**COURSE GUIDE**

# FMT 204
# INTRODUCTION TO MATHEMATICAL ECONOMICS


**Course Team**

Shogunro Samuel (Course Developer/Writer) - Lagos State Polytechnic, Ikorodu, Lagos
Anthonio Yisa (Course Editor) - Lagos State Polytechnic, Ikorodu, Lagos
Dr. Saheed O. Ajibola (Course Editor/Programme Leader)-NOUN



**NATIONAL OPEN UNIVERSITY OF NIGERIA**

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island, Lagos

Abuja Office
5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e-mail: centralinfo@nou.edu.ng
URL: www.nou.edu.ng

Published by
National Open University of Nigeria

Printed 2015

ISBN: 978-058-354-X

**CONTENTS** **PAGE**

## INTRODUCTION

You are holding in your hand the course guide for *FMT 204: Introduction to Mathematical Economics*.

The purpose of the course guide is to relate to you the basic structure of the course material you are expected to study. Like the name 'course guide' implies, it is to guide you on what to expect from the course material and at the end of your study of the course material.

## COURSE CONTENT

Logarithms, Exponential and Growth Mathematics, Production functions, Differential and Total derivatives, Matrix Algebra, Input-Output Analysis, Comparative Statistics, Linear Programming, Dual Programming and Games Theory.

## COURSE AIM

The aim of the course is to bring to your cognisance the *Introduction to Mathematical Economics* as mentioned in the course content to enable you solve financial problems and calculations.

## COURSE OBJECTIVES

At the end of the course material, among other objectives, you should be able to:

- explain the concept of Logarithms, Exponential and Growth Mathematics
- contextualise the use of Production functions, Differential and Total derivatives, Matrix Algebra to handle financial problems
- discuss the introduction and insight to Comparative Statistics, Linear Programming, Dual Programming and Games Theory.

## COURSE MATERIALS

The course material package is composed of:

- The Course Guide
- Study Units
- Assignment File
- Tutor-Marked Assignments
- Textbooks and References

## STUDY UNITS

There are 6 modules broken into 7 study units as listed below:

**Module 1**

Unit 1 Indices, Exponential Equations and Logarithms

**Module 2**

Unit 1 Growth Mathematics

**Module 3**

Unit 1 Matrix Algebra and Vector

**Module 4**

Unit 1 Comparative Statics and the Concept of Derivative
Unit 2 Application to Comparative Static Analysis

**Module 5**

Unit 1 Games Theory

**Module 6**

Unit 1 Linear Programming

Each unit of the course has a self-assessment exercise. You will be expected to attempt them as this will enable you learn the facts about the unit.

## TUTOR-MARKED ASSIGNMENTS (TMAs)

The tutor-marked assignments (TMAs) at the end of each unit are designed to test your knowledge and application of the concepts learned. Besides the preparatory TMAs in the course material to test what has been learnt, it is important that you know that at the end of the course, you must have done your examinable TMAs as they fall due, which are marked electronically. They make up to 30% of the total score for the course.

## SUMMARY

*Financial Mathematics* can be compared to a cathedral. We wish to visit a small part of this cathedral of human ideas of quantities and space. We wish to learn how financial mathematics can be built. Financial Mathematics spans a very wide spectrum, from the simple arithmetic operations a pupil learns in primary school to the sophisticated and difficult research which only a specialist can understand after years of long and hard postgraduate study. We place ourselves somewhere higher up in the lower half of this spectrum. This can also be roughly described as where University mathematics starts. In natural sciences, the criterion of validity of a theory is experiment and practice.

Financial mathematics is very different. Experiment and practice are insufficient for *establishing* mathematical truths. Mathematics is *deductive*; the only means of ascertaining the validity of a statement is logic. However, the chain of logical arguments cannot be extended indefinitely: inevitably there comes a point where we have to accept some basic propositions without proofs.

The era which huge and complex calculations take eternity to arrive has passed and technology has made so many things easy for us.

Many aspects of business and accounting say depreciation, loans, interest calculations, investment appraisals, have as their basis some relatively simple formula. Our goal is to be able to answer such typical questions like: A firm rents its premises and the rent agreement provided for a regular annual increase of N2, 550. If the rent in the first year is N9, 500, what is the rent in the tenth year? A building cost N500, 000 and it depreciate at 10% per annum on the reducing balance method. What will its written down value be after 25 years? If N1, 000 is invested at 18% interest compounded semi-annual, what will be its worth in 5 years? How long does it take an investment to double at an interest rate of 8%? If I buy a N200, 000 house, put N40, 000 down, and obtain a 30 year mortgage for the balance at a 9% annual interest rate, what will be my monthly repayment?

Good luck in your studies!

**COURSE GUIDE**

## CONTENTS PAGE

# MODULE 1

Unit 1 Indices, Exponential Equations and Logarithms

## UNIT 1 INDICES, EXPONENTIAL EQUATIONS AND LOGARITHMS

**CONTENTS**



## 1.0 INTRODUCTION

The prime factor of 64 is 2, meaning that of all the factors of 64 (i.e 1, 2, 4, 8, 16, 32, 64) only 2 is a prime number. Hence we can express 64 as $2 \times 2 \times 2 \times 2 \times 2 \times 2$. This expression can be written in a shorter form as $2^6$.

Therefore, $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$ . Here, we have written 64 in index form (i.e. $2^6$ )

In this case 2 is referred to as the base while 6 is called the index or power or degree or exponent.

For example, 100,000 can be written in index form as $10^5$ where 10 is the base and 5 is the index.

There are some rules that guides indices and these rules are sometimes referred to as laws or properties. (In fact these rules or laws are simply definitions).

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify the laws of indices
- solve problems relating to indices
- explain related problems on logarithms.

## 3.0 MAIN CONTENT

## 3.1 Indices

**Laws of Indices**

Six laws of indices will be considered**.**

**Law I:**

$$a^m \times a^n = a^{m+n}$$

The interpretation of this law is that whenever you are multiplying two or more numbers written in index form and having common base then, you add their powers as indicated above.

**Example 1:**
Simply the following

(a) $2a^2 \times 3a^2$ (b) $4b^3 \times 7b^{-6}$ (c) $10^5 \times 10^2$

**Solution:**

(a) $2a^2 \times 3a^2$

We can re-write this expression as

$$(2 \times a^2) \times (3 \times a^2)$$
$$= 2 \times a^2 \times 3 \times a^2$$
$$= 2 \times 3 \times a^2 \times a^2$$
$$= 6 \times a^2 \times a^2$$
$$= \quad 6 \times a^{2+2} \text{ Here we have applied the rule}$$
$$= 6 \times a^{12} = 6a^2 = 6a^3$$

(b) $4b^3 \times 7b^{-6}$

$$= (4 \times b^3) \times (7 \times b^{-6})$$
$$= 4 \times 7 \times b^3 \times b^{-6}$$
$$= 28 \times (b^3 \times b^{-6})$$

$= 28 \times (b^{3+(-6)})$ i.e applying the rule

$$= 28 \times (b^{-3})$$
$$= 28b^{-3}$$

(c) $10^5 \times 10^2 = 10^{5+2} = 10^7$

**Note:**

$$10^5 \times 10^2 = (10 \times 10 \times 10 \times 10 \times 10) \times (10 \times 10)$$
$$= 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^7$$

**Law II:**

$$a^m \div a^n = a^{m-n}$$

The interpretation of these laws is that whenever you are to divide two numbers written in index form where their base(s) are equal, what you do is to subtract their powers from each other as demonstrated in the above definition.

**Example 2:**
Simply the following

(a) $x^{-2}y^3z^{-4} \div x3y^{-3}z^4$ (b) $2^{-3} \div 2^4$ (c) $125 \times 5^4 \div 5^5$

**Solution**:

(a) $x^{-2}y^3z^{-4} \div x3y^{-3}z^4$

$$= (x^{-2}y^3z^{-4}) \div (x^3y^{-3}z^4)$$
$$= (x^{-2} \times y^3 \times z^{-4}) \div (x^3 \times y^{-3} \times z^4)$$
$$= (x^{-2} \div x^3) \times (y^3 \div y^{-3}) \times (z^{-4} \times z^4)$$
$$= x-2-3 \times y^{3-(-3)} \times z^{-4-4}$$
$$= x-5 \times y^{3+3} \times z^{-8}$$
$$= x^{-5} \times y^6 \times z^{-8} = x^{-5}y^6z^{-8}$$

(b) $2^{-3} \div 2^4$

$$= 2^{-3-4}$$
$$= 2^{-7}$$

(c) $125 \times 5^4 \div 5^5$

**Note**: $125 = 5^3$(index form)

$$\text{Hence } 125 \times 5^4 \div 5^5$$
$$= 5^3 \times 5^4 \div 5^5$$
$$= (5^{3+4}) \div 5^5$$

$$= 5^7 \div 5^5$$
$$5^{7-5}$$
$$= 5^2 = 25$$

**Note**:

$$5^7 \div 5^5$$

$$= 5^5$$
$$\frac{5\times5\times5\times5\times5\times5\times5}{5\times5\times5\times5\times5}$$
$$= 5\times5 = 5^2 = 25$$

**Law III:**

$$(a^m)^n = a^{m\times n}$$

This rule states that, if an index number is raised to a power, then, we multiply the two powers as defined above.

**Example 3:**

Simplify the following.

(a) $(10^3)^2$ (b) $(10^x)^2$ (c) $32^x$ (d) $16^{3/4}$ (e) $(\frac{8}{27})^3$

**Solution**:

(a) $(10^3)^2 = 10^{3\times2} = 10^6$

Alternatively,

$$(10^3)^2 = 10^3 \times 10^3 = 10^{3+3} = 10^6$$

or $(10^3)^2 = (10 \times 10 \times 10) \times (10 \times 10 \times 10) = 10 \times 10 \times 10 \times 10 \times 10 \times 10$

$= 10^6$

(b) $(10^x)^2 = 10^{x\times2}$

$= 10^{2x} = (10^2)^x$

$= 100^x$

(c) $32^x = (2^5)^x$

$= 25\times x = 2^{5x}$

$= (2^x)^5$

(d) $16^{3/4}$

$= (2^4)^{3/4}$

$= 2^{4\times 3/4}$

$= 2^{12/4} = 2^3 = 8$

(e)

$$\left(\frac{8}{27}\right)^{\frac{2}{3}}$$

$$= \left(\frac{2^3}{3^3}\right)^{\frac{2}{3}} \qquad \textbf{(i.e writing } \frac{8}{27} \textbf{ index form)}$$

$$= \left[\left(\frac{2}{3}\right)^3\right]^{\frac{2}{3}}$$

$$= \left(\frac{2}{3}\right)^{3\times\frac{2}{3}} \qquad \textbf{(Applying rule 3)}$$

$$= \left(\frac{2}{3}\right)^{\frac{6}{3}} = \left(\frac{2}{3}\right)^2$$

$$= \frac{2^2}{3^2} = \frac{4}{9}$$

**Law IV:**

$$a^0 = 1$$

This law indicates that any number raised to the power of zero will give:

e.g. $10^0 = 1$, $2^0 = 1$, $3^0 = 1$, $7^0 = 1$.

Hence $10^0 = 2^0 = 3^0 = 7^0 = 1$.How?
Consider $10^2 \div 10^2 = 10^{2-2} = 10^0 = 1$ Now, $10^2 \div 10^2$

$$= \frac{10^2}{10^2}$$

$$= \frac{10\times10}{10\times10} = \frac{1}{1} = 1$$

**Example 4:**
Simplify $3^{1-n} \times 3^{n-1}$

**Solution**:

$$3^{1-n} \times 3^{n-1}$$
$$= 3^{1-n+n-1}$$
$$= 3^0 = 1$$

**Law V:**

$$a^{-n} = \frac{1}{a^n}$$

This law relates to a situation whereby the index/power is negative. Changing the index to positive, then, we have the inverse of the index number (but with positive index) as written above. e.g

$$2^{-1} = \frac{1}{2}$$

$$5^{-3} = \frac{1}{5^3}$$

$$\textbf{Also} \quad \frac{1}{32}$$

$$= \frac{1}{2^5}$$

$$= 2^{-5}$$

**Example 5:**

Simplify the following (a) $32^{-1/5} \times 81^{-1/4}$ (b) $(8a^{-6})^{-1/3}$
**Solution**:

(a) $32^{-1/5} \times 81^{-1/4}$

$$32^{-1/5} \times 81^{-1/4}$$

$$= (2^5)^{-1/5} \times (3^4)^{-1/4}$$

$$= 2^{-5/5} \times 3^{-4/4}$$

$$= 2^{-1} \times 3^{-1}$$

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

(b) $(8a^{-6})^{-1/3}$

$$= (8 \times a^{-6})^{-1/3}$$

$$= 8^{-1/3} \times (a^{-6})^{-1/3}$$

$$= 2^{3\times -1/3} \times a^{-6\times -1/3}$$

$$= 2^{-3/3} \times a^{6/3}$$

$$= 2^{-1} \times a^2$$

$$= \frac{1}{2} \times a^2 = \frac{a^2}{2}$$

**Law VI:**

$$a^{m/n} = (^{n}\sqrt{a})^{m}$$

This rule talk about a fractional power, that is where the index/power is a quotient say $^{m/n}$ (i.e a $^{m/n}$). In this case, we find the $n^{th}$ root of "a," raised the result to power m and then evaluate.

**Example 6:**
Simplify $8^{3}$

**Solution**:

$$8^{3} = (\sqrt[3]{8})^{2} \text{ (i.e. finding the cube root of 8)}$$
$$= 2^{2} = 4$$

**Note**:
In most cases we can also apply rule III in solving problems involving rule VI if the base can be written in index form.
e.g. $8^{3} = (2^{3})^{3}) = 2^{3\times} 3 = 2^{3} = 2^{2} = 4$

**NOTE:**
1. $\sqrt{x} = x^{\frac{1}{2}}$ is called square root of x
2. $\sqrt[3]{x} = x^{\frac{1}{3}}$ is called the cube root of x
3. $\sqrt[4]{x} = x^{\frac{1}{4}}$ is called $4^{th}$ root of x
4. $\sqrt[5]{x^2} = (x^2)^{\frac{1}{5}}$ is called $5^{th}$ root of $x^2$
5. $(a \times b)^n = a^n \times b^n$
6. $(\frac{a}{b})^n = \frac{a^n}{b^n}$
7. $(a \pm b)^n \neq a^n \pm b^n$

**Example 7:**
Simplify $4a^3b \times (3ab)^{-2}$

**Solution**:

$4a^3b \times (3ab)^{-2}$

$$= (4 \times a^3 \times b) \times (3 \times a \times b)^{-2}$$

$$= (4 \times a^3 \times b) \times (3^{-2} \times a^{-2} \times b^{-2})$$

$$= 4 \times a^3 \times b \times 3^{-2} \times a^{-2} \times b^{-2}$$

$$= (4 \times 3^{-2}) \times (a^3 \times a^{-2}) \times (b \times b^{-2})$$

$$= (4 \times \tfrac{1}{3^2} \times (a^{3+(-2)}) \times (b^{1+(-2)})$$

$$= (4 \times \tfrac{1}{9}) \times (a^{3-2}) \times b^{1-2})$$

$$= (4/9) \times (a^1) \times (b^{-1}) = \tfrac{4}{9} \times a \times b^{-1}$$

$$= \tfrac{4}{9} \times a \times \tfrac{1}{b} = \tfrac{4a}{9b}$$

**Example 8:**

**Evaluate** $\sqrt{(125^2)^{-\frac{1}{3}}}$

**Solution:**

$\sqrt{(125^2)^{-\frac{1}{3}}} = \left[(125^2)^{-\frac{1}{3}}\right]^{\frac{1}{2}}$ **(Here we change the square root sign and replace it with power** $-\frac{1}{3}$**)**

$= \left[((5^3)^2)^{-1/3}\right]^{1/2}$ **(Here we write 125 in index form as** $5^3$ **)**

$= 5^{3\times2\times-\frac{1}{3}\times\frac{1}{2}} = 5^{3\times-\frac{1}{3}\times2\times-\frac{1}{2}}$ **(Applying rule III)**

$= 5^{-\frac{6}{6}} = 5^{-1} = \frac{1}{5}$ **( Applying rule V)**

**Example 9:**

**Simplify** $9^{-\frac{n}{2}} \times 3^{n+2} \times \times 81^{-\frac{1}{4}}$

**Solution:**

$$9^{-\frac{n}{2}} \times 3^{n+2} \times 81^{-\frac{1}{4}}$$

$= (3^2)^{-\frac{n}{2}} \times 3^{n+2} \times (3^4)^{-\frac{1}{4}}$ **(Here, we have written 9 and 81 in index form )**

$= 3^{-\frac{2n}{2}} \times 3^{n+2} \times 3^{-\frac{4}{4}}$ **(i.e applying law III of indices)**

$$= 3^{-n} \times 3^{n+2} \times \times 3^{-1}$$

$$= 3^{-n+n+2+(-1)}$$

$= 3^{-n+n+2-1}$**(since they have the same base and also applying indices law I)**

$$3^0 = 3^1 = 3$$

**Example 10:**

Simplify $16^{\frac{3n}{4}} \div 8^{\frac{5n}{3}} \times 4^{n-1}$

**Solution:**

$$16^{\frac{3n}{4}} \div 8^{\frac{5n}{3}} \times 4^{n-1}$$

$$= (2^4)^{\frac{3n}{4}} \div (2^3)^{\frac{5n}{3}} \times (2^2)^{n-1}$$

$$= 2^{\frac{12n}{4}} \div 2^{\frac{15n}{3}} \times 2^{2n-2} \quad \text{(law 3 applied)}$$

$$= 2^{3n} \div 2^{5n} \times 2^{2n-2} \quad \text{(base are now equal, then apply the laws )}$$

$$= 2^{3n-5n+2n-2} = 2^{-2}$$

$$= \frac{1}{2^2} = \frac{1}{4}$$

**Example 11:**

Simplify $(2a)^{\frac{1}{2}} \times (2a^3)^{\frac{3}{2}}$

**Solution:**

$$(2a)^{\frac{1}{2}} \times (2a^3)^{\frac{3}{2}} = (2 \times a)^{\frac{1}{2}} \times (2 \times a^3)^{\frac{3}{2}}$$

$$= (2^{\frac{1}{2}} \times a^{\frac{1}{2}}) \times (2^{3/2} \times (a^3)^{\frac{3}{2}})$$

$$= 2^{\frac{1}{2}} \times a^{\frac{1}{2}} \times 2^{\frac{3}{2}} \times a^{\frac{9}{2}}$$

$$= 2^{\frac{1}{2}} \times 2^{\frac{3}{2}} \times a^{\frac{1}{2}} \times a^{\frac{9}{2}} \text{ (By re-arrangement)}$$

$$= 2^{\frac{1}{2}+\frac{3}{2}} \times a^{\frac{1}{2}+\frac{9}{2}}$$

$$= 2^{\frac{4}{2}} \times a^{\frac{10}{2}}$$

$$= 2^2 \times a^5$$

$$= 4 \times a^5 = 4a^5$$

**Example 12:**

Simplify $\dfrac{12^{\frac{1}{3}} \times 6^{1/3}}{81^{\frac{1}{6}}}$

**Solution:**

$$\frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{81^{\frac{1}{6}}}$$

Writing 81 in index form (i.e $9^2$) then the expression becomes

$$\frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{(9^2)^{\frac{1}{6}}} = \frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{9^{\frac{2}{6}}} = \frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{9^{\frac{1}{3}}}$$

Now all the terms in the expression has the power of $^{13}$ hence, the expression becomes

$$\left(\frac{12 \times 6}{9}\right)^{\frac{1}{3}} = \left(\frac{72}{9}\right)^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}}$$

$$= 2^{\frac{3}{3}} = 2^1 = 2$$

Alternative method:

$$\frac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{81^{\frac{1}{6}}} = (12^{\frac{1}{3}} \times 6^{\frac{1}{3}}) \div 81^{\frac{1}{6}}$$

$$= (4^{\frac{1}{3}} \times 3^{\frac{1}{3}}) \times (2^{\frac{1}{3}} \times 3^{\frac{1}{3}}) \div (3^{\frac{4}{6}})$$

$$= (4 \times 3)^{\frac{1}{3}} \times (2 \times 3)^{\frac{1}{3}} \div (3^4)^{\frac{1}{6}}$$

$$= (2^2)^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 3^{\frac{1}{3}} \div 3^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}} \times 2^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \div 3^{\frac{1}{3}} \textbf{ (By re-arrangement)}$$

$$= 2^{\frac{2}{3}+\frac{1}{3}} \times 3^{\frac{1}{3}+\frac{1}{3}-\frac{2}{3}}$$

$$= 2^{\frac{3}{3}} \times 3^{\frac{2}{3}-\frac{2}{3}}$$

$$= 2^1 \times 3^0 = 2 \times 1 = 2$$

**Example 13:**

**Simplify** $\dfrac{2^{n+1} - 4(2^{n+1})}{2^n - 2^{n+1}}$

**Solution:**

$$\frac{2^{n+2}-4(2^{n+1})}{2^n-2^{n+1}}$$

$= \frac{2^n.2^2-2^2.2^n2^1}{2^n-2^n.2^1}$ $(2^n)$ is common to both numerator and denomination

$$= \frac{2^n\left(2^2-2^22^1\right)}{2^n(1-2)} \quad \text{(hence } 2^n \text{ cancel out)}$$

$$= \frac{2^2-2^3}{-1} = \frac{4-8}{-1}$$

$$= \frac{-4}{-1} = 4$$

NB: (”.” multiplication)
Alternative method:

$$\frac{2^{n+2}-4(2^{n+1})}{2^n-2^{n+1}}$$

$$= \frac{2^n.2^2-4(2^n2^1)}{2^n-2^n.2^1}$$

$$= \frac{2^n.4-4(2^n2^1)}{2^n-2^n.2^1}$$

$$= \frac{4(2^n)[1-2]}{2^n[1-2]} = \frac{4(-1)}{-1} = 4$$

**Example 14:**

Simplify $\frac{2^n\left(2^n-3.2^{2n-2}\right)(3^n-2.3^{n-2})}{3^{n-2}(8^{n+3}-4^n)}$ where (.) stands for multiplication

**Solution:**

$$2^n\frac{\left(2^n-3.2^{2n-2}\right)(3^n-2.3^{n-2})}{3^{n-2}(8^{n+3}-4^n)}$$

$$= 2^n\frac{\left(2^n-3.2^{2n}2^{-2}\right)(3^n-2.3^n.3^{-2})}{3^n.3^{-2}\left(2^{3(n+3)}-(2^2)^n\right)}$$

$$= 2^n \frac{\left(2^n - 3.2^{2n}.\frac{1}{2^2}\right)(3^n - 2.3^n.\frac{1}{3^2})}{3^n.\frac{1}{3^2}\left(2^{3(n+3)} - 2^{2n}\right)}$$

$$= \frac{\left(2^n 2^n - 3.2^{2n}.\frac{1}{4}\right)(3^n - 2.3^n.\frac{1}{9})}{3^n.\frac{1}{9}\left(2^{3(n+3)} - 2^{2n}\right)}$$

$$= \frac{\left(2^n 2^n - 3.2^{2n}.\frac{1}{4}\right)(3^n - 2.3^n.\frac{1}{9})}{3^n.\frac{1}{9}(2^{3n}2^9 - 2^{2n})}$$

$$= \frac{2^{2n}\left(1 - \frac{3}{4}\right)3^n(1 - \frac{2}{9})}{3^n.\frac{1}{9}(2^{2n}2^n2^3 - 2^{2n})}$$

$$= \frac{2^{2n}\left(\frac{1}{4}\right)3^n(\frac{7}{9})}{3^n.\frac{1}{9}2^{2n}(2^n2^9 - 1)}$$

$$= \frac{\left(\frac{1}{4}\right)(\frac{7}{9})}{\frac{1}{9}(2^{n+9} - 1)}$$

$$= \frac{\left(\frac{7}{36}\right)}{\frac{1}{9}(2^{n+9} - 1)}$$

$$= \frac{7}{36} \times \frac{9}{2^{n+9} - 1} = \frac{7}{4(2^{n+9} - 1)}$$

**Example 15:**

**Simplify** $\left(4a^{\frac{7}{3}} - 8a^{\frac{4}{3}} + 4a^{\frac{1}{3}}\right) \div 4a^{\frac{1}{3}}$
**Solution:**

$$\left(4a^{\frac{7}{3}} - 8a^{\frac{4}{3}} + 4a^{\frac{1}{3}}\right) \div 4a^{\frac{1}{3}} = \frac{4a^{\frac{7}{3}} - 8a^{\frac{4}{3}} + 4a^{\frac{1}{3}}}{4a^{\frac{1}{3}}}$$

$$= \frac{4a^{\frac{7}{3}}}{4a^{\frac{1}{3}}} - \frac{8a^{\frac{4}{3}}}{4a^{\frac{1}{3}}} + \frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}}}$$

$$= \frac{a^{\frac{7}{3}}}{a^{\frac{1}{3}}} - \frac{2a^{\frac{4}{3}}}{a^{\frac{1}{3}}} + \frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}}}$$

$$= a^{\frac{(7-1)}{3}} - 2a^{\frac{(4-1)}{3}} + a^{\frac{(1-1)}{3}}$$

$$= a^{\frac{6}{3}} - 2a^{\frac{3}{3}} + a^{\frac{0}{3}}$$

$$= a^2 - 2a^1 + a^0$$

$$= a^2 - 2a^1 + 1$$

$$= a^2 - 2a + 1 = a^2 - a - a + 1$$

$$= a(a-1) - 1(a-1) \quad \text{(factorization by grouping method)}$$

$$= (a-1)(a-1) = (a-1)^2$$

**Example 16:**

**Simplify** $\frac{1}{1-x^2} - \frac{1}{x^{-2}-1}$

**Solution:**

$$\frac{1}{1-x^2} - \frac{1}{x^{-2}-1} = \frac{1}{1-x^2} - \frac{1}{\frac{1}{x^2}-1}$$

$$= \frac{1}{1-x^2} - \frac{1}{\frac{1-x^2}{x^2}}$$

$$= \frac{1}{1-x^2} - \frac{x^2}{1-x^2}$$

$$= \frac{1-x^2}{1-x^2}$$

$$= \frac{(1-x)(1+x)}{(1-x)(1+x)} = 1$$

**Example 17:**

**Show that** $\frac{16(32^x)-4^{x+1}.2^{3x-2}}{15(16^x).2^{x-1}} - \frac{5^x}{\sqrt{25^x}} = 1$

**Note: ".” means multiplication**

**Solution:**

**From LHS**

$$\frac{16(32^x)-4^{x+1}.2^{3x-2}}{15(16^x).2^{x-1}} - \frac{5^x}{\sqrt{25^x}}$$

$$= \frac{2^4(2^5)^x-2^{2(x+1)}.2^{3x-2}}{15(2^4)^x.2^{x-1}} - \frac{5^x}{(5^x)^{1/2}}$$

$$= \frac{2^4(2^{5x})-2^{(2x+2)}.2^{3x-2}}{15(2^{4x}).2^{x-1}} - \frac{5^x}{(5^2)^{1/2}}$$

$$= \frac{2^{4+5x}-2^{2x+2+3x-2}}{15(2^{4x+x-1})} - \frac{5^x}{5^{2x/2}}$$

$$= \frac{2^{4+5x}-2^{5x}}{15(2^{5x-1})-1}$$

$$= \frac{2^{4+5x}-2^{2(x+1+3x-2)}}{15.2^{4x+x}2^{-1}-1}$$

$$= \frac{2^{5x}(2^4-1)}{15(2^{5x}2^{-1})-1}$$

$$= \frac{2^{5x}(16-1)}{2^{5x}(15/2)} - 1$$

$$= \frac{(15)}{(\frac{15}{2})-1}$$

$$= (15 \times \frac{15}{2}) - 1$$

$$= \frac{30}{15} = 2 - 1 = 1$$

## 3.2 Exponential Equations

**Introduction**

Equations such as $3^x = 27$, $16^{x-1} - 8^{2x+1} = 0$ $2^x = \frac{1}{32}$ e.t.c are examples of exponential equations. To solve such equation the left and right hand side of the equation must be written in the index form (with outright simplification and conformity). If the two sides of the equation are of the same base then their powers will be same and if the powers of both sides are the same and both have single expression as the base, then their bases will also be equal. It should be noted that an exponential function can NEVER be negative.

**Example 18:**

If

$$2^x = 2^3$$

$$\Rightarrow x = 3$$

If

$$x^2 = 7^2 \Rightarrow x = 7$$

Also If

$$3^{x+5} = 3^{7-x}$$

$$\Rightarrow x + 5 = 7 - x$$

Similarly, If

$$(3x+1)^4 = (2x-1)^4$$

$$\Rightarrow 3x + 1 = 2x - 1$$

**Example 19:**
Solve the equation $3^x = 81$

**Solution**:

$$3^x = 81$$

$$\Rightarrow 3^x = 3^4$$

$$x = 4$$

**Example 20:**
Solve the equation $9^x = \frac{1}{729}$

**Solution**:

$$9^x = \frac{1}{729} \Rightarrow 9^x = \frac{1}{9^3}$$

$$\Rightarrow 9^x = 9^{-3}$$

$$\Rightarrow x = -3$$

**Example 21:**

Solve the equation $125^{x-1} = 25^{2x-3}$

**Solution**:

$$125^{x-1} = 25^{2x-3}$$

$$\Rightarrow (5^3)^{x-1} = (5^2)^{2x-3}$$

$$\Rightarrow 5^{3(x-1)} = 5^{2(2x-3)}$$

$$\Rightarrow 3(x-1) = 2(2x-3)$$

$$\Rightarrow 3x - 3 = 4x - 6$$

$$\Rightarrow -3 + 6 = 4x - 3x$$

$$\Rightarrow 3 = x$$

Therefore,

$$x = 3$$

**Example 22:**

Solve the simultaneous equation $2^{x+y} = 8$, $3^{2x-y} = 27$

**Solution:**

$$2^{x+y} = 8$$

$$\Rightarrow 2^{x+y} = 2^3$$

$$\Rightarrow x + y = 3 \qquad (1)$$

Also

$$3^{2x-y} = 27$$

$$\Rightarrow 3^{2x-y} = 3^3$$

Therefore,

$$\Rightarrow 2x - y = 3 \qquad (2)$$

Now, solve the equations (1) and equation (2) simultaneously to get

$$x+y=3$$

$$2x-y=3$$

By adding equations (1) and (2) to get

$$3x = 6$$

$$x = \frac{6}{3} = 2$$

Substitute x = 2 in either equation (1) or (2)
Now using equation (1) i.e.

$$x + y = 3$$

$$\Rightarrow 2 + y = 3$$

$$y = 3 - 2 = 1$$

Hence x=2, y=1

**Example 23**:

Solve the equation $2^{2x} - 5(2^x) = 4 = 0$

**Solution**:

$$2^{2x} - 5(2^x) + 4 = 0$$

$$\Rightarrow (2^x)^2 - 5(2^x) + 4 = 0$$

$$\textbf{Let } 2^x = a$$

then, the equation becomes $a^2 - 5a + 4 = 0$ which has turned to a quadratic equation. Now, solving this equation by factorization method through groupings, we have

$$a^2 - a - 4a + 4 = 0$$

$$\Rightarrow a(a-1) - 4(a-1) = 0$$

$$\Rightarrow (a-1)(a-4) = 0$$

$$\Rightarrow (a-1) = 0 \textbf{ or } (a-4) = 0$$

$$\Rightarrow a = 1 \textbf{ or } a = 4$$

but $2^x = a$,

$$\text{when } a = 1, \Rightarrow 2^x = 1$$

$$\Rightarrow 2^x = 2^0$$

$$x = 0$$

$$\text{Also, when } a = 4, \Rightarrow 2^x = 4$$

$$\Rightarrow 2^x = 2^2$$

$$x = 2$$

Hence x=0, or x=2

**Example 24**:
Solve the equation $3^x + 3^{1-x} = 4$

**Solution**:

$$3^x + 3^{1-x} = 4$$

$$\Rightarrow 3^x + (3^1 \times 3^{-x}) = 4$$

$$\Rightarrow 3^x + (3^1 3^{-x}) = 4$$

$$\Rightarrow 3^x + \left(\frac{3^1}{3^x}\right) = 4$$

$$\text{Let } 3^x = b$$

then, the equation becomes

$$b + \frac{3}{b} = 4$$

multiplying through by b, we have

$$b^2 + 3 = 4b$$

$\Rightarrow b^2 - 4b + 3 = 0$ (then now form a quadratic equation)

$$\Rightarrow b^2 - b - 3b + 3 = 0$$

$\Rightarrow b(b-1) - 3(b-1) = 0$ (By factorization )

$$\Rightarrow (b-1)(b-3) = 0$$

$$\Rightarrow (b-1) = 0, \text{ or } (b-3) = 0$$

$$\Rightarrow b = 1, \text{ or } b = 3$$

but b = $3^x$
if b=1,

$$3^x = 1$$

$$\Rightarrow 3^x = 3^0$$

$$\Rightarrow x = 0$$

**Also, if b=3,**

$$,3^x = 3$$

$$\Rightarrow 3^x = 3^1$$

$$x = 1$$

Hence x=0, or x=1

**Example 25:**
**Solve the equation $3x^{\frac{2}{3}} = 12$**
**Solution:**

$$3x^{\frac{2}{3}} = 12$$

**Divide both sides by 3 to get**

$$3x^{\frac{2}{3}} = \frac{12}{3}$$

$$\Rightarrow x^{\frac{2}{3}} = 4$$

**Raise both sides to power $\frac{3}{2}$**

$$\text{i.e } (x^{\frac{2}{3}})^{\frac{3}{2}} = 4^{\frac{3}{2}}$$

$$\Rightarrow x^{\frac{6}{6}} = (2^2)^{\frac{3}{2}}$$

$$\Rightarrow x^1 = 2^3$$

$$\Rightarrow x = 8$$

**Example 26:**

**Solve the equation** $(\frac{1}{2})^x = 2\frac{1}{2}$

**Solution:**

$$(\tfrac{1}{2})^x = 2\tfrac{1}{2}$$

$$\Rightarrow (2^{-1})^x = \tfrac{5}{2}$$

$$\Rightarrow 2^{-x} = \tfrac{5}{2}$$

$$2(2^{-x}) = 5$$

$$2^{1+(-x)} = 5$$

$$2^{1-x} = 5$$

Now take log of both sides to base 10

$$\text{i.e } log_{10}^{2^{x-1}} = log_{10}^{5}$$

$$(1-x)log_{10}^{2} = log_{10}^{5}$$

$$(1-x) = \frac{log_{10}^{5}}{log_{10}^{2}}$$

$$1 - x = \frac{0.6990}{0.3010}$$

$$1 - x = 2.3223$$

$$\Rightarrow 1 - 2.3222 = x$$

$$x = -1.3222 \ \textbf{(4dp)}$$

**Example 27:**

Solve the equation $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 9 = 0$

**Solution:**

$$2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 9 = 0$$

$$2(x^{\frac{1}{3}})^2 + 3x^{\frac{1}{3}} - 9 = 0$$

Let $x^{\frac{1}{3}} = a$

$\Rightarrow 2a^2 + 3a - 9 = 0$ (this now becomes a quadratic equation)

$$\Rightarrow 2a^2 - 6a + 3a - 9 = 0$$

$$\Rightarrow (a+3) - 3(a+3) = 0 \text{ (By factorization )}$$

$$\Rightarrow (a+3)(2a-3) = 0$$

$$\Rightarrow (a+3) = 0 \text{ or } (2a-3) = 0$$

$$a = -3 \text{ or } a = 3/2$$

but $a = x^{1/3}$
when $a = -3 \Rightarrow x^{1/3} = -3$
cube both sides or raise both sides to power 3

$$i.e(x^{1/3})^3 = (-3)^3$$

$$\Rightarrow (x^{3/3}) = (-3)^3$$

$$(x^1) = -27$$

$$x = -27$$

Also when $a = 3/2, \Rightarrow x^{1/3} = 3/2$ cube both side to get

$$\Rightarrow (x^{1/3})^3 = (3/2)^3$$

$$\Rightarrow x^1 = \frac{27}{8}$$

$$\Rightarrow x = \frac{27}{8}$$

Hence x=-27, or $\frac{27}{8}$

**Example 28**:

Given that $(3^{x+1})(5^{x}) =$ 675 find x
**Solution**:

$$(3^{x+1})(5^{x}) = 675$$

$$(3^{x+1})(5^{x}) = 25 \times 27$$

$$\Rightarrow (3^{x+1})(5^{x}) = 5^{2} \times 3^{3}$$

$$\Rightarrow 3^{x+1} = 3^{3} \text{ or } 5^{x} = 5^{2}$$

$$\Rightarrow x + 1 = 3 \text{ or } x = 2$$

$$\Rightarrow x = 3 - 1 \text{ or } x = 2$$

$$\Rightarrow x = 2 \text{ or } x = 2$$

$$x = 2$$

**Alternatively:**

$$(3^{x+1})(5^{x}) = 675$$

$$\Rightarrow (3^{x}3^{1})(5^{x}) = 675$$

$$\Rightarrow\Rightarrow 3^{x} \times 3^{1} \times 5^{x} = 675$$

$$\Rightarrow 3^{x} \times 5^{x} = \frac{675}{3}$$

$$\Rightarrow (3 \times 5)^{x} = 225$$

$$\Rightarrow (15)^{x} = 15^{2}$$

$$x = 2$$

**Example 29:**

Solve $25^{x} - 3.5^{x} + 2 = 0$

$$25^{x} - 3.5^{x} + 2 = 0$$

$$\Rightarrow (5^{2})^{x} - 3.5^{x} + 2 = 0$$

$$\Rightarrow (5^{x})^{2} - 3.5^{x} + 2 = 0$$

put $5^x = a$

$\Rightarrow a^2 - 3a + 2 = 0$ (this now becomes a quadratic equation)

$$\Rightarrow a^2 - a - 2a + 2 = 0$$

$$\Rightarrow a(a-1) - 2(a-1) = 0$$

$$\Rightarrow (a-1)(a-2) = 0 \qquad \text{(by factorization method)}$$

$$\Rightarrow a - 1 = 0 \ or \ a - 2 = 0$$

Therefore, $a = 1$ $or$ $a = 2$
but $a = 5^x$

when $a = 1, \Rightarrow 5^x = 1 = 5^0$

Therefore, $x = 0$
Also when $a = 2, \Rightarrow 5^x = 2$
Take log of both sides to base 10

$$\text{i.e } \log_{10} 5^x = \log_{10} 2$$

$$\Rightarrow x = \frac{\log_{10} 2}{\log_{10} 5} = \frac{0.3010}{0.6990}$$

$$x = 0.4306$$

Therefore, $x = 0$ or $x = 0, 43$

**Example 30:**

Solve $2^{x^2-2} = 16(2^{5x})$

**Solution :**

$$2^{x^2-2} = 16(2^{5x})$$

$$\Rightarrow 2^{x^2-2} = 2^4(2^{5x})$$

$\Rightarrow 2^{x^2-2} = 2^{4+5x}$ **(Since both sides are of the same base)**

$$\Rightarrow x^2 - 2 = 4 + 5x$$

$$\Rightarrow x^2 - 5x - 2 - 4 = 0$$

$\Rightarrow x^2 - 5x - 6 = 0$ **(by factorization using group method)**

$$x^2 + x - 6x - 6 = 0$$

$$\Rightarrow x(x+1) - 6(x+1) = 0$$

$$\Rightarrow (x+1)(x-6) = 0$$

$$\Rightarrow x + 1 = 0 \ or \ x - 6 = 0$$

**Therefore,** $x = -1 \ or \ x = 6$

**Example 29:**

**Given that** $4^x = 0.125$**,find x**
**Solution :**

$$4^x = 0.125$$

$$\Rightarrow 4^x = \frac{125}{1000} = \frac{1}{8}$$

$$(2^2)^x = 2^{-3} = \frac{1}{2^3}$$

$$2x = -3$$

**Therefore,** $x = -\frac{3}{2}$

**Example 31:**

**Find x, if** $2(3^{2x+1}) + 3(3^{x-1} - 4) = 0$
**Solution:**

$$2(3^{2x+1}) + 3(3^{x-1} - 4) = 0$$

$$\Rightarrow 2(3^{2x}.3^1) + 3(3^x.3^{-1} - 4) = 0$$

$$\Rightarrow 6(3^{2x}) + 3(\tfrac{3^x}{3^1} - 4) = 0$$

$$\Rightarrow 3\left[(2(3^x)^2 + (\tfrac{3^x}{3^1} - 4)\right] = 0$$

$\Rightarrow 2(3^x)^2 + (\frac{3^x}{3^1} - 4) = 0$ **(i.e dividing both sides by 3)**
**Let** $3^x = a$, **hence, we have**

$$2a^2 + (\tfrac{a}{3} - 4) = 0$$

$$\Rightarrow 6a^2 + a - 12 = 0$$

$\Rightarrow 6a^2 + 9a - 8a - 12 = 0$ **(now a quadratic equation)**

$\Rightarrow 3a(2a+3) - 4(2a+3) = 0$**(by factorization using group method)**

$$\Rightarrow (2a+3)(3a-4) = 0$$

$$\Rightarrow 2a + 3 = 0 \ or \ 3a - 4 = 0$$

$$\Rightarrow 2a = -3 \ or \ 3a = 4$$

$$\Rightarrow a = \tfrac{-3}{2} \ or \ a = \tfrac{4}{3}$$

**But** $a = 3^x$

**when** $a = -\frac{3}{2} \Rightarrow 3^x = \frac{-3}{2}$ **(unsolveable)**

**Also when** $a = \frac{4}{3}$

$$\Rightarrow 3^x = \tfrac{4}{3}$$

**Take the log of both sides to base 10**

$$\text{i.e } \log_{10} 3^x = \log_{10} \tfrac{4}{3}$$

$$\Rightarrow x \log_{10} 3 = \log_{10} 1.3333$$

$$\Rightarrow x = \frac{\log_{10} 1.3333}{\log_{10} 3} = \frac{0.124938}{0.477121}$$

$$x = 0.261858$$

**Therefore,** $x \approx 0.2619$

**Example 31:**

**Given that** $K = \frac{a}{(1+t)^2}$ **and** $L = \frac{a(1-t)}{(1+t)}$,
**Show that** $(L+a)^2 = 4aK$

**Solution:**

**To show that** $(L+a)^2 = 4aK$
**Consider the LHS**

$$\text{i.e } (L+a)^2 = \left[\frac{a(1-t)}{(1+t)} + a\right]^2$$

$$= \left[a\left(\frac{1-t}{1+t} + 1\right)\right]^2$$

$$= \left[a^2\left(\frac{1-t+(1+t)}{1+t}\right)\right]$$

$$= a^2\left[\frac{1-t+1+t}{1+t}\right]^2$$

$$= a^2\left(\frac{2}{1+t}\right)^2$$

$$= a^2 \times \left(\frac{4}{(1+t)^2}\right)^2$$

$$= 4a^2\left(\frac{a}{(1+t)^2}\right)$$

$$\text{But } K = \frac{a}{(1+t)^2}$$

$$\text{Hence, } (L+a)^2 = \frac{4a.a}{(1+t^2)^2}$$

$$4a\left(\frac{a}{(1+t)^2}\right) = 4aK = RHS$$

## 3.3 Logarithms

You will be familiar with the use of logarithms for multiplication and division, but there are certain properties of logarithm that are very useful in more advanced work. Having just considered indices, this is the appropriate place to discuss logarithms because logarithm can also be written or expressed in index form.

Hence, the logarithm of a number (say k) to a particular base (say x) is the power (say t) to which that particular number (x) must be raised to give the number (k)

**i.e if** $log_x k = t$
$\Rightarrow x^t = k$

Note:
1. The logarithm of w to base x is written as $\log_x$ w.
2. Whenever the base of a logarithm is not indicated, in most cases, we assumed it is in base 10.

**Therefore, if** $log_{10}100 = 2 \Rightarrow 10^2 = 100.$
**Also,** $log_2 32 = 5$
$\Rightarrow 2^5 = 32.$
**Similarly,** $log_{16}32 = x$
$\Rightarrow 16^x = 32$
**Hence, in general, if** $log_a b = x$
$\Rightarrow a^x = b$

**Example 32:**
Find the value of the following:

(i) $log_3 81$ (ii) $log_{32}16$
(iv) $\log_{0.25} 32$ (v) $log_{100}0.001$
(vii) $log_6\sqrt{6}$ (viii) $log_3 9\sqrt{3}$

(iii) $log_{\sqrt{27}}1/9$
(vi) $log_{0.4}0.064$
(ix) $5log_8\sqrt{0.5}$ (x) $log_3(\sqrt[4]{3}\sqrt[3]{3})$

**Solution:**

(i) Let $log_3 81 = x \Rightarrow 3^x = 81$

Now, write 81 in index form to get

$$3^x = 3^4 \Rightarrow x = 4$$

Therefore, $log_3 81 = 4$

(ii) Let $log_{32} 16 = y \Rightarrow 32^y = 16$ Now, write both 32 and 16 in index form

$$\text{i.e } (2^5)^y = 2^4$$

$$\Rightarrow 2^{5y} = 2^4$$

$$\Rightarrow 5y = 4$$

$$\Rightarrow y = 45$$

Therefore, $log_{32} 16 = 45$ or $0.8$

(iii) Let $\log_{\sqrt{27}} \frac{1}{9} = k$

$$\Rightarrow (\sqrt{27})^k = \frac{1}{9}$$

$$(27^{\frac{1}{2}})^k = 9^{-1}$$

Now, write both 27 and 9 in index form to have

$$(27^{\frac{k}{2}}) = 9^{-1}$$

$$(3^3)^{\frac{k}{2}}$$

$$= (3^2)^{-1}$$

$$\Rightarrow 3^{\frac{3k}{2}} = 3^{-2}$$

$$\Rightarrow \frac{3k}{2} = -2$$

$$\Rightarrow 3k = -4$$

$$k = -\frac{4}{3}$$

**Therefore,** $\log_{\sqrt{27}} \frac{1}{9} = -\frac{4}{3}$

**(iv) Let** $log_{0.25}32 = t$

$$\Rightarrow (0.25)^t = 32$$

**Convert 0.25 to fraction then, we have**

$$(\frac{25}{100})^t = 32 \Rightarrow (\frac{1}{4})^t = 32$$

$$\Rightarrow (4^{-1})^t = 32 \Rightarrow 4^{-t} = 32$$

**Now, write both 4 and 32 in index form to get**

$$(2^{-2})^t = 2^5$$

$$\Rightarrow 2^{-2t} = 2^5$$

$$\Rightarrow -2t = 5$$

$$\text{or } t = \frac{5}{-2} = -\frac{5}{2}$$

**Therefore,** $log_{0.25}32 = -\frac{5}{2}$ **or** $-2.5$

**(v) Let** $log_{100}0.001 = p$

$$\Rightarrow 100^p = 0.001$$

$$\Rightarrow 100^p = 1/1000$$

$$\Rightarrow (10^2)^p = \frac{1}{10^3} = 10^{-3}$$

$$\Rightarrow 10^{2p} = 10^{-3}$$

$$\Rightarrow 2p = -3 \text{ or } p = -\frac{3}{2}$$

**Hence,** $log_{100}0.001 = -\frac{3}{2}$

**(vi) Let** $log_{0.4}0.064 = w$

$$\Rightarrow (0.4)^w = 0.064$$

**Now change both 0.4 and 0.064 to fraction to get**

$$(\tfrac{4}{10})^w = 64/1000$$

$$\Rightarrow (\tfrac{4}{10})^w = \tfrac{4^3}{10^3}$$

$$\Rightarrow (\tfrac{4}{10})^w = (\tfrac{4}{10})^3$$

$$\Rightarrow w = 3$$

**Therefore,** $log_{0.4} 0.064 = 3$

**(vii) Let** $log_6 \sqrt{6} = x$

$$\Rightarrow 6^x = \sqrt{6}$$

$$\Rightarrow 6^x = 6^{\frac{1}{2}}$$

$$\Rightarrow x = \tfrac{1}{2}$$

**Therefore,** $\Rightarrow log_6 \sqrt{6} = \frac{1}{2}$

**(viii) Let** $log_3 9\sqrt{3} = y$

$$\Rightarrow 3^y = 9\sqrt{3}$$

$$\Rightarrow 3^y = 3^2(3^{\frac{1}{2}})$$

$$\Rightarrow 3^y = 3^{2+\frac{1}{2}} = 3^{2\frac{1}{2}}$$

$$\Rightarrow 3^y = 3^{\frac{5}{2}}$$

$$\Rightarrow y = \tfrac{5}{2}$$

**Hence,** $log_3 9\sqrt{3} = \frac{5}{2}$

**(ix)Let** $5 log_8 0.5 = k$

$\Rightarrow log_8 (0.5)^5 = k$ **(see rule 3 of logarithm in the next dicussion)**

$$\Rightarrow 8^k = (0.5)^5$$

$$\Rightarrow 8^k = (\tfrac{5}{10})^5 = (\tfrac{1}{2})^5$$

$$\Rightarrow (2^3)^k = (2^{-1})^5$$

$$\Rightarrow 2^{3k} = 2^{-5}$$

$$\Rightarrow 3k = -5$$

$$\Rightarrow k = -\tfrac{5}{3}$$

Therefore, $5log_8 0.5 = -\frac{5}{3}$

(x) Let $log_3(\sqrt[4]{3}\sqrt[3]{3}) = t$

$$\Rightarrow 3^t = (\sqrt[4]{3})(\sqrt[3]{3})$$

$$\Rightarrow 3^t = (3^{\frac{1}{4}})(3^{\frac{1}{3}}) = 3^{\frac{1}{4}+\frac{1}{3}} = 3^{\frac{7}{12}}$$

$$\Rightarrow 3^t = 3^{\frac{7}{12}}$$

$$\Rightarrow t = \tfrac{7}{12}$$

Therefore, $log_3(\sqrt[4]{3}\sqrt{3}) = \frac{7}{12}$

## 3.4 Rules of Logarithms

There are some rules that guides logarithm which includes
1. $log_x(ab) = log_x a + log_x b$ (Note: ab means $a \times b$)
2. $log_x(a/b) = log_x a - log_x b$
3. $log_x a^b = b log_x a$
4. $log_a a = 1$
5. $log_x 1 = 0$

These five rules governs logarithms and can be proved.

**Theorem 1:**

**Prove that** $log_x(ab) = log_x a + log_x b$
**Proof:**
**Let** $log_x a = n$ **and** $log_x b = m$ (1)
**Then,** $x^n = a$ (2)
**and** $x^m = b$ (3)
**Multiplying equations (2) and (3), we have**

$$x^{n+m} = ab \text{ or } ab = x^{n+m}$$

$\Rightarrow ab = x^{n+m}$ **(first law of indices)**

**Now, take log of both sides to base x to get**

$$log_x(ab) = log_x x^{m+n}$$

$\Rightarrow log_x(ab) = (n+m)log_x x$ **(third law of logarithm)**

$\Rightarrow log_x(ab) = (n+m)$ **(since** $log_x x = 1$ **fourth law of logarithm )**

**But** $log_x a = n$ **and** $log_x b = m$ **from equation (1)**

**Therefore,** $log_x(ab) = log_x a + log_x b$ **proved.**

**Theorem 2:**

**Prove that** $log_x \frac{a}{b} = log_x a - log_x b$

**Proof:**

**Let** $log_x a = n$ **and** $log_x b = m$ (1)

**Then,**

$$x^n = a \qquad (2)$$

**and**

$$x^m = b \qquad (3)$$

**Dividing equation (2) and (3), we have large**

$$\frac{x^m}{x^n} = \frac{a}{b}$$

$\Rightarrow x^m \div x^n = \frac{a}{b} = x^{m-n}$ **(second law of indices)**

**Now, take log of both sides to base x to get**

$$log_x\frac{a}{b} = log_x x^{m-n}$$

$$\Rightarrow log_x(\frac{a}{b}) = (n-m)log_x x \text{ (third law of logarithms)}$$

$\Rightarrow log_x(\frac{a}{b}) = (m-n)$ **(Since** $log_x x = 1$, **fourth law of logarithms )**

**But** $log_x a = n$ **and** $log_x b = m$ **(from (1) above)**

**Hence,** $log_x(\frac{a}{b}) = log_x a - log_x b$ **proved**

**Theorem 3:**

**Prove that** $log_x a^b = blog_x$
**Proof:**

**Let** $\log_x a = k$

$$\Rightarrow x^k = a \qquad (1)$$

**Now, raise both sides to power b to get**

$$(x^k)^b = a^b$$

$x^{bk} = a^b \text{ or } a^b = x^{kb}$ **(third law of indices )**

**We now take log of both sides to base x to have**
\

$$\log_x a^b = \log_x x^{bk}$$

$$\Rightarrow \log_x a^b = bk \log_x x$$

$$\Rightarrow \log_x a^b = bk \text{ (since } \log_x x = 1 \text{ )}$$

**But** $k = log_x a$

**Therefore,** $log_x a^b = blog_x a$ **proved.**

**Theorem 4:**
**Prove that** $\log_a a = 1$
**Proof:**

**Let** $log_a a = x$ **(1)**

$$a^x = a$$

**We now take log of both sides to base a**

$$\log_a a^x = \log_a a$$

$$\Rightarrow x \log_a a = \log_a a$$

$$\Rightarrow x = \frac{\log_a a^x}{\log_a a} = 1$$

**But** $x = \log_a a = 1$ **from equation (1) above**

**Theorem 5:**
**Prove that** $x = \log_x 1 = 0$
**Proof:**

$$\textbf{Let } \log_x 1 = 0$$

$$\Rightarrow x^0 = 1$$

**We now take log of both sides to base x**

$$\text{i.e } \log_x x^0 = \log_x 1$$

$$\Rightarrow 0 \times (\log_x x)$$

$$\Rightarrow 0 = \log_x 1$$

**Therefore,** $\log_x 1 = 0$

## Example 33:

**Simplify the following**

(i) $\log_4 9 + \log_4 21 - \log_4 7$ **(ii)** $\log_5(\frac{3}{8}) + 2\log_5(\frac{4}{5}) - \log_5(\frac{2}{5})$

(iii) $\log_{10} 12 + 2\log_{10} 0.75 - \log_{10} 0.675$ **(iv)** $\log_{10}\sqrt{15} - \log_{10}\sqrt{6} + \log_{10} 2$

(v) $\frac{\log\sqrt{5}}{\log 5}$ (vi) $\log x^2 + 2\log(xy) - \log y^2$ **(vii)** $\log_{\frac{2}{3}}(\frac{8}{27})$

(viii) $\frac{\log 8 - \log 4}{\log 4 - \log 2}$ (ix) $\log_4 8 - \log_4 2$

**Solution:**

(i) $\log_4 9 + \log_4 21 - \log_4 7$

**Since the base are equal, then we can apply the logarithm rules. Hence we have**

$$\log_4(9 \times \frac{21}{7})$$

$$\log_4(9 \times 21 \div 7)$$

$$\log_4(9 \times 3)$$

$$\log_4 27$$

$$\log_4 3^3$$

$$= 3\log_4 3$$

(ii) $\log_5(\frac{3}{8}) + 2\log_5(\frac{4}{5}) - \log_5(\frac{2}{5})$

**Here, we need to re-write the second term i.e $2\log(4/5)$ to be in conformity with others by using third rule of logarithms which bring the expression to be:**

$$\log_5(\frac{3}{8}) + \log_5(\frac{4}{5})^2 - \log_5(\frac{2}{5})$$

$= \log_5(\frac{3}{5}) + \log_5(\frac{16}{25}) - \log_5(\frac{2}{5})$ Since they are all of the same base then, we have

$$\log_5(\frac{3}{8} \times \frac{16}{25} \div \frac{2}{5})$$

$$= \log_5(\frac{3}{8} \times \frac{16}{25} \times \frac{5}{2})$$

$$= \log_5(3/5)$$

$$= \log_5 3 - log_5 5$$

$$= \log_5 3 - 1 \qquad (Note: \ \log_5 5 = 1)$$

**(iii)** $\log_{10} 12 + 2\log_{10} 0.75 - \log_{10} 0.675$

The first thing we do here is to change the decimal to fraction, hence we have

$$\log_{10} 12 + 2\log_{10}(\tfrac{75}{100}) - \log_{10}(\tfrac{675}{1000})$$

Re-writing the second term using 3rd law of logarithm, we have

$$\log_{10} 12 + \log_{10}(\tfrac{75}{100})^2 - \log_{10}(\tfrac{675}{1000})$$

By applying the logarithms rules, we have

$$\log_{10} 12 \times (\tfrac{75}{100})^2 \div \tfrac{675}{1000}$$

$$\log_{10} 12 \times \tfrac{75}{100} \times \tfrac{75}{100} \div \tfrac{1000}{675}$$

$$= \log_{10} 10 = 1$$

**(iv)** $\log x^2 - 2\log(xy) - \log y^2$

$$\log x^2 - 2\log(xy) + \log y^2$$

$$\log x^2 - \log(xy)^2 + \log y^2$$

$$= \log x^2 - \log x^2 y^2 + \log y^2$$

$$= \log(x^2 \div x^2 y^2 \times y^2)$$

$$= \log\left(\tfrac{x^2 \times y^2}{x^2 \times y^2}\right)$$

$$= \log\left(\tfrac{x^2 y^2}{x^2 \times y^2}\right)$$

$$= \log 1 = 0$$

(v) $\log_{10}\sqrt{15} - \log_{10}\sqrt{6} + \log_{10} 2$

$= \log_{10}(\sqrt{15} \div \sqrt{6} \times 2)$ **(i.e by the rules of logarithm)**

$\left(\sqrt{15} \times 2\sqrt{6}\right)$

$= \log_{10}\left(\frac{15^{\frac{1}{2}} \times 2}{6^{1/2}}\right)$

$= \log_{10}\left(\frac{15^{\frac{1}{2}} \times 4^{\frac{1}{2}}}{6^{\frac{1}{2}}}\right)$ **Note** $2 = 4^{\frac{1}{2}}$

$= \log_{10}\left(\frac{15 \times 4}{6}\right)^{\frac{1}{2}}$ **(Since the power** $\frac{1}{2}$ **affects both numerator and denominator)**

$= \log_{10}(\frac{60}{6})^{1/2}$

$= log_{10} 10^{1/2}$

$= \frac{1}{2}\log_{10} 10 = \frac{1}{2}$ **(since** $\log_{10} 10 = 1$)

(vi) $\frac{\log\sqrt{5}}{\log 5}$

$= \frac{\log(5)^{\frac{1}{2}}}{\log 5}$

$= \frac{1}{2}\frac{\log 5}{\log 5}$ **(since** $\log 5$ **cancel out)**

$= \frac{1}{2}(1) = \frac{1}{2}$

(vii) $\log_{\frac{2}{3}}(\frac{8}{27})$

$\log_{\frac{2}{3}}\frac{2^3}{3^3} = \log_{\frac{2}{3}}(\frac{2}{3})^3$

$= 3\log_{\frac{2}{3}}(\frac{2}{3})$ **(i.e using third law of logarithm,** $\log_{\frac{2}{3}}\frac{2}{3} = 1$)

**Hence,** $\log_{\frac{2}{3}}(\frac{8}{27}) = 3$

(viii) $\frac{\log 8-\log 4}{\log 4-\log 2}$

$$=\frac{\log\frac{8}{4}}{\log\frac{4}{2}} \quad \text{(i.e using second law of logarithm)}$$

$$=\frac{\log 2}{log 2}=1$$

**Alternative solution:**

$$\frac{\log 8-\log 4}{\log 4-\log 2}$$

$$\frac{\log 2^3-\log 2^2}{\log 2^2-\log 2}$$

$$\frac{3\log 2-2\log 2}{2\log 2-\log 2}$$

$$\frac{\log 2}{\log 2}=1$$

(ix) $\frac{\log 27+\log 8-\log 125}{\log 5-\log 6}$

$$=\frac{\log(27\times 8\div 125)}{log(5\div 6)}$$

$$=\log\left(\frac{27\times 8}{125}\right)\div\log(5\div 6)$$

$$=\frac{\log\left(\frac{216}{125}\right)}{\log\left(\frac{5}{6}\right)}$$

$$=\frac{\log\left(\frac{6^3}{5^3}\right)}{\log\left(\frac{5}{6}\right)}$$

$$=\log\left(\frac{6}{5}\right)^3\div\log\left(\frac{5}{6}\right)$$

$$\frac{\log\left(\frac{6}{5}\right)^3}{\log\left(\frac{6}{5}\right)^{-1}} \quad \text{since } \frac{5}{6}=\left(\frac{6}{5}\right)^{-1}$$

$$=3\frac{\log\left(\frac{6}{5}\right)}{-\log\left(\frac{6}{5}\right)}=-\frac{3}{1}=-3$$

(x) $\log_4 8 - \log_4 2$

$$\text{Let } \log_4 8 = x \text{ and } -\log_4 2 = y$$

$$\Rightarrow 8^x = 4 \text{ and } 4^y = 2$$

$$\Rightarrow 2^{3x} = 2^2 \text{ and } 2^{2y} = 2^1$$

$$\Rightarrow 3x = 2 \text{ and } 2y = 1$$

$$\Rightarrow x = \tfrac{2}{3} \text{ and } y = \tfrac{1}{2}$$

$$\text{Hence} \quad \log_4 8 = \tfrac{2}{3} \text{ and } \log_4 2 = \tfrac{1}{2}$$

$$\text{Therefore, } \log_4 8 - \log_4 2 = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{4-3}{6} = \tfrac{7}{6}$$

**Example 34:**

**Express as a single logarithm**
(i) $-\log_{10} 2$ (ii) $2 - 2\log_{10} 2$
**Solution:**

(i)

$$-\log_{10} 2 = \log_{10} 2^{-1}$$

$$= \log_{10}(\tfrac{1}{2})$$

(ii)

$$2 - 2\log_{10} 2$$

$$= 2 - \log_{10} 2^2$$

$$= \ 2 - \log_{10} 4$$

$$\textbf{But } \log_{10} 100 = 2$$

$$\text{Hence, } \ 2 - 2\log_{10} 2 = \log_{10} 100 - \log_{10} 4$$

$$= \log_{10}(\tfrac{100}{4}) - \log_{10} 25$$

## 3.5 Logarithmic Equations

**Example 36:**

**Solve the following equation**

**(i)** $\log_4(x^2+x+10) = \frac{1}{2} + \log_4(x^2+x-5)$ **(ii)** $\log_{10}(2x^2+5x-2) = 1$

**(iii)** $\log_x \frac{1}{6} = \frac{1}{2}$ **(iv)** $\log_2(x^2+7x-6) - \log_2 3 = \log_2(x^2+3x-6)$

**Solution:**

**(i)** $\log_4(x^2+x+10) = \frac{1}{2} + \log_4(x^2+x-5)$

$$\log_4(x^2+x+10) = \tfrac{1}{2} + \log_4(x^2+x-5)$$

**Collecting like terms, we obtain**

$$\log_4(x^2+x+10) - \log_4(x^2+x-5) = \tfrac{1}{2}$$

**using the second law of logarithm we have**

$$\log_4 \frac{(x^2+x+10)}{(x^2+x-5)} = \frac{1}{2}$$

$$\Rightarrow \frac{(x^2+x+10)}{(x^2+x-5)} = 4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{(x^2+x+10)}{(x^2+x-5)} = 2$$

$$\Rightarrow (x^2+x+10) = (x^2+x-5)$$

$$\Rightarrow x^2+x+10 = 2x^2+2x-10)$$

$$\Rightarrow 2x^2+2x-10-x^2-x-10 = 0$$

$$\Rightarrow x^2+x-20 = 0$$

Now, by factorization using grouping method, we have

$$x^2+5x-4x-20 = 0$$

$$\Rightarrow x(x+5) - 4(x+5) = 0$$

$$\Rightarrow (x+5)(x-4) = 0$$

$$\Rightarrow x+5 = 0 \quad \text{or} \quad x-4 = 0$$

**Therefore,**

$$x = -5 \quad \text{or} \quad x = 4$$

**(ii)** $\log_{10}(2x^2 + 5x - 2) = 1$

$$\Rightarrow \log_{10}(2x^2 + 5x - 2) = \log_{10} 10 \qquad (\log_{10} 10)$$

$\Rightarrow 2x^2 + 5x - 2 = 10^1$ **(from previous examples)**

$$\Rightarrow 2x^2 + 5x - 2 = 10$$

$$\Rightarrow 2x^2 + 5x - 2 - 10 = 0$$

$$\Rightarrow 2x^2 + 5x - 12 = 0$$

**By factorization method, we have**

$\Rightarrow 2x^2 + 8x - 3x - 12 = 0$

$\Rightarrow 2x(x + 4) - 3(x + 4) = 0$

$\Rightarrow (x + 4)(2x - 3) = 0$

$\Rightarrow (x + 4) = 0$ or $(2x - 3) = 0$

$\Rightarrow x = -4$ or $2x = 3$

**Therefore** $\Rightarrow x = -4$ **or** $x = \frac{3}{2}$

**(iii)** $\log_x(\frac{1}{6}) = -\frac{1}{2}$

**Solution:**

$$\log_x(\tfrac{1}{6}) = -\tfrac{1}{2}$$

$$\Rightarrow x^{-\frac{1}{2}} = \tfrac{1}{6}$$

**Raise both to power of -2 , we have**

$$\Rightarrow (x^{-\frac{1}{2}})^{-2} = (\tfrac{1}{6})^{-2}$$

$$\Rightarrow x^{-\frac{2}{2}} = (6^{-1})^{-2}$$

$$\Rightarrow x^{-1} = 6^2$$

Therefore, $x = 36$

(iv) $\log_2(x^2 + 7x - 6) - \log_2 3 = \log_2(x^2 + 3x - 6)$

$$\Rightarrow \log_2(x^2 + 7x - 6) - \log_2 3 - \log_2(x^2 + 3x - 6) = 0$$

$$\Rightarrow \log\left((x^2 + 7x - 6) \div 3 \div (x^2 + 3x - 6)\right) = 0$$

$$\Rightarrow \log_2\left(\frac{x^2+7x-6}{3(x^2+3x-6)}\right) = 0$$

$$\Rightarrow \left[\frac{x^2+7x-6}{3(x^2+3x-6)}\right] = 0$$

$$\Rightarrow \left[\frac{x^2+7x-6}{(x^2+9x-18)}\right] = 1$$

$$\Rightarrow x^2 + 7x - 6 = 1((3x^2 + 9x - 18)$$

$$\Rightarrow x^2 + 7x - 6 = 3x^2 + 9x - 18$$

$$\Rightarrow 3x^2 + 9x - 18 - x^2 - 7x + 6 = 0$$

$$\Rightarrow 2x^2 + 2x - 12 = 0$$

Dividing through by 2, we have

$x^2 + x - 6 = 0$

Using factorization method, we have

$$x^2 + 3x - 2x - 6 = 0$$

$$\Rightarrow x(x + 3) - 2(x - 3) = 0$$

$$\Rightarrow (x + 3)(x - 2) = 0$$

$$\Rightarrow x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$

$$\Rightarrow x = -3 \quad \text{or} \quad x = 2$$

## 4.0 CONCLUSION

The students can identify the laws of indices and logarithms as well as apply it to problems.

## 5.0 SUMMARY

This unit highlighted the six laws of indices and logarithms, and how they can be used to resolve mathematical problems.

## 6.0 TUTOR-MARKED ASSIGNMENT

Simplify the following:

1. $\dfrac{\left(\frac{16}{18}\right)^{-\frac{1}{4}} \times \left(\frac{27}{8}\right)^{-\frac{4}{3}}}{\left(\frac{8}{27}\right)^{-\frac{1}{3}}}$
2. $6^{\frac{a}{2}} \times 12^{a+1} \times 27^{-\frac{9}{2}} \div 32^{\frac{a}{2}}$

Solve the following equation.

3. $8^{a+1} = 16(2^{a-1})$
4. $3^{2k-1} - 3^{k+1} - 3^{k+1} = 0$
5. $\log_2(a^2 + 7a - 6) = \log_2(a^2 + 3a - 6) + \log_2 8$
6. $\log_{10}(y + 9) = 1 + \log_{10}(y + 1) - \log_{10}(y - 2)$

Simplify the following

7. $\log_4 2 + \log_8 4$
8. $\log_7 98 + \log_7 30 + \log_7 15$
9. $2^{2n-1} - 3^{2-a} = 0$
10. $2^b \times 2^{b+1} = 10$

## 7.0 REFERENCE/FURTHER READING

Sogunro, S.O. (1999). *Basic Business Mathematics Elementary Mathematic*s. Lagos: University of Lagos Press.

# MODULE 2

Unit 1 Growth Mathematics

## UNIT 1 GROWTH MATHEMATICS

**CONTENTS**



## 1.0 INTRODUCTION

Many aspects of business and accounting say depreciation, loans, interest calculations, investment appraisals, have as their basis some relatively simple formula. Our goal is to be able to answer such typical questions like:

1. A firm rents its premises and the rent agreement provided for a regular annual increase of N2, 550. If the rent in the first year is N9, 500, what is the rent in the tenth year?

2. A building cost N500, 000 and it depreciate at 10% per annum on

the reducing balance method. What will its written down value be after 25 years?

3. If N1, 000 is invested at 18% interest compounded semi-annual, what will be its worth in 5 years?

4. How long does it take an investment to double at an interest rate 8%?

5. If I buy a N200, 000 house, put N40, 000 down, down, and obtain a 30 years mortgage for the balance at a 9% annual interest rate, what be my monthly payment?

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine simple and compound interest
- undertake discount and commission
- explain the principle of annuity
- state how to calculate percentage and proportion.

## 3.0 MAIN CONTENT

### 3.1 Series

In many financial calculations is the concept of allocating or paying out receiving money at some regular interval (i.e. weekly, monthly, every three months or quarterly, every four months and even annually). Typical examples are depreciation calculations, investing funds, loan repayment and cash flow analysis. We represent these situations by series of which the two most common types are arithmetic and geometric progressions.

#### 3.1.1 Arithmetic Progression

This is also called. This is a series of quantities where each new value is obtained by adding a constant amount to the previous value. The constant amount is sometimes called the common difference.

An arithmetic progression is of the form:
$a, a + d, a + 2d, a + 3d, \quad , a+(n-1)d$
where a is the first term
d is the common difference
n is the number of terms in the series.

**Example 1:**

Mr Jacob buys equipment for N32, 500 which is expected to last for 20 years and to have a scrap of N7, 500. If it depreciates on the straight line method, how much would be provided for in each year?

In this problem, number of terms in the series is one more than the number of years because the cost is the value at the beginning of the first year and the scrap value is at the end of the year.

**Solution:**

$L_n = a + (n - 1)d$, is the $n^{th}$ term of the progression.

$$n=21, a=32,500, L_n = 7,500$$
$$7,500 = 32,500 + (21 - 1)d$$
$$7,500 = 32,500 + (20)d$$

$$7,500 - 32,500 = (20)d$$
$$-2,500 = 205$$
$$-\frac{25,000}{20} = d$$
$$d = -1250$$

The straight line depreciation is N1250 per annum.

### 3.1.2 Sum of Arithmetic Progression

To find the sum of arithmetic progression is to evaluate each of the successive terms and sum them up. So we have the formula as
$S_n = \underline{n}2(2a + (n - 1) d)$ where $S_n$ is the sum of arithmetic progression

**Example 2:**

An employee, who received fixed annual increment, had a final salary of N90, 000 per annual after 10 years, if total salary was N650, 000 over 10years, what was his initial salary?

**Solution**:

Note that $S_n = S_{10}$ = N 650, 000, a =? and d= ?

$$S_n = 650,000 = \frac{10}{2}(2a + (10-1)d)$$
$$650,000 = 5(2a + 9d)$$
$$130,000 = (2a + 9d) \quad (1)$$

**From above statement**

$$L_n = 90,000 = (a + (10-1)d)$$
$$L_n = 90,000 = (a + 9d) \quad (2)$$

**Then, solving above equations smultanously, we have**

**130, 000=2a +9d**

**90, 000=a + 9d**

**40, 000=a**

**Substitute for a in (2)**

**90, 000=40, 000 +9d**

**50, 000=9d**

$$d = \frac{50,000}{9} = 5555.5555556$$

**d=N5, 556 approximately**

**The annual increment is N5, 556**

## 3.2 Geometric Progression

A series of quantities where each value is obtained by multiplying the previous value by a constant value known as common ratio called a *geometric progression* or *exponential progression*. A geometric progression is of the form

a, ar, $ar^2$, $ar^3$, $ar^4$, $ar^{n-1}$

where a is the first term.

r is the common ratio

n is the number of terms in the series

A geometric progression has a general formula given as $ar^{n-1}$

**Example 3:**

A building cost N500, 000 and depreciates at 10% per annum on the reducing balance method. What will its value be after 25 years?

**Solution**:

a=500,000 n=26 and r=(1−d)=(1−0.10) where d is the depreciation rate

Value after 25 years = $ar^{n-1}$

= 500, 000$(1- 0.10)^{26-1}$

= 500, 000$(0.90)^{25}$

= 500, 000

=N35, 894.90 approximately

### 3.2.1 Sum of Geometric Progression

In a similar way as in arithmetic progression, the terms of geometric progression could be evaluate and added together with a final simple formula as:

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r < 1$$

$$S_n = \frac{a(r^n-1)}{r-1} \qquad r > 1$$

$$S_n = \infty \qquad r = 1$$

**Example 4:**

A company sets up a sinking fund and invests N20, 000 each year for 25 years at 9% compound interest. What will the fund be worth after 5 years?

Note:
**Solution:**
From above questions, it can be inferred that N20, 000 is invested at the end of the each year and so last allocation earns no interest, we have the whole series set up as

$$S_n = 20,000\left(1 + (1+0.09) + (1+0.09)^2 + (1+0.09)^3(1+0.09)^4\right)$$

**Applying**

$$S_n = \frac{a(r^n-1)}{r-1} \qquad r > 1$$

**a=20,000, r= 1.09 and n=5**

$$S_5 = \frac{20,000((1+0.09)^5-1)}{((1+0.09)-1)}$$

$$S_5 = \frac{20,000((1+0.09)^5-1)}{0.09}$$

$$S_5 = \frac{20,000(1.53863-1)}{0.09} = N19,69421 \qquad approximately$$

## 3.3 Simple and Compound Interest

The concept of simple interest has to do with the problems involving the basic concepts of progressions just discussed. Common practice uses the following terminology:

P is the sum at the present time or principal
S is the sum arising in the future
T is the number of the interest hearing period usually, but not exclusively, expressed in years
I is the total amount of interest.

Suppose p (naira) called the principal are invested in an enterprise (which may be a bank bond or a common stock) with an annual interest rate of r, simple interest is the amount earned on the p naira over a period of time.

Hence if P naira are invested for n years, then the simple interest I is given by I=Prt

In other words, the process whereby interest only accrues on the principal is known as simple interest. In this case the interest is not re interest to earn more interest.

S=P+I
S=P+Prt
S=P(1+rt) which is the total amount at the end of transaction

**Example 5:**
How much will N15, 000 amount to at 10% simple interest over 20 years?

**Solution**:

$$\mathbf{P=N15{,}000\ n=20,\ r=}8\% = 0.08$$

$$\mathbf{S=P(1+rt)}$$
$$=15,000(1+(0.08\times 20))$$
$$\mathbf{=15,\ 000(1+1.6)}$$
$$\mathbf{=15,\ 000(2.60)}$$
$$\mathbf{=N39,\ 000}$$

The second method of paying interest is the compound interest method. Here the interest for each of time period is added to the principal before interest is computed for the next time period.

This method applies whenever the period interest payment are not withdrawn compound interest is the interest paid on the interest previously earned as well as on the original investment. In this case interest is reinvested to earn more interest. If that interest is paid annually then, suppose P naira are invested the interest after one year is rp naira and original invested is now worth

$P + rP = P(1+r)$ after first year.
$P(1 + r) + rP(1 + r) = P(1 + r)^2$ after second year.
$P(1 + r)^2 + rP(1 + r)^2 = P(1 + r)^3$ after third year.
If it continues for say n years
$S = P(1+r)^n$ where S denotes the amount of the investment after n year with an interest rate of r.

**Example 6:**
What is the value after 20 years of a N5, 0000 invested earning 10% interest compounded annually

**Solution:**
We have p=5, 000, r=0.10 and n=20
$S = P(1+r)^n$ $S = 5{,}000(1+0.10)^{20}$
$S = 5{,}000(1.10)^{20}$
S =5,000(6.7275) = ₦ 33.637.50 approximately

**Example 7**:
What compound rate of interest will be required to produce N5, 000 after 5 years which an investment of N4, 000?

**Solution**:

$S = P(1+r)^n$
$\frac{S}{P} = (1+r)^n$ **(3)**
**Take $n^{th}$ root of both sides**

$$\left(\frac{S}{P}\right)^{\frac{1}{n}} = ((1+r)^n)^{\frac{1}{n}} = (1+r)$$

**n=5, S=5. 000, P=4, 000**
**We find r as**

$$S_n = \left(\frac{S}{P}\right)^{\frac{1}{n}} - 1 = r$$

**or**

$$S_n = \sqrt[5]{\frac{S}{P}} - 1 = r$$

$$S_n = \sqrt[5]{\frac{5,000}{4,000}} - 1 = r$$

$$S_n = \sqrt[5]{1.25} - 1 = r$$

$$S_n = (1.25)^{0.20}$$

$$S_n = 0.04564$$

$$S_n = 4.56\% \qquad approximately$$

## 3.4 Discounting

The basic compounding principle of $S_n = P(1 + r)^n$ may be used in making P ( the principal invested ) the subject. It will be apparent that there are occasions when the future values are known and it's required to calculate the present value (P). The formula can be put as

$$P = \frac{S}{(1+r)^n}$$

which is restated in term of discounting to a present value.

Note :
This formula is the basis of all discounting method and is particularly useful as the basis of discounting cash flow techniques.

In practice interest is compounded more frequently than annually. If its paid k times a year, then interest is $\frac{r}{k}$ and in n years, there are nk periods.
we apply

$$S = P\left(1+\frac{r}{k}\right)^{nk}$$

for

$$P = \frac{S}{\left(1+\frac{r}{k}\right)^{nk}} = S\left(1+\frac{r}{k}\right)^{-nk}$$

The value P is known as the present value of an investment worth S for nk periods at interest rate of r% compounded k times a years.

**Example 8:**
If interest is compounded quarterly at 6%?

(i) How long will it take an investment to double?

(ii) How much will have to be invested now to produce N20, with a 10% compound interest rate?

**(i)Solution:**

$$S = P\left(1+\frac{r}{k}\right)^{nk}$$

**S=2P and k=4, r=0.06,** $\frac{r}{k} = \frac{0.06}{4} = 0.015$

$$2P = P(1+0.015)^{4n}$$

$$2 = (1+0.015)^{4n}$$

Take the log of both sides

$$\log 2 = 4n\log(1+0.015)$$

$$\log 2 = 4n\log(1.015)$$

$$\frac{\log 2}{4\log(1.015)} = n$$

$$n = \frac{0.3010}{0.0259} = 11.62$$

**(ii)Solution:**
**S=20, 000 r=0.10 and n=6**

$$P = \frac{S}{\left(1+\frac{r}{k}\right)^{nk}}$$

$$P = \frac{20,000}{(1+0.1)^6}$$

$$P = \frac{20,000}{(1.7716)} = N11,289.23 \quad approximately$$

## 3.5 Discounting a Series

There are many problems dealing with discounting one value and may need to do with involvement of a whole series of cash flows required to be discounted to a present value. In such a case the formula

$$P = \frac{S}{(1+r)^n}$$

Becomes

$$P = \sum_{i=1}^{i=n} \frac{A_i}{(1+r)^i}$$

$A_i$ represents the cash flow arising at the end of year 1, 2, 3 ,4.......n i.e. 1,2, 3 ,4........ n

Example 9:

What is the present value of receiving N1,500 in one year's times N3,000 in 2 years time and N4, 500 in 3 years time when the discount rate is 10%.

**Solution:**

$$P = \sum_{i=1}^{i=n} \frac{A_i}{(1+r)^i}$$

$i = 1, 2, 3$ and $r = 0.01$

$A_1 = 1,500$ $A_2 = 3,000$ $A_3 = 4,500$

$$P = \frac{A_1}{(1+r)^1} + \frac{A_2}{(1+r)^2} + \frac{A_3}{(1+r)^3}$$

$$P = \frac{1,500}{(1+0.01)^1} + \frac{3,000}{(1+0.01)^2} + \frac{4,500}{(1+0.01)^3}$$

$$P = \frac{1,500}{(1.01)^1} + \frac{3,000}{(1.01)^2} + \frac{4,500}{(1.01)^3}$$

$P = 1,500(0.909) + 3,000(0.826) + 4,500(0.751)$

$P = 1363.5 + 2478 + 3379.5$

## 3.6 Bank Discount

The charge of interest on a loan is not calculated on amount borrowed but the balance of amount to be repaid later.

A charge for a loan computed in this manner is called the *Bank Discount*. The amount the borrower receives is called the *proceeds* of a loan, and is the amount received now.

P=S(1-dt)where NS is the future amount to be paid back and borrower receives NP( proceed is P)rate of interest is d , where d is the bank discount rate and the period is t years.

## 3.7 Compound Interest with Growing Annual Investment

Suppose a sum of NP is invested at the beginning of the year and each subsequent year, an additional sum of Na is added to the investment. If no withdrawals are made and the whole sum invested is allowed to accumulate at a compound interest rate r, then the balance B(t) after t years is given by

$$B(t) = \left(P + \frac{a}{t}\right)(1+i)^t - \frac{a}{t}$$

**Example 10:**

(i) Find the proceeds for a N4, 000 two year loan from a bank, if the discount rate is 10%

(ii) Ade invested N15, 000 at the beginning of 1990, it remains invested and on first January each subsequent year, another N500 is added is. What sum will be available to Ade 31st 1998 if interest is compounded annually at the rate of 5% per annum.

**Solution:**

(i) P=S(1-dt), where S=4, 000, d=0.10 and t=2, dt=0.2, 1-0.2=0.8 P=4,000(0.8)

(ii) P=N15, 000, a=N500, d=5% =0.10 t=9years

$$B(t) = \left(P + \frac{a}{t}\right)(1+d)^t - \frac{a}{d}$$

$$B(9) = \left(15,000 + \frac{500}{0.05}\right)(1+0.05)^9 - \frac{500}{0.05}$$

$B(9) = (15,000 + 10,000)\,(1.05)^9 - 10,000$
$(25,000)(1.05)^9 - 10,000$
**=N28, 783.21**

## 3.8 Annuities

An *annuity* is a sequence of fixed equal payments (or receipts) made over uniform interval, and some common examples of annuities are:
Weekly or monthly salaries, insurance premiums, house purchase, mortgage payment and hire purchase payment. Annuities are used in all areas of business and commerce. Loans are normal repaid with an annuity investment funds are set up to meet fixed future commitments (for example, asset replacement) by the [payment of an annuity, perpetual annuities can be purchased with (single) lump sum payment to enhance pensions.

**Types of Annuity:**

a. Annuities may be paid
i. at the end of payment intervals (ordinary annuity)
ii. at the beginning of payment intervals (a due annuity)
b. The terms of an annuity may
i. beginning and end of fixed dates (ascertain annuity )
ii. depend on some event that cannot be fixed (a contingent annuity)

c. A perpetual annuity is one that carries on indefinitely Calculations involving annuity are:
i. Accrued amount (compound interest ) $A = P (1 + i)^r$
ii. Sum of the first a term of a geometrical progression

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

**Example 11:**
Suppose N1, 000 is invested in a saving plan at the end of each year that 8% interest is paid compound annually, how much will be in the account after 5 years?

**Solution:**

$$B(t) = B + B(1+i) + B(1+i)^2 + B(1+i)^3 + B(1+i)^4 + \ldots\ldots + +B(1+i)^{n+1}$$

$$B(t) = B\left(1 + (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4 + \ldots\ldots + (1+i)^{n+1}\right)$$

$$\frac{(1+i)^n - 1}{i}$$ **Note:** *this is by applying the sum of $i^{st}$ n term of a GP*

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$S_n = A_n, a = B, r = (i+1)$ **We have**

$$A_n = \frac{B((1+i)^n - 1)}{(1+i) - 1}$$

**Given that B=1000 i=0.008 n=5**

$$A_n = \frac{1000((1+0.08)^5 - 1)}{(1+0.008) - 1}$$

$$A_n = \frac{1000((1.08)^5 - 1)}{(0.008)} = N5,866.60$$

## 3.9 Present Value of an Annuity

The present value of an ordinary annuity is given by

$$P = \frac{B}{i}\left[1 - \left(\frac{1}{1+i}\right)^n\right] \quad or \quad \frac{B}{i}\left[1 - (1+i)^{-n}\right]$$

where B is periodic payment of an annuity
I is the interest rate paid each period
n is the number of periods

**Example 12:**
What is the present value of an annuity that would be N4, 000 a year for 15 years assuming an interest rate 6% compound annually?

**Solution**:

$$P = \frac{B}{i}\left[1 - \left(\frac{1}{1+i}\right)^{n}\right] \quad or \quad \frac{B}{i}\left[1-(1+i)^{-n}\right]$$

Here P = initial lump sum to be put in the saving account or any other types of investment where the interest is compounded annually at the rate of 6% to giving room for yearly withdrawal/payment of N4,000 for five years B=4,000 I=6%=0.06 n=15

$$P = \frac{4,000}{0.06}\left[1-(1+0.06)^{-15}\right]$$

$$P = \frac{4,000}{0.06}\left[1-(1.06)^{-15}\right]$$

$$P = \frac{4,000}{0.06}\left[1-(0.4172)\right] = N38,848 \quad approximately$$

it would be observed that the amount is less than N6,000 which is supposed to be the total withdrawals at the end of 15years. This is due to accumulation of interest compounding rate periodically.

$a_{n-1}$=present the value of an ordinary annuity consisting of payment of N1 with interest rate High pay at the end of each period. By setting B=I

$$a_{n-1} = \frac{(1-(1+i)^{n})}{i}$$

Present value of an ordinary annuity with payment of N B = $Ba_{n-1}$
Note :

An annuity due is an annuity in which payments are made at the beginning of the time periods. Examples of annuity due are deposit in a saving account, rents payment, payment of an insurance premium. This means that an annuity due drams interest from one more period than the corresponding ordinary annuity.

## 3.10 Amortisation and Sinking Funds

An interest bearing debts is set to be amortised when all liabilities (both principal and interest) are discharged by sequence of (usually) equal payment made at equal interval of time.

**Example 13:**

A debt of N5, 000 with at interest of 5% compounded is set to be amortised by equal semi annual payment of R over the next three years, the first due is six months. Find the payment, six payment of R from an ordinary annuity whose present value is N5, 000 then.

**Solution**:

$R_{n-1}$=5, 000

$$R = \frac{5,000}{a_{6-0.025}}$$

$$= \frac{5,000}{1-(1+0.025)^6}$$

$$= \frac{0.025 \times 5,000}{1-(1+0.025)^6}$$

$$= \frac{125}{1-0.8623} = \frac{125}{0.1377} = N907.77 \qquad approximately$$

**Example 14:**

Mr x buys a new car which sell for N180, 000. He agrees to pay for the car over 4 years by making 48payments, one at the rate of 12% compounded monthly. What will his payment?

**Solution:**

P=180, 000, 0.1212 n=48, the, we are to find B which is the monthly payment

$$B = \frac{iP}{1-(1+i)^n}$$

$$B = \frac{1,800}{1-(1.01)^{49}}$$

$$B = \frac{1,800}{1-(1.6203)}$$

$$B = \frac{1,800}{0.3797} = N4,740.58 \qquad monthly \quad approximately$$

## 3.11 Sinking Funds

A Sinking Fund is an account into which periodic deposits are made so that a fixed sum of money may be paid on the due maturity. It is an ordinary annuity with fixed future value.

$$A_n = B(S_{n-1} - 1) = B\left(frac(1+i)^n - 1i\right)$$

we have

$$B = \frac{A_n}{S_n}, \qquad = \frac{iA_n}{(1+i)^n - 1}$$

Where B is the regular payment for the required sinking fund to have a future value $A_n$ paid at the end of each period.

**Example 15:**
How much will have to be invested at the end of each year at 8% compounded annually to pay of N75, 000 after 10 years?

**Solution**:
$A_n$ = 75, 000 n=10 I=8%=0.08

$$B = \frac{iA_n}{(1+i)^n - 1}$$
$$B = \frac{0.08 \times 75,000}{(1+0.08)^{10} - 1}$$
$$B = \frac{0.08 \times 75,000}{(1.08)^{10} - 1}$$
$$B = \frac{6,000}{(2.1589) - 1}$$
$$B = \frac{6,000}{1.1589} = N5,177.32$$

**Example 16:**
A company set aside a sum of N18, 000 annually to enable it to pay off a debenture issue of N220, 000 at the end of 10years. Assuming that the sum accumulates at 4% per annum compounded interest, fund the surplus after paying off the debenture stock.

**Solution:**
At the end of 10 years N1, 800 will amount to 18, 000 × $(1.04)^{10}$
The second sum of N18, 000 will amount to 18, 000 × $(1.04)^{9}$
Then total money will be

$$18,000\left(\frac{1+(1.04)^{10}}{1-(1.04)}\right)\times(1.04)$$

$$18,000\left(1+(1.04)^{10}-1\right)\times\frac{1.04}{(0.04)}$$

$$\frac{18,000(1.04)}{0.04}\times(1.47990-1)$$

$$\frac{18,000(1.04)}{0.04}\times(0.47990)=N224.174$$

The required surplus is N (224172-220, 000) =N4, 712

## 3.12 Percentages

There are really kinds of ratios which are very useful in making comparisons. The fractions or more correctly ratios with 100 as denominations are known as percentages, the term meaning "per hundred" the denominator of such fractions is always omitted and numerator is called the rate percent which may written as percent (PC) or often %

### 3.12.1 Percentage Gain or Loss

**We define gain as selling price-costprice**
**Percentage gain**$=\frac{actualgain}{costprice}\times 100$
**where selling - cost price** $>0$ **is known as gain**
**but when the selling price - cost pric** $<0$ **is a loss**
**We can also define loss as cost price -selling price or loss=CP-SP**
**Percentage loss**$=\frac{actualloss}{costprice}\times 100$
**Percentage error**$=\frac{actual}{truevalue}\times 100$
**We denotes CP =cost price SP=selling price**

**Example 17:**

A man bought 1000 oranges for N1040, 160 of them were bad, and he sold the rest N20 a dozen. What percentage profit or loss did he make?

**Solution**:
160 oranges were bad.
840 oranges were sold
at 1 dozen=N20
1 orange was sold at = N then SP for 840

**1 orange was sold at** $= N\left(\frac{20}{12}\right)$
**then SP for 840**
$2012 \times 840 = N1,400$
**CP=N1, 240**
**Gain=SP -CP=1, 400-1, 240=N160**
**Gain**$\% = \frac{160}{1400} \times 100 = 11.43\%$

**Example 18:**
A trader buy some goods all at the same price. He sells profit of 16%, and has to sell the remaining 10 at a loss of percentage profit on the deal?

**Solution:**
Let the goods cost Nx each
Total cost profit for 30 goods=N30x
20 sold at profit of 165
Cost prices for 20 goods =N20x Selling price for 20
=(CP + prof it%) =(100% + 16%)
=116%
SP for 20=1.16 of CP 20(1.16)x=23.2x
Then 10 of them at loss of 4% CP for 10 =10x
SP for 10= CP-loss% =(100-4%)
=96%
=0.96
**SP for 10 of 0.96 of CP**
**10x(0.96)=9.6x**
**Total selling price=23.2x+9.6x=N32.8x**
**Total profit=total selling price-total cost price**
**32.8x-30x=2.8x**
**percentage profit=**$\frac{(2.8x)100}{30x} = 0.083 \times 100 = 9.3\%$

**Example 18:**
A shopkeeper marks his goods to 45% but allow 5% discount for cash. By selling a purse he makes a profit of N18.875 on a cash deal. Find what the shopkeeper paid for the purse.

**Solution**:
Marked price=45% of CP SP=Marks Price
Let x be the cost price of the good.
The marked pric is 45% of the cost price
Marked price =(100+45)x100
Marked price =145x100 = 1.45x

SP=1.45x
Dicount of 5% mean 5% off the selling price or marked price

i.e $\frac{(100-5)x}{100} \times 1.45x$
$\frac{(95x}{100} \times 1.45x = 0.95(1.45) = 1.3775(\text{SP in cash})$
profit=SP(in cash) -CP
=1.3775x-x
18.875=0.3775x
$x = \frac{18.875}{0.3775}$ =N50

i.e. the shopkeeper paid N50 for the goods

A common scale for a detailed map is 2cm to 5km. This means that 2cm on the map represents an actual horizontal distance of 5km. The scale can also be given as 1 in 250:500 or 1:250,000, which compares the distance on the map with actual horizontal distance.

The different quantities of the same kind may always be compared in this way. If one of the quantities is expressed as a fraction of the other quantity, this fraction is said to be the ratio of their sizes. It should be noted that the quantities that a ration express in the form a/b is written as being a:b
Thus $\frac{8}{36} = \frac{2}{9} = 2:9$

**Example 19:**
Express a length of 8cm to a length of 3m as a ratio

Solution:
$3m = (3 \times 100)cm = 300cm$
$8cm : 3cm = 8cm : 300cm$
$\frac{8cm}{300cm} = \frac{8}{300} = \frac{2}{75} = 2:75$

**Example 20:**
Express a speed of 12km/h to a speed of 10m/s as a ratio.

Solution:

$12km/h = \frac{12km}{h} = \frac{12 \times 1000m}{60 \times 60} = \frac{10}{3}m/s$
$12km/h : 10m/s = \frac{10m/s}{3} : 10m/s$
$\frac{\frac{10m/s}{3}}{10m/s} = \frac{10}{3} \times \frac{1}{10} = \frac{1}{3} = 1:3$

**Example 21:**

If the price of two commodities A and B are N450 and N600 respectively, find the ratio of Price of commodity A to B. Price of commodity B to A.

**Solution**:

Price of Commodity A $= N450$

Price of Commodity B $= N600$

(i) Ratio of Price of Commodity A to B

$= \frac{Price of Commodity A}{Price of Commodity B}$

$= \frac{N450}{N600} = \frac{3}{4} = 3:4$

(ii) Ratio of Price of Commodity B to A

$= \frac{Price of Commodity B}{Price of Commodity A}$

$= \frac{N600}{N450} = \frac{4}{3} = 4:3$

**Example 22**:

A sales agent allows the list price of his goods a trade discount of 20% and a cash discount of 5%. What is the ratio of the cash price to the list price?

**Solution**:

Let the list price = x

20% discount $= \frac{20}{100} \times x = 0.2x$

The new list price$=(x - x = 0.2x)$

$(1 - 0.2)x = 0.8x$

5% cash discount on the new list price:

$= \frac{5}{100} \times x = 0.8x$

$= 0.05 \times 0.8x$

$= 0.04x$

The cash price = New list price - cash discount

$0.8x - 0.04x$

$0.76x$

Ratio of cash price to list price

$= \frac{Cash Price}{list Price}$

$= \frac{0.76}{x} = \frac{0.76}{1} = \frac{76}{100}$

$= \frac{19}{25} = 19.25$

**Example 23:**

Find the ratio between the selling price which will give a profit of 20% in the cost price and the selling price which will give a profit of cost price is the same in both cases.

**Solution**:

Let the C = Cost price and s = selling price

20% profit in cost price = $\frac{120}{100}$C

= $\frac{120}{100}$C⇒ S ⇒ 100S=120C

$\frac{S}{C} = \frac{120}{100} = \frac{1}{2}$ (1)

Similarly, 25% profit in cost price = $\frac{125}{100}$C

= $\frac{125}{100}$C⇒ S ⇒ 100S=125C

$\frac{S}{C} = \frac{125}{100} = \frac{5}{4}$ (2)

The required ratio = Ratio of equation (1) to equation (2)

$\frac{120}{100}C = \frac{120}{100}C$

120 : 125

24 : 25

## 3.13 Proportion

Proportion is the equality of ratios. Suppose a given quantity of value of money, assets is to be shared among two or more individuals, then the ratio or proportion of sharing must be stated. In solving problem on proportion, the individual ratios are added together to obtain the general ratio.

**Example 24**:
Find A: B: C. Given that A:B=4:3 and A:C=4:5 **Solution**:
A:B=4:3
A:C=4:5
A:B:C=4:3:5

**Example 25**:
Kola, Tola and Shola share N4800. If the ratio of Kola: of Kola:Sola = 4:5 Find the individual's share.

**Solution**
Kola: Tola = 4:3
Kola:Shola = 4:5
Kola: Tola:Shola = 4:3:5
Total ratio = 4+3+5=12
(i)Kola share=$\underline{4}$
12× N 4800 = N 1600
(ii)Tola share =$\underline{3}$12× N 4800 = N 1200
(iii Shola share = $\underline{5}$ N 4800 = N 2000
Tola = 4:3 and that
12×

**Example 26:**
If A:B: = 4 : 3 and B : C := 4 : 5 Find A:B:C

**Solution**:
A:B = 4 : 3
B:C: = 4 : 5
⇒A:B=4:3×4=16:12 ⇒B:C =4:5×3=12:15 A:B:C=16:12:15

**Example 27:**
Kunle, Tunde and Dele share the sum of N34, 000. If the ratio of Kunle of Tunde is 4:3 and that of Tunde of Dele is 4:5. Find the share of each of them.

**Solution:**
Kunle : Tunde=4:3
Tunde : Dele=4:5
⇒ Kunle : Tunde = 4 : 3 × 4 = 16 : 12
⇒Tunde : Dele = 4 : 5 × 3 = 12 : 15
Kunle : Tunde : Dele=16: 12 : 15
Total ratio=16+ 12 + 15= 43
(i) Kunle share=16 43× N 43, 000 = N 16, 000
(i i)Tunde share =12
43× N 43, 000 = N 12, 000
(iii Dele share =15 43× N 43, 000 = N 15, 000

**Example 28:**
If A:B: = 4 : 2 and B : C := 2 : 5 Find A:B:C
**Solution:**
A:B = 4 : 2
B:C: = 2 : 5
⇒A:B=4:2×2=8:4
⇒B:C =2:5×4=18:20 A:B:C= 8:4:20=2:1:5

**Example 29**:
If N6400 is to be shared among Wale, Tade and Ade such that the ratio of Wale of Tade is 4:2 and that of Wale to Ade is 2:5. Find the share of each one of them.

**Solution:**
Wale: Tunde = 4:2
Wale: Ade = 2:5
⇒Wale:Tade = 4 : 2 × 2 = 8 : 4
⇒Tunde: Ade = 2 : 5 × 4 = 8 : 20
⇒Wale:Tade : Ade = 8 : 4 : 20 = 2 : 1 : 5 Total ratio=8
(i)Wale's share= $^{28}$×N6400=N1600

(ii)Tade's share = $^{18}\times$N6400=N800
(iii Ade's share = $^{58}\times$N6400=N4000

## 4.0 CONCLUSION

Conclusively, Students could understand simple and compound interest and an annuity and sinking fund as a well relationship that exist between them. At the end they are able to determine the percentage and proportion.

## 5.0 SUMMARY

This unit focused on the use of arithmetic and geometric progression to solve financial calculations for business and accounting purposes.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Find in what time a sum of money trebles itself at 5 percent per annum compound interest.

2. The sum of N20, 000 is borrowed at 4 percent per annum compound interest. Principal and interest are to be repaid in 25 equal, annual instalments beginning one year hence. Find the yearly payment.

3. A sum of money was invested by Ajibola at compound interest amount to N21, 632 at the end of the second year and to N22, 497.28 at end of the third year. Find the rate of interest and sum invested.

4. A machine costs a company N100, 000 and its effective life is estimated to be 20 years. If the scrap is expected to realise N5000 only find the sum to be invested every year at 5 percent per annum compound interest for 20 years, to replace the machine which is expected to cost them 25 percent more over the its percent cost. Assume that the sale of scrap would be utilised for meeting the cost of the machine.

5. (a) KAMAH (Nig) Limited decides to invest N10,000 at the beginning of 1992 in a fund earning 12% per annum. KMAH (Nig) Limited will add further N2, 500 to the fund at the beginning of each year, commencing in 1993. What will be the value of the total investment in the fund at the end of 1996?

(b) As an alternate form of investment the company decide to make equal annual instalments starting at the beginning of 12%. Calculate to the nearest Naira the annual investments necessary for the fund to have the same value at the end of 1996 in (a)

(c) Calculate the present value of perpetual annuity of N10, 000 at 12% per annum first payable in one year.

6. A proposal has come before the Management Board of LASPOTECH for the purchase of machine for processing of palm oil in the school of Agricultural at Ikorodu. Anticipated results for the expected five year life of the machine are supplied by the coordinator of the project as follows:
   - i. If the Management board cost of capital is 12% per annum.
   - ii. Would you advise the management to invest in the machine?

7. A certain project is expected to yield the returns given below over the next five years. It would require an initial investment of N3, 500. Determine its internal rate of return. State how would use this in deciding whether or not to invest in the project.

| $Year$ | $Returns(N)$ |
|---|---|
| 1 | 2000 |
| 2 | 4000 |
| 3 | 6000 |
| 4 | 5000 |
| 5 | 3000 |

8. An investment opportunity has the following expected cash flows: The discount rate is 12%. You are required:
   a. Calculate the payback period
   b. Calculate the opportunity’s Internal Rate return
   c. Calculate the opportunity’s Net Present value.

9. A gari processing industry is considering the replacement of its processing plant which could not cope with the present processing demand of gari. The company is given the choice, the profitability of the plants. A discount rate of 10 percent is to used. Determined which of the plant to be bought, earning after taxation are expected to as follows:

| $Year$ | $Plant\ A\ Cash\ inflow$ | $Plant\ B\ Cash\ inflow$ |
|---|---|---|
| 1 | 15,0000 | 5,000 |
| 2 | 20,000 | 15,000 |
| 3 | 25,000 | 20,000 |
| 4 | 15000 | 30,0000 |
| 5 | 10,000 | 20,000 |

10. The management of KAMAH (Nig.) Ltd. is considering two mutually exclusive projects X and Y, investment is N10, 000 on each one of them. The life of the asset is expected to be with residual value. Net profit is expected to be as follows:

| $Year$ | $X(N)$ | $Y(N)$ |
|---|---|---|
| 1 | – | 2000 |
| 2 | 2000 | 3000 |
| 3 | 3000 | 4000 |
| 4 | 3000 | 50000 |
| 5 | 4000 | 1000 |
| 6 | 4000 | 5000 |
| 7 | 3000 | 1000 |
| 8 | 2000 | 1000 |
| 9 | 2000 | 1000 |
| 10 | 1000 | 1000 |

Using the discount rate of 20% determine which of the project is more profitable investment.

## 7.0 REFERENCE/FURTHER READING

Sogunro, S.O. (1999). *Basic Business Mathematics and Elementary Mathematic*s. Lagos: University of Lagos Press.

## MODULE 3



## UNIT 1 MATRIX ALGEBRA AND VECTOR

### CONTENTS

## 1.0 INTRODUCTION

This unit will discuss Matrix, types, Crammer's rules, Gaussian elimination and solutions to simultaneous equations using Matrix Approach.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify rows and columns of a Matrix
- classify different types of Matrices
- perform operation on Matrix
- describe Determinant
- solve simultaneous equations using Matrix and Crammer's rule.

## 3.0 MAIN CONTENT

### 3.1 Matrix

**Definition 1**:
A Matrix is a triangular or rectangular or square array of objects or items or numbers (real, complex, rational, irrational, natural numbers) in rows and columns enclosed within brackets, subject to certain rules of operations.

A Matrix having m rows and n columns is called an "m by n " or m× n and is referred to as having order m × n.

**Example 1:**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & . & a_{mn} \end{pmatrix}_{m \times n}$$

In the above Matrix, the numbers or functions $a_{ij}$ (n = $a_{ij}$ ) called its elements. In the, double subscript notation, the first subscript indicates the row and the second subscript indicates the column in which the element stands. Matrix or a Matrix of order m × n.
Suppose

$$B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}_{2\times 2} \quad and \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{2\times 2}$$

We say that B is a 2 × 2 Matrix while C is a 3 × 3 Matrix.

## 3.2 Types of Matrix

### 3.2.1 Square Matrix

**Definition 2:**
This is the type of Matrix in which the number of rows equal number of columns.

**Example 2:**
**If**

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2} \quad is\ a\ 2 \times 2\ Matrix$$

**and**

$$D = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}_{3\times 3} \quad is\ a\ 3 \times 3\ Matrix$$

### 3.2.2 Zero Matrix

**Definition 3:**

A zero or null or Void Matrix is Matrix each whose elements is zero is called zero or null or Void Matrix

**Example 3:**

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

### 3.2.3 Diagonal Matrix

**Definition 4:**

It is a square Matrix that has its diagonal elements to be non-zero while other elements re zero. In order words the elements $a_{ij,}$ are called diagonal elements of a square matrix ($a_{ij}$ ).

**Example 4:**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{3\times 3}$$

Is a diagonal Matrix.
It should be noted that, the diagonal elements is a diagonal matrix may also be zero as shown below:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}_{2\times 2} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2} \quad are\ also\ digonal\ Matrices.$$

### 3.2.4 Identity Matrix

**Definition 5:**
A diagonal matrix whose diagonal elements are equal to 1 (unit) is called identity matrix or unit matrix.

**Example 5:**

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$

## 3.3 Triangular Matrix

**Definition 6:**

A triangular matrix is a square Matrix $a_{ij}$ elements $a_{ij} = 0$.
It is referred to as LOWER TRIANGULAR Matrix whenever $i < j$ and UPPER TRIANGULAR MATRIX whenever $i > j$

**Example 6:**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 3 & 6 & 2 \end{pmatrix}_{3\times3} \qquad B = \begin{pmatrix} 0 & 3 \\ 4 & 2 \end{pmatrix}_{2\times2} \textit{are lower triangular matrices}$$

**and**

$$C = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{3\times3} \qquad D = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}_{2\times2} \textit{are upper triangular matrices}$$

## 3.4 Scalar Matrix

**Definition 7:**

This is a diagonal matrix whose diagonal elements are equal.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}_{2\times2} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}_{3\times3} \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times3} \textit{are scalar matrices}$$

## 3.5 Row Matrix

**Definition 8:**
It is a matrix which has exactly one row.

**Example 7 :**
A=(1 2 3)

## 3.6 Column Matrix

**Definition 9:**
It is a matrix which has exactly one column

**Example 8:**

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

## 3.7 Algebra of Matrices

### 3.7.1 Equality of Matrices

**Definition 10:**

Two matrices A and B are said to be equal if:

(a) Both A and B are of the same order.

(b) Corresponding elements in both A and B are the same.

**Example 9:**

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 10 & 2 & 5 \\ 3 & 4 & 2 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 4 & 1 & 3 \\ 10 & 2 & 5 \\ 3 & 4 & 2 \end{pmatrix}_{3\times3} \quad then, A = B$$

### 3.7.2 Sum and Difference of Matrices

**Definition 11:**

If A = ($a_{ij}$ ) and B = ($b_{ij}$ ) are two × Matrices, their sum or difference (A ± +B) is definition as the Matrix C = ($c_{ij}$ ), where each element of C is the sum or difference of the corresponding elements of A and B. Thus, A ± B = ($a_{ij} \pm b_{ij}$ ) . Two matrices of the same order are said to be conformable for addition or subtraction.

**Example 10:**

Suppose

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 3 \end{pmatrix}_{3\times3} \quad and \quad B = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 4 \end{pmatrix}_{3\times3}$$

**Solution**:

**Find (i) A+B (ii)A-B**

$$(i)\quad A+B=\begin{pmatrix}1&0&3\\4&1&3\end{pmatrix}_{2\times 2}+\begin{pmatrix}1&4&2\\3&1&4\end{pmatrix}_{3\times 3}$$

$$=\begin{pmatrix}1+1&0+4&3+2\\4+3&1+1&3+4\end{pmatrix}_{3\times 3}=\begin{pmatrix}2&4&5\\7&2&7\end{pmatrix}_{3\times 3}\quad and$$

**(ii)**

$$A-B=\begin{pmatrix}1&0&3\\4&1&3\end{pmatrix}_{3\times 3}-\begin{pmatrix}1&4&2\\3&1&4\end{pmatrix}_{3\times 3}$$

$$=\begin{pmatrix}1-1&0-4&3-2\\4-3&1-1&3-4\end{pmatrix}_{3\times 3}=\begin{pmatrix}0&-4&1\\1&0&-1\end{pmatrix}_{3\times 3}$$

### 3.7.3 Multiplication of Matrix

**Definition 12:**

**Scalar Multiplication:**

If K is any complex number and A , a given matrix then KA is the Matrix obtained from A by multiplying each element of A by K. The number K is called scalar.

**Example 11:**

If

$$A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}_{3\times 3}$$

**and K=4. Find KA**

**Solution:**

$$KA=4\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}_{3\times 3}=\begin{pmatrix}4&8&12\\16&20&24\end{pmatrix}_{3\times 3}$$

**Example 12:**

**If**

$$A=\begin{pmatrix}1&2&3\\4&1&2\end{pmatrix}\qquad B=\begin{pmatrix}3&6&7\\1&4&5\end{pmatrix}_{3\times 3}$$

**Find the value of 2A+3B**
**Solution:**

$$2A = 2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}_{3\times3} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 2 & 4 \end{pmatrix}_{3\times3}$$

$$3\begin{pmatrix} 3 & 6 & 7 \\ 1 & 4 & 5 \end{pmatrix}_{3\times3} = \begin{pmatrix} 9 & 18 & 21 \\ 3 & 12 & 15 \end{pmatrix}_{3\times3}$$

$$2A + 3B = \begin{pmatrix} 2+9 & 4+18 & 6+21 \\ 8+3 & 2+10 & 4+15 \end{pmatrix}_{3\times3} = \begin{pmatrix} 11 & 22 & 27 \\ 11 & 12 & 19 \end{pmatrix}_{3\times3}$$

**Example 13**:

If

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 6 & 5 \end{pmatrix}_{2\times2}$$

Find Matrix B such that A × B

**Solution**:

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$A = \begin{pmatrix} 2+b_{11} & 1+b_{12} \\ 4+b_{21} & 4+b_{22} \\ 6+b_{31} & 5+b_{32} \end{pmatrix}$$

**It implies**

$2 + b_{11} = 0 \Rightarrow b_{11} = -2$
$1 + b_{12} = 0 \Rightarrow b_{11} = -1$
$4 + b_{21} = 0 \Rightarrow b_{21} = -4$
$3 + b_{22} = 0 \Rightarrow b_{22} = -3$
$6 + b_{31} \Rightarrow b_{31} = -6$
$5 + b_{32} \Rightarrow b_{32} = -5$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -4 & -3 \\ -6 & -5 \end{pmatrix}$$

### 3.7.4 Multiplication of Two Matrices

**Definition 13:**

The product of AB of two Matrices A and B is defined only when the number of columns of A is the same as the number of rows in B. If A and B were order m × n and n × p respectively, then the product AB is a matrix of order n × p.

**Example 14:**

If

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix}_{2\times 3} \qquad and B = \begin{pmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}_{3\times 2}$$

Find (i) AB (ii) BA

**Solution:**

$$AB = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} (2\times 7)+(4\times -2)+(6\times 4) & (2\times 1)+(4\times 9)+(6\times 3) \\ (3\times 7)+(9\times -2)+(5\times 4) & (3\times 1)+(9\times 9)+(5\times 3) \end{pmatrix}$$

$$= \begin{pmatrix} 14-8+24 & 6-8+4 \\ 21-18+20 & 3+81+15 \end{pmatrix} = \begin{pmatrix} 30 & 56 \\ 23 & 99 \end{pmatrix}$$

**Example 15:**

If $A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \\ 3 & 4 \end{pmatrix}$ find $A^a + 3A + 5I$ where I is unit Matrix of order 2.

**Solution**:

$$A^2 = \begin{pmatrix} 1 & 2 \\ 5 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & -2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} (1 \times 1) + (2 \times 3) & (1 \times 2) + (2 \times 0) \\ (-3 \times 1) + (0 \times 3) & (-3 \times 2) + (0 \times 0) \\ (3 \times 3) + (4 \times 6) & (3 \times 4) + (4 \times 2) \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & 2+0 \\ -3+0 & -6+0 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ -3 & -6 \end{pmatrix}$$

## 3.8 Transpose Matrix

**Definition 14**:

The transpose of a matrix is where the rows change to columns and column change to rows.

If A = $(a_{ij})_{m,n}$ then $A^T$ or $A^t = (a_{ij})$ where $A^T = A^t$ =transpose of A.

**Example 16:**

$$A = \begin{pmatrix} -2 & 3 & 5 \\ 11 & 2 & 6 \end{pmatrix}_{2\times3} \qquad A^T = \begin{pmatrix} -2 & 11 \\ 3 & 2 \\ 5 & 6 \end{pmatrix}_{3\times2}$$

Also if

$$B = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 8 \\ 7 & 6 & 9 \end{pmatrix}_{3\times3} \qquad B^T = \begin{pmatrix} 2 & 4 & 7 \\ 3 & 5 & 6 \\ 1 & 8 & 9 \end{pmatrix}_{3\times3}$$

## 3.9 Symmetric Matrix

**Definition 15:**

Let A be a square Matrix. If $A = A^T$ , then we say that A is a symmetric Matrix.

**Example 17**: If

$$A = \begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & 8 \\ -1 & 8 & 7 \end{pmatrix} \qquad A^T = \begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & 8 \\ -1 & 8 & 7 \end{pmatrix}$$

Hence $A^T = A$ which implies that A is a symmetric Matrix.

## 3.10 Skew Symmetric Matrix

**Definition 16:**

Suppose A is a square Matrix, then A is said to be a Skew symmetric Matrix if $A = -A^T$ They are only valid for square Matrix ( i.e. symmetric and skew symmetric Matrices).

**Example 18:**

Let

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

Solution:
Then,

$$A^T = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

$$-A^T = -\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = A$$

## 3.11 Determinant of a Matrix

**Definition 17:**

With any square Matrix there is associated a number Δ which is calculated from products of the elements of the Matrix . Thus if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then, determinant of A is

$$\Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a \times d) - (c \times b) = ad - bc|A|$$

represent determinant of A or Δ.

**Example 19:**

Find the determinant of Matrix A. Given that

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} |A| = (1 \times 7) - (3 \times 5) = 7 - 15 = -8$$

Consider a 3 × 3 Matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Using first row to expand

$$\Delta = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \qquad 1$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg) = aei - afh - bdi + bfg + cdh - ceg$$

Alternatively, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} f & d \\ i & g \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \qquad 2$$

$$= a(ei - fh) + b(di - fg) + c(dh - eg) = aei - afh + bdi - bfg + cdh - ceg$$

**Note:**
The minor of the element $a_{ij}$ is the Matrix obtained by deleting the $i^{th}$ row and $j^{th}$ column.

**Example 20**: The minor of the element

$$a_{22} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$$

**minor of the element**

$$a_{23} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

**minor of the element**

$$a_{31} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

**Example 21:**

Calculate the value of the determinant of Matrix A, if

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 5 & -1 & 2 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 5 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 \\ -11 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix}$$

$$= 2(0 + 3) + 1(2 - 15) + 3(-1 - 0) = 2(3) + 1(13) + 3(-1) = 6 + 13 - 3 = 16$$

Using the first column to expand

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} c & b \\ i & h \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

$$= a(ei - fh) - d(ch - bi) + g(bf - ce) = aei - afh - dch + dbi + gbf - gce$$

Alternatively, if

$$A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

$$= a(ei - fh) + d(ib - ch) + g(bf - ce) = aei - afh - dch + dbi + gbf - gce$$

## 3.12 Sarru's Rule

**Definition 18:**

Sarru's rule can be applied in getting determinant of a Matrix.
If Suppose Matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

then, using Sarru's rule, we have

$$|A| = \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix} \downarrow_{+} \quad \uparrow^{-}$$

Note:
Here we have added first two columns in each row to the row now giving us a 3×5 matrix the map (or cross) as indicated to get.

$$|A| = aei + bfg + cdh - ceg - afh - bdi$$

**Example 22**: Given that Matrix

$$A = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 5 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 & 2 & 1 \\ 1 & 0 & 3 & 1 & 0 \\ 5 & -1 & 2 & 5 & -1 \end{vmatrix}$$

$$|A| = (2\times0\times2) + (1\times3\times5) + (3\times1\times-1) - (5\times0\times3) - (-1\times3\times2) - (2\times1\times1)$$
$$0 + 15 + (-3) - 0 - (-6) - 2 = 15 - 3 + 6 - 2 = 16$$

Note: Sarru's rule is the only applicable to 3 × 3 Matrix

## 3.13 Properties of Determinants

(1) If we add (or subtract) a scalar multiple of a row or column to another then the determinant does not change.

e.g.

Let

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad and \quad B = \begin{vmatrix} a+tc & b+td \\ c & d \end{vmatrix}$$

$$|B| = (a+tc)d - (b+td)c = ad + tdc - cb - tcd = ad - bc = |A|$$

**(2)** $|A| = |A^T|$

(3) If we interchange two rows (or column) then the sign of the determinant changed.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = |A| = ad - cd$$

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \quad |B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) = -|A|$$

**(4)**

$$\begin{pmatrix} a \\ c \end{pmatrix} \quad and \quad \begin{pmatrix} b \\ d \end{pmatrix}$$

**or (a, b) and (c, d)are linearly dependent if and only if the determinant of**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

(5) If any two rows or two columns of a square matrix are the same, then the determinant will be zero .e.g. if

$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, |A| = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

(6) The determinant of a diagonal Matrix A is equal to the product of its diagonal elements.

(7) The determinant of the product of two Matrices is equal to the product of the determinant of the two Matrices.e.g |AB| = |A|.|B| where A and B are two given Matrices.

## 3.14 Singular and Non-Singular Matrix

Suppose A is a square Matrix A is said to a singular Matrix if its determinant is equal to zero i.e. |A| = 0. Matrix A is said to be non-singular when its determinant is not equal to zero i.e. if |A| = 0.

**Example 23**: Show that A is a singular Matrix given that |A| = 0.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$|A| = 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$|A| = 2(0-1) - 1(2-2) + 2(1-0) = 2(-1) - 1(0) + 2(1) = -2 - 0 + 2 = 0$$

**Example 24:**

Given that

$$A = \begin{vmatrix} 2x-5 & -1 \\ 2x & x^2 \end{vmatrix} \qquad (0.1)$$

where A is a Singular Matrix, determine the values of x.

Solution:
Since Matrix A is singular

$$A = \begin{vmatrix} x-5 & -1 \\ 2x & x^2 \end{vmatrix} = 0$$

$$x^2(x-5) - (-2x) = 0$$
$$\Rightarrow x^3 - 5x^2 + 2x = 0$$
$$x(x^2 - 5x + 2) = 0$$

dividing both sides by x, we have $x^2 - 5x + 2 = 0$ which is a quadratic equation using factorisation, we have:

⇒ x2− x − 4x + 2 = 0
⇒ x(2x − 1) − 2(2x − 1) = 0
⇒ (x − 2) (2x − 1) = 0
⇒ (x − 2) = 0 or (2x − 1) = 0
⇒x=2 or x = 12

## 3.15 Cofactor of a Matrix

Given an n-square Matrix A. i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23}. & .a_{2n} & \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & a_{mn} \end{pmatrix}$$

The scalar $C_{ij} = (-1)^{i+j} |M_{ij}|$ is called the cofactor of the element $a_{ij}$ of the Matrix A, where M and $(-1)^{i+j}$ are called minor and scalar respectively.

Hence the cofactor of $a_{11} = c_{11} = (-1)^{1+1}$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

and the cofactor

$a_{12} = c_{12} = (-1)^{1+2}$

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Note: cofactor is a scalar while the minor is a Matrix.

## 3.16 Adjoint of a Matrix

The transpose of the cofactor of a Matrix is known as the adjoint of the Matrix.

**Example 25:**
Given that Matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

(a) Determine the cofactor of A, and hence find
(b) Its Adjoint.

**Solution**:

(a) $C_{ij} = (-1)^{i+j}|M_{ij}|$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = (-1)^2(-3-2) = 1(-5) = -5$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = (-1)^3(-1-6) = -1(-7) = 7$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = (-1)^4(1-9) = 1(-8) = -8$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = (-1)^3(-1-0) = -1(-1) = 1$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = (-1)^4(-2-0) = 1(-2) = -2$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = (-1)^5(2-3) = -1(-1) = 1$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = (-1)^4(2-0) = 1(2) = 2$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = (-1)^5(4-0) = -1(4) = -4$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = (-1)^6(6-1) = 1(5) = 5$$

$$Cofactor of A = \begin{pmatrix} -5 & 7 & -8 \\ 1 & -2 & \\ 2 & -4 & 5 \end{pmatrix}$$

**(b) Adjoint of A is the transpose of the cofactor of A**

$$A = \begin{pmatrix} -5 & 1 & 2 \\ 7 & -2 & -4 \\ -8 & 1 & 5 \end{pmatrix}$$

## 3.17 Inverse of a Matrix

(Using adjoint and determinant)
Suppose A is a square Matrix, the inverse of A denoted $A^{-1}$ is given by

$A^{-1} = \frac{Adj(A)}{|A|}$.

**It can be shown that** $AA^{-1} = A^{-1}A = I$

$A^{-1} = \frac{1}{A}$ *or* $A = \frac{I}{A^{-1}}$

**Also** $A(Adj(A)) = (Adj(A))A = |A|I$

**where I is the identity Matrix and A and I are of the same order.**

**Example 26:**
Given that Matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

show that the inverse of Matrix A is denoted by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

**Solution:**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

**Now to find the cofactor of A**

$C_{11} = (-1)^2|d| = 1(d) = d$

$C_{12} = (-1)^3|c| = -1(c) = -c$

$C_{21} = (-1)^3|b| = -1(b) = -b$

$C_{22} = (-1)^4|a| = 1(a) = a$

**Hence, cofactor of**

$$A = \begin{pmatrix} d & c \\ -b & a \end{pmatrix} \qquad Adj(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

**Example 27:**

Find the inverse of A, given that

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

Solution:

$$A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 3 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= (-3-2) - 1(-1-6) + 0(1-9)$$

$$= 2(-5) - 1(-7) + 0(-8)$$

$$= -10 + 7 + 0 = -3$$

$$|A| = -3$$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = \begin{pmatrix} -5 & 1 & 2 \\ 7 & -2 & -4 \\ -8 & 1 & 5 \end{pmatrix}$$

**as obtained in Example 27: above.**
**Hence**

$$A^{-1} = \frac{1}{-3}\begin{pmatrix} -5 & 1 & 2 \\ 7 & -2 & -4 \\ -8 & 1 & 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & \frac{1}{-3} & \frac{2}{-3} \\ \frac{7}{-3} & \frac{2}{3} & \frac{4}{3} \\ \frac{8}{3} & \frac{1}{-3} & \frac{5}{-3} \end{pmatrix}$$

**Example 28:**

A and B are Matrix such that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad and \quad \begin{pmatrix} b1 \\ b2 \end{pmatrix}$$

where $b_1$, $b_2$ are non-zero numbers. Determine the values of K which satisfies AB = KB

**Solution:**

$$AB = KB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} b1 \\ b2 \end{pmatrix} = K \begin{pmatrix} b1 \\ b2 \end{pmatrix}$$

$b_1 + 2b_2 = Kb_1$ (1)

$2b_1 + b_2 = Kb_2$ (2)

from equation (1) we have

$$b_1 - Kb_1 + 2b_2 = 0$$
$$\Rightarrow (1 - K)b_1 + 2b_2 = 0 \qquad (3)$$

from equation (2)we have

$$2b_1 + b_2 - Kb_2 = 0$$
$$\Rightarrow 2b_1 + (1 - K)b_2 = 0 \qquad (4)$$

Multiplying equation (3) by 2, we have

$$2(1 - K)b_1 + 4b_2 = 0 \qquad (5)$$

Multiplying equation (4) by $(1 - K)$, we have

$$2(1 - K)b_1 + (1 - K)(1 - K)b_2 = 0 \qquad (6)$$

Subtract equation (6)from to equation (5) to get

$$4b_2 - (1 - K)(1 - K)b_2 = 0$$
$$\Rightarrow 4b_2 = (1 - K)(1 - K)b_2$$

Dividing both sides by $b_2$, we have

$$4 = (1 - K)(1 - K)$$
$$\Rightarrow 4 = 1 - K - K + K_2$$
$$\Rightarrow K^2 - 2K - 3 = 0$$
$$\Rightarrow (K + 1)(K - 3) = 0$$
$$K + 1 = 0 \ or \ K - 3 = 0$$
$$K = -1 \ or \ K = 3$$

Therefore the values of K for which $AB = Kb$ are $-1$ and $3$

**Example 29:**

Given that

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} B = \begin{pmatrix} a \\ b \end{pmatrix}$$

$a \neq 0, b \neq 0$ and $AB = KB$ show that $(5-K)a + 2b = 0$ By finding another equation satisfied by a and b,show that $K^2 - 6K + 1 = 0$

Solution

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = K \begin{pmatrix} a \\ b \end{pmatrix}$$

$$5a + 2b = Ka$$

$$\Rightarrow\Rightarrow 5a - Ka + 2b = 0$$

$$\Rightarrow (5-K)a + 2b = 0(required) \qquad (1)$$

$$Also 2a + b = Kb$$

$$\Rightarrow 2a + b - Kb = 0 \qquad (2)$$

$$2a + (1-K)b = 0$$

Multiply equation (1) by $(1-k)$ to get

$$(5-K)(1-K)a + 2(1-K)b = 0 \qquad (3)$$

Multiply equation (2) by 2 to get

$$4a + 2(1-K)b = 0$$

Subtracting equation (4) from equation (3), we have

$$(5-K)(1-K)a - 4a = 0$$

$$\Rightarrow [(5-K)(1-K) - 4]a = 0$$

Dividing both sides by a to get

$$(5-K)(1-K) = 4$$

$$5 - 5K - K + K^2 - 4 = 0$$

$$K^2 - 6K + 1 = 0 \quad (proved).$$

**Example 30:**

Solve for x, y, z in the matrix equation

$$\begin{pmatrix} z & x \\ -y & z \end{pmatrix} + \begin{pmatrix} 7 & y \\ 6 & x \end{pmatrix} = \begin{pmatrix} 4y & -y \\ 4 & -3 \end{pmatrix}$$

**Solution:**

$$z + 7 = 4y$$

$\Rightarrow z - 4y = -7$ (1)

$$x + y = -y$$

$\Rightarrow x + 2y = 0$ (2)

$$-y + 6 = 4$$

$$y = 6 - 4y = 2$$

and

$$z + x = -3$$

$$x + z = -3$$

substitute $\mathbf{y} = 2$ in equation (2), we have

$$x + 2(2) = 0$$

$$x = -4$$

also substitute $\mathbf{y} = 2$ equation (1)in to get

$$z - 4(2) = -7$$

$$z = -7 + 8$$

$$\Rightarrow z = 1$$

Therefore $\mathbf{x} = -4$, $\mathbf{y} = 2$ and $\mathbf{z} = 1$

**Example 31**:

Given that Matrix

$$A = \begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}$$

Find $A^2 - 3A + 5$

Solution

$$A^2 = AA = \begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 4+20 & 8-4 \\ 10-5 & 20+1 \end{pmatrix} = \begin{pmatrix} 24 & 4 \\ 5 & 21 \end{pmatrix}$$

$$A^2-3A+5=\begin{pmatrix}24&4\\5&21\end{pmatrix}-3\begin{pmatrix}2&4\\5&-1\end{pmatrix}+5$$
$$=\begin{pmatrix}24&4\\5&21\end{pmatrix}-\begin{pmatrix}6&12\\15&-3\end{pmatrix}+5$$
$$=\begin{pmatrix}18&-8\\-10&24\end{pmatrix}+5=\begin{pmatrix}18&-8\\-10&24\end{pmatrix}+5I$$

where

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$=\begin{pmatrix}18&-8\\10&24\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}18&-8\\-10&24\end{pmatrix}+\begin{pmatrix}5&0\\0&5\end{pmatrix}$$
$$=\begin{pmatrix}23&-8\\-10&29\end{pmatrix}$$

Therefore

$$A^2-3A+5=\begin{pmatrix}23&-8\\10&29\end{pmatrix}$$

**Example 32:**

**Let A and B be $3\times3$ Matrix with A $=(a_{ij})$ and B $=(b_{ij})$ where $a_{ij}=3i-j$ and $b_{ij}=2i+3j$ Find (i) $A+2B$ (ii) $A^T$ (iii) $3A+B-20$ (iv) $B^{-1}$(v) $(A+B)^T$+ I, where I is identity Matrix.**

**Solution:**

Consider

$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\qquad(0.2)$$

**then, since A $=(a_{ij})$ and** $a_{ij}=3i-j$

$a_{11}=3(1)-1=3-1=2,\ a_{12}=3(1)-2=3-2=1,\ a_{13}=3(1)-3=3-3=0$

$a_{21}=3(2)-1=6-1=5,\ a22=3(2)-2=6-2=4,\ a_{23}=3(2)-3=6-3=3$

$a_{31}=3(3)-1=9-1=8,\ a_{32}=3(3)-2=9-2=7,\ a_{33}=3(3)-3=9-3=6$

**Hence**

$$A=\begin{pmatrix}2&1&0\\5&4&3\\8&7&6\end{pmatrix}$$

also Let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

**Since then, B** $= (b_{ij})$ **and** $b_{ij} = 2i + 3j$ $b_{11} = 2(1) + 3(1) = 2 + 3 = 5,$ $b_{12} = 2(1) + 3(2) = 2 + 6 = 8,$ $b_{13} = 2(1) + 3(3) = 2 + 9 = 11$

$b_{21} = 2(2)+3(1) = 4+3 = 7,$ $b_{22} = 2(2)+3(2) = 4+6 = 10,$ $b_{23} = 2(2)+3(3) = 4+9 = 13$

$b_{31} = 2(3)+3(1) = 6+3 = 9,$ $b_{32} = 2(3)+3(2) = 6+6 = 12,$ $b_{33} = 2(3)+3(3) = 6+9 = 15$

**Therefore,**

$$B = \begin{pmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{pmatrix}$$

**(i)**

$$A + 2B = \begin{pmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 8 & 7 & 6 \end{pmatrix} + 2\begin{pmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 5 & 14 & 3 \\ 8 & 7 & 6 \end{pmatrix} + 2\begin{pmatrix} 10 & 16 & 22 \\ 14 & 20 & 26 \\ 18 & 24 & 30 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 17 & 22 \\ 19 & 24 & 29 \\ 26 & 31 & 36 \end{pmatrix}$$

**(ii)If**

$$\begin{pmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 8 & 7 & 6 \end{pmatrix}$$

**the**

$$A^T = \begin{pmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 0 & 3 & 6 \end{pmatrix}$$

**(ii)**

$$3A + B - 20 = 3\begin{pmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 8 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{pmatrix} - 20$$

$$= \begin{pmatrix} 6 & 3 & 0 \\ 15 & 12 & 9 \\ 24 & 21 & 18 \end{pmatrix} + \begin{pmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{pmatrix} - 20 = \begin{pmatrix} 11 & 11 & 11 \\ 22 & 22 & 22 \\ 33 & 33 & 33 \end{pmatrix} - 20$$

$$= \begin{pmatrix} 11 & 11 & 11 \\ 22 & 22 & 22 \\ 33 & 33 & 33 \end{pmatrix} - 20I$$

Where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 11 & 11 \\ 22 & 22 & 22 \\ 33 & 33 & 33 \end{pmatrix} - \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = \begin{pmatrix} -9 & 11 & 11 \\ 22 & 2 & 22 \\ 33 & 33 & 13 \end{pmatrix}$$

(iv) and (v) are to be taken as exercises. Answer are at the end of the chapter

## Example 33:

**Given that**

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & x & 4 \end{pmatrix}$$

**(i) Find x, if AB (ii) If AB =**

$$AB = \begin{pmatrix} 4 & 4 & 8 \\ 15 & -3 & 30 \\ 14 & 2 & 28 \end{pmatrix}$$

**find x**
**Solution**

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & 6 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 2 & x & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} (2\times 1)+(1\times 2) & (2\times 3)+(1\times x) & (2\times 2)+(1\times 4) \\ (3\times 1)+(6\times 2) & (3\times 3)+(6\times x) & (3\times 2)+(6\times 4) \\ (4\times 1)+(5\times 2) & (4\times 3)+(5\times x) & (4\times 2)+(5\times 4) \end{pmatrix}$$

$$AB = \begin{pmatrix} 2+2 & 6+x & 4+4 \\ 3+12 & 9+6x & 6+24 \\ 4+10 & 12+5x & 8+20 \end{pmatrix} = \begin{pmatrix} 4 & 6+x & 8 \\ 15 & 9+6x & 30 \\ 14 & 12+5x & 28 \end{pmatrix}$$

**But**

$$AB = \begin{pmatrix} 4 & 6+x & 8 \\ 15 & 9+6x & 30 \\ 14 & 12+5x & 28 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 8 \\ 15 & -3 & 30 \\ 14 & 2 & 28 \end{pmatrix}$$

$\Rightarrow 6 + x = 4$ **or** $9 + 6x = -3$ **or** $12 + 5x = 2$
$\Rightarrow x = 4 - 6$ **or** $6x = -3 - 9$ **or** $5x = 2 - 12$
$\Rightarrow x = -2$ **or** $6x = -12$ **or** $5x = -10$
$\Rightarrow x = -2$ **or** $x = \frac{-12}{6}$ **or** $x = \frac{-10}{5}$
$x = -2$ **or** $x = -2$ **or** $x = -2$
**Trivially,the Matrix above has shown that** $x = -2$

**Example 34:**
Suppose Matrix

$$A = \begin{pmatrix} 2 & x & 2 \\ -3 & 1 & 4 \\ -1 & 0 & 6 \end{pmatrix}$$

(i) Find the value of x for which A is a singular Matrix and (ii) Hence determine the Adjoint of A.

**Solution**
(i) If A is a singular Matrix then, |A| = 0

$$\Rightarrow A = \begin{vmatrix} 2 & x & 2 \\ -3 & 1 & 4 \\ -1 & 0 & 6 \end{vmatrix} = 0$$

$$\Rightarrow 2\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix} - x\begin{vmatrix} -3 & 4 \\ -1 & 6 \end{vmatrix} + 2\begin{vmatrix} -3 & 1 \\ -1 & 0 \end{vmatrix} = 0$$

$\Rightarrow 2(6 - 0) - x(-18 + 4) + 2(0 + 1) = 0$
$\Rightarrow 2(6) - x(-14) + 2(1) = 0$
$\Rightarrow 12 + 14x + 2 = 0$
$\Rightarrow 14 + 14x = 0$
$14x = -14$
$x = \frac{-14}{14} = -1$
$x = -1$
**(ii) Now**

$$A = \begin{pmatrix} 2 & -1 & 2 \\ -3 & 1 & 4 \\ -1 & 0 & 6 \end{pmatrix}$$

**To determine the cofactor of A**

$$a_{11} = (-1)^{1+1} = \begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix} = (-1)^2(6-0) = (1)(6) = 6$$

$$a_{12} = (-1)^{1+2} \begin{vmatrix} -3 & 4 \\ -1 & 6 \end{vmatrix} = (-1)^3(-18+4) = (-1)(-14) = 14$$

$$a_{13} = (-1)^{1+3} \begin{vmatrix} -3 & 1 \\ -1 & 0 \end{vmatrix} = (-1)^3(0+1) = (1)(1) = 1$$

$$a_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix} = (-1)^3(-6-0) = (-1)(-6) = 6$$

$$a_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 2 \\ -1 & 6 \end{vmatrix} = (-1)^4(12+2) = (1)(14) = 14$$

$$a_{23} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = (-1)^5(0-1) = (-1)(-1) = 1$$

$$a_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} = (-1)^4(-4-2) = (1)(-6) = -6$$

$$a_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 2 \\ -3 & 4 \end{vmatrix} = (-1)^5(8+6) = (-1)(14) - 14$$

$$a_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} = (-1)^6(2-3) = (1)(-1) = -1$$

**Hence, cofactor of**

$$A = \begin{pmatrix} 6 & 14 & 1 \\ 6 & 14 & 1 \\ -6 & -14 & -1 \end{pmatrix}$$

**Therefore adjoint of**

$$A = \begin{pmatrix} 6 & 6 & -6 \\ 14 & 14 & -14 \\ 1 & 1 & -1 \end{pmatrix}$$

## 3.18 Solutions to Simultaneous Equations using Matrix Approach

### Using Inverse of a Matrix

The inverse of Matrix can also be use to provide solution to a simultaneous linear equation. In this section we are going to consider simultaneous equation with two and three unknowns.

**Example 35:**

(a) Given that

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 8 & 4 & -4 \\ 5 & -5 & -5 \\ 7 & 1 & -11 \end{pmatrix}$$

(a) (i) Find AB, (ii) Find the inverse A

(b) Using the inverse of A in (a)(ii)above, or otherwise, solve the following simultaneous equation.

3x + 2y − 2z + 8 = 0
x−3y+3=0
2x + y − 3z + 9 = 0

**Solution:**

**(a) (i)**

$$AB = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 4 & -4 \\ 5 & -5 & -5 \\ 7 & 1 & -11 \end{pmatrix}$$

$$= \begin{pmatrix} 24+10-14 & 12-10-2 & -12-10+22 \\ 8-15+7 & 4+15+1 & -4+15-11 \\ 16+5-21 & 8-5-3 & -8-5+33 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = 20 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 20I$$

**(ii)**

$$A^{-1}A = I \Rightarrow A^{-1} = \frac{I}{A}$$

**but**

$$AB = 20I \Rightarrow A = \frac{20I}{B}$$

**Now**

$$A^{-1} = \frac{\frac{I}{20I}}{B}$$

**Since**

$$A = \frac{I}{20B}$$

$$A^{-1} = \frac{B}{20} = B\frac{1}{20}$$

$$A^{-1} = \frac{I}{20}\begin{pmatrix} 8 & 4 & 4 \\ 5 & -5 & -5 \\ 7 & 1 & -11 \end{pmatrix} = \begin{pmatrix} \frac{8}{20} & \frac{4}{20} & \frac{-4}{20} \\ \frac{5}{20} & \frac{-5}{20} & \frac{-5}{20} \\ \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} & \frac{-1}{5} \\ \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{pmatrix}$$

(b) Re-writing the equation, we have:

$3x + 2y - 2z = -8$
$x-3y+z=-3$
$2x + y - 3z = -9$

Write the equations in Matrix form we have:

$$\begin{pmatrix} 3 & 2 & -2 \\ 1 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \\ -9 \end{pmatrix}$$

**Let**

$$A = \begin{pmatrix} 3 & -2 & -2 \\ 1 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix} C = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad and \quad K = \begin{pmatrix} -8 \\ -3 \\ -9 \end{pmatrix}$$

**Hence**

$$A^{-1}K = C$$

**But**

$$A^{-1} = \begin{pmatrix} \frac{8}{20} & \frac{4}{20} & \frac{-4}{20} \\ \frac{5}{20} & \frac{-5}{20} & \frac{-5}{20} \\ \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{pmatrix} \text{ as obtained above.}$$

**Hence**

$$\begin{pmatrix} \frac{8}{20} & \frac{4}{20} & \frac{-4}{20} \\ \frac{5}{20} & \frac{-5}{20} & \frac{-5}{20} \\ \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{pmatrix} \begin{pmatrix} -8 \\ -3 \\ -9 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{-64}{20} - \frac{12}{20} + \frac{36}{20} \\ \frac{-40}{20} + \frac{15}{20} + \frac{45}{20} \\ \frac{-56}{20} - \frac{3}{20} + \frac{99}{20} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{-64-12+36}{20} \\ \frac{-40+15+45}{20} \\ \frac{-56-3+99}{20} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{-40}{20} \\ \frac{20}{20} \\ \frac{40}{20} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$x = -2, \quad y = 1 \quad$ **and** $\quad z = 2$

**Example 36:**

Using the inverse of a Matrix solve the system of the equations

2x + 3y = 7
x+2y=3

**Solution**

**Writing the equation in Matrix form, we have**

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

**Let**

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} x \\ y \end{pmatrix} \quad and \quad C = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$|A| = 2(2) - 1(3) = 4 - 3 = 1$

**To find the cofactor of A we have**

$$a_{11} = (-1)^{1+1}|2| = (-1)^2(2) = 2, a_{12} = (1)^{1+2}|1| = (-1)^3|1| = -1$$

$$a_{21} = (-1)^{2+1}|3| = (-1)^3|3| = -3, a_{22} = (-1)^{2+2}|2| = (-1)^{2+2}|2| = 2$$

**Hence co-factors of**

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad Adj(A) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}Adj(A) = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

**Now to solving the equation**

$$A^{-1}C = B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 14-9 \\ -7+6 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

**x= 5, y = −1**

**Using Matrix inversion method via adjoint of a Method to solve the following system of equation completely.**

$2x_1 - 2x_2 + x_3 + x_4 = 1$

$x_1 + 3x_2 - x_3 + 2x_4 = 2$

$-x_1 + 2x_2 - 2x_3 - x_4 = -3$

$5x_1 + x_2 - 2x_4 = -9$

**Writing the above equations in matrix form, we have**

$$A = \begin{pmatrix} 2 & -2 & 1 & 1 \\ 1 & 3 & -1 & 2 \\ -1 & 2 & -2 & -1 \\ 5 & 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ -9 \end{pmatrix}$$

**Let**

$$A = \begin{pmatrix} 2 & -2 & 1 & 1 \\ 1 & 3 & -1 & 2 \\ -1 & 2 & -2 & -1 \\ 5 & 1 & 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 2 \\ -3 \\ -9 \end{pmatrix}$$

**Now, we find determinant of A, that is**

$$|A| = \begin{vmatrix} 2 & -2 & 1 & 1 \\ 1 & 3 & -1 & 2 \\ -1 & 2 & -2 & -1 \\ 5 & 1 & 0 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & -1 & 2 \\ 2 & -2 & -1 \\ 1 & 0 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -1 & 2 \\ -1 & -2 & -1 \\ 5 & 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 & 2 \\ -1 & 2 & -1 \\ 5 & 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 & -1 \\ -1 & 2 & -2 \\ 5 & 1 & 0 \end{vmatrix}$$

$$= 2\left(3\begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix}\right)$$

$$+2\left(1\begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 5 & -2 \end{vmatrix} + 2\begin{vmatrix} -1 & -2 \\ 5 & 0 \end{vmatrix}\right)$$

$$+1\left(1\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ 5 & -2 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix}\right)$$

$$-1\left(1\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} -1 & -2 \\ 5 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix}\right)$$

$$= 2\,(3(4+0) + 1(-4+1) + 2(0+2))$$

$$+2\,(1(4+0) + 1(2+5) + 2(0+10))$$

$$+1\,(1(-4+1) - 3(2+5) + 2(-1-10))$$

$$-1\,(1(0+2) - 3(0+10) - 1(-1-10))$$

$$2\,((3(4) + 1(-3) + 2(2))$$

$$+2\,(1(4) + 1(7) + 2(10))$$

$$+1\,(1(-3) - 3(7) + 2(-11))$$

$$-1\,(1(2) - 3(10) - 1(-11))$$

$$= 2\,((12-3)+4) + 2\,(4+7+20)$$

$$+1\,(-3-21-22)$$

$$-1\,(2-30+11)$$

$$= 2(13) + 2(31) + 1(-46) - 1(-17) = 26 + 62 - 46 + 17 = 59$$

**Therefore,** $|A| = 59$
**Cofactor** $C_{ij} = (-1)^{i+j}|M_{ij}|$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 & 2 \\ 2 & -2 & -1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= (-1)^2 \left( 3 \begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} - 1 \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + 2 \begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} \right)$$

$$= 1\,(3(40 - 0) + 1(-4 + 1) + 2(0 + 2))$$

$$= 1\,(3(4) + 1(-3) + 2(2))$$

$$= (1)(12 - 3 + 4) = (12 - 3 + 4) = 13$$

$$c_{12} = (-)^{1+2} \begin{vmatrix} 1 & -1 & 2 \\ -1 & -2 & -1 \\ 5 & 0 & -2 \end{vmatrix}$$

$$= (-1)^3 \left( 1 \begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 5 & -2 \end{vmatrix} + 2 \begin{vmatrix} -1 & -2 \\ 5 & 0 \end{vmatrix} \right)$$

$$= (-1)\,(1(4 + 0) + 1(2 + 5) + 2(0 + 10))$$

$$= (-1)(1(4) + 1(7) + 2(10)) = (-1)(4 + 7 + 20)$$

$$= (-1)(31) = -31$$

$$c_{13} = (-)^{1+3} \begin{vmatrix} 1 & 3 & 2 \\ -1 & 2 & -1 \\ 5 & 1 & -2 \end{vmatrix}$$

$$= (-1)^4 \left( 1 \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} - 3 \begin{vmatrix} -1 & -1 \\ 5 & -2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix} \right)$$

$$= (1)\,(1(-4 + 0) + 3(2 + 5) + 2(-1 - 10))$$

$$= (1)(1(-3) - 3(7) + 2(11))$$

$$= (1)(-3 - 21 - 22) = (1)(-46) = -46$$

$$c_{14} = (-)^{1+4}\begin{vmatrix} 1 & 3 & -1 \\ -1 & 2 & -2 \\ 5 & 1 & 0 \end{vmatrix}$$

$$= (-1)^5\left(1\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} -1 & -2 \\ 5 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix}\right)$$

$$= (-1)\,(1(0+2) - 3(0+10) - 1(-1-10))$$

$$= (-1)\,(1(2) - 3(10) - 1(11))$$

$$(-1)(2 - 30 + 11) = (-1)(-17) = 17$$

$$c_{21} = (-)^{2+1}\begin{vmatrix} -2 & 1 & 1 \\ 2 & -2 & -1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= (-1)^3\left(-2\begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix}\right)$$

$$= (-1)\,(-2(4+0) - 1(-4+1) + 1(0+2))$$

$$= (-1)(-2(4) - 1(-3) - 1(2))$$

$$= (-1)(-8+3+2) = (-1)(-3) = 3$$

$$c_{22} = (-)^{2+2}\begin{vmatrix} 2 & 1 & 1 \\ -1 & -2 & -1 \\ 5 & 0 & -2 \end{vmatrix}$$

$$= (-1)^4\left(-2\begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix} - 1\begin{vmatrix} -1 & -1 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 \\ 5 & 0 \end{vmatrix}\right)$$

$$= (1)\,(2(4+0) - 1(2+5) + 1(0+10))$$

$$= (1)\,(2(4) - 1(7) - 1(10))$$

$$= (1)(8 - 7 + 10) = (1)(11) = 11$$

$$c_{23} = (-)^{2+3}\begin{vmatrix} 2 & -2 & 1 \\ -1 & 2 & -1 \\ 5 & 1 & -2 \end{vmatrix}$$

$$= (-1)^5\left(2\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} -1 & -1 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix}\right)$$

$$= (-1)\,(2(-4+1) + 2(2+5) + 1(-1-10))$$

$$= (-1)(2(-3) + 2(7) - 1(-11))$$

$$= (-1)(-6 + 14 - 11) = (-1)(-3) = 3$$

$$c_{24} = (-)^{2+4} \begin{vmatrix} 2 & -2 & 1 \\ -1 & 2 & -2 \\ 5 & 1 & 0 \end{vmatrix}$$

$$= (-1)^6 \left( 2 \begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & -2 \\ 5 & 0 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix} \right)$$

$$= (1)\left(2(0+2) + 2(0+10) + 1(-1-10)\right)$$

$$= (1)(2(2) + 2(10) + 1(-11))$$

$$= (1)(4 + 20 - 11) = (1)(13) = 13$$

$$c_{31} = (-)^{3+1} \begin{vmatrix} -2 & 1 & 1 \\ 3 & -1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= (-1)^4 \left( -2 \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} \right)$$

$$= (1)(-2(2-0) - 1(-6-2) + 1(0+1))$$

$$= (1)(-2(-3) - 1(-8) - 1(1)) = (1)(-4 + 8 - 1) = (-1)(5) = 5$$

$$c_{32} = (-)^{3+2} \begin{vmatrix} -2 & 1 & 1 \\ 3 & -1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= (-1)^5 \left( +2 \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} \right)$$

$$= (-1)\left(2(2-0) - 1(-2-10) + 1(0+5)\right)$$

$$= (-1)\left(2(2) - 1(-12) + 1(5)\right)$$

$$= (-1)(4 + 12 + 5) = (-1)(21) = -21$$

$$c_{33} = (-)^{3+3} \begin{vmatrix} 2 & -2 & 1 \\ 1 & 3 & 2 \\ 5 & 1 & -2 \end{vmatrix}$$

$$= (-1)^6 \left( +2\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} \right)$$

$$= (1)\,(2(-6-2) + 2(-2-10) + 1(1-15))$$

$$= (1)(2(-8) + 1(-12) + 1(-14))$$

$$= (1)(-16-24-14) = (1)(-54) = -54$$

$$c_{34} = (-)^{3+4}\begin{vmatrix} 2 & -2 & 1 \\ 1 & 3 & -1 \\ 5 & 1 & 0 \end{vmatrix}$$

$$= (-1)^6 \left( 2\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} \right)$$

$$= (-1)\,(2(0+1) + 2(0+5) + 1(1-15))$$

$$= (-1)\,(2(1) + 2(5) + 1(-14))$$

$$= (-1)(2+10-14) = (-1)(-2) = 2$$

$$c_{41} = (-)^{4+1}\begin{vmatrix} -2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & -2 & -1 \end{vmatrix}$$

$$= (-1)^5 \left( -2\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} \right)$$

$$= (-1)\,(-2(1+4) - 2(-3-4) + 1(-6+2))$$

$$= (-1)(-2(5) - 1(-7) + 1(-4))$$

$$= (-1)(-10+7-4) = (-1)(-7) = 7$$

$$c_{42} = (-)^{4+2}\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & -1 \end{vmatrix}$$

$$= (-1)^6 \left( 2\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} \right)$$

$$= (1)\,(2(1+4) - 1(-1+2) + 1(-2-1))$$

$$= (1)(2(5) - 1(-1) + 1(-3))$$

$$= (1)(10-1-3) = (1)(6) = 6$$

$$c_{43} = (-)^{4+3}\begin{vmatrix} 2 & -2 & 1 \\ 1 & 3 & 2 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= (-1)^7\left(2\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix}\right)$$

$$= (-1)\,(2(-3-4) + 2(-1+2) + 1(2+3))$$

$$= (-1)\,(2(-7) + 2(1) + 1(5))$$

$$= (-1)(-14+2+5) = (-1)(-7) = 7$$

$$c_{44} = (-)^{4+4}\begin{vmatrix} 2 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & 2 & -2 \end{vmatrix}$$

$$= (-1)^8\left(2\begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix}\right)$$

$$= (1)\,(2(-6+2) + 2(-2-1) + 1(2+3))$$

$$= (1)(2(-4) + 2(-3) + 1(5))$$

$$= (1)(-8-6+5) = (1)(-9) = -9$$

**Hence cofactor of**

$$A = \begin{pmatrix} 13 & -31 & -46 & 17 \\ 3 & 11 & 3 & 13 \\ 5 & -21 & -54 & 2 \\ 7 & 6 & 7 & -9 \end{pmatrix}$$

**Adjoint of**

$$A = \begin{pmatrix} 13 & 3 & 5 & 7 \\ -31 & 11 & -21 & 6 \\ -46 & 3 & -54 & 7 \\ 17 & 13 & 2 & -9 \end{pmatrix}$$

**Therefore,**

$$A^{-1} = \frac{1}{|A|}(Adjoint\ of\ A) = \frac{1}{59}\begin{pmatrix} 13 & 3 & 5 & 7 \\ -31 & 11 & -21 & 6 \\ -46 & 3 & -54 & 7 \\ 17 & 13 & 2 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{13}{59} & \frac{3}{59} & \frac{5}{59} & \frac{7}{59} \\ \frac{-31}{59} & \frac{11}{59} & \frac{-21}{59} & \frac{6}{59} \\ \frac{-46}{59} & \frac{3}{59} & \frac{54}{59} & \frac{7}{59} \\ \frac{17}{59} & \frac{13}{59} & \frac{2}{59} & \frac{-9}{59} \end{pmatrix}$$

**Now, solving the system of equation completely**

$$A^{-1}C = B \Longrightarrow \begin{pmatrix} \frac{13}{59} & \frac{3}{59} & \frac{5}{59} & \frac{7}{59} \\ \frac{-31}{59} & \frac{11}{59} & \frac{-21}{59} & \frac{6}{59} \\ \frac{-46}{59} & \frac{3}{59} & \frac{54}{59} & \frac{7}{59} \\ \frac{17}{59} & \frac{13}{59} & \frac{2}{59} & \frac{-9}{59} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \\ -9 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{13}{59} + \frac{6}{59} - \frac{15}{59} - \frac{63}{59} \\ \frac{-31}{59} + \frac{22}{59} + \frac{63}{59} - \frac{54}{59} \\ \frac{-46}{59} + \frac{6}{59} + \frac{169}{59} - \frac{63}{59} \\ \frac{17}{59} + \frac{26}{59} - \frac{6}{59} + \frac{81}{59} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{-59}{59} \\ \frac{0}{59} \\ \frac{59}{59} \\ \frac{118}{59} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

**Therefore,** $x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad and \quad x_4 = 2$

## 3.19 Crammer's Rules

Crammer's rule is another Matrix method of solving a simultaneous linear equation. This approach is applicable only for square Matrix.
Given A X = K where while K is coefficient.

$$i.e A = \begin{pmatrix} a_{11} & a_{12} & a13 & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & . & a_{mn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \\ . \\ . \\ K_n \end{pmatrix}$$

**Then**

$$X_i = \frac{1}{|A|} \sum_{j=1}^{n} K_j A_{ij}$$

where the summation is the expansion of determinant of A by its ith column if the element of the $i^{th}$ column of A are replaced by $K_1K_2K_3$,......Kn i.e.

$$X_i = \frac{1}{|A|} \begin{pmatrix} K_1 & a_{12} & a_{13} & . & . & a_{1n} \\ K_2 & a_{22} & a_{23} & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ K_n & a_{m2} & a_{m3} & . & . & a_{mn} \end{pmatrix}$$

**Example 37:**

(a) Find the values of w for which

$$\begin{pmatrix} w & 0 & 2 \\ 3 & -1 & 4 \\ 6 & 2w & 0 \end{pmatrix} = 16$$

(b) By substituting the integral value of w as obtained in (a) above, use Crammer's rule to solve the system of equation.

wx + 2z = 3
3x − y + 4z = 4
6x + 2wy = −4

**Solution** (a)

$$\begin{vmatrix} w & 0 & 2 \\ 3 & -1 & 4 \\ 6 & 2w & 0 \end{vmatrix} = 16$$

$$\Rightarrow \begin{vmatrix} w & 0 & 2 \\ 3 & -1 & 4 \\ 6 & 2w & 0 \end{vmatrix} = w \quad w\begin{vmatrix} -1 & 4 \\ 2w & 0 \end{vmatrix} - 0\begin{vmatrix} 3 & 4 \\ 6 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 6 & 2w \end{vmatrix} \quad = 16$$

w(0 − 8w) − 0(0 − 24) + 2(6w + 6) = 16
⇒ w(−8w) − (−24) + 12w + 12 = 16 ⇒ $-8w^2 + 24 + 12w + 12 = 16$
⇒ $-8w^2 + 12w + 12 - 16 = 0$
⇒ $-8w^2 + 12w - 4 = 0$ or
⇒ $8w^2 - 12w + 4 = 0$
⇒ $4(2w^2 - 3w + 1) = 0$
Dividing both side by 4, we have ⇒ $2w^2 - 3w + 1 = 0$
⇒ $2w^2 - 2w - w + 1 = 0$
⇒ 2w(w − 1) − 1(w − 1) = 0 ⇒ (w − 1)(2w − 1) = 0
w=1 or w=½

(b) The integral value of w is 1 with w =1the system becomes wx+2z= 3
3x − y + 4z = 4
6x + 2wy = −4

Writing this equation in Matrix form, we have

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 6 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix}$$

**Let**

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 6 & 2 & 0 \end{pmatrix} \quad \overline{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad and \quad K = \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix}$$

$|A| = 16$ **(given) Using Crammer's ruler**
**Replacing column 1 with the values of K, we have**

$$x = \frac{1}{16}\begin{pmatrix} 3 & 0 & 2 \\ 4 & -1 & 4 \\ -4 & 2 & 0 \end{pmatrix} = \frac{1}{16}\left( (3\begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 0\begin{vmatrix} 4 & 4 \\ -4 & 0 \end{vmatrix} + 2\begin{vmatrix} 4 & -1 \\ -4 & 2 \end{vmatrix} \right)$$

$$= \frac{1}{16}(3(0-8) - 0(0+16) + 2(8-4))$$
$$= \frac{1}{16}(-24 - 0 + 8) = \frac{1}{16}(-16)$$
$$= \frac{16}{-16} = -1$$

**Replacing column 2 with the values of K, we get**

$$y = \frac{1}{16}\begin{vmatrix} 1 & 3 & 2 \\ 3 & 4 & 4 \\ 6 & -4 & 0 \end{vmatrix} = \frac{1}{16}\left( 1\begin{vmatrix} 4 & 4 \\ -4 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 6 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 6 & -4 \end{vmatrix} \right)$$

$$= \frac{1}{16}((0-(-16)) - 3(0-24) + 2(-12-24))$$
$$= \frac{1}{16}(1(16) - 3(-24) + 2(-36))$$
$$= 16(16 + 72 - 72) = \frac{16}{16} = 1$$

**Replacing column third with the values of K**

$$z = \frac{1}{16}\begin{vmatrix} 1 & 0 & 3 \\ 3 & -1 & 4 \\ 6 & 2 & -4 \end{vmatrix}$$

$$= \frac{1}{16}\left( (1\begin{vmatrix} -1 & 4 \\ 2 & -4 \end{vmatrix} - 0\begin{vmatrix} 3 & 4 \\ 6 & -4 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 6 & 2 \end{vmatrix} \right)$$

$$= \frac{1}{16}(1(4-8) - 0(-12-24) + 3(6+6))$$

$$= \frac{1}{16}(1(-4) - 0(-36) + 3(12))$$

$$= \frac{1}{16}(-4 - 0 + 36)$$

$$= \frac{16}{32} = 2$$

**Therefore,** $x = -1, \quad y = 1$ **and** $z = 2$

## 3.20 Gaussian Elimination

This is another method of finding the inverse of a non-singular Matrix. It is a situation whereby a square Matrix been converted to the identity Matrix by series of row operations or by a series of column operations. The same series of operations performed on the identity Matrix will change it (identity Matrix) to the inverse of the square matrix.
The procedures of obtaining the inverse of a square Matrix using Gaussian elimination are as follows.

**Step 1:**
Put the square Matrix and its equivalent identity in the form [A/I] i.e adjoining the Matrices A and I, where A is a square Matrix and I is the identity Matrix and label $R_i$(where i = 1, 2, 3 ,R = row) on the left hand sides e.g$R_1$, $R_2$, $R_3$ if A is a 3×3 Matrix.

**Step 2:**
Divide the first row of the Matrices (i.e. that of the square and identity) by the element in the first column of the square Matrix (i.e. $a_{11}$), then use the result obtained (for this new row) to obtain zeros in the first column of each of the other rows. Continue the labelling of the rows on the left hand side from where you stop i.e. $R_1$, $R_2R_3$ if A is a 3×3 Matrix for example.

**Step 3:**
Divide the second row (in step 2 above i.e. $R_5$ for 3 × 3 Matrix)by the element in its second column to get R8 and use the result obtained for this

new $R_8$ to obtain zero in the second column of each of the other rows ($R_7$ and $R_9$ for a 3×3 Matrix).

**Step 4:**
Divide the third row (in step 3 above, i.e. $R_9$ )by the element in its third column to get $R_{12}$ and use the result obtained in $R_{12}$ to get zero in the third column of each of the other rows (i.e. $R_{10}$ and $R_{11}$ for a 3 × 3 Matrix).

The process will continue till the nth row depending on the type of Matrix one is considering. In other words step (n + 1) will be: Divide the nth row (in step n) by the element in its and use the new row obtained to get zero in the $n^{th}$ column of each of the other rows.

**NOTE:**
(1) it should be noted that when considering the above procedure, if for examples at the(r + 1) step the element in the rth column is zero, you need to interchange that particular row with subsequent row that is having a non-zero element in the rth column and proceed with the (r + 1) step.
(2) The result will be in the form A and I is the identity Matrix.

**Example 38:**

**Given that Matrix**

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

Find the inverse of A, using Gaussian elimination method.
Solution

**Step 1:**

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left(\begin{array}{ccc|ccc} 3 & 2 & -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

**Step 2:**

$$\begin{array}{c} R_4 \\ R_5 \\ R_6 \end{array} \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{11}{3} & \frac{-5}{3} & \frac{1}{3} & -1 & 0 \\ 0 & \frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 0 & -1 \end{array}\right) \begin{array}{c} \frac{1}{3}R_3 \\ R_{4-R_2} \\ 2R_4 - R_3 \end{array}$$

**Step 3:**

$$\begin{array}{c} R_7 \\ R_8 \\ R_9 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{-12}{33} & \frac{3}{11} & \frac{2}{11} & 0 \\ 0 & 1 & \frac{-5}{11} & \frac{1}{11} & \frac{-3}{11} & 0 \\ 0 & 0 & \frac{-20}{11} & \frac{-7}{11} & \frac{-1}{11} & 1 \end{array}\right) \begin{array}{c} \frac{-2}{3}R_8 + R_4 \\ \frac{3}{11}R_5 \\ \frac{1}{3}R_8 - R_6 \end{array}$$

**Step 4:**

$$\begin{array}{c} R_{10} \\ R_{11} \\ R_{12} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{array}\right) \begin{array}{c} \frac{12}{33}R_{12} + R_7 \\ \frac{5}{11}R_{12} + R_8 \\ \frac{-11}{20}R_9 \end{array}$$

**Hence**

$$A^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \\ \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{pmatrix}$$

Alternatively (for better understanding)

**Step 1:**

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 3 & 2 & -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

**Step 2:**

$$\begin{array}{c} R_4 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \frac{1}{3}R_1 \quad \begin{array}{c} R_4 \\ R_5 \\ R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{11}{3} & \frac{-5}{3} & \frac{1}{3} & -1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right) R_4 - R_2$$

$$\begin{array}{c} R_4 \\ R_5 \\ R_6 \end{array} \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{11}{3} & \frac{-5}{3} & \frac{1}{3} & -1 & 0 \\ 0 & \frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 0 & 1 \end{array}\right) 2R_4 - R_3$$

**Step 3:**

$$\begin{array}{c} R_4 \\ R_7 \\ R_6 \end{array} \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{-5}{11} & \frac{1}{11} & \frac{-3}{11} & 0 \\ 0 & \frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 0 & -1 \end{array}\right) \frac{3}{11}R_5 \Longrightarrow \begin{array}{c} R_8 \\ R_7 \\ R_6 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{-12}{33} & \frac{3}{11} & \frac{2}{11} & 0 \\ 0 & 0 & \frac{-5}{11} & \frac{1}{11} & \frac{-3}{11} & 0 \\ 0 & \frac{1}{3} & \frac{5}{3}R_1 & 0 & 0 & 1 \end{array}\right) \frac{-2}{3}R_7 + R_4$$

$$\begin{array}{c} R_8 \\ R_7 \\ R_9 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{-12}{33} & \frac{3}{11} & \frac{2}{11} & 0 \\ 0 & 1 & \frac{-5}{11} & \frac{1}{11} & \frac{-3}{11} & 0 \\ 0 & 0 & \frac{-20}{11} & \frac{2}{3} & 0 & 1 \end{array}\right) \frac{1}{3}R_7 - R_6$$

**Step 4:**

$$\begin{array}{c} R_8 \\ R_7 \\ R_{10} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{-12}{33} & \frac{3}{11} & \frac{2}{11} & 0 \\ 0 & 1 & \frac{-5}{11} & \frac{1}{11} & \frac{-3}{11} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{array}\right) \frac{11}{20}R_9 \Longrightarrow \begin{array}{c} R_{11} \\ R_7 \\ R_{10} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \\ 0 & 1 & \frac{-5}{11} & \frac{1}{11} & \frac{-3}{11} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{array}\right) \frac{12}{33}R_{10} + R_8$$

$$\begin{array}{c} R_{11} \\ R_{12} \\ R_{10} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{array}\right) \frac{5}{11}R_{10} + R_7$$

**Hence**

$$A^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{-1}{5} \\ \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{7}{20} & \frac{1}{20} & \frac{-11}{20} \end{pmatrix}$$

**Example 39:**

Find the inverse of Matrix A, if

$$A = \begin{pmatrix} 0 & -4 & 2 \\ 1 & -1 & 5 \\ 2 & 3 & 4 \end{pmatrix}$$

using Gaussian elimination method.

Hence solve the system of the equations:

$$\begin{aligned} 2z - 4y &= 2 \\ x + 5z &= y + \tfrac{8}{3} \\ 2x + 3y + 4z - \tfrac{5}{3} &= 0 \quad \textbf{completey} \end{aligned}$$

Solution:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left(\begin{array}{ccc|ccc} 0 & -4 & 2 & 1 & 0 & 0 \\ 1 & -1 & 5 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

It can be noticed that the element in the first row, first column of Matrix A is zero, hence there is need to interchange the first and second rows, which will now look like:

$$\begin{matrix} R_4 \\ R_5 \\ R_3 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & -4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{matrix} R_4 \\ R_5 \\ R_6 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & -4 & 2 & 0 & 1 & 0 \\ 0 & -5 & 6 & 0 & 2 & -1 \end{array}\right) \quad 2R_4 - R_3$$

$$\Rightarrow \begin{matrix} R_7 \\ R_8 \\ R_9 \end{matrix} \left( \begin{matrix} 1 & 0 & \frac{9}{2} \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & \frac{7}{2} \end{matrix} \middle| \begin{matrix} \frac{-1}{4} & 0 & 0 \\ \frac{-1}{4} & 1 & 0 \\ \frac{-5}{4} & 2 & -1 \end{matrix} \right) \begin{matrix} R_8 + R_4 \\ \frac{-1}{4} R_5 \\ 5R_4 + R_6 \end{matrix}$$

$$\begin{matrix} R_{10} \\ R_{11} \\ R_{12} \end{matrix} \left( \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} \frac{19}{14} & \frac{-11}{7} & \frac{9}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{-15}{7} & \frac{4}{7} & \frac{-2}{7} \end{matrix} \right) \begin{matrix} \frac{-9}{2} R_{12} + R_7 \\ \frac{1}{2} R_{12} + R_8 \\ \frac{2}{7} R_9 \end{matrix}$$

$$A^{-1} = \begin{pmatrix} \frac{19}{14} & \frac{-11}{7} & \frac{9}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{-5}{14} & \frac{4}{7} & \frac{-2}{7} \end{pmatrix}$$

**CHECKS:**

**To confirm whether the inverse is correct or not**

**NOTE**

$AA^1 = A^{-1}A = I$ i.e

$$\begin{pmatrix} 0 & -4 & 2 \\ 1 & -1 & 5 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} \frac{19}{14} & \frac{-11}{7} & \frac{9}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{-5}{14} & \frac{4}{7} & \frac{-2}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Also**

$$\begin{pmatrix} \frac{19}{14} & \frac{-11}{7} & \frac{9}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{-15}{7} & \frac{4}{7} & \frac{-2}{7} \end{pmatrix} \begin{pmatrix} 0 & -4 & 2 \\ 1 & -1 & 5 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now to solve the system of equations, firstly, we need to rewrite the equations, that is, equations.

$2z - 4y = 2$
$x+5z=y+^{83}$
$2x + 3y + 4z - ^{53} = 0$
becomes
$-4y + 2z = 2$
$x+5z=y+^{83}$
$2x + 3y + 4z = ^{53}$

**Now, writing this in Matrix form,we have**

$$\begin{pmatrix} 0 & -4 & 2 \\ 1 & -1 & 5 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{8}{3} \\ \frac{5}{3} \end{pmatrix}$$

**Let**

$$A = \begin{pmatrix} 0 & -4 & 2 \\ 1 & -1 & 5 \\ 2 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad C = \begin{pmatrix} 2 \\ \frac{8}{3} \\ \frac{5}{3} \end{pmatrix}$$

It can be seen that A is the same as the original Matrix A given and from the working above, we have

$$A^{-1} = \begin{pmatrix} \frac{19}{14} & \frac{-11}{7} & \frac{9}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{-5}{14} & \frac{4}{7} & \frac{-2}{7} \end{pmatrix}$$

**Hence, solving the system of the equations completely, we have**

$$A^{-1}C = B \Rightarrow \begin{pmatrix} \frac{19}{14} & \frac{-11}{7} & \frac{9}{7} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \\ \frac{-15}{7} & \frac{4}{7} & \frac{-2}{7} \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{8}{3} \\ \frac{5}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{38}{14} + (-\frac{88}{21}) + \frac{45}{21} \\ \frac{-6}{7} + \frac{16}{21} - \frac{5}{21} \\ \frac{-10}{14} + \frac{32}{21} - (\frac{-10}{21}) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{114-176+90}{42} \\ \frac{-36+32-10}{42} \\ \frac{-30+64-20}{42} \end{pmatrix} \begin{pmatrix} \frac{28}{42} \\ \frac{-14}{42} \\ \frac{14}{42} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

**Therefore, x= $\frac{2}{3}$, y= $\frac{-1}{3}$ and z= $\frac{1}{3}$**

**Example 40:**

Using Gaussian elimination method to solve the equations

$2x_1 - 2x_2 + x_3 + x_4 = 1$
$x_1 + 3x_2 - x_3 + 2x_4 = 2$
$-x_1 + 2x_2 - 2x_3 - x_4 = -3$
$5x_1 + x_2 - 2x_4 = -9$

**Solution:**

Writing the above equations in Matrix form, we have

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ 1 & 3 & -1 & 2 \\ -1 & 2 & -2 & -1 \\ 5 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \\ -9 \end{pmatrix}$$

**Let**

$$A = \begin{pmatrix} 2 & -2 & 1 & 1 \\ 1 & 3 & -1 & 2 \\ -1 & 2 & -2 & -1 \\ 5 & 1 & 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 2 \\ -3 \\ -9 \end{pmatrix}$$

**Now, to find the inverse of A we follow the following steps**

**Step 1:**

$$\begin{array}{c} R_1 \\ R_2 \\ R_4 \\ R_5 \end{array} \left( \begin{array}{cccc|cccc} 2 & -2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & -1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 5 & 1 & 0 & -2 & 0 & 0 & 0 & 1 \end{array} \right)$$

**Step 2:**

$$\begin{array}{c} R_5 \\ R_6 \\ R_7 \\ R_8 \end{array} \left( \begin{array}{cccc|cccc} 1 & -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -4 & \frac{3}{2} & \frac{-3}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 1 & \frac{-3}{2} & \frac{-1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 6 & \frac{-5}{2} & \frac{-9}{2} & \frac{-5}{2} & 0 & 0 & 1 \end{array} \right) \begin{array}{c} \frac{1}{2}R_1 \\ R_5 - R_2 \\ R_5 + R_3 \\ 5R_9 + R_4 \end{array}$$

**Step 3:**

$$\begin{array}{c} R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{array} \left( \begin{array}{cccc|cccc} 1 & 0 & \frac{1}{8} & \frac{7}{8} & \frac{3}{8} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{-3}{8} & \frac{3}{8} & \frac{-1}{8} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{9}{8} & \frac{7}{8} & \frac{-5}{8} & \frac{1}{4} & -1 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{27}{4} & \frac{7}{4} & \frac{3}{2} & 0 & -1 \end{array} \right) \begin{array}{c} R_{10} + R_5 \\ \frac{-1}{4}R_6 \\ R_{10} - R_7 \\ 6R_{10} - R_8 \end{array}$$

**Step 4:**

$$\begin{array}{c} R_{13} \\ R_{14} \\ R_{15} \\ R_{16} \end{array} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{7}{9} & \frac{4}{9} & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & \frac{7}{9} & \frac{-5}{9} & \frac{2}{9} & \frac{-8}{9} & 0 \\ 0 & 0 & 0 & \frac{-59}{9} & \frac{-17}{9} & \frac{-13}{9} & \frac{-2}{9} & 1 \end{array} \right) \begin{array}{c} \frac{-1}{8}R_{15} + R_9 \\ \frac{3}{8}R_{15} + R_{10} \\ \frac{8}{9}R_{11} \\ \frac{1}{4}R_{15} - R_{12} \end{array}$$

**Step 5:**

$$\begin{array}{c} R_{17} \\ R_{18} \\ R_{19} \\ R_{20} \end{array} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{13}{59} & \frac{3}{59} & \frac{5}{59} & \frac{7}{59} \\ 0 & 1 & 0 & 0 & \frac{-31}{59} & \frac{11}{59} & \frac{-21}{59} & \frac{6}{59} \\ 0 & 0 & 1 & 0 & \frac{-46}{59} & \frac{3}{59} & \frac{-54}{59} & \frac{7}{59} \\ 0 & 0 & 0 & 1 & \frac{17}{59} & \frac{13}{59} & \frac{2}{59} & \frac{-9}{59} \end{array} \right) \begin{array}{c} \frac{-2}{9}R_{20} + R_{13} \\ \frac{-2}{3}R_{20} + R_{14} \\ \frac{-7}{9}R_{20} + R_{15} \\ \frac{-5}{59}R_{16} \end{array}$$

**Hence**

$$A^{-1} = \begin{pmatrix} \frac{13}{59} & \frac{3}{59} & \frac{5}{59} & \frac{7}{59} \\ \frac{-31}{59} & \frac{11}{59} & \frac{-21}{59} & \frac{6}{59} \\ \frac{-46}{59} & \frac{3}{59} & \frac{-54}{59} & \frac{7}{59} \\ \frac{17}{59} & \frac{13}{59} & \frac{2}{59} & \frac{-9}{59} \end{pmatrix}$$

**Now, to solve for the variables** $x_1, x_2, x_3$ **and** $x_4$**. We should let** $A^{-1}C = B$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{13}{59} & \frac{3}{59} & \frac{5}{59} & \frac{7}{59} \\ \frac{-31}{59} & \frac{11}{59} & \frac{-21}{59} & \frac{6}{59} \\ \frac{-46}{59} & \frac{3}{59} & \frac{-54}{59} & \frac{7}{59} \\ \frac{17}{59} & \frac{13}{59} & \frac{2}{59} & \frac{-9}{59} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \\ -9 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{13}{59}+\frac{6}{59}-\frac{15}{59}-\frac{63}{59} \\ \frac{-31}{59}+\frac{22}{59}+\frac{63}{59}-\frac{54}{59} \\ \frac{-46}{59}+\frac{6}{59}+\frac{162}{59}-\frac{63}{59} \\ \frac{17}{59}+\frac{26}{59}-\frac{2}{59}+\frac{81}{59} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{13+6-15-63}{59} \\ \frac{-31+22+63-54}{59} \\ \frac{-46+6+162-63}{59} \\ \frac{17+26-6+81}{59} \end{pmatrix} = \begin{pmatrix} \frac{-59}{59} \\ \frac{0}{59} \\ \frac{59}{59} \\ \frac{118}{59} \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

**Therefore,** $x_1 = -1, x_2 = 0, x_3 = 1,$ **and** $x_4 = 2$

## 3.21 Vectors

Vectors can be in column form or row form. A Matrix that has one column,
that is, an m × 1 Matrix is called column Vector. An example of a column vector is

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ . \\ .u_m \end{pmatrix}$$

where $u_i$ are real numbers called the components of the Vector. It should be noted that $u_i$ is the $i^{th}$ component of the Vector U. The example given above is also called an m-component or an m-dimensional Vector.

**Example 41:**

$$\begin{pmatrix} 3 \\ 2 \\ -4 \\ 5 \end{pmatrix}$$

This is a 4-component or a 4-dimensional column Vector which can be re-ferred to as a 4×1 Matrix. Also a matrix with n rows that is a 1 × n Matrix is called a row Vector. An examples of a row vector is $V = (V_1, V_2, V_3, \ldots\ldots V_n)$ .where $V_j$ are real numbers which the components of the Vector. The Vj is the $j^{th}$ component of the Vector V. The example of a row Vector given above is also called an n-component or an n-dimensional Vector.

**Example 42**:
[3 2 4 5**]**
This is a 4-component row Vector which can also be called a 1×4.Matrix. Two row Vectors with the same number of rows or two columns Vector with the same number of column are said to be equal if and only if all the corresponding elements are equal that is if the vectors are identical. It should be noted that a Matrix is composed of series of row or column Vector.

**Example 43:**

**The Matrix**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$$

**can be regarded as consisting of the two column Vector:**

$$\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} \quad and \quad \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

**It can be regarded as consisting of the row vectors.**
[12][34][56][78]

**Example 44:**

$A = [35\ 60\ 45\ 25]$

$$B = \begin{pmatrix} 20 \\ 30 \\ 45 \\ 60 \end{pmatrix} \quad and \quad C = \begin{pmatrix} 6 \\ 9 \\ 24 \\ 15 \end{pmatrix}$$

Determine: (i) B + C (ii)B - C (iii) A(B - C) (iv) AB - AC (v) what algebraic law satisfies (iii) and (iv) above?

**Solution**:

(i)

$$B + C = \begin{pmatrix} 20 \\ 30 \\ 45 \\ 60 \end{pmatrix} + \begin{pmatrix} 6 \\ 9 \\ 24 \\ 15 \end{pmatrix} = \begin{pmatrix} 20+6 \\ 30+9 \\ 45+24 \\ 60+15 \end{pmatrix} = \begin{pmatrix} 26 \\ 39 \\ 69 \\ 75 \end{pmatrix}$$

(ii)

$$B - C = \begin{pmatrix} 20 \\ 30 \\ 45 \\ 60 \end{pmatrix} - \begin{pmatrix} 6 \\ 9 \\ 24 \\ 15 \end{pmatrix} = \begin{pmatrix} 20-6 \\ 30-9 \\ 45-24 \\ 60-15 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \\ 21 \\ 45 \end{pmatrix}$$

(iii)

$$B - C = \begin{pmatrix} 14 \\ 21 \\ 21 \\ 45 \end{pmatrix} \qquad A = (35\ 60\ 45\ 25)$$

**Then, A(B-C)**

$$(35\ 60\ 45\ 25)\begin{pmatrix} 14 \\ 21 \\ 21 \\ 45 \end{pmatrix}$$

$$= (35 \times 14) + (60 \times 21) + (45 \times 21) + (25 \times 45)$$

$$= 490 + 1260 + 945 + 1125 = 3820$$

(iv)

$$AB = (35\ 60\ 45\ 25)\begin{pmatrix} 20 \\ 30 \\ 45 \\ 60 \end{pmatrix}$$

$$= (35 \times 20) + (60 \times 30) + (45 \times 45) + (35 \times 60))$$

$$= 700 + 1800 + 945 + 1500 = 6025$$

$$AC = (35\ 60\ 45\ 25)\begin{pmatrix} 6 \\ 9 \\ 24 \\ 15 \end{pmatrix}$$

$= (35 \times 6) + (60 \times 9) + (45 \times 24) + (25 \times 15)$
$= 210 + 540 + 1080 + 375 = 2205$
Therefore, $AB - AC = 6025 - 2205 = 3820$
(v) $A(B - C) = 3820$ and $AB - AC = 3820$ . Since (iii) and (iv) are equal, therefore, the algebraic law of distribution has been established.

**Example 45:**
One unit of commodity A is produced by combining 1 unit of land, 2 units of labour and 5 units of capital. Also 1 unit of commodity B is produced by 2 units of land, 3 units of labour and 1 unit of capital. Similarly, 1 unit of commodity C results from the use of 3 units of land, 1 unit of labour and 2 units of capital. Assume that the prices of commodity A, B and C are respectively $P_A$ = N 270, $P_B$ = N 160 and $P_C$ = N 190.Find the total rent(R), the wages (W) and interest(I) of the three resources.

**Solution:**

Let D=Land, L=Land, and C=Capital.
The information given can be expressed in form thus:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} P_A \\ P_B \\ P_C \end{pmatrix} = \begin{pmatrix} R \\ W \\ I \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 270 \\ 160 \\ 190 \end{pmatrix} = \begin{pmatrix} R \\ W \\ I \end{pmatrix}$$

$$\begin{pmatrix} (1 \times 270) + (2 \times 160) + (3 \times 190) \\ (2 \times 270) + (3 \times 160) + (1 \times 190) \\ (5 \times 270) + (1 \times 160) + (2 \times 190) \end{pmatrix} = \begin{pmatrix} R \\ W \\ I \end{pmatrix}$$

$$\begin{pmatrix} 270 + 320 + 570 \\ 540 + 480 + 190 \\ 1350 + 160 + 380 \end{pmatrix} = \begin{pmatrix} R \\ W \\ I \end{pmatrix}$$

$$\begin{pmatrix} 1160 \\ 1210 \\ 1890 \end{pmatrix} = \begin{pmatrix} R \\ W \\ I \end{pmatrix}$$

Therefore, Total Rent = N 1160, Total Wages = N 1210 and Total Interest =N 1890

Alternative Solution:

**Total Rent :**

$$= (1\ 2\ 3)\begin{pmatrix} 270 \\ 160 \\ 190 \end{pmatrix}$$

$$= (2 \times 270) + (3 \times 160) + (1 \times 190) = 270 + 320 + 5700 = N1160$$

**Total wage:**

$$(2\ 3\ 1)\begin{pmatrix} 270 \\ 160 \\ 190 \end{pmatrix}$$

$$= (2) + (3 \times 160) + (1 \times 190)) = 540 + 480 + 190 = N1210$$

**Total interest :**

$$(5\ 1\ 2)\begin{pmatrix} 270 \\ 160 \\ 190 \end{pmatrix}$$

$$= (5) + (1 \times 160) + (2 \times 190)) = 1350 + 160 + 380 = N1890$$

## 4.0 CONCLUSION

At the end of the module students are able to differentiate between Matrix and Determinant and solve difference problems related to matrix and determinant.

## 5.0 SUMMARY

This unit analysed the use of the Matrix Approach focusing on Crammer's rules, Gaussian elimination, and solutions to simultaneous equations.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. Determine the range of values of x for which the determinant of the matrix A is:

   greater than or equal to 1, where

$$\begin{pmatrix} 2x - 1 & 1 \\ 2 & x \end{pmatrix}$$

   Show the range of value of x on a number line.

2. The matrix $\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$ is denoted by A and the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by X. It is given that AX=KX where K is any integers. Form a pair of equation connecting x, y and K and hence find two different expressions involving K for the fraction $^{yx}$. Find the two possible values of K and the two corresponding values of $^{yx}$

3. Given the simultaneous equations
$2x_1 + 3x_2 - x_3 = -3$
$x_1 + x_2 + x_3 = 2$
$x_1 - x_2 - x_3 = 0$
(i) Write the above equations in matrix form
(ii) Find the inverse of the $3 \times 3$ matrix so formed, and
(iii) Hence solve the ststems of the equation given.

## 7.0 REFERENCES/FURTHER READING

Stroud, K.A. (1992). *Engineering Mathematics*.

Sogunro, S.O. (1996). *Basic Business Mathematics*.

# MODULE 4

Unit 1 Comparative Statics and the Concept of Derivative
Unit 2 Applications to Comparative Static Analysis

## UNIT 1 COMPARATIVE STATICS AND THE CONCEPT OF DERIVATIVE

**CONTENTS**



## 1.0 INTRODUCTION

*Comparative statics*, as the name suggests, is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables. For purposes of such a comparison, we always start by assuming a given initial equilibrium state. In the isolated-market model, for example, such an initial equilibrium will be represented by a determinate price P and a corresponding quantity

Q. Similarly, in the simple national-income model, the initial equilibrium will be specified by a determinate Y and a corresponding Y. Now if we let a disequilibrating change occur in the model-in the form of a variation in the value of some parameter or exogenous variable-the initial equilibrium will, of course, be upset. As a result, the various endoge-nous variables must undergo certain adjustments. If it is assumed that a new equilibrium state relevant to the new values of the data can be defined and attained, the question posed in the comparative-static analysis is: How would the new equilibrium compare with the old?

It should be noted that in comparative statics we again disregard the process of adjustment of the variables; we merely compare the initial (prechange) equilibrium state with the final (postchange) equilibrium state. Also, we again preclude the possibility of instability of equilibrium, for we assume the new equilibrium to be attainable, just as we do for the old.

A comparative-static analysis can be either qualitative or quantitative in nature. If we are interested only in the question of, say, whether an increase in investment % will increase or decrease the equilibrium income f, the analysis will l > e qualitative because the direction of change is the only matter considered. But if we are concerned with the magnitude of the change in Y resulting from a given change in % (that is, the size of the investment multiplier), the analysis will obviously be quantitative. By obtaining a quantitative answer, however, we can automatically tell the direction of change from its algebraic sign. Quantitative analysis always embraces the qualitative.

It should be clear that the problem under consideration is essentially one of finding a rate of change: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable. For this reason, the mathematical concept of derivative takes on preponderant significance in comparative statics, because that concept-the most fundamental one in the branch of mathematics known as differential calculus is directly concerned with the notion of rate of change Later on, moreover, we shall find the concept of derivative to be of extreme importance for optimisation problems as well.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define comparatives statics
- describe derivatives of a function
- state the limits of function
- discuss the techniques of differentiation
- explain partial differentiation
- discuss applications of derivatives.

## 3.0 MAIN CONTENT

### 3.1 The Nature of Comparative Statics

#### 3.1.1 Rate of Change and the Derivative

Even though our present context is concerned only with the rates of change of the equilibrium values of the variables in a model, we may carry on the discussion in a more general manner by considering the rate of change of any variable y in response to a change in another variable x, where the two variables are related to each other by the function

$$y=f(x)$$

Applied in the comparative-static context, the variable y will represent the equilibrium value of an endogenous variable, and x will be some parameter. Note that, for a start, we are restricting ourselves to the simple case where there is only a single parameter or exogenous variable in the model. Once we have mastered this simplified case, however, the extension to the case of more parameters will prove relatively easy.

#### 3.1.2 The Difference Quotient

Since the notion of "change" figures prominently in the present context, a special symbol is needed to represent it. When the variable x changes from the value $x_0$ to a new value $x_1$, the change is measured by the difference $x_1 - x_0$. Hence, using the symbol A (the Greek capital delta, for "difference") to denote the change, we write $X = x_1 - x_0$. Also needed is a way of denoting the value of the function f(x) at various values of x. The standard practice is to use the notation f(x;) to represent the value of f(x) when x =X;. Thus, for the function $f(x) = 5 + x^2$, we have $f(0) = 5 + 0^2 = 5$ and similarly, $f(2) = 5 + 2^2 = 9$, etc. When x changes from an initial value $x_0$ to a new value $(x_0 + x)$, the value of the function y = f(x) changes from $f(x_0)$ to $f(x_0 + x)$. The change in y per unit of change in x can be represented by the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta) - f(x_0)}{\Delta x}$$

**Example 1:**

Given $y = f(x) = 3x^2 - 4$, we can write: $f(x_0) = 3(x_0)^2 - 4$ $f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4$

Therefore, the difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{(3x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x}$$
$$= \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x} = 6x_0 + 3\Delta x$$

which can be evaluated if we are given $x_0$ and Δx. Let $x_0$ = 3 and Δx = 4; then the average rate of change of y will be 6(3) + 3(4) = 30. This means that, on the average, as x changes from 3 to 7, the change in y is 30 units per unit change in x.

## 3.2 The Derivatives

Frequently, we are interested in the rate of change of y when Δx is very small. In such a case, it is possible to obtain an approximation of $\frac{\Delta y}{\Delta x}$ by dropping all the terms in the difference quotient involving the expression Δx. In (6.1), for instance, if Δx is very small, we may simply take the term $6x_0$ the right as an approximation of $\frac{\Delta y}{\Delta x}$. The smaller the value of x, of course, the closer is the approximation to the true value of $\frac{\Delta y}{\Delta x}$.

As x approaches zero (meaning that it gets closer and closer to, but never actually reaches, zero), $6x_0$ + 3Δx will approach the value $6x_0$ , and by the same token, $\frac{\Delta y}{\Delta x}$ will approach 6x0 also. Symbolically, this fact is expressed either by the statement $\frac{\Delta y}{\Delta x} \rightarrow 6x0$ as $\Delta x \rightarrow 0$.

Several points should be noted about the derivative. First, a derivative is a function; in fact, in this usage the word derivative really means a derived function. -The original function y = f( x) is a primitive function, and the derivative is another function derived from it. Whereas the difference quotient is a function of $x_0$ and Δx, observe derivative is a function of $x_0$ only. This is because Δx is already compelled to approach zero, and therefore it should not be regarded as another variable in the function. Let us also add that so far we have used the subscripted symbol $x_0$ only in order to stress the fact that a change in x must start from some specific value of x. Now that this is understood, we may delete the subscript and simply state that the derivative, like the primitive function, is itself a function of the independent variable x. That is, for each value of x, there is a unique corresponding value for the derivative function.

Second, since the derivative is merely a limit of the difference quotient, which measures a rate of change of y, the derivative must of necessity also be a measure of some rate of change. In view of the fact that the change in x envisaged in the derivative concept is infinitesimal (that is, $\Delta x \to 0$), however, the rate measured by the derivative is in the nature of an instantaneous rate of change.

Third, there is the matter of notation. Derivative functions are commonly denoted in two ways. Given a primitive function y = f(x), one way of denoting its derivative (if it exists) is to use the symbol f'( x ), or simply f'; this notation is attributed to the mathematician Lagrange. The other common notation is $\frac{\Delta y}{\Delta x}$, devised by the mathematician Leibniz. [Actually there is a third notation, Dy, or DF(x), but we shall not use it in the following discussion.] The notation f'(x), which resembles the notation for the primitive function f(x), has the advantage of conveying the idea that the derivative is itself a function of x. The reason for expressing it as f'(x)-rather than, say, φ(x)-is to emphasise that the function f' is derived from the primitive function f. The alternative notation, $\frac{\Delta y}{\Delta x}$ serves instead to emphasise that the value of a derivative measures a rate of change. The letter d is the counterpart of the Greek Δ, and $\frac{\Delta y}{\Delta x}$x differs from Δx chiefly in that the former is the limit of the latter as Δx approaches zero. In the subsequent discussion, we shall use both of these notations, depending on which seems the more convenient in a particular context.

Using these two notations, we may define the derivative of a given function y = f(x) as follows:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

## 3.3 The Derivatives and Slope of a Curve

Elementary economics tells us that, given a total-cost function C = f(Q), where C denotes total cost and Q the output, the marginal cost (MC) is defined as the change in total cost resulting from a unit increase in output: that is, MC = $\frac{\Delta C}{\Delta Q}$ . It is understood that ΔQ is an extremely small change. For the case of a product that has discrete units (integers only), a change of one unit is the smallest change possible; but for the case of a product whose quantity is a continuous variable,ΔQ will refer to an infinitesimal change. In this

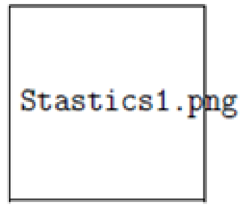


Figure 1:

latter case, it is well known that the marginal cost can be measured by the slope of the total-cost curve. But the slope of the total cost curve is nothing but the limit of the ratio $\frac{\Delta C}{\Delta Q}$, when ΔQ approaches zero. Thus the concept of the slope of a curve is merely the geometric counterpart of the concept of the derivative. Both have to do with the "marginal" notion so extensively used in economics.

In Fig 1, we have drawn a total-cost curve C, which is the graph of the (primitive) function C=f(Q).Suppose that we consider $Q_0$ as the initial output level from which an increase in output is measured, then the relevant point on the cost curve will be A. If output is to be raised to $Q_0 + \Delta Q = Q_2$, the total cost will be increased from $C_0$ to $C_0 + \Delta C = C_2$; $\frac{\Delta C}{\Delta Q} = \frac{(C_2 - C_0)}{(Q_2 - Q_0)}$

Geometrically, this is the ratio of two line segments, $\frac{EB}{AE}$ or the slope of the A B. This particular ratio measures an average rate of change the average.

Marginal cost for the particular ΔQ pictured-and represents a difference quotient. As such, it is a function of the initial value $Q_0$ and the amount of change ΔQ. What happens when we vary the magnitude of ΔQ? If a smaller output increment is contemplated (say, from $Q_0$ to $Q_1$ only), then the average marginal cost will be measured by the slope of the line AD inste11d. Moreover, as we reduce the output increment further and further, flatter and flatter lines will result until, in the limit (as $\Delta Q \rightarrow 0$), we obtain the line KG (which is the tangent line to the cost curve at point A) as the relevant line. The slope of KG (= $\frac{HG}{KH}$) measures the slope of the total-cost curve at point A and represents the limit of $\frac{\Delta C}{\Delta Q}$ as $\Delta Q \rightarrow 0$, when initial output is at Q = Q0. Therefore, in terms of the derivative, the slope of the C = f (Q) curve at point A corresponds to the particular derivative value f' (Q0).

What if the initial output level is changed from $Q_0$ to, say, $Q_2$? In that case, point B on the curve will replace point A as the relevant point, and the slope of the curve at the new point B will give us the derivative value f'($Q_2$ ). Analogous results are obtainable for alternative initial output levels. In general, the derivative $f'(Q)$ a function of Q will vary as Q changes.

## 3.4 The Concept of Limit

The derivative $\frac{dy}{dx}$ has been defined as the limit of the difference quotient $\frac{\Delta C}{\Delta Q}$ as $\Delta x \rightarrow 0$. If we adopt the shorthand symbols q = $\frac{\Delta C}{\Delta Q}$ (q for quotient) and v = $\Delta x$ (v for variation),

we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{v \to 0} q$$

In view of the fact that the derivative concept relies heavily on the notion of limit, it is imperative that we get a clear idea about that notion.

**Left-Side Limit and Right-Side Limit**

The concept of limit is concerned with the question: "What value does one variable (say, q) approach as another variable (say, v) approaches a specific value (say, zero)?" In order for this question to make sense, q must, of course, be a function of v: say, q = g (v). Our immediate interest is in finding the limit of q as v → 0, but we may just as easily explore the more general case of v -+ N, where N is any finite real number. Then,

$$\lim_{v \to 0} q$$

will be merely a special case of

$$\lim_{v \to N} q$$

where N = 0. In the course of the discussion, we shall actually also consider the limit of q asv →∞ (plus infinity) or as v →−∞ (minus infinity).

When we say v → N , the variable v can approach the number N either from values greater than N, or from values less than N. If, as v → N from the left side (from values less than N), q approaches a finite number L, we call L the left-side limit of q. On the other hand, if L is the number that q tends to as v → N from the right side (from values greater than N), we call L the right-side limit of q. The left- and right-side limits may or may not be equal.

The left-side limit of q is symbolised by

$$\lim_{v \to N^-} q$$

(the minus sign signifies from v → N − values less than N), and the right-side limit is written as

$$\lim_{v \to N^+} q$$

When and only when-the two limits have a common finite value (say, L), we consider the limit of q to exist and write it as

$$\lim_{v \to N} q = L$$

Note that L must be a finite number. v → N If we have the situation of

$$\lim_{v \to N} q = \infty$$

(or - ∞), we shall consider q to possess no v → N limit, because lim q = ∞ means that q →∞ as v → N, and if q will assume v → N ever increasing values as v tends to N, would be contradictory to say that q has a limit. As a convenient way of expressing the fact that q →∞ as v → N , however, people do indeed write

$$\lim_{v \to N} q = \infty$$

and speak of q as having an "infinite limit."

In certain cases, only the limit of one side needs to be considered. In taking the limit of q as v → +∞, for instance, only the left-side limit of q is relevant, because v can approach +∞ only from the left. Similarly, for the case of v →−∞, only the right-side limit is relevant. Whether the limit of q exists in these cases will depend only on whether q approaches a finite value as v v → +∞, or as v v →−∞.

It is important to realise that the symbol ∞ (infinity) is not a number, and therefore it cannot be subjected to the usual algebraic operations. We cannot have 3 + ∞ or 1∞; nor can we write q = ∞, which is not the same as q ∞. However, it is acceptable to express the limit of q as " = " (as against →) ∞, for this merely indicates that q →∞.

## 3.5 Graphical Illustrations

Let us illustrate, in above figure several possible situations regarding the limit of a function q = g( v ). Figure 2 shows a smooth curve. As the variable v tends to the value N from either side on the horizontal axis, the variable q tends to the value L. In this case, the left-side limit is identical with the right-side limit; therefore we can write

$$\lim_{v \to N} q = L$$

The curve drawn in above figure is not smooth; it has a sharp turning point directly above the point N. Nevertheless, as v tends to N from either side, q again tends to an identical value L. The limit of q again exists and is equal to L.

It shows what is known as a step function.* In this case, as v tends to N, the left-side limit of q is Lp but the right-side limit is L 2, a different number. Hence, q does not have a limit as v -+ N. Lastly, in above figure, as v tends to N, the left-side limit of q is −∞, whereas the right-side limit is +∞, because the two parts of the (hyperbolic) curve will fall and rise indefinitely while approaching the broken vertical line as an asymptote.
Again,

$$\lim_{v \to N} q$$

does not exist. On the other hand, if we are considering different sort of limit in diagram d, namely,

$$lim_{v \to +\infty} q$$

then only the left-side limit has relevance, and we do find that limit to exist:

$$\lim_{v \to +\infty} q = M$$

Analogously, you can verify that

$$\lim_{v \to +\infty} q = M$$

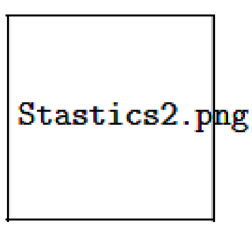


Figure 2:

as well.
It is also possible to apply the concepts of left-side and right-side limits to the discussion of the marginal cost in Fig 1. In that context, the

variables q and v will refer, respectively, to the quotient $\frac{\Delta C}{\Delta Q}$ and to the magnitude of ΔQ, with all changes being measured from point A on the curve. In other words, q will refer to the slope of such lines as AB, AD, and KG, whereas v will refer to the length of such lines as$Q_0$, $Q_2$ (= line AE) and $Q_0$, $Q_1$ (= line AF). We have already seen that, as v approaches zero from a positive value, q will approach a value equal to the slope of line KG. Similarly, we can establish that, if ΔQ approaches zero from a negative value (i.e., as the decrease in output becomes less and less), the quotient $\frac{\Delta C}{\Delta Q}$ as measured by the slope of such lines as RA (not drawn), will also approach a value equal to the slope of line KG. Indeed, the situation here is very much akin to that illustrated in Fig. 6.2a. Thus the slope of KG in Fig 1 (the counterpart of L in Fig 2) is indeed the limit of the quotient q as v tends to zero, and as such it gives us the marginal cost at the output level $Q = Q_0$.

**Evaluation of a Limit**

Let us now illustrate the algebraic evaluation of a limit of a given function q = g(v). Example 2:
Given $q = 2 + v^2$, find

$$\lim_{v \to 0} q.$$

To take the left-side limit, we substitute the series of negative values $-1, -\frac{1}{10}, -\frac{1}{100}, \ldots\ldots$ (in that order) for v and find that $(2 + v^2)$ will decrease steadily and approach 2 (because $v^2$ will gradually approach 0). Next, for the right-side limit, we substitute the series of positive values $1, \frac{1}{10}, \frac{1}{100}, \ldots\ldots$ (in that order) for v and find the same limit as before. In as much as the two limits are identical, we consider the limit of q to exist and write

$$\lim_{v \to 0} q = 2.$$

It is tempting to regard the answer just obtained as the outcome of setting v = 0 in the equation $q = 2 + v^2$, but this temptation should in general be resisted. In evaluating

$$\lim_{v \to N} q$$

we only let v tend to N but, as a rule, do not let v = N. Indeed, we can quite legitimately speak of the limit of q as $v \to N$ , even if N is not in the domain of the function q = g(v). In this latter case, if we try to set v = N, q will clearly be undefined.

### 3.5.1 Formal View of the Limit Concept

The above discussion should have conveyed some general ideas about the concept of limit. Let us now give it a more precise definition. Since such a definition will make use of the concept of neighbourhood of a point on a line (in particular, a specific number as a point on the line of real numbers), we shall first explain the latter term. For a given number L, there can always be found a number $(L - a_1) < L$ and another number $(L + a_2) > L$, where $a_1$ and $a_2$ are some arbitrary positive numbers. The set of all numbers falling between $(L - a_1) < L$ and $(L + a_2) > L$ is called the interval between those two numbers. If the numbers $(L - a_1) < L$ and $(L + a_2) > L$ are included in the set, the set is a closed interval; if they are excluded, the set is an open interval. A closed interval between $(L - a_1) < L$ and $(L + a_2) > L$ is denoted by the bracketed expression

$$[(L - a_1), (L + a_2) > L] = q\{(L - a_1) \leq q \leq (L + a_2) > L\}$$

and the corresponding open interval is denoted with parentheses: (6.4)

$$((L - a_1), (L + a_2) > L) = q\{(L - a_1) < q < (L + a_2) > L\}$$

Thus, [ ] relate to the weak inequality sign $\leq$ , whereas ( ) relate to the strict inequality sign $<$ . But in both types of intervals, the smaller number $(L - a_1)$ is always listed first. Later on, we shall also have occasion to refer to half-open and half-closed intervals such as (3, 5] and [6, ∞), which have the following meanings:

$$(3, 5] = \{x|3 < x \leq 5\} \qquad [6, \infty) = \{x|6 \leq x < \infty\}$$

Now we may define a neighborhood of L to be an open interval as defined in (6.4), which is an interval "covering" the number L* Depending on the magnitudes of the arbitrary numbers $a_1$ and $a_2$ , it is possible to construct various neighborhoods for the given number L. Using the concept of neighbourhood, the limit of a function may then be defined as follows:

As v approaches a number N, the limit of q = g( v) is the number L, if, for every neighbourhood of L that can be chosen, however small, there can be found a corresponding neighbourhood of N (excluding the point v = N) in the domain of the function such that, for every value of v in that N-neighbourhood, its image lies in the chosen L-neighbourhood. This statement can be clarified with the help of Fig.3, which resembles Fig. 2a. From what was learned about the latter figure, we know that

$$\lim_{v \to N} q = L$$

in

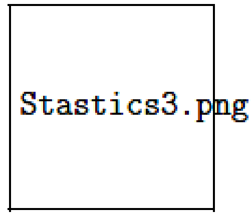


Figure 3:

Figure 3 show that L does indeed fulfill the new definition of a limit. As the first step, select an arbitrary small neighbourhood of L, say, $(L - a_1, L + a_2)$. (This should have been made even smaller, but we are keeping it relatively large to facilitate exposition.) Now construct a neighbourhood of N, say, $(N - b_1, N + b_2)$, such that the two neighbourhoods (when extended into quadrant I) will together define a rectangle (shaded in diagram) with two of its corners lying on the given curve. It can then be verified that, for every value of v in this neighbourhood of N (not counting v = N), the corresponding value of q = g( v) lies in the chosen neighbourhood of L. In fact, no matter how small an L-neighbourhood we choose, a (correspondingly small) N neighbourhood can be found with the property just cited. Thus L fulfills the definition of a limit, as was to be demonstrated.

We can also apply the above definition to the step function of Fig 2c in order to show that neither $L_1$ nor $L_2$ qualifies as $\lim_{v \to N} q$. If we choose a very small neighborhood of $L_1$ say,just a hair's width on each side of $L_1$-then, no matter what neighbourhood we pick for N, the rectangle associated with the two neighbourhoods cannot possibly enclose the lower step of the function. Consequently, for any value of v > N , the corresponding value of q (located on the lower step) will not be in the neighbourhood of $L_1$, and thus $L_1$ fails the test for a limit. By similar reasoning, $L_2$ must also be dismissed as a candidate for

$$\lim_{v \to N} q$$

In fact, in this case no limit exists for q as $v \to N$.

### 3.5.2 Limit Theorem

Our interest in rates of change led us to the consideration of the concept of derivative, which, being in the nature of the limit of a quotient, in turn prompted us to study questions of the existence and evaluation of a limit. The basic process of limit evaluation, as illustrated, involves letting the variable v approach a particular number (say, N) and observing the value which q approaches. When actually evaluating the limit of a function, however, we may draw upon certain established limit theorems, which can materially simplify the task, especially for complicated functions.

### 3.5.3 Theorems involving a Single Function

When a single function q = g(v) is involved, the following theorems are applicable.

**Theorem 1:**
If q = av + b, then $\lim_{v \to N} q = aN + b$ (a and b are constants). Example 2: Given q = 5v + 7, we have $\lim_{v \to 2} q = 5(2) + 7 = 17$. Similarly, $\lim_{v \to 0} q = 5(0) + 7 = 7$.

**Theorem 2:**
Ifq=g(v)=b,then $\lim_{v \to N} q = b$.
This theorem, which says that the limit of a constant function is the constant in that function, is merely a special case of Theorem 6.1, with a=0.

**Theorem 3:**
If q = v, then $\lim_{v \to N} q = N$.

If $q = v^k$, then $\lim_{v \to N} q = N k$.

**Example 2**:
Given $q = v^3$ , we have $\lim_{v \to 2} q = (2)3 = 8$.

You may have noted that, in the above three theorems, what is done to find the limit of q as v → N is indeed to let v = N. But these are special cases, and they do not vitiate the general rule that "v → N " does not mean "v = N."

### 3.5.4 Theorems involving Two Functions

If we have two functions of the same independent variable v, $q_1$= g(v), and if both functions possess limits as follows

$$\lim_{v \to N} q_1 = L_1 \qquad \lim_{v \to N} q_2 = L_2$$

where $L_1$ and $L_1$ are two finite numbers, the following theorems are applicable.

**Theorem 4**: (sum-difference limit theorem)

$$\lim_{v \to N}(q_1 \pm q_2) = L_1 \pm L_2$$

The limit of a sum (difference) of two functions is the sum (difference) of their respective limits. In particular, we note that

$$\lim_{v \to N} 2q_1 = \lim_{v \to N}(q_1 + q_1) = L_1 + L_1 = 2L_1$$

which is in line with Theorem 1.

**Theorem 5:** (product limit theorem):

$$\lim_{v \to N}(q_1 q_2) = L_1 L_2$$

The limit of a product of two functions is the product of their limits. Applied to the square of a function, this gives

$$\lim_{v \to N}(q_1 q_1) = L_1 L_1 = L_1^2$$

which is in line with Theorem 3.

**Theorem 6** (quotient limit theorem):

$$\lim_{v \to N} \frac{q_1}{q_2} = \frac{L_1}{L_2} (L_1 \neq 0)$$

The limit of a quotient of two functions is the quotient of their limits. Naturally, the limit $L_2$ is restricted to be non-zero; otherwise the quotient is undefined.

**Example 3**
Find

$$\lim_{v \to 0} \frac{(I + v)}{(2 + v)}$$

Since we have here

$$\lim_{v \to 0}(I + v); = I$$

and -$\lim_{v \to 0}(2 + v) = 2$, the desired limit is 1.

Remember that $L_1$ and $L_2$ represent finite numbers; otherwise these theorems do not apply. In the case of Theorem 6, furthermore, $L_2$ must be nonzero as well. If these re-strictions are not satisfied, we must fall back on the method limit evaluation illustrated in examples above, which relate to the cases, respectively, of L being zero and of L being infinite.

### 3.5.4 Limit of a Polynomial Function

With the above limit theorems at our disposal, we can easily evaluate the limit of any polynomial function

$$q = g(v) = a_0 + a_1 v + a_2 v + ..... + a_n v^n \qquad 2$$

as v tends to the number N. Since the limits of the separate terms are respectively.

$$\lim_{v \to N} a_0 = a_0 \qquad \lim_{v \to N} a_1 v = a_1 N \qquad \lim_{v \to N} a_2 v^2 = a_2 N^2$$

$$\lim_{v \to N} q = a_0 + a_1 v + a_2 v + ..... + a_n v^n \qquad 3$$

This limit is also, we note, actually equal to g( N ), that is, equal to the value of the function in (2) when v = N. This particular result will prove important in discussing the concept of continuity of the polynomial function.

## 3.6 Continuity and Differentiability of a Function

The preceding discussion of the concept of limit and its evaluation can now be used to define the continuity and differentiability of a function. These notions bear directly on the derivative of the function, which is what interests us.

### 3.6.1 Continuity of a Function

When a function q = g( v) possesses a limit as v tends to the point N in the domain, and when this limit is also equal to g(N)-that is, equal to the value of the function at v = N the function is said to be continuous at N. As stated above, the term continuity involves no less than three requirements:

(1) the point N must be in the domain of the function; i.e. g(N) is defined.

(2) the function must have a limit as v → N i.e.

$$\lim_{v \to N} g(v)$$

exists and

(3) that limit must be equal in value to g( N ),

$$\lim_{t \to N} g(v) = g(N)$$

It is important to note that while-in discussing the limit of the curve in above Figure. Above the point (N, L) was excluded from consideration, we are no longer excluding it in the present context. Rather, as the third requirement specifically states, the point ( N, L) must be on the graph of the function before the function can be considered as continuous at point N.

Let us check whether the functions shown in above figure are continuous. In diagram a, all three requirements are met at point N. Point N is in the domain, q has the limit L as $v \rightarrow N$ ; and the limit L happens also to be the value of the function at N. Thus, the function represented by that curve is continuous at N. The same is true of the function depicted in above figure, since Lis the limit of the function as v approaches the value N in the domain, and since L is also the value of the function at N. This last graphic example should suffice to establish that the continuity of a function at point N does not necessarily imply that the graph of the function is "smooth" at v = N, for the point (N, L) in above figure is actually a "sharp" point and yet the function is continuous at that value of v. When a function q = g( v) is continuous at all values of v in the interval (a, b), it is said to be continuous in that interval. If the function is continuous at all points in a subset S of the domain (where the subset S may be the union of several disjoint intervals), it is said to be continuous in S. And, finally, if the function is continuous at all points in its domain, we say that it is continuous in its domain. Even in this latter case, however, the graph of the function may nevertheless show a discontinuity (a gap) at some value of v, say, at v = 5, if that value of v is not in its domain.

Again referring to above figure, we see that in diagram c the function is discontinuous at N because a limit does not exist at that point, in violation of the second requirement of continuity. Nevertheless, the function does satisfy the requirements of continuity in the interval (0, N) of the domain, as well as in the interval $[N, \infty)$. Diagram d obviously is also discontinuous at v = N. This time, discontinuity emanates from the fact that N is excluded from the domain, in violation of the first requirement of continuity.

On the basis of the graphs in above figure, it appears that sharp points are consistent with continuity, as in diagram b, but that gaps are taboo, as in diagrams c and d. This is indeed the case. Roughly speaking, therefore, a function that is continuous in a particular interval is one whose graph can be drawn for the said interval without lifting the pencil or pen from the paper-a feat which is possible even if there are sharp points, but impossible when gaps occur.

### 3.6.2 Polynomial and Rational Functions

Let us now consider the continuity of certain frequently encountered functions. For any polynomial function, such as q = g( v) in above, we have found from above that $\lim_{v \to N}$ q exists and is equal to the value of the function at N. Since N is a point (any point) in the domain of the function, we can conclude that any polynomial function is continuous in its domain. This is a very useful piece of information, because polynomial functions will be encountered very often.

What about rational functions? Regarding continuity, there exists an interest ing theorem (the continuity theorem) which states that the sum, difference, product, and quotient of any finite number of functions that are continuous in the domain are, respectively, also continuous in the domain. As a result, any rational function (a quotient of two polynomial functions) must also be continuous in its domain.

**Example 4:**
The rational function

$$q = g(v) = \frac{4v^2}{(v^2 + I)}$$

is defined for all finite real numbers; thus its domain consists of the interval (−∞, ∞). For any number N in the domain, the limit of q is (by the quotient limit theorem)

$$\lim_{v \to N} q = \frac{\lim_{v \to N}(4v^2)}{\lim_{v \to N}(v^2 + I)} = \frac{4N^2}{(v^2 + I)}$$

which is equal to g(N). Thus the three requirements of continuity are all met at N. Moreover, we note that N can represent any point in the domain of this function; consequently, this function is continuous in its domain.

### 3.6.3 Differentiability of a Function

The previous discussion has provided us with the tools for ascertaining whether any function has a limit as its independent variable approaches some specific value. Thus we can try to take the limit of any function y = f(x) as x approaches some chosen value, say, $x_0$. However, we can also apply the "limit" concept at a different level and take the limit of the difference quotient of that function $\frac{dy}{dx}$, as x approaches zero. The outcomes of limit-taking at these two different levels relate to two different, though related, properties of the function.

Taking the limit of the function y = f(x) itself, we can, in line with the discussion of the preceding subsection, examine whether the function f is continuous at $x = x_0$. The conditions for continuity are:

(1) $x = x_0$ must be in the domain of the function,
(2) y must have a limit as $x \rightarrow x_0$ , and
(3) the said limit must be equal to f $(x_0)$. When these are satisfied, we can write

$$\lim_{x \to x0} ( \quad continuity \quad condition \quad )$$

When the "limit" concept is applied to the difference quotient $\frac{dy}{dx}$: as x → x0, on the other hand, we deal instead with the question of whether the function f is differentiable at x = x0 , i.e., whether the derivative $\frac{dy}{dx}$: exists at x = x0 , or whether f′(x0) exists.

The term "differentiable" is used here because the process of obtaining the derivative $\frac{dy}{dx}$: is known as differentiation (also called derivation). Since f′(x0) exists if and only if the limit of $\frac{\Delta y}{\Delta x}$ exists at $x \rightarrow x^0$ as $\Delta x \rightarrow 0$, the symbolic expression of the differentiability off is

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} (differentiability condition)$$

These two properties, continuity and differentiability, are very intimately related to each other-the continuity of f is a necessary condition for its differentiability (although, as we shall see later, this condition is not sufficient). What this means is that, to be differentiable at $x = x^0$, the function must pass the test of being continuous at $x = x_0$. To prove this, we- shall demonstrate that, given a function y = f(x), its continuity at x = x 0 follows from its differentiability at $x = x_0$, i.e differentiability condition. Before doing this, however, let us simplify the notation somewhat by:

(1) Replacing $x_0$ with the symbol N and
(2) Replacing $(x_0 + \Delta x)$ with the symbol x. The latter is justifiable because the postchange value of x can be any number (depending on the magnitude of the change) and hence is a variable denotable by x. This is the equivalence of the two notation systems, where the old notations appear (in brackets) alongside the new. Note that, with the notational

change, Δx now becomes (x - N), so that the expression "Δx → 0" becomes "x → N ," which is analogous to the expression v → N ,used before in connection with the function q = g(v). We can now be rewritten, respectively, as

$$\lim_{x \to N} f(x) = f(N) \qquad (4)$$

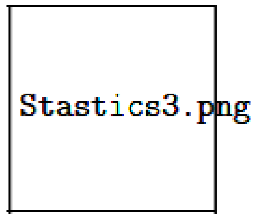


Figure 4:

$$f'(x) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \qquad (5)$$

What we want to show is, therefore, that the continuity condition follows from the differentiability condition. First, since the notation x → N implies that x ± N , so that x – N is a non-zero number, it is permissible to write the following identity:

$$f(x) - f(N) = \frac{f(x) - f(N)}{x - N}(x - N) \qquad (6)$$

Taking the limit of each side of (6) as x → N yields the following results:

$$left\ side \lim_{x \to N} f(x) - \lim_{x \to N} f(N)(difference\ limit\ theorem)$$
$$\lim_{x \to N} f(x) - f(N)\ (f(N)\ constant)$$

$$righr\ side \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \lim_{x \to N}(x - N)(product\ limit\ theorem)$$
$$= f'(N)\left(\lim_{x \to N} x - \lim_{x \to N} N\right)$$
$$= f'(N)(N - N) = 0$$

Note that we could not have written these results, if condition had not been granted, for if f’ ( N) did not exist, then the right-side expression (and hence also the left-side expression) in (6) would not possess a limit.

If f' (N) does exist, however, the two sides will have limits as shown above. Moreover, when the left-side result and the right-side result are equated, we get

$$\lim_{x \to N} f(x) - f(N) = 0$$

which is identical. Thus we have proved that continuity, as shown in above equation, follows from differentiability, as shown in above. In general, if a function is differentiable at every point in its domain, we may conclude that it must be continuous in its domain. Although differentiability implies continuity, the converse is not true. That is, continuity is a necessary, but not a sufficient, condition for differentiability. To demonstrate this, we merely have to produce a counter-example. Let us consider the function

$$y = f(x) = |x - 2| + 1 \qquad (7)$$

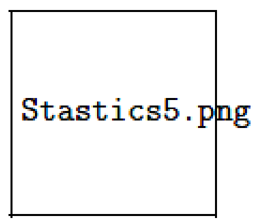


Figure 5:

which is graphed above. As can be readily shown, this function is not differentiable, though continuous, when x = 2. That the function is continuous at x = 2 is easy to establish. First, x = 2 is in the domain of the function. Second, the limit of y exists as x tends to 2; to be specific, $\lim_{x\to 2^-} y = \lim_{x\to 2^+} = 1$. Third, f (2) is also found to be
1. Thus all three requirements of continuity are met. To show that the function is not differentiable at x = 2, we must show that the limit of the difference quotient

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{|x - 2| - 1 + 1}{x - 2} = \lim_{x \to 2} \frac{|x - 2|}{x - 2}$$

does not exist. This involves the demonstration of a disparity between the left-side and the right-side limits. Since, in considering the right-side limit, x must exceed 2, we have $|x - 2| = x - 2$ Thus the right-side limit is

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2}=\lim_{x\to 2^+}\frac{x-2}{x-2}=\lim_{x\to 2^+}1=1$$

On the other hand, in considering the left-side limit, x must be less than 2; thus,|x − 2| = −(x − 2). Consequently, the left-side limit is

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2}=\lim_{x\to 2^-}\frac{-(x-2)}{x-2}=\lim_{x\to 2^-}(-1)=-1$$

which is different from the right-side limit. This shows that continuity does not guarantee differentiability. In sum, all differentiable functions are continuous, but not all continuous functions are differentiable.

In above figure, the non differentiability of the function at x = 2 is manifest in the fact that the point (2, 1) has no tangent line defined, and hence no definite slope can be assigned to the point. Specifically, to the left of that point, the curve has a diagramm slope of - 1, but to the right it has a slope of + 1, and the slopes on the two sides display no tendency to approach a common magnitude at x = 2. The point (2, 1) is, of course, a special point; it is the only sharp point on the curve. At other points on the curve, the derivative is defined and the function is differentiable. More specifically, when above function can be divided into two linear functions as follows:

Left part: $y = -(x - 2) + 1 = 3 - x$ $(x \le 2)$
Right part: y $y = (x - 2) + 1 = x - 1 (x > 2)$

The left part is differentiable in the interval $(-\infty, 2)$, and the right part is differentiable in the interval $(2, \infty)$ in the domain. In general, differentiability is a more restrictive condition than continuity, because it requires something beyond continuity. Continuity at a point only rules out the presence of a gap, whereas differentiability rules out "sharpness" as well. Therefore, differentiability calls for "smoothness" of the function (curve) as well as its continuity. Most of the specific functions employed in economics have the property that they are differentiable everywhere. When general functions are used, moreover, they are often assumed to be everywhere differentiable, as we shall do our in the subsequent discussion.

## 3.7 Rules of Differentiation and their Use in Comparative Statics

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem of finding the derivative of some function y = f(x), provided only a small change in x is being considered. Even though the derivative $\frac{dy}{dx}$ is defined as the limit of the difference quotient q = g( v) as v → 0, it is by no means necessary to undertake the process of limit taking each time the derivative of a function is sought, for there exist various rules of differentiation (derivation) that will enable us to obtain the desired derivatives directly. Instead of going into comparative-static models immediately, therefore, let us begin by learning some rules of differentiation.

## 3.8 Rules of Differentiation for a Function of One Variable

First, let us discuss three rules that apply, respectively, to the following types of function of a single independent variable: y = k (constant function),y = x", andy = ex" (power functions). All these have smooth, continuous graphs and are therefore differentiable everywhere.

### 3.8.1 Constant-Function Rule

The derivative of a constant function y = f(x) = k is identically zero, i.e., is zero forall values of x. Symbolically, this may be expressed variously as $\frac{dy}{dx}$ = 0 or $\frac{dk}{dx}$ = 0 or f′(x) = 0

In fact, we may also write these in the form

$$\frac{d}{dx}y = \frac{d}{dx}k = \frac{d}{dx}f(k) = 0$$

where the derivative symbol has been separated into two parts, $\frac{d}{dx}$ on the one hand, and y (or f(x) or k) on the other. The first part $\frac{d}{dx}$, may be taken as an operator symbol, which instructs us to perform a particular mathematical operation. Just as the operator symbol; instructs us to take a square root, the symbol $\frac{dk}{dx}$ represents an instruction to take the derivative of, or to differentiate, (some function) with respect to the variable x. The function to be operated on (to be differentiated) is indicated in the second part; here it is y = f(x) = k.

The proof of the rule is as follows. Given f(x) = k, we have f(N) = k for any value of N.

Thus the value of f’ (N) the value of the derivative at x = N as defined above will be

$$f'(x) = \lim_{x\to N}\frac{f(x)-f(N)}{x-N} = \lim_{x\to N}\frac{k-k}{x-N} = \lim_{x\to N} 0 = 0$$

Moreover, since N represents any value of x at all, the result $f'(N) = 0$ can be immediately generalised to $f'(x) = 0$. This proves the rule.

It is important to distinguish clearly between the statement $f'(x) = 0$ and the similar looking but different statement $f'(x_0) = 0$. By $f'(x) = 0$, we mean that the derivative function $f'$ has a zero value for all values of x; in writing $f'(x_0) = 0$, on the other hand, we are merely associating the zero value of the derivative with a particular value of x, namely, $x = x_0$.
As discussed before, the derivative of a function has its geometric counterpart in the slope of the curve. The graph of a constant function, say, a fixed-cost function CF = f(Q) = N1200, is a horizontal straight line with a zero slope throughout. Correspondingly, the derivative must also be zero for all values of Q:

$$\tfrac{d}{dQ}C_f = \tfrac{dk}{dQ}N1200 = 0 \text{ or } f'(Q) = 0$$

### 3.8.2 Power-Function Rule

The derivative of a power function $y = f(x) = x^n$ is $nx^{n-1}$. Symbolically, this is expressed as

$$\tfrac{dx^n}{dx} = nx^{n-l} \text{ or } f'(x) == nx^{n-l}$$

**Example 6:**

The derivative of y = $x^3$ is $\frac{dy}{dx} = \frac{dx^3}{dx} = 3x^2$

**Example 7:**

The derivative of y = $x^9$ is $\frac{dy}{dx} = \frac{dx^9}{dx} = 9x^8$

This rule is valid for any real-valued power of x; that is, the exponent can be any real number. But we shall prove it only for the case where n is some positive integer. In the simplest case, that of n = 1, the function is f(x) = x, and according to the rule, the derivative is

$$f'(x) = \frac{d}{dx}x = 1(x^0) = 1$$

The proof of this result follows easily from the definition of f′(N ). Given f(x) = x, the derivative value at any value of x, say, x = N, is

$$f'(x) = \lim_{x\to N}\frac{f(x) - f(N)}{x - N} = \lim_{x\to N}\frac{k - k}{x - N} = \lim_{x\to N} = 1$$

Since N represents any value of x, it is permissible to write $f'(x) = 1$. This proves the rule for the case of n = 1. As the graphical counterpart of this result, we see that the function y = f(x) = x plots as a 45 line, and it has a slope of + 1 throughout. For the cases of larger integers, n = 2, 3, ... , let us first note the following identities:

$$\frac{x^2 - N^2}{x - N} = \frac{(x+N)(x-N)}{x-N} = x + N \qquad (2\ term\ on\ the\ right)$$

$$\frac{x^3 - N^3}{x - N} = \frac{(x-N)(x^2+Nx+N^2)}{x-N} = (x^2+Nx+N^2) \qquad (3\ term\ on\ the\ right)$$

$$\frac{x^n - N^2}{x - N} = (x^{n-1} + N^{n-2} + N^2x^{n-3} + \dots\dots + N^n) \qquad (n\ term\ on\ the\ right)$$

We can express the derivative of a power function f (x) = $x^n$ at x = N as follows:

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x^2 - N^2}{x - N}$$

$$= \lim_{x \to N} (x^{n-1} + N^{n-2} + N^2x^{n-3} + \dots\dots + N^n)$$

$$= \lim_{x \to N} x^{n-1} + \lim_{x \to N} N^{n-2} + \lim_{x \to N} N^2x^{n-3} + \dots\dots + \lim_{x \to N} N^n)$$

$$= N^{n-1} + N^{n-1} + N^{n-1}$$

$$= nN^{n-1}$$

Again, N is any value of x; thus this last result can be generalised to $f'(x) = nx^{n-1}$ which proves the rule for n, any positive integer.

As mentioned above, this rule applies even when the exponent n in the power expression $x^n$ is not a positive integer. The following examples serve to illustrate its application to the latter cases.

**Example 5:**

Find the derivative of $y = x^0$, we find $\frac{dy}{dx}$

$$\frac{d}{dx}x0 = 0(x^1) = 0$$

## 3.9 Power Function Rule Generalised

When a multiplicative constant c appears in the power function, so that f (x) = $cx^n$ its derivative is

$$\frac{d}{dx}cx^n = cnx^{n-1} \qquad or \qquad f'(x) = cnx^{n-1}$$

This result shows that, in differentiating $cx^n$, we can simply retain the multiplicative constant c intact and then differentiate the term $x^n$.

**Example 8:**
The derivative of $f(x) = 3x^{-2}$ is $f'(x) = -6x^{-3}$ For a proof of this new rule, consider the fact that for any value of x, say, x = N, the value of the derivative of $f(x) = cx^n$ is

$$f'(N) = \lim_{x\to N} \frac{f(x)-f(N)}{x-N} = \lim_{x\to N} \frac{cx^2 - cN^2}{x-N} = c\left(\lim_{x\to N} \frac{x^2-N^2}{x-N}\right)$$

$$= \lim_{x\to N} c \lim_{x\to N} \frac{x^2-N^2}{x-N}$$

$$c \lim_{x\to N} \frac{x^2-N^2}{x-N}$$

$$= cN^{n-1}$$

In view that N is any value of x, this last result can be generalised immediately to $f'(x) = cnx^{n-1}$, which proves the rule.

### 3.9.1 Rules of Differentiation involving Two or more Function of the same Variable

The three rules presented in the preceding section are each concerned with a single given function f (x ). Now suppose that we have two differentiable function of the same variable x, say, f(x) and g(x ), and we want to differentiate the sum difference, product, or quotient formed with these two functions. In such circumstances, are there appropriate rules that apply? More concretely, given two functions-say, $f(x) = 3x^2$ and $g(x) = 9x^{12}$ how do we get the derivative of, say, $3x^2 + 9x^{12}$ or the derivative of $(3x^2)(9x^{12})$?

### 3.9.2 Sum-Difference Rule

The derivative of a sum (difference) of two functions is the sum (difference) of the derivatives of the two functions:

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

The proof of this again involves the application of the definition of a derivative and of the various limit theorems. We shall omit the proof and, instead, merely verify its validity and illustrate its application.

**Example 11 :**

From the function $y = 14x^3$, we can obtain the derivative_ $\frac{dy}{dx} = 42x^2$ But $14x^3 = 5x^3 + 9x^{3,}$ so that y may be regarded as the sum of two functions f (x) = $5x^3$ and g(x) = $9x^3$.

According to the sum rule, we then have $\frac{dx}{dy} = \frac{d}{dx}$ $(5x^3 + 9x^3) = 15x^2 + 27x^2 = 42x^2$ which is identical with our earlier result.

This rule, stated above in terms of two functions, can easily be extended to more functions. Thus, it is also valid to write $\frac{d}{dx}(J(x) \pm g(x) \pm h(x)) = f'(x) \pm g'(x) \pm h'(x)$

**Example 12:**

$$\frac{d}{dx}(7x^4 + 2x^3 - 3x + 37) = 28x^3 + 6x^2 - 3 + 0 = 28x^3 + 6x^2 - 3$$

Note that in the last two examples the constants c and 37 do not really produce any effect on the derivative, because the derivative of a constant term is zero. In contrast to

Stastics6.png

Figure 6:

the multiplicative constant, which is retained during differentiation, the additive constant drops out. This fact provides the mathematical explanation of the well-known economic principle that the fixed cost of a firm does not affect its marginal cost. Given a short-run total-cost function

$$C = Q^3 - 4Q^2 + lOQ + 75$$

the marginal cost function (for infinitesimal output change) is the limit of the quotient Q, or the derivative of the C function:

$$\frac{dC}{dQ} = 3Q2 - 8Q + 10$$

whereas the fixed cost is represented by the additive constant 75. Since the latter drops out during the process of deriving $\frac{dC}{dQ}$, the magnitude of the fixed cost obviously cannot affect the marginal cost.

In general, if a primitive function y = f(x) represents a total function, then the derivative function $\frac{dx}{dy}$ is its marginal function. Both functions can, of course, be plotted against the variable x graphically; and because of the correspondence between the derivative of a function and the slope of its curve, for each value of x the marginal function should show the slope of the total function at that value of x. A linear (constant slope) total function is seen to have a constant marginal function. On the other hand, the nonlinear (varying slope) total function gives rise to a curved marginal function, which lies below (above) the horizontal axis when the total function is negatively (positively) sloped. And, finally, the reader may note that "nonsmoothness" of a total function will result in a gap (discontinuity) in the marginal or derivative function. This is in sharp contrast to the everywhere smooth total function in which gives rise to a continuous marginal function. For this reason, the smoothness of a primitive function can be linked to the continuity of its derivative function. In particular, instead of saying that a certain function is smooth (and differentiable) everywhere, we may alternatively characterise it as a function with a continuous derivative function, and refer to it as a continuously differentiable function.

### 3.9.3 Product Rule

The derivative of the product of two (differentiable) functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function:

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)g(x) = f'(x)g(x) + g'(x)f(x)$$

**Example 13:**

Find the derivative of y = $(2x + 3)(3x^2)$. Let f(x) = 2x + 3 and g(x) =$3x^2$. Then it follows that f'(x) = 2 and g'(x) = 6x the desired derivative is

$$\frac{d}{dx}(2x+3)(3x^2) = (3x^2)\frac{d}{dx}(2x+3) + (2x+3)\frac{d}{dx}(3x^2) = (3x^2)2 + (2x+3)6x = 18x^2 + 18x$$

This result can be checked by first multiplying out f(x)g(x) and then taking the The important point to remember is that the derivative of a product of two functions is not the simple product of the two separate derivatives.

$$\frac{d}{dx}(f(x)g(x))\,|_{x=N} = \lim_{x\to N}\frac{f(x)g(x) - f(N)g(N)}{x - N} \qquad (8)$$

But, by adding and subtracting f(x)g(N) in the numerator (thereby leaving the original magnitude unchanged), we can transform the quotient on the right of (6.11) as follows:

$$\frac{f(x)g(x) - f(x)g(N) + f(x)g(N) - f(N)g(N)}{x - N}$$
$$= f(x)\frac{g(x) - g(N))}{x - N} + g(x)\frac{f(x) - f(N))}{x - N}$$

Substituting this result into (6.11) and taking the limit, we then have

$$\frac{d}{dx}(f(x)g(x))\,|_{x=N} = \lim_{x\to N} f(x)\lim_{x\to N}\frac{g(x) - g(N))}{x - N} + \lim_{x\to N} g(x)\lim_{x\to N}\frac{f(x) - f(N))}{x - N} \qquad (9)$$

The four limit expressions in (6.12) are easily evaluated. The first one is f(N) and the third is g(N) (limit of a constant). The remaining two are f' (N) and g' (N). Thus, the above equation can be written as

$$\frac{d}{dx}(f(x)g(x))\,|_{x=N} = f(x)\lim_{x\to N} f(N)g'(N) + g(N)f'(N) \qquad (10)$$

Since N represents any value of x, (6.13) remains valid if replace every N as x. Hence then prove the Theorem which can be generalised by replacing the symbol N with x, because N represents any value of x. This proves the quotient rule.

### 3.9.4 Relationship between Marginal Cost and Average Cost Functions

As an economic application of the quotient rule, let us consider the rate of change of

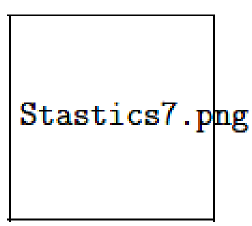


Figure 7:

average cost when output varies.
Given a cost function C= C(Q), the average cost (AC) the function will

be quotient of two functions Q. Since AC= $\frac{C(Q)}{Q}$ defined as long as Q > 0. Therefore, the rate of change of AC with respect to Q can be found by differentiating AC:

$$\frac{d}{DQ}\frac{C(Q)}{Q} = \frac{C'(Q)Q - C(Q)1}{Q^2} = \frac{1}{Q}\left(C'(Q) - \frac{C(Q)}{Q}\right)$$

from this follows that, $Q > 0$

$\frac{d}{DQ}\frac{C(Q)}{Q} \geq 0$ iff $C'(Q) \geq \frac{d}{DQ}\frac{C(Q)}{Q}$

Since the derivative C'(Q) represents the marginal-cost (MC) function, and $\frac{C(Q)}{Q}$ represents the AC function, the economic meaning of the above graph is the slope of the AC curve will be positive, zero, or negative if and only if the marginal-cost curve lies above, intersects, or lies below the AC curve. This is illustrated above where the MC and AC functions plotted are based on the specific total-cost function

$$C = Q^3 - 12Q^2 + 60Q$$

To the left of Q = 6, AC is declining, and thus MC lies below it; to the right, the opposite is true:At Q =6 , AC has a slope of zero, and MC and AC have the same value. The qualitative conclusion in above is stated explicitly in terms of cost functions. How-ever, its validity remains unaffected if we interpret C (Q) as any other differentiable total function with $\frac{C(Q)}{Q}$ and $C'(Q)$as its corresponding average and marginal functions.

Thus this result gives us general marginal-average relationship. In particular, we may point out, the fact that MR lies below AR when AR is downward-sloping, as discussed in con-nection with above is nothing but a special case of the general result of the above.

### 3.9.5 Rules of Differentiation involving Two or more Function of Difference Variable

In the preceding section, we discussed the rules of differentiation of a sum, difference, product, or quotient of two (or more) differentiable functions of the same variable. Now we shall consider cases where there are two or more differentiable functions, each of which has a distinct independent variable.

### 3.9.6 Chain Rule

If we have a function z = f (y), where y is in turn a function of another variable x, say, y = g(x), then the derivative of z with respect to x is equal to the derivative of z with respect to y, times the derivative of y with respect to x. Expressed symbolically,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x) \qquad (11)$$

This rule known as the chain rule appeals easily to intuition. Given a Δx, there must result a corresponding Δy via the function y = g(x), but this Δy will in turning bring about a Δz via the function z := f (y). Thus there is a "chain reaction follow

$$\Delta x \quad \underrightarrow{via\ g} \quad \Delta y \quad \underrightarrow{via\ f} \quad \Delta z$$

The two links in this chain entail two difference quotients, $\frac{\Delta y}{\Delta x}$ and $\frac{\Delta z}{\Delta y}$, but when they are multiplied, the y will cancel each other out, and we end up with

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x}$$

a difference quotient that relates Δz to Δx. If we take the limit of these difference quotients as Δx (which implies Δy → 0), each difference quotient will turn into a derivative; i.e., we shall have $\frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dx}$.This is precisely the result in (9). In view of the function y = g( x ), we can express the function z = f (y) as z = f(g(x)), where the contiguous appearance of the two function symbols f and g indicates that this is a composite function (function of a function). It is for this reason that the chain rule is also referred to as the composite-function rule or function of a function rule.

The extension of the chain rule to three or more functions is straightforward. If we have z = f (y), y = g(x), and x = h(w), then

$$\frac{dz}{dw} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw} = f'(x)g'(y)h'(w)$$

and similarly for cases in which more functions are involved.

**Example 14:**

If $z = 3y^2$ , where y = 2x + 5, then

$\frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dx}$

$6y(2) = 12y = 12(2x+5)$

**Example 10**: Given a total-revenue function of a firm R = f(Q), where output Q is a function of labor input L, or Q = g(L), finddRdL . By the chain rule, we have

$$\frac{dR}{dQ}\frac{dQ}{dL} = \frac{dR}{dL} = f'(Q)g'(L)$$

Translated into economic term $\frac{dR}{dQ}$ is the MR function $\frac{dQ}{dL}$ marginal physical labour M P PL functions. Similarly, $\frac{dQ}{dL}$ has the connotation of the marginal-revenue-product-of-labor $MRP_L$ function. Thus the result shown above constitutes the mathematical statement of the well-known result in economic thatM $RP_L$ = M.RM P $P_L$.

## 3.10 Total Derivatives

With the notion of differentials at our disposal, we are now equipped to answer the question posed at the beginning of the chapter, namely, how we find the rate of change of the function C(Y, $T_0$) with respect to $T_0$ when Y and $T_0$ are related.

As previously mentioned, the answer lies in the concept of total derivative. Unlike a partial derivative, a total derivative does not require the argument Y to remain constant as $T_0$ varies, and can thus allow for the postulated relationship be the two arguments

### 3.10.1 Finding the Total Derivatives

To carry on the discussion in a more general framework, let us consider any function

y =f(x, w) where x= g( W)

with the three variables y, x, and w related. In this, which we shall refer to as a channel map, it is clearly seen that w-::-the ultimate source of change in this case-can affect y through two channels: OJ indirectly, via the function g and then f (the straight arrows), and (2) directly, via the fun tion f (the curve arrow). Whereas the partial derivatie $f_w$ is adequate for expressing the direct efffect along, a total derivative it needs to express both jointly.

To obtain this total derivative, we first differentiate y totally, to get the total erential dy = $f_x$dx + $f_w$dy .When- both sides of this equation are divided by the differential dw, the result is

$$\frac{dy}{dw} = f_x\frac{dx}{dw} + f_w\frac{dw}{dw} = \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}\left(\frac{\partial w}{\partial w} = 1\right)$$

Since the ratio of two differentials may be interpreted as a derivative, the e sion $\frac{dy}{dw}$ on the left rriiy b regarded as some measure of the rate of change of y with respect. Moreover, if the two terms on the right side of above equation can be identified, respectively, as the indirect and direct effects of w on y, then $\frac{dy}{dw}$ will indeed be total derivative we are seeking. Now, the second term $(\frac{\partial y}{\partial w})$ is already known to measure the direct effect, and it tl:n!s. corresponds tonthe curved arrow, That the first term $(\frac{\partial y}{\partial x}\frac{dx}{dw})$ measures the indirect ∂xdw) effect will also become evident when we analyze it with the help of some arrows as follows:

$$(\frac{\partial y}{\partial x}\frac{dx}{dw})$$

The change in w (namely, dw) is in the first instance transmitted to the variable x, and through the resulting change in x (namely, dx) it is relayed to the variabley. But this is precisely the indirect effect, as depicted by the sequence of straight arrows in above.

Hence, the expressions in above does indeed represent the desired total derivative. The process of finding the total derivative $\frac{dy}{dw}$ is referred to as total differentiation of y with respect to w.

## 3.11 Partial Differentiation

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter.

Therefore, asa final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

### 3.11.1 Partial Derivatives

Let us consider a function

$$y=f(x_1,x_2,x_3, \quad x_n) \qquad (12)$$

There the variables X; ( i = 1, 2, ...n) all independent of one another, so that each can vary by itself without affecting the others. If the variable $x_1$ undergoes a change $\Delta x_1$ while $x_2$, $x_n$ all remain fixed, there will be a corresponding change in y, namely, $\Delta y$. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_2, x_2, .......x_n) - f(x_1, x_2, .......x_n)}{\Delta x_1} \qquad (13)$$

If we take the limit of $\frac{\Delta y}{\Delta x_1}1$ as $\Delta x1 \rightarrow 0$, that limit will constitute a derivative. We call it the partial derivative of y with respect to x1, to indicate that all the other independent variables in the- function are held constant when taking tills particular derivative. Similar partial derivatives can be defined for infinitesimal changes in the other independent variables. The process of taking partial derivatives is called partial differentiation.

Partial derivatives are assigned distinctive symbols. In lieu of the letter d (as in $\frac{dy}{dx}$) we employ the symbol which is a variant of the Greek (δ) (lower case delta). Thus we shall now write $\frac{\partial y}{\partial x}$ the partial derivative y with respect to $x_i$." The partial-derivative symbol sometimes is also written as $\frac{\partial}{\partial x_i}y$ in that case, its $\frac{\partial}{\partial x_i}$ part can be regarded as an operator symbol instructing us to take the partial derivative of (some function) with respect to the variable $x_i$ Since the function involved here is denoted in (9) by f, it is also permissible to write $\frac{\partial}{\partial x_i}$.

Is there also a partial-derivative counterpart for the symbol f'(X) that we used before?

The answer is yes. Instead of f', however, we now-use $f_1$, $f_1$, etc where the subscript indicates independent variable (alone) is being allowed to vary. If the function in (13) happens to be written in terms of unsubscripted variables, such as Y − = f (u, vw), then the partial derivatives may be denoted by $f_u$, $f_v$ and $f_w$ rather than $f_1$, $f_2$ and $f_3$.

In line with these notations, and on the basis of (12) and (13), we can now define

$$f_1 = \frac{\partial}{\partial x_i} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x_i}$$

as the first in the set of n partial derivatives of the function f.

### 3.11.2 Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold (n - 1) independent variables constant while allow ing one variable to vary. Inasmuch as we have learned how to handle constants in differentiation, the actual differentiation should pose little problem.

**Example 14:**
Given y = f ($x_1$, $x_2$) = $3x_2f + x_1x_2 + 4x$,

find the partial derivatives. When finding $\frac{\partial}{\partial x_i}$ (or f1), we must bear in mind that $x_2$ is to be treated as a constant during differentiation. As such, $x_2$ will drop out in the process if it is an additive constant (such as the term $4x^2$ but will be retained if it is as multiplicative constant (such as in term of $x_1x_2$ ). Thus we have

$$\frac{\partial}{\partial x_1} = f_1 = 6x_1 + x_2$$

Similarly, by treating $x_1$ as a constant, we find that

$$\frac{\partial}{\partial x_2} = f_1 = x_1 + 8x_2$$

Note that, like the primitive function f, both partial derivatives are themselves functions of the variables $x_1$ and $x_2$. That is, we may write them as two derived functions $f_1 = f_1(x_1, x_2)$ and $f_3 = f_2(x_1, x_2)$. For the point $(x_1, x_2) = (1, 3)$ in the domain of the function f example, the partial derivatives will take the following specific values:

$f_1(1, 3) = 6(1) + 3 = 9)$ and $f_2(1, 3) = 1 + 8(3) = 25$

### 3.11.3 Geometric Interpretation of Partial Derivatives

As a special type of derivative, a partial derivative is a measure of the instanta neous rates of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.
Let us consider a production function Q = Q(K, L), where Q, K, and L denote output, capital input, and labor input, respectively. This function is a particular two-variable version of (12), with n = 2. We can therefore

define two partial derivatives $\frac{\partial Q}{\partial K}(or Q_K)$ and $\frac{\partial Q}{\partial L}(or Q_L)$. The partial derivative QK relates to the rates of change in output with respect to infinitesimal changes in capital, while labor input is held constant. Thus QK symbolizes the marginal physical-product-of-capital (M P PK) function. Siinilarly, the partial derivative QL is the mathematical representation of the MPPL function.

Geometrically, the production function Q = Q( K, L) can be depicted by a production surface in a 3 space, such as is shown in fig 5 The variable Q is plotted vertically, so that for any point ( K, L) in the base plane (KL plane), the height of the surface will indicate

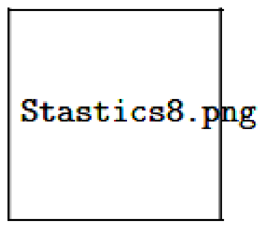


Figure 8:

the output Q. The domain of the function should consist of the entire nonnegative quadrant of the base plane, but for our purposes it is sufficient to consider a subset of it, the rectangle $OK_0BL_0$. As a consequence, only a small portion of the production surface is shown in the figure.

Let us now hold capital fixed at the level $K_0$ and consider only variations in the input L. By setting $K = K_0$ , all points in our (curtailed) domain become irrelevant except those on the line segment $K_0B$. By the same token, only the curve $K_0CDA$ (a cross section of the production surface) will be germane to the present discussion. This curve represents a total physical product of labor (TPPL).

See the diagram below for better understanding curve for a fixed amount of capital $K = K_0$, thus we may read from its slope the rate of change of Q with respect to changes in L while K is held constant. It is clear, therefore, that the slope of a curve such as $K_0CDA$ represents the geometric counterpart of the partial derivative QL. Once again, we note that the slope of a total ($TPP_L$) curve is its corresponding marginal ($MPPL = Q_L$) curve.

It was mentioned earlier that a partial derivative is a function of all the independent variables of the primitive function. That QL is a function of L is immediately obvious from the K0CDA curve itself. When $L = L_1$, the value of QL is equal to the slope of the curve at point C; but when $L = L_2$ , the relevant slope is the one at point D. Why is $Q_L$ also a function of K? The answer is that K can be fixed at various levels, and for each fixed level of K, there will result a different TPPL curve (a different cross section of the production surface), with inevitable repercussions on the derivative $Q_L$. Hence $Q_L$ is also a function of K.

An analogous interpretation can be given to the partial derivative QK. If the labor input is held constant instead of K (say, at the level of $L_0$ ), the line segment $L_0$ B will be the relevant subset of the domain, and the curve $L_0$ A will indicate the relevant subset of the production surface. The partial derivative $Q_K$ can then be interpreted as the slope of the curve $L_0$ A-bearing in mind that the K axis extends from southeast to northwest in Figure above. It should be noted that $Q_K$ is again a function of both the variables L and K.

## 4.0 CONCLUSION

At end of this module students are able differentiate and determine the limit of func-tions. Also differentiate between partial and total differentiation with their applications.

## 5.0 SUMMARY

This unit highlighted Comparative Statics, described Derivatives and their applications, discussed Function and its limits, and the techniques of Differentiation.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. A function y=f(x) is dicontinuous at $x = x_0$ when any of the three requirements forcontinuity is violated at $x = x_0$. Construct three graphs to illustrate the violation of each of each requirement.
2. Given the function $q=g(v)=\underline{{}^{v}2^{+2}}+2$
   (a) Use limit theorem to find $\lim_{v\to N} q$, N being a finite ral number
   (b) Check whether this is equal g(N)
   (c) Check continuity of the function g(v) at N and in its domain $(-\infty, \infty)$

## 7.0 REFERENCES/FURTHER READING

Binmore, K. (1991). *Fun and Games: A Text on Game Theory*. D. C. Heath: Lexington, MA.

Dixit, A. K. & Nalebuff, B. J. (1991). *Thinking Strategically: The Competitive Edge in Business, Politics, and Everyday Life*.

Norton: New York.

Fudenberg, D. & Tirole, J. (1991). *Game Theory*. Cambridge, MA. Gibbons: MIT Press.

Robert (1992), *Game Theory for Applied Economists*. Princeton, NJ: Princeton University Press.

Myerson, R. B. (1991). *Game Theory: Analysis of Conflict*. Cambridge, MA: Harvard University Press.

Nasar, S. (1998). *A Beautiful Mind: A Biography of John Forbes Nash, Jr., Winner of the Nobel Prize in Economics, 1994.* New York: Simon and Schuster.

Rasmusen, E. (2001). *Games and Information: An Introduction to Game Theory*. (3rd ed.) Blackwell: Oxford.

# UNIT 2 APPLICATIONS TO COMPARATIVE STATIC ANALYSIS

CONTENTS



## 1.0 INTRODUCTION

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- illustrate the application of differentiation
- describe how to determine market model
- demonsrtate how to determine national income and model
- discuss input and output model.

## 3.0 MAIN CONTENT

### 3.1 Market Model

First let us consider again the simple one-commodity market model of (3.1). That model can be written in the form of two equations: Q = a−bP (a, b > 0)(c, d > 0) (demand)
(supply)
$Q = -c + dP (a, b > 0)(c, d > 0)$ (demand) (supply)
with solutions

$$\bar{P} = \frac{a+c}{b+d} \quad (1)$$

$$\bar{Q} = \frac{ad-be}{b+d} \quad (2)$$

These solutions will be referred to as being in the reduced form: the two endogenous variables have been reduced to explicit expressions of the four mutually independent parameters a, b, c, and d.

To find how an infinitesimal change in one of the parameters will affect the value of P, one has only to differentiate (2) partially with respect to each of the parameters. If the sign of a partial derivative, say, $\frac{\partial \bar{P}}{\partial a}$ can be determined from the given information about the parameters, we shall know the direction in which P will move when the parameter a changes; this constitutes a qualitative conclusion. Ifthe magnitude of$^{\partial}$ $\frac{\partial \bar{P}}{\partial a}$ can be ascertained, it will constitute quantitative conclusion.

Similarly, we can draw qualitative or quantitative conclusions from the partial derivatives of P with respect to each parameter, such as $\frac{\partial \bar{Q}}{\partial a}$. To avoid misunderstanding, however, a clear distinction should be made between the two derivatives $\frac{\partial \bar{Q}}{\partial a}$. and $\frac{\partial \bar{P}}{\partial a}$. The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative $\frac{\partial \bar{Q}}{\partial a}$. pertains, on the other hand, to the equilibrium quantity in (3) which, being in the nature of a solution of the model, takes into account the interaction of demand and supply together. To emphasize this distinction, we shall refer to the partial derivatives of P and Q with respect to the parameters as comparative-static derivatives.

Concentrating on P for the time being, we can get the following four partial derivatives from (1):

$$\frac{\partial \bar{P}}{\partial a} = \frac{1}{a+b} \qquad (parameters\ a\ has\ the\ coefficient\ of\ \frac{1}{a+b})$$

$$\frac{\partial \bar{P}}{\partial b} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2}$$

$$\frac{\partial \bar{P}}{\partial c} = \frac{1}{b+d}\left(\frac{\partial \bar{P}}{\partial a}\right)$$

$$\frac{\partial \bar{P}}{\partial d} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2}\left(\frac{\partial \bar{P}}{\partial a}\right)$$

Since all the parameters are restricted to being positive in the present model, we can conclude that

$$\frac{\partial \bar{P}}{\partial a} = \frac{\partial \bar{P}}{\partial c} \qquad and \qquad \frac{\partial \bar{P}}{=}\partial d\frac{\partial \bar{P}}{\partial b} \tag{3}$$

In figure below pictures an increase in the parameter a (to $a'$) This means a higher vertical intercept for the demand curve, and inasmuch as the parameter b (the slope parameter) is unchanged, the increase in a results in a parallel upward shift of the demand curve from D to $D'$ . The intersection of $D'$ and the supply curve S determines an equi-librium price $P'$, which is greater than the old equilibrium price P. This corroborates the result that $\frac{\partial \bar{P}}{\partial a} > 0$, although for the sake of exposition we have shown in figure below a much larger change in the parameter a than what the concept of derivative implies.

The situation figure below has a similar interpretation; but since the increase takes place in the parameter c, the result is a parallel shift of the supply curve instead. Note that this shift is downward because the supply curve has a vertical intercept of -c; thus an increase in c would mean a change in the intercept, say, from -2 to -4. The graphi-cal comparative static result, that P′ exceeds P, again conforms to what the positive sign

Stastics9.png

Figure 9:

of the derivative $\frac{\partial \bar{P}}{\partial a}$ expect.

The below illustrate the effects of changes in the slope parameters b and d of the twofunctions in the model. An increase in b means that the slope of the demand curve willassume a larger numerical (absolute) value; i.e., it will become steeper. In accordancewith the result $\frac{\partial \bar{P}}{\partial a} < 0$, we find a decrease in P in this diagram. The increase in d thatmakes the supply curve steeper also results in a decrease in the equilibrium price. This is, of course, again in line with the negative sign of the comparative-static derivative $\frac{\partial \bar{P}}{\partial a}$ .

Thus far, all the results in (3) seem to have been obtainable graphically. If so, why should we bother to learn differentiation at all? The answer is that the differentiation approach has at least two major advantages. First, the graphical technique is subject to a dimensional restriction, but differentiation is not. Even

When the number of endogenous variables and parameters is such that the equilibrium state cannot be shown graphically, we can nevertheless apply the differentiation techniques to the problem. Second, the differentiation method can yield results that are on a higher level of generality. The results in (2) will remain valid, regardless of the specific values that the parameters a, b, c, and d take, as long as they satisfy the sign restrictions.

So the comparative-static conclusions of this model are, in effect, applicable to an infinite number of combinations of (linear) demand and supply functions. In contrast, the graphical approach deals only with some specific members of the family of demand and supply curves, and the analytical result derived therefrom is applicable, strictly speaking, only to the specific functions depicted.

The above serves to illustrate the application of partial differentiation to comparativestatic analysis of the simple market model, but only half of the task has actually been accomplished, for we can also find the comparative-static derivatives pertaining to Q. This we shall leave to you as an exercise.

## 3.2 National-Income Model

Let us study a slightly enlarged model with three endogenous variables, Y (national income), C (consumption), and T (taxes):

$$\left.\begin{array}{ll} Y = C + 1_0 + G_0 & \\ C = \alpha + \beta(Y - T) & (\alpha > O; O < \beta < l) \\ T = \gamma + \delta Y & (\gamma > O; 0 < \delta < 1) \end{array}\right\} \qquad (4)$$

The first equation in this system gives the equilibrium condition for national income, while the second and third equations show, respectively, how C and T are determined in the model.

The restrictions on the values of the parameters α, β, γ, and δ can be explained thus: a is positive because consumption is positive even if disposable income (Y- T) is zero; β is a positive fraction because it represents the marginal propensity to consume; y is positive because even if Y is zero the government will still have a positive tax revenue (from tax bases other than income); and finally, l) is a positive fraction because it represents an income tax rate, and as such it cannot exceed 100 percent.

The exogenous variables (investment) and $G_0$ (government expenditure) are, of course, nonnegative. All the parameters and exogenous variables are assumed to be independent of one another, so that any one of them can be assigned a new value without affecting the others.

This model can be solved for Y by substituting the third equation of (4) into the second and then substituting the resulting equation into the first. The equilibrium income (in reduced form) is

$$\bar{Y} = \frac{\alpha - \beta Y + 1_0 + G_0}{1 - \gamma + \beta\delta} \qquad (5)$$

Similar equilibrium values can also be found for the endogenous variables C and T, but we shall concentrate on the equilibrium income.

From (5), there can be obtained six comparative-static derivatives. Among these, the following three have special policy significance:

$$\frac{\partial Y}{\partial G_0} = \frac{1}{1 - \gamma + \beta\delta} > 0 \qquad (6)$$

$$\frac{\partial \bar{Y}}{\partial \gamma} = \frac{-\beta}{1 - \gamma + \beta\delta} < 0 \qquad (7)$$

$$\frac{\partial \bar{Y}}{\partial \delta} = \frac{-\beta(\alpha - \beta Y + 1_0 + G_0)}{(1 - \gamma + \beta\delta)^2} = \frac{\beta \bar{Y}}{1 - \gamma + \beta\delta} \qquad (8)$$

The partial derivative in (8) gives us the government-expenditure multiplier. It has a positive sign here because β is less than 1, and βδ is greater than zero. If numerical values are given for the parameters β and δ, we can also find the numerical value of this multiplier from (6). The derivative in (5) may be called the non income-tax multiplier, because it shows how a change in y, the government revenue from non income-tax sources, will affect the equilibrium income. This multiplier is negative in the present model be-cause the denominator in (4) is positive and the numerator is negative.-Lastly, the partial derivative in (8) represents an income-tax-rate multiplier. For any positive equilibrium income, this multiplier is also negative in the model.

Again, note the difference between the two derivatives a $\frac{\partial \bar{Y}}{\partial G_0}$ **and** $\frac{\partial Y}{\partial G_0}$ The former is derived from (5), the expression for the equilibrium income. The latter, obtainable from the first equation in (4), is $\frac{\partial Y}{\partial G_0} = 1$, which is altogether different in magnitude and in concept.

## 3.3 Input-Output Model

The solution of an open input-output model appears as a matrix equation $x = (I - A)^{-1}d$. If we denote the inverse matrix $x = (I - A)^{-1}$ by $B = (b_{ij})$, then, for instance, the solution for a three-industry economy can be written as x = Bd or

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

What will be the rates of change of the solution values x1 with respect to the exogenous final demands $d_1$, $d_2$, and $d_3$? The general answer is that

$$\frac{\partial \bar{x}}{\partial d_k} = b_{jk} \qquad (j, k = 1, 2, 3, ..........n)$$

To see this, let us multiply out Bd in above and express the solution as

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} b_{11}d_1 + b_{12}d_2 + b_{13}d_3 \\ b_{21}d_1 + b_{22}d_2 + b_{23}d_3 \\ b_{31}d_1 + b_{32}d_2 + b_{33}d_3 \end{pmatrix}$$

In this system of three equations, each one gives a particular solution value as a function of the exogenous final demands. Partial differentiation of these will produce a total of nine comparative-static derivatives:

$$\frac{\partial \bar{x}_1}{\partial d_1} = b_{11} \qquad \frac{\partial \bar{x}_1}{\partial d_2} = b_{12} \qquad \frac{\partial \bar{x}_1}{\partial d_3} = b_{13}$$

$$\frac{\partial \bar{x}_2}{\partial d_1} = b_{21} \qquad \frac{\partial \bar{x}_2}{\partial d_2} = b_{22} \qquad \frac{\partial \bar{x}_2}{\partial d_3} = b_{23}$$

$$\frac{\partial \bar{x}_3}{\partial d_1} = b_{31} \qquad \frac{\partial \bar{x}_3}{\partial d_2} = b_{32} \qquad \frac{\partial \bar{x}_3}{\partial d_3} = b_{33}$$

This is simply the expanded version of above structure.
Reading above structure as three distinct columns, we may combine the three deriva tives in each column into a matrix (vector) derivative:

$$\frac{\partial \bar{x}}{\partial d_1} = \frac{\partial}{\partial d_1}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} \qquad \frac{\partial \bar{x}}{\partial d_2} = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} \qquad \frac{\partial \bar{x}}{\partial d_3} = \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \end{pmatrix}$$

Since the three column vectors in (7.23") are merely the columns of the matrix B, by further consolidation we can summarize the nine derivatives in a single matrix derivative axj ad. Given x = Bd, we can simply write

$$\frac{\partial \bar{x}}{\partial d}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = B$$

This is a compact way of denoting all the comparative-static derivatives of our open inputoutput model. Obviously, this matrix derivative can easily be extended from the present three-industry model to the general n-industry case.

Comparative-static derivatives of the input-output model are useful as tools of economic planning, for they provide the answer to the question: If the planning targets, as reflected in ($d^1$, $d_2$, $d_{3.............}$ $d_n$), are revised, and if we wish to take care of all direct and indirect requirements in the economy so as to be completely free of bottlenecks, how must we change the output goals of the n industries?

## 4.0 CONCLUSION

We can conclude that, here students are able to handle any applications problems on differentiation, partial differential equations and modelling and able to solve them relatively.

## 5.0 SUMMARY

This unit focused on Comparative Static Analysis as a platform to determine how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters. It is useful in determining market model, national income and model, input and output model.

## 6.0 REFERENCES/FURTHER READING

Binmore, K. (1991). *Fun and Games: A Text on Game Theory*. D. C. Heath: Lexington, MA.

Dixit, A. K. & Nalebuff, B. J. (1991). *Thinking Strategically: The*

*Competitive Edge in Business, Politics, and Everyday Life*. Norton: New York.

Fudenberg, D. & Tirole, J. (1991). *Game Theory*. Cambridge, MA. Gibbons: MIT Press.

Robert (1992), *Game Theory for Applied Economists*. Princeton, NJ: Princeton University Press.

Myerson, R. B. (1991). *Game Theory: Analysis of Conflict*. Cambridge, MA: Harvard University Press.

Nasar, S. (1998). *A Beautiful Mind: A Biography of John Forbes Nash, Jr., Winner of the Nobel Prize in Economics, 1994.* New York: Simon and Schuster.

Rasmusen, E. (2001). *Games and Information: An Introduction to Game Theory*.(3rd ed.) Blackwell: Oxford.

# MODULE 5



## UNIT 1 GAMES THEORY

**CONTENTS**



## 1.0 INTRODUCTION

**Backward induction**

Backward induction is a technique to solve a game of perfect information. This process first considers the moves that are the last in the game and determine the best move for the player in each case. Then, taking these as given future actions, it proceeds backwards in time, again determining the best move for the respective players, until the beginning of the game is reached.

**Common knowledge**

A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.

**Dominating strategy**

A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing. It weakly dominates the other strategy if it is always at least as good.

**Extensive game**

An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves, and the information each player has at each decision point.

Game A game is a formal description of a strategic situation.

**Game theory**

Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

**Mixed strategy**

A mixed strategy is an active randomisation, with given probabilities that determine the players' decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

**Nash equilibrium**

Nash equilibrium, also called strategic equilibrium, is a list of strategies, one for each player, which has the property that no player can unilaterally change his strategy and get a better payoff.

**Payoff**

A payoff is a number, also called utility that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the players' attitude towards risk.

**Perfect information**

A game has perfect information when at any point in time only one player makes a move, and knows all the actions that have been made until then.

**Player**

A player is an agent who makes decisions in a game.

**Rationality**

A player is said to be rational if he seeks to play in a manner which maximises his own payoff. It is often assumed that the rationality of all players is common knowledge.

**Strategic form**

A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

**Strategy**

In a game in strategic form, a strategy is one of the given possible actions of a player. In an extensive game, a strategy is a complete plan of choices, one for each decision point of the player.

**Zero -sum game**

A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one players gain is the other player's loss, so their interests are diametrically opposed.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- describe the techniques of games theory equilibrium
- explain zero sum and computation in games theory
- discuss bidding and auction in games theory.

## 3.0 MAIN CONTENT

### 3.1 Game Theory

The earliest example of a formal game theoretic analysis is the study of a duopoly by Antoine Cournot in 1838. The mathematician Emile Borel suggested a formal theory of games in 1921, which was furthered by the mathematician John von Neumann in 1928 in a theory of parlour games. Game theory was established as a field in its own right after the 1944 publication of the monumental volume Theory of Games and Economic Behaviour by von Neumann and the economist Oskar Morgenstern. This book provided much of the basic terminology and problem setup that is still in use today.

In 1950, John Nash demonstrated that finite games have always have an equilibrium point, at which all players choose actions which are best for them given their opponents choices. This central concept of noncooperative game theory has been a focal point of analysis since then. In the 1950s and 1960s, game theory was broadened theoretically and applied to problems of war and politics. Since the 1970s, it has driven a revolution in economic theory. Additionally, it has found applications in sociology and psychology, and established links with evolution and biology. Game theory received special attention in 1994 with the awarding of the Nobel Prize in economics to Nash, John Harsanyi, and Reinhard Selten.

At the end of the 1990s, a high-profile application of game theory has been the design of auctions. Prominent game theorists have been involved in the design of auctions for allocating rights to the use of bands of the electromagnetic spectrum to the mobile telecommunications industry. Most of these auctions were designed with the goal of allocating these resources more efficiently than traditional governmental practices, and additionally raised billions of dollars in the United States and Europe.

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms, or any combination of these. The concepts of game theory provide a language to formulate structure, analyse, and understand strategic scenarios.

### 3.1.1 Game Theory and Information Systems

The internal consistency and mathematical foundations of game theory make it a prime tool for modelling and designing automated decision-making processes in interactive environments. For example, one might like to have efficient bidding rules for an auction website, or tamper-proof automated negotiations for purchasing communication bandwidth. Research in these applications of game theory is the topic of recent conference and journal papers but is still in a nascent stage. The automation of strategic choices enhances the need for these choices to be made efficiently, and to be robust against abuse. Game theory addresses these requirements.

As a mathematical tool for the decision-maker the strength of game theory is the methodology it provides for structuring and analysing problems of strategic choice. The process of formally modelling a situation as a game requires the decision-maker to enumerate explicitly the players and their strategic options, and to consider their preferences and reactions. The discipline involved in constructing such a model already has the potential of providing the decision-maker with a clearer and broader view of the situation. This is a prescriptive application of game theory, with the goal of improved strategic decision making. With this perspective in mind, this article explains basic principles of game theory, as an introduction to an interested reader without a background in economics.

### 3.1.2 Definitions of Games

The object of study in game theory is the game, which is a formal model of an inter-active situation. It typically involves several players; a game with only one player is usually called a decision problem. The formal definition lays out the players, their preferences, their information, and the strategic actions available to them, and how these influence the outcome.

Games can be described formally at various levels of detail. A coalitional (or cooperative) game is a high-level description, specifying only what payoffs each potential group, or coalition, can obtain by the cooperation of its members. What is not made explicit is the process by which the coalition forms. As an example, the players may be several parties in parliament. Each party has a different strength, based upon the number of seats occupied by party members. The game describes which coalitions of parties can form a majority, but does not delineate, for example, the negotiation process through which an agreement to vote en bloc is achieved.

Cooperative game theory investigates such coalitional games with respect to the relative amounts of power held by various players, or how a successful coalition should divide its proceeds. This is most naturally applied to situations arising in political science or international relations, where concepts like power are most important. For example, Nash proposed a solution for the division of gains from agreement in a bargaining problem which depends solely on the relative strengths of the two parties bargaining position. The amount of power a side has is determined by the usually inefficient outcome that results when negotiations break down. Nash model fits within the cooperative framework in that it does not delineate a specific time line of offers and counteroffers, but rather focuses solely on the outcome of the bargaining process.

In contrast, non cooperative game theory is concerned with the analysis of strategic choices. The paradigm of non cooperative game theory is that the details of the ordering and timing of players' choices are crucial to determining the outcome of a game. In contrast to Nash's cooperative model, a non cooperative model of bargaining would post a specific process in which it is pre-specified who gets to make an offer at a given time. The term non cooperative means this branch of game theory explicitly models the process of players making choices out of their own interest. Cooperation can, and often does, arise in non-cooperative models of games, when players find it in their own best interests. Branches of game theory also differ in their assumptions. A central assumption in many variants of game theory is that the players are rational. A rational player is one who always chooses an action which gives the outcome he most prefers, given what he expects his opponents to do. The goal of game-theoretic analysis in these branches, then, is to predict how the game will be played by rational players, or, related, to give advice on how best to play the game against opponents who are rational. This rationality assumption can be relaxed, and the resulting models have been more recently applied to the analysis of observed behaviour. This kind of game theory can be viewed as more descriptive than the prescriptive approach taken here.

This article focuses principally on non cooperative game theory with rational players. In addition to providing an important baseline case in economic theory, this case is designed so that it gives good advice to the decision-maker, even when or perhaps especially when one's opponents also employ it.

### 3.1.3 Strategic and Extensive Form Games

The strategic form (also called normal form) is the basic type of game studied in non cooperative game theory. A game in strategic form lists each player's strategies, and the outcomes that result from each possible combination of choices. An outcome is represented by a separate payoff for each player, which is a number (also called utility) that measures how much the player likes the outcome.

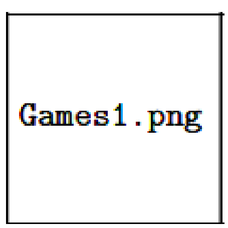


Figure 10:

The extensive form, also called a game tree, is more detailed than the strategic form of a game. It is a complete description of how the game is played over time. This includes the order in which players take actions, the information that players have at the time they must take those actions, and the times at which any uncertainty in the situation is resolved. A game in extensive form may be analysed directly, or can be converted into an equivalent strategic form.

## 3.2 Dominance

Since all players are assumed to be rational, they make choices which result in the outcome they prefer most, given what their opponents do. In the extreme case, a player may have two strategies A and B so that, given any combination of strategies of the other players, the outcome resulting from A is better than the outcome resulting from B. Then strategy A is said to dominate strategy B. A rational player will never choose to play a dominated strategy. In some games, examination of which strategies are dominated results in the conclusion that rational players could only ever choose one of their strategies. The following examples illustrate this idea.

**Example 1: Prisoners Dilemma**

The Prisoners Dilemma is a game in strategic form between two players. Each player has two strategies, called cooperate and defect, which are labeled C and D for player I and c and d for player II, respectively. (For simpler identification, upper case letters are used for strategies of player I and lower case letters for player II.)

Figure 1 shows the resulting payoffs in this game. Player I chooses a row, either C or D, and simultaneously player II chooses one of the columns c or d. The strategy combination (C; c) has payoff 2 for each player, and the combination (D; d) gives each player payoff 1. The combination (C; d) results in payoff 0 for player I and 3 for player II, and when (D; c) is played, player I gets 3 and player II gets 0.

Any two-player game in strategic form can be described by a table like the one in Figure 1, with rows representing the strategies of player I and columns those of player II. (A player may have more than two strategies.) Each strategy combination defines a payoff pair, like (3; 0) for (D; c), which is given in the respective table entry. Each cell of the table shows the payoff to player I at the (lower) left, and the payoff to player II at the (right) top. These staggered payoffs, due to Thomas Schelling, also make transparent when, as here, the game is symmetric between the two players. Symmetry means that the game stays the same when the players are exchanged, corresponding to a reflection along the diagonal shown as a dotted line in Figure2. Note that in the strategic form, there is no order between player I and II since they act simultaneously (that is, without knowing the others action), which makes the symmetry possible.

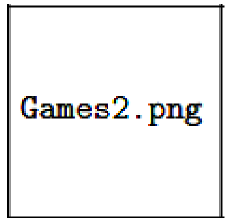


Figure 11:

In Figure 2, the game of Figure 1 with annotations is implied by the payoff structure. The dotted line shows the symmetry of the game. The arrows at the left and right point to the preferred strategy of player I when player II plays the left or right column, respectively. Similarly, the arrows at the top and bottom point to the preferred strategy of player II when player I play top or bottom.

In the Prisoners Dilemma game, defect is a strategy that dominates and cooperates. Strategy D of player I dominate C since if player II chooses c, then player 1s payoff is 3 when choosing D and 2 when choosing C; if player II chooses d, then player I receives 1 for D as opposed to 0 for C. These preferences of player I are indicated by the downward pointing arrows in Figure 8. 2. Hence, D is indeed always better and dominates C. In the same way, strategy d dominates c for player II.

No rational player will choose a dominated strategy since the player will always be better off when changing to the strategy that dominates it. The unique outcome in this game, as recommended to utility-maximising players, is therefore (D; d) with payoffs (1;1). Somewhat paradoxically, this is less than the payoff (2; 2) that would be achieved when the players chose (C; c).

The story behind the name Prisoners Dilemma is that of two prisoners held suspect of a serious crime. There is no judicial evidence for this crime except if one of the prisoners testifies against the other. If one of them testifies, he will be rewarded with immunity from prosecution (payoff 3), whereas the other will serve a long prison sentence (payoff 0). If both testify, their punishment will be less severe (payoff 1 for each).

However, if they both cooperate with each other by not testifying at all, they will only be imprisoned briefly, for example for illegal weapons possession (payoff 2 for each). The defection from that mutually beneficial outcome is to testify, which gives a higher payoff no matter what the other prisoner does, with a resulting lower payoff to both. This constitutes their dilemma.

Prisoners Dilemma games arise in various contexts where individual defections at the expense of others lead to overall less desirable outcomes.

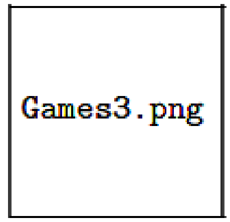


Figure 12:

Examples include arms races, litigation instead of settlement, environmental pollution, or cut-price marketing, where the resulting outcome is detrimental for the players. Its game-theoretic justification on individual grounds is sometimes taken as a case for treaties and laws, which enforce co-operation.

Game theorists have tried to tackle the obvious inefficiency of the outcome of the Prisoners Dilemma game. For example, the game is fundamentally changed by playing it more than once. In such a repeated game, patterns of cooperation can be established as rational behaviour when players fear of punishment in the future outweighs their gain from defecting today.

**Example 1: Quality choice**

The next example of a game illustrates how the principle of elimination of dominated strategies may be applied iteratively. Suppose player I is an internet service provider and player II a potential customer. They consider entering into a contract of service provision for a period of time. The provider can, for himself, decide between two levels of quality of service, High or Low. High-quality service is more costly to provide, and some of the cost is independent of whether the contract is signed or not. The level of service cannot be put verifiably into the contract. High-quality service is more valuable than low-quality service to the customer, in fact so much so that the customer would prefer not to buy the service if she knew that the quality was low. Her choices are to buy or not to buy the service.

Figure 3 shows high-low quality game between a service provider (player I) and a customer (player II).

Figure 3 gives possible payoffs that describe this situation. The customer prefers to buy if player I provide high-quality service, and not to buy otherwise. Regardless of whether the customer chooses to buy or not, the provider always prefers to provide the low-quality service. Therefore, the strategy Low dominates the strategy High for player I. Now, since player II believes player I is rational, she realises that player I always prefers Low, and so she anticipates low quality service as the providers choice. Then she prefers not to buy (giving her a payoff of 1) to buy (payoff 0). Therefore, the rationality of both players leads to the conclusion that the provider will implement low-quality service and, as a result, the contract will not be signed.

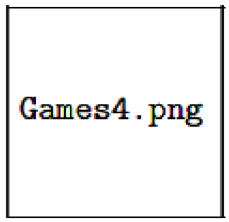


Figure 13:

This game is very similar to the Prisoners Dilemma in Figure 1. In fact, it differs only by a single payoff, namely payoff 1 (rather than 3) to player II in the top right cell in the table. This reverses the top arrow from right to left, and makes the preference of player II dependent on the action of player I. (The game is also no longer symmetric.) Player II does not have a dominating strategy. However, player I still does, so that the resulting outcome, seen from the flow of arrows in Figure 3, is still unique. Another way of obtaining this outcome is the successive elimination of dominated strategies: First, High is eliminated, and in the resulting smaller game where player I has only the single strategy Low available, player IIs strategy buy is dominated and also removed. As in the Prisoners Dilemma, the individually rational outcome is worse for both players than another outcome, namely the strategy combination (High, buy) where high quality service is provided and the customer signs the contract. However, that outcome is not credible, since the provider would be tempted to renege and provide only the low quality service.

## 3.3 Nash Equilibrium

In the previous examples, consideration of dominating strategies alone yielded precise advice to the players on how to play the game. In many games, however, there are no dominated strategies, and so these considerations are not enough to rule out any outcomes or to provide more specific advice on how to play the game.

The central concept of Nash equilibrium is much more general. Nash equilibrium recommends a strategy to each player that the player cannot improve upon unilaterally, that is, given that the other players follow the recommendation. Since the other players are also rational, it is reasonable for each player to expect his opponents to follow the recommendation as well.

**Example 8.4: Quality choice revisited**

A game-theoretic analysis can highlight aspects of an interactive situation that could be changed to get a better outcome. In the quality game in Figure 3, for example, increasing the customers' utility of high-quality service has no effect unless the provider has an incentive to provide that service. So suppose that the game is changed by introducing an opt-out clause into the service contract. That is, the customer can discontinue subscribing to the service if she finds it of low quality. The resulting game is shown in Figure 4. Here, low-quality service provision, even when the customer decides to buy, has the same low payoff 1 to the provider as when the Figure 4 shows a high-low quality game with opt-out clause for the customer. The left arrow shows that player I prefers High when player II chooses to buy. Customer does not sign the contract in the first place, since the customer will opt out later. However, the customer still prefers not to buy when the service is Low in order to spare her the hassle of entering the contract.

The changed payoff to player I means that the left arrow in Figure 4 points upwards. Note that, compared to Figure 8.3, only the providers' payoffs are changed. In a sense, the opt-out clause in the contract has the purpose of convincing the customer that the high-quality service provision is in the providers own interest. This game has no dominated strategy for either player. The arrows point in different directions. The game has two Nash equilibrium in which each player chooses his strategy deterministically. One of them is, as before, the strategy combination (Low, don't buy). This is equilibrium since Low is the best response (payoff-maximising strategy) to don't buy and vice versa.

The second Nash equilibrium is the strategy combination (High, buy). It is an equilibrium since player I prefers to provide high-quality service when the customer buys, and conversely, player II prefers to buy when the quality is high. This equilibrium has a higher payoff to both players than the former one, and is a more desirable solution. Both Nash equilibriums are legitimate recommendations to the two players of how to strategy combination that is not Nash equilibrium is not a credible solution. Such a strategy combination would not be a reliable recommendation on how to play the game, since at least one player would rather ignore the advice and instead play another strategy to make

him better off.
As this example shows, Nash equilibrium may be not unique. However, the previously discussed solutions to the Prisoners Dilemma and to the quality choice game in Figure 3 are unique Nash equilibriums. A dominated strategy can never be part of equilibrium since a player intending to play a dominated strategy could switch to the dominating strategy and be better off. Thus, if elimination of dominated strategies leads to a unique strategy combination, then this is Nash equilibrium. Larger games may also have unique equilibria that do not result from dominance considerations.

## 3.4 Equilibrium selection

If a game has more than one Nash equilibrium, a theory of strategic interaction should guide players towards the most reasonable equilibrium upon which they should focus. Indeed, a large number of papers in game theory have been concerned with equilibrium refinements that attempt to derive conditions that make one equilibrium more plausible or convincing than another. For example, it could be argued that an equilibrium that is better for both players, like (High, buy) in Figure 8.4, should be the one that is played.

However, the abstract theoretical considerations for equilibrium selection are often more sophisticated than the simple game-theoretical models they are applied to. It may be more illuminating to observe that a game has more than one equilibrium, and that this is a reason that players are sometimes stuck at an inferior outcome.

One and the same game may also have a different interpretation where a previously undesirable equilibrium becomes rather plausible. As an example, consider an alternative scenario for the game in Figure 8.4. Unlike the previous situation, it will have a symmetric description of the players, in line with the symmetry of the payoff structure.

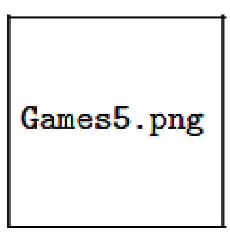


Figure 14:

Two firms want to invest in communication infrastructure. They intend to communicate frequently with each other using that infrastructure, but they decide independently on what to buy. Each firm can decide between High or Low bandwidth equipment (this time, the same strategy names will be used for both players). For player II, High and Low replace buy and don't buy in Figure 8. 4. The rest of the game stays as it is.

The (unchanged) payoffs have the following interpretation for player I (which applies in the same way to player II by symmetry): A Low bandwidth connection works equally well (payoff 1) regardless of whether the other side has high or low bandwidth. However, switching from Low to High is preferable only if the other side has high bandwidth (payoff 2), otherwise it incurs unnecessary cost (payoff 0).

As in the quality game, the equilibrium (Low, Low) (the bottom right cell) is inferior to the other equilibrium, although in this interpretation it does not look quite as bad. Moreover, the strategy Low has obviously the better worst-case payoff, as considered for all possible strategies of the other player, no matter if these strategies are rational choices or not. The strategy Low is therefore also called a max-min strategy since it maximises the minimum payoff the player can get in each case. In a sense, investing only in low bandwidth equipment is a safe choice. Moreover, this strategy is part of equilibrium, and entirely justified if the player expects the other player to do the same.

### 3.4.1 Evolutionary games

The bandwidth choice game can be given a different interpretation where it applies to a large population of identical players. Equilibrium can then be viewed as the outcome of a dynamic process rather than of conscious rational analysis.

Figure 5 shows the bandwidth choice game where each player has the two strategies High and Low. The positive payoff of 5 for each player for the strategy combination (High, High) makes this an even more preferable equilibrium than in the case discussed above. In the evolutionary interpretation, there is a large population of individuals, each of which can adopt one of the strategies. The game describes the payoffs that result when two of these individuals meet. The dynamics of this game are based on assuming that each strategy is played by a certain fraction of individuals. Then, given this distribution of strategies, individuals with better average payoff will be more successful than others, so that their proportion in the population increases over time. This, in turn, may affect which strategies are better than others. In many cases, in particular in symmetric games with only two possible strategies, the dynamic process will move to equilibrium. In the example of Figure 5, a certain fraction of users connected to a network will already have High or Low bandwidth equipment. For example, suppose that one quarter of the users has chosen High and three quarters have chosen Low. It is useful to assign these as percentages to the columns, which represent the strategies of player II. A new user, as player I, is then to decide between High and Low, where his payoff depends on the given fractions. Here it will be $^{14}\times5+4^{\times0 = 1.25}$ when player I chooses High, and 1 $1+^{34}\times4\times1=1$ when player I chooses Low. Given the average payoff that player I can expect when interacting with other users, player I will be better off by choosing High, and so decides on that strategy. Then, player I joins the population as a High user. The proportion of individuals of type High therefore increases, and over time the advantage of that strategy will become even more pronounced. In addition, users replacing their equipment will make the same calculation, and therefore also switch from Low to High. Eventually, everyone plays High as the only surviving strategy, which corresponds to the equilibrium in the top left cell in Figure 5.

The long-term outcome where only high-bandwidth equipment is selected depends on there being an initial fraction of high-bandwidth users that is large enough. For example, if only ten percent have chosen High, then the expected payoff for High is $0.1\times5+0.9\times0 = 0.5$ which is less than the expected payoff 1 for Low (which is always 1, irrespective of the distribution of users in the population). Then by the same logic as before, the fraction of Low users' increases, moving to the bottom right cell of the game as the equilibrium. It is easy to see that the critical fraction of High users so that this will take off as the better strategy is 15. (When new technology makes high-bandwidth equipment cheaper, this increases the payoff 0 to the High user who is meeting Low, which changes the game.)

The evolutionary, population-dynamic view of games is useful because it does not require the assumption that all players are sophisticated and think the others are also rational, which is often unrealistic. Instead, the notion of rationality is replaced with the much weaker concept of reproductive success: strategies that are successful on average will be used more frequently and thus prevail in the end. This view originated in theoretical biology with Maynard Smith (Evolution and the Theory of Games, Cambridge University Press, 1982) and has since significantly increased in scope.

## 3.5 Mixed Strategies

A game in strategic form does not always have a Nash equilibrium in which each player deterministically chooses one of his strategies. However, players may instead randomly select from among these pure strategies with certain probabilities. Randomising one's own choice in this way is called a mixed strategy. Nash showed in 1951 that any finite strategic-form game has equilibrium if mixed strategies are allowed. As before, equilibrium is defined by a (possibly mixed) strategy for each player where no player can gain on average by unilateral deviation. Average (that is, expected) payoffs must be considered because the outcome of the game may be random.


Figure 15:

**Example 5: Compliance inspections**

Suppose a consumer purchases a license for a software package, agreeing to certain restrictions on its use. The consumer has an incentive to violate these rules. The vendor would like to verify that the consumer is abiding by the agreement, but doing so requires inspections which are costly. If the vendor does inspect and catches the consumer cheating, the vendor can demand a large penalty payment for the noncompliance. Figure 6 shows possible payoffs for such an inspection game. The standard outcome, defining the reference payoff zero to both vendor (player I) and consumer (player II).

Figure 6 Inspection game between a software vendor (player I) and consumer (player II) is that the vendor chooses don't inspect and the consumer chooses to comply. With-out inspection, the consumer prefers to cheat since that gives her payoff 10, with resulting negative payoff N10 to the vendor. The vendor may also decide to Inspect. If the consumer complies, inspection leaves her payoff 0 unchanged, while the vendor incurs a cost resulting in a negative payoff N1. If the consumer cheats, however, inspection will result in a heavy penalty (payoff N90 for player II) and still create a certain amount of hassle for player I (payoff N6).

In all cases, player I would strongly prefer if player II complied, but this is outside of player offs control. However, the vendor prefers to inspect if the consumer cheats (since -6 is better than -10), indicated by the downward arrow on the right in Figure 6. If the vendor always preferred don't inspect, then this would be a dominating strategy and be part of a (unique) equilibrium where the consumer cheats.

The circular arrow structure in Figure 6 shows that this game has no equilibrium in pure strategies. If any of the players settles on a deterministic choice (like Don't inspect by player I), the best response of the other player would be unique (here cheat by player II), to which the original choice would not be a best response (player I prefers Inspect when the other player chooses cheat, against which player II in turn prefers to comply). The strategies in Nash equilibrium must be best responses to each other, so in this game this fails to hold for any pure strategy combination.

### 3.6 Mixed Equilibrium

What should the players do in the game of Figure 6? One possibility is that they prepare for the worst, that is, choose a max-min strategy. As explained before, a max-min strategy maximises the player offs worst payoff against all possible choices of the opponent. The Max-min strategy for player I is to Inspect (where the vendor guarantees himself payoff 6), and for player II it is to comply (which guarantees her payoff 0). However, this is not a Nash equilibrium and hence not a stable recommendation to the two players, since player I could switch his strategy and improve his payoff. A mixed strategy of player I in this game is to Inspect only with a certain probability. In the context of inspections, randomising is also a practical approach that reduces costs.

Even if an inspection is not certain, a sufficiently high chance of being caught should deter from cheating, at least to some extent. The following considerations show how to find the probability of inspection that will lead to equilibrium. If the probability of inspection is very low, for example one percent, then player II receives (irrespective of that probability) payoff 0 for comply, and payoff $0.99 \times 10 + 0.01 \times (-90) = 9$, which is bigger than zero, for cheat. Hence, player II will still cheat, just as in the absence of inspection. If the probability of inspection is much higher, for example 0:2, then the expected payoff for cheat is $0.8 \times 10 + 0.2 \times (-90) = -10$, which is less than zero, so that player II prefers to comply. If the inspection probability is either too low or too high, then player II has a unique best response. As shown above, such a pure strategy cannot be part of equilibrium. Hence, the only case where player II herself could possibly randomise between her strategies is if both strategies give her the same payoff, that is, if she is indifferent. It is never optimal for a player to assign a positive probability to playing a strategy that is inferior, given what the other players are doing. It is not hard to see that player II is indifferent if and only if player I inspects with probability 0.1, since then the expected payoff for cheat is$0.9 \times 10 + 0.1 \times (-90) = 0$, which is then the same as the payoff for comply.

With this mixed strategy of player I (Dont inspect with probability 0.9 and Inspect with probability 0.1), player II is indifferent between her strategies. Hence, she can mix them (that is, play them randomly) without losing payoff. The only case where, in turn, the original mixed strategy of player I is a best response is if player I is indifferent. According to the payoffs in Figure 6, this requires player II to choose comply with probability 0.8 and cheat with probability 0.2. The expected payoffs to player I are then for Don't inspect $0.8 \times 0 + 0.2 \times (-10) = -2$, and for Inspect $0.8 \times (-1) + 0.2 \times (-6) = -2$, so that player I is indeed indifferent, and his mixed strategy is a best response to the mixed strategy of player II. This defines the only Nash equilibrium of the game. It uses mixed strategies and is therefore called a mixed equilibrium. The resulting expected payoffs are -2 for player I and 0 for player II.

### 3.6.1 Interpretation of Mixed Strategy Probabilities

The preceding analysis showed that the game in Figure 6 has a mixed equilibrium, where the players choose their pure strategies according to certain probabilities. These probabilities have several noteworthy features.

The equilibrium probability of 0.1 for Inspect makes player II indifferent between comply and cheat. This is based on the assumption that an expected payoff of 0 for cheat, namely $0.9 \times 10 + 0.1 \times (-90)$, is the same for player II as when getting the payoff 0 for certain, by choosing to comply. If the payoffs were monetary amounts (each payoff unit standing for one thousand naira, say), one would not necessarily assume such a risk neutrality on the part of the consumer. In practice, decision-makers are typically risk averse, meaning they prefer the safe payoff of 0 to the gamble with an expectation of 0. In a game-theoretic model with random outcomes (as in a mixed equilibrium), however, the payoff is not necessarily to be interpreted as money. Rather, the players' attitude towards risk is incorporated into the payoff figure as well. To take our example, the con sumer faces a certain reward or punishment when cheating, depending on whether she is caught or not. Getting caught may not only involve financial loss but embarrassment and other undesirable consequences.

However, there is a certain probability of inspection (that is, of getting caught) where the consumer becomes indifferent between comply and cheat. If that probability is 1 against 9, then this indifference implies that the cost (negative payoff) for getting caught is 9 times as high as the reward for cheating successfully, as assumed by the payoffs in Figure 6. If the probability of indifference is 1 against 20, the payoff -90 in Figure 6 should be changed to N200. The units in which payoffs are measured are arbitrary. Like degrees on a temperature scale, they can be multiplied by a positive number and shifted by adding a constant, without altering the underlying preferences they represent.

In a sense, the payoffs in a game mimic a play offs (consistent) willingness to bet when facing certain odds. With respect to the payoffs, which may distort the monetary amounts, players are then risk neutral. Such payoffs are also called expected-utility values. Expected-utility functions are also used in one-player games to model decisions under uncertainty.

The risk attitude of a player may not be known in practice. A game-theoretic analysis should be carried out for different choices of the payoff parameters in order to test how much they influence the results. Typically, these parameters represent the political features of a game-theoretic model that most sensitive to subjective judgement, compared to the more technical part of a solution. In more involved inspection games, the technical part often concerns the optimal usage of limited inspection resources, whereas the political decision is when to raise an alarm and declare that the inspector has cheated.

Secondly, mixing seems paradoxical when the player is indifferent in equilibrium. If player II, for example, can equally well comply or cheat, why should she gamble? In particular, she could comply and get payoff zero for certain, which is simpler and safer. The answer is that precisely because there is no incentive to choose one strategy over the other, a player can mix, and only in that case there can be equilibrium. If player II would comply for certain, then the only optimal choice of player I is do not inspect, making the choice of complying not optimal, so this is not equilibrium.

The least intuitive aspect of mixed equilibrium is that the probabilities depend on the opponent payoffs and not on the players own payoffs (as long as the qualitative preference structure, represented by the arrows, remains intact). For example, one would expect that raising the penalty -90 in Figure 8.6 for being caught lowers the probability of cheating in equilibrium. In fact, it does not. What does change is the probability of inspection, which is reduced until the consumer is indifferent.

This dependence of mixed equilibrium probabilities on the opponents' payoffs can be explained in terms of population dynamics. In that interpretation, Figure 6 represents an evolutionary game. Unlike Figure 8.5, it is a non-symmetric interaction between a vendor who chooses Don't Inspect and Inspect for certain fractions of a large number of inter-actions. Player IIs actions comply and cheat are each chosen by a certain fraction of consumers involved in these interactions. If these fractions deviate from the equilibrium probabilities, then the strategies that do better will increase. For example, if player I chooses Inspect too often (relative to the penalty for a cheater who is caught), the fraction of cheaters will decrease, which in turn makes

Don't Inspect a better strategy. In this dynamic process, the long-term averages of the fractions approximate the equilibrium probabilities.

### 3.6.3 Extensive Games with Perfect Information

Games in strategic form have no temporal component. In a game in strategic form, the players choose their strategies simultaneously, without knowing the choices of the other players. The more detailed model of a game tree, also called a game in extensive form, formalises interactions where the players can over time be informed about the actions of others. This section treats games of perfect information. In an extensive game with perfect information, every player is at any point aware of the previous choices of all other players. Furthermore, only one player moves at a time, so that there are no simultaneous moves.

**Example 6 Quality choice with commitment**

Figure 7 shows another variant of the quality choice game. This is a game tree with perfect information. Every branching point, or node, is associated with a player who makes a move by choosing the next node. The connecting lines are labelled with the players choices. The game starts at the initial node, the root of the tree, and ends at a terminal node, which establishes the outcome and determines the players' payoffs. In Figure 8 and 7, the tree grows from left to right; game trees may also be drawn top-down or bottom-up.

The service provider, player I, makes the first move, choosing High or Low quality of service. Then the customer, player II, is informed about that choice. Player II can then decide separately between buy and don't buy in each case. The resulting payoffs are the Figure 7. Quality choice game where player I commits to High or Low quality, and player II can react accordingly. The arrows indicate the optimal moves as determined by backward induction same as in the strategic-form game in Figure 3. However, the game is different from the one in Figure 3, since the players now move in sequence rather than simultaneously. Extensive games with perfect information can be analysed by backward induction.

This technique solves the game by first considering the last possible choices in the game. Here, player II moves last. Since she knows the play will end after her move, she can safely select the action which is best for her. If player I has chosen to provide high quality service, then the customer prefers to buy, since her resulting payoff of 2 is larger than 1 when not buying. If the provider has chosen Low, then the customer prefers not to purchase. These choices by player II are indicated by arrows in Figure 7.

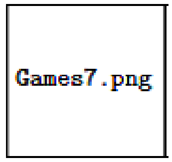


Figure 16:

Once the last moves have been decided, backward induction proceeds to the players making the next-to-last moves (and then continues in this manner). In Figure 7, player I makes the next-to-last move, which in this case is the first move in the game. Being ra tional, he anticipates the subsequent choices by the customer. He therefore realises that his decision between High and Low is effectively between the outcomes with payoffs (2; 2) or (1; 1) for the two players, respectively. Clearly, he prefers High, which results in a payoff of 2 for him, to Low, which leads to an outcome with payoff 1. So the unique solution to the game, as determined by backward induction, is that player I offers high quality service, and player II responds by buying the service.

### 3.6.2 Strategies in Extensive Games

In an extensive game with perfect information, backward induction usually prescribes unique choices at the players' decision nodes. The only exception is if a player is indifferent between two or more moves at a node. Then, any of these best moves, or even randomly selecting from among them, could be chosen by the analyst in the backward induction process. Since the eventual outcome depends on these choices, this may affect a player who moves earlier, since the anticipated payoffs of that player may depend on the subsequent moves of other players. In this case, backward induction does not yield a unique outcome; however, this can only occur when a player is exactly indifferent between two or more outcomes.

The backward induction solution specifies the way the game will be played. Starting from the root of the tree, play proceeds along a path to an outcome. Note that the analysis yields more than the choices along the path. Because backward induction looks at every node in the tree, it specifies for every player a complete plan of what to do at every point in the game where the player can make a move, even though that point may never arise in the course of play. Such a plan is called a strategy of the player. For example, a strategy of player II in Figure 7 is buy if offered high-quality service; don't buy if offered low quality service. This is player IIs strategy obtained by backward induction. Only the first choice in this strategy comes into effect when the game is played according to the backward-induction solution.


Figure 17:

Figure 8 is a strategic form of the extensive game in Figure 7.

With strategies defined as complete move plans, one can obtain the strategic form of the extensive game. As in the strategic form games shown before, this tabulates all strategies of the players. In the game tree, any strategy combination results into an outcome of the game, which can be determined by tracing out the path of play arising from the players adopting the strategy combination. The payoffs to the players are then entered into the corresponding cell in the strategic form. Figure 8 shows the strategic form for our example. The second column is player IIs backward induction strategy, where buy if offered high-quality service, don't buy if offered low-quality service is abbreviated as H: buy, L: don't.

A game tree can therefore be analyzed in terms of the strategic form. It is not hard to see that backward induction always defines a Nash equilibrium. In Figure 8, it is the strategy combination (High; H: buy, L: don't).

A game that evolves over time is better represented by a game tree than using the strategic form. The tree reflects the temporal aspect, and backward induction is succinct and natural. The strategic form typically contains redundancies. Figure 8, for example, has eight cells, but the game tree in Figure 7 has only four outcomes. Every outcome appears twice, which happens when two strategies of player II differ only in the move that is not reached after the move of player I. All move combinations of player II must be distinguished as strategies since any two of them may lead to different outcomes, depending on the action of player I.

Not all Nash equilibria in an extensive game arise by backward induction. In Figure 8, the rightmost bottom cell (Low; H: don't, L: don't) is also an equilibrium. Here the customer never buys, and correspondingly Low is the best response of the service provider to this anticipated behaviour of player II. Although H: don't is not an optimal choice (so it disagrees with backward induction), player II never has to make that move, and is therefore not better off by changing her strategy. Hence, this is indeed equilibrium. It prescribes a suboptimal move in the sub game where player II has learned that player I has chosen High. Because a Nash equilibrium obtained by backward induction does not have such a deficiency, it is also called sub game perfect.

The strategic form of a game tree may reveal Nash equilibria which are not sub game perfect. Then a player plans to behave irrationally in a sub game. He may even profit from this threat as long as he does not have to execute it (that is, the sub game stays unreached). Examples are games of market entry deterrence, for example the so-called Chain store game.

The analysis of dynamic strategic interaction was pioneered by Selten, for which he earned a share of the 1994 Nobel Prize. First-mover advantage a practical application of game-theoretic analysis may be to reveal the potential effects of changing the rules of the game. This has been illustrated with three versions of the quality choice game, with the analysis resulting in three different predictions for how the game might be played by rational players. Changing the original quality choice game in Figure 3 to Figure 4 yielded an additional, although not unique, Nash equilibrium (High, buy). The change from Figure 3 to Figure 7 is more fundamental since there the provider has the power to commit himself to high or low quality service, and inform the customer of that choice. The backward induction equilibrium in that game is unique, and the outcome is better for both players than the original equilibrium (Low, don't buy). Many games in strategic form exhibit what may be called the first-mover advantage. A player in a game becomes a first mover or leader when he can commit to a strategy, that is, choose a strategy irrevocably and inform the other players about it; this is a change of the rules of the game.

The first-mover advantage states that a player who can become a leader is not worse off than in the original game where the players act simultaneously. In other words, if one of the players has the power to commit, he or she should do so. This statement must be interpreted carefully. For example, if more than one player has the power to commit, then it is not necessarily best to go first. For example, consider changing the game in Figure 3 so that player II can commit to her strategy and player I moves second. Then player I will always respond by choosing Low, since this is his dominant choice in Figure 3. Backward induction would then amount to player II not buying and player I offering low service, with the low payoff 1 to both. Then player II is not worse off than in the simultaneous-choice game, as asserted by the first-mover advantage, but does not gain anything either. In contrast, making player I the first mover as in Figure 7 is beneficial to both.

If the game has antagonistic aspects, like the inspection game in Figure 6, then mixed strategies may be required to find Nash equilibrium of the simultaneous-choice game. The first-mover game always has equilibrium, by backward induction, but having to commit and inform the other player of a pure strategy may be disadvantageous. The correct comparison is to consider commitment to a randomised choice, like to a certain inspection probability. In Figure 6, already the commitment to the pure strategy Inspect gives a better payoff to player I than the original mixed equilibrium since player II will respond by complying, but a commitment to a sufficiently high inspection probability (anything above 10 per cent) is even better for player I.

**Example 9: Duopoly of chip manufacturers**

The first-mover advantage is also known as Stackelberg leadership, after the economist Heinrich von Stackelberg who formulated this concept for the structure of markets in 1934. The classic application is to the duopoly model by Cournot, which dates back to 1838.


Figure 18:

Figure 9- Duopoly game between two chip manufacturers who can decide between high, medium, low, or no production, denoted by H;M; L;N for firm I and h; m; l; n for firm II. Prices fall with increased production. Payoffs denote profits in millions of dollars. As an example, suppose that the market for a certain type of memory chip is dominated by two producers. The firms can choose to produce a certain quantity of chips, say either high, medium, low, or none at all, denoted by H;M; L;N for firm I and h; m; l; n for firm II. The market price of the memory chips decreases with increasing total quantity produced by both companies. In particular, if both choose a high quantity of production, the price collapses so that profits drop to zero. The firms know how increased production lowers the chip price and their profits. Figure 9 shows the game in strategic form, where both firms choose their output level simultaneously. The symmetric payoffs are derived from Cournots model, explained below.

The game can be solved by dominance considerations. Clearly, no production is dominated by low or medium production, so that row N and column n in Figure 9 can be eliminated. Then, high production is dominated by medium production, so that row H and column h can be omitted. At this point, only medium and low production remains. Then, regardless of whether the opponent produces medium or low, it is always better for each firm to produce medium. Therefore, the Nash equilibrium of the game is (M;m), where both firms make a profit of N16 million.

Consider now the commitment version of the game, with a game tree (omitted here) corresponding to Figure 9 just as Figure 7 is obtained from Figure 8.3. Suppose that firm I is able to publicly announce and commit to a level of production, given by a row in Figure 9.

Then firm II, informed of the choice of firm I, will respond to H by l (with maximum payoff 9 to firm II), to M by m, to L also by m, and to N by h. This determines the backward induction strategy of firm II. Among these anticipated responses by firm II, firm I do best by announcing H, a high level of production. The backward induction outcome is thus that firm I makes a profit 18million, asopposedtoonly16 million in the simultaneous-choice game. When firm II must play the role of the follower, its profits fall from N16 million to N9 million. The first-mover advantage again comes from the ability of firm I to credibly commit itself.

After firm I has chosen H, and firm II replies with l, firm I would like to be able switch to M, improving profits even further from N18 million to N20 million. However, once firm I is producing M, firm II would change to m. This logic demonstrates why, when the firms choose their quantities simultaneously, the strategy combination (H; l) is not an equilibrium. The commitment power of firm I, and firm IIs appreciation of this fact, is crucial.

The payoffs in Figure 9 are derived from the following simple model due to Cournot. The high, medium, low, and zero production numbers are 6, 4, 3, and 0 million memory chips, respectively. The profit per chip is 12 Q dollars, where Q is the total quantity (in mil-lions of chips) on the market. The entire production is sold. As an example, the strategy combination (H; l) yields Q = 6 + 3 = 9, with a profit of N3 per chip. This yields the payoffs of 18 and 9 million dollars for firms I and II in the (H; l) cell in Figure 9. Another example is firm I acting as a monopolist (firm II choosing n), with a high production level H of 6 million chips sold at a profit of N6 each. In this model, a monopolist would produce a quantity of 6 million even if other numbers than 6, 4, 3, or 0 were allowed, which gives the maximum profit of N36 million. The two firms could cooperate and split that amount by producing 3 million each, corresponding to the strategy combination (L; l) in Figure 9. The equilibrium quantities, however, are 4 million for each firm, where both firms receive less. The central four cells in Figure 9, with low and medium production in place of a cooperate h and defect, have the structure of a Prisoner Dilemma game (Figure 1), which arises here in a natural economic context. The optimal commitment of a first mover is to produce a quantity of 6 million, with the follower choosing 3 million. These numbers, and the equilibrium (g Cournot) quantity of 4 million, apply even when arbitrary quantities are allowed.

## 3.7 Extensive games with imperfect information

Typically, players do not always have full access to all the information which is relevant to their choices. Extensive games with imperfect information model exactly which information is available to the players when they make a move. Modelling and evaluating strategic information precisely is one of the strengths of game theory. John Harsanyis pioneering work in this area was recognised in the 1994 Nobel awards. Consider the situation faced by a large software company after a small start up has announced deployment of a key new technology. The large company has a large research and development operation, and it is generally known that they have researchers working on a wide variety of innovations. However, only the large company knows for sure whether or not they have made any progress on a product similar to the start-ups new technology.

The start up believes that there is a 50 per cent chance that the large company has developed the basis for a strong competing product. For brevity, when the large company has the ability to produce a strong competing product, the company will be referred to as having a g strong position, as opposed to a g weak one.

The large company, after the announcement, has two choices. It can counter by announcing that it too will release a competing product. Alternatively, it can choose to cede the market for this product. The large company will certainly condition its choice upon its private knowledge, and may choose to act differently when it has a strong position than when it has a weak one. If the large company has announced a product, the start up is faced with a choice: it can either negotiate a buyout or sell itself to the large company, or it can remain independent and launch its product. The start up does not have access to the large firm private information on the status of its research.

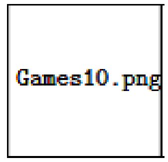


**Figure 19:**

However, it does observe whether or not the large company announces its own product, and may attempt to infer from that choice the likelihood that the large company has made progress of their own. When the large company does not have a strong product, the start up would prefer to stay in the market over selling out. When the large company does have a strong product, the opposite is true, and the star t up is better off by selling out instead of staying in. Figure 10 shows an extensive game that models this situation. From the perspective of the start up, whether or not the large company has done research in this area is random. To capture random events such as this formally in game trees, chance moves are introduced. At a node labelled as a chance move, the next branch of the tree is taken randomly and non-strategically by chance, or g nature, according to probabilities which are included in the specification of the game.

The game in Figure 8.10 starts with a chance move at the root. With equal probability 0.5, the chance move decides if the large software company, player I, is in a strong position (upward move) or weak position (downward move). When the company is in a weak position, it can choose to cede the market to the start up, with payoffs (0, 16) to the two players (with payoffs given in millions of dollars of profit). It can also announce a competing product, in the hope that the start up company, player II, will sell out, with payoffs 12 and 4 to players I and II. However, if player II decides instead to stay in, it will even profit from the increased publicity and gain a payoff of 20, with a loss of -4 to the large firm. Figure 8 explains extensive game with imperfect information between player I, a large soft-ware firm, and player II, a start up company. The chance move decides if player I is strong (top node) and does have a competing product, or weak (bottom node) and does not. The ovals indicate information sets. Player II sees only that player I chose to announce a competing product, but does not know if player I is strong or weak.

In contrast, when the large firm is in a strong position, it will not even consider the move of ceding the market to the start up, but will instead just announce its own product. In Figure 10, this is modelled by a single choice of player I at the upper node, which is taken for granted (one could add the extra choice of ceding and subsequently eliminate it as a dominated choice of the large firm). Then the payoffs to the two players are (20, -4) if the start up stays in and (12, 4) if the start up sells out.

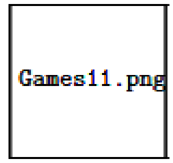


Figure 20:

In addition to a game tree with perfect information as in Figure 7, the nodes of the players are enclosed by ovals which are called information sets. The interpretation is that a player cannot distinguish among the nodes in information set, given his knowledge at the time he makes the move. Since his knowledge at all nodes in an information set is the same, he makes the same choice at each node in that set. Here, the start up company, player II, must choose between stay in and sell out. These are the two choices at player II's information set, which has two nodes according to the different histories of play, which player II cannot distinguish.

Because player II is not informed about its position in the game, backward induction can no longer be applied. It would be better to sell out at the top node, and to stay in at the bottom node. Consequently, player Ifs choice when being in the weak position is not clear: if player II stays in, then it is better to Cede (since 0 is better than -4), but if player II sells out, then it is better to Announce.

The game does not have equilibrium in pure strategies: The start-up would respond to Cede by selling out when seeing an announcement, since then this is only observed when player I is strong. But then player I would respond by announcing a product even in the weak position. In turn, the equal chance of facing a strong or weak opponent would induce the start up to stay in, since then the expected payoff of $0.5(-4) + 0.5 \times 20 = 8$ exceeds 4 when selling out.

Figure 11 Strategic form of the extensive game in Figure 8 with expected payoffs resulting from the chance move and the players choices.

The equilibrium of the game involves both players randomising. The mixed strategy probabilities can be determined from the strategic form of the game in Figure 11. When it is in a weak position, the large firm randomises with equal probability 1/2 between Announce and Cede so that the expected payoff to player II is then 7 for both stay in and sell out.

Since player II is indifferent, randomisation is a best response. If the start up chooses to stay in with probability 3/4 and to sell out with probability 1/4, then player I, in turn, is indifferent, receiving an overall expected payoff of 9 in each case. This can also be seen from the extensive game in Figure 10: when in a weak position, player I is indifferent between the moves Announce and Cede where the expected payoff is 0 in each case. With probability 1/2, player I is in the strong position, and stands to gain an expected payoff of 18 when facing the mixed strategy of player II. The overall expected payoff to player I is 9.

## 3.8 Zero-sum Games and Computation

The extreme case of players with fully opposed interests is embodied in the class of two player zero-sum (or constant-sum) games. Familiar examples range from rock-paper scissors, to many parlour games like chess, go, or checkers.

A classic case of a zero-sum game, which was considered in the early days of game theory by von Neumann, is the game of poker. The extensive game in Figure 10, and its strategic form in Figure 11, can be interpreted in terms of poker, where player I is dealt a strong or weak hand which is unknown to player II. It is a constant-sum game since for any outcome; the two payoffs add up to 16, so that one players gain is the other player's loss. When player I choose to announce despite being in a weak position, he is colloquially said to be bluffing. This bluff not only induces player II to possibly sell out, but similarly allows for the possibility that player II stays in when player I is strong, increasing the gain to player I.

Mixed strategies are a natural device for constant-sum games with imperfect information. Leaving one's own actions open reduces ones vulnerability against malicious responses. In the poker game of Figure 10, it is too costly to bluff all the time and better to randomise instead. The use of active randomisation will be familiar to anyone who has played rock-paper-scissors.

Zero-sum games can be used to model strategically the computer science concept of demonic non determinism. Demonic non determinism is based on the assumption that, when an ordering of events is not specified, one must assume that the worst possible sequence will take place. This can be placed into the framework of zero-sum game theory by treating nature (or the environment) as an antagonistic opponent. Optimal randomisation by such an opponent describes a worst-case scenario that can serve as a benchmark.

A similar use of randomisation is known in the theory of algorithms as Raos theorem, and describes the power of randomised algorithms. An example is the well-known quick sort algorithm, which has one of the best observed running times of sorting algorithms in practice, but can have bad worst cases. With randomisation, these can be made extremely unlikely.

Randomised algorithms and zero-sum games are used for analysing problems in online computation. This is, despite its name, not related to the internet, but describes the situation where an algorithm receives its input one data item at a time, and has to make decisions, for example in scheduling, without being able to wait until the entirety of the input is known. The analysis of online algorithms has revealed insights into hard optimisation problems, and seems also relevant to the massive data processing that is to be expected in the future. At present, it constitutes an active research area, although mostly confined to theoretical computer science.

## 3.9 Bidding in Auctions

The design and analysis of auctions is one of the triumphs of game theory. Auction Theory was pioneered by the economist William Vickrey in 1961. Its practical use became apparent in the 1990s, when auctions of radio frequency spectrum for mobile telecommunication raised billions of dollars. Economic theorists advised governments on the design of these auctions, and companies on how to bid. The auctions for spectrum rights are complex. However, many principles for sound bidding can be illustrated by applying game-theoretic ideas to simple examples.

### 3.9.1 Second-price Auctions with Private Values

The most familiar type of auction is the familiar open ascending-bid auction, which is also called an English auction. In this auction format, an object is put up for sale. With the potential buyers present, an auctioneer raises the price for the object as long as two or more bidders are willing to pay that price. The auction stops when there is only one bidder left, who gets the object at the price at which the last remaining opponent drops out. A complete analysis of the English auction as a game is complicated, as the extensive form of the auction is very large. The observation that the winning bidder in the English auction pays the amount at which the last remaining opponent drops out, suggests a simpler auction format as the second-price auction for analysis. In a second-price auction, each potential buyer privately submits, perhaps in a sealed envelope or over a secure computer connection, his bid for the object to the auctioneer. After receiving all the bids, the auctioneer then awards the object to the bidder with the highest bid, and charges him the amount of the second-highest bid. Vickrey's analysis dealt with auctions with these rules.

How should one bid in a second-price auction? Suppose that the object being auctioned is one where the bidders each have a private value for the object. That is, each bidder's value derives from his personal tastes for the object, and not from considerations such as potential resale value. Suppose this valuation is expressed in monetary terms, as the maximum amount the bidder would be willing to pay to buy the object. Then the optimal bidding strategy is to submit a bid equal to ones actual value for the object. Bidding ones private value in a second-price auction is a weekly dominant strategy.

That is, irrespective of what the other bidders are doing, no other strategy can yield a better outcome. (Recall that a dominant strategy is one that is always better than the dominated strategy; weak dominance allows for other strategies that are sometimes equally good.) To see this, suppose first that a bidder bids less than the object was worth to him. Then if he wins the auction, he still pays the second-highest bid, so nothing changes. However, he now risks that the object is sold to someone else at a lower price than his true valuation, which makes the bidder worse off. Similarly, if one bids more than ones value, the only case where this can make a difference is when there is, below the new bid, another bid exceeding the own value. The bidder, if he wins, must then pay that price, which he prefers less than not winning the object. In all other cases, the outcome is the same. Bidding ones true valuation is a simple strategy, and, being weakly dominant, does not require much thought about the actions of others.

While second-price sealed-bid auctions like the one described above are not very common, they provide insight into a Nash equilibrium of the English auction. There is a strategy in the English auction which is analogous to the weekly dominant strategy in the second price auction. In this strategy, a bidder remains active in the auction until the price exceeds the bidders' value, and then drops out. If all bidders adopt this strategy, no bidder can make himself better off by switching to a different one. Therefore, it is Nash equilibrium when all bidders adopt this strategy.

Most online auction websites employ an auction which has features of both the English and second-price rules. In these auctions, the current price is generally observable to all participants. However, a bidder, instead of frequently checking the auction site for the current price, can instead instruct an agent, usually an automated agent provided by the auction site, to stay in until the price surpasses a given amount. If the current bid is by another bidder and below that amount, then the agent only bids up the price enough so that it has the new high bid. Operationally, this is similar to submitting a sealed bid in a second-price auction. Since the use of such agents helps to minimise the time investment needed for bidders, sites providing these agents encourage more bidders to participate, which improves the price sellers can get for their goods.

**Example 9: Common values and the winners curse**

A crucial assumption in the previous example of bidding in a second-price auction is that of private values. In practice, this assumption may be a very poor approximation. An object of art may be bought as an investment, and a radio spectrum license is acquired for business reasons, where the value of the license depends on market forces, such as the demand for mobile telephone usage, which have a common impact on all bidders. Typically, auctions have both private and common value aspects.

In a purely common value scenario, where the object is worth the same to all bidders, bidders must decide how to take into account uncertainty about that value. In this case, each bidder may have, prior to the auction, received some private information or signals about the value of the object for sale. For example, in the case of radio spectrum licenses, each participating firm may have undertaken its own market research surveys to estimate the retail demand for the use of that bandwidth. Each survey will come back with slightly different results, and, ideally, each bidder would like to have access to all the surveys in formulating its bid. Since the information is proprietary, that is not possible.

Strategic thinking, then, requires the bidders to take into account the additional information obtained by winning the auction. Namely, the sheer fact of winning means that one's own, private information about the worth of the object was probably overly optimistic, perhaps because the market research surveys came back with estimates for bandwidth demand which were too bullish. Even if everybody's estimate about that worth is correct on average, the largest (or smallest) of these estimates is not. In a procurement situation, for example, an experienced bidder should add to his own bid not only a mark up for profit, but also for the likely under-estimation of the cost that results from the competitive selection process. The principle that winning a common-value auction is bad news for the winner concerning the valuation of the object is called the winners curse.

The following final example, whose structure was first proposed by Max Bazerman and William Samuelson, demonstrates the considerations underlying the winners curse not for an auction, but in a simpler situation where the additional information of winning is crucial for the expected utility of the outcome. Consider a potential buyer who is preparing a final, take it or leave it offer to buy out a dot-com company.

Because of potential synergies, both the buyer and the seller know that the assets of the dot-com are worth 50 percent more to the buyer than to the current owner of the firm. If the value of the company were publicly known, the parties could work out a profitable trade, negotiating a price where both would profit from the transaction.

However, the buyer does not know the exact value of the company. She believes that it is equally likely to be any value between zero and ten million dollars. The dotcoms current owners know exactly the value of retaining the company, because they have complete information on their company's operations. In this case, the expected value of the company to the current owners is five million dollars, and the expected value of the company to the prospective buyer is seven and a half million dollars. Moreover, no matter what the value of the company truly is, the company is always worth more to the buyer than it is to the current owner. With this in mind, what offer should the buyer tender to the dot-com as her last, best offer, to be accepted or rejected?

To find the equilibrium of this game, note that the current owners of the dot-com will accept any offer that is greater than the value of the company to them, and reject any offer that is less. So, if the buyer tenders an offer of five million dollars, then the dotcom owners will accept if their value is between zero and five million. The buyer, being strategic, then realises that this implies the value of the company to her is equally likely to be anywhere between zero and seven and a half million. This means that, if she offers five million, the average value of the company, conditioning upon the owners of the dot-com accepting the offer, is only three and three-quarters million less than the value of the offer. Therefore, the buyer concludes that offering five million will lead to an expected loss.

The preceding analysis does not depend on the amount of the offer. The buyer soon realizes that, no matter what offer she makes, when she takes into account the fact that the offer will be accepted only when the value of the dot-com turns out to be on the low end. The expected value of the company to the buyer, conditional on her offer being accepted, is always less than her offer. It is this updating of the buyers beliefs, shifting her estimation of the dot-coms value to the low end, which embodies the winners curse in this example. Having her offer accepted is bad news for the buyer, because she realises it implies the value of the dot-com is low. The equilibrium in this game involves the buyer making an offer of zero, and the offer never being accepted.

This example is particularly extreme, in that no transaction is made even though everyone involved realises that a transaction would be profitable to both sides. As is generally the case with non cooperative game theory, the equilibrium does depend on the details of the rules of the game, in this case, the assumption that one last, best offer is being made, which either will be accepted or rejected. In general, the winners curse will not always prohibit mutually profitable transactions from occurring. This example demonstrates the importance of carefully taking into account the information one learns during the course of play of a game. It also shows how a game-theoretic model that incorporates the information and incentives of others helps promote sound decision-making.

## 4.0 CONCLUSION

Students could understand what games theory is all about, and they are able to determine the equilibrium of games theory. They could understand also determine bidding and auctions as well as zero sum and computation. Hence students are able to give in detail some principles of game theory.

## 5.0 SUMMARY

This unit discussed Games Theory and its various segments- Backward induction, Common knowledge, Dominating strategy, Extensive game, Mixed strategy, Nash equilibrium, Payoff, Perfect information, Player, Rationality, Strategic form, Strategy and Zero sum game. This provides a context for understanding equilibrium, computation, bidding and auction in Games Theory.

## 6.0 REFERENCES/FURTHER READING

Binmore, K. (1991). *Fun and Games: A Text on Game Theory*. D. C. Heath: Lexington, MA.

Dixit, A. K. & Nalebuff, B. J. (1991). *Thinking Strategically: The Competitive Edge in Business, Politics, and Everyday Life*. Norton: New York.

Fudenberg, D. & Tirole, J. (1991). *Game Theory*. Cambridge, MA. Gibbons: MIT Press.

Robert (1992), *Game Theory for Applied Economists*. Princeton, NJ: Princeton University Press.

Myerson, R. B. (1991). *Game Theory: Analysis of Conflict*. Cambridge, MA: Harvard University Press.

Nasar, S. (1998). *A Beautiful Mind: A Biography of John Forbes Nash,* *Jr., Winner of the Nobel Prize in Economics, 1994.* New York: Simon and Schuster.

Rasmusen, E. (2001). *Games and Information: An Introduction to Game Theory*.(3rd ed.) Blackwell: Oxford.

## MODULE 6



## UNIT 1 LINEAR PROGRAMMING

**CONTENTS**



## 1.0 INTRODUCTION

Many management decisions involve trying to make the most effective use of organisational resources. These resources include Machinery, Labour, Money, Time, Warehouse space or Raw materials to produce goods (machinery, furniture, food or cooking) or service (schedules for machinery and production advertising polices or investment decision). Linear programming (LP) is a widely used mathematical technique designed to help managers in planning and decision making relative to resource allocations.

## 2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply Linear Programming to Business
- list the properties of Linear Programming
- make assumptions on Linear Programming
- provide solutions to Linear Programming models.

## 3.0 MAIN CONTENT

### 3.1 Application of Linear Programming to Business

1. **Product -Mix:** Use in the selection of the product-mix in a factory to make best use of machine and machine hours available while maximising profit, that is, to find out which product to include in the production plan and in what quantities that should be produced.

2. **Blending Problems:** Use for the selection of different blends of raw materials to produce the best combination at minimum cost e.g. food drinks, etc.

3. **Production Schedule:** Use to develop a production schedule that will satisfy future demands for a firm's product and at the same time minimise production and inventory cost.

4. **Production Quantity:** Use in the determination of how much quantity to produce of different grades of petroleum product (say) to yield maximum.

5. **Distribution System:** Use in determining a distribution system that will minimise total shipping cost from several warehouses to various market locations.

6. **Limited Advertisement:** Use in the allocation of limited advertising budget among radio, TV and newspaper spots in order to maximise the returns on investment.

7. **Investment:** Use in selecting investment port-folio from a variety of stocks and bonds available in such a way as to maximise the returns on investment.

8. **Work Schedule:** Use in the development of a work schedule that allows a large restaurant to meet staff needs at all hours of the day, while minimising the total number of employees.

### 3.2 Properties of Linear Programming Model

All linear programming models have four basic properties in common. They are:

i. All LP models seek to maximise or minimise some quantity, usually profit or cost.
ii. All LP models have constraints or limitations that limit the degree to which the object can be purse. E.g. deciding how many

units of product in a product line to be produced is restricted to the manpower and machinery available.

iii. There must be alternative course of action to choose from e.g. if there are four (4) different products, management may decide (using LP) how to allocate limited resources among them.

Objectives and constraints in LP model must be express in linear equations and inequalities.

## 3.3 Assumption of Linear Programming

**Certainty:** We assume that numbers in the objective and constraints are known with certainty and do not change during the period under study.

**Proportionality:** We are sure that proportionality exists in the objective and the constraints. This mean that, if production of one unit of product uses two of a particular scare resource; then making five units of that product uses ten resources.

**Additivity:** This means that the total of all activities equals the sum of each individual activity.

**Divisibility:** This means that solution may take fractional values and need not be in whole numbers (integers). If a fraction of a product cannot be produced, integer programming problem exist.

**Non-negativity:** We assume that all answers or variables are non-negative. Negative values of physical quantities are an impossible solution.

## 3.4 Cover Material

5x = Amount of cover material used for half-upholstered. 5y = Amount of cover material used for full-upholstered. The total cover material cannot exceed 35.

This is the maximum available: $5x + 5y \leq 35$ Thus, the linear programming model is:
Maximise: $5x + 5y \leq 35$
Subject to: P=N80x + N90y
$2x + y \leq 12$ (Wood material)
$2x + 4y \leq 24$ (Foam material)
$5x + 5y \leq 35$ (Cover material)
$x \geq 0, x \geq 0$ (Non-negative)

**Example 1: (Diet Problem)**

A convalescent hospital wishes to provide at a minimum cost, a diet that has a minimum of 200g of carbohydrates, 100g of protein and 120g of fats per day can be met with two foods.

| $Food$ | $Carbonhydrates$ | $protein$ | $Fats$ |
|---|---|---|---|
| $A$ | $10g$ | $2g$ | $30g$ |
| $B$ | $5g$ | $5g$ | $4g$ |

If food A cost 29k per ounce and food B cost 15k per ounce, how many ounces of each food should be purchased for each patient per day in order to meet the minimum requirements at the lowest cost? Required: formulate the LP model.

**Solution:**

Let, x = Number of ounces of food A.
Y=Number of ounces of food B.
The minimum cost, C, is found by
Cost of food A = 0.29x
Cost of food B =0 .15y
C=0.29x+0.15y
The constraints are:
$x \geq 0, y \geq 0$

The amounts of food must be non - negative
The table gives a summary of nutrients provided

| $Food$ | $AmountNaira(N)$ | $Carbonhydrates$ | $protein$ | $Fats$ |
|---|---|---|---|---|
| $A$ | $x$ | $10x$ | $2x$ | $3x$ |
| $B$ | $y$ | $5y$ | $5y$ | $4y$ |
| $TOTAL$ | | $10x + 5y$ | $2x + 5y$ | $3x + 4y$ |

Daily requirements:
$10x + 5y \geq 200$
$2x + 5y \geq 100$
$3x + 4y \geq 120$
The LP model is :
Minimise : $C = 0.29x + 0.15y$ Subject to :
$10x + 5y \geq 200$ (Carbohydrates)
$2x + 5y \geq 100$ (Protein)
$3x + 4y \geq 120$ (Fats)
$X \geq 0, y \geq 0$ (Non-Negativity)

## 3.5 Solution of a Linear Programming Model

Having formulated the linear programming model, we shall now at this stage solve the model using any of the following methods:

- Graphical method.
- Simplex method.

However, the simplex method has advantage over the graphical method that; it can be used for problem involving two or more decision variables while graphical method cannot.

## 3.6 Graphical Solution of Linear Programming Problems

**Example 2:**
Maximise: P= 4x + 5y Subject to:
$2x + 5y \leq 25$
$6x + 5y \leq 45$
xgleq0, $y \geq 0$

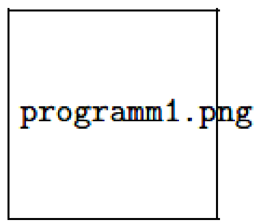


Figure 21:

**Solution:**

To solve the above linear programming model using the graphical method, we shall turn each constraints inequality to equation and set each variable equal to zero (0) to obtain twp (2) coordinate points for each equation ( i.e using double intercept form).

Having obtained all the coordinate points, we shall determine the range of our variables which enables us to known the appropriate scale to use for our graph. Thereafter, we shall draw the graph and join all the coordinate points with require straight line.

$2x + 5y = 25$ (Constraint 1)
When x = 0, y = 5 and when y = 0, x = 12.5
$6x + 5y = 45$ [Constraint 2 ]
When x = 0, y = 9 and when y = 0, x = 7.5
Minimum value of x = 0
Maximum value of x = 12.5
Range of x is $0 \leq x \leq 12.5$

Minimum value of y is y = 0
Maximum value of y is y = 9
Range of y is $0 \le y \le 9$.

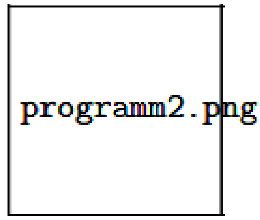


Figure 22:

The constraint give a set of feasible solutions as graphed above. To solve the linear programming problem, we must now find the feasible solution that makes the objective function as large as possible. Some possible solutions are listed below:

| $Feasiblesolution(Apointinthesolutionsetofthesystem)$ | $ObjectivefunctionR = 4x + 5y$ |
|---|---|
| $(2,3)$ | $4(2) + 5(3) = 8 + 15 = 23$ |
| $(4,2)$ | $4(4) + 5(2) = 16 + 10 = 26$ |
| $(5,1)$ | $4(5) + 5(1) = 20 + 5 = 25$ |
| $(7,0)$ | $4(7) + 5(0) = 28 + 0 = 28$ |
| $(0,5)$ | $4(0) + 5(5) = 0 + 25 = 25$ |

In this list, the point that makes the objective function the largest is (7,0). But, is this the largest for all feasible solutions? How about (6,1)? or (5,3)? It turns out that (5,3) provide the maximum value: 4(5) + 5(3) = 20 + 15 = 35.

**Example 3:**

Find the corner points for :
$2x + 5y \le 25$ $6x + 5y \le 45$ $x \ge 0$, $y \ge 0$

**Solution**:
The graph for Example 6 is repeated here and shows the corner points

Some corner points can usually be found by inspection. In this case, we can see A = (0,0) and D = (0,5). Some corner points may require some work with boundary lines (uses equation of boundaries not the inequalities given the region)

**Point C**

System: $2x + 5y = 25$ (1)

$6x + 5y = 45$ (2)

- (2)- $4x = -20$

$x = 5$

if $x = 5$, then from (1) or (2)

$y = 3$

**Point B**

System $y = 0$ (1)

$6x + 5y = 45$ (2)

Solve by substitution

$6x + 5(0) = 45$

$x = \frac{45}{6} = 7.5$

The corner points for example 7 are: (0,0), (0,5), (7,5,0) and (5,3).
Convex sets the corner points lead us to a method for solving certain linear programming problems.

**Example 4:**
Maximise: P = 143x + 60y
Subject to:
$X+y \leq 100$
$120x + 210y \leq 15000$
$110x + 30y \leq 4000$
$x \geq 0, y \geq 0$

**Solution:**
$x + y = 100$ (Constraint 1)
When $x = 0$, $y = 100$ and when y 0, $x = 100$ ($x = 100$, $y = 100$)
$120x + 210y = 15000$ (Constraint 2)
When $x = 0$, $y = \frac{500}{1}$ and when $y = 0$, $x = 125$ $\left(x = 125, y = \frac{500}{1}\right) = 71.43$
When $x = 0$, $y = \frac{400}{7}$ and when $y = 0$, $x = \frac{400}{11}$ ($x = \frac{400}{11}$ , $y = \frac{300}{3}$) = 133

Next, find the corner points. By inspection,

Point A = (0,0)

**Point B:**

programm3.png

Figure 23:

System: 120x + 210y = 15000 (1)

x = 0 (2)

Solve (1) and (2) simultaneously by substituting

For x = **0** in (1):

120 (0) + 210y = 1500

$Y = \frac{15000}{210} = \frac{500}{7}71.43$

Point B: $(0, \frac{500}{7}$

**Point B:**

System: 110x + 30y = 4000 (1)

120x + 210y = 15000 (2)

7 (1) - (2) 650x = 13000

x = 20

substitute for x = 20 in (1)

110(20) + 30y = 4000

30y = 1800

Y = 60

**Point C:** (20,60).

**Point C:** System: 110x + 30y = 4000 .....(1)

120x + 210y = 15000..... (2)

7 (1) - (2) 650x = 13000

x = 20

substitute for x = 20 in (1)

110(20) + 30y = 4000

30y = 1800

Y = 60

Point C: (20,60).

**Point D:**

System: 110x + 30y = 4000 . .. (1)

Y = 0...... (2)

Solve (1) and (2) simultaneously by substituting for y = **0** in(1)

110x + 30 (0) = 4000

110x = 4000

$x = \frac{4000}{11}$

Point D: $(\frac{4000}{11}, 0)$

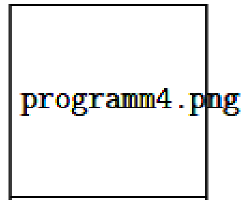


Figure 24:

Use the linear programming theorem and check the corner points:

| $Cornerpoint$ | $Objective function P = 143x + 60y$ |
|---|---|
| $(0,0)$ | $143(0) + 60(0) = 0$ |
| $(0, \frac{400}{11})$ | $143(0) + 60(\frac{400}{11}) = 4,286$ |
| $(\frac{400}{11}, 0)$ | $143(\frac{400}{11}) + 60(0) = 5,200$ |
| $(20, 60)$ | $143(20) + 60(60) = 6,460$ |

The maximum value of P is 6,460 at (20,60). This means that to maximum profit, the farmer should plant 20 acres in corn, plant 609 acres in wheat and leave 20 acres unplanted.

Notice from the graph in Example 8 that some of the constraints could be eliminated from the problem and everything else would remain unchanged. For example, the boundary x +y = 100 was not necessary in finding the maximum value of P. such a condition is said to be a superfluous constraint. It is not uncommon to have superfluous constraints in a linear programming problem. Suppose, however, that the farmer in Example 1 contracted to have the grain stored at neighbouring farm and now the contract calls for at least 4,000 bushels to be stored. This change from 110x + 30y? 4000 to 110x 30y? 4000 now makes the condition x + y? 100 an important to the solution of the problem. Therefore, you must be careful about superfluous constraints even though they do not affect the solution at the present time.

**Example 5:**
Solve the following linear programming problem:
Minimise: C = 60x + 30y
Subject to:
$2x + 3y \leq 120$
$2x + y \leq 80$
$x \geq 0, y \geq 0.$

**Solution:**
Corner points A = (0, 80) and C = (60, 0) are found by inspection.
bc **Point B**

System: 2x + 3y = 120...................... (1)
2x + y = 80....................... (2)
- (2)2y = 40
Y=20.
Substitute for y = 20 in (2):
2x + 20 = 80
2x = 60
X=30
**Point B**: (30, 20)
Extreme values.

Corner point Objective function C=60x+30y
(0, 80)
(30, 20)
(60, 0) 60(0) + 30(80) = 2400
60(30) + 30(20) = 2400
60(60) + 30(0) = 3600

From the table above, there are two minimum Values for the objective function: A = (0,80) and B = (30,20). In this situation, the objective function will have the same minimum value (2,400) at all points along the boundary line segment A and B.

## 4.0 CONCLUSION

By now you are familiar with all the issues concerning Linear Programming.

## 5.0 SUMMARY

This unit focused on Linear Programming (LP) as a mathematical technique helpful to managers in planning and decision-making vis-a-vis resource allocations. In the process, LP's applications, properties, assumptions and solutions were highlighted.

## 6.0 TUTOR-MARKED ASSIGNMENT

1. An oil company manufacturer two brands of lubricants namely A and Z, lubricant A valued at N50 needs 15 kilograms of raw materials and 9 hours of machine time. Lubricant Z also valued at N50 needs 10 kilogram of the same raw materials and 12hours of machine time. Establish the maximum value of the products that can be made from 360 hours of machine and 375 kilograms of raw materials and the respective quantities of lubricants A and Z.

2. A caterer has 1600 grams, 1100 grams an 1500 grams of yam, fish and meat respec-tively. She requires 100 grams of fish, 100 grams of meat and 200 grams of yam to prepare a plate of pounded yam. To prepare a plate of porridge, she requires 200 grams of fish, 300 grams of meat and 100 grams of yam. If a plate of porridge sells for N3.00 and a plate of pounded yam for N5.00, how many plates of each should she prepare of maximize her sales?

3. Berger Paints Nigeria Limited manufacturer's two types of paints Emulsion and Gloss. 'Emulsion' valued at N12.50 per gallon needs 5 kilograms of raw materials and 9hours of machine time.'Gloss' valued at N15.00 per gallon needs 6 kilograms of the same raw materials and 12hours of machine time. Establish the maximum value of each of the products that can be made from 400 hours of machine time and 500 kilograms of raw materials.

4. Ajasco Nigeria Limited manufacturers plastic and zinc buckets. 1 hour, 2 hours and 1hour of time on machines A, B and C respectively; are required to manufacture 1,000 plastic buckets. 2 hours, 1hour, 1hour and 1hour of time on machines C, A and B respectively; are required to manufacture 1,000 zinc buckets. In a given period, the available hours on machine A, B and C are 8, 12 and 14 respectively. The profit per unit on plastic bucket is N50 and on zinc bucket is N60. Find the optimum allocation i.e the product mix and the resulting profits.

5. Mrs, Viju Milk is a small scale business woman who has just started a mini catering outfit in Lagos. She has decided to produce two types of cakes, namely chocolate cakes (x) and fruit cakes (Y). The two types of cakes go through two main processes. i.e baking and decorating. In order to produce a chocolate cake, she needs 2 hours for baking and 6hours for decorating. To produce a fruit cake, she needs 4 hours of baking but only 2 hours of decorating. She has available 400 man hours of baking and 600 hours of decorating. From market research she calculates that she will make a profit of N200 on each chocolate cake and N300 each on fruit cake.
   Required:
   i. Formulate the problem as Linear Programming.
   ii. Use the graphical approach to determine how many chocolate and fruit cake Iya Ibeji should produce to maximise her profit.

## 6.0 REFERENCES/FURTHER READING

Binmore, K. (1991). *Fun and Games: A Text on Game Theory*. D. C. Heath: Lexington, MA.

Dixit, A. K. & Nalebuff, B. J. (1991). *Thinking Strategically: The Competitive Edge in Business, Politics, and Everyday Life*. Norton: New York.

Fudenberg, D. & Tirole, J. (1991). *Game Theory*. Cambridge, MA. Gibbons: MIT Press.

Robert (1992), *Game Theory for Applied Economists*. Princeton, NJ: Princeton University Press.

Myerson, R. B. (1991). *Game Theory: Analysis of Conflict*. Cambridge, MA: Harvard University Press.

Nasar, S. (1998). *A Beautiful Mind: A Biography of John Forbes Nash, Jr., Winner of the Nobel Prize in Economics, 1994.* New York: Simon and Schuster.

Rasmusen, E. (2001). *Games and Information: An Introduction to Game Theory*.(3rd ed.) Blackwell: Oxford.