NATIONAL OPEN UNIVERSITY OF NIGERIA (NOUN)

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: FMT 215

COURSE TITLE: QUANTITATIVE ANALYSIS IN FINANCE

MODULE 1

MODULE 2

MODULE 1
CONTENTS

1.0 INTRODUCTION

The aim of Quantitative Analysisi in finance is to apply the techniques of the concept of series to the major concepts in finance.
These major concepts in finance include:
(i) Calculation of depreciation values of the assets of companies.
(ii) Calculation of both the simple and compound interests on cash inflows.
(iii) Calculation of annuity.

A series is the sum of certain well ordered numbers called sequences. The succession of numbers in a sequence will be defined by certain rules (functions) of their postions or values following each other.
We shall be interested in the general term of the sequence, as well as the sum of the first in terms of the series.

2.0 OBJECTIVES

At the end of this unit, you should be able to:
(a) define and recognise a sequence.
(b) find the n^{th} term of a sequence.
(c) determine the sum of the first n terms of a sequences.

3.0 MAIN CONTENT

General

3.1 Series

When we mention a number in this text, we shall mean a real number.
Real numbers are the numbers you use everyday for usual transactions. These numbers are the integers and fractions. The positive integers are called "the counting numbers".

3.1.1 Definition : A sequence of numbers is an endless succession of numbers placed in a certain order. The numbers in the sequence are called terms of the sequences. The terms of the sequence are got by certain well defined rules, which will be clear to you very soon.
Consider the sequence 2, 4, 6, 8, 10, 12, 14............
In the above sequence

2 is called the first term.
4 is called the second term.
6 is called the third term.
8 is called the fourth term.

Usually, you will denote the n^{th} term by the symbol U_n. Thus $U_1 = 2,\ U_2 = 4,\ U_3 = 6,\ U_4 = 8$ e.t.c. Now let us consider the defined rule that determines the terms

$$U_1 = 2 = 2.1$$
$$U_2 = 4 = 2.2$$
$$U_3 = 6 = 2.3$$
$$U_4 = 2 = 2.4$$

It becomes clear that the general $n^{th} term, U_n$ will be given by $U_n = 2.n$
You will write such a sequence as $U_n = 2n, n = 1, 2, 3, 4, 5, ...$
Let us consider two more examples.

3.1.2 **Example :** Determine the n^{th} term in the following sequences.

(i) $1, 2, 3, 4, 5, ...$
(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$

Solution :
(i) Since $U_1 = 1, U_2 = 2, U_3 = 3, U_4 = 4, U_5 = 5, U_6 = 6$. It folows that $U_n = n$
(ii) Since $U_1 = 1, U_2 = \frac{1}{2}, U_3 = \frac{1}{3}, U_4 = \frac{1}{5}, U_5 = \frac{1}{6}$.It folows that $U_n = \frac{1}{n}$
You will denote a sequence whose n^{th} term (general term) is U_n by $\{U_n\}$.
In (3.1.2), the two sequences can be written as
$\{n\}$ and $\{\frac{1}{n}\}$
Remark : Let us consider a property of sequences which may not be obvious so far. Consider the sequence.

$$U_n = (-1)^n n = 1, 2, 3, 4, ...$$

This sequence is -1, 1, -1, 1,-1, 1, 1........
It is therefore clear that it is not necessary for the term to be distinct, hence a sequence is **NOT** a set since the elements of a set must be distinct.
Example : Determine the general term in the following sequences.
(i) 1, 4, 9, 16, 25, 36,............
(ii) 1, 8, 27, 64, 125, 216,.......
Solution :
(i) You have $U_1 = 1, U_2 = 4, U_3 = 9, U_4 = 16, U_5 = 25, U_6 = 36$. You now think of a definite rule that links each term to its position in the sequence. Thus $U_1 = 1 = 1^2, U_2 = 4 = 2^2, U_3 = 9 = 3^2, U_4 = 16 = 4^2, U_5 = 25 = 5^2, U_6 = 36 = 6^2$. Hence, you can conclude that $U_n = n^2$
(ii) If you apply a similar reasoning of the first example, you will find $U_1 = 1 = 1^3, U_2 = 8 = 2^3, U_3 = 27 = 3^3, U_4 = 64 = 4^3, U_5 = 125 = 5^3, U_6 = 216 = 6^3$. Hence you have the general term $U_n = n^3$

Exercises

(1) Define a set and show that a sequence is not a set.
(2)Justify that there is no need to define a finite sequence.

3.1.3 Series

Definition :
A series is the sum of the terms of a sequence. The sum of the first n terms of a sequence is denoted by S_n, i,e

$$U_1 + U_2 = 2 + U_3 + ... + U_n = \sum_{i=1}^{n} U_r.$$

(Read as "sums U_r, r=1 to r=n")
Let us try some examples.

3.1.3.1 Example

In the above notation, you have the following
(i)

$$\sum_{i=1}^{6} r = 1 + 2 + 3 + 4 + 5 + 6$$

(ii)

$$\sum_{i=1}^{4} 2r^2 = 2 + 8 + 18 + 32$$

(iii)

$$\sum_{i=1}^{6} r^3 = 1 + 8 + 27 + 64 + 125 + 216$$

(iv)

$$\sum_{i=1}^{6} \frac{2}{r+3} = \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \frac{2}{8} + \frac{2}{9}$$

3.1.3.2 Example

Give the short form for the following series
(i)$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$
(ii)$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + ...\frac{n}{n+1}$
Solution :
(i)$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \sum_{r=1}^{n} \frac{1}{r}$
(ii)$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + ...\frac{n}{n+1} = \sum_{r=1}^{n} \frac{r}{r+1}$

Exercise
Write out the following series in full
(1)

$$\sum_{r=1}^{10} r(r+1)(r+2)$$

(2)

$$\frac{1}{4} \sum_{n=1}^{6} n^2(1+n^2)^2$$

(3)

$$\sum_{k=1}^{5} k(k^2-1)$$

(4)

$$\sum_{k=1}^{6} \frac{3}{K^3}$$

3.1.4 Special Remarks :

In (3.1.1), we defined the term of a sequence as being a function of its position in the sequence, i.e a sequence is of the form f(1), f(2), f(3), ...,f(n), ... where f(n) is the n^{th} term and f(.) is a definite rule.
In some special occassions, the terms of the sequence are relatedto each other. Some of these special sequencies are
(i) Arithmetic sequences and
(ii) Geometric sequeneces which will be discussed in unit 2 and unit 4

4.0 CONCLUSION

You now the general idea of sequences and series, which are the basic concepts needed for this course. As noted in (3.1.4) , the remaining matewrial in this cours are just special cases of the topic just developed.

5.0 SUMMARY

In this unit , you have learnt the definitions of sequence and series. you can now recognise a sequence and determine the n^{th} term. You can also write down the short form for a series.

6.0 TUTOR MARKED ASSIGNMENT

(1) Write down the following series

$$(a) \quad \sum_{k=1}^{n} \frac{K}{K+1} \qquad\qquad (b) \quad \sum_{k=1}^{10}(K-1)$$

(2) Determine the n^{th} term of the following sequences.
(a) 1, -2, 3, -4, 5
(b) $\frac{1}{3}, 3(\frac{1}{3})^2, (\frac{1}{3})^3, 3(\frac{1}{3})^4$

7.0 REFERENCES/ FURTHER READINGS

(1) Andre Francis : Business Mathematics and Statistics. 6^{th} Edition 2004.
(2) Green, S.G. : Advanced level Mathematics printed in Great Briton by University Tutorial press. LTD.
(3) Qazi Zameerudin, Vijay K.Khama and S.K Bhambri: Business Mathematics 2^{nd} Edition.2010. Vikas Publishing House PVT LTD.

Unit 2 : Arithemic Series

CONTENTS

1.0 INTRODUCTION

In this section, you will learn about arithmetic series, which is one of the special series mentioned in unit one.
Arithmetic Series is a Series in which each preceding term differs from the succeeding term by a constant value. Many financial problems are of this nature, Examples include:
Straight line depreciation and
Annual salary increment
In such problems, there are three parameters: The initial value, The constant increment and The number of equal periods. These parameters allow you to determine,
(1) The value of the A.P at any time and
(2) The sum of the A.P.
The use of the A.P model allows you to give a concise solution to problems in finance.

2.0 OBJECTIVES

At the end of this study you should be able to formulate certain problems in finance as an arithmetic progression model:
Solve the A.P model by determining any or all of the three parameters.
Interpret the solution to suit the requirements in the finance problem.

3.0 MAIN CONTENT

3.1 Arithmetic Progression

3.1.1 Definition: A sequence is said to be an arithmetic sequence if every succeed-

ing term is obtained by adding a constant number (value) to the preceeding term. The constant value is called the "Common difference" and represented by d.
Thus U_{n+1}.........$U_n = d \ \ \forall n$. For special sequence, We shall denote the first term by "a" i.e $U_1 = a$.
Thus, an Arithmetic sequence is of the form
$a, a+d, a+2d, a+3d, a+4d, a+5d, ...$ and the Arithmetic series or Arithmetic progression (A.P) is of the form
$S_n = a + [a + d] + [a + 2d] + [a + 3d] .... + [a + (n - 1)d]$...............................(3.3.2)
which is the sum of the first n terms of the A.P

Examples:
The Series 10+40+70+100+130 Is in A.P
Here the first term, $U_1 = 10$ and the common difference =40-10=70-40=30.

3.1.2 Evaluation of A.P

Let
$$S_n = a + [a + d] + [a + 2d] + ... + [a + (n - 1)d] \qquad (3.1.2.1)$$
If you rewrite the above in reverse order you get
$$S_n = [a+(n-1)d]+[a+(n-2)d]+[a+(n-3)d]+[a+(n-3)d]+..........[a+d]+a \quad (3.1.2.2)$$
Adding (3.3.4) and (3.3.5) gives
$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + ...[2a + (n - 1)d] = n[2a + (n - 1)d]$$
$$S_n = \frac{n}{2}[2a + (n - 1)d] \qquad (3.1.2.3)$$
Hence, the sum of the first n terms of an A.P with $U_1 = a$ and d=common difference is given by (3.1.2.3)

Aternatively, if you know the first term a and the last term U_n then the sum of the first n terms is given by
$$S_n = (U_1 + U_n)$$
$$S_n = \frac{n}{2}(U_1 + U_n)$$
$$S_n = \frac{n}{2}(a + U_n)$$
Example :
Evaluate the sum $10 + 40 + 70 + 100 + 130$
Solution:
The series in an A.P with U_1=a=10
Common difference =d=30 and n=5

Alternatively, if you know the first term U_1 and the last term U_n, then the sum of the first n terms is given by

$$S_5 = \frac{5}{2}[2(10) + (5-1)30]$$
$$S_5 = \frac{5}{2}[20 + (4)30]$$
$$S_5 = \frac{5}{2}[20 + 120]$$
$$S_5 = \frac{5}{2}(140)$$
$$S_5 = 5(70) = 350$$

3.1.3 Application To Finance

Arithmetic sequence is well adapted to certain problems in finance where a fixed amount of money is added regularly to an amount of money, as in the determination of a rent collected over a period of time with a fixed annual increment.

Example:
Mr. John arranges to pay off a debt of N9, 600 in 48 annual installments which form an Arithmetic Progression. When 40 of these installments had been paid, Mr. John died and his creditor finds that N2,400 still remains unpaid. Determine the value of each of the first two installments of Mr. John.
Solution: Let the installment have "a" first value of "a" with a common difference d. Then N9,600 is the sum of all the installments for 48 years i.e

$$9,600 = \frac{48}{2}[2a + (48-1)d]$$
$$= 24[2a + 47d]$$
$$2a + 47d = \frac{9600}{24} = 400 \quad (3.3.4)$$

When Mr. John died , he had only paid N9, 600-N2, 400=N7, 200 in 40 years. Thus the sum of all the instalmental in 40 years is N7, 200

$$7,200 = \frac{40}{2}[2a + (40-1)d]$$
$$7,200 = 20[2a + 39d]$$
$$7,200 = 40a + 780d$$

$$360 = 2a + 780d \tag{3.3.5}$$

Solving (3.3.4) and (3.3.5) simultanously gives

$$8d = 40$$

$$d = 5$$

and

$$2a = 360 - 39(5)$$

$$2a = 360 - 195$$

$$a = 165$$

So the first instalment a=N82.50 and the second instalment a+d=N87.50

Example: Show that the sequence 3.75, 3.50, 3.25 forms an Arithmetic sequence and hence determine the 16th term.
Solution:
$U_1 = 3.75, U_2 = 3.50$ and $U_3 = 3.25$
Now $U_2 - U_1 = 3.50 - 3.75 = -0.25$
Also $U_2 - U_1 = 3.25 - 3.50 = -0.25$
Since the difference $U_2 - U_{n-1}$ =d , a constant, the sequence is an arithmetic sequence.
The 16^{th} term

$$U_{16} = U_1 + (16 - 1)d$$

$$3.75 - 15(0.25)$$

$$3.75 - 3.75 = 0$$

Example:
Which term of the A.P 49+44+39+....is 9?
Solution: Let the required term be U_n
$U_1 = 49, U_2 = 44$, $U_3 = 39$ and $U_2 - U_1 = U_3 - U_2 = -5 = d$
$U_n = 49 + (n-1)d = 49 - 5(n-1)$
$9 = 49 - 5(n-1)$.
$\Rightarrow 5(n-1) = 49 - 9 = 40$
$5n = 40 + 5 = 45$
n-9
The 9^{th} term is 9 .

Example:
Show that the series $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} +$ .... form an A.P and hence determine the sum of the 19 terms.
Solution:

$U_1 = \frac{3}{4}, U_2 = \frac{2}{3}$ and $U_3 = \frac{7}{12}$. Now $\frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$ also $\frac{2}{3} - \frac{7}{12} = -\frac{1}{12}$
Thus $U_2 - U_1 = U_3 - U_2 = -\frac{1}{12}$ a constant.Hence, the series is A.P
The sum of the first 19 terms is

$$S_{19} = \frac{19}{2}[2\left(\frac{3}{4}\right) + (19-1)\left(-\frac{1}{12}\right)]$$

$$S_{19} = \frac{19}{2}[\left(\frac{3}{2}\right) - 18 \times \frac{1}{12}] = \frac{19}{2}[\left(\frac{3}{2}\right) - \frac{3}{2}] = 0$$

Hence, the sum is 0.

Example:
Show that the sequence -9, -6, -3 forms an arithmetic sequence and hence determine the number of terms whose sum will be 66
Solution:
$U_1 = -9, U_2 = -6$ and $U_3 = -3$
Since $U_2 - U_1 = U_3 - U_2 = 3$, is constant.
It implies that the sequence is an arithmetic sequence. Let the required number of terms be n. Then,

$$66 = \frac{n}{2}[2(-9)) + (n-1) \times (3)]$$

$$132 = n[-18 + 3(n-1)]$$

$$132 = -18n + 3n^2 - 3n$$

$$132 = -21n + 3n^2$$

Dividing through by 3, gives
$n^2 - 7n - 44 = 0$
which is a quadratic equation. By using the factorization method, we have

$$(n-1)(n-4) = 0$$

$$\Rightarrow n = 11 \quad or \quad n = -1$$

Since the number of terms cannot be negative, we have n=11. Therefore, the number of terms is 11

Exercise:
Determine the number of terms of the arithmetic sequence 4, 2, 0, -2, -4, -6,........... which will have a sum of 6.

Example:
The monthly salary of Mr. Koko was N320 for each of the first three years of his service period. He got annual increment of N40 per month for each of the following successive 12 years and his salary remained constant till retirement. During the calculation of his retirement benefit it was observed that Mr. Koko's average monthly salary during the service years was 698. You are required to determine the number of years he spent in service.
Solution:
Let M years be the period when Mr. Koko salary was constant. The total salary when he was in service is divided into three categories:
(i). The first three years in service.
(ii). The next 12 years when salary increased annually by N40 per month. (iii). The last M years when salary was constant.
The salary in each of the above period is calculated as follows
(i). The salary for the first 3 years was

$$12 \times 3 \times 320 = N12 \times 960 \qquad (*)$$

(ii). In the next 12 years, the salary followed an Arithmetic progression with an annual constant increment of N40 per month. So salary earned was

$$12 \times \frac{12}{2}[2 \times 360 + (12 - 1) \times 40]$$

$$= N12 \times 6960$$

(iii). For the remaining M years, salary earned was

$$12M[360 + (12 - 1) \times 40]$$

$$= 12M \times 800$$

Therefore, the total salary earned was

$$12 \times 960 + 12 \times 6960 + 12M \times 800 \qquad (**)$$

Now total number of years spent in service was (M + 15) with an average salary of N698 per month. So, the total salary earned during the period was

$$N698 \times 12(M + 15) \qquad (***)$$

Equating (***)and(**) you get

$$N698 \times 12(M \times 15) = 12 \times 960 + 12 \times 6960 + 12M \times 800$$

$$698(M + 15) = 7920 + 800M$$

$$698M + 698 \times 15 = 7920 + 800M$$

$$800M - 698M = 7920 = 10470 = 7920$$
$$102m = 2550$$
$$M = 25$$

So the number of years in service is $M + 15 = 40$ years.

Example

A staff of Lasu, ojo earned a salary of N96,000 per month. After 2 years in service, he was promoted to a salary scale with an annual increment of N12,000 per month.
If the staff left Lasu after 10 years, determine the value of his last monthly salary and the total salary earned during the period.

Solution

The salary for the first two years was $N96,000 \times 2 \times 12$
Later, He spent 8 years on an initial salary of N96,000 per month, with an increment of N12,000 per month, so his 10^{th} year monthly salary was

$$(96,000 + (8-1) \times 12,000) = N180,000$$

The amount earned in the last 8 years was

$$12 \times \frac{8}{2}[2 \times 96,000 + (8-1) \times 12,000)]$$
$$= 12 \times 4[192,000 + 84,000]$$
$$= N48 \times 276,000$$

The total amount earned in the 10 years was

$$96,000 \times 2 \times 12 + 48 \times 276,000$$

$$2,304,000 + 13,248,000 = N15,632,000$$

3.1.4 Depreciation

A very important concept in business is "depreciation" Depreciation is an allowance made in estimates, Valuations, Machine operations or balance sheets normally for "Wear and tear".
It is normal accounting practice to depreciate the value of certain assets e.g Machines used during manufacturing and vehicles used for daily company operations.
We shall treat two methods of calculating depreciation i.e
(i). Straight line (or equal installment) depreciation and
(ii. Reducing balance depreciation.

Straight Line Depreciation Technique:

In this technique, a constant value is taken-away from the original value at a regular interval, Thus the value of the asset will form an arithmetic sequence .
We can then use the formulas for an A.P to calculate the depreciated value with the constant decrement being negative increment. Let the value of the asset be P. If the depreciation is d, then the value after
1 years is p-d
2 years is p-2d
3 years is p-3d e.t.c
So after n years, the value of the asset $A = P - nd$

Example :
A machine was bought for N840,000 which is expected to last for 10 years with a scrap value of N200,000. Calculate the yearly depreciation value, using the method of straight line depreciation.
Solution
Initial value = N840,000
Time = n = 10 years
Scrap value = A = 200,000
Let the yearly depreciation be d, Then

$$200,000 = 840,000 - (10)d.$$
$$= 840,000 - 10d$$
$$10d = 640,000$$
$$d = N64,000$$

Example:
A man saved N16,500 in 10years, in each year after the first, he saved N100 more than he did in the preceeding year. Determine the amount he saved in the first year.

Solution:
Let the amount saved in the first year be a. In the second year He saved a+100, and in the third year a+200 e.t.c.

The amount saved will then form an A.P with a common difference of N100.
Since the total amount saved in 10years is 16,500, you will have

$$16,500 = \frac{10}{2}[2a + (10 - 1) \times 100]$$
$$16,500 = 5[2a + 9 \times 100]$$

$$16,500 = 5[2a + 900]$$
$$16,500 = 10a + 4500$$
$$1650 = a + 450$$
$$1650 - 450 = a$$
$$N1,200 = a$$

Therefore the amount saved in the first year is N1,200.

Example:
An employee produced 600 units in the 3^{rd} year of its existence and 700 units in its 7^{th}year, What was the initial production in the first year, if production follows an A.P.

Solution:
Let the initial production level be a. with an annual constant increment of d units. Then

$$600 = U_3 = a + 2d \tag{3.3.6}$$

$$700 = a + 6d = U_7 \tag{3.3.7}$$

Solving by substracting (3.3.6) from (3.3.7)gives

$$4d = 100$$

d = 25 units Then $a = 600 - 2d = 600 - 50 = 550$ units. So the initial production in the first year is 550 units.

Example:
A man secured an interest free loan of N14,500 from a friend and agrees to repay it in 10 installments. He paid N1000 as first installment and then increased each installment by equal amount over the preceeding inatallments.
Determine the value of the last installment.

Solution:
This problem is an application of A.P. let the constant equal increment be d. Since the sum of all repayments was N14,500 and the first installment was N1,000.
Then, For 10 years

$$14,500 = S_n = \frac{10}{2}[2(1,000) + 9d]$$
$$14,500 = 10,000 + 45d$$
$$45d = N4,500$$
$$\Rightarrow d = N100$$

the last installment in the 10th year will be

$$U_{10} = 1000 + 9d$$

$$1000 + 9(100) = N1,900$$

Remark:

You will learn the reducing balance depreciation technique after the treatment of geometric progression in unit 3.

4.0 CONCLUSION

You now know how to

1. formulate certain problems in finance as A.P Model
2. Solve the A.P model by determining any or all the parameters of the A.P model
3. Interpret the solution of the A.P model to suit the requirement of the given problem in finance.

5. 0 SUMMARY

1. An arithmetic progression has three parameters namely.
(a) The first value, denoted by a.
(b) The common difference denoted by d.
(c) The number of the periods denoted by n.
2. An A.P is a series in which the difference between any two consecutive term is a constant called the common difference
3. The terms of the A.P are of the form

$$a, a + d, a + 2d, a + 3d, ...$$

4. The value of the n^{th} term is

$$U_n = a + (n - 1)d$$

5. The sum of the first n terms is

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Which is equivalent to

$$S_n = \frac{n}{2}[a + U_n]$$

6. Using the straight line depreciating technique, the depreciation value A in n period with an initial value P and constant depreciation d is given by A = P-(nd)
7. An A.P model can be applied to any problem in finance where equal constant amount is added or subtracted from a given amount in equal interval or time.

6.0 Tutor Marked Assignments

1. Show that the following sequences are A.P and hence determine the 7^{th} and 13^{th} terms (a) $3, 5, 7, ...$(b) $1, 4, 7, 10, ...$
c) $5, 11, 17, ...$ (d) $8, 3, -2, -7, ...$
2. The 3^{rd} and 13^{th} terms of an A.P are respectively equal to -40 and 0. Determine the A.P and the 20^{th}term.
3. Is 203 any term of the A.P $13, 28, 43, 58, 73, ...$?
4. Mr. John takes loan of N2,000 from Jide and agrees to repay in number of installments, each installment. (beginning with the second) exceeding the previous one by N10. If the first installment is N5, find how many installments will be necessary to wipe the loan completely.
5. A piece of equipment costs N600,000. If it depreciates in value 15% the first years, $13\frac{1}{2}\%$ the second year, 12% the third year and so on. What will be its value at the end of 10years, all percentages applying to the original cost?

7.0 REFERENCES/ FURTHER READINGS

(1) Andre Francis : Business Mathematics and Statistics. 6^{th} Edition 2004.
(2) S.G. Green: Advanced level Mathematics printed in Great Briton by University Tutorial press. LTD.
(3) Qazi Zameerudin, Vijay K.Khama and S.K Bhambri: Business Mathematics 2^{nd} Edition.2010. Vikas Publishing House PVT LTD.

Unit 3 : Geometric Series(or Progression)(G.P)

CONTENTS

1.0 INTRODUCTION

In this unit, you are introduced to another special series called Geometric series (or pregression) (G.P) This is a series in which the ratio of any two consecutive terms is a constant, called the common ratio.
Many series of values in Finance are of this type, A very good examples of such series of values is the series of depreciated values of an asset. When an asset depreciates as a percentageof the original value, the series of depreciated value forms a G.P.
You will learn the concept of compound interest in module 2 which is a direct opposite to the method of reducing balance depreciation method.

2.0. OBJECTIVES

At the end of this study, you should be able to:
(a) Define and recognize a geometric sequence and a geometric series.
(b) Determine the general term, usually called the nth term of a G.P.
(c) Apply a formula to evaluate the sum of the first n term of a G.P.
(d) Determine the depreciated value of an asset using the reducing balance depreciation method.

3.0 MAIN CONTENT

3.1 Definitions:

3.1 Definition:

A sequence of value is said to be geometric sequence if a succeeding value is obtained by

multiplying the preceding value by a constant value, called the common ratio.
Denote the common ratio by r and the first value by a. then a geometric sequence is of the form

$$a, ar, ar^2, ar^3, ar^4, ar^5, ...$$

The nth term is a geometric sequence is

$$U_n = ar^{n-1}. \qquad n = 1, 2, 3, 4, ...$$

A geometric series (Or progression) (G.P) is the sum of the values of the geometric sequence, Denote the sum of the first n value by S_n, then

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ... + ar^{n-1} \qquad (3.2.1)$$

3.1.2 Evaluate of the sum the of a G. P

You will now derive a concise formular for S_n. Now, let

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1} \qquad (*)$$

Multiply (3.1.2) by r to give

$$rS_n = ar + ar^2 + ar^3 + ar^4 + ar^5 + ... + ar^n \qquad (**)$$

Subtract (3.1.2) and (3.1.3), to get

$$rS_n - S_n = ar^n - a$$

$$\Rightarrow (r-1)S_n = a(r^n - 1)$$

Thus

$$S_n = a\frac{(r^n - 1)}{r - 1} \qquad (3.2.1)$$

We normally apply (3.2.1) as

$$S_n = \begin{cases} a\frac{(r^n-1)}{r-1} & if \quad r > 1 \\ a\frac{(1-r^n)}{1-r} & if \quad r < 1 \\ na \quad if \quad r = 1 \end{cases} \qquad (3.2.2)$$

3.3 Examples

(A) Examples
The following sequences are in G.p
(i) $1, 2, 4, 8, 16, ...$with a=1 and r=2
(ii) $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, ...$ a=3 and r=$\frac{-1}{3}$
(iii) $1, \sqrt{3}, 3, 3\sqrt{3}, 9, ...$ with a=1 and r=$\sqrt{3}$

(B) Examples

Determine the sum of the first 6 terms in the above.

(i)

$$S_6 = 1\frac{(2^6-1)}{2-1} = \frac{64-1}{1} = 63$$

(ii)

$$S_6 = 3\frac{(1-(\frac{-1}{3})^6)}{1-(\frac{-1}{3})} = 3\frac{(1-(\frac{1}{3})^6)}{1\frac{1}{3}}$$

$$3\frac{(1-(\frac{1}{32}))}{\frac{4}{3}} = \frac{9}{4}\left(\frac{31}{32}\right) = \frac{279}{128}$$

(iii)

$$S_6 = 1\frac{(\sqrt{3}^6-1)}{\sqrt{3}-1} = \frac{3^3-1}{\sqrt{3}-1} = \frac{26}{\sqrt{3}-1} = \frac{26(\sqrt{3}+1)}{3-1} = 13(\sqrt{3}+1)$$

(Hint: you used the laws of idices and surd rationalisation)

(C) Examples

Determine the n^{th} termin each of the problems in the above example.

Solution

(i)

$$U_n = ar^{n-1} = 1.2^{n-1} = 2^{n-1}$$

(ii)

$$U_n = 3\left(\frac{-1}{3}\right)^{n-1} = \begin{cases} (\frac{-1}{3})^{n-2} & for \;\; n \;\; even \\ (\frac{1}{3})^{n-2} & for \;\; n \;\; odd \end{cases}$$

(iii)

$$U_n = 1.\left(3^{\frac{1}{2}}\right)^{n-1} = \sqrt{3^{n-1}}$$

3.4 Self Assessment Exercises

You are to try the following three problems before checking on the solutions that follow. The problems are meant for better understanding of the topic.

(1) Show that $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, ...$ is a geometric sequnence and hence determine the sum of the first 10 terms.

(2) Determine the value of the 14^{th} term and the sum of the first 14 terms of the G.P 3, 9, 27, 81, 243, 729, ...

(3) Find the sum of the first 11 term of the G.P given by $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, ...$. Determine the general n^{th} terms.

Solution to Exercise

(1) Here $U_1 = 1, U_2 = \frac{1}{5}, U_3 = \frac{1}{25}$ and $U_2 = \frac{1}{125}$
since

$$\frac{U_2}{U_1} = \frac{1}{5} \div 1 = \frac{1}{5}$$

$$\frac{U_3}{U_2} = \frac{1}{25} \div \frac{1}{5} = \frac{1}{25} \times \frac{1}{5}$$

and

$$\frac{U_4}{U_3} = \frac{1}{125} \div \frac{1}{25} = \frac{1}{125} \times \frac{25}{1} = \frac{1}{5}$$

$$\Rightarrow \frac{U_2}{U_1} = \frac{U_3}{U_2} = \frac{U_4}{U_3} = \frac{1}{5} \qquad Constant.$$

$\Rightarrow$ The sequence is a G.P.
The sum of the first 10 terms is given by

$$S_{10} = \frac{a\left(1 - (\frac{1}{5})^{10}\right)}{1 - \frac{1}{5}}$$

$$= 1.\left(\frac{5^{10} - 1}{5^{10}}\right) \times \frac{5}{4}$$

$$= \frac{1}{4}\left(\frac{5^{10} - 1}{5^9}\right)$$

(2) The common ratio of the G.P is r=3 and $U_1 = 3 = a$
Thus, the value of the 14^{th} is

$$U_{14} = ar^{14-1}3 \times 3^{13} = 3^{14}$$

Secondly, the sum of first 14 term.

$$S_n = 3\frac{3^{14} - 1}{3 - 1}$$

$$S_n = 3\frac{3^{14} - 1}{2}$$

(3) First, you should determine the common ration, which is r= $\frac{1}{2}$
Then, you apply the formula for the sum

$$S_n = \frac{(1 - r^n)}{1 - r} \qquad r < 1$$

$$= 1.\frac{\left(1 - (\frac{-1}{2})^{11}\right)}{1 - (\frac{-1}{2})} = \frac{1 + (\frac{1}{2})^{11}}{\frac{3}{2}}$$

$$\frac{2}{3}\left(\frac{2^{11}+1}{2^{11}}\right)$$

$$\frac{1}{3}\left(\frac{2^{11}+1}{2^{10}}\right)$$

3.5 Application of G.P to Finance

You will now consider certain problems in finance which are acceptable to G.P As a start, let us complete the methods of depreciation, introduced in Unit 2.

Reducing Balance Depreciation

Recall that in the straight line depreciation method, the depreciation value was a constant and it was deducted from the actual present value at a fixed regular interval. In the reducing balance depreciation method, the depreciation value is a percentage of the present value.
Let the depreciation rate be r%and let the present value be P. Then the depreciation value is $\frac{pr}{100}$ and the new value of the asset will

$$P - \frac{Pr}{100} = P\left(1 - \frac{r}{100}\right)$$

In the next period, the depreciation value will be,

$$\frac{r}{100} \times P\left(1 - \frac{r}{100}\right)$$

and the new value of the asset will be

$$P\left(1 - \frac{r}{100}\right) - P\left(1 - \frac{r}{100}\right) \times \frac{r}{100}$$

$$= P\left(1 - \frac{r}{100}\right)\left(1 - \frac{r}{100}\right)$$

$$= P\left(1 - \frac{r}{100}\right)^2$$

After n period intervals, the new value of the asset will be

$$P\left(1 - \frac{r}{100}\right)^n \qquad (*)$$

where P is the original value of the asset

Thus, the value, A, of the asset after n period interval with depreciation rate r% is giving by

$$A = \left(1 - \frac{r}{100}\right)^n$$

where P is the original value of the asset.
Remark:

You will notice that the reducing balance depreciation method is a direct application of a G.P where the successive depreciated value of the asset are

$$P\left(1 - \frac{r}{100}\right), P\left(1 - \frac{r}{100}\right)^2, ...P\left(1 - \frac{r}{100}\right)^n$$

and P is the original value of the asset
In conclusion, reducing balance depreciation is the technique of depreciating the book value of an asset by a conatamt percentage.

Example:
A cutting machine was purchased for N25,000. What will be the value of the machine after four years at the depreciating rate of 15% using the reducing balance depreciating method.

Solution:
Original value =N25,500
Depreciation rate = 15%
Number of periods (years) = 4
Value of machines

$$= 25,500 \times (\frac{1 - 15}{100})^4 = N13,311.20$$

Remarks:
On the other hand, if the depreciation value of an asset is known after n year at r% depreciating rate, then the original value of the asset can be found by re-arranging the formula in (3.5.1) as

$$p = \frac{A}{\left(1 - \frac{r}{100}\right)^n} \qquad (3.5.2)$$

(3.5.3) Example:
Calculate the original value of an asset which depreciated to N5378.91 after three years at 25% depreciation rate.
Solution:
Let the ariginal value=P
Here A= N5378.91 r=25% n=3

$$P=\frac{A}{\left(1-\frac{r}{100}\right)^3}=\frac{5378.91}{\left(1-\frac{25}{100}\right)^3}$$
$$=\frac{5378.91}{(0.75)^3}=N12,750$$

You can now solve a problem that involves both methods of depreciation.

Example:
A super microcomputer purchased for N220,000 will depreciate to a scrap value N120,000 in 5 years.
Required
(1) if the reducing balance method of depreciation is used, determine the depreciation rate.
(2) What is the book value of the super microcomputer at the end of the third years?
(3) How much more would the book value be at the end of the third year if the straight line method of depreciation had been used ?
Solution:
(1) From the problem
P=N220,000, A= 12,000 and n=5
from
$$A=P\left(1-\frac{r}{100}\right)^n$$
$$\Rightarrow 12,000=220,000\times\left(1-\frac{r}{100}\right)^5$$
$$\Rightarrow \frac{12}{220}=\left(1-\frac{r}{100}\right)^5$$
$$\Rightarrow 1-\frac{r}{100}=\sqrt[5]{\frac{6}{110}}=\sqrt[5]{0.05456}=0.5589 \qquad (*)$$
$$\Rightarrow \frac{r}{100}=0.4411$$
$$r=44.11\%$$

The depreciated rate is 44.11%.
Remarks:
The calculation in (*) can be carried out easily using the logarithm table or directly with the scientific calculator.
(2) The book value after the third year is given by
$$A=p\left(1-\frac{r}{100}\right)^3$$
$$=220,000\left(1-\frac{44.11}{100}\right)^3$$

$$= 220,000\,(0.5689)^3 = N38,408$$

(3) Using the straight line depreciation method, you need to first determine the constant depreciation value, using the formula

$$A_n = a - nd$$

$$\Rightarrow A = A_5 = P - 5d$$

$$d = \frac{P - A}{5} = \frac{220,000 - 12,000}{5} = \frac{208,000}{5} = 41,600$$

The depreciation value (i.e Book value) after 3 years is

$$A = A_3 = P - 3d$$

$$220,000 - 3 \times (41600)$$

$$220,000 - 124,800 = N95,200$$

Therefore, after three years the book value using the straight line depreciation method would (95, 200-38, 408)=N56, 792 more than using the reducing balance method.

Example:

A machine purchsed for Ni8, 750 will depreciate each year by 20%. find the estimate value at the end of 5 years.

Solution:

Here P=N18750 r=20 n=5

Therefore, the estimated (depreciation) value

$$A = P\left(1 - \frac{r}{100}\right)^n$$

$$18750\left(1 - \frac{2}{100}\right)^5$$

$$16750\,(0.8)^5 = N6,144$$

4.0 CONCLUSION

In this section, you learnt the definition of a G.P and you have acquired enough skill to recognise a G.P.

You can also determine the general term in G.P as well as evaluate the sum of the first n terms in a G.P

Finally, youn learnt the reducing balancing depreciation method of calculating the depreciation value of an asset.

5.0 SUMMARY

A geometry series ia s special type of series in which the ratio of any two consecutive terms is a constant.
Usually denoted the first term by a and the common ratio by r. The n^{th} term,

$$U_n = ar^{n-1}$$

The sum of the first n terms

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1} & r > 1 \\ \frac{a(1-r^n)}{1-r} & r < 1 \end{cases}$$

The n^{th} depreciated value of an asset with intitial value P at $r\%$ deprecition is given by

$$A = \left(1 - \frac{1}{100}\right)^n$$

6.0 Tutor Marked Assignments

(1) Show that the following sequences are G.P and hence determine the 6^{th} term and the sum of the first six terms

(a) $4, 12, 36, ...$

(b) $-21, 14, \frac{-28}{3}$

(c) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, ...$

(d) $1, -3, 9, -27, ...$

(e) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{-28}{2^3}, ...$

(2) The sum of first six terms of a G.P is 9 times the sum of the first three terms. Determine the common ratio.

(3) The sum of three numbers in G.P is 18 and their production is 1728. Find the numbers.

7.0 REFERENCES/ FURTHER READINGS

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(2) S.G.Green: Advanced level Mathematics printed in Great Briton by University Tutorial press. LTD.

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MODULE 2

Unit 4: Interest

CONTENTS

1.0 INTRODUCTION

The aim of this section is to utilise the series model in the formulation and solution of standard problems in Business and commerce. The daily activities in finance involves both inflow and outflow of money.
When Mr. A borrows money from Mr. B then A has to pay certain amount of money to B for use of the money. The amount paid by A is called interest. The amount borrowed by A from B is called principal and the sum of the principal and interest is called Amount. There are two ways of calculating interest paid: namely simple interest and compounding interest methods.The simple interest methds uses the arithmetic progrssion technique while the compound interest method uses geometric progression technique. The major dfference between the simple and compund interest method is that:
In simple interest , interest is calculated based on the principal while in compound interest, interest in one period is calculated based on the calculated amount in the just previous period.
Interst is therefore treated as the direct opposite of deprecition treated in unit 3.
We dfferentiate between quoted norminal interest rates and calculated actual percentage interest rate (APR). The procedure for calculation of APR is given.
A strking feature of the course is the inclusion of self Assessment exercises meant for the students but solutions are provided. We employ the students to tackle the problems before checking the solutions.

2.0 OBJECTIVE

At the end of the section you should be able to:
(1) define interest, interest rate, principal and amount.
(2) explain simple and compound interests.
(3) recognise p[roblems in finance leading to simple and compound interests calculation.
(4) determine APR.
(5) Calculate accrue amount.
(6) calculate principal if amount is given and
(7) calculate interest and interest rate.

3.0 MAIN CONTENT

3.1. Defination of Basic Terms

Let us start with the main concept of interest.
Interest is a fundametal concept in finance and it can be defined simply as :
Interest, denoted by I, is an amount of money earned on money inested or paid on money borrowed after some time interest.
To fully understand how interest is determined and calculated, you will need the definition of some basic terms in finance, which are introduced below:
(1) Principal amount(P): is the initial amount of money consider in a business. It might be: an amount of money to be invested oveer a period of time;
(a) An amount of the money loaned;
(b) Initial value of an asset (e.g plant or machinery).
Example : If Ade borrowed N1, 000 from John. The N1, 000 is consider as the principal amount.
(2) (Accrued) Amount(A) : is the principal (P) plus the interest added after some time period. It is important to specify how interest will be caculated at the begining of the business. Thus

$$A = P + I \tag{3.1.1}$$

For consistency, with the notations used in series, you will denoted the amount earned after n period of time by A_n (if necessary).
(3) Rate of interest (or interest rate):
Interest is the proportionate amount of money which is added to some principal (amount) (invested or borrowed)interest is normally expressed as a percentage rate per annum. Denote the interest percentage rate per annum by r%. Thus, interest in one year $\frac{pr}{100}$.
Example:
If N1, 000 is invested at 10%(i.e interest rate of 10%) per annum (p.a).
Here, the principle, P=N1, 000. The interest, I, after one year

$$\frac{10}{100} \times 1000 = N100$$

Thus, the amount,

$$A = A_1 = P + I = 1000 + 100 = N1100$$

Thus, given an interest rate, the interest, I, is a percentage of the principal, P.
(4) Number of time period (n). The number of time period over which interest is being calculated will be denoted by n. It is usually in years,but other time period like quarter or mouths are used.
Let c be the number of times interest in being calculated and added /subtracted in a year. Than the total number of time periods in n years is nc. For example,if interest is calculated quarterly,the number of time periods in 5 years will be $4 \times 5 = 20$ (since there are 4 quarters in a year).

Remark :
From the above, it is clear that there are three parameters involved in the calculation of an accrued amount. These parameters are P,r and n.
Exercise :
From our study on series (either arithmetic or geometric series),how many parameter are needed to determine the general term.

3.2 Method of Interest Calculation: Simple and Compound Interest

There basically two ways of determining and calculating on the interest on a principal. These two types of interests are:
(1) Simple interest.
(2) Compound interest.

3.2.1 Simple Interest Method

This is a method where interest earned is based on the principal only over the time interval i.e any interest earned in one period is NOT added back to the principal inorder to calculate the next interest.

Example:
Determine the accrued amount on N1, 000 invested at 10% simple interest over 5 years.
Solution:
Here P=N, 1000 r=10% n=5years.
So, interest, I is given by

$$I = \frac{Prn}{100} = \frac{1,000 \times 10 \times 5}{100} = N500$$

Amount,

$$A = P + Prn = 1,000 + 500 = N1,500$$

Remark:
(1) When amount is calculated at simple interest the question may specify interest instead of simple interest. Any other type of interest will be mentioned.

(2) Interest rate is always per annum whether specified or not.
(3) The accrued amount $A = A_n$ in n years follows an arithmetic progression with incremental (interest) d=I and the initial amount a=P.
Thus

$$A = A_n = P + \frac{Pnr}{100} = P\left(1 + \frac{nr}{100}\right) \qquad (3.2.2)$$

Thus, the accrued amount,A, at simple interest is simply an application of arithmetic progression.
(4)You can now see that simple interest calculation is the direct opposite of straight line depreciation. Thus, amount

$$A_n = P\left(1 + \frac{nr}{100}\right) = P + \frac{Pnr}{100} \qquad \textit{at simple interest}$$

$$A_n = P - nd \qquad \textit{straight line depreciation}$$

straiht line depreciation where d = depreciation value.
Let us consider two examples .

Example:
A man invest N200,000 over 10 years period at simple interest and collected N840,000 at the end. Calculate the total interest and the interest rate.
Solution: Here
A=N840, 000 P= 200, 000 n=10years.

$$A = A_n = P + \frac{Pnr}{100} = P\left(1 + \frac{nr}{100}\right) \qquad (3.2.2)$$

$$840,000 = 200,000\left(1 + \frac{10r}{100}\right)$$

$$\Rightarrow \frac{840,000}{200,000} = \left(1 + \frac{r}{10}\right)$$

$$4.2 = 1 + \frac{r}{10}$$

$$3.2 = \frac{r}{10}$$

$$r = 32\%$$

So the interest rate= 32%. Now , the total interest, I is given by

$$A = P + I$$

$$\Rightarrow I = A - P = 844,000 - 200,000 = N640,000$$

Total interest due is N640, 000.
The next example has been done in unit 2 (see 3.1.4).
Example:

A machine was bought for N840,000 which is expected to last for 10years with a scrap value of N200,000. Calculate the yearly depreciation value,using the method of straight line depreciation.

solution:

Here, initial value=P=840, 000 Scrap value, A=200, 000. Time, n=10years.

From;

$$A = P - nd$$

$$200,000 = 840,000 - 10d$$

$$d = N64,000$$

Remarks:

The two problems one direct opposite.The deprciation value d, which is negative increment(interest)

$$d = \frac{Pr}{100}$$

3.2.2 Compound Interest Method

This is a method of calculating total interest where interest calculation in one period is based on the amount in the previous period.

For example: for a principal P, the interest in the first year is

$$\frac{Pr}{100}$$

and the amount

$$A_n = P + \frac{Pr}{100}$$

The interest in the second year

$$\Rightarrow \left(P + \frac{Pr}{100}\right)\frac{r}{100} = \frac{Pr}{100}\left(1 + \frac{r}{100}\right)$$

while the amount in the second year.

$$A = P + \frac{Pr}{100}\left(1 + \frac{r}{100}\right)$$

$$= P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$$

$$= P\left(1 + \frac{r}{100}\right)^2$$

You can continue in this manner to calculate the amount after n years as

$$A = P\left(1 + \frac{r}{100}\right)^n \qquad (3.2.2.1)$$

from (3.2.2.1), you get

$$P = \frac{A}{\left(1 + \frac{r}{100}\right)^n} \tag{3.2.2.2}$$

Which gives the principal(i.e an amount of money) which will amount to A in n years at $r\%$ compound interest.

Remark:
It is important to emphasis that when interest is computed at compound interest, the adjective " compound" **MUST** be inserted.

Example:
Determine the amount on N5, 000 for 3 years invested at 9% per annum with interest compounded yearly.
Solution:
Here, P=5, 000 n=3 , Therefore, the amount after 3 years

$$A = P\left(1 + \frac{r}{100}\right)^n = 5,000(1 + 0.09)^3$$

$$= 5,000(1 + 0.09)^3 = N6475.15$$

Example:
Mr.John invested N100 at the end of 2001, N200 at the end of 2002 end N300 at theend of 2003, N400 at end end of 2004, N500 at the end of 2005. If all interest accummulated at 5% per annum and compounded yearly. What will be the amount of Mr. John's investment at the end of 2006?
Solution:

N100	is	invested	for	5 years
N200	is	invested	for	4 years
N300	is	invested	for	3 years
N400	is	invested	for	2 years
N500	is	invested	for	1 year

So the sum of the investment is

$$A = 100(1.05)^5 + 200(1.05)^4 + 300(1.05)^3 + 400(1.05)^2 + 500(1.05)$$

Using calculator, you get

$$A = 100(1.276) + 200(1.215) + 300(1.158) + 400(1.103) + 500(1.05)$$

$$= 127.6 + 243.0 + 347.4 + 441.2 + 525.0 = N1684.20$$

Mr. John's investment will be N164.20

Remark :
In the last example, you calculatedthe amount realised on each of pricipal. Let us consider another problem where interest are withdrawn.

Example:
A man inested in AJE bank Ny on 1 Jan 1996. In the subsequent years on 1 Jan, he deposits money double that of the previous year after withdrawing the interest only on the same day. It was found that balance in his account on jan.2,2006 was N2046. Determine the money invested in 1996.

Solution:
You must realise that no interest accrues in this problem since any accrued investment from the previous year is withdrawn just as the new amount is to be deposited. The amount deposited in
1996 is y
1997 is 2y
1998 is 4y e.t.c
So the sum is

$$A = y + 2y + 4y + 8y + 16y + ... + 10\ terms$$
$$A = y(1 + 2 + 4 + 8 + 16 + ... + 10^{th}\ terms$$
$$= y\left(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + ... + 2^9\right)$$

This forms a G.P with r=2. So

$$A = \frac{y\,(2^{10} - 1)}{2 - 1}$$

Since this sum was in the account in 2006. Then

$$A = y\left(2^{10} - 1\right) = 2046$$
$$i.e\ \ 1023y = 2046$$
$$\Rightarrow y = 2$$

So the money invested in 1996 was N2

Remark:
You will now consider a problen when interest is added at regular intervals less than one year.

Example:
Find the interst on N1,000 for 10 years at 4% per annum interest being paid quarterly.

Solution:

Here, each time interval in three months (1 quarter). so there are $4 \times 10 = 40$ time intervals . The interest rate in each interval is

$$\frac{4}{100} \times \frac{1}{4}$$

Thus the amount on N1,000 for 40 time interval at 1% per interval is

$$A = 1,000\left(1 + \frac{1}{100}\right)^{40} = 1,000(1 + 0.01)^{40} = 1,000(1.01)^{40} = N1,486$$

Therefore, the interest, I is given by

$$A = P + I$$

$$\Rightarrow I = A - P = 1,486 - 1,000 = N486$$

Remark:
In general, if P amount of money is invested at $r\%$ per annum and interest is compounded c equal times per year. The amount A after n years is

$$A = P\left(1 + \frac{r}{c}\right)^{nc} \tag{3.2.2.3}$$

This is the general formula for the amount realised at compount interest.
In some crucial continuous investments, you need care in the calculation of the amount realised since money may be paid at regular intervals different from the regular intervals in which interest is being added.

Example:
Mr. James pays his saving of N25 every 6 months into AJE bank which pays interest quarterly at the rate of $2\frac{1}{2}\%$ per annum. Determine the amount in hisaccount after 10 years.

Solution: You need care !!!
The first, you need N25 will accrue to

$$25\left(1 + \frac{\frac{1}{2}}{100} \times \frac{1}{4}\right)^{4\times10} = 25\left(1 + \frac{5}{800}\right)^{40}$$

After 6 months, the second N25 will accrue to

$$25\left(1 + \frac{5}{800}\right)^{38}$$

In one year, the third N25 will accrue to

$$25\left(1 + \frac{5}{800}\right)^{36}$$

the last N25 after $9\frac{1}{2}$years will accrue to

$$25\left(1+\frac{5}{800}\right)^2$$

Thus the sum of amounts

$$A = 25\left[\left(1+\frac{5}{800}\right)^{40} + 25\left(1+\frac{5}{800}\right)^{38} + ... + 25\left(1+\frac{5}{800}\right)^{2}\right]$$

$$= 25(1.00625)^{40} + (1.00625)^{38} + ... + (1.00625)^2$$

$$= 25(1.00625)^2\left[1 + (1.00625)^2 + ... + (1.00625)^{36} + (1.00625)^{38}\right]$$

This is a G.P with sum

$$A = 25(1.00625)^2\frac{((1.00625)^{2\times 20} - 1)}{(1.00625)^2 - 1}$$

$$= 25(1.00625)^2\frac{((1.00625)^{40} - 1)}{0.0125}$$

$$= 25(1.00625)^2\left[1 + (1.00625)^2 + ... + (1.00625)^{36} + (1.00625)^{38}\right]$$

$$= 25(1.0125)\frac{(0.2823)}{0.0125} = N572$$

I can now hear you breath down heavily. You have done very well.

Remark:

From the formula

$$A = P\left(1+\frac{r}{100}\right)^{nc}$$

you get

$$P = \frac{A}{\left(1+\frac{r}{100}\right)^{nc}}$$

which gives the principal that will accrue to A in n years.

Example:

Determine the principal amount that will accrue to N4917.25 after 3years at 12% per annum compound interest.

Solution:

A=4917.25 n=3 c=1 and r=12

$$P = \frac{4917.25}{\left(1+\frac{12}{100}\right)^{3\times 1}}$$

$$P = \frac{4917.25}{(1.12)^3}$$

$$P = \frac{4917.25}{(1.12)^3} = \frac{4,917.25}{1.4019} = N3,500$$

Example:
A firm plan to invest an amount at the begining of every year inorder to accrue a sum of N100, 000 at the end of 5 years priod. Determine the value of the amount, if the investment rate 14% per annum.
Solution:
You will notice that the money is being invested as a series of equal payments and not as a single principal amount.
Let the amount invested at the begining of every year be P. then,
The first payment accrues to p$(1.14)^5$
The second payment accrues to p$(1.14)^4$
and the last payment accrues to p(1.14)
So, the sum of the separate amountS

$$A = P(1.14)^5 + P(1.14)^4 + P(1.14)^3 + P(1.14)^2 + P(1.14)$$

This amount is N100, 000, so

$$100,000 = P(1.14)\left[1 + (1.14)^4 + (1.14)^3 + (1.14)^2 + (1.14)\right]$$

This is a G.P with r= (1.14). Therefore,

$$100,000 = \frac{P(1.14)\left[(1.14)^5 - 1\right]}{1.14 - 1}$$
$$100,000 = \frac{P(1.14)\left[(1.14)^5 - 1\right]}{0.14}$$
$$100,000 = P(7.5355)$$
$$P = \frac{100,000}{7.5355} = N13270.52$$

i.e the form will need to invest N13270.52 at the begining of each year.

3.3 Student Self Assessment Exercises 1

(1) What value will N450 accrued to at 12% per annum in 3 years.
(2) What principal amount will accrue to N8, 500. If it is compounded at 14.5% per annum 6 years.
(3) Find the compound interest rate necessary for N20, 00 to accrue to N50, 000 in 12 years.
Remark :
Do not check the solutions untill you have attempted the question.
Solution
(1)

$$A = 450\left(1 + \frac{12}{100}\right)^3$$

$$= 450(1.12)^3 = N632.22$$

(2)

$$P = \frac{A}{\left(1 + \frac{r}{100}\right)^n}$$

$$P = \frac{8,500}{\left(1 + \frac{29}{100}\right)^6}$$

$$P = \frac{8,500}{(1.145)^6} = N3772.12$$

(3)From

$$A = P\left(1 + \frac{r}{100}\right)^n$$

You have

$$50,000 = 20,000\left(1 + \frac{r}{100}\right)^{12}$$

$$\Rightarrow \frac{50,000}{20,000} = \left(1 + \frac{r}{100}\right)^{12}$$

$$\Rightarrow \frac{5}{2} = \left(1 + \frac{r}{100}\right)^{12}$$

$$\sqrt[12]{2.5} = \left(1 + \frac{r}{100}\right)$$

$$1.08 = \left(1 + \frac{r}{100}\right)$$

$$1.08 - 1 = \frac{r}{100}$$

$$0.08 = \frac{r}{100}$$

$$0.08 \times 100 = r$$

$$8\% = r$$

3.4 Norminal and Effective Interest Rates

You have noticed that interest rate for problems in finance is expressed as per annual even though interest may be compounded over time periods of less than one year. In this types of problem, the given annual interest rate is called a norminal rate which you have denoted by r.

Since compounding may be at any interval less than one year, e.g half yearly. quarterly or monthly as the case may be, the actual annual rate of interest is called the effective rate or actual.

Percentage rate (APR), denoted by i . Infact i will always be greater than $(i > r)$.

Remark :

The standard method of determining the APR is to make the effective time period equal the compounding period and actually compound over a period of one year.

Example :

If N100 is invested at a nominal interest at 10% compound half yearly for 1 year. Determine the APR.

Solution:

The amount, A, after i year is

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$A = P\left(1 + \frac{r}{100}\right)^2$$

$$A = 100\left(1 + \frac{10}{100}\right)^2 = 100(1.05)^2 = 110.25$$

(Recall that there are two half year in 1 year) Let $i\%$= APR then using

$$A = P(1 + APR)^n = 100\left(1 + \frac{i}{100}\right)$$

$$\Rightarrow 110.25 = 100\left(1 + \frac{i}{100}\right)$$

$$\Rightarrow \frac{110.25}{100} = \left(1 + \frac{i}{100}\right)$$

$$\Rightarrow 1.1025 = \left(1 + \frac{i}{100}\right)$$

$$\Rightarrow 1.1025 - 1 = \left(\frac{i}{100}\right)$$

$$\Rightarrow 0.1025 = \left(\frac{i}{100}\right)$$

$$0.1025 \times 100 = i$$

$$10.25 = i$$

$$i = APR = 10.25\%$$

3.4.2 Formular for Calculating APR

Consider a principal P=1 invested at $r\%$ and compounded at n intervals in 1 year. Then, the amount, A is given by

$$A = P\left(1 + \frac{r}{n}\right)^n = \left(1 + \frac{r}{n}\right)^n$$

If APR=$i\%$ then,

$$A = P(1 + i) = (1 + r)$$

Therefore,

$$(1+i) = \left(1+\frac{r}{n}\right)^n$$

$$APR = i = \left(1+\frac{r}{n}\right)^n - 1 \tag{3.4.3}$$

(3.4.3) gives the formular for calculating the effective percentage rate if the nominal rate, r, is given.

Remark :

From now on, you will not insist on inserting per annum with the nominal interest rate. This will always be assumed.

Example :

N100 is invested at the 12% compounded quarterly. Determine the accrued amount after one year and the APR.

Solution:

Here P=100, r=12%, n=4 (four quarters= 1 year) using

$$A = P\left(1+\frac{r}{100}\right)^n = 100\left(1+\frac{3}{100}\right)^4$$

$$A = 100(1,03)^4 = N112.55$$

Thus the accrued amount is the N112.55.

Let APR=i then,

$$APR = \frac{i}{100} = \left(1+\frac{12}{100}\times\frac{1}{4}\right)^4 - 1 = (1.03)^4 - 1$$

$$\Rightarrow \frac{i}{100} = 1.1255 - 1 = 0.1255$$

$$\Rightarrow i = 0.1255 \times 100 = 12.55$$

$$APR = 12.55\%$$

Example :

A finance house advertises money at 22% nominal interest but compounds monthly. Determine the APR.

Solution:

$$APR = \frac{i}{100} = \left(1+\frac{22}{100}\times\frac{1}{12}\right)^{12} - 1$$

$$\Rightarrow \frac{i}{100} = (1.0183)^{12} - 1 = 1.24 - 1$$

$$\Rightarrow i = 0.24 \times 100 = 24$$

$$APR = 24\%$$

3.5 Student Self Assessment Exercises 2

(1) Determine the APR in the following problesms.
(a) 10% norminal, compounded quarterly.
(b) 24% norminal, compounded monthly.
(2) A company intends to spend N300,000 on a new plant in two years. The current investment norminal rate is 10%. Determine the single sum to be invested if compounding is six monthly and hence calculate APR.
(**Hint:** Work the problems before looking at the solution)
Solution
(1a)

$$APR = \left(1 + \frac{10}{100} \times \frac{1}{4}\right)^4 - 1 = (1.025)^4 - 1 = 10.38\%$$

(1b)

$$APR = \left(1 + \frac{24}{100} \times \frac{1}{12}\right)^{12} - 1 = (1.02)^4 - 1 = 26.82\%$$

(2)

$$P = \frac{300,000}{\left(1 + \frac{10}{100} \times \frac{1}{2}\right)^4} = \frac{30,000}{(1.05)^4} = N2468810.75$$

The single sum is N246810.75

$$APR = \left(1 + \frac{10}{100} \times \frac{1}{2}\right)^2 - 1 = (1.05)^4 - 1 = 10.25\%$$

4.0 CONCLUSION

You can now
(1) define interest, principal and amount.
(2) recognise and differentiate between simple and compound interests.
(3) solve problems involving simple interest.
(4) solve problems involving compound interest.
(5) determine the accrued amount from successive investment amounts.
(6) calcul;ate APR.

4.0 SUMMARY

(1) Simple interest method is when interest is calculated on the principal only.
(2) Compound interest method is when interest calculation is based on the previous interest.

(3) The amount, A and the principal P arte related by

$$A = P(1 + rn)$$

Using simple interest
(4)

$$A = P\left(1 + \frac{r}{c}\right)^{nc}$$

when interest is compounded c times in a year.
(5)

$$APR = \left(1 + \frac{r}{c}\right)^{nc} - 1$$

(6) Interest is the direct opposite of depreciation.
(7) $A = P + I$ where I=interest.

4.0 TUTOR MARKED ASSIGNMENTS

(1)N1200 is invested at 12% simple interest . How much will be realised after 3 years ?
(2)Find the amount of:
(a) N480 compounded at 14% over 5 years.
(b)N1240 compound at 11.5%
over 12 years.
(3)After 3 years, what principal value will amount to N1100 at 8% compounded ?
(4)N10, 000 was compound at 12.5% and amounted to N52, 015.80. How many years did it take ?
(5)What percentage compound interest rate (to 2dp) will treble the value of an investment over a period of 5 years?
(6)A firm borrows N6000 from AJE bank at 12% compounded semi-annually. If no repayments are made, how much is owned after 4 years ?
(7)A loan company advertises money at 18% (norminal). What is the APR (to 2dp), if compounding is (a)12%(b)12.4%(c)12.7%(d)

7.0 REFERENCES/ FURTHER READINGS

(1) Andre Francis : Business Mathematics and Statistics. 6^{th} Edition 2004.
(2) S.G. Green: Advanced level Mathematics printed in Great Briton by University Tutorial press. LTD.
(3) Qazi Zameerudin, Vijay K.Khama and S.K Bhambri: Business Mathematics 2^{nd} Edition.2010. Vikas Publishing House PVT LTD.

MODULE 2

Unit : Annuity

CONTENTS

1.0 INTRODUCTION

This section describes various techniques associated with fixed payments (or receipts) over time, otherwise known as annities. The invested value, Present Value (PV) and the sum of present values of a series of annuities, known as net Present Value (NPV) are treated together with the Amortisation and Sinking fund methods of debt repayment. We also treated how annuities can be used to depreciate an asset.

2.0 SUBJECTIVES

At the end of this unit, you should be able to:

* give the formula of present Value (PV) of a futher amount of money.
* use the discounting table.
* define and recognise annuities.
* calculate the present Value (PV).
* work problems on Amortisation.
* work problems on sinking fund.

3.0 MAIN CONTENTS

3.1 Present Value (PV)

Let us atart with a formular we had before.
Let P be invested at $r\%$ compound interest for n years. The accrued amount A is given by

$$A = P(1 + r)^n \tag{3.1.1}$$

which is rewriten as

$$P = \frac{A}{(1 + r)^n} \tag{3.1.2}$$

P is called the present value (PV) of A.
The idea is if N100 is invested for 2years at 10% compound interest. The amount A is

$$a = 100\left(1 + \frac{10}{100}\right)^2 = N121$$

Then N100 is called the present value of N121 after 2years at the investment rate of 10%. The above demonstrates the concept of the present value of a future sum. The investment rate, used in this context, is called the discount rate.

3.2 Use of Discounting Tables

We have intentionally avoided the use of a particular mathematical equipment for the quick evaluation or simplification of calculations. We introduce the use of discounting table here to ease your calculation. You have to purchase a discounting table now. In (3.1.2), the factor

$$\frac{1}{(1 + r)^n} \tag{3.2.1}$$

is the present value factor or discount factor. There is a range of values of n and r. You have to learn how to use the table. The main column has the n values while the row has the r values.

Example : To evaluate

$$\frac{1}{(1 + \frac{14}{100})^4}$$

look at the column for n=4 and then go along the row to r=14, the intersection gives the value 0.5921.

$$P = \frac{1}{(1 + 14)^4} = 0.5921$$

Then the PV of N100 in 4 years at 14% is

$$P = \frac{100}{(1.14)^4} = 100 \times \frac{1}{(1.14)^4}$$

$$= 100 \times 0.5921 = N59.21$$

Example :
Use the discounting table to determine the PV of N1, 500 in 6 years time at a discount rate of 19% .
Solution:
On the table, for n=6 and r=19%, the value is 0.3521. So

$$PV = \frac{1,500}{(1+\frac{19}{100})^6} = 1,500 \times (0.3521)$$
$$= N528.15$$

3.3 Annuity

We shall now consider certain sseries of equal payments or receipts at equal intervals. This is not new to us we had considered the sum of such a sequence in unit 4.

3.3.1 Definition:

An annuity is a sequence of fixed equal payments (or receipts) made over uniform time intervals.
Common examples of annuities are
* weekly wages or monthly salaries.
* insurance premiums.
* house purchase morgage payment (see morgage Bank policy).
* hire purchase payments.

Remark:
Annuity are use in all areas of business and commerce as: * loans are normally repaid with an annuity.
* investment funds are setup to meet fixed further commitment .
(e.g asset replacement) by the payment of annuity.

3.3.2 Types of Annuity

(a) Annuity may be paid:
(i) at the end of payment intervals (an ordinary annuity) or
(ii) at the begining of payment intervals (a due annuity).
(b) The term of an annuity may:
(i) begin and end on fixed dates (a certain annuity)
(ii)depend on some event that cannot be fixed (a contingent annuity).

(c) a perpertual annuity is one that carries on indefinitely.

3.3.3 Some Aspects and Examples of annuities

(a) The most common form of an annuity is certain and ordinary. That is, the annuity will be paid at the end of the payment interval (or "in arrears") and will begin and end on fixed dates. For example, some domestic hire purchase contracts will involve the payment of an intial deposit and then equal payments, payable at the end of each month, up to a fixed date.
Persoal loans from banks or finance houses are paid off in a similar manner, but certainly without the intial deposite.
(b) Annuities that are being invested however are often due, that is, paid "in advance of the interval. For example a saving scheme, paid as an annuity (with a bank , building society or insurance company) will not be deemed to have started untill the first payment has been made.
(c) Standaerd persion or superannuiation schemes can be thought of as two stages annuities. the first stage involves a due, certain annuitu (i.e regular payment into the fund up to retirement age), the second stage being the receipt of a contingent annuity (i.e regular receipts untill death).

Example :
Determine the value of an annuity of three annual payments of N12,000 invested in a fund that pays 12%.

Solution :
The value of the first payment is

$$12,000\,(1.12)^3$$

The value of the second payment is

$$12,000\,(1.12)^2 \qquad and$$

The value of the third payment is

$$12,000\,(1.12)$$

The total value of the invested monthly is

$$A = 12,000\,(1.12)^3 + 12,000\,(1.12)^2 + 12,000\,(1.12)$$

$$12,000(1.12)\left[1 + (1.12) + (1.12)^2\right]$$

This is a G.P with sum

$$12,000(1.12)\left[\frac{(1.12)^3 - 1}{0.12}\right] = N45,351.60$$

The accrued amount is N45, 351.60

Example :
Determine accrued amount of 12 monthly payments of N100 into a building society account which pays a fixed norminal rate of 10.75% compounded monthly.
Solution : (You have worked a similar problem before).
The accreued amounts of the monthly payments is

$$A = 100\left(1 + \frac{0.1075}{12}\right)^{12} + 100\left(1 + \frac{0.1075}{12}\right)^{11} + ... + 100\left(1 + \frac{0.1075}{12}\right)$$

$$A = 100\,(1.00896)^{12} + 100\,(1.00896)^{11} + ... + 100\,(1.00896)$$
$$= 100(1.00896)\left[1 + (1.00896) + (1.00896)^2 .... + (1.00896)^{11}\right]$$

This is a G.P with sum

$$= 100(1.00896)\left[\frac{(1.00896)^{12} - 1}{0.0896}\right] = 100(12.7223) = N1272.23$$

So the total accrued amounti N1272.23 **3.4 Sum of Present Values**

We consider the sum of PVs. Suppose A is the annual payment with present value P at r% discounting rate.
Denote $\left(1 + \frac{r}{100}\right)$ by R .
After n years, the sum of the present values is

$$P = \frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + ... + \frac{A}{R^n} \tag{3.4.1}$$

$$= \frac{A}{R}\left(1 + \frac{1}{R} + \frac{1}{R^2} + ... + \frac{1}{R^{n-1}}\right)$$

This is a G.P with common ratio $\frac{1}{R}$. Thus

$$P = \frac{A}{R}\left[1 - \left(\frac{1}{R}\right)^n\right]1 - \frac{1}{R} = \frac{A}{R}\frac{\left[1 - \frac{1}{R^n}\right]}{\frac{R-1}{R}}$$

$$= \frac{A}{R-1} = \left[1 - \frac{1}{R^n}\right] \tag{3.4.2}$$

$$= \frac{A}{(r+1) - 1} = \left[1 - \frac{1}{(r+1)^n}\right]$$

$$\frac{A}{r} = \left[1 - \frac{1}{(r+1)^n}\right] \tag{3.4.3}$$

(3.4.3) is called the Net present Value (NPV) of annuity A over n years and subject to a discounting rate of r%.
Definition :

An annuity which is payment forever is called perpetuity.
In this case, you will consider the number of years to be very large i.e you consider n to tend to infinity. Denoted infinity by the symbol ∞ and write $n \to \infty$.
From (3.4), the PV of an annuity A is given as

$$PV = \frac{A}{r} = \left[1 - \frac{1}{(r+1)^n}\right] \tag{3.9.1}$$

When $n \to \infty$, $\frac{1}{(r+1)^n} \to 0$ So (3.9.1) becomes

$$PV = \frac{A}{r}$$

which is then the total present Value of A to infinity.
Remark :
For conciseness, you have $r = \frac{r}{100}$ in (3.4.3)
Example :
Determine the NPV of an annuity N125 payable at the end of each of fiveyears and subject to a discount rate of 8%.
Solution : Here r=8%, so let

$$R = \left(1 + \frac{r}{100}\right) = 1.08$$

Since A=125, using (3.4.1)

$$\begin{aligned} NPV &= \frac{A}{R-1}\left[1 - \frac{1}{R^n}\right] \\ &= \frac{125}{0.08}\left[1 - \frac{1}{(1.08)^5}\right] \\ &= 125(3.9926) = N499.08 \end{aligned}$$

Example :
The tenants of a rented house have their rent fixed at N1650 per year in advance with immediate effect. They plan to stay in the property for 15 years. Find the total value of the payment (in today's terms). If the average discount rate is estimated at 10%.
Hint: Notice that this is a due anniuty.
Solution :
The NPV of the rents is

$$1650 + \frac{1650}{(1.10)} + \frac{1650}{(1.10)^2} + \ldots\ldots\ldots\ldots \frac{1650}{(1.10)^{14}}$$

$$= 1650\left[1 + \frac{1}{(1.10)} + \frac{1}{(1.10)^2} + \ldots\ldots\ldots\ldots \frac{1}{(1.10)^{14}}\right]$$

This is a G.P with common ratio $\frac{1}{1.10}$. So

$$NPV = 1650\frac{\left[1 - \frac{1}{(1.10)^{15}}\right]}{1 - \frac{1}{1.10}}$$

$$1650\left[1 - \frac{1}{(1.10)^{15}}0.10\right] \times 1.10$$

$$\frac{1650(1.10)}{0.10}\left[1 - \frac{1}{(1.10)^{15}}\right] = 1650(8.3671) = N13805.72$$

So the total present value of the payment is N13, 805.72.
Let us consider a debt paying problem where the value of the initial payment is differenct from the value of the instalmental payment.

Example :
A department store advertises goods at N700 deposiote and three further equal annual payment of N500. If the discount rate is 7.5%, calculate the present value of the value of the goods.
Solution :
There are four payment to consider, but the first (N700) is payable now and thus its present value is the same as its face value . The other three payments of N500 have to be discounted over 1, 2 and 3 years respectively, using discount rate 7.5%. Using the formular in (3.4.1)

$$PV = 700 + \frac{500}{\left(1 + \frac{7.5}{100}\right)} + \frac{500}{\left(1 + \frac{7.5}{100}\right)^2} + \frac{500}{\left(1 + \frac{7.5}{100}\right)^3}$$

$$= 700 + \frac{500}{(1.075)} + \frac{500}{(1.075)^2} + \frac{500}{(1.075)^3}$$

$$= 700 + 500(0.9302) + 500(0.8653) + 500(0.8050)$$

$$= 700 + 500\left[(0.9302) + (0.8653) + (0.8050)\right]$$

$$700 + 500(2.6005) = N2,000.25$$

Thus, the present value isN2, 00.25. i.e N2, 000.25 could be considered as the present cash price of the goods.

3.5 Present Value of an Interest-Bearing Debt

Everybody is in business in order to make some profit. Giving loan to people is a form of business. So when an amount of money is borrowed. It will attracted interest at an appropriate borrowing rate. If no intermadiate payment are made, at the end of a period, the debt plus interest must be repaid as a lump sum. The PV of such amount is often

required.
Let P be the amount borrowed at r% compounded for n years. The debt will amount to $P(1+r)^n$. If the investment (discounted) rate is R%, the debt will have to be discounted at this rate back over the n years.
Thus the present value of the debt is

$$PV = P\frac{(1+r)^n}{(1+R)^n} \tag{3.5.1}$$

Note that, since the borrowing rate will always be greater than the investment (discount) rate, $r > R$, hence

$$\frac{(1+r)^n}{(1+R)^n} > 1 \Rightarrow PV > P$$

Example :
Find the PV of a debt of N2, 500 taken out over 4 years (with no intermediate repayment) where the borrowing rate is 12% and the worth of money (discount rate) is 9.5%˙ **Solution :**
The value of the debt in 4 years is

$$2,500\left(1+\frac{12}{100}\right)^4 = 3933.80$$

The present value of N3933.80 is

$$PV = \frac{3933.80}{\left(1+\frac{9.5}{100}\right)} = \frac{3933.80}{(1.095)^4}$$

$$N2736.25$$

In otherwords, the original debt of N2,500 will cost N2736.25 (in today's money terms) to repay.
Thus, it can be considered that the real cost of the debt is N(2736.25-2, 500)=N236.25 (in today's money terms).

3.6 Amortasation Annuity

In the last example, you considered the PV of a debt in which there are no intermediate repayment. Most debt repayments do not follow this procedure.
If an amount of money is borrowed over a period of time, one way of repaying the debt is by paying an amortisation anniuty. This consists of a regular annuity (ordinary and certain) in which each payment accounts for both repayment of capital and interest.
The debt is said to be amortised if this method is used.
Many of the loan issued by banks and building societies for house purchase are of this type where it is known as a repayment mortgage.
Consider a borrowed amount P subject to interest at r%. The problem is to determine

the annuity payment, A, which will amortise the debt in exactly n years.
In this csae, you allow P to be the Present Value of the series of repayment A over n years. Thus

$$P = \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + ... + \frac{A}{(1+r)^n}$$
$$= \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

Thus, the annual repayment Ais given by

$$A = \frac{Pr}{\left[1 - \frac{1}{(1+r)^n}\right]} \tag{3.6.1}$$

Example:
ABC company negotiates a loan of N200, 000 over 15 years at 10.5% p.a. Calculate the annual payment necessary to amortise the debt.
Solution: Here p=200, 000 and R=10.5%. Let A be the amortisation annuity, then

$$A = \frac{200,000 \times 0.105}{\left[1 - \frac{1}{(1.105)^{15}}\right]} = N27,050$$

Thus the annual payment necessary to amorsatise the debt is N27, 050.
Definition :
The dfference between amount of installment and interes is called Amortisation.

3.7 Sinking Fund

Definition :
A sinking fund is the money accumulated at compound interest by setting aside a fixed amount at regular intervals. That is sinking fund is an annuity interest in order to meet a known commitment at some future date.
Remark :
Sinking funds are commnly used for the following purposes:
(a) repayment of debts.
(b) to provide funds to purchase a new asset when the existing one is fully depreciated.
Example : (Debt repayment using a sinking fund)
Suppose a debt P incurred over n years subject to a given borrowing rate R%. A sinking fund must be set up to mature to the outstanding amount of the debt.
The value of the debt is $P(1+R)^n$. Set up an annuity A at investment rate r%. then

$$P(1+R)^n = A(1+r)^{n-1} + A(1+r)^{n-2}A(1+r)^{n-3} + ... + A(1+r)^2 + A(1+r) + A$$
$$= A\left[1 + (1+r) + ... + A(1+r)^{n-1}\right]$$
$$= A\frac{[(1+r)^n - 1]}{(1+r) - 1} = \frac{A}{r}\left[(1+r)^n - 1\right]$$

$$\Rightarrow A = \frac{P(1+R)^n r}{[(1+r)^n - 1]}$$

Example :
The sum of N25, 000 is borrowed over 3 years at 12% compounded. Determine the regular fixed amount paid into a sinking fund to pay the sum, at investment rate of 9.5%.
Solution :
The value of te debt in 3 years is

$$25,000\left(1+\frac{12}{100}\right)^3 = 35123.20$$

If payment into the sinking fund are in arrears with an annuity A, then

$$35123 = A(1.093)^2 + A(1.093) + A$$
$$= A\left[(1.095)^2 + (1.095) + 1\right]$$
$$= A\left[3.2940\right]$$
$$\Rightarrow \frac{35123.20}{3.2940} = 10662.78$$

That is, the annuity payment into the sinking fund is N10, 662.78 (which will produce, at 9.5%, N35, 123.20 at the end of 3 years).

Example :
ABC company borrows N46, 000 which is compounded at 15% to finance a new production line. The debt will be discharged at the annual payment into a sinking fund which pays 11.25%. Calculate the annual payment into the fund , assuming that the first payment into the fund is made at the end of the first year.
Solution:
The payments into the the fund in this case form an ordinary annuity, at the end of 5% years, the debt amount to

$$46,000\left(1+\frac{15}{100}\right)^5 = 46,000(1.15)^5 = 92522.43$$

The payment into the fund will mature to (with annuity A)

$$A(1.1125)^4 + A(1.1125)^3 + A(1.1125)^2 + A(1.1125) + A$$
$$A\left[1.53179 + 1.37689 + 1.23766 + 1.1125 + 1 + 1\right] = A(6.25884)$$
$$\Rightarrow 92522.43 = A(6.25884)$$

$$\Rightarrow \frac{92522.43}{6.25884} = N14782.68$$

3.8 Sinking Fund Method of Depreciation

You will recall that we had treated two forms of depreciation methods : straight line and reducing balance method. Now
(a) one reason for depreciating an asset is to take proper account of its replacement. Thus one can consider the periodic book payments in respect of depreciation, the depreciation charge, as forming a pool that, at the end of the assets useful life, will fund a replacement or alternative.
(b) The sinking fund method of depreciation consider the depreciation charge payment as being available for investment into a fund (a depreciation fund, paying a market rate of interest) which will mature to some predetermine value. The book value of the respective assets at the end of any year can be determined by subtracting the curent amount in fund from the origional book value of the asset.
Example :
A mechane value at N12, 500, wth a 6 years life is estimated to have a scrap value of N450. If the depreciation fund earns 8% : You are required to use the sinking fund method based on an ordinary annuity to fund the annual deposit into the fund (Depreciation charge).
Example :
The difference between the original value=12, 500=450=12, 050
which must be the value of the depreciation fund after 6 years, interest paid at 8%.
Let the annuity=A. Then

$$12,050 = A\left[1 + 1.08 + (1.08)^2 + (1.08)^3 + (1.08)^4 + (1.08)^5\right]$$

$$= A\frac{[(1.08)^6 - 1]}{(1.08) - 1} = A(7.36)$$

$$\Rightarrow A = \frac{12050}{7.36} = 164.60$$

Thus the annuity depreciation large is N1642.60

3.9 Student Self Assessment Exercise

(1) Sanya, the son of chief Ajetunmobi is just 5 years old. In order to provide a sum of N50, 000 when sanya is 21 years, Chief Ajetunmobi has decided to invest a fixed sum of money every year begining immmediately and the last payment will be when sanya is 21 years . Find the fixed sum of money assuming investment rate is 8%.

(2) Find the interest on N1,000 for 10 years at 4%, interest being paid quarterly.
(3) A man borrrows N1, 000 and repays the loan by yearly instalment of N100, the first instalment being paid one year afyter the loa. After how many years, will he be out of debt, interest being reckoned through at 4%.
(4) a man borrows N20, 00 and agrees to pay the borrowed amount in 10 equal annual instalment at the rate of 6% . Find the amount of each instalments, the first being paid one years after the money was borrowed.
Hint : You nmust work through the above problems before looking at the prefered

Solution :
(1) Let the fixed sum of money = Ny.
The money will be paid for (21-4)=17 year. The sum of the value of the instalment is

$$y(1.08)^{16} + y(1.08)^{15} + y(1.08)^{14} + \ldots\ldots\ldots\ldots\ldots + y$$

This sum must equal N50, 000. So

$$50,000 = y\left[(1.08)^{16} + (1.08)^{15} + (1.08)^{14} + \ldots\ldots.. + 1\right]$$

This is now a G.P. So

$$50,000 = \frac{y\left[(1.08)^{17} - 1\right]}{(0.08}$$

$$\Rightarrow y = \frac{50,000 \times (0.08)}{(1.08)^{17} - 1} = \frac{50,000 \times (0.08)}{3.70 - 1}$$

$$\Rightarrow y = \frac{50,000 \times (0.08)}{2.70} = N1481$$

So, the fixed sum of money is N1, 481.

(2) The amount, A is given by

$$A = P(1 + r)^n$$

where r=$\frac{4\%}{4} = 1\%$

$$A = 1,000(1.01)^{40}$$

since there are 40 quarters i 10 years, A=N1486
Interest =A-P=1486-1, 000=N480.00
(3) The sum of the present values of the instalment must equal to N1, 000, after n years, so,

$$1,000 = \frac{100}{1.04} + \frac{100}{(1.04)^2} + \ldots\ldots\ldots\frac{100}{(1.04)^n}$$

$$= \frac{100}{1.04}\left[1 + \frac{100}{(1.04)} + ...\frac{100}{(1.04)^{n-1}}\right]$$

which is now a G.P . Hence

$$1,000 = \frac{100}{1.04}\frac{\left[1 - \frac{1}{1.04)^n}\right]}{1 - \frac{1}{(1.04)}}$$

$$1,000 = \frac{100}{1.04}\left[1 - \frac{1}{1.04)^n}\right]$$

$$\Rightarrow \frac{1000 \times 0.04}{100} = 1 - \frac{1}{(1.04)^n}$$

$$\Rightarrow \frac{1}{(0.04)^n} = 0.6$$

$$\Rightarrow (0.04)^n = \frac{10}{6} = 1.667$$

$$\Rightarrow n \log 0.04 = \log 1.667$$

$$\Rightarrow n = \frac{\log 1.667}{\log 0.04}$$

$$\Rightarrow n = 13.05 \quad years$$

Since n is integral

$$\Rightarrow n = 14 \quad years$$

(4) Let the amount of each instalment=y.
The sum of the PV of each instalment must equal to 20, 000. So

$$20,000 = \frac{y}{(1.06)} + \frac{y}{(1.06)^2} + ... + \frac{y}{(1.06)^{10}}$$

$$= \frac{y}{(1.06)}\left[1 + \frac{1}{(1.06)^2} + ... + \frac{1}{(1.06)^9}\right]$$

This is again a G.P, so

$$20,000 = \frac{y}{1.06} \times \frac{\left[1 - \frac{1}{(1.06)^{10}}\right]}{1 - \frac{1.06}{(0.06)^2}} = \frac{y}{1.06}\left[1 - \frac{}{(1.06)^{10}}\right] \times 1 - \frac{1}{(1.06)^2}$$

$$\frac{y}{1.06}\left[1 - \frac{1}{(1.06)^{10}}\right]$$

$$\Rightarrow y\left[1 - (1,06)^{-10}\right] = 20,000 \times 0.06 = 1200$$

$$y = \frac{1200}{1 - (1.06)^{-10}} = \frac{1200}{1 - 0.5584}$$

$$\frac{1200}{0.4416} = 2718.$$

4.0 CONCLUSION :

You can now

*** define and give the formular for present Value (Pv) of a future amount of money.**

*** use the discounting table.**

*** calculate NPV.**

*** define annuities as a series of fixed amount of money.**

*** work problems on Amortisation.**

*** work problems on sinking fund.**

More importantly, you have only used the formula for G.P.

SUMMARY

Recall that

(1) Amount,

$$A = P(1+r)^n$$

(2)

$$P = \frac{A}{(1+r)^n} \quad is \quad the \quad PV$$

(3)

$$\frac{1}{(1+r)^n}$$

is the discounting factor which is got from the discounting table for various values of n and r .

(4)

$$NPV = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

(5)

$$PV = \frac{P(1+r)^n}{(1+R)^n}$$

is the present value of an interest-bearing debt P.

5.0 TUTOR MARKED ASSIGNMENT :

(1)A man puts N10 at the end of every year in the saving Bank at $2\frac{1}{2}\%$ compound interest. How much will his saving amount to , in 15 years.

(2) A loan of N1,000 is to be paid in five equal annual instalments, interest being at 6% compound interest and first payment being made after one year. Determine the amortisation annuity.

(3) A company sets aside a sum of N20, 000 annually to enable it to pay off a debenture issue of N230, 000 at rhe end of 10 years.

Assuming that the sum accumulates at 4% compound interest, find the surplus after paying off the debenture stock.
(4) A machine costs a company N52, 000 and its effective life is estimated to be 25 years. A sinking fund is created to replace the machine by new model at the end of its life time, when its scrap realises a sum of N2500 only. The price for a new model is estimated to be 25% higher than the present one. Find what amount should be set aside.

(5) A lends B N320. B is to pay interest on whatever amount he has not paid back at the rate of 5% for the first year, 6% for the second year and 7% for the third year. B pays A N100 at the end of first year, N100 at the end of second year and enough to pay off completely the debt and interest at the end of the third year. How much is the last payment.

(6) A man buy a car for N16, 000. He estimates that its value will depreciate each year by 20% of its value at the begining of a year. Find the depreciated value Nx(correct to the nearest naira) of the car at the end of five years. If the man sets aside at the end of each of the five years a certain fixed sum Ny to accumulate at 4% compound interest in order to be able to buy at the end of the five years another car costly N22, 000 (after allowing the above depreciated value Nx for the old car in past exchangre), find to nearest naira the value Ny of each payment.

7.0 REFERENCES/ FURTHER READINGS

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(3) Qazi Zameerudin, Vijay K.Khama and S.K Bhambri: Business Mathematics 2^{nd} Edition.2010. Vikas Publishing House PVT LTD.