures below for the graphs.

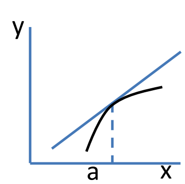


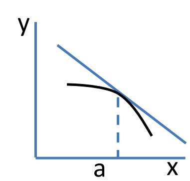
(a) $g'(x) > 0$
$g''(x) > 0$



(b) $g'(x) < 0$
$g''(x) > 0$

Convex at x = a





(d) $g'(x) < 0$
$g''(x) < 0$

Concave at x = a

Supposing we have two functions $g = -2x^3 + 4x^2 + 9x - 15$ and $g = (5x^2 - 8)^2$, test for their convexity and concavity at $x = 3$.

Solution

i) $g = -2x^3 + 4x^2 + 9x - 15$

$g' = -6x^2 + 8x + 9$

$g'' = -12x + 8$

$g''(3) = -12(3) + 8 = -36 + 8 = -28 < 0$ (in line with the rules, this is concave).

ii) $g' = 2(5x^2 - 8)(10x) = 20x(5x^2 - 8) = 100x^3 - 160x$
$g'' = 300x^2 - 160$ $g''(3) = 300(3)^2 - 160 = 300*9 - 160 = 2700 - 160 = 2540 > 0$ (convex).

SELF-ASSESSMENT EXERCISE

i. From your understanding of the discussion on optimisation, in your own language; explain what you understand by optimisation to mean.
ii. State if the function $g(x)= 2x^2 - 11x + 3$ is rising, falling or static at $x = 6$.
iii. Also, state if the function $g(x) = - 3x^3 + 5x^2 + 11x - 100$ is either convex or concave at $x = 4$

3.2 Relative Extrema

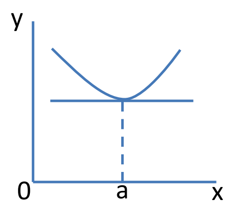
Recall that graph and relation are not the right tools to solve optimality problems, except with the methods of 'relative extreme and inflection.'(See optimisation of functions above).To start with, what is relative extrema all about? It is a point at which any economic model or function as we shall soon observe is at a relative maximum or minimum. To be at either maximum or minimum at any point say *a*, the economic model or function must be at a relative table like form (i.e. flat) where the model or function is not rising, and is not falling at *a*. If that is the case, the first derivative of the function at *a*is either zero or be undefined function or equation. Any point in the region of a function where the first differentiation is equal to zero or is undefined is known as '*critical point* or *stationary value*.' this point is a standstill position. It is a point on the function or the equation graph where its first derivative is always zero. The first test for stationarity or critical point can be established on either the peak or bottom of the business cycle (see explanation below).

Now, it is vital to differentiate between a relative maximum and minimum functions. In doing this, a mathematical approach will be required, and this is by taking the second-order derivative of the initial economic model or function. For instance, if $g'(x) = 0$ at *a* (that is at initial or first differentiation), at second differentiation commonly referred to as the second-order derivative, one of the followings would be observed: a) If $g''(x) > 0$ at *a*. It shows that the model or function is convex and that, the graph of the function or equation stays wholly above the tangential line at $x = a$, the function or model is at a relative minimum (like the bottom of a business cycle graph), b) If $g''(x) < 0$ at *a*, it signifies that, the model or function is concave and that the graph of the function is wholly underneath the tangential line at $x = a$, the function or model is at a relative maximum (like the peak of a business cycle graph), and c) If we have a situation different from the two instances already discussed, that is where $g''(x)$= 0, it means that the test is inconclusive.

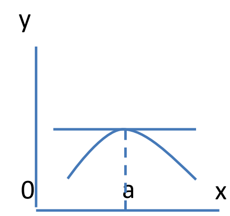
However, it is important for us to have it in our minds that, only functions whose first differentiable value at all values of *x* is equal to zero ($f'(x)$ = 0) are to be considered, if there is the need to determine the critical points of the functions or equations. In other words, if we subject it to further test the situation where the first derivative is zero, we will arrive at any of the conclusions summarised thus mathematically

$g'(x) = 0$ $g''(x) > 0$: $\longrightarrow$ relative minimum at $x = a$

$g'(x) = 0$ $g''(x) < 0$: $\longrightarrow$ relative maximum at $x = a$.



g'(x) = 0
g" (x) > 0
Relative Minimum at x = a
(a)

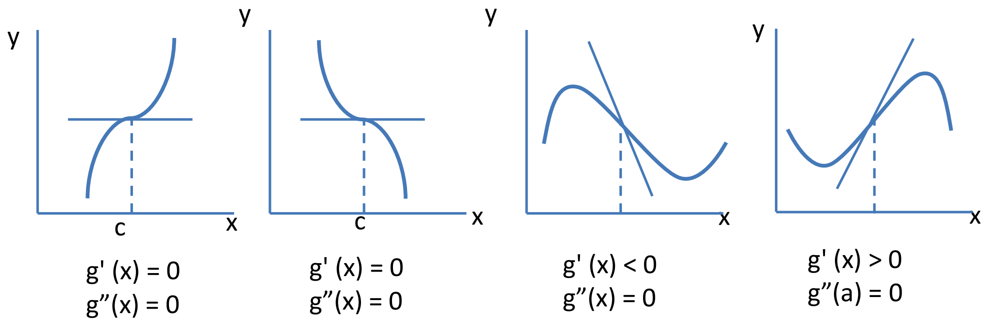


g'(x) = 0
g" (x) < 0
Relative Maximum at x = a
(b)

3.2.1 Inflection Points

Another possible method of looking at optimisation apart from relative extrema just discussed is '*Infection Point*.' Most economic models or functions that are been studied are often convex or concave in an interval at the second derivative. In the process of testing for convexity or concavity, an economic model or function may change from either convex to concave or vice versa. The point at which a function or model changed from being concave to convex or vice versa are referred to as *inflection point*. By definition, an inflection point is a point on the graph where the function crosses its tangential line and changes from concave to convex, or vice versa. Or, an inflection point for a function *g(x)* if there exists an interval about any point on the graph such that: i) $g''(x) \geq 0$ in say (a, c)and $g''(x) \leq 0$ in say (c, b), or ii) $g''(x) \leq 0$ in say (a, c) and $g''(x) \geq 0$ in say (c, b). Let us assume that $x = c$ and it is the inflection point.

Inflection point occurs only where the second derivative equals zero or is undefined. However, note as stated earlier in this section that, when the first derivative is equal to zero or undefined as in the second derivative, it calls for further test. This time, it is to determine either the convexity or concavity of the graph. When the inflection point is to be determined, the indication of the first derivative is irrelevant. In sum, for an inflection point at *c*, as seen in Figure below, the followings are observed: 1) $g''(x)$ = 0 at *c* or is undefined, 2) concavity changes at $x = c$, and 3) graph crosses its tangent line at $x = c$.



(a) (b) (c) (d)

Example 1: Determine the relative extrema for the following equations by (a) finding the critical value(s) and (2) determining if it is at the critical value(s) (the equation is at relative maximum or minimum).

a) $g(x) = -5x^2 + 130x - 50$

Solution

This problem will be solved by way of following some steps. That is moving from one step to the other.

i) Differentiate the primitive equation or function (take the first derivative), set it equal to zero, and then solve for *x* to determine the critical value(s).

$g'(x) = -10x + 130 = 0$

$x = 13$ this is the critical value

ii) Take the second-order derivative of the first derivative and estimate it using the critical value already known.

$g''(x) = -10$

$g''(13) = -10 < 0$ this is concave, and is a relative maximum.

b) $h(x) = 3x^3 - 36x^2 + 135x - 20$

Solution

Imitate the earlier steps followed in the solution to the above,

i) Take the first derivative, and set it equal to zero. solve for *x*

$$h'(x) = 9x^2 - 72x + 135 = 0$$
$$= 9(x^2 - 8x + 15) = 0$$
$$= 9(x - 3)(x - 5) = 0$$

$x = 3$ $x = 5$ these are the critical values

ii) Take the second-order derivative and estimate based on the critical values determined.

$h''(x) = 18x - 72$

$h''(3) = 18(3) - 72 = -18 < 0$ concave, and is relative maximum

$h''(5) = 18(5) - 72 = 18 > 0$ convex, and is relative minimum

3.3 Fundamental Clues on Optimisation Analysis

As we round-off this section on introduction to optimisation, it is essential that students should familiarize themselves with some facts on optimisation which will assist them in analyzing optimisation problems. This is what we refer to as "fundamental clues on optimisation analysis." Assuming you are given a primitive function, model or equation that has an optimisation problem and is differentiable, know these:

- Differentiate the primitive function, model or equation (1st derivative), set it equal to zero, and solve for the critical value(s). This step is known as the *necessary condition* or the *first-order condition*. It locates the points at which the function, model, or equation is neither rising nor falling, but at a leveled ground. The points arrived at are possible relative extrema.

- After the step above, you then again differentiate the first derivative (2nd derivative), estimate the result using the critical value(s), and check for the sign(s). Note the followings: (i) if the estimated result is less than zero, the function is concave hence it is a relative maximum, (ii) if it is greater than zero, the function is convex hence it is a relative minimum and (iii) if it is equal to zero, it shows that the analysis is inconclusive.

- Where the *necessary or first-order condition* is met, the second step, called the second-order derivative analysis is known as the *sufficiency condition*. In sum,

Relative maximum	Relative minimum
$g'(x) = 0$	$g'(x) = 0$
$g''(x) < 0$	$g''(x) > 0$

- Where the estimated result of 2nd derivative is equal to zero as mentioned in step two, the second-derivative analysis is inconclusive. In such cases, the continuous differentiation analysis is helpful: (a) If the first non-zero value of a higher-order differentiation, when evaluated at a critical value, is an odd-number say 3, 5, 7 etc., then the model, function or equation is at an inflection point. (b) Also, if the first non-zero value of a higher-order differentiation, when evaluated at a critical value is an even number say 2, 4, 6, etc., then the model, function, or equation is at a relative extrema. But, if the even number is negatively signed, it is shows that the function or equation is concave and is relative maximum, and if vice versa it shows that the equation is convex and is relative minimum.

SELF-ASSESSMENT EXERCISE

Determine the critical values of the equation $q = - p^3 + 4.5^2 – 6p + 6$, and test if it is relative maximum or minimum or inflection points.

4.0 CONCLUSION

Under differential calculus in module one, we saw a case of a function of a single variable, how to find the first derivative and that of the second derivative treated under higher-ordered derivatives. In this part, which is about optimality of values by economic agents, we seen how the two (the first and second derivatives) are combined to be used as tools to resolve optimisation problems.

5.0 SUMMARY

In a nutshell, in this unit, we have considered the following:

- We have explained that optimisation is about the quest for the best amongst given alternatives by economic agents, such as households, firms, nations, and many more.
- We also discussed convexity and concavity as a case in optimisation. That at the second derivative of any model or equation and when evaluated at their critical value(s) it could be either positive or negative. If the sign is positive, it shows that the

model or equation is convex but, if otherwise, it means that the model or equation is concave.

- We equally discussed relative extrema and inflection points. You will recall that we said relative extrema and inflection points are modes of analysing optimisation problems. That, relative extrema is of two extremes, the maximum and the minimum. At a particular instance in the life of a function or model, there is always a transformation that is, minimum changing to maximum, or vice versa. At that very point where the transformation occurs on model or function, is referred to as inflection point.

6.0 TUTOR-MARKED ASSIGNMENT

1. Based on your understanding of optimisation, sketch graphs to show the followings: i) a rising function, ii) a falling function, iii) a concave model, and iv) a convex equation.
2. Given that $y = 2x^3 – 3x^2 + 11x – 30$. Examine the equation at $x = 4$, and show if it is falling, rising, or static.
3. Determine the critical values, relative extrema (i.e. maximum or minimum), or possible inflection point of the equation $z = - (a – 8)^4$.

7.0 REFERENCES / FURTHER READING

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UNIT 2 FUNCTION OF SEVERAL VARIABLES

CONTENTS

1.0 INTRODUCTION

In module one, where we discussed basically derivatives and integration, and in the preceding unit to this unit, you will discover that all the functions, equations or models treated thus far, are mainly a case of one parameter or a variable, that is, $y = g(x)$ or $\int f(x)dx$, and so on. From the equations we have seen so far, there is always one independent variable on the right-hand side of the equation or function deciding the fate of the dependent variable on the left-hand side. This is what the author referred to as a *'mono-variable/parameter function.'* In this section, we shall be looking at a situation where we shall be having more than one variable or parameter as independent variables on the right-hand side of the function or equation. Where we have *two variables*, say $z = g(x, y)$, it is called *'functions of two variables'*, where it is more than two variables like $Q^d = g(p_x, i, p_s, n)$, is referred to as, 'functions of several variables. 'When cases like these are treated, any problem with multi-product organisation can successfully be resolved by applying it.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the difference between a single-variable function and a multi-variable function
- discuss the essence of partial derivative in multi-variable models

- explain the procedures for resolving problems with multi-variable models.

3.0 MAIN CONTENT

3.1 Functions of Several Variables

To restate the obvious, what you have done so far in the preceding units as mentioned in the introduction of this unit were restricted to a function of a single variable (i.e. exogenous variable) such as $y = g(x)$. In the field of economics, a lot of economic activities involve models of more than one exogenous variable or parameter. For instance, in production theory, the production function has more than one exogenous parameter, $Q = g\ (L, K)$. This model is defined as a model of two exogenous or independent variables if there exist one and only one value of Q in the range of g for the pair of real numbers (L, K) in the region of g.Q is the *endogenous variable* (variable that is determined within the model); L and K are *exogenous variables* (variable that are determined outside the model). This is a typical example of 'function of two variables.' In the main time, we will be discussing more of this.

Supposing we have a function or an equation like the one stated earlier in the introduction, where $z = g(x, y)$. This equation or function can be analysed in a way that, we can investigate the impact of the exogenous variables individually on the endogenous variable. This analysis is most often referred to measurement of changes between the endogenous and the exogenous variables. One basic mode of carrying out this measurement is the use of a mathematical method called *"Partial Differentiation."* It measures the effect of a change in an individual exogenous variable *(x or y)* on the endogenous variable *(z)* in a two-variable function. The partial differentiation of z with respect to x measures the sudden rate of change of z with respect to x while the exogenous variable y remains unchanged.

3.1.1 Partial Differentiation with Two Variables

When the rate of change between the endogenous and the exogenous variables in a two-variable function is to be measured, partial differentiation is of paramount importance. Unlike what we saw when we had $y = f(x)$, a situation of one-variable function. The notations for partial differentiation or derivative are as follows:

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ f_x(x, y),\ f_x \text{ or } z_x \quad \rightarrow \text{where } y \text{ is held constant}$$

$$\frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y},\ f_y(x,y),\ f_y \text{ or } z_y \rightarrow \text{ where } x \text{ is held constant}$$

Therefore, if $z=f(x, y)$, then $\partial z/\partial x$ ($\partial z/\partial y$) shows the derivative of $f(x, y)$ with respect to $x(y)$ when $y(x)$ is held constant. Note, to find the partial differentiation of a function as given above, say $\frac{\partial z}{\partial x}$, assumes that the variable y does not exist in the function, and differentiate the function with respect to x (w.r.t. x) only. The same operation goes for $\frac{\partial z}{\partial y}$. Partial differentiation is mathematically expressed thus:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \text{With } y \text{ held constant}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \quad \text{With } x \text{ held constant}$$

Partial differentiation with respect to one of the exogenous variables in any function follows the same rules as ordinary differentiation while the other exogenous variables are treated as constant. Consider the examples below:

Example 1: The partial derivatives of a two-variable function such as $z = 5x^2y^4$ are determined thus:

Solution

(i) When differentiating with respect to x, treat the y variable as a constant term, then:

$$\frac{\partial z}{\partial x} = 10xy^4$$

(ii) Now, differentiate with respect to y, treat the x variable as a constant term also, and you will have:

$$\frac{\partial z}{\partial y} = 20x^2y^3$$

Example 2: To find the partial derivatives for this two-variable function $z = 7x^3 - 3x^2y^2 + 6y^4$:

Solution

(i) When differentiating with respect to x, let the variable y be held constant as in example 1, then:

$$\frac{\partial z}{\partial x} = 21x^2 - 6xy^2$$

(ii) Now, differentiate with respect to *y*, treat the *x* variable as a constant term also, and you will have

$$\frac{\partial z}{\partial y} = -6x^2 + 24y^3$$

SELF-ASSESSMENT EXERCISE

Determine the partial differentiation of the following:

i. $g(w, x, y) = 4w^3 + 10wxy^2 – y^2 + x^4$
ii. $h(p, n) = 10p^3 + 6pn^2 + 7n^3$

3.2 Rules of Partial Differentiation

In the worked examples just treated above, one part of the basic rules was taken into cognizance. These rules are referred to as rules of partial differentiation, just as we have in differentiation proper in unit one and two of the preceding module. Now, we are going to consider these rules one after the other.

Rule1: Product rule

This rule is similar to the earlier treated version in the previous module. The only and remarkable difference is that, while this version is done with caution because of its nature of multivariable case, the former is not.

Given $z = g(x, y).\ h(x, y)$,

$$\frac{\partial z}{\partial x} = g(x, y).\frac{\partial h}{\partial x} + h(x, y).\frac{\partial g}{\partial x}$$ Where *y* is held constant

$$\frac{\partial z}{\partial y} = g(x, y).\frac{\partial h}{\partial y} + h(x, y).\frac{\partial g}{\partial y}$$ Where *x* is held constant

Example 1: if $z = (5x + 2)(4x + 3y)$, partially differentiate the model.

Solution

Using product rule as given above, first differentiate w.r.t *x*, then w.r.t *y*.

$$\frac{\partial z}{\partial x} = (5x+2)(4)+(4x+3y)(5) = 40x + 8 + 15y$$

$$\frac{\partial z}{\partial y} = (5x + 2)(3) + (4x + 3y)(0) = 15x + 6$$

Rule 2: Quotient rule

Again, the variation between this rule and the previous one treated in module one is in the mode of application. Like we mentioned under rule one in this section, its application is done with some measure of caution because of its multivariable nature. In this case, at times a variable is held constant while the other is allowed to change and its rate of change measured.

Given that $z = \frac{g(x,y)}{h(x,y)}$ where $h(x,y)$ is not equal to zero.

$$\frac{\partial z}{\partial x} = \frac{h(x,y).\partial g/\partial x - g(x,y).\partial h/\partial x}{[h(x,y)]^2}$$ where y is held constant

$$\frac{\partial z}{\partial y} = \frac{h(x,y).\partial g/\partial y - g(x,y).\partial h/\partial y}{[h(x,y)]^2}$$ where $x$is held constant

Example 2: Given that $z = (7x + 6y)/(3x + 5y)$, use rule 2 to partially differentiate the model or equation.

Solution

Using the quotient rule as already indicated in the question; first differentiate w.r.t to x, and then w.r.t to y thus:

$$\frac{\partial z}{\partial x} = \frac{(3x+5y)(7)-(7x+6y)(3)}{(3x+5y)^2}$$

$$\frac{21x+35y-21x-18y}{(3x+5y)^2} = \frac{17y}{(3x+5y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(3x+5y)(6)-(7x+6y)(5)}{(3x+5y)^2}$$

$$\frac{18x+30y-35x-30y}{(3x+5y)^2} = \frac{-17x}{(3x+5y)^2}$$

Rule 3: Power-function rule universalised

This rule is similar in operation to the version already seen in the previous module. The main difference is in the number of exogenous variables in the function, in which the operation will be done based on one variable in use, while the other is held constant as we have observed in the already discussed rules.

If we have$z = [h(x, y)]^n$,

$$\frac{\partial z}{\partial x} = n[h(x,y)]^{n-1} . \frac{\partial h}{\partial x} \quad \text{where } y \text{ is held constant}$$

$$\frac{\partial z}{\partial y} = n[h(x,y)]^{n-1} . \frac{\partial h}{\partial y} \quad \text{where } x \text{ is held constant}$$

Example 3: Supposing, $z = (x^2 - 5y^3)^5$, find the partial derivatives?

Solution

We can only find the partial derivatives of this function except the power-function rule universalized is applied. First differentiate w.r.t x and then w.r.t y. thus we have:

$$\frac{\partial z}{\partial x} = 5(x^2 - 5y^3)^4 * (2x) = 10x\ (x^2 - 5y^3)^4$$

$\frac{\partial z}{\partial y} = 5(x^2 - 5y^3)^4 * (15y^2) = 75y^2(x^2 - 5y^3)^4$(Note that, the asterisk is multiplication sign)

SELF-ASSESSMENT EXERCISE

Use any of the rules just discussed to determine the 1st-order partial differentiation of the following equations:

i. $q = 4p^2(4p + 9i)$

ii. $q = \dfrac{p^2 + i^2}{2p - 3i}$

iii. $z = \dfrac{(x - 9y)^4}{2x + 4y}$.

3.3 Second-Order Partial Derivatives

What we have done so far in this unit is the first-order derivative of the functions, models or equations given. Now in this part, we want to see how the initial derivative (i.e. the 1st derivative result) can be differentiated further (i.e. second time) to get another result that will be different from the initial result. The differentiation of the first-order derivative's result is called second-order derivative. Given a function $z = h(x, y)$, the second-order (direct) partial derivative shows that the equation has been differentiated partly with respect to one of the exogenous variables *(x, y)* twice while the other exogenous variable has been held constant:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}$$

The notations above show that, in effect, h_{xx} $(\partial x,\partial x)$ measures the rate of change of the first-order partial derivative h_x with respect to *x* while *y* is held constant. Also, h_{yy} $(\partial y,\partial y)$ is exactly opposite of the previous one.

However, there are instances where we have what is fondly referred to as crossed partial derivative h_{xy} and h_{yx} which means the first the initial equation has been partly differentiated w. r. t one variable and then their partial derivative, have in turn been partially differentiated with respect to the other exogenous variable:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\partial x}$$

In short, a crossed partial, derivative measures the rate of change of a 1st-order partial derivative w. r. t the other exogenous variable. Let us see one or two worked examples.

Example 1: Determine the i) 1st -order, ii) 2nd -order, and iii) crossed partial derivatives of the equation $q = 5p^3 + 11pi + 2i^4$.

Solution

In tackling this problem, what is required on the part of the learners' (students) is patience and sense of reasoning.

i) $\frac{\partial q}{\partial p} = q_p = (15p^2 + 11i)$ $\qquad$ $\frac{\partial q}{\partial i} = q_i = 11p + 8i^3$

ii) $\frac{\partial^2 q}{\partial p^2} = q_{pp} = 30p$ $\qquad$ $\frac{\partial^2 q}{\partial i^2} = q_{ii} = 24i^2$

iii) $\frac{\partial^2 q}{\partial i \partial p} = \frac{\partial}{\partial i}\left(\frac{\partial q}{\partial p}\right) = \frac{\partial}{\partial i}(15p^2 + 11i) = q_{ip} = 11$

iv) $\frac{\partial^2 q}{\partial p \partial i} = \frac{\partial}{\partial p}\left(\frac{\partial q}{\partial i}\right) = \frac{\partial}{\partial p}(11p + 8i^3) = q_{pi} = 11$

So, $q_{pi} = q_{ip}$.

3.4 Multivariable Functions and Optimisation

In unit one of this module, we discussed optimisation of functions in which the exogenous variable [i.e. variable(s) on the right hand side of an equation] was just one, that is, $y = f(x)$. In that scenario, we have been able to determine the relative extrema, inflection point, etc. The same goes for a function or equation with more than one exogenous variable. For such a function where $z = g(x, y)$, to be at a relative extrema, three conditions must be fulfilled. These are:

a) The 1st-order partial must equal zero at the same time. This shows that at the given point, often referred to as critical, the function is neither rising nor falling but is at a relative table like form.

b) The 2nd-order partial, when estimated at the critical point, must both be negative which shows it is at maximum and minimum when both are positive. This indicates that from that table-like form, the function is concave when the curve bends downward in the case of relative maximum and convex when it bends upward in the case of minimum.

c) The product of the 2nd-order partials estimated at the critical point should surpass the product of the crossed partials also calculated at the critical point. This is required to rule out an inflection point. In summary:

Relative maximum	Relative minimum
$g_x, g_y = 0$	$g_x, g_y = 0$
$g_{xx}, g_{yy} < 0$	$g_{xx}, g_{yy} > 0$
$g_{xx} \cdot g_{yy} > (g_{xy})^2$	$g_{xx} \cdot g_{yy} > (g_{xy})^2$

d) If $g_{xx} \cdot g_{yy} < (g_{xy})^2$, when g_{xx} and g_{yy} have the same signs, the function is at an inflection point; when g_{xx} and g_{yy} have different signs, the function is at an inflection point.

e) If $g_{xx} \cdot g_{yy} = (g_{xy})^2$, the test is inconclusive.

f) If the function is strictly concave (convex) in *x* and *y*, there will be only one maximum (minimum), called an absolute or global maximum (minimum). If the function is simply concave (convex) in *x* and *y* on an interval, the critical point is a relative or local maximum (minimum).

SELF-ASSESSMENT EXERCISE

Determine i) the 1st -order, ii) the 2nd -order, and iii) the crossed partial derivatives for the equation

$q = 4p^2i^3$

Also, determine the critical values and if the function below is at a relative maximum or minimum, given that,

$q = 2i^3 - p^3 + 147p - 54i + 12.$

4.0 CONCLUSION

We stated earlier at the starting of this unit that, what we studied in unit one was a case of one endogenous parameter being impacted by just a single exogenous parameter too. However, there are extreme cases where a single endogenous parameter is affected by more than one exogenous parameter. That had led us to the study of two-variable/multivariable functions. With what we have discussed thus far in this topic, we can confidently apply this to solve problems associated with multi-products ventures.

5.0 SUMMARY

In sum, we have in this unit, considered the following:

- We have studied functions of several variables. We realised that, a single endogenous variable could be influenced by more than one exogenous variable. This will assist researchers to tackle optimisation problems in a multi-products organisation
- We also have discussed partial derivatives. That is, it is about observing the influence of the exogenous variables on the endogenous variable, while the rest are held constant, and vice versa. In the application of partial derivatives, some fundamental rules were studied.

- We equally discussed relative extrema and inflection points in multivariable functions. We have discussed the conditions to be met to be able to determine the relative extrema and inflection points in a function that has more than one exogenous parameter.

6.0 TUTOR-MARKED ASSIGNMENT

Find the 2nd-partial and crossed partials of the following:

1) $q = 3pi^4 + 5p^3i$
2) $q = p^{0.7}i^{0.3}$

Find the critical values, and determine if these values/points in the function is at relative extrema, inflection point, or inconclusive. Given that,

$q(p, i) = 3p^3 - 8i^2 - 120p + 40i + 100$

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UNIT 3 OPTIMISATION WITH CONSTRAINTS

CONTENTS

1.0 INTRODUCTION

In this section, we shall still continue our discussion on the concept of optimisation, but with a difference. In the preceding sections (that is, units one and two) of this module, we studied optimisation where we determined the relative extrema of an objective function of two or more choice variables. One basic feature of this form of optimisation is that all the choice variables were independent of one another. Where this form of undependability is found among the choice variables in optimisation, this form is generally referred to as free or unconstrained optimisation.

However, in the field of economics, certain problems needed to be optimised. In some cases, variables involved are often required to satisfy certain constraints. For instance, the amounts of different items demanded by a buyer must fulfill the budget constraint of the buyer (in this case, the buyer's income can be seen as a constraint). This unit will introduce us to optimisation with constraint. In particular, the method of Lagrange multipliers will be studied to understand how problems in optimisation are resolved using it.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- state the difference between unconstrained optimisation and the constrained type
- describe the module operands pertaining to optimisation with constraint
- express the usefulness of Lagrange multipliers in optimisation discussion.

3.0 MAIN CONTENT

3.1 Constraint: Its Effect

You will recall that, we have mentioned at the starting of this module that, optimisation denotes the quest for the best. No matter how bad any economic activity may look, there will always be an optimum point, if we apply the concept of optimisation. Basically in economics, every economic activity is all about limiting factors which hinder or constraint the ability or power of any economic agent to do certain things. So, in concept of organisation these limiting factors are duly recognised as constraints.

Let us consider the popular utility concept in the theory of demand. This concept is about the study of the satisfaction of consumer(s) subject to their income level, which is always referred to as Budget constraint. Let us study a buyer with simple utility function:

$$U = x_1x_2 + 2x_1$$

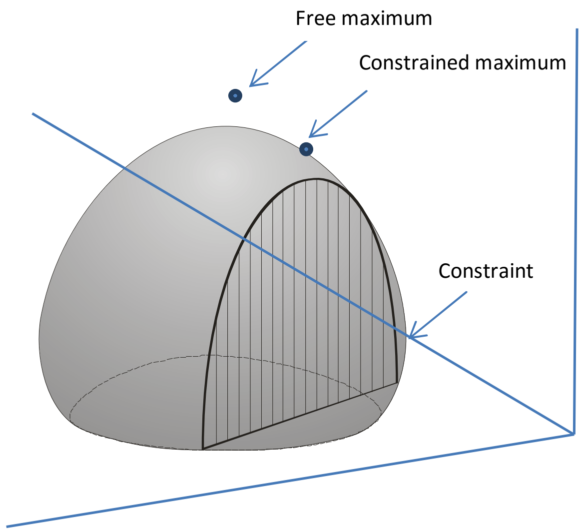
What we have stated above is a total utility function of a buyer. The marginal utilities can be determined by taking the partial derivatives of the total utility function *(U)* w. r.t x_1 and x_2. In this case, the buyer's utility is maximised without any constraint. Which means, the buyer is able to any of good he/she deems fit. This scenario did not consider the income of the buyer as a limiting factor. The case as presented has very little or no pragmatic relevance in economics. To make the case relevant in economics, the income of the buyer should be incorporated into the function. Assuming, the buyer intends to expend a total amount of ₦ 120, on the two items $(x_1 and\ x_2)$, and if the prices of the items are:$x_1 = 8$ and $x_2 = 2$, we can now express this limiting factors (income and prices) that have been incorporated in a linear form thus:

$$8x_1 + 2x_2 = 120$$

With this constraint, the items x_1 and x_2 are now mutually dependent. The next thing is to maximise the utility function stated earlier, subject it to the constraint just derived.

In trying to observe the relative extrema of both the constrained and unconstrained organisation, the difference between the two can be illustrated in a three-dimensional graph (see graph below). The unconstrained extremum is the peak point of the entire dome, while, the peak point of the inverse u-shaped curve denote that of the constrained extremum. Also, in terms of values, the value of the unconstrained maximum is expected to be bigger than that of the constrained maximum. At rear instance, the two maxima may have the same value.

However, the constrained maximum is not expected to surpass that of the unconstrained maxima.



(Adopted from Chiang and Wainwright, 2005).

SELF-ASSESSMENT EXERCISE

i. The organisation of a consumer's utility without constraint amount to little or no economic sense, discuss.
ii. State the limiting factors in the study of utility concept.

3.2 Constrained Organisation with Lagrange Multiplier

From our discussion under constraint, by now we should be able to explain what constrained organisation is all about. In sum, constrained organisation is the quest for the best in the midst of available (limited) resources. That is, a consumer who tries to maximise his/her level of satisfaction bearing in mind the limiting factors. There are many ways constrained problem organisation can be resolved. Ways such as through the method of substitution and elimination of variables, etc. However, some constraints could be complicated function, or when there are several constraints to be considered, the methods of substitution and elimination of variables become ineffective. To resolve problems of this nature, a method that is best applicable is Lagrange multiplier.

3.2.1 The Method of Lagrange Multiplier

The essence of the Lagrange multiplier is to convert a constrained-extremum problem into a form that can be resolved applying the 1^{st}-order condition. A typical economic instance of a constrained organisation problem is about a buyer who decides how much of his/her income I is to be spent on an item say x whose price is p_x, and how much income is to be left over for expenditure $p_y y$ on other item y. In this situation the consumer is faced with some limiting factors, which in this case is represented with budget constraint ($P_x X + P_y Y = I$). Assuming the buyer's preferences is represented by the utility function $U(x, y)$. Therefore, there is a problem of preference organisation among the available items by the buyer subject to the limiting factor (budget constraint). This can be expressed in mathematical terms as:

$$\text{Max } U(x, y) \quad \text{subject to} \quad P_x X + P_y Y = I$$

Where $U(x, y)$ is the utility function, $P_x X + P_y Y = I$ is the budget constraint. What we have just seen, is a classic constrained maximisation problem. In this instance, we can solve for either y or x. When this happens, the problem becomes unconstrained maximisation which can be solved by the method of substitution and elimination.

However, where the constraint is complicated, or where there are several constraints to be considered as mentioned earlier, the substitution technique becomes ineffective. In such circumstance, other systems should be employed. Of particular interest to economists is the Lagrange multiplier approach. Essentially, the same mode is often employed by economists even for problems that are quite easy to express as unconstrained problems. The major reason behind the use of Lagrange multipliers by economists is its clarity in economic interpretation. Besides that, the introduction of Lagrange multipliers can be modified in a number of more complicated constrained organisation problems, such as those expressed in terms of inequalities.

Now, assuming we have the function $f(x, y)$ subject to a constraint $h(x, y) = k$ where k is a constant item. From these, we can combine the objective function and the constraint function to get a new function. This new function is arrived at thus:

(a) Set the constraint function to equal to zero, then
(b) Multiply the function in step (a) by λ (the Lagrange multiplier), and
(c) Add the product (the result) to the objective function.

(Note that the sign λ is a Greek letter often called *lambda* in mathematical sciences, and is the Lagrange multiplier).

Thus we have:

$$F(x, y, \lambda) = f(x, y) + \lambda\,[k - h(x, y]$$

Objective function

Lagrange function

Constraint function

From the equation stated above, $F(x, y, \lambda)$ is the Lagrange function, $f(x, y)$ is the objective/ original function, and $h(x, y)$ is the constraint function. Note that the constraint function is all the time equal to zero, with the multiplication of λ to form $\lambda[k - h(x, y)]$ is equally set to zero and its addition does not change the value of the original function. Meanwhile, x_0, y_0, and λ_0 are the critical values that optimise the function. They (the values) are determined by partly (partial) differentiating the function F w. r. t all the three mutually dependent variables (x_0, y_0, and λ_0), setting them equal to zero, and resolve simultaneously, that is:

$$F_x(x, y, \lambda) = 0 \qquad F_y(x, y, \lambda) = 0 \qquad F_\lambda (x, y, \lambda) = 0.$$

Let us see one or two worked examples.

Example 1: Optimise the function $q = 8p^2 + 6pi + 12i^2$subject to the constraint $p + i = 112$.

Solution

To optimise the function, the steps given above need be followed strictly.

(i) Set the constraint equal to zero by subtracting the variables from the constant value

$p + i = 112$, set this equal to zero thus, we have $112 – p– i = 0$

Multiply the result in step (i) by λ and add the outcome of the two to the original function to form the Lagrange function q. that is,

$$q = 8p^2 + 6pi + 12i^2 + \lambda (112 – p - i) \qquad (a)$$

(ii) Now, take the 1st-order partials, set each equal to zero, and solve simultaneously

$$q_p = 16p + 6i – \lambda = 0 \qquad (b)$$
$$q_i = 6p + 24i – \lambda = 0 \qquad (c)$$
$$q_\lambda = 112 - p - i = 0 \qquad (d)$$

Subtracting (c) from (b) to get rid of λ gives

$10p – 18i = 0$

∴ $p= 1.8i$

Substitute $p = 1.8i$ in (d),

$112 – 1.8i – i = 0$

$2.8i = 112$

$\therefore i = \frac{112}{2.8} = 40 \;\; i_0 = 40$

We then work with the value of i_0 to get p_0 and λ_0, therefore

$p_0 = 72$ $\lambda_0 = 1,392$ (this measures the effect of increasing the constraint constant by a unit on *q)*.

Now that we have determined our critical values, substitute them in (a) we will have,

$q = 8(72)^2 + 6(72)(40) + 12(40)^2 + (1,392)(112 - 72 - 40)$

$= 8(5,184) + 6(2,880) + 12(1,600) + 1,392(0) = 77,952.$

Example 2: Optimise the function $x^2 + y^2 + z^2$ subject to

$$\begin{cases} x+2y+z=30 & (a) \\ 2x-y-3z=10 & (b) \end{cases}$$

Solution

Again, to optimise this function, follow the steps given above.

(i) Let the constraints be equal to zero

(ii) $30 - x - 2y - z = 0$

$10 - 2x + y + 3z = 0$

Multiply the result in step (i) by λ and add the outcome of the two to the original function to form the Lagrange function *F*. that is,

$$F(x, y, z) = x^2 + y^2 + z^2 - \lambda_1(x + 2y + z - 30) - \lambda_2(2x - y - 3z - 10) \qquad (c)$$

(iii) Now, take the 1st-order partials, set each equal to zero, and solve simultaneously

$$\frac{\partial F}{\partial x} = 2x - \lambda_1 - 2\lambda_2 = 0 \qquad (d)$$

$$\frac{\partial F}{\partial y} = 2y - 2\lambda_1 + \lambda_2 = 0 \qquad (e)$$

$$\frac{\partial F}{\partial z} = 2z - \lambda_1 + 3\lambda_2 = 0 \qquad (f)$$

In this instance, there are five unknowns *x, y, z,* λ_1 and λ_2 to be determined, so simultaneously solving (d) and (e) for λ_1 and λ_2 gives

$$\lambda_1 = \frac{2}{5}x + \frac{4}{5}y, \qquad \lambda_2 = \frac{4}{5}x - \frac{2}{5}y$$

Inserting these expressions of λ_1 and λ_2 into (f) and rearranging yields

$$x - y + z = 0$$

This equation together with (i) and (ii) constitutes a system of three linear equations in the unknowns *x*, *y*, and *z*. Solving this system by elimination gives

$$x^o = 10, \quad y^o = 10, \quad z^o = 0, \quad \lambda^o{}_1 = 12 \text{ and } \lambda^o{}_2 = 4$$

SELF-ASSESSMENT EXERCISE

i. Optimise the function $f(x, y) = 8x^2 - 4xy + 12y^2$ subject to $x + y = 36$
ii. Optimise the function $f(x, y, z)=2xyz^2$ subject to $x + y + z = 112$

3.3 Lagrange Multiplier: Its Implication to Economists

We have discussed extensively Lagrange multiplier a tool for resolving organisation with constraint. Some examples have seen solved using this tool (Lagrange multiplier). The question now is, of what interest is this method to economists in the analysis of economic issues? The Lagrange multiplier λ as is commonly referred to helps to estimate the marginal impact on the objective function as a result of any small change in the constant of the constraint. In the worked examples, especially example one, our λ equals to 1,392.Means for instance, that any unit rise (drop) say a unit in the constant of the constraint will influence *q* to rise (fall) by 1,392 units (see worked example below). Lagrange multipliers are often referred to as shadow prices. In production organisation subject to inputs constraint, Lagrange multiplier λ will help to estimate the marginal productivity of an additional input.

Example 1: To prove that a unit change in the constant of the constraint will cause a change of approximately 1,392 units in *Q* in the earlier worked example.

Solution

Reverse to the initial objective function $q = 8p^2 + 6pi + 12i^2$ and optimise it subject to a new constraint $x + y = 113$ in which the constant of the constraint has risen by one unit.

$Q = 8p^2 + 6pi + 12i^2 + \lambda(113 - p - i)$
$Q_x = 16p + 6i - \lambda = 0$
$Q_y = 6p + 24i - \lambda = 0$
$Q_\lambda = 113 - p - i = 0$

By solving simultaneously, this gives:

$p_0 = 73.28$ $i_0 = 40.72$ $\lambda_0 = 1{,}416.8$

Substituting these values in the Lagrange function gives Q = 79,351.6 which is 1,399.6 l bigger than the old constrained optimum of 77,952, close to the approximation of the 1,392rise suggested by the Lagrange multiplier (λ).

SELF-ASSESSMENT EXERCISE

Discuss the impact of one unit change in constant constraint on the value of the objective function.

4.0 CONCLUSION

We have seen the case of constraint organisation, and how the Lagrange multiplier was used to resolve the problem. What we have discussed in this part, is a pure instance of pragmatism. In economics, indeed human wants are many, but, the resources to meet them are limited (the limiting factors). With what we have studied thus far about Lagrange multiplier, one can submit that, Lagrange multiplier is indeed a vital mathematical tool useful in economics, and it is a measure of marginal impact in applied economics.

5.0 SUMMARY

In sum, we have in this unit, considered the following:

- We have studied constraint and its effect. That, constraints are the limiting factors that would not allow a consumer or buyer to purchase all he/she desire to get. For instance, the price of an item, the consumer's income, and a lot more constitute constraints.
- Also, we have discussed constrained organisation. What we have studied before this very unit were cases of free or unconstrained organisation, so as to ascertain the relative extrema. However, we have found out that that is not practicable. So, with our discussion on constrained organisation, we have seen how real life economic issues can be resolved.
- Lastly, we studied a method often used by Economists to resolve constrained organisation problems. The method is Lagrange multiplier. It is about subjecting the objective function to constant

constraint, and by this mode, the variables are made mutually independent with one another. This method helps Economists to ascertain the marginal effect of a phenomenon.

6.0 TUTOR-MARKED ASSIGNMENT

1. Optimise $U(x, y) = xy$ subject to $2x + y = 50$
2. State the Lagrange function of $f(K, L)$ subject to$rK + wL = I$

7.0 REFERENCES/FURTHER READING

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UNIT 4 DIFFERENTIALS

CONTENTS

1.0 INTRODUCTION

We have seen in the just concluded unit how real life issues can be mathematically modeled based on our understanding of economics, and resolved some issues using a mathematical method like Lagrange multiplier that has been incorporated in it these real life issues. In unit two, our study was mainly on partial differentiation and its modus operand. This has enabled us resolve simpler comparative-static issues, where equilibrium of a model, function or equation can be stated in a reduced form. That is, with partial differentiation, there exists no relationship among the exogenous parameters.

Still discussing comparative-static analysis, there are limits in which the partial differentiation will no longer be effective. For instance, in a simple national-income equation with two endogenous parameters (variables) *Y & C*:

$$Y = C + I_o + G_o$$

$$C = C(Y, T_o)$$

These equations can be reduced to a single equation

$$Y = C(Y, T_o) + I_o + G_o.$$

In this form of equilibrium condition where an explicit solution is not possible, partial derivative will be ineffective in its application. Instead, total differential will be more appropriate in resolving economic problem of this kind.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain what differentials is all about
- apply the use of differentials in resolving economic issues
- distinguish between total and partial differentiation.

3.0 MAIN CONTENT

3.1 Differentials and Derivatives

At the start of module one precisely in unit one, derivative *dy/dx* was expressed as a single notation representing the limit of $\Delta y/\Delta x$ as Δx approaches zero. The derivative *dy/dx* can as well be expressed as a ratio of differentials whereby *dy* is the differential of *y* and *dx* the differential of *x*. supposing we have a model of one exogenous parameter/variable $y = f(x)$, the differential of *y* is *dy*, and it measures the rate of change in *y*as a result of a small change in *x*, which is denoted by *dx*.

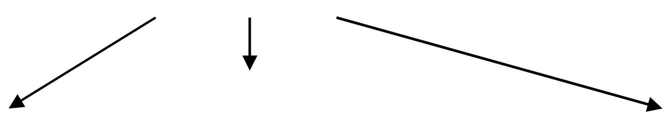
Differentials as a mathematical method as the some fundamental characteristics that would help our understanding of its operations. (i) In differentials, while *dy* is an endogenous variable, *dx* is exogenous. In this instance, *dy* is dependent on *x* and *dx*. This is because the variable *x* has changed location. (ii) Since *dy* is dependent on *dx*, it therefore means that if $dx = 0$, then $dy = 0$. But, where $dx \neq 0$, we then divide *dy* by *dx* to obtain *f'(x)* or multiply *dx* by *f'(x)* to have *dy*. (iii) the differential *dy* can only be expressed in terms of other differential(s) like *dx*.

Now, assuming we have the equation $q = 4p^2 + 11p + 4$, the differential of *q* will be obtained by first differentiating *q* w. r. t*p*, this measures the rate at which variable *q* is influenced by a small change in the variable *p*.

$$\frac{dq}{dp} = 8p + 11$$

Then multiply this outcome by a specific change in *p* (*dp*) to get the actual change in *q*(*dq*).

$dq = (8p + 11)\, dp$ a differential or simple change



Change in *q* = rate at which *q* changes for a small Δ in $p \times$a small change in *p*.

Let us see a worked example.

Example 1: Assuming we have the function $q = 5p^3 + 4p^2 + 20$, find the differentials?

Solution

To find the differentials, note that, it is the derivative of *q* w. r. t *p* multiply by the little change in *p*.

Therefore,

$$\frac{dq}{dp} = 15p^2 + 8p \qquad \text{derivative of } q \text{ w. r. t } p,$$

Then multiply it with the little change in *p* denoted by *dp*, and we have

$$\frac{dq}{dp} = dq = (15p^2 + 8p)\ dp.$$

SELF-ASSESSMENT EXERCISE

Find the differential of the following functions:

i. $q = (8p - 30)^4$

ii. $q = \dfrac{18p - 4}{10p}$

iii. $q = (6p + 3)(5p - 12)$

3.2 Total and Partial Differentials

Total differentials as a concept can be discussed in reference to a function of two or more exogenous variables. Total differentials measure the rate of change in the endogenous variable occasioned by a little/small change in the individual exogenous variables. Supposing we have an economic model of the form Q = $g(K, L)$, that is a production function, where Q is output, K is capital, L is labour. We can measure the influence of K or L on Q which is the endogenous variable individually by a process of partial derivative ($\partial Q/\partial K$ or $\partial Q/\partial L$) (see unit 2). This gives the marginal impact of the variables K and L on the endogenous variable Q. For any change in any of the two exogenous variables, the resulting change in Q can be estimated as $(\partial Q/\partial K)dK$, which is the same as $dq = g'(k)\ dk$. The same goes for the variable L. therefore; the total influence of the two exogenous variables on Q is their differentials, and is mathematically expressed thus:

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL$$

Or, state differently as

$$dQ = Q_K dK + Q_L dL$$

Where Q_K and Q_L are the partial derivatives of Q with respect to (w. r. t)K and L respectively, and d_K and d_L are small changes in K and L. The partial derivatives have played the role of a converter which helped in arriving at the total change in Q occasioned by the individual change in the two exogenous variables. In sum, the total differentials can thus be determined by partially differentiating the function w. r. t the individual exogenous variable and substituting these values in the objective function.

Example 2: Determine the total differential for production functions below:

(i) $Q(z_1, z_2) = xz_1 + yz_2$ and (ii)$Q(z_1, z_2) = z_1^2 + z_2^3 + z_1 z_2$. Note, x, $y$are greater than zero

Solution

To find the total differential, take the partial derivative of the individual exogenous variables, multiply it with little/small change in the individual left-side variables, and then sum the outcomes thus,

(i) $\frac{\partial Q}{\partial z_1} = Q_1 = x$ and $\frac{\partial Q}{\partial z_2} = Q_2 = y$

Therefore,

$$dQ = Q_1 dz_1 + Q_2 dz_2 = xdz_1 + ydz_2.$$

(ii) $\frac{\partial Q}{\partial z_1} = Q_1 = 2z_1 + z_2$ again, $\frac{\partial Q}{\partial z_2} = Q_2 = 3z_2^2 + z_1$

Therefore,

$$dQ = Q_1 dz_1 + Q_2 dz_2 = (2z_1 + z_2)dz_1 + (3z_2^2 + z_1)dz_2.$$

However, if one of the right hand variables in the production function earlier treated is held constant (for instance, $dL = 0$), we then have a partial differential:

$$dQ = Q_K dK.$$

By definition, a partial differential measures the rate of influence on the endogenous variable for a two or multivariable function occasioned by a small change in one of the exogenous variables, assuming other right-hand variables are held constant.

SELF-ASSESSMENT EXERCISE

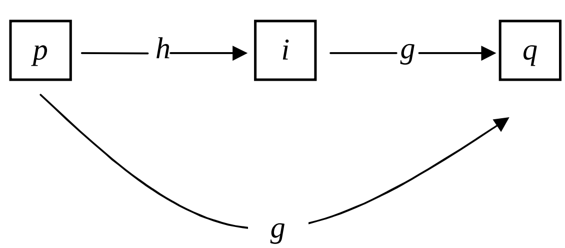
Find the total differentials of the followings:

i. $q = 7p^2 + pi - 3i^3$
ii. $z = (2x - y)/(x + 2)$

3.3 Total Derivatives

We have discussed instances of partial derivative, partial differential, and total differential. Individually, these term mean different things in mathematical operations though, their pronunciations may sound similar. For total derivative, its modules operandi is not in any form similar to that of total differential. Let us assume we have a model or function where $q = g(p, i)$ and $i= h(p)$, that is, when p and i are not exogenous, a change in p will influence q directly through the function g and indirectly through the function h. This process is diagrammatically represented below for better understanding. To measure the influence of a change in p on q when p and i are not exogenous, the total derivative must be determined. The total derivative measures the direct effect of p on q, plus the indirect influence of p on q through i, that is, $\frac{\partial q}{\partial i} * \frac{di}{dp}$. In a nutshell, the total derivative is:

$$\frac{dq}{dp} = q_p + q_i \frac{di}{dp}$$



A channel map showing the directions of impacts between variables

Example 3: Find the total derivative *dq/dp*, given the function $q = f(p, i) = 12p^3 + 14i$

Where $i= g(p) = 8p^2 + 6p + 16$.

Solution

We will apply the total derivative methodology to resolve this problem.

Therefore, the total derivative of *dq/dp* with respect to *p* is

$$\frac{dq}{dp} = q_p + q_i \frac{di}{dp}$$

Where $q_p = 36p^2$, $q_i = 14$, *and* $di/dp = 16p + 6$. Substitute all these into the model above, we have,

$$\frac{dq}{dp} - 36p^2 + 14(16p + 6) = 36p^2 + 224p + 84$$

As a check, we substitute the function *g* into the function *f*, to get

$$q = 12p^3 + 14(8p^2 + 6p + 16) = 12p^3 + 112p^2 + 84p + 224$$

Thus: $\frac{dq}{dp} = 36p^2 + 224p + 84.$

SELF-ASSESSMENT EXERCISE

Find the total derivative of *dz/dp* for these functions:

i. $z = 12p^2+30pi+3i^2$ where $i = 14p^2$
ii. $z = (11p-16i)^2$ where $i = p+6$

3.4 Derivatives of Implicit and Inverse Functions

Recall that in the beginning of this module, we treated the explicit function in which the endogenous variable is on the left hand side of the equal sign, and the exogenous variable is situated on the right hand side. For instance, the function or equation of the form $q = f(p)$ express *q* explicitly in terms of *p*. However, as we advanced in the module often, we came across implicit functions, in which the variables including the constant term are all on the left hand side of the equal to sign (=). Consider the function of the form $f(p, q) = 0$. If an implicit function $f(p, q) = 0$ exists and $f_p \neq 0$ at the point around which the implicit function is defined, the total differential is simply $f_p dp + f_q\, dq = 0$.

We have stated earlier that a derivative is a ratio of differentials. Having said that, we can then rearrange the total differentials to get the implicit function rule:

$$\frac{dq}{dp} = \frac{-f_p}{f_q}$$

Notice that the derivative *dq/dp* is the negative of the reciprocal of the corresponding partials.

$$\frac{dq}{dp} = \frac{-f_p}{f_q} = -\frac{1}{f_q/f_p}$$

If we have the function $q = f(p)$, the inverse function will be $p = f^{-1}(q)$,if individual value of *q* produces one and only one value of *p*. Assuming we have an inverse function, *the inverse function rule states that the derivative of the inverse function is the reciprocal of the derivative of the original function.* So, if $Q = f(P)$ is the original function, the derivative of the original function is *dQ/dP*, the derivative of the inverse function $[P = f^{-1}(Q)]$ is *dP/dQ*, and

$$\frac{dP}{dQ} = \frac{1}{dQ/dP} \quad \text{only if} \frac{dQ}{dP} \neq 0.$$

Example 4: Find the derivative of the implicit function: $14x^2 - 2y = 0$

Solution

Remember the implicit rule of derivative. Therefore, the derivative of the function is found as follow:

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Here $f_x = 28x$ and $f_y = -2$. Substitute this in the model above,

$$\frac{dy}{dx} = -\frac{28x}{(-2)} = 14x.$$

Example 4: Find the derivative for the inverse of the function $Q = 80 - 5P$.

Solution

Know that the derivative of an inverse function is the reciprocal of the derivative.

Therefore,

$$\frac{dP}{dQ} = \frac{1}{dQ/dP}$$

Where $dQ/dP = -5$. Thus,

$$\frac{dP}{dQ} = \frac{1}{-5} = -\frac{1}{5}$$

4.0 CONCLUSION

In this unit, we have seen that differentials are not derivatives. While derivative is the ratio of differentials, differential is the differential of the variables. For example, *dy* is the differential of *y* and *dx* is the differential of *x*. However, in a situation where an explicit solution is not possible, partial derivative will be ineffective in its application. Differential becomes more appropriate in resolving such economic problem. In sum, differential is the mathematical method that assists economists to resolve economic issues with implicit form.

5.0 SUMMARY

Thus far in this section, we have treated the following:

- Differentials and derivatives. While derivatives are basically employed to tackle functions, models or equations that are largely explicit in form, differentials work where derivative is less effective.
- The derivative of implicit and inverse functions. Where the derivative of the implicit function (*dq/dp*) is the negative of the reciprocal of the corresponding partials, the derivative of the inverse function $(p = f^{-1}(Q))$ is the reciprocal of the derivative of the original function.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the derivative of the inverse function *dx/dy*. Where $y = 1000 – 3x^2$
2. Find the derivative of the function $f(x, y) =4x^2 + 3xy + 6y^3$, using implicit rule.

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MODULE 3 LINEAR ALGEBRA

UNIT 1 MATRIX

CONTENTS

1.0 INTRODUCTION

In the just concluded module, that is module two, which centered on organisation, we have learnt that organisation is about the quest for the best among the available few. Organisation is basically of two categories, the constrained and the unconstrained. The former is about factoring what is generally referred to as the limiting factors into the organisation model in order to arrive at the solution. While the latter looks at organisation with the view that the quest for the best can be gotten without recourse to any limiting factor. This category of organisation is not practicable in the real sense of life. We have seen organisation in terms of two or more variables. We have also discussed Lagrange multiplier as a tool that is always applied to resolving issues which has to do with constrained organisation.

In this part of the module, we shall be discussing matrix algebra. A lot of economic models used in analysing economic issues are derived from mathematical models that ultimately involve a system of several equations. If eventually these equations are all linear, the study of such systems of equations belongs to an area of mathematics called linear (or matrix) algebra. Also in this part of the module, we shall be treating topics like matrix and vector, matrix operations, matrix inversion, and many more.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain what linear equation is all about
- define certain matrix terms such as rows, columns, etc.
- state the laws in matrix.

3.0 MAIN CONTENT

3.1 Systems of Equations (Linear)

Perhaps in your elementary mathematics or in your mathematics for economists I, you might have been introduced to systems of two simultaneous linear equations in two variables. For the purpose of refreshing your memory, here is an example of such a system, whereby the two unknowns are designated by x_1 and x_2,

$$2x + 4y = 4$$
$$4x - 2y = 8$$

A solution to this system is a pair of numbers (x, y) which satisfies both equations. One known technique of resolving this form of mathematical issue is by *elimination*. Take on the first equation, and make x the subject of the equation. Then we have $x = - 2y + 2$. Then insert x into the second equation, and we have $4(- 2y + 2) - 2y = 8$. By way of simple mathematical manipulation, we shall have $y = 0$, and then $x_1 = - 2(0) + 2 = 2$. The only solution is therefore $(x, y) = (2, 0)$.

In sum, the notation for any system which has two linear equations with unknown parameters x and y is always in the form below:

$$b_{11}x + b_{12}y = c_1$$

$$b_{21}x + b_{22}y = c_2$$

What we have seen is a case of two equations with two unknown parameters. However, there are instances in matrix operations as we shall soon see in which we consider a large number of equations with unknown parameters or variables, and then we shall be looking at a notation that will be suitable for that instance. In such instance, we shall be exploring a linear system of equations with m equations and n unknowns. In such a system of equations, the m is always larger than, equal to, or less than n. Assuming we have a case where the unknowns are represented by $x_1 \ldots\ldots x_n$, such a system of equation will be denoted thus:

$$b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n = c_1$$

$$b_{12}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n = c_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_{m1}x_1 + b_{m2}x_2 + \ldots + b_{mn}x_n = c_m$$

In this elaborate notation, b_{11}, b_{22}…., b_{mn} are referred to as coefficients, while c_1….c_m are known as the right-hand sides, and they are always real numbers or constants. It is very important we notice the subscripts used in the notation. Universally, b_{ij} is the coefficient in the *i*th equation of the *j*th parameters/variables (*i* stands for the row while *j* stands the column). At times, most of these coefficients are always zero (0). Meanwhile, as we progress in this study, the subscripts b_{ij} or b_{mn} will be better explained in matrix terminology.

3.2 Definitions and Terms

We have mentioned in section 3.1 of this unit that *matrix* is another name for linear algebra. In the real sense, what is matrix and its essence in mathematical sciences and economics? By simple definition, a *matrix* is a rectangular array of numbers, parameters, or variables, each *m* of which is placed orderly in the matrix. The numbers (parameters or variables) are called *elements* of the matrix. These elements (numbers) are orderly placed in rows and in columns (that is *m* and *n* as seen in section 3.1). The elements orderly placed horizontally are referred to as *row elements*; and the ones placed vertically are known as *column elements*. The number of rows *m* and columns *n* shows the scopes of the matrix ($m \times n$), this is pronounced *m by n*, meaning *m* rows by *n* columns. It is vital to note that in matrix, the row number always come before the column number. That is what we have seen by the term *m* by *n* matrix.

However, there are instances where the rows and the columns are the same (that is equal). A *square matrix* is a good example of such a case. Here (square matrix) the number of rows equal the number of columns (that is, $m = n$). Another instance in matrix is a case of single column matrix where the dimension is such that it is *m* by *1*. In this case, it is termed a *column vector*. Also, we can have a case of a single row, with *1* by *n* dimension; this is a case of *row vector*. There are times we need to make some manipulations in mathematics. Supposing in a matrix, we decide to make some manipulations by way of transforming the rows to columns and the columns to rows, what we have just done is known as *transpose matrix*. Let us see one or two matrix examples.

Examples 1: Given

$$A = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 4 & 5 \end{bmatrix}$$

3×3 2×3 3×1 1×3

The matrices as shown above have given us some scopes or dimensions of matrices to be exposed to, even larger ones not included in the scenario. Matrix A is more or less a special one with *3by3* dimensions. By this, the matrix is thus a square matrix (see the definition above).You will notice that all the elements in the matrix are arranged orderly in horizontal and vertical dimensions having subscripts attached which gives the address or placement of the individual element. The subscripts are the $m by n$ earlier discussed. They tell us the row and column an element is placed. For instance, b_{32} is an element which is placed in row 3, and column 2; b_{13} is an element sited in row 1, and column 3.

In the case of matrix B, its dimension is *2 by 3*. That is, it has two rows and three columns. The matrix has real numbers as it elements; therefore the site of the numbers can easily be ascertained using the explanations above. For example, its b_{23} element is 7; its b_{11} element is 1, and many more. The matrices C and D are column and row vectors with dimensions of *3 by 1* and *1 by 3* respectively. Notice that, the composition or the numbers of element in a matrix can be determined via the dimension of the matrix. A matrix of *2 by 2* has a composition of 4 elements or the numbers of element in that matrix are 4.

Recall that, we have mentioned that a matrix is transposed when the rows are transformed into columns, and columns into rows. Let us see practical instance using matrix B that has a *2 by 3* dimension. The notation for transpose matrix in the case of matrix B is written B' (or B^T). The transpose of matrix B as given above is:

$$B' = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix} 3\times 2$$

In this situation (that is B'), we can see that the dimension of the matrix has changed upon the transformation that took place. Instead of the initial *2 by 3*, what we now have is a matrix of *3 by 2* dimensions. Also, the positions of the elements have changed except for b_{11} that still maintained it initial site. Apart from a square matrix, the dimension of a matrix changes anytime the matrix is transposed.

SELF-ASSESSMENT EXERCISE

i. What is matrix, and how is it different from a simple linear equation?
ii. On your own, design a five by four matrix, transpose it and state its dimension.

3.2.1 Linear Algebra: Its Roles

Recall we have stated before now that, we shall be discussing the importance of matrix or Linear algebra in mathematical studies. One can reliably or confidently state that, matrix operation is a higher version of simple linear equation. As a result of this, matrix as a part of mathematics fills the following gaps in mathematical studies:

a) It allows complicated system of equations expressed in a form that is understandable. That is, equations that are more than two or three system of equations which look clumsy and difficult can be made simplified with matrix.[ ]
b) With matrix or linear algebra, one can easily determine if a particular mathematical problem is resolvable before it is attempted. Like the case where we have numerous systems of equations to solve could appear not solvable. But with the understanding of matrix, one could determine if the situation is solvable.
c) Matrix provides a way by which systems of equations are dealt with. It can only be applied to systems of linear equations. In most case, many relationships in economics are basically linear equations and where they are not, can be easily transformed into a form that is linear.

A good instance where matrix is applicable is a situation where a dealer in all forms of airtime or recharge cards has a lot of shops, say 5 where he/she sells. He/she can have a concise stock keeping of the items in those shops by means of matrix operation. See details below:

SHOPS	GLO	MTN	AIRTEL	ETISALAT
A	100	150	80	200
B	250	300	100	150
C	234	200	120	220
D	350	270	300	400
E	450	250	275	356

From the matrix above, the dealer can have an ideal of all the stock of each network airtime he/she has in the five shops. For instance, across a row of the matrix, the dealer can ascertain the level of stock of each item in a particular shop. Then down and up of a column in the matrix, gives the total stock level of any item in the five shops.

3.3 Matrix and Vector Operations

You will recall that matrix is about the arrangement of numbers horizontally and vertically (that is, in rows and in column). Vectors are special forms of matrix that has its feature direction and magnitude. As a result, vectors are suitable for the application of all the algebraic tasks already discussed. There are two forms of vector matrix. These are the row and the column vectors. The former is one in which the numbers are arranged laterally. While the latter, is the arrangement of numbers in a matrix in vertical order.

3.3.1 Multiplication of Vectors

An $m \times 1$ column vector *a*, and a $1 \times n$ row vector *b*, yield a product matrix *ab* of dimension $m \times n$.

For example, supposing that:

$a = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$, going by the principle of multiplication, we have matrix *a* times matrix *b*. we therefore have,

$$ab = \begin{bmatrix} 5(2) & 5(4) & 5(6) \\ 2(2) & 2(4) & 2(6) \end{bmatrix} = \begin{bmatrix} 10 & 20 & 30 \\ 4 & 8 & 12 \end{bmatrix}$$

Now, the dimension of outcome *ab* is a *2 by 3* matrix. Remember that matrix *a* is a*2 by 1*, while matrix *b*is a *1 by 3* matrix. When we compare these matrices dimensions with the outcome's dimension, we can see a sort of resemblance in the dimensions. In matrix multiplication, the dimension/scope/magnitude of the resultant matrix is a combination of the rows *(m)* and columns *(n)* of the individual matrix. Notice that in vector operation, a *1 by n* row vector *a* and an *n by 1* column vector *b*, the product *ab* will produce a matrix of dimension *1 by 1*.

It is important to note that for matrices to be conformable, the number of columns in the *lead matrix* (matrix that comes before the other matrix in any matrix operation, e.g. Matrix*a* as in the example above) must be equal to the number of rows in the *lag matrix* (a matrix that comes after the lead matrix, e.g. matrix *b*).

Supposing we have

a = $\begin{bmatrix} 5 & 2 \end{bmatrix}$ and b = $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$, by matrix multiplication, the resultant matrix will be

ab = $[5(6) + \quad 2(9)] = [48]$. The scope of this resultant matrix is *1 by 1* as already indicated. This outcome (that is, the *1 by1*), is a good example of a scalar matrix that has just magnitude only (matrix *ab* is known as singleton).

SELF-ASSESSMENT EXERCISE

i. What are vector matrices?
ii. Distinguished between a vector matrix and a scalar matrix.

3.4 Laws in Matrix

In matrix, there are laws that guide its operations. These are universally referred to as laws in matrix. These laws are commutative, associative, and distributive in nature. Let us understand how these laws work. It is important to note that both multiplication and addition in matrix are done in line with commutative, associative, and distributive laws. Firstly, with commutative law, matrix addition is ($a + b = b + a$). However, since the addition is merely the summation of the corresponding elements of the matrices involved, the order of their summation is immaterial. Still discussing commutative law, matrix multiplication with vector is not commutative (that is, $ab \neq ba$), just with few exemptions. But, in case of scalar multiplication, the law (i.e. commutative) is followed (that is, $cd = dc$). Secondly, in the case of associative law, matrix addition is $(a+b) + c = a + (b+c)$. In the case of multiplication, the law(that is associative) is applied only when the matrices are in order of conformability. That is $(xy)\, z = x\, (yz)$. Lastly, as we have seen in the laws already discussed, in the case of multiplication, the distributive law is $x(y + z) = xy + xz$.

However, it is worthy of note to state that, matrix subtraction is also cumulative and associative. This is so because of the convertibility principle involved. That is, matrix subtraction (a – b) can be transformed to matrix addition a + (-b).

Let us consider some worked examples applying those laws.

Example 1: Given that:

$$a = \begin{bmatrix} 11 & 4 \\ 3 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 8 & 2 \\ 1 & 3 \end{bmatrix}$$

Show that the matrix addition and subtraction are commutative by (i) a + b = b + a, and (ii) a – b = - b + a.

Solution: Note the calculations are done in conformity with law of commutative. See workings below.

(i) a + b $\begin{bmatrix} 11+8 & 4+2 \\ 3+1 & 5+3 \end{bmatrix}$ = $\begin{bmatrix} 19 & 6 \\ 4 & 8 \end{bmatrix}$ = b + a = $\begin{bmatrix} 8+11 & 2+4 \\ 1+3 & 3+5 \end{bmatrix}$ =

$\begin{bmatrix} 19 & 6 \\ 4 & 8 \end{bmatrix}$

(ii) a – b $\begin{bmatrix} 11-8 & 4-2 \\ 3-1 & 5-3 \end{bmatrix}$ = $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ = - b + a = $\begin{bmatrix} -8+11 & -2+4 \\ -1+3 & -3+5 \end{bmatrix}$

= $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$

From the workings above, the matrix addition and subtraction in line with the commutative law as discussed earlier has been shown using numbers.

Again, we want to show that the multiplication operation in matrix is not commutative as stated at the beginning.

Example 2: Supposing we have:

a = $\begin{bmatrix} 9 & 10 & 11 \\ 5 & 7 & 14 \end{bmatrix}$ b = $\begin{bmatrix} 2 & 1 \\ 4 & 6 \\ 3 & 8 \end{bmatrix}$. The dimensions of the two matrices are very vital in matrix multiplication. For matrix a, it is a *2by3*, while matrix b is *3by 2*.

Solution

It is always advisable to check if the matrices involved are conformable. In this case the two matrices are conformable in that, the numbers of columns in matrix *a* is equal to the numbers of rows in matrix *b*, therefore:

ab = $\begin{bmatrix} 9(2)+10(4)+11(3) & 9(1)+10(6)+11(8) \\ 5(2)+7(4)+14(3) & 5(1)+7(6)+14(8) \end{bmatrix}$ = $\begin{bmatrix} 92 & 157 \\ 66 & 159 \end{bmatrix}$

2by2 *2by2*

ba = $\begin{bmatrix} 2(9)+1(5) & 2(10)+1(7) & 2(11)+1(14) \\ 4(9)+6(5) & 4(10)+6(7) & 4(11)+6(14) \\ 3(9)+8(5) & 3(10)+8(7) & 3(11)+8(14) \end{bmatrix}$ = $\begin{bmatrix} 23 & 27 & 36 \\ 66 & 82 & 128 \\ 58 & 56 & 137 \end{bmatrix}$

3by3 *3by3*

Considering these outcomes, we have been able to confirm that $ab \neq ba$. In multiplication operations, there is what we refer to as *pre-multiply* and *post-multiply*. In this instance (*ab*), the matrix *b* is pre-multiplied by matrix *a*, while matrix *a* is post-multiplied by matrix *b*.

We have said concerning the associative law of matrix that, in multiplication operation, it is applicable if the matrices order is in conformity. See below the applicability of the law with numeric example.

Example 3: Given that we have,

$$x = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 6 & 4 \end{bmatrix} \quad y = \begin{bmatrix} 2 & 13 & 4 \\ 3 & 7 & 9 \end{bmatrix} \quad z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3by2 *2by3* *3by1*

Solution

$$xy = \begin{bmatrix} 1(2)+5(3) & 1(13)+5(7) & 1(4)+5(9) \\ 3(2)+2(3) & 3(13)+2(7) & 3(4)+2(9) \\ 6(2)+4(3) & 6(13)+4(7) & 6(4)+4(9) \end{bmatrix} = \begin{bmatrix} 17 & 48 & 49 \\ 12 & 53 & 30 \\ 24 & 106 & 60 \end{bmatrix}$$

$$(xy)\,z = \begin{bmatrix} 17 & 48 & 49 \\ 12 & 53 & 30 \\ 24 & 106 & 60 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 17(1)+48(2)+49(3) \\ 12(1)+53(2)+30(3) \\ 24(1)+106(2)+60(3) \end{bmatrix} = \begin{bmatrix} 260 \\ 208 \\ 416 \end{bmatrix}$$

$$yz = \begin{bmatrix} 2(1)+13(2)+4(3) \\ 3(1)+7(2)+9(3) \end{bmatrix} = \begin{bmatrix} 40 \\ 44 \end{bmatrix}$$

$$x\,(yz) = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 40 \\ 44 \end{bmatrix} = \begin{bmatrix} 1(40)+5(44) \\ 3(40)+2(44) \\ 6(40)+4(44) \end{bmatrix} = \begin{bmatrix} 260 \\ 208 \\ 416 \end{bmatrix}$$

From the calculation above, it therefore holds that the associative law in the case of multiplication is true, if order of the matrices conforms.

SELF-ASSESSMENT EXERCISE

i. Determine (i) a – b and (ii) – b + a if

$$a = \begin{bmatrix} 5 & 8 \\ 4 & 6 \\ 9 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 12 & 2 \\ 3 & 11 \\ 6 & 9 \end{bmatrix}$$

ii. Find (i) *xy* and (ii) *yx* if

$x = \begin{bmatrix} 7 & 10 & 13 \end{bmatrix}$ $y = \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}$

4.0 CONCLUSION

At the start of this part, we defined matrix as an array of numbers in rows and in columns. We further stated that, these numbers are located in a particular position in the matrix. This location can be found by simply tracing the rows and the columns in the matrix. This shows that the placement or location of number in a matrix is very vital in matrix operation.

5.0 SUMMARY

In this module, which is basically about linear algebra, we started by treating matrix as one of the units that forms part of the entire module. In sum, in the unit we have considered the following:

- Systems of linear equations. We discussed in this part simultaneous equation with two models or functions, and how issues in it could be resolved by the method of elimination. However, there are instances where the equations are more than two, and three; in that case, matrix algebra comes handy as a way out.
- We have defined matrix and some basic terms in line with matrix operations. Terms such as square matrix, rows and columns, transpose matrix, and a lot more.
- Also, we discussed the roles of matrix in economic studies. We emphasized that matrix amongst others helps economists to simplify complicated systems of equations.
- We equally considered the laws in matrix operations. These laws are commutative, associative and distributive in nature. And we also have discussed their applicability to economic issues.

6.0 TUTOR-MARKED ASSIGNMENT

1. Discuss the *rows* and *column* in matrix. What *roles* do they perform in matrix operation?
2. What does *dimension/scope/size* of matrix mean in matrix operation?
3. In a *four by three* matrix, how many columns and rows are there in the matrix, and what does this say about the matrix?
4. Design a *3by2* matrix on your own using numbers, find the *transpose* of the matrix, and state its *dimension*.

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UNIT 2 MATRIX OPERATIONS

CONTENTS

1.0 INTRODUCTION

We rounded up the preceding unit, by considering the laws in matrix. These laws are commutative, associative and distributive in nature. These laws guide whatever any mathematical operation that needs to be done in matrix operation. We have treated some worked examples in line with these laws. In this unit, we shall continue our discussion looking at more mathematical explorations under matrix operations. Under this topic, we shall be considering sub-topics such as linear dependence, scalar and vector operations, transpose of matrix, and many more.

You will recall that, we had earlier defined matrix as a rectangular array of numbers in rows and in columns. That a matrix will always have an *m-by-n*(also written as $m \times n$) order or dimension. Recall also that, in matrix operations, *m* stands for the number of rows a particular matrix is having, while *n* indicates the number of columns in that same matrix. For instance, a *3-by-2* (3×2) order matrix has three rows and two columns. With the order or dimension of matrix, we can compare two or more matrices to determine if they are equal or not. This brings us to equality of matrices.

Equality of matrices states that, two or more matrices are equal if they possess the same order/dimension and if all their corresponding elements are equal. That is if we have two matrices X and Y, and their orders are *3- by- 3* (i.e. $X_{3\times3}$and $Y_{3\times3}$), it then means that matrix X is equal to matrix Y(X = Y).

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- discuss linear dependence amongst matrices

- explain and conduct operations as regard transpose of a matrix
- distinguish between identity matrix and null matrix.

3.0 MAIN CONTENT

3.1 Linear Dependence

In matrix operation, a set of vectors x_1 to $x_n$can be said to be *linearly dependent* if and only if any of the set of vectors can be expressed as a linear combination of the other vectors; if not, the set of vectors will be linearly independent. In other words, if two or more vectors (either row or column) are examined mathematically to be equal to a particular vector matrix, it means that there is linear dependence among the set of vectors. See the example below:

Supposing we have three vector matrices *x*, *y* and *z*, where one say *z* is a linear combination of *x* and *y*. if:

$x = \begin{bmatrix}4\\8\end{bmatrix}$, $y = \begin{bmatrix}3\\7\end{bmatrix}$, and $z=\begin{bmatrix}5\\9\end{bmatrix}$. Matrix *z* is linearly dependent on the remaining two matrices *x* and *y*. I know for sure someone will ask how:

$2x – y = \begin{bmatrix}8\\16\end{bmatrix} - \begin{bmatrix}3\\7\end{bmatrix} = \begin{bmatrix}5\\9\end{bmatrix} = z$. to further verify our result, we can again work on these matrices to have a zero or null matrix, which shows that these matrices are linearly dependent. How?

$2x – y – z = 0$ that is $\begin{bmatrix}0\\0\end{bmatrix}$. This is a null or zero vector matrix.

Also, we can have row vector matrices say two that can be linearly dependent. Assuming,

$a = \begin{bmatrix}2 & 5\end{bmatrix}$ and $b = \begin{bmatrix}8 & 20\end{bmatrix}$, these vector matrices are linearly dependent because

$4a = 4\begin{bmatrix}2 & 5\end{bmatrix} = \begin{bmatrix}8 & 20\end{bmatrix} = b$.

The concept of linear dependence has a simple interpretation. Two vectors *a* and *4a* –one being a multiple of the other –are obviously dependent. When more than two vectors in the 2-space are considered, there emerges this significance conclusion: once we have found two linearly independent vectors in the 2-space (say, *x* and *y*) all the other vectors in the space will be expressed as a linear combination of these (*x* and *y*). Furthermore, by extending, shortening, and reversing the given vectors *x* and *y* and then combining these into various parallelograms, we can generate a vast number of new vectors, which will exhaust the set of all 2-vectors. Because of this, any set of three or more 2-vectors (three or more vectors in a 2-space) must be linearly dependent. Two of them can be independent, but then the third must be a linear combination of the first two.

3.2 Addition and Subtraction of Matrices

Addition (subtraction) of two matrices *x* + *y* (*x* – *y*) entails that the matrices be of equal dimensions or order. For instance two or three matrices to be added or subtracted must be uniform in their scope (i.e. $X_{3\times3}$, $Y_{3\times3}$, etc.). In this case, the matrices have equal numbers of rows and columns. Therefore, this enables each entry of one matrix to be added to (subtracted from) the corresponding element of the other matrix. In the worked example below, x_{11} in X will be added to (subtracted from) y_{11} in Y; x_{12} to y_{12}; etc.

Example 1: Given matrices *x* and *y* below, determine the sum of matrices *x* and *y*.

Solution: Ensure that the matrices are of the same dimensions or order.

$$x = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 5 & 9 \\ 3 & 6 & 10 \end{bmatrix} \quad y = \begin{bmatrix} 2 & 6 & 8 \\ 5 & 3 & 4 \\ 7 & 2 & 1 \end{bmatrix} \text{ then } x+y = \begin{bmatrix} 1+2 & 4+6 & 8+8 \\ 2+5 & 5+3 & 9+4 \\ 3+7 & 6+2 & 10+1 \end{bmatrix} =$$

3by3 *3by3* *3by3*

$$\begin{bmatrix} 3 & 10 & 16 \\ 7 & 8 & 13 \\ 10 & 8 & 11 \end{bmatrix}$$

3by3

Example 2: Given two matrices *k* and *j* of the same order, the entries of the two matrices can be subtracted thus:

$$k = \begin{bmatrix} 5 & 8 & 2 \\ 3 & 4 & 6 \end{bmatrix} \quad j = \begin{bmatrix} 2 & 7 & 4 \\ 1 & 3 & 8 \end{bmatrix} \text{ then } k\text{-}j = \begin{bmatrix} 5-2 & 8-7 & 2-4 \\ 3-1 & 4-3 & 6-8 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & -2 \end{bmatrix}$$

Note that in matrix operations for addition and subtraction, the order/dimension of the matrices involved, and the place of the elements in the matrices are important.

3.3 Matrices Multiplication

In matrix operations, the multiplication of two or more matrices with dimensions $(mbyn)_1$, $(mbyn)_2$ . . . , $(mbyn)_n$ necessitates that the matrices concerned be conformable. That is, $n_1 = m_2$, or the number of columns in matrix 1, the lead matrix, equal the number of rows in matrix 2, the lag matrix. The analysis in matrix operations as regard multiplication is simple. Each row vector in the lead matrix is then multiplied by each column vector of the lag matrix, in accordance with the rules for multiplying row and column vectors already discussed in unit one Section 3.3.1. The row-column products, referred to as inner products or dot products, are then used as entries to form the product or resultant matrix, such that every entry of the product of the new matrix is a scalar derived from the multiplication of the *i*th row of the lead matrix and the *j*th column of the lag matrix.

One quick way for test for conformability in matrix operations as regards multiplication, before undertaking any operation, is to place the dimensions/scope in the order in which matrices are to be multiplied, then see if the number of columns in the lead matrix is the same with the number of rows in the lag matrix. If the numbers are equal (i.e. the same), it means the two matrices are conformable. Then, the multiplication can take place as proposed.

For instance if two matrices *x* and *y* of order *4by3* and *3by4* respectively, are to the multiplied. The first thing is to determine the conformability of the matrices. In this case of matrices *x* and *y*, the number of columns of the lead matrix *x* is equal to the number of rows in the lag matrix *y*. Therefore, they are conformable, and are said to be *defined*. However, if we have a case of matrices *x* and *y* having a dimension *3by4* and *2by3*respectively, here, these matrices are not conformable because the lead matrix number of columns is not equal in number to the number of rows in the lag matrix. These forms of matrices are not right for multiplication, and *not defined*. For worked example as regard matrices multiplication, see worked example 3 in unit one section 3.4.

SELF-ASSESSMENT EXERCISE

i. Find the sums of matrix *a* and matrix *b* below:

$$a = \begin{bmatrix} 9 & 4 \\ 2 & 6 \\ 8 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 2 & 3 \\ 3 & 5 \\ 5 & 7 \end{bmatrix}$$

ii. Find the difference between matrix *c* and matrix *d* below:

$$c = \begin{bmatrix} 10 & 2 \\ 2 & -2 \end{bmatrix} \qquad d = \begin{bmatrix} 6 & -3 \\ 2 & -2 \end{bmatrix}$$

iii. Show if *xy* is defined, indicate the scopes of the product matrix, and determine the product of the matrix if possible.

$x = \begin{bmatrix} 4 & 5 \\ 3 & 2 \\ 1 & 4 \end{bmatrix}$ $y = \begin{bmatrix} 1 & 5 \\ 2 & 0 \\ 3 & 6 \end{bmatrix}$

3.4 Identity and Null Matrices

Identity Matrix

An identity matrix is a square matrix with *1s* in its principal diagonal from left to right and *0s* everywhere else. It is designated by the sign *I*, or I_n, where the subscript *n* assists to show its row (as well as column) dimension (*nbyn*). Therefore,

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. However, we can simply represent the two by *I* instead of I_2 and I_3.

This class matrix is very special because of the fact that it plays a role similar to that of the number 1 in scalar algebra. For instance, for any number say *x*, we have $1(x) = x(1) = x$. in the same vain. For any matrix say *Z*, if we multiply by an identity matrix, we have

$IZ = ZI = Z$

Example 1: if $Z = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, estimate *IZ*

Solution

Remember that matrix *I* is an identity matrix and a special form of matrix that has the features of number one in algebra. Therefore,

$$IZ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1(2)+0(2) & 1(3)+0(1) & 1(4)+0(5) \\ 0(2)+1(2) & 0(3)+1(1) & 0(4)+1(5) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix} = Z$$

This has proved that special feature about identity matrix is right as stated. That, any matrix multiply by an identity matrix, equals that matrix.

Identity matrix, because of its special nature, during process of multiplication, it can be *inserted* or *deleted* from the matrices, and the product matrix will not be affected. For instance,

$$\underset{(mxn)}{A}\underset{(nxn)}{I}\underset{(nxp)}{B} = (AI)B = \underset{(mxn)}{A}\underset{(nxp)}{B}$$

This shows that the inclusion or absence of an identity matrix in the operation above did not change product matrix. Observe that dimension conformability it preserved whether or not an identity matrix appears in the product.

Assuming we have Z to be equal to I_n, then we will have:

$$ZI_n = (I_n)^2 = I_n$$

This tells us that an identity matrix squared is equal to itself. Universally written as,

$$(I_n)^k = I_n \qquad (k = 1,2,\ldots,n)$$

This is a case of *idempotent matrix*. A situation where an identity matrix remains the same when multiplied by itself any number of times ($ZZ = Z$).

Null Matrix

Just as seen in matrix operation with identity matrix where it plays the role of the number *1* in algebra, a *null matrix* or *zero matrix* represented by *0*, also plays a role in matrix operations. In this matrix, all elements are*0* unlike that of identity matrix which is *1*. A *null matrix* is simply a matrix whose entries are all zero. Unlike identity matrix, the zero matrix is not a square matrix. Therefore,

$$0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad 0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A square null is idempotent, but a non-square one is not. This is because null matrices obey the rules of operation (subject to conformability) as stated below with regard to addition and multiplication:

$$\underset{(m\times n)}{Z} + \underset{(m\times n)}{0} = \underset{(m\times n)}{0} + \underset{(m\times n)}{Z} = \underset{(m\times n)}{Z}$$

$$\underset{(m\times n)}{Z}\underset{(m\times p)}{0} = \underset{(m\times p)}{0} \quad \text{and} \quad \underset{(q\times m)}{0}\underset{(m\times n)}{Z} = \underset{(q\times m)}{0}$$

Note that, in multiplication, the null matrix to the left of the equals sign and the one to the right may be of different dimensions.

Example 1: $Z + 0 = \begin{bmatrix} 8 & 2 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 5 & 4 \end{bmatrix} = Z$

Example 2$Z_{(2\times 3)}0_{(3\times 1)} = \begin{bmatrix} 2 & 5 & 7 \\ 9 & 11 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0_{(2\times 1)}$

SELF-ASSESSMENT EXERCISE

State the main distinguishing differences between a *null matrix* and an identity matrix.

4.0 CONCLUSION

Basically, this unit is wholly about exploring fully the operations as regards the earlier stated laws in matrix in unit one. We have also looked at some special forms of matrix such as identity and null matrices. Indeed going by what we have discussed concerning these classes of matrices, one can conclude that these matrices are in a class of theirs.

5.0 SUMMARY

In this unit, which centres on matrix operations, we have treated the following:

- We discussed linear dependence. We discovered that certain matrices are somehow a product of the combinations of one or more matrices through a process of some mathematical manipulations.
- We looked at addition and subtraction in matrix. This we did bearing in mind the laws in matrix. We saw that for addition or subtraction to be possible in matrix, the dimensions/orders of the matrices matter. The same applies to multiplication in matrix. The order of the matrices *must* conform before any mathematical computation can start.
- Furthermore, we treated identity and null matrices. That an identity matrix is one in which the diagonal of the matrix from left to right is *1* and zero anywhere else. On the other hand, a null matrix unlike an identity matrix has all the entries to be zeros.

Lastly, we looked at transpose and inverse matrices. In the case of transpose, it is like having the direct opposite of the initial matrix. While an inverse matrix, when multiplied by any matrix results in an identity matrix.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given $x = \begin{bmatrix} 3 & 7 \\ 6 & 5 \end{bmatrix}$, $y = \begin{bmatrix} 1 & 6 \\ 2 & 9 \end{bmatrix}$, and $z = \begin{bmatrix} 4 & 7 \\ 5 & 6 \end{bmatrix}$, verify that

(i) $(x + y) + z = x + (y + z)$
(ii) $(x + y) - z = x + (y - z)$

2. Given $x = \begin{bmatrix} -1 & 4 & 6 \\ 0 & -2 & 5 \end{bmatrix}$, $y = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$, and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$:

3. Determine: a) *XI* b) *IX*c) *IZ*, and d) *ZI*. Where *'I'* is an identity matrix.

7.0 REFERENCES/ FURTHER READING

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UNIT 3 MATRIX INVERSION

CONTENTS

1.0 INTRODUCTION

In unit two, we discussed basically the mathematical operations (addition, subtraction, and multiplication) involved in matrices. We have seen the intricacies involved in matrix operations. For instance, we saw the case of the special forms of matrix (that is, the identity and null matrices) where their inclusion or exclusion from matrix operation make the product matrix unchanged or changed. Still treating linear algebra, we shall continue our discussion by looking at “matrix inversion”.

Matrix inversion in linear algebra is primarily about determinants and Non singularity of matrices. We do know that matrix inversion or inverse matrix is the reciprocal of the matrix in question. And it is (i.e. matrix inversion) required to put mathematical operations right in linear algebra for easy application. For example in economics, some models are required to be formulated by means of matrix inversion before any matrix operation can be fully applied.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define and explain determinants
- discuss what non singularity means in matrix
- explain other terms in matrix such as minor, cofactor, adjoint, and many more
- apply Cramer’s rule to matrix operations.

3.0 MAIN CONTENT

3.1 Determinants and Non Singularity

A determinant is a scalar specially associated with a square matrix. For a given matrix, say *Z*, The determinant of it has a notation $|\boldsymbol{Z}|$. Determinant is only operational when we have a square matrix. The determinant $|\boldsymbol{Z}|$for a *2by2* matrix, which is a square matrix, also refers to as second-order determinant. This is arrived at by multiplying the two entries on the principal diagonal and subtracting from it the product of the two entries off the principal diagonal. Assuming we have a *2by2* matrix

$$\boldsymbol{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

The determinant is $$|\boldsymbol{Z}| = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} = z_{11}z_{22} - z_{12}z_{21}$$

From the calculation above, the matrix determinant is gotten by the product of z_{11} and z_{22}, minus the product of z_{12} and z_{21}. Note that z_{11} and z_{22} elements are *on* the principal diagonal while z_{12} and z_{21} are on the *off* the principal diagonal.

The determinant is a scalar and is found only for square matrices. If the determinant of a matrix is zero, it means that the determinant vanishes and such matrix is referred to as *singular matrix*. By definition, a *singular matrix* is one in which there exists linear dependence between at least two rows or columns. So, if the determinant of matrix is not equal to zero, the matrix is said to be *nonsingular* and all matrix rows and columns are not linearly dependent.

If linear dependence exists in a system of equations, it shows that the system has more than one possible solution, making a unique solution impossible. Hence, our duty is to see that linearly dependence does not arise in our models. To do this, we follow the simple determinant test to finish out potential hitches. Assuming we have a system of equations with coefficient matrix ***Z***, to test for linear dependence of the equation, find the determinant, and observe the followings:

(i) If $|\mathbf{Z}| = 0$, the matrix is singular and there is linear dependence among the equations. Therefore, solution to it is not possible.

(ii) However, If $|\mathbf{Z}| \neq 0$, the matrix is nonsingular and there is no linear dependence among the equations. Therefore, the matrix is solvable through a unique solution.

This brings us to the issue of matrix ranking. For understanding and clarity, the *rank of a matrix* is defined as the highest number of linearly independent rows or columns in a matrix. The rank of a matrix can also be tested for too. To do this, we assume a square matrix of dimension *n*, we therefore observe the followings:

(i) If $\rho(\mathbf{Z}) = n$, **Z** is nonsingular and there is no linear dependence.

(ii) If $\rho(\mathbf{Z}) < n$, **Z** is singular and there is linear dependence.

Example 1: Evaluate the following matrices below, and determine the singularity and non singularity of the matrices:

$$|\boldsymbol{X}| = \begin{bmatrix} 12 & 8 \\ 14 & 18 \end{bmatrix} \qquad |\mathbf{Y}| = \begin{bmatrix} 8 & 12 \\ 12 & 18 \end{bmatrix},$$

Solution: Going be the rules of determinant,

$$|\mathbf{X}| = 12(18) - 8(14) = 104$$

The determinant of the matrix is non-zero, meaning $|\mathbf{X}| \neq 0$. Therefore, the matrix is nonsingular, and there is no existence of linear dependence between any of its rows, or columns.

However,

$$|\mathbf{Y}| = 8(18) - 12(12) = 0$$

In this case, the determinant of matrix Y is zero, indicating $|\mathbf{Y}| = 0$.It therefore shows that matrix **Y** is singular, and there is the existence of linear dependence between its rows and columns.

3.1.1 Determinant of 3rd Order

We have just treated how to find the determinant of a *2by2*matrix, otherwise known as second-order determinant. Now, we want to consider a*3by3* matrix, commonly called '*third-order determinant*.' If we have matrix Y as shown below,

$$\begin{matrix} & + & - & + \end{matrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

It is called a third-order determinant, which is the addition of three products. These three products are derived thus:

(i) Take the first entry of the first row, y_{11}, and mentally erase the rest entries in the row and column in which y_{11} is located. See (1) below. Then multiply y_{11} by the determinant of the remaining entries, which is a *2 by 2* matrix.

(ii) Take the second entry of the first row, y_{12}, again mentally erase the remaining entries in the row and column in which y_{12} is sited. See (2) below. Then multiply y_{12} by -1 times the determinant of the rest entries.

(iii) Finally, take the third entry of the first row, y_{13}, mentally delete as usual the rest entries in the row and column in which y_{13} appears. Again multiply y_{13} by the determinant of the remaining entries.

$$+\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} - \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

(1) (2) (3)

From the outcome we have above the determinant of the matrix Y is estimated thus:

$$|\boldsymbol{Y}| = y_{11}\begin{vmatrix} y_{22} & y_{23} \\ y_{32} & y_{33} \end{vmatrix} + y_{12}(-1)\begin{vmatrix} y_{21} & y_{23} \\ y_{31} & y_{33} \end{vmatrix} + y_{13}\begin{vmatrix} y_{21} & y_{22} \\ y_{31} & y_{32} \end{vmatrix}$$

$$= y_{11}(y_{22}y_{33} - y_{23}y_{32}) - y_{12}(y_{21}y_{33} - y_{23}y_{31}) + y_{13}(y_{21}y_{32} - y_{22}y_{31})$$

= a scalar.

Warning: In calculating the determinant of any matrix of third-order, before commencing the estimation of the products, you are to mentally allocate the signs (+ and -) to the entries on the first row of the main matrix such as you see it done in case of matrix Y as shown above. The same goes for the case of *4by4* and so on. Also, the determinants of *4by4* and *5by5* matrices are the sum of four and five products respectively.

Example 1: Given matrix X, find the determinant

$$\mathbf{X} = \begin{bmatrix} 4 & 1 & 5 \\ 6 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution

This a *3by3* matrix, the warning holds. Remember to mentally allocate the plus (+) and minus (-) signs. Therefore, the determinant is determined thus below:

$$|X| = 4\begin{vmatrix}2 & 3\\3 & 1\end{vmatrix} + 1\,(-1)\begin{vmatrix}6 & 3\\2 & 1\end{vmatrix} + 5\begin{vmatrix}6 & 2\\2 & 3\end{vmatrix}$$

$$= 4[2(1) – 3(3)] – [6(1) – 3(2)] + 5[6(3) – 2(2)]$$

$$= 4(-7) – (0) + 5(14) = 42.$$

The matrix X's determinant is a non-zero number, which indicates that the matrix is nonsingular.

SELF-ASSESSMENT EXERCISE

i. Distinguish between a second-order matrix and a third-order matrix.
ii. What is the rank of a matrix?

3.1.2 Main Properties of Determinants

We can now define and discuss the term with our friends and colleagues. We have equally seen how determinants are estimated from a given matrix. The onus on us now, is to discuss the main properties of determinants. These properties will provide us with the ways in which matrices can be explored to simplify the entries to zero, before determining the determinant:

(i) The interchange of rows and columns does not have implication on the determinant's value. That is, if matrix *Y*is given, the determinant of matrix *Y*is not different from the determinant of matrix *Y* transpose.
(ii) When any two columns or rows are interchanged, the value of the determinant remains the same, except for the signs that will change.
(iii) Multiplying the entries of any column or row by a constant will alter the value of the determinant by the constant.
(iv) Adding (subtracting) any non –zero multiple of one column or row from another column or row, will not affect the value of the determinant.
(v) The determinant of a triangular matrix, that is a matrix with zero entries everywhere above or below the principal diagonal, is the same as the product of the entries on the principal diagonal.
(vi) If in a matrix, the entries of any row or column are zero, the value of that matrix determinant will be zero.

3.2 Minors and Cofactors

In section 3.1 subsection 3.1.1, we discussed that, in finding the determinant of a third-order matrix, the entry in the first row of the matrix as we have in matrix *Y* will be taken, while the rest entries in the row and column where the first entry (y_{11}) as in matrix *Y* above will be deleted. The entries of the matrix remaining after the deletion form a sub-determinant of the matrix often referred to as *minor*. And the first entry y_{11}, will therefore turn M_{11}, $M_{12}$and so on, see detail below. Thus, a minor is the determinant of the sub-matrix formed by deleting the *i*th row and *j*th column of the matrix. Using the matrix *Y* in section 3.1 subsection 3.1.1 as our guide, we have:

$$|\mathbf{M}_{11}| \equiv \begin{vmatrix} y_{22} & y_{23} \\ y_{32} & y_{33} \end{vmatrix} \quad |\mathbf{M}_{12}| \equiv \begin{vmatrix} y_{21} & y_{23} \\ y_{31} & y_{33} \end{vmatrix}$$

$$|\mathbf{M}_{13}| \equiv \begin{vmatrix} y_{21} & y_{22} \\ y_{31} & y_{32} \end{vmatrix}.$$

Observe that, $|\mathbf{M}_{11}|$ is the minor of y_{11}, $|\mathbf{M}_{12}|$ the minor ofy_{12}, and $|\mathbf{M}_{13}|$ the minor ofy_{13}. The determinant is therefore as stated below,

$|\mathbf{Y}| = y_{11}|\mathbf{M}_{11}| + y_{12}(-1)\ |\mathbf{M}_{12}| + y_{13}|\mathbf{M}_{13}| = \quad y_{11}|\mathbf{M}_{11}| - y_{12}|\mathbf{M}_{12}| \quad + y_{13}|\mathbf{M}_{13}|$

A cofactor is a concept that is related to the concept of minor. The notation for a cofactor is$|\mathbf{C}_{ij}|$, and is a minor with a prescribed sign. The rule for the sign of a cofactor is

$$|\mathbf{C}_{ij}| \equiv (-1)^{i+j}|\mathbf{M}_{ij}|$$

Thus if the sum of the subscripts is an even number,$|\mathbf{C}_{ij}| \equiv |\mathbf{M}_{ij}|$, since -1 raised to an even power is positive. If $i + j$ is equal to an odd number,$|\mathbf{C}_{ij}| \equiv -|\mathbf{M}_{ij}|$, since – 1 raised to an odd power is negative.

The cofactors ofC_{11},C_{12}, andC_{13} for the matrix *Y* in Section 3.1 subsection 3.1.1 are as follows:

1) $|\mathbf{C}_{11}| \equiv (-1)^{1+1}|\mathbf{M}_{11}|$

Since $(-1)^{1+1}$= $(-1)^2$ = 1,

$$|\mathbf{C}_{11}| \equiv |\mathbf{M}_{11}| = \begin{vmatrix} y_{22} & y_{23} \\ y_{32} & y_{33} \end{vmatrix}$$

Note the superscript 1+1 above. It means row 1, column 1.

2) $|C_{12}| = (-1)^{1+2}|M_{12}|$

Since $(-1)^{1+2} = (-1)^3$ = -1,

$|C_{12}| = -|M_{12}| = -\begin{vmatrix} y_{21} & y_{23} \\ y_{31} & y_{33} \end{vmatrix}$

Also, the superscript 1+2, means row 1, column 2.

3) $|C_{13}| = (-1)^{1+3}|M_{13}|$

Since $(-1)^{1+3} = (-1)^4$ = 1

$|C_{13}| = |M_{13}| = \begin{vmatrix} y_{21} & y_{22} \\ y_{31} & y_{32} \end{vmatrix}$

The superscript 1+3, again means row 1, column 3.

SELF-ASSESSMENT EXERCISE

Define a cofactor and a minor; state the main difference between the two.

3.3 Cofactor and Adjoint Matrices

In section 3.2, we looked at the concepts of minor and cofactor as they relate to matrix determinant. At the end of the section, we saw how a cofactor was worked from a minor. There is just a tiny line of difference between the two concepts. Still we continue our study on matrix inversion; we shall be looking at cofactor and adjoint matrices. A *cofactor matrix* is a matrix in which every entries x_{ij} is substituted with its cofactor $|C_{ij}|$. If it happens that this matrix (the cofactor matrix) is transposed, whereby the rows are transformed to columns, and the columns to rows, the new matrix so formed is known as *adjoint matrix*. See example below:

$\mathbf{C} = \begin{bmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{bmatrix}$ Adj $\mathbf{X} = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & |C_{31}| \\ |C_{12}| & |C_{22}| & |C_{32}| \\ |C_{13}| & |C_{32}| & |C_{33}| \end{bmatrix}$

Example 1: Given matrix X, estimate the cofactor and the adjoint of the matrix.

$$\mathbf{X} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

Solution

Hints:

In find the cofactor of a matrix, the allocation of the signs is very important. Meanwhile, see the section on minor and cofactor concerning signs determination and allocation. There is a short cut as regards signs distribution in a cofactor matrix. Once the minors (M_{ij}) are determined using the first row of the matrix, and the signs of these minors are known via the cofactor, needless calculating the minors of the remaining rows. The sign already determined will be used. Supposing we used for the first row + - +, the second row will definitely be, - + -, and so on.

Therefore, replacing the elements with their cofactors according to the laws of cofactors,

$$\mathbf{C} = \begin{bmatrix} \begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 14 & -5 & -8 \\ -8 & -2 & 7 \\ -5 & 3 & -2 \end{bmatrix}$$

The adjoint matrix Adj **X** is the transpose of **C** (C^{τ}).

$$\text{Adj } \mathbf{X} = C^{\tau} = \begin{bmatrix} 14 & -8 & -5 \\ -5 & -2 & 3 \\ -8 & 7 & -2 \end{bmatrix}$$

SELF-ASSESSMENT EXERCISE

Structure a third-order matrix; find the minors, the cofactor matrix, and the adjoint matrix.

3.4 Transposes and Inverses Matrices

Recall we had mentioned in the starting of the module, precisely in the first unit a little about matrix transpose, and we equally mentioned that it will be discussed fully in the latter part of this module. When the rows and columns of any matrix are interchanged, such that its first row becomes the first column, and vice versa, the transpose of that matrix is obtained. Universally, the notation for any transposed matrix say A is represented by A' orA^{τ}.

Example 1: Find the transpose of the matrix below:

$$A = \begin{bmatrix} 3 & -5 & 0 \\ -1 & 2 & 4 \end{bmatrix} 2by3$$

Solution

The transpose of this matrix is to transform the rows to columns, and columns to rows. See the transposed matrix below:

$$A^T \text{or} A' = \begin{bmatrix} 3 & -1 \\ -5 & 2 \\ 0 & 4 \end{bmatrix} \textit{3by2}$$

From the solution, the initial *2by3* matrix, after transposed, is now a matrix of dimension *3by2*. This shows that, if a matrix is of the order *mbyn,* after transpose, the matrix must be *nbym*. For a square matrix, the dimensions remain the same.

We have seen how a matrix can be transposed. We shall continue our study in this part by looking at the *inverse of a matrix*. For any given matrix, the transpose is always derivable. In the same vein, the inverse of a matrix also a "derived" matrix, may or may not exist. The inverse of a matrix has a notation represented thus, A^{-1} assuming we are dealing with matrix *A*, and is defined only if *A* is a square matrix, in which case the inverse is the matrix that satisfies the condition.

$$AA^{-1} = A^{-1} A = I$$

That is, whether *A* is pre- or post multiplied by A^{-1}, the product will be the same identity matrix. This is another exception to the rule that matrix multiplication is not commutative.

4.0 CONLUSION

This section is concerned about matrix inversion. In the start of this part, recall we mentioned that in economics, certain models could be cumbersome, making evaluations difficult. However, with inversion of matrix, these models are made less cumbersome, and estimation becomes easy. We have seen how determinants of matrices are determined, minors of matrix estimated, cofactors, adjoint matrix, and many more. One can emphatically submit that, this has made simple the cries about matrix inversion.

5.0 SUMMARY

In summary, in this unit, we have done justice to the following:

- We started by looking at determinant of a nonsingular matrix. That a matrix is nonsingular when there is no linear dependence between the rows or columns. Only a matrix that is nonsingular has determinant.

- We treated the concepts of minor and cofactors. We found out that, the difference between the two concepts lies in the algebra signs.
- Finally, we took a swift at cofactor matrix and adjoint matrix. In that part, we saw that adjoint matrix is a transpose of a cofactor matrix.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the determinants for the matrices below:
2. **X** = $\begin{bmatrix} 7 & 11 \\ 13 & 16 \end{bmatrix}$ **(ii)** **X** = $\begin{bmatrix} 5 & 8 \\ 6 & 9 \\ 7 & 1 \end{bmatrix}$
3. Find the minors and the cofactors of the entries in the second row of the matrix below:

$$Z = \begin{bmatrix} 7 & 8 & 4 \\ 2 & 3 & 6 \\ 5 & 10 & 3 \end{bmatrix}.$$

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UNIT 4 ECONOMICS APPLICATIONS OF MATRIX

CONTENTS

1.0 INRODUCTION

The unit we just rounded off was mainly on matrix inversion. It basically opened up the way to resolve real issues in matrix algebra. It is in fact, the climax in linear or matrix algebra discussion. In that section, we started by looking at how to find the determinants of matrices, such matrix as 2^{nd} order and 3^{rd} order. We also discussed Non singularity matrix, and discovered that, only a nonsingular matrix has determinant. We equally treated other vital areas like minors and cofactors, cofactor and Adjoint matrices, etc.

However, we shall continue our study in this last unit of this module by applying what we have learnt so far in the module, most especially in unit three to solving some basic economic issues. Meanwhile, we shall be starting our discussion in this section by looking at inverse matrices, and its applications.

2.0 OBJECTIVES

At the end of this unit, you should be able to :

- discuss inversion of matrix based on their understanding of the topic apply it (matrix inversion) into solving matrix equation
- define and explain Cramer's rule
- demonstrate the use of matrices to resolve simple systems equations.

3.0 MAIN CONTENT

3.1 Matrix Inversion

For a given matrix, *A*, an inverse of the matrix which is represented by A^{-1} may or may not be found. A necessary but not sufficient condition

for the existence of an inverse matrix is that such a matrix say matrix *A* must be a square matrix, which means that the dimension or order of matrix *A* should be $A_{mxn,}$ where $m = n$ (that is, the number of rows equal the number of columns in matrix *A*). We have seen a case of necessary but not sufficient condition for the existence of an inverse matrix. However, there is a necessary and sufficient condition for the existence of an inverse matrix requires that a matrix must be squared, and that the set of vectors that make up the matrix must be linearly independent.
When a matrix has no inverse, the matrix is referred to as a singular matrix. Recall that, for singular matrix, the determinant is always zero. The formula for deriving the inverse of a matrix is

$$A^{-1} = \frac{1}{|A|} Adj \text{ A.}$$

Note that if the determinant of an inverse matrix is zero, that is $|A| = 0$. It therefore shows that a singular matrix has no inverse. The inverse of matrix *A*, is represented by A^{-1}, is defined only if *A* is a square matrix, in which case the inverse is the matrix that satisfies the condition.

$$AA^{-1} = A^{-1} A = I$$

That is, whether *A* is pre- or post-multiplied by A^{-1}, the product will be the same identity matrix *I*. This is another exception to the rule that matrix multiplication is not commutative.

About four (4) steps are involved in the estimation of an inverse matrix. These steps are as follows:

i) Estimate the determinant of the given matrix, such as matrix $|A|$
ii) Evaluate the minors and cofactors of the matrix in question
iii) Based on the outcome of step two above, get the Adjoint matrix from the cofactor matrix.
iv) Finally, divide the Adjoint matrix so derived in step three by the determinant in step one to arrive at the inverse of the matrix in question (i.e. $\frac{adjA}{|A|} = A^{-1}$).

Let us see one or two worked examples.

Example 1: Assuming $Z = \begin{bmatrix} 4 & 2 & -2 \\ 0 & 2 & 3 \\ 2 & 0 & 6 \end{bmatrix}$ determine Z^{-1}

Solution

What we shall be doing here is to determine the inverse of the matrix by following the steps stated above:

(i) $|Z| = 4\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \quad -2\begin{bmatrix} 0 & 3 \\ 2 & 6 \end{bmatrix} \quad -2\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

= 4(12-0) -2(0-6) -2(0-4)

= 4(12) -2(-6) -2(-4)

= 48 + 12 + 8

= 68

(ii) Minor & cofactor matrix =

$$\begin{bmatrix} M_{11}=\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} & M_{12}=-\begin{bmatrix} 0 & 3 \\ 2 & 6 \end{bmatrix} & M_{13}=\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \\ M_{21}=-\begin{bmatrix} 2 & -2 \\ 0 & 6 \end{bmatrix} & M_{22}=\begin{bmatrix} 4 & -2 \\ 2 & 6 \end{bmatrix} & M_{23}=-\begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix} \\ M_{31}=\begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix} & M_{32}=-\begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix} & M_{33}=\begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} \end{bmatrix}$$

$$C_{ij} = \begin{bmatrix} 12 & 6 & -4 \\ -12 & 28 & 4 \\ 10 & -12 & 8 \end{bmatrix}$$

(iii) Adjoint matrix $Z = \begin{bmatrix} 12 & -12 & 10 \\ 6 & 28 & -12 \\ -4 & 4 & 8 \end{bmatrix}$

$$Z^{-1} = \frac{adjZ}{|Z|} = \frac{1}{68}\begin{bmatrix} 12 & -12 & 10 \\ 6 & 28 & -12 \\ -4 & 4 & 8 \end{bmatrix} = \begin{bmatrix} \frac{12}{68} & \frac{-12}{68} & \frac{10}{68} \\ \frac{6}{68} & \frac{28}{68} & \frac{-12}{68} \\ \frac{-4}{68} & \frac{4}{68} & \frac{8}{68} \end{bmatrix} =$$

$$\begin{bmatrix} 0.1764 & -0.1764 & 0.1471 \\ 0.0882 & 0.4118 & -0.1764 \\ -0.0588 & 0.0588 & 0.1176 \end{bmatrix}$$

SELF-ASSESSMENT EXERCISE

i. Find the inverse of the matrix below:

$X = \begin{bmatrix} 22 & 13 \\ 7 & 4 \end{bmatrix}$.

ii. State how the outcome of an inverse matrix can be confirmed.

3.2 Linear Equations with Matrix Algebra

We have just discussed inverse matrix, and how it works. One vital role of inverse matrix in matrix algebra is that, it is used to resolve matrix equations. Assuming

$$Z_{nxn}X_{nx1} = Y_{nx1}$$

If the inverse of matrix *Z* exists, then multiply both sides of the equation by the matrix inverse (Z^{-1}), having in mind the conformability laws and we will have

$Z^{-1}_{nxn}Z_{nxn}X_{nx1} = Z^{-1}_{nxn}Y_{nxn.}$ Remember that, the multiplication of an inverse matrix with the matrix produces an identity matrix. Therefore,

$I_{nxn}X_{nx1} = Z^{-1}_{nxn}Y_{nxn.}$ Also recall that, an identity matrix multiply by a matrix equal to that matrix. In that sense,

$X_{nx1} = Z^{-1}_{nxn}Y_{nxn}$ or $X_{nx1} = (Z^{-1}Y)_{nx1.}$ This implies that the coefficient of the inverse matrix multiplied by the column vector of Y produces the equation's solution.

Supposing we have a system of equations as shown below, with more than two unknown, how do you go about this? The way out is simple, express the equation in a matrix form, determine the Adjoint of the matrix, and multiply it by the column vector on the right hand side of the equation.

$$12x_1 - 12x_2 + 10x_3 = 10$$

$$6x_1 + 28x_2 - 12x_3 = 14$$

$$-4x_1 + 4x_2 + 8x_3 = 9$$

Therefore, in matrix form, we have

$$\begin{bmatrix} 12 & -12 & 10 \\ 6 & 28 & -12 \\ -4 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$$

If the determinant is 68, therefore:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{68} \begin{bmatrix} 12 & -12 & 10 \\ 6 & 28 & -12 \\ -4 & 4 & 8 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$$

$$= \frac{1}{60} \begin{bmatrix} 12(10) & -12(14) + & 10(9) \\ 6(10) + & 28(14) - & 12(9) \\ -4(10) + & 4(14) + & 8(9) \end{bmatrix}$$

$$= \frac{1}{68} \begin{bmatrix} 120 - & 168 + & 90 \\ 60 + & 392 - & 108 \\ -40 + & 56 + & 72 \end{bmatrix}$$

$$= \frac{1}{68} \begin{bmatrix} 42 \\ 344 \\ 88 \end{bmatrix} = \begin{bmatrix} 0.6176 \\ 5.0588 \\ 1.2941 \end{bmatrix}$$

$\therefore x_1 = 0.6176$, $x_2 = 5.0588$ and $x_3 = 1.294$.

3.3 Cramer's Rule for Matrix Solutions

Cramer's rule provides a simplified method of solving a system of linear equations through the use of determinants. It is also employed to solve *n* linear equations in *n* unknowns. The rule states

$$\bar{x}_i = \frac{|A_i|}{|A|}$$

Where x_i is the *i*th unknown variable in a series of equations, $|\mathbf{A}|$ is the determinant of the coefficient matrix, and $|\mathbf{A}_i|$ is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_i with the column vector of constants.

Example 1: Cramer's rule is used below to solve the system of equations

$$6x_1 + 5x_2 = 49$$

$$3x_1 + 4x_2 = 32$$

1. Express the equations in matrix form
   AX = B

$$\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 49 \\ 32 \end{bmatrix}$$

2. Find the determinant of **A**

$$|\mathbf{A}| = 6(4) - 5(3) = 9$$

3. Then to solve for x_1, replace column 1, the coefficients of x_1, with the vector of constants **B,** forming a new matrix $\mathbf{A}_1$.

$$\mathbf{A}_1 = \begin{bmatrix} 49 & 5 \\ 32 & 4 \end{bmatrix}$$

Find the determinant of $\mathbf{A}_1$,

$$|\mathbf{A}_1| = 49(4) - 5(32) = 196 - 160 = 36$$

and use the formula for Cramer's rule,

$$\bar{x}_1 = \frac{|A_1|}{|A|} = \frac{36}{9} = 4$$

4. To solve for x_2, replace column 2, the coefficients of x_2, from the original matrix, with the column vector of constants **B**, forming a new matrix $\mathbf{A}_2$.

$$\mathbf{A}_2 = \begin{bmatrix} 6 & 49 \\ 3 & 32 \end{bmatrix}$$

Take the determinant,

$|\mathbf{A}_2|$ = 6(32) – 49(3) = 192 – 145 = 45

and use the formula

$$\bar{x}_2 = \frac{|A_2|}{|A|} = \frac{45}{9} = 5$$

SELF-ASSESSMENT EXERCISE

Use Cramer's Rule to solve the following equations:

i. 2x + y = 4
x + 2y = 8

ii. 4x + 3y = 12
3x – y = 10

iii. 2x + y – z = 10
x- 2y +3z = 14
2x – 2y – z = 9

4.0 CONCLUSION

This unit concludes our study of linear or matrix algebra. We have seen that our statement at the start of this module that, the essence of linear algebra is to proffer solutions to cases where some systems of equations maybe complex and applying simple elimination method to solving it may be impossible. However, with matrix operations, that complex systems of equations, is solvable. That is basically what matrix inversion and Cramer's Rule is all about.

5.0 SUMMARY

In summary, in this unit, we have done justice to the following:

- We started by treating inversion of matrix. In our discussion, we found out that the necessary and sufficient condition for the existence of an inverse matrix is that a matrix must be squared, and that the set of vectors that make up the matrix must be linearly independent.

- We considered linear equation with matrix algebra. In our study, we discover that, to solve systems of equations with matrix algebra, the inverse of the matrix derived from the equation is crucial to solving the problem.
- Finally, our discussion in this unit culminated in studying of Cramer's rule. The method is one of the several methods of solving equations in algebra. The rule is a simplified method of solving a system of linear equations through the use of determinants. In short it is commonly referred to as a rule of '*determinant over determinant.*'

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the inverse of matrix *D* below:

$$D = \begin{bmatrix} 4 & 3 & 4 \\ 3 & 1 & 6 \\ 8 & 7 & 9 \end{bmatrix}$$

2. Apply Cramer's rule to solve the unknown variables in the equation below:

$10p_1 - p_2 - p_3 = 30$

$-p_1 + 4p_2 - 2p_3 = 26$

$-p_1 - 3p_2 + 9p_3 = 20$

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