NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

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COURSE TITLE: INTRODUCTION TO MATHEMATICAL MODELLING IN FINANCE

Introduction to Mathematical Modelling in Finance

FMT 313

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MODULE 1

UNIT 1 INTRODUCTION TO FINANCIAL MATHEMATICS

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1.0 INTRODUCTION

In the life of every human being, finance is very important for his existence. This implies that every activity of human beings rest on how to raise fund and equally spend it. Perhaps this fund is not readily available and as such requires strategies to maximize its use. No better ways have been identified for its maximum utilization except by using mathematics. Hence our aim is to adopt mathematical modeling to the optimum uses of available fund and this we have done in this block. The terms Finance and Financial System would be discussed along with the use of some fundamental tools of Mathematics. This thus afford us the opportunity to develop a methodology for the study of Mathematical Modeling in Finance.

2.0 OBJECTIVES

After you must have read this unit, you should be able to:

- understand the difference between finance and financial system
- explain the extent to which Mathematics is veritable to the analysis of Finance
- establish the extent of the relationship between Finance and Mathematics
- identify the approaches to the study of Mathematical Modeling in Finance

3.0 MAIN CONTENT

3.1 Review of Basic Concept – What is Finance and Financial system?

Finance is very essential in every aspect of human's activities, hence the need to understood what it is all about. A scientist, an engineer, a medical practitioner and all professions need to plan with their limited resources in other to make the best investment decision. Irrespective of the profession one belongs to, the need to have financial knowledge arises out of the fact that it aids in planning effectively, solving problem and making reasonable decision.

Finance is therefore the application of series of financial and economic principles to maximize the shareholders or stockholders wealth. It is also a means to maximize the overall value of a business.

Usually organization goals are more than just maximizing wealth of shareholders, other goals abound are: profit maximization, sales revenue maximization, managerial reward maximization, behavioural and social responsibility goals maximization. Out of all these goals, shareholders' wealth maximization and profit maximization are the most important, though the later supersedes the former. While the stockholders wealth maximization is a long-term goal, the profit maximization is a single-period or short-term goal.

For a long-term goal, wealth maximization takes cognizance of risk and uncertainty, the timing of return and the shareholders return.

A firm or corporate organization thus invest in projects or involves in acquisition of assets whose yields or returns have the highest possible profit at the minimum risk.

The more the risk that is undertaken the more the profit or return. In this wise, there is a risk/ return trade –off involved. A project with higher risk is expected to yield higher return. Whereas, a project with lower risk is expected to yield a lower return. This is demonstrated in figure I below:

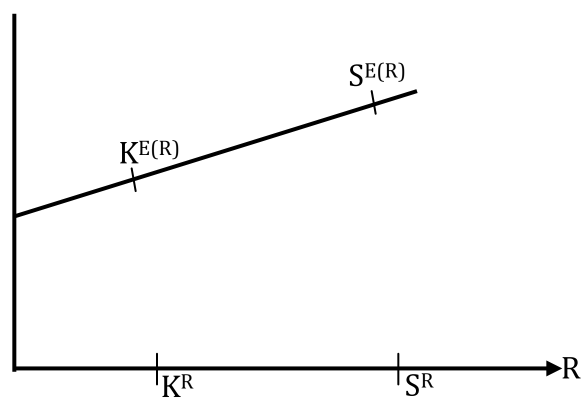

Fig. I.I: Relationship between Risk(R) and Expected Returns E(R)

Figure 1.1 has been able to establish the fact that a higher return is associated with a higher risk. For Project K^R with lower risk, the return is also low $K^{E(R)}$ compared with the project S^R with higher risk that has higher return – $S^{E(R)}$. For this reason, wealth maximization is always concerned with risk unlike the profit maximization which is more concerned with immediate or single-period gain.

The significance of finance therefore arises out of the need to determine which source of funding (whether debt or equity) is more beneficial in realizing the company's or the firm's capital goals. To achieve this, the costs and benefits of each option of financing are weighted against each other.

Further to this, finance enhances making intelligent financial decisions through the analysis of financial information for competitors based on their financial statement. Through this, a financial manager is able to plan and identify which project suitably maximizes the shareholders wealth.

In reality therefore, finance involves many interrelated and coordinated functions which include raising of funds, using of these funds, monitoring and evaluation of performance, and providing solutions to current and future problems.

All the functions of finance are interrelated in an operating environment called the financial system. A financial system is a mechanism through which people who need funds (Borrowers) are brought in contact with those who have funds (Lenders or Savers).

The financial system is made up of the financial markets and the financial intermediaries.

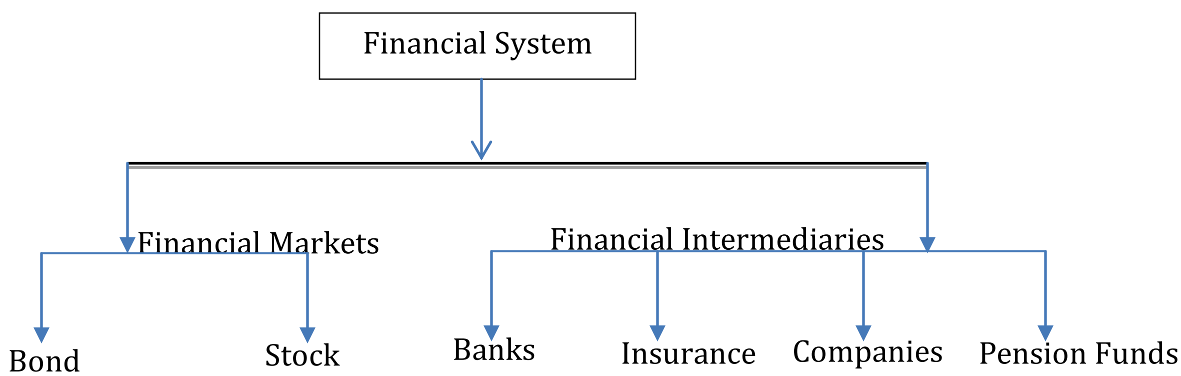

Figure I.2: Structure of a Financial System

Figure 1.2 explains that the financial markets provide the funds for long-term, medium term and short-term borrowing (bond and stock) with the banks and non-bank and non bank financial institutions facilitating the transaction. On this note, the financial system performs the basic function of getting people together by moving funds from those who have surplus of funds to those who have shortage of funds. It is very crucial and important to have a sound financial system in order to enhance economic stability and efficiency. For these two reasons, the financial system is usually regulated so as to: provide information to the investors, ensure a virile and strong financial system along with improving the control of monetary policy.

The total activity in the financial system is thus demonstrated in the process of funds flow through the system as shown in figure 1.3 below:

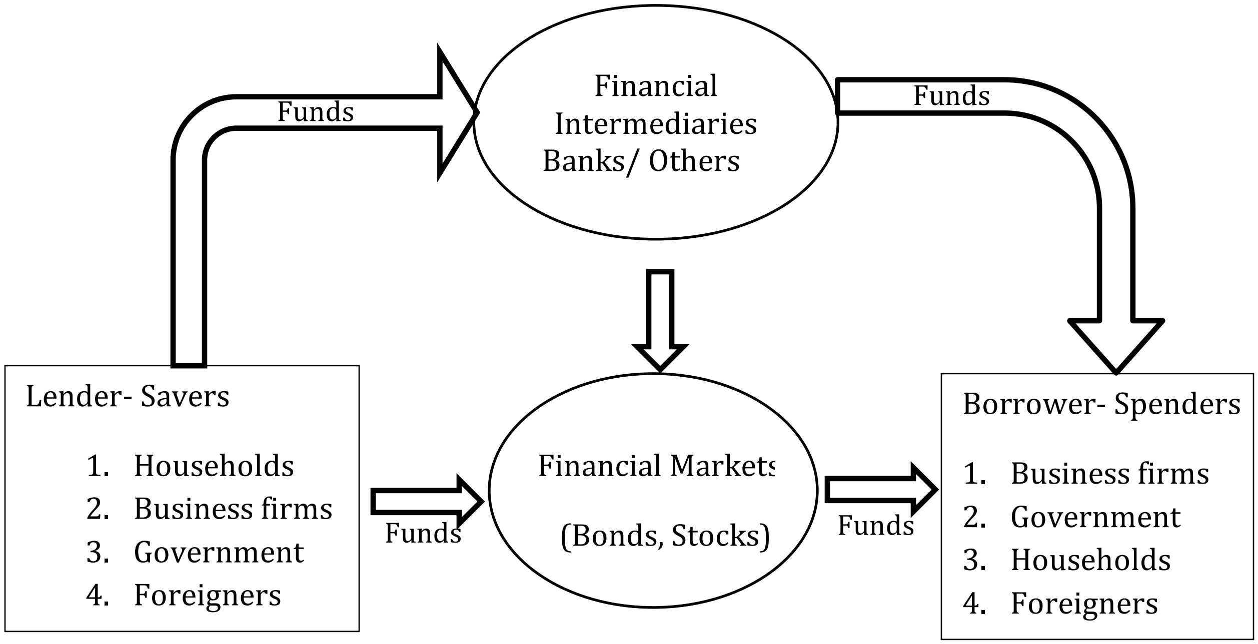

Figure 1.3: Flows of funds through the financial system

The activities in the entire financial system can be direct or indirect. From figure 1.3, those that borrow funds directly from lenders in the financial markets are involved in direct financing. On the other hand, those that borrow through the financial intermediaries are involved in indirect financing as shown in the figure 1.3.

In summary, borrowing through these mechanisms makes funds easily and readily available to the investors. This thus ensures higher production and economic efficiency. Once production efficiency is attained, consumption efficiency is stimulated and overall welfare is enhanced.

SELF-ASSESSMENT EXERCISE 1

Risk taken is significant to return in any business. Hence the higher the risk, the higher the return and vice versa. It should be noted however that engaging in any business depends on the availability of funds and these can be facilitated by the financial system. The extent of risk taken can be said to rely on how strong an economy's financial system is.

3.2 Basics of Mathematics

Mathematics as a discipline is a common method or approach to evaluate a real life situation. In most cases ideas are only turned into reality only when they can be expressed in simple and concise manner.

For instance, financial managers would often measure organizational performance by calculating fractions called financial ratios. Several financial ratios are used but depend on whether they are meant to calculate the organizational growth, level of profit, level of debt, inventory turnover, to mention but few.

In measuring the overall performance of an organization for instance, its level of productivity or efficiency rate is calculated. This involves finding the ratio of total input to total output:

$$P_T = \frac{\gamma}{\propto L + \beta K} \quad X\ 100$$

Where: P_T = rate of efficiency (Total factor productivity)

L & K = Total inputs (Labour & Capital respectively)

Y = Output.

$\propto +$ß = rate of efficiency of each input

$\propto +$ß > 0

The rate of efficiency will be high if the ratio of inputs used is less than the rate of output realized and vice-versa.

Again for a small business to continue existing from time to time, it depends on its inventory turnover ratio:

$$\text{Inventory turnover ratio} = \frac{\text{net sales}}{\text{average inventory}}$$

Where inventory is measured in total naira value at a particular point of sale. In an explicit but broad perspective the inventory turnover ratio becomes:

$$\text{Inventory turnover ratio} = \frac{\text{Gross sales – return and allowances}}{\frac{\text{Opening inventory + closing inventory}}{2}}$$

Thus the inventory turnover ratio measures how quickly the retailer' stocks in and out. The higher the ratio, the faster the turnover.

Significantly, financial analysis draws from known mathematical symbols to aid in its reasoning in dealing with financial problems. In doing this, geometrical methods and mathematical techniques like matrix algebra, differential and integral calculus, differential equations, difference equations etc. are frequently utilized to derive theoretical results. Apart from these techniques, some fundamental ingredients of mathematics that need to be familiar with are to wit: Variables, Constants and Parameters; Equations and identities; the real number system; the concept of sets; relations and functions and types of functions and graphs.

i) Variables, Constants and Parameters

A variable is anything whose magnitude or direction can change at any point in time. This means that any point in time, a variable can take on any value.

Variables that are frequently encountered in finance are shares, bonds, risk and return, interest, dividend etc. Other variables that are commonly known to everybody irrespective of profession or discipline are price, profit, revenue, cost, income, consumption, investment, imports and exports. All these are represented by symbols and they may assume values. For example, price can be represented by P_x, profit by π, revenue by R, cost by C and so forth. All these variables may however be freezed or restricted when they take on specific values for example, P=3, or π =10. Variable can be endogenous and exogenous. It is endogenous when it originates within the equation but exogenous when it originates outside the equation.

A constant is a magnitude that does not change but frequently appears as a fixed value with a variable. It is rather an antithesis of a variable. For example, if the cost of a project is given as 0.5C, it means that the project would always incurre a fixed value of 0.5. This however can be represented also as a symbol like $\propto$.

Hence, the cost of the project can be represented by $\propto C$. The symbol $\propto$ is a parametric constant.

A parameter is therefore a symbol used to represent a constant value attached to a variable. It does not change and therefore fixed. Parametric constants are usually represented by symbols a, b, c or their counterparts in Greek alphabets such as $\propto$, β, λ and Υ.

ii) Equations and Identities

An Equation is a mathematical statement setting two algebraic expressions equal to each other. This makes variables to be more interesting and meaningful once they are related to each other.

An equation can be definitional, behavioural or conditional in nature. A definitional equation sets up an identity between two alternate expressions that have exactly the same meaning. Instead of using strictly equals to sign (=), it uses identically equals to sign ($\cong$).

For example, $R \cong GPV - C$

Where R = Cash flow on a project/ return

GPV = Gross Present Value of a project

C = Initial cost of capital

The equation is thus referring to the total profit (return) as the excess of revenue (GPV) over the total cost (C).

A behavioural equation however specifies the manner in which a variable behaves in response to changes in other variables. It can be linear and non-linear in nature. It is linear when there is a one-one relationship between the variables but non-linear when the relationship is multifaceted such as in quadratic equation.

For example where $C = 60 + 5Q \quad (1)$

$C = 50 + bQ + cQ^2 = 0 \quad (2)$

Equation (1) is linear because it is raised to the first power while equation (2) is quadratic because it is raised to the second power.

Both equations (1) and (2) explain the behaviour of cost (C) with respect to the level of output (Q).

A conditional equation is premised on the need for certain requirements to be satisfied. For example a phenomenon that requires equilibrium condition, involves setting up an equation that reflects the equilibrium situation. This is for instance reflected in a phenomenon where:

$$S=I \qquad (3)$$

Equation (3) is a phenomenon that strictly states that intended saving(s) is equals to intended investment (I). This means that all that are saved are invested. Hence, the capital or money market (financial market) is cleared or at equilibrium. This is further graphically represented in figure 1.4 below:

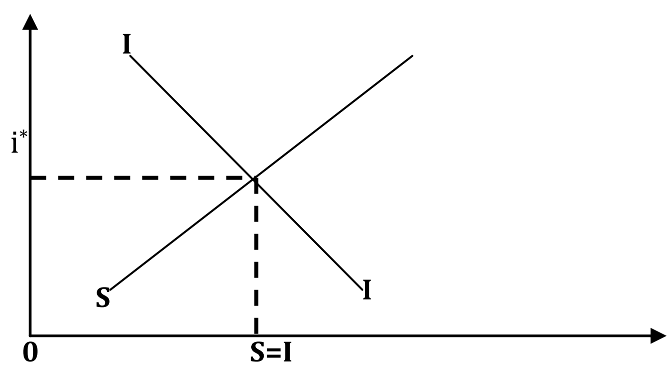

Figure 1.4: Equilibrium Savings and Investment

The clearance in the financial market arises out of the attainment of equilibrium interest rate (i^*) acceptable to the savers (lenders) and the investors (borrowers). Notice that while the saving is positively related to interest rate by sloping upward from left to right; investment is negatively related to interest rate by sloping downward from left to right. This implies that when the interest rate is high, savings will be high and vice versa. Likewise when the interest rate is high, investment will be low and vice-versa.

iii) Functions, Graphs, Slopes and Intercepts

A function is one of the most basic tools in all mathematics and was introduced into the study of calculus by Gottfried Wilhelm Leibniz.

It is referred to as a special type of relation that expresses how one quantity (the output) depends on another quantity (the input). Hence, a function (*f*) is a rule that assigns to each value of a variable (x) called the argument of the function, one and only one value (*f*(x)) referred to as the value of the function at x.

By so doing, a function is a rate that assigns to each input number exactly one output number. The set of all input numbers to which the rule applies is called the domain of the function. The set of all output numbers is called the range.

For example, when money is invested in a business at some interest rate, the interest I (output) depends on the length of time t (input) that the money is invested.

This means that I depend on t. This can be illustrated by simply using symbols such as I= f (t). Furthermore, suppose =N=100 earns simple interest at an annual rate of 6%, using the interest rate-time function,

$$I = 100(0.6)t \qquad (1)$$

where: I = Naira value of the investment

T= time frame expressed in years.

If the time frame is 6 months or half of the years,

i.e. If t =½ ; then I=100(0.6)(1/2) = 3 (2)

With the formula (1), the interest rate (output 3) is assigned to the length of time t (input 1/2), thus defining the rule: multiply t by 100(0.06). The rule assigns to each input number t exactly one output number I, which is symbolized by the following arrow notation: $t \rightarrow$ I or t $\rightarrow$ 100(0.06) t.

The domain is therefore a set of all non negative numbers – that is, all $t \geq 0$. So when the input is ½ , the output is 3, therefore the range is 3 as shown in the equation (2) above.

A function is therefore basically a correspondence whereby to each input number in the domain there is exactly one output number assigned in the range.
For the correspondence given by $f(x) = x^2$; figure 1.5 exhibits the relationship between domain and the range.

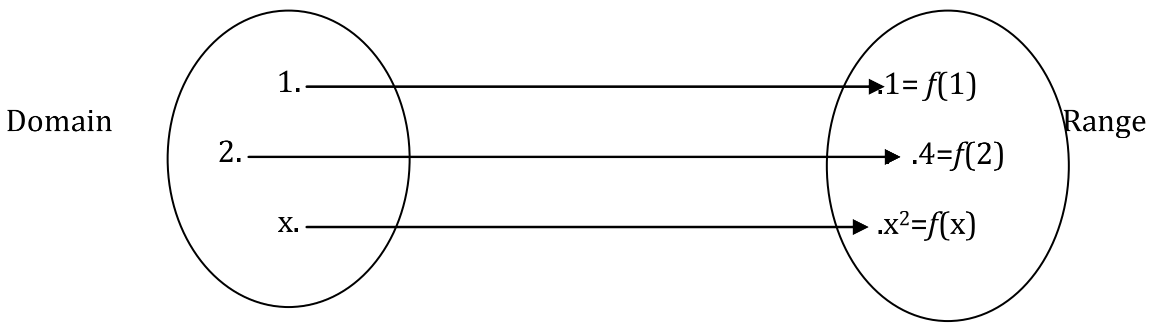

Figure 1.5: Functional Correspondence for $f(x) = x^2$

Functions normally encountered in finance are Linear and Quadratic function.

Linear Function : $f(x) = mx + b$

Quadratic Function : $f(x) = ax^2 + bx + c$ $(a \neq 0)$

However, other functions abound in economics are:

Polynomial function of degree n: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a0$

(n= nonnegative integer; $a_n \neq 0$)

Rational Function :

$$f(x) = \frac{g(h)}{h(x)}$$

where g(x) and h(x) are both polynomials and $h(x) \neq 0$. (Rational arises from ratio).

Power Function : $f(x) = ax^n$ (n= any real number).

A graph is a geometric means of representing equations in two variables as well as functions. For instance, a function such as $y = f(x)$ can be graphically be illustrated by placing x on the horizontal axis otherwise known as the independent variable with y placed on the vertical axis otherwise also known as the dependent variable.

The graph of a linear function is a straight line, for instance the risk / return trade-off is a linear relationship since a change in the risk taken leads to a change in the rate of return. (See figure 1.1).

The slope of a line which is indicated by "m" in the linear function $f(x) = mx + b$ or as the case may be in $y = f(x)$, measures the change in y (Δy)due to change in x (Δx). So for a line passing through the coordinate points (x, y) and (x_2, y_2), the slope "m" is computed as

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for all } x, \neq x_2$$

So given that y= -1/4 x + 3, if initially x= 0, then y= 3 and if later on y= 0, then ¼ x = 3, and x= 12. Therefore the line would pass through the coordinate (0, 3) and (12, 0). Therefore:

$$m = \frac{\Delta y}{\Delta x} = \frac{0 - 3}{12 - 0} = \frac{-3}{12} = -\tfrac{1}{4}$$

The slope thus indicates the steepness and direction of a line as shown in the figure 1.6:

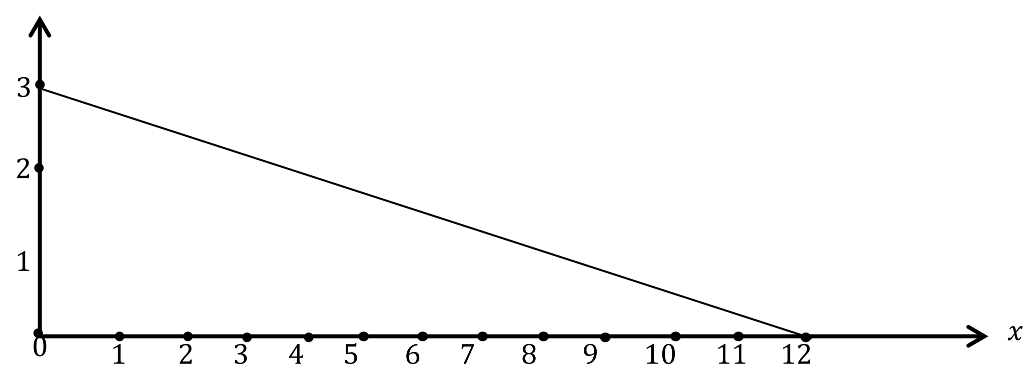

Figure 1.6: Graph of a linear equation

Arising from figure 1.6 are the intercepts of the equation y=$f(x)$ or $mx + b$. The y – intercept is the point where the graph crosses the y – axis which occurs when $x = 0$. The x– intercept is the point where the line intercepts the x -axis which occurs when y = 0.

From all indications, mathematical approach to finance has proved more advantageous in the sense that:

1. The "language" used is more concise and precise.
2. It provides copious of mathematical theorems for usage in analyzing financial and other business problems.
3. By ensuring an explicit statement of assumptions, it prevents the problem arising from an unintentional adoption of unwarranted implicit assumptions.
4. It gives room for treatment of several financial problems at a time otherwise known as n-variable case.
5. It assists in providing a logical approach to financial problems and speed up decision-making.

SELF-ASSESSMENT EXERCISE 2

The significance of mathematical tools in expressing the financial activities is over whelming. Mathematics simplifies complex situations of real life by being precise and straight to the point. It further reduces the time involved in expression by the use of functions and equations which are also depending on the use of dependent and independent variables.

3.3 Methodology of Finance and Mathematics

Going by the brief insights into what finance and mathematics are, it is expedient to critically look at the point of connection between the two disciplines. Hence this section treats the processes involved in the nexus between finance and mathematics.

It perhaps discusses the methods of reasoning in theoretical finance and mathematics and how they are connected. These methods thus help to establish the truth of the subject matter. There are two forms of this method – deductive and inductive.

The Deductive method is a form of reasoning or inference from the general to the particular or from the universal to the individual phenomenon.
Inductive method on the other hand is the process of reasoning from a part to the whole, from particulars to the whole or from the individual to the universal.

While the deduction involves the following steps- selection of problem, statement of assumptions, statement of hypothesis and testing and verification of hypothesis, the induction involves identification of the problem, data gathering and analysis, observation and generalization.

The inductive method is more applicable in the case of finance since it studies how an organization maximizes stockholders' wealth at the least risk. On this note, the stages involved in getting to the facts are diagrammatically expressed in figure 1.7 below:

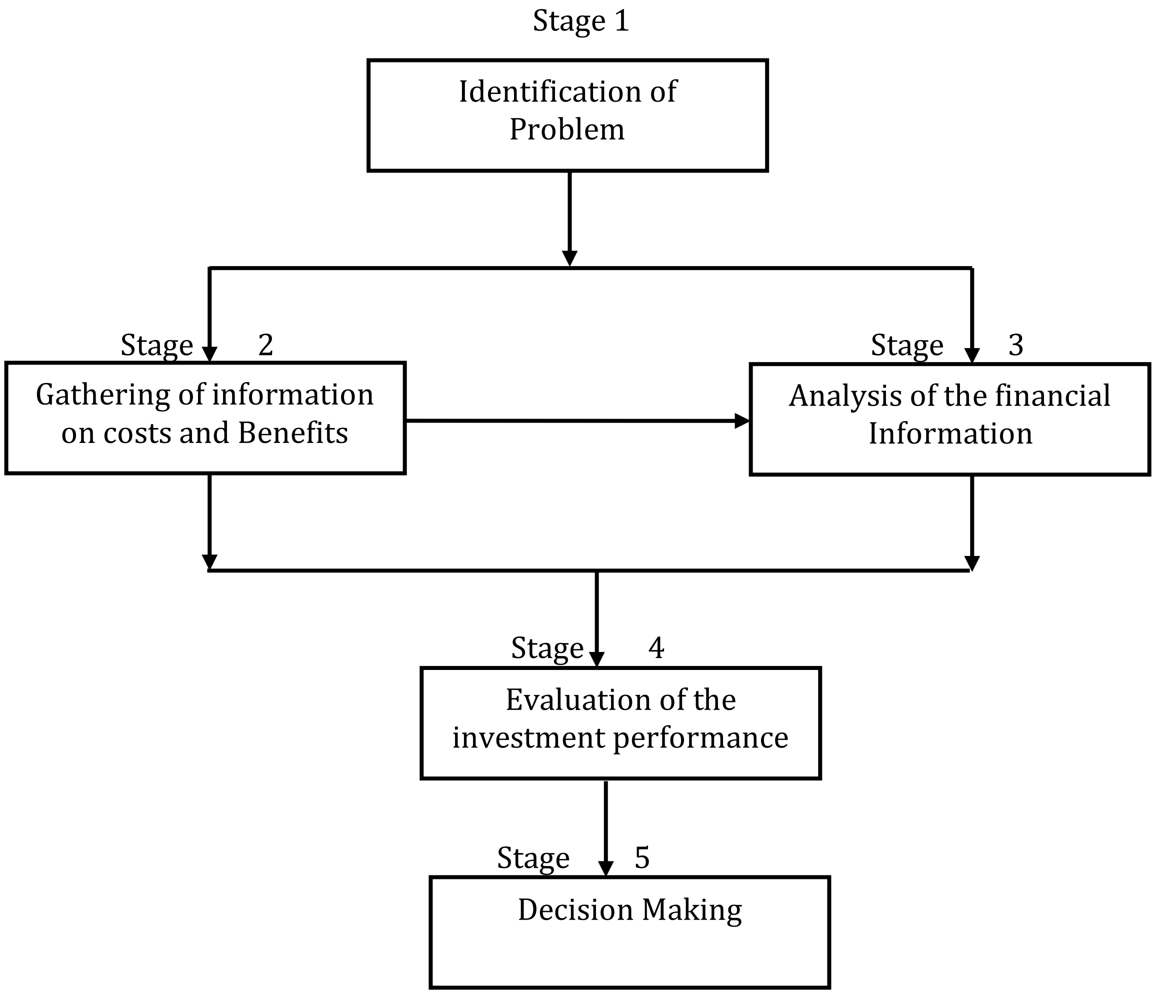

Figure 1.7: Methodology of finance and Mathematics

The major problem which many organizations that wish to stay in business always encounter is how to maximize stockholders' wealth. In doing so, selecting the best investment faces the challenge of how best to fund it. Should it be through debt or equity financing? It is not easy to determine the best option at this stage 1, except to gather information on costs and benefits of the investment. This process of stage 2 then enumerates, classifies and analyses the data collected using various statistical and financial techniques at stage 3.

Some of the statistical techniques that can be used are descriptive or inferential while financial techniques can include income statements, flow of funds account, and balance sheet to mention but few.

Stage 4 which is the evaluation of the investment performance is to determine the worth of the business. The techniques would include the use of discounted and non-discounted methods of evaluation. The discounted methods are Net present Value (NPV), Internal Rate of Return (IRR), Average Rate of Return (ARR), Profitability Index while the non-discounted methods include Payback period and cost and benefit ratio. Ratios like acid test, Earnings per share, Asset utilization ratios can also be used to evaluate the performance of an investment.

It is at stage 5, based on the evaluation that determines whether to invest or not depending on the outcome of the evaluation.

Notice that stages 2, 3 and 4 more or less involves calculation of values. Implicitly therefore, the methodology of finance tilts towards mathematical expression thus making it impossible to divorce mathematics from finance.
In fact, mathematics makes finance to be more meaningful and easily understood. This is more glaring in the development of mathematical models in financing.

4.0 CONCLUSION

The summary of what we have covered so far in this unit is highlighted in the next section below.

5.0 SUMMARY

In this unit we have been able to cover the following points:

1. Finance deals with how to maximize the overall value of a business vis-à-vis the maximization of shareholders' or stockholders' wealth.
2. The functions of finance depend on the operating environment through the financial system
3. There is a very strong nexus between mathematics and finance, thus making it possible to draw a methodology of understanding the mathematical modeling in finance.

4. The basic tools of mathematics like functions and equations, variables, constants and parameters have been used to explain finance in precise and cogent manner,
5. The focus of finance is also to identify risk and return of a business or project.

6.0 TUTOR-MARKED ASSIGNMENT

1. Explain the term risk/return trade off. Of what significance is the financial system to any developing country's economy?
2. Explain the pitfalls identified with mathematical approach to the study of finance.
3. If a principal of P Naira is invested at a simple annual interest rate of r for t years, express the total accumulated amount of the principal and interest as a function of t. Is your result a linear function of t?
4. The following table is a demand schedule for XYZ shares. It gives a correspondence between the price (p) of a unit and the quantity (q) that investors are willing to purchase at that price.
   (i) If P=$f(q)$, list the numbers in the domain of f, find $f(2900)$ *and* $f(3000)$.
   (ii) If $q = g(p)$, list the numbers in the domain of g, find $g(10)$ *and* $g(17)$.

Price/Unit (P)₦	Quantity Demanded / week (q)
10	3,000
12	2,900
17	2,300
20	2,000

5. Is finance a science or an Art? Discuss with reasonable examples as much as you can.
6. Why is finance and mathematics viewed from the perspective of inductive reasoning?

7.0 REFERENCES / FURTHER READINGS

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UNIT 2 MODELLING IN FINANCE AND MATHEMATICS

CONTENTS

1.0 INTRODUCTION

In this unit we shall be making use of the various mathematical tools and the basic knowledge of finance discussed in Unit 1 to explain what a model. Further to this, the basic types of model are identified and explained using financial concepts. You should however notice that since a model is an abstraction of a real life situation, it relies heavily on mathematical tools for its expression through the use of functions and equations. This is shown in the various examples that are show cased within this block.

2.0 OBJECTIVES

After reading through this unit, you should be able to:

- Understand what a model is all about
- How a model is constructed from a real life phenomenon
- The basic types of modeling
- Identify the relevance and the need for modeling in finance

3.1 Model – Concept and Meaning

A model is a representation of a real life situation. It is an approach to simplify the complexities involved in human endeavours. Perhaps, it provides a means to an end whereby one moves from generalization to particular. For instance, the immense complexities in the real financial system make it impossible to understand all the interrelationships at once. Hence, it is essential to divide the entire system into segments like financial markets and financial intermediaries.

This procedure as sensible as it is would make it possible to pick out what appeals to our reason to be the primary factors and relationships relevant to our problem and to focus our attention on these alone.

This deliberate simplified analytical framework is referred to as a model as it is only a societal and rough representation of the actual financial system.

A model is thus an abstraction from a real life situation. Models are therefore used to weed away unnecessary detail and reduce the complexity of reality. In a simple two-variable model, the variable to be explained is connected to one believed to be largely responsible for its behaviour. For instance, $I=f(i)$ as a model implies that any change in interest rate (i) will bring about a change in investment (I).

So many factors that could also influence the behaviour of investment apart from interest rate like profit, marginal efficiency of capital, government policy etc. have all been suppressed. This invariably reduces the complexity that may arise and more so, to actually show case the significant influence of interest (cost of capital) on investment.

In another sense, a model can be regarded as a representation of a theory or a part of a theory, most at times used to gain insight into cause and effect. This is because theory is simply a generalization or abstraction of experience and observation.

The term “model” is well known to children, economists, scientists and all alike who use the term in much of the some way that children do.

A child's model automobile or airplane looks and operates much like the real thing, but it is much smaller and much simpler, and so it is easier to manipulate and understood. It can therefore be deducted that in every model, there will always be endogenous and exogenous variables. Hence, models display how a change in an exogenous variable affects the value of the endogenous variable.

3.2 Types of Model

Because of the fact that modeling is familiar to virtually all professions, it can generally be classified into three main categories: Iconic model, Analogue model and Hybrid model.

i) An Iconic Model is a physical representation of a real life situation. For instance, a toy car is an example of an iconic model. Though it is compared to the real car but it is also operated virtually in the same way. Another example that is readily comes to mind is the architectural design of a building usually show cased at the entrance of the building. The construction looks exactly like the building you are about entering and it shows all the various parts and ways into the building. Furthermore, a baby doll is an iconic model of a girl.

Analogue Model is a diagrammatic or graphical representation of an event. For example, the relationship between demand and supply of commodity at a given price can be represented in diagrammatic form given their schedules. This is more glaring if there is an equilibrium between the two variables as shown in figure 1.8

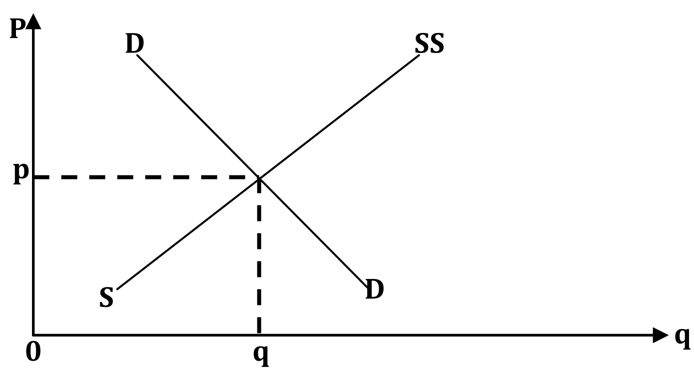

Figure 1.8: A Linear Partial Market Equilibrium Model

The figure 1.8 exhibits the attainment of partial equilibrium in the market where there exists a consensus price and quantity upon which the market is cleared.

Another example of an analogue model is the response of investors to fluctuations in the share prices. The values of shares depend on daily trading at the stock exchange. Hence, the values are subject to change from time to time and as such this can be captured with the use of graph. The graph will show its peak, rising level and the falling level. This can be seen in form of the volatility of returns for a company as exhibited in figure 1.9 below:

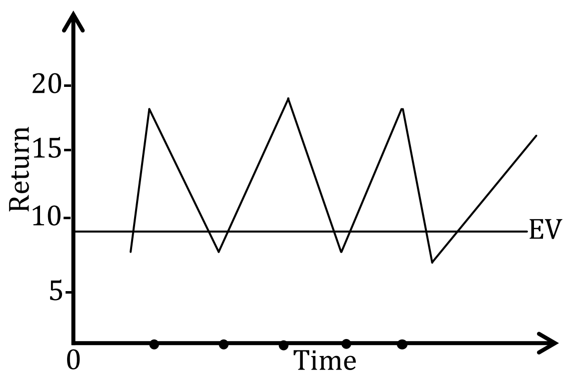

Figure 1:9: Volatility of Returns of Company XYZ.

With the graphical illustration of figure 1.9, it is easy to see from a far that the returns on the investment of company XYZ have not been stable overtime. Thus analogue model aids quick glance of a phenomenon.

ii) Hybrid Model on the other hand is a combination of iconic and analogue models. In most cases, it uses mathematical approach in its analysis. In fact, it is the analogue combined with the use of mathematics that is commonly used in financial analysis and economics. The reason is because, most of the relationships among several variables are expressed in set of equations designed to describe the structure of the model. Abinitio, the model is based on sets of assumptions which make application of relevant mathematical operations to these equations inevitable. This eventually enhances the possibility of deriving a logical conclusion.

It is due to this role of mathematics that the third type of model is sometimes referred to as mathematical model. A mathematical model is thus a quantitative representation of a reality. These models are often used for decision making in all areas of business. It thus gives room for simulation or "what-if" analysis. For example, the mathematical formula or equation can be used to compute the breakeven point for a business:

$$X_b = \frac{FC}{(P-V)}$$

Where: X_b= break-even point

P= Price or average revenue per unit

V= Unit Variable Cost

FC= Total Fixed Cost

Given that NOUN is considering its privatization by offering a unit of its share for ₦250. Variable costs are ₦50 per unit. Total fixed costs per year are ₦650,000. What is the break even number of shares that should be subscribed for? Using the formula:

$$X_b = \frac{FC}{(P-V)} = \frac{650,000}{250-50} = 3,250 \text{ Shares}$$

In general, this mathematical modeling also extends to statistical techniques used for forecasting demand and production planning. It is equally adopted in the optimization techniques such as linear programming used for providing the best possible solution to a problem at hand.

SELF ASSESSMENT EXERCISE

A model is a physical representation of a real life situation. For a model to be meaningful and be accurate, it should be able to gather information concerning the phenomenon in question. Apart from this the phenomenon should be critically observed such that if the model is not real, it should be nearer to being real. Examples abound in the equilibrium situation of a risk taker and the volatility that follows the stock market.

3.0 CONCLUSION

This unit will now be ended by briefly summarizing what we have been able to cover.

4.0 SUMMARY

In this unit we have covered the following:

1. A model represents the real phenomenon and that it relies on gathering of information through collection of data and observation.
2. A model helps in converting complex situation into a simplified and meaningful concept.
3. Models are used for decision making in all spheres of businesses.
4. Three major types of model often come in contact with are the iconic, analogue and mathematical modeling.

6.0 TUTOR-MARKED ASSIGNMENT

1. List and explain the types of models that are useful to operations managers.
2. Given that P=N3, AVC (Average Variable Cost) = N1.80 and the TFC (Total Fixed Cost) = N60,000, (i) What is the breakeven level of output? (ii) Graphically illustrate your answer.

7.0 REFERENCES / FURTHER READINGS

Dowling, E. T (1992). *Shaum's Outlines of Theory and Problems to Mathematical Economics*. New York: McGraw – Hill Publishers

Haeussler, E. F. Jr:, Paul, R. S and Wood, R (2000). *Introductory Mathematical Analysis for Business Economics and the Life and Social Sciences.* Boston: Pearson / Custain Publishing.

MODULE 2

Unit 1 Differentiation

Unit 2 Integration

UNIT 1 DIFFERENTIATION

CONTENTS

1.0 INTRODUCTION

You will recall that in the Module 1 the basics of Mathematics and Finance were extensively discussed. In this unit we shall be following up on how best to use these basics especially by treating the extent of the relationship among the various variables and functions earlier identified. The effects of these tools on finance are thus presented in this unit.

2.0 OBJECTIVES

After reading this unit, you should be able to:

- Understand what a derivative of a function is.
- Familiarize with the various rules of differentiations
- Identify the various types of differentiation
- How differentiation is applicable to decision making processes in businesses

3.0 MAIN CONTENT

3.1 Derivative of a function

The derivative of a function is the gateway to the understanding of differentiation calculus. It is often described as a painless way to understanding what a differentiation is. Using the slope of a straight line as an example, the slope or gradient of a line is the change in y (vertical axis) divided by the corresponding change in x (horizontal axis) as we move between any two points on the line.

Generally it is written as:

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Given for instance that the slope of a line passes through A (1,2) and B(3,4); using the formula above,

$$m = \frac{\Delta y}{\Delta x} = \frac{4-2}{3-1} = \frac{2}{2} = 1$$

This means that as we more from A to B, the y coordinate changes from 2 to 4 which is an increase in 2 units brought about by the movement of x from 1 to 3 which is also an increase of 2 units. Hence, the change in x, brought about a change in y as exhibited in figure 2.1

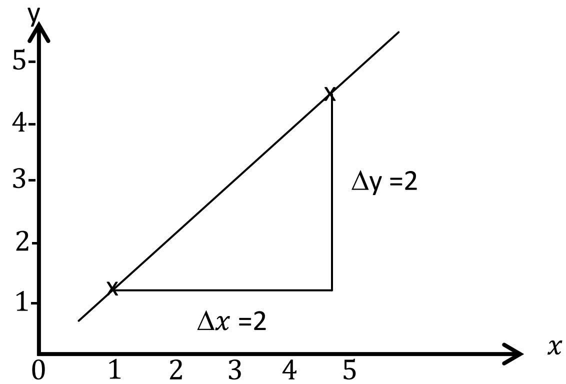

Figure 2.1: Graphical Illustration of a gradient.

It should be noted from the figure 2.1 above that the derivative is a derived function. For instance, given that $y = f(x)$, the $f'(x)$ which is derived from the function is also a function of (x). This implies that for any value of x, there is a unique corresponding value for the derivative function $\frac{dy}{dx}$ or $\frac{\Delta y}{\Delta x}$.

The slope of the graph in figure 2.1 of a function is called the derivative of the function. In another sense, the rule f' and the slope of the graph of f at x defines a function. It should be noted however that it is not always every time that functions are linear in nature. Hence, it becomes imperative to extend the definition of the slope to include more general curves through the use of a tangent as illustrated in figure 2.2.

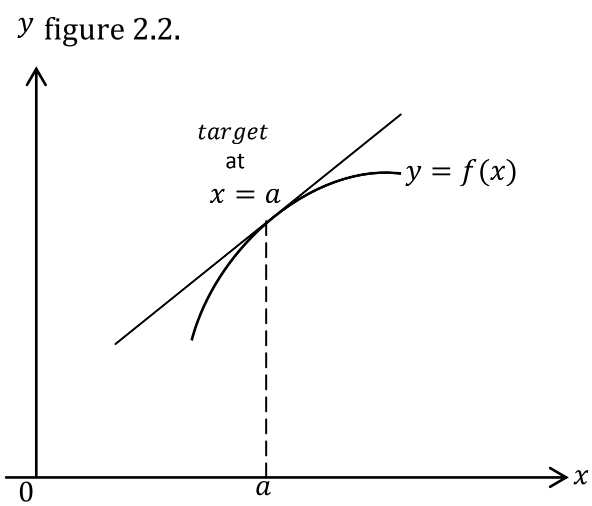

Figure 2:2: Graphical illustration of a Tangent

Figure 2:2 illustrates that a straight line that passes through a point on the above and just touches the curve at a point, "a" is called a tangent. Hence, the slope, or gradient of a curve at "$x = a$" is thus defined to be that of the tangent at "$x = a$".

Example 1:

Complete the following table of function values and hence sketch an accurate graph of $f(x) = x^2$.

x	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0
$f(x)$									

Draw the tangents to the graph at x =-1.5, -0.5, 0,0.5 and 1.5. Hence estimate the values of $f'(-1.5), f'(-0.5), f'(0), f'(0.5), and\ f'(1.5)$

Answer to example 1:

x	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0	1.5	2.0
$f(x)$	4	2.25	1	0.25	0	0.25	1	2.25	4

Graphing the square function in figure 2.3

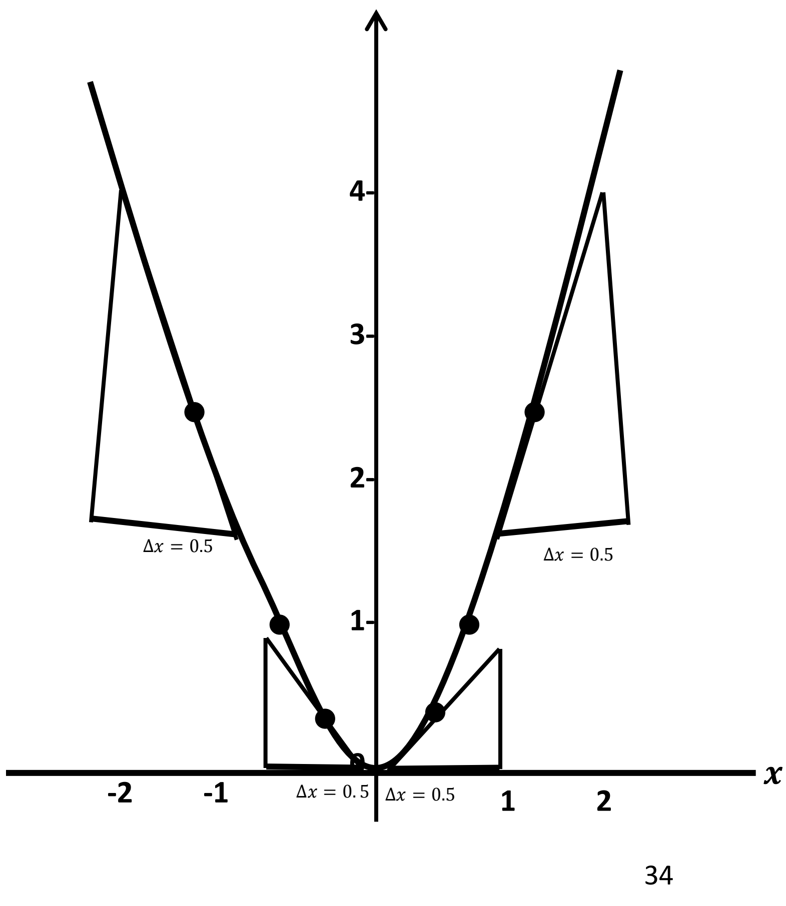

Figure 2.3: Graphical Illustration of Square Function

The slopes of the tangents are:

$f'(-1.5) = \frac{-1.5}{0.5} = -3$

$f'(-0.5) = \frac{-0.5}{0.5} = -1$

$f'(0) = 0$

$f'(0.5) = \frac{0.5}{0.5} = 1$

$f'(1.5) = \frac{1.5}{0.5} = 3$

Note that the slopes of the tangents to the left of the y axis have the same size as those of the corresponding tangents to the right. However, they have opposite signs since the curve slopes downhill on one side and uphill on the other.

SELF ASSESMENT EXERCISE 1

a. Given the function $y = 4x^2 + 9$

(i) Find the derivative $\frac{dy}{dx}$s

(ii) Find $f'(3)$ and $f'(4)$

(iii) Find $f'(2)$ and $f'(3)$

3.2 Differentiation- Meaning, Types and Rules

The term differentiation is based on the derivation of a function such as $y = f(x)$. Hence, differentiation is a process of finding the derivative of a function. The process of obtaining the derivative $\frac{dy}{dx}$ is premised on the concept of limit theorems. For instance, the $f'(x0)$ exists if and only if the limit of $\frac{\Delta y}{\Delta x}$ exists at x= xo as $\Delta x \to 0$, as expressed in: $f'(xo)$=$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$$\cong \lim_{\Delta x \to 0} \frac{f(xo + \Delta x) - f(xo)}{\Delta x} \qquad 2.1$$

Despite the fact that the procedure of taking limit is essential to differentiation as shown in equation 2.1, it is stressful, tedious and time consuming.

Based on these problems, special rules of differentiation have been put in place to fast track the process of obtaining desired derivatives of explicit functions.

On this note, the process of finding the derived function symbolically (rather than using graphs such figures 2.1 and 2.2) is known as differentiation with the applications of its rules. Apart from the usual $y = f(x)$, we are familiar with, other common functions are $g(x)$ and $h(x)$ where g and h are both unspecified functions of x.

There are about seven basic rules of differentiation of which only few of them will be applicable in the study of finance.

Rule 1: The Constant Rule

The derivative of a constant given that $f(x)$=c, where c is a constant and equals to zero.

That is, $f(x)$=c, $f'(x)$=0,

Example 2:

Given that $f(x)$=8, $f'(x)$=0

However this can be extended to the derivative of a constant multiple of a function such as:

$h(x) = cf(x)$, then, $h'(x) = cf'(x)$.

The process involved is to differentiate the function and multiply by the constant.

Example 3:

Differentiate (i) $y = 3x^4$; (ii) $y = 3x$

Answer:

(i) $y = 3x^4$, then $\frac{dy}{dx}$= $3(4x^{4-1}) = 3(4x^3)$

∴ $\frac{dy}{dx}$ = $12x^3$,

(ii) $y = 5x$, then $\frac{dy}{dx}$ = $5(x^{1-1})$= $5(x^0)$

Since any element raised to the power of zero is equal to one, hence $x^0 = 1$, and $\frac{dy}{dx}$= 5.

In a nutshell, constants differentiate to zero.

Rule 1: The sum rule

The derivative of the sum of two differentiable functions is the sum of the derivatives of the two functions.

If $h(x) = f(x) + g(x)$

Hence, $h'(x) = f'(x) + g'(x)$

This rule thus informs us how the derivatives of the sum of two functions are arrived at as illustrated above. Hence, each function is differentiated separately and latter added together.

Example 4:

Differentiate: (i) $y = x^2 + x^{50}$, (ii) $y = x^3 + 3$

Answer: (i) $y = x^2 + x^{50}, then\ \frac{dy}{dx} = 2x + 50x^{49}$

(ii) $y = x^3 + 3, then\ \frac{dy}{dx} = 3x^2 + 0 = 3x^2$

Rule 3: The Difference Rule

This rule is meant to demonstrate how to find the derivative of the difference of two functions. This is more or else the opposite of the sum rule since it is also just differentiating each function separately and then subtract each from each other.

If $h(x) = f(x) - g(x), then\ h'(x) = f'(x) - g'(x)$

Example 5:

Differentiate (i) $y = x^3 - 2x^2$, (ii) $y = e^{2x} - x$

Answer:

(i)$y = x^3 - 2x^2, then\ \frac{dy}{dx} = 3x^2 - 4x$

(ii)$y = e^{2x} - x, then\ \frac{dy}{dx} = 2e^{2x} - 1$

SELF ASSESSMENT EXERCISE 2

(a) Find $\frac{dy}{dx}$ if (i) $y = x^3 - 2x^2 + 5$; (ii) $y = (2x - a)^3$ (iii) $y = (x^2 - 4)^4$

(b) Differentiate $y = 2x^4 + 12x^3 - 7x - 400$

These rules have been based on using two variables, $y = f(x)$, but in reality, there can be many other symbols that can be used. For example, given a supply function:

$Q = P^2 + 3P + 1$

${}^{dQ}/_{dP} = 2P + 3$

In conclusion, differentiation is just a mechanical process that depends on the layout of the function and not on the labels used in identifying the variables.

3.2.1 Some Extensions of Rules of Differentiation

Rule 4: The Chain Rule

The chain rule is meant to differentiate a function of a function. For instance, given that y is a function of u *and* u in turn is a function x, that is $y = f(u)$ and $u = g(x)$, then = $y = f[g(x)]$

In this case, the derivative of y with respect to x is equal to the derivative of the first function with respect to u multiplied by the derivative of the second function with respect to x:

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

Example 6:

(i) Given that $y = (2x + 3)^{10}$

$y = u^{10}$ and $u = (2x + 3)$

$\frac{dy}{du} = 10u^9 = 10(2x + 3)^9$

$\frac{du}{dx} = 2; \frac{dy}{du} = 10; \frac{dy}{dx} = \frac{dy}{du} x \ \frac{du}{dx}$

∴ $\frac{dy}{du} = 10(2\,x + 3)^9 x\ 2 = 10(2x + 3)^9\,(2)$

Hence $\frac{dy}{du} = 20(2x + 3)^9$

(ii) Consider the function $y = (4x^2 + 3)^4$

$y = u^4; u = (4x^2 + 3)$

$\frac{dy}{du} = 4u^3; \frac{du}{dx} = 8x$

∴$\frac{dy}{dx} = 4(4x^2 + 3)(8x)$

Hence, $\frac{dy}{dx} = 4u^3 . 8x = 32xu^3$

∴ $\frac{dy}{dx} = 32x(4x^2 + 3)^3$

Rule 5: The Product Rule

This involves the differentiation of the product of two functions by multiplying each function by the derivative of the other and add.

If $y = uv, then\ \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Example 7:

Differentiate (i) $y = x^2(2x + 1)^3$

(ii) $y = x^4e^{2x}$

Answer:

(i) The function $x^2(2x + 1)^3$ involves the product of two simple functions, namely x^2 and $(2x + 1)^3$, while $(u = x^2, v = (2x + 1)^3$. It may be expressed the other way round i.e. v = x^2 and $u = (2x+1)^3$. It will give the same answer.

Hence:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

= $x^2\{6(2x+1)^2\}$ +$(2x+1)^3(2x)$

$\frac{dy}{dx} = 2x(2x+1)^2\{3x + (2x+1)\}$

= $2x(2x+1)^2(5x+1)$

(ii) $U = x^4$ *and* $V = e^{2x}$ for which

$\frac{du}{dx} = 4x^3$ and $\frac{dv}{dx} = 2e^{2x}$

Using the product rule;

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^4\,(2e^{2x}) + e^{2x}\,(4x^3)$$

$= 2x^4\,e^{2x} + 4x^3\,e^{2x}$

$= 2e^{2x}\,e^{2x}\,(x^4 + 2x^3)$

Rule 6: The Quotient Rule:

This rule involves how to differentiate the quotient of two functions, that is:

$y = {}^{u}/_{v}$, then $\frac{dy}{dx}$ = $\frac{\underline{vdu/dx - udv/dx}}{V^2}$

Example 8:

Differentiate y = ${}^{x}/_{1+x}$

Answer:

Here $u = x$, and $v = 1 + x$ for which $\frac{du}{dx}$ = 1, $\frac{dv}{dx}$ = 1

By quotient rule:

$$\frac{du}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$$

$$\frac{dy}{dx} = \frac{(1+x)(1) - x(1)}{(1+x)^2}$$

$$= \frac{1 + x - x}{(1+x)^2}$$

(.) $\quad = \quad \dfrac{1}{(1+x)^2}$

SELF ASSESSMENT EXERCISE 3

a. Use the chain rule to differentiate $y = (3x^2 - 5x + 4)^2$

b. Use the product rule to differentiate $y = x/1 + x$

3.3 Applications of differentiations in businesses

The significance of differentiation in finance is to some extent limited. This is because the subject matter – finance deals with actual data on sources and uses of fund. But be as it may, differentiation can still be used as guiding principles, upon which a successful business can be based. For it is more useful for a business to consider the economic efficiency of its decision especially those concerning its revenue and cost as well as when to produce and when not to produce. Hence, this section will briefly discuss revenue and cost.

Revenue and Cost Relationship

This involves the relationship that exists between the two concepts. The two concepts cannot be separated from each other since it is just as good as discussing the relationship between benefits (return) and the cost involved in generating the return. On this note at what point will an investor breaks even?

This is possible whenever the extra return earned for increasing additional subscription for shares or bonds exactly equals the extra cost incurred in bringing about this transaction.

Going by this explanation, the properties of revenue and cost would need to be defined. The total revenue (return) TR is defined as the multiplication of price of bond or share (P) by the unit subscribed for (Q), that is PQ.
PQ thus gives rise to the total return and which can also be regarded as an equation of subscription for bond or share.

So if $P = 50 - 3Q$
Then $TR = PQ$
$= (50 - 3Q)Q$
$= 50Q - 3Q^2$
Hence, this formula can be used to calculate the value of TR corresponding to any value of Q. However it is not enough to find the value but also to determine the effect of a change in the value of Q on TR. On the basis of this, there is need to incorporate the concept of marginal return.
Marginal return is therefore the derivative of total revenue (return) with respect to subscriptions, that is:

$$MR = \frac{d(TR)}{dQ}$$

For example the MR function coreesponding to:
$TR = 50Q - 3Q^2$ is given by

$$= \frac{d(TR)}{dQ} = 50 - 6Q$$

So if the current subscription for bond or share is 5, then:
$MR = 50 - 6(5) = 20.$

Example 9:

If the total benefit function of a bond is given by $100Q - Q^2$, write down an expression for the marginal return function. If the current subscription is 60 estimate the change in the value of total benefit due to a two unit increase in Q.

Answer:

If $TR = 100 - Q^2$

Then $MR = \frac{d(TR)}{dQ}$

M $= 100 - 2Q$

So when $Q = 60, MR = 100 - 2(60) = -20$.

If Q increases by two units, $\Delta Q = 2$ and the formula $\Delta(TR) = MR\ x\ \Delta Q$, indicates that change in total return is approximately $(-20)x\ 2 = -40$.

This means that a unit increase in Q therefore leads to a decrease in TR of about 40.

Notice that the marginal return (revenue) can assumes both negative and positive values due to the fact that the total revenue (return) function is quadratic and its graph is the familiar parabolic shape as shown in figure 2.4.

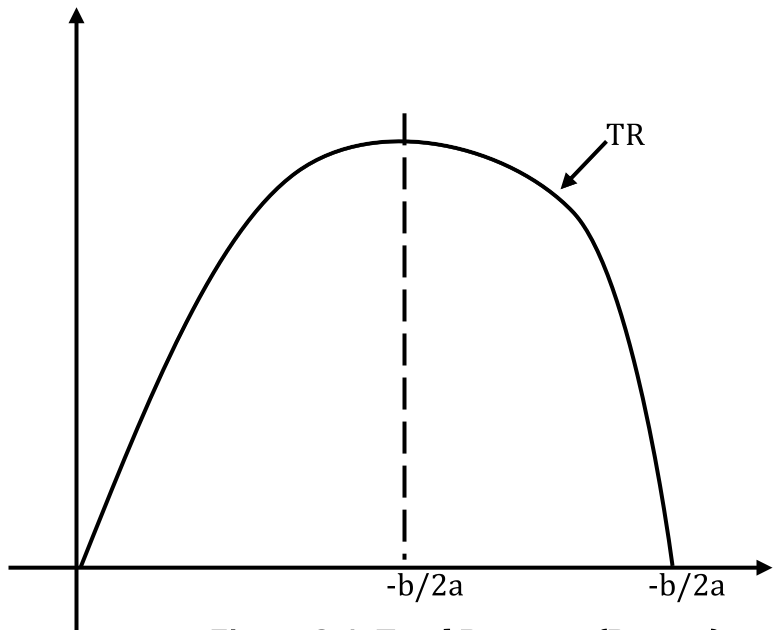

Figure 2:4: Total Revenue (Return)

Figure 2.4 explains that when the graph is uphill, corresponding to a positive value of marginal revenue (return) but downhill to the right of this point gives a negative value of marginal revenue (return). But at the peak point of the TR Curve, the tangent is horizontal with zero slope and so, MR is zero.

Analogous to our discussion is the extra cost incurred for putting up bonds/shares for subscription as earlier pointed out. So therefore marginal cost (MC) which is the derivative of total cost (TC) with respect to volume/unit of bonds/ shares is:

MC = $\frac{d\,(TC)}{dQ}$

So if the TC = $2Q^2 + 6Q + 13$, using the formula MC = $\frac{d\,(TC)}{dQ}$; MC = 4Q + 6Q.

So if Q = 15, MC = 4(15) + 6 = 66

Finally therefore, an investor as earlier pointed out will breakeven whenever MC = MR in the stock exchange market.

Example 10:

Given the total cost incurred by an investor for putting up shares for subscription as:

$TC = 120q - q^2 + 0.02q^3$

and the volume of subscriptions as:

$P = 114 - 0.25q$,

(i) Obtain the marginal cost and marginal revenue (return) functions

(ii) At what levels of volume is MC = MR?

Answer:

(i) Using the MC = $\frac{d\ (TC)}{dQ}$

$$= 120 - 2q + 0.06q^2$$

For MR = $\frac{d\ (TR)}{dQ}$; where TR = PQ

:. TR= $(114 - 0.25q)q = 114q - 0.025q^2$

Hence $MR = 114 - 0.5q$

(ii) To find the volume of breakeven point for the investor, $MC = MR$, therefore:

$$120 - 2q + 0.6q^2 = 114 - 0.5q$$

$$120 - 114 - 2q + 0.5q + 0.06q^2$$

$$6 - 1.5q + 0.06q^2$$

Using the quadratic formula, the volume of bonds/shares (q) is:

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where a =0.06, b=-1.5, c=6 from MC=MR

$$\frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(0.06)(6)}}{2(0.06)}$$

$$\frac{1.5 \pm \sqrt{2.25 - 1.44}}{0.12}$$

$$q = \frac{1.5 \pm \sqrt{0.81}}{0.12} = \frac{1.5 \pm 0.9}{0.12}$$

$\therefore$ $$q = \frac{2.4}{0.12} \text{ or } = \frac{0.6}{0.2}$$

Hence q= 20 or 3. The highest volume is thus picked as the breakeven point.

4.0 CONCLUSION

We will now summarize what we have covered in the above unit.

5.0 SUMMARY

In this unit, the following have been discussed:

1. The types of differentiation
2. The rules of differentiation
3. Some special cases of differentiation and their rules
4. Usefulness of Calculus to businesses.

6.0 TUTOR-MARKED ASSIGNMENT

1. Explain the concepts of total and marginal revenues. How are these applicable to finance?
2. Given the following total revenue (return) and total cost functions for different firms,

   $TR = 1400Q - 6Q^2$

   $TC = 1500 - 80Q$

   (i) Obtain the marginal cost and marginal revenue (return) functions.

   (ii) Hence or otherwise, determine the volume of subscriptions that will make the firms to break even.
3. Complete the following table of function values and hence sketch an accurate graph of $f(x) = x^3$

x	-1.50	-1.25	-1.00	0.75	-0.50	-0.75	0.00
$f(x)$		-1.95			-0.13		

x	0.25	0.50	0.75	1.00	1.25		1.50
$f(x)$			0.13			0.95	

Draw the tangents to the graph at x =-1.0, and 1. Hence estimate the values of $f'(-1), f'(0) and\ f'(1)$

7.0 REFERENCES / FURTHER READING

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UNIT 2 INTEGRATION

CONTENTS

1.0 INTRODUCTION

Recall that in the unit 1 we have been able to study how to differentiate functions and its applications. But how do we get back to original functions before differentiation? This can be achieved through the use of integration. Hence we shall be covering integral calculus in this unit 2 of Module 2.

Integral calculus is therefore the reverse of differentiation which involves just the determination of a function when its derivative is given. It is nothing but anti-differentiation. In other words, integration is the opposite process of differentiation. With integration, it is possible to retrace back from any given marginal revenue (return) function to total revenue (return) function.

2.0 OBJECTIVES

At the end of this unit you should be able to:

- Understand how to reverse from differentiation back to the original function
- Identify the various rules of integration
- Distinguish between integration and differentiation
- Apply integration in the decision making processes in businesses

3.0 MAIN CONTENT

3.1 Types of integration

There are two types of integral calculus namely indefinite integration and definite integration.

The definite integration is very useful in economics and finance. This is because it is used to find the area under the graph of a function which can be applied to supply and demand in order to calculate the producer's and consumer's surpluses. Also, it is used to determine capital stock and to discount a continuous revenue stream. The inverse of differentiation is therefore called integration.

Given an ordinary function $y = f(\text{x})$, the derivative is $\frac{dy}{dx} = f'(x)$. Further if $y = 2x^2$, $dy/dx = 4x$. From this derivative, $dy/dx = 4x$, we can derive the integration. Hence using "∫" notation of integration,

$\int f'(x)dx = f(x) + C$

$\therefore \int 4xdx = 2x^2 + C$

In all, $\int f(x)dx = f(x) + C$ means that "the indefinite integral of f of x with respect to x". The $\int$ symbol is an integral sign, $f(x)$ is the integrand, and c is the constant of integration, the arbitrary constant must always be added. "In effect, the most general equation leading to $dy/dx = 2x$ on differentiation is $y = x^2 + C$, which represents a family of curves parallel to $y = x^2$ as shown in figure 2.5.

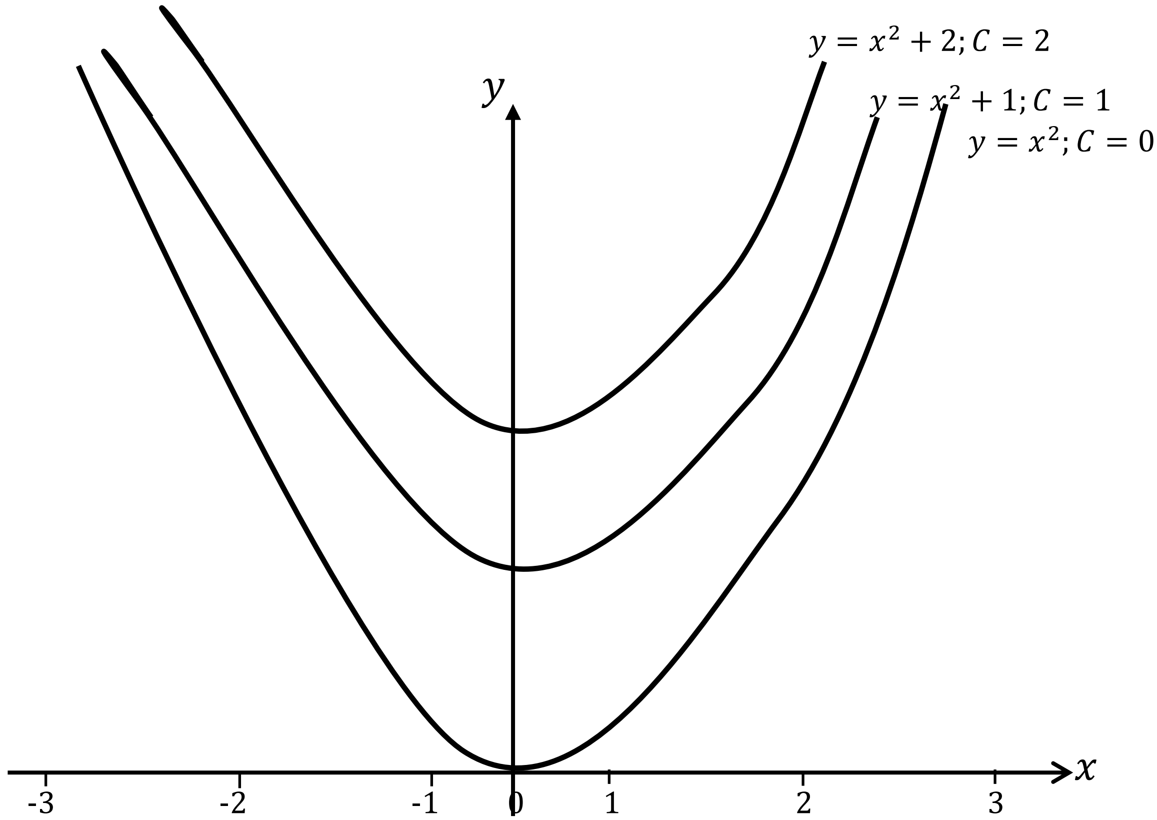

Figure 2.5 Family Curves of Y = X²

Example 1:

Find the curve whose derivative is $3x^2$, i.e. $f(x) = 3x^2$.

Answer:

$y = \int 3x^2 dx = x^3 + C$

Or $\frac{3}{3}x^{2+1} + C = x^3 + C$

Example 2:

Find the function which differentiates to $f(x) = x^7$

Answer:

$y = \int f(x)dx = \int x^7 dx$

Using $\frac{8x^{7+1}}{8} = x^7$

:. $y(x) = 1/8x^8$

3.2 Rules of Integration

The rules that govern differentiation are also applicable to integration except that they are also reversed. It is easy to check for their accuracy since the derivative of the integral must equal the integrand.

Rule 1: The Power Rule

The integral of a power function x^n, whenever $n \neq -1$ is given by the power rule as:

$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \qquad n \neq -1$

From this, you can integrate a power function by simply adding one to the power and divide by the number you get as the coefficient. This formula holds for all functions whenever a is positive, negative, or a whole number or a fraction. But whenever $n = -1$, such as $1/x$, since it is impossible to divide by zero, the formula cannot be used. Hence to integrate $1/x$, you use the natural logarithm that is:

$\int 1/x \, dx = ln\, x + C \qquad x > 0$

Rule 2: Integral of an exponential Function

Given that $\int em^x dx$, to differentiate an exponential you only need to multiply by the coefficient of x. To integrate you do exactly the opposite and divide by the coefficient of x, hence:

$\int e^{mx} dx = \frac{1}{m} e^{mx} + C$

Therefore for easy checking if $f(x) = 1/m\, e^{mx}$

Then $f'(x) = \frac{m}{m} e^{mx} = e^{mx}$

Rule 3: The integral of a Constant times a function equals the constant multiplied by the integral of the function.

Rule 4: Sum or difference Rule

The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$\int [(fx) + g(x)]dx = \int f(x)dx + \int f(x)dx$

$\int [(fx) - g(x)]dx = \int f(x)dx - \int g(x)dx$

Rule 5: Integral of Negative Function

The integral of the negative of a function equals the negative of the integral of that:

$$\int - f(x)dx = -\int f(x)dx$$

Example 3:

Find (i) $\int (2x^2 - 4x^6)dx$ (ii) $\int (7e^{-x} - 2/x)dx$

(iii) $\int (5x^2 + 3x + 2)\, dx$

Answer:

(i) Using the difference rule;

$$\int (2x^2 - 4x^6)\, dx$$

$$= 2\int (x^2 dx - 4\int x^6 dx$$

by substituting for $n = 2$ and $n = 6$ in the formula

$$= \int (x^n dx = \frac{1}{n+1} x^{n+1}$$

It gives $= \int x^2 dx = \frac{1}{3}x^3$ and $\int x^4 dx = \frac{1}{7}x^7$

Hence: $\int (2x^2 - 4x^6) dx = 2/3x^3 - 4/7x^7$+C

(ii) $\int \left(7^{e-x} + \frac{2}{x}\right) dx$ $=7\int(e^{-x}dx + 2\int \frac{1}{x} dx$

So if $\int e^{mx} dx = 1/m^{emx}$ putting $m = -1$ gives

$$\int e^{-x} dx = 1/1 - 1^{e-x} - e^{-x}$$

Hence: $\int \frac{1}{x} dx = lnx$

Therefore: $\int \left(7e^{-x} + \frac{2}{x}\right) dx = -\, 7e^{-x} + 2lnx + C$

(iii) $\int (5x^2 + 3x + 2) dx = 5\int x^2 dx + \ 3\int x dx + 2\int 1 dx$

By substituting for $n = 2, 1, and\ 0$ into

$$\int x^n dx = \frac{1}{n+1} x^{n+1},$$

It gives: $\int x^x dx = \frac{1}{3}x^3, \int x dx = \frac{1}{2}x^2, \int 1 dx = x,$

Hence: $\int (5x^2 + 3x + 2) dx = \frac{5}{3x^3} + \frac{3}{2x^2} + 2x + C$

SELF ASSESSMENT EXERCISE 5

1. Explain what an integration is and briefly distinguish with the aid of adequate diagram(s) between the types known to you.

3.3 Applications of Integration in Businesses

The application of integration in economics and finance really assists in understanding some concepts and ensure taking accurate decision. This is more typical of definite integration whenever the results produce a single number.

Examples 4: Given an investor's marginal cost function $MC = Q^2 + 2Q + 4$, find the total cost function if the fixed costs are 100.

Answer:

There is need to find the total cost from the marginal cost function:

$MC = Q^2 + 2Q + 4$

Therefore: $MC = \frac{d(TC)}{dQ}$

So: TC=$\int mcdQ$

=$\int (Q^2 + 2Q + 4)dx$

Using the formula where $n = 2, 1\ and\ 0$;

It will be $1/3Q^3 + 1/2Q^2 + 1/1Q + C$

:.$\frac{Q3}{3} + Q^2 + 4Q + C$

The constant of integration is therefore equal to the fixed costs of production, so $C = 100$, therefore:

$$TC = \frac{Q^3}{3} + Q^2 + 4Q + 100$$

Using the consumer's surplus a definite integration will provide information about the benefits derived by the consumer and the producer. For a consumer's surplus, with a demand equation of $P = f(Q)$ as shown in figure 2.6.

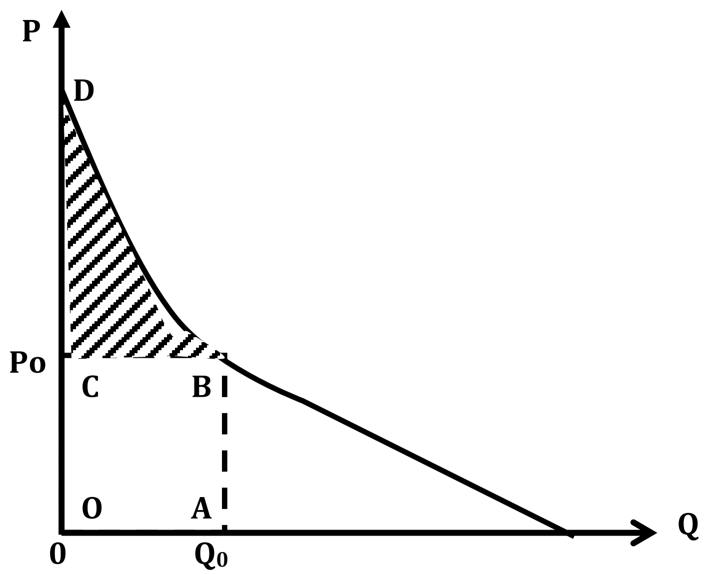

Figure 2.6: Graphical Illustration of a Consumer's Surplus

A consumer surplus is when the consumer is willing to pay a price over and above the actual price for a commodity rather than do without it.

From figure 2.6, consumer surplus (CS) is the shaded area that is BCD=OABD-OABC.

The area OABD is the area under the demand curve, P=f (Q), between $Q = 0$ and $Q = Q_{0,}$ and therefore equal to:

$$\int_{0}^{Qo} f(Q)dQ$$

The region OABC is a rectangle with base Q_0 and height P_0, so area OABC=Q_0P_0, Hence:

CS= $\int_0^{Qo} f(Q)dQ$-Q_0P_0

Example 5

Find the consumer's surplus at Q=5 for the demand function

$P = 30 - 4Q$ with $Q_0 = 5$

Answer:

$f(Q) = 30 - 4Q$

Substitute for Q, using 5 into $P = 30 - 4Q$,

$P_0 = 30 - 4(5) = 10$

Using the formula for consumer's surplus:

CS= $\int_0^{Qo} f(Q)dQ - Q_0P_0$

CS= $\int_0^5 (30 - 4Q)dQ - 5(10)$

CS= $[30Q - 2Q^2]0 - 50$

CS= $[30(5) - 2(5)^2][30(0) - 2(0)^2] - 50$

CS= $100 - 0 - 50$

CS= 50

Using the investment flow example, the interest in this concept is to investigate the rate of change in capital stock otherwise known as Net Investment (IC) that is : I=$\frac{dk}{dt}$

In this case, I(t) means the flow of money measured in naira per year, and K(t) is the amount of capital accumulated at time t as a result of this investment flow and is measured in naira.

From the above, the rate of change in capital is time based since the change can only happen over a period of time, hence making the capital stock a flow. Therefore, to calculate the capital formation during the time period from $t = t_1 \; to \; t = t_2$, we evaluate the definite integral:

$$\int_{t1}^{t2} I(t)dt$$

Example 6

Given that the investment flow is: $I(t) = 9000\sqrt{t}$, calculate

(a) The capital from the end of the first year to the end of the fourth year

(b) The number of the years required before the capital stock exceeds ₦100,000

Answer:

(a) Calculate the capital formation from $t = 1\ to\ t = 4$, to find the definite integral:

$\int_1^4 \sqrt{tdt}$=9000 $\int_1^4 t^{1/2}\ dt$

$= 9000[\frac{2}{3}t^{3/2}]_1^4$

$= 9000[\frac{2}{3}(4)^{\frac{3}{2}} - 2/3(1)^{\frac{3}{2}}]$

$= 9000(\frac{16}{3} - \frac{2}{3})$

= ₦42, 000

(b) Calculate the number of years required to accumulate a total of ₦100,000. Since the number of years is known, it is assumed that the total number of years is T. hence after T years the capital stock is:

$\int_0^T 9000\sqrt{tdt}$=9000 $\int_0^T t^{1/2}\ dt$

To find the value for T,

=$\int_0^T t^{1/2}\ dt$=100,000

$= 9000[\frac{2}{3}t^{\frac{3}{2}}] = 9000(2/3T^{\frac{3}{2}}(0)^{3/2}$

$= 6000T^{3/2}$

Hence $6000T^{3/2}$ = 100,000

∴ $T^{3/2} = \frac{100,000}{6000}$

$T^{3/2} = 16.67$

$\therefore \quad T = 6.5$

In conclusion the capital stock reaches ₦100,000 level about halfway through the seventh year.

The integral calculus can also be applied to discounting in other to determine the future value of income stream. So if the fund is to provide a continuous revenue stream for n years at an annual rate of s naira per year then the present value can be found by evaluating the definite integral.

$$P = \int_0^a Se^{-rt/100}\, dt$$

Example 7

Calculate the present value of a continuous revenue stream for 5 years at a constant rate of ₦1,000 per year if the discount rate is 9%

Answer

Using the formula to find the present value,

$$P = \int_0^a Se^{-rt/100}\, dt$$

with $S = 1000, r = 9, and\ n = 5$

$$P = \int_0^5 1000e^{-9t}\, dt$$

$$= 1000 \int_0^5 e^{-0.09t}\, dt$$

$$= 1000[-\frac{1}{0.09} e^{-0.09t} dt]_0^5$$

$$\frac{-1000}{0.09}[e^{-0.09t}]_0^5 = -\frac{1000}{0.09}(e^{-0.45} - 1)$$

= ₦4,026.35

4.0 CONCLUSION

We shall be ending this unit by giving summary of what we have covered in it.

5.0 SUMMARY

In this unit we have been able to cover the following points:

1. That integration is the reverse of differentiation.
2. That almost all the rules that are found in differentiation are also found in integration.
3. That integration is of two types – definite and indefinite integral.
4. Integration is more applicable in finance especially in the area of investigating the rates of change in capital stock.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the indefinite integrals of

   (i) $\int(2x - 4x^3)dx$ (b) $\int(10x^4 + 5/x^2)dx$ (c) $\int(7x^2 - 3x + 2)dx$

2. Find the consumer's surplus at $Q = 8$ for the demand function $P = 100 - Q^2$
3. If the net investment function is given $I\ (t) = 800t^{1/3}$

   Calculate

   (a) The capital formation from the end of the first year to the end of the eighth year.

   (b) The number of years required before the capital stock exceeds ₦48,600.

4. Calculate the present value of a continuous revenue stream for 10 years at a constant rate of ₦5,000 per year if the discount rate is 6%.

7.0 REFERENCES / FURTHER READING

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MODULE 3

Unit 1 Simple and Compound Interests and Ratio Analysis

Unit 2 Present Values, Annuities and Amortization

UNIT 1 SIMPLE AND COMPOUND INTERESTS AND RATIO ANALYSIS

CONTENTS

1.0 INTRODUCTION

Throughout Module 2 you will recall that we discussed differentiation and integration. Within this module some applications were introduced most especially by analyzing the effect(s) of time on investment decision. This was covered mostly in the integration exercise. However, we shall be focusing more in details on the impacts of time on investment analysis in this unit.

2.0 OBJECTIVES

The aim of this unit is to be able to:

- Explain what simple and compound interests are.

- Understand how simple and compound interests can be used by a lay man in taken business decision.
- Compute financial ratios and apply them in day to day monitoring of stock exchange transactions and businesses.
- Identify the effect of time on cash holdings.

3.0 MAIN CONTENT

3.1 Simple Interest and its Applications

Simple interest is a fee paid by a borrower to the lender for the fact that the lender had sacrificed his fund for another person's usage (borrower). It is therefore a compensation to the lender. Usually it is a fixed percentage of the loan amount.

For instance, when Ajibola borrows money from Ogundipe, then Ajibola has to pay certain amount to Ogundipe for the use of the money. The amount paid by Ajibola is called interest. The amount borrowed by Ajibola from Ogundipe is called the Principal. The sum of interest and principal is therefore the total amount. The rate percent per annum is the interest payable for one year.

In another sense, it can be paid to a lender by a person depositing money into a bank account. When you deposit money into a bank, you are essentially loaning it to the bank. Simple interest is the amount initially charged on a loan and does not take into account the compounding of interest over time. This implies that simple interest is the percentage calculated each time and not the amount actually accrued.

Automatically the amount of interest received will be the same all over the years the fund is borrowed. Hence the interest is not compounded since the interest on an investment is calculated once per period usually per year on the amount of capital alone and not on any interest already earned or accrued.

The formula is:

$$I = Prt$$

Where: I = Interest

P = Principal

r = Percentage rate of intrest

t = time

Example 1

Find the interest payable on an invested capital of N4,500 at 9.5% for the period of 6 years, and find the new total amount.

Answer

Using the formula $I = Prt$

$I = (4{,}500)\ (0.095)\ (6)$

$I =$ ₦2,565.

Total Amount = Principal + Interest

$=$ ₦$(4{,}500 + 2{,}565)$

$=$ ₦7,065

***Example* 2**

A bank is prepared to offer a 90% house loan to first time buyer at 10% interest rate per annum. If the worth of the house is ₦300,000.

(a) Determine the amount the bank will lend to a new couple willing to buy the house.

(b) Calculate the interest payable on the loan to the couple

(c) What is the total amount the couple will pay at the end of 3 years?

Answer:

(a) Find the equivalence of 90% of N300,000

Hence $\frac{90}{100}$ x 300,000 or 0.9 (300,000) $=$ ₦270,000

(b) For the first year, the interest payable on the loan using the formula,

$I = Prt$

Note that the loan given to the couple by the bank is ₦270,000, therefore :

$I = (270{,}000)\ (0.1)\ (1)$

$I \ = ₦27{,}000$

(c) Using the formula again, the interest the couple will pay at the end of 3 years is:

$I \ = \ (270{,}000)\ (0.1)\ (3)$

I = ₦81,000 or simply multiply ₦27,000 per year by 3

The total amount the couple will pay at the end of three years will be:

Principal Amount + Interest on the loan

$= ₦270{,}000.00 + 81{,}000$

$Total\ A$mount $= ₦351{,}000.00$

SELF ASSESSMENT EXERCISE 1

Simple interest is the profit earned for parting away with ones liquidity. This implies that an individual will be motivated to sacrifice its present suitable condition for his future. Hence he would need to be compensated for the sufferings encountered during the period of sacrifice. Perhaps he may also have suffered from the problem of price changes due to the effect of time of parting with his cash.

3.2. Compound Interest and its Applications

A Compound interest is the interest earned by an invested amount of money (principal) that is reinvested so that it, too earns interest. This means that the interest is converted (or compounded) at principal and hence, there is "interest on interest". For example if ₦100 is invested at the rate of 5% compounded annually. At the end of the first year the value of the investment is the original principal (₦100) plus the interest on the principal [100(0.05)]:

$100 + 100\ (0.5) = ₦105$

This is the amount on which interest is earned for the second year. At the end of the second year, the value of the investment is the principal at the end of the first year(₦105), plus the interest on the sum $[100(0.05)]: = ₦110.25$

Thus each year the principal increases by 5%. The ₦110.25 represents the original principal, plus all accrued interest; it is called the accumulated amount or compound amount. The difference between the compound amount and the original principal is called the compound interest. Here the compound interest is $110.25 - 100 = 10.25$.

This process continues as long as the number of years continued and this rather complex and labourious. Hence there is need for a method of calculating the investment after, say, 10 years without having to determine the amount for nine intermediate years.

Therefore, for each year the investment gets multiplied by $1 + r/100$ such that after n years,

$$S = P\left(1 + \frac{r}{100}\right)^{n}$$

or

$$S = P(1 + r)^{n}$$

Example 3

Given that ₦1000 is invested for 10 years at 6% compounded annually;

(a) Find the compound amount

(b) Find the compound interest

Answer

(a) Using the formula

$S = P(1+r/100)^{n}$, P = N1000, r = 0.06 n =10;

Therefore $S = 1000\ (1+0.06)^{10} = 1000\ (1.06)^{10}$ = ₦1790.85.

This is can further be graphically represented in figure 3.1 which indicates that as time goes on, the compound amount grows systematically

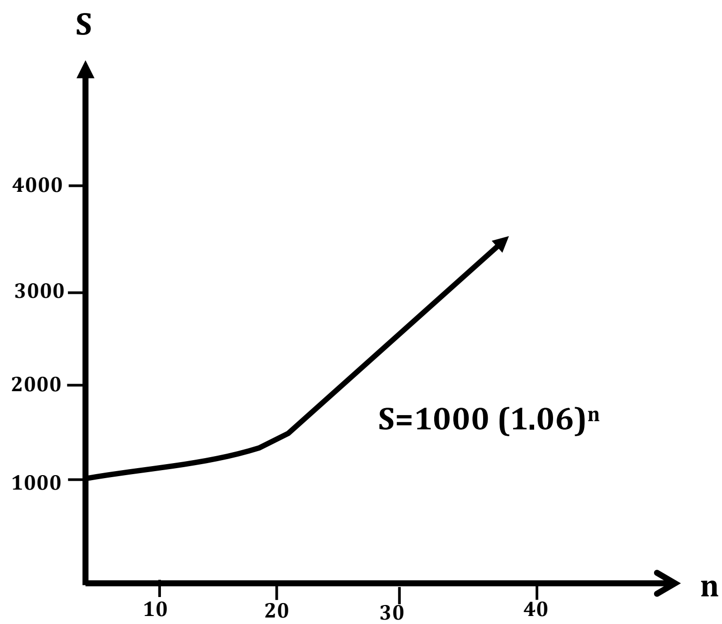

Figure 3.1 Graph of $S = 1000\ (1.06)^n$

(b) From the result in part (a) above,

compound interest = S – P

= ₦1790.85 – 1000 = ₦790.85

Please note that the interest is generally compounded annually but is also possible for interest to be added to the investment more frequently than this.

***Example* 4**

A principal of ₦10 is invested at 12% interest for one year.

Determine the future value if the interest is compounded.

(a) Annually

(b) semi annually

(c) quarterly

(d) monthly

(e) weekly

Answer:

Using the formula for compound interest:

$S = 10(1+r/n)^n$

(a) If the interest is compounded annually than r=12, n=1, so $S=10\ (10+12)^1 = 10(1.1/12)^1$ = ₦11.20

(b) If the interest is compounded semi-annually then an interest of 12/2 = 6% is added on every six months and since there are two six month periods in a year,

S = ₦$10(1.06)^2$ = ₦11.24

(c) If the interest is compounded quarterly then an interest of 12/4 = 3% is added on every three months and since there are four three –month periods in a year.

S = ₦$10\ (1.03)^4$ = ₦11.26

(d) If the interest is compounded monthly then an interest of 12/12 = 1% is added on every month and since, there are 12 months in a year

S = ₦$10(1.01)^{12}$ = ₦11.27

(e) If the interest is compounded weekly then an interest of 12/52 = 0.23% is added on every week and, since there are 52 weeks in a year.

$S = N10\ (1.0023)^{52}$ = ₦11.27

Following example 4 above, it is interesting to note that the future value rises as the frequency of compounding rises due to the fact that interest is charged on interest. This is a situation of continuous compounding of which can be presented in another special formula instead of going through the pains in example 4. This special formula is used to compute the future value directly. The future value S, of a principal, P, compounded continuously for t years at an annual rate of r% is:

$$S = Pe^{rt/100}$$

Where letter e = 2.718281828459045235536

Example 5

If r = 12, t = 1 and p = 10, to use the formula $S=pe^{rt}100$, first multiply the interest rate r, by the number of years t, and divide by 100.

***Answer*:**

S = ₦$10e^{12x1/100} = 10e^{0.12}$ = ₦11.27

3.2.1 Effective Interest Rate

An effective interest rate is a method that is used to appraise different investment opportunities, and it is just the rate of simple interest earned over a period of one year. This is common in use since it is compounded annually. The effective annual rate is often referred to as the annual percentage rate (APR). The APR is the rate of interest which, when compounded annually produces the same yield as the nominal (i.e the stated) rate of interest.

The effective rate (r_e) that is equivalent to a nominal rate r compounded n times a year is given by:

$$r_e = (1 + r/n)^n - 1$$

Example 6

What effective rate is equivalent to a nominal rate of 6% compounded (a) Semi annually and (b) quarterly?

Answer

(a) The effective rate (r_e)

$r_e = (1+^{0.06}/2)^2 - 1 = (1.03)^2 - 1 = 0.0609$

$= 6.09\%$

(b) $r_e = (1+^{0.06}/4)^4 - 1 = (1.015)^4 - 1 = 0.061364$

$= 6.14\%$

SELF ASSESSMENT EXERCISE 2

It should be noted in the compound interest that the interest earned initially from parting away with ones liquidity through (may be) the purchase of stocks or shares can further be reinvested. Once this occurs, there will be additional interest. The implication of this is that there will be interest upon interest. All the same, it is also time bound. Time is very essential in the computation of compound interest.

3.3 Types and Uses of Financial Ratios

A ratio is an established proportional relationship between two different numbers or quantities. Sometimes, it is obtained by dividing one number by another number usually of the same kind. In fact, it indicates the quotient of two mathematical expressions.

Example 7

Calculate the ratio between fourteen days and four weeks.

Answer

Note that there are 28 days in four weeks, so the ratio can be expressed as 14:28 and reduced to 2: or better still 2/4 = ½ or 1 to 2.

The significance of ratio in financial analysis is represented as it is used as a benchmark for evaluating the financial position and performance of a firm.

This is more useful to trade creditors, debenture holders, investors and management of any firm.

In this case, we are not just dealing with ratio ordinarily but since it is connected to financial statement, we refer to it as financial ratio.

Financial ratio is therefore a relationship between two accounting figures meant to summarize copious financial data in order to make qualitative judgment in respect of firm's financial performance.

In summary, the results arising from financial ratio assist in evaluating the state of a business, identify the weaknesses of the business and make projections as well as forecasting the course of future operations.

How useful financial ratios are depend on Inter industry comparison and Trend analysis.

Financial ratios can be grouped into five major categories. These are:

(a) Liquidity Ratio

(b) Asset utilization Ratio (Activity Ratio)

(c) Solvency Ratio (Leverage and Debt Service Ratio)

(d) Profitability Ratio

(e) Market Value Ratio

For the effective use of these ratios, a company's profit and loss account and balance sheet are needed else the exercise of financial analysis using financial ratio will be useless.

(a) **The liquidity ratios:** They are used in measuring firm's ability to meet short term maturing debt obligations. This is very important to be survival of any business most especially during economic crisis. It gives an indication on the ability of the company's in fulfilling its obligations of which if not possible, it might increase its cost of financing and can render it unable to pay bills and dividends.

The liquidity ratios are: (i) net working capital (ii) the current ratio and (c) the quick (acid-test) ratio.

(i) Net working capital = current assets – current liabilities for given year (1982)

= ₦124 - ₦56

= ₦68

(ii) Current ratio = $\frac{\text{(Current Assets)}}{\text{Current Liabilities}}$

Thus for a given year

(1982) current ratio = $\left(\frac{₦124}{56}\right)$ = 2:21

(iii) Using the more liquid assets by deducting inventory and prepaid expenses, the quick (acid-test) ratio is obtained by dividing the more liquid assets by current liabilities

$$\text{Acid Test Ratio (ATR)} = \frac{\text{Cash + Marketable Security}}{\text{Current Liabilities}}$$

The ATR for 1982 = $\left(\frac{₦28 + 21 + 22}{56}\right)$ = 1.27

(b) **The Asset utilization ratios:** These explain the extent to which a firm uses its assets. It measures the firm's efficiency in its activity level to obtain revenue and profit. These ratios can also be referred to as turnover ratio since they indicate the speed with which assets are being converted or turned over into sales.

On this note, it implies that there is a well established relationship between sales and assets. So whenever there is a balance, it means that the assets are well managed. The Activity ratios are among others includes the following:

(i) Account Receivable Ratios: This comprises of Account Receivable Turnover and Average Collection Period. Furthermore, the Account Receivable turnover is the number of times accounts receivables are collected in the years.

$$\text{Account receivable turnover} = \frac{\text{Net Credit Sales (NCS)}}{\text{Average Account Receivable (ACR)}}$$

Note average accounts receivable (ACR) is derived by adding the beginning and ending balances and then divided by 2. So if the net credit sale is ₦(80.30/18.50) = 4.34 in 1982.

The average collection period on the other hand measures the length of time it takes to collect receivables or the number of days receivables are held.

$$\text{Average Collection Period} = \frac{\text{365 days}}{\text{Accounts Receivable Turnover}}$$

$$\text{ACP} = \frac{365}{4.34} = 84.1 \text{ days}$$

In summary, it will take the firm about 84 days to convert receivables to cash.

(ii) Inventory Ratios: The ratios are mathematical expression to determine the extent of inventory investment. Inventory investment refers to goods produced but not yet sold. They are rather warehoused and these tie down cash, hold down profit and increased storage costs. Hence the extension of credit or money lending to a business firm is tied to its inventory turnover and average age of inventory. It thus indicates the efficiency of the firm in producing and selling its product.

$$\text{Inventory Turnover (IT)} = \frac{\text{Cost of Goods sold}}{\text{Average Inventory}}$$

For 1982, the inventory turnover is: ₦ (52/49.5)=1.05

$$\text{Average age of inventory} = \frac{365}{\text{Inventory turnover}}$$

$$= 365/1.05 = 347.6 \text{ days}$$

Note that it is essential to determine the total asset turnover in order to find out the extent to which the firm is efficiently employing its total assets to obtain sales revenue generated.

A low ratio may indicate that too many assets are being held compared to the sales revenue generated.

$$\text{Total Asset Turnover} = \frac{\text{Net Sales}}{\text{Average Total Assets}}$$

$$= \frac{\text{N}80}{\text{₦} (204 + 227)/2} = \left(\frac{\text{₦} 80.3}{215.5}\right) = .37$$

(c) **The Solvency ratios:** These are otherwise referred to as leverage and debt service ability relates to the capacity of a firm to fulfill its long-term and short term debt obligations. Hence, it is concerned with the financial and operating structure of any company. It is also important for considering the financial leverage that is, the size of debt in the company's capital structure. To a greater extent also, solvency depends on earning power in terms of long-run profit. The higher the profit earned the more the ability of the company to offset its debt. The solvency ratios thus include:

(i) Debt Ratio - This refers to the amount of money a firm owes to all its creditors, and it is measured as

$$\text{Debt – Ratio} = \frac{\text{Total Liabilities}}{\text{Total Assets}}$$

$$\text{For 1982 the ratio} = \left(\frac{₦139}{227}\right) = 0.61$$

(ii) Debt-Equity Ratio (D/E): It indicates the percentage of a firm's debt in its capital structure. The higher the percentage the more the risk of running out of cash in difficult times, and the more the interest as well as principal paid. This ratio depends on the ratio of other firms in the industry, the degree of access to additional debt financing and the stability of operations. This is measured as:

$$D/E = \frac{\text{Total Liabilities}}{\text{Stockholders' Equity}}$$

$$= \left(\frac{₦139}{88}\right) = 1.58$$

(d) **Profitability Ratios**: Refers to the extent to which a firm earns profit measured as the difference between total revenue and total cost. In most cases every firm seeks to maximize profit and this enhances its financial well-being and efficiency. It can be measured through

(i) $$\text{Gross Profit Margin} = \frac{\text{Gross Profit}}{\text{Net Sales}}$$

$$\left(\frac{₦28.3}{0.35}\right) = 80.3$$

This gross profit margin indicates the percentage of each naira remaining once the firm has paid for goods acquired.

(ii) Profit Margin indicates the earning generated from revenue and is the key indicator of operating performance.

It determines the firm's pricing, cost structure and efficiency and it is measured as:

$$\text{Profit margin} = \frac{\text{Net Income}}{\text{Net Sales}}$$

$$= ₦\ (8/80.3) = 0.10$$

(iii) Return on investment is another means of measuring the profitability ratio and this include return on total assets and Return on Equity (ROE).

- Return on total assets indicates whether management is efficient in using available resources to get profit and is measured as:

- Return on Total Assets $= \frac{\text{Net income}}{\text{Average total assets}}$

$$= \frac{₦8}{₦\ (227 + 200/2} = 0.37$$

Return on Equity (ROE) is the rate of return earned on the stockholders' investment and it is measured as:

$$\text{ROE} = \frac{\text{net income available to stockholder}}{\text{average stockholders' equity}}$$

$$= \frac{₦\ 8}{₦\ (88 + 80)/2} = 0.095$$

(e) **Market Value Ratios:** These ratios relate the firm's stock price to its dividends, earnings, or book value per share. These include:

(i) Earnings per share (EPS) - this is the ratio commonly watched by investors since it gauges corporate operating performance and of expected future dividends.

$$EPS = \frac{\text{net income - preferred dividend}}{\text{Common shares outstanding}}$$

$$= \frac{₦8,000}{4,600 \text{ shares}} = ₦1.74$$

ii. The Price/Earning (P/E) Ratio otherwise called earnings multiple, reflects the company's relationship to its stockholders.

$$P/E = \frac{\text{Market Price per share}}{\text{Earnings per share}}$$

$$= ₦\ (12/1.74) = 6.9$$

iv. Book value per share equals the net assets available to common stockholders divided by shares outstanding, it is measured as:

$$\text{Book Value per Share} = \frac{\text{total shareholders' equity – preferred stock}}{\text{Common shares outstanding}}$$

$$= \frac{₦88,000 – 0}{4,600}$$

$$= ₦19.13$$

SELF ASSESSMENT EXERCISE 3

1. Distinguish between finance analysis and financial management
2. Explain the types and uses of finance ratios known to you
3. Enumerate the ratios that are of interest to the shareholders and why?
4. How would you calculate the following Ratios?
   (a) EPS (b) DPS (c) Price/Earning

4.0 CONCLUSION

At the end of this unit we will be summarizing what we have covered in it.

5.0 SUMMARY

You are expected to have covered the following points in this unit:

1. The uses of simple and compound interests through their computation.
2. How to extract information from financial statement of firms in computing financial ratios.
3. Identify the various types of financial ratios
4. How to use these ratios in forecasting future investment decision

6.0 TUTOR-MARKED ASSIGNMENT

1. Explain the term simple interest and critically examine its advantages.
2. Determine the interest payable on an invested capital of ₦20,500 at 9.5% for the period of five years. Hence or otherwise find the total amount involved.
3. Explain the terms: Compound Interest and Effective interest rate. Is there any difference between the two?

4, Find the value in 10 years' time of ₦100 interest at 8% interest compounded annually.

5. A small business promises a profit of ₦8,000 on an initial investment of ₦20,000 after five years, calculate the effective annual rate of interest.

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UNIT 2 PRESENT VALUES, ANNUITIES AND AMORTIZATION

CONTENTS

1.0 INTRODUCTION

Recall that in the unit 1 we have been able to cover how time affects business decisions and investments in stocks and shares or what can be regarded as nominal purchases. In this unit we will be advancing in identifying how the time value of money affects real purchases such as investment in real projects. We will also go further to determine whether undertaken a particular project real worth it or not. Hence we shall be coming across the terminologies like accept or reject.

2.0 OBJECTIVES

At the end of this unit you should be able to:

- Understand the two basic methods of computing present values of returns
- Identify the components of discounted and non discounted techniques of project evaluation
- Determine whether to accept or reject a project
- Appraise whether the entire business is making progress and be able to determine future investment.

3.0 MAIN CONTENT

3.1 Present Value - Concept and Amortization

The concept of present value has a significant connection with money that has a time value. The saying that: “a bird in hand worth two in the bush” comes to mind by virtue of the fact that a naira today is worth more than naira in the future.

This implies that money changes over time because its value decreases progressively due to inflation, risk and preference for liquidity.

On this note, time value of money is a critical consideration in financial and investment decisions. This is because it is used to evaluate the future cash flow associated with capital budgeting projects.

By the nature of this concept, it involves once again the element of compound of interest and of discounting.

Emphases will be laid here on the impact of discount rate to determine whether an investment is worth undertaken or not.

Based on the above, the discounted cash flow (DCF) technique will put into use the simple and the compound interests.

These techniques are the Net present value, internal rate of return and the terminal value.

They are referred to as DCF because they are calculated based on discounting (or compounding) the cash flows.

The techniques are however based on some set of assumptions and these are:

(a) Future cash flows are certain at initial stage

(b) The value of money is stable

(c) There is also perfect competition in the capital market; that is the forces of demand for and supply of funds are determined through their intersection.

(i) The Net Present Value Technique (NPV)

This is a method that involves the addition of all the discounted values that are later added to give gross present value. (GPV)

$$GPV = \frac{B_1}{1+r} + \frac{B_2}{(1+r)^2} + \frac{B_3}{(1+r)^3} + \ldots\ldots\ldots\ldots\ldots + \frac{B_n}{(1+r)^n}$$

$$= \sum_{t=1}^{n} \frac{Bt}{(1+r)t}$$

Where Bt = Cash flow in period t

t = time period and t = 1, 2, 3,……., n.

r = discount rate (i.e. the opportunity cost of capital)

In order to obtain the NPV, the initial (cost) outlay (denoted by C) is subtracted from the GPV, hence

NPV = GPV - C

$$= \sum_{t=1}^{n} \frac{Bt}{(1+r)t} - C$$

$$\text{or } NPV = \frac{B_1}{1+r} + \frac{B_2}{(1+r)^2} + \frac{B_3}{(1+r)^3} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{B_n}{(1+r)^n} - C$$

The decision to undertake the project or not depends on if NPV > 0 = positive, the project (investment) is profitable, hence it should be accepted. But if NPV < 0 = negative, it is not profitable hence it should be rejected but if NPV=0, it is either accepted or rejected. However, for a mutually exclusive projects the investment with the highest net present value is selected.

Example 7:

The cash flows associated with buying of shares in Lamurudu PLC are:

t_o	t_1	t_2	t_3	t_4
-20,000	5,000	8,000	7,000	8,000

You are required to determine the economic profitability of the project, if the cost of capital is

(a) 10 per cent

(b) 20 per cent

Answer

$$GPV = \frac{5,000}{1.10} + \frac{8,000}{(1.10)^2} + \frac{7,500}{(1.10)^3} + \frac{8,000}{(1.10)^4}$$

GPV = 4,545 + 6,611.20 + 5,634.75 + 5,464

GPV = 22,255.00

NPV = GPV - C

NPV = ₦(22,255- 20,000) = ₦2,255

Since NPV > 0; the investment is worth undertaken, hence the project should be accepted.

(b) $$GPV = \frac{5}{1.20} \quad \frac{8,000}{(1.20)^2} + \frac{7,500}{(1.20)^3} + \frac{8,000}{(1.20)^4}$$

= 4,166.50+ 5,555.20 + 4,340.25 + 3,858.40

= 17,920.3

∴ NPV = ₦(17,920.35 – 20,000)=

= - ₦2,079.65

Since NPV <0; the investment is not worth undertaken hence the project should be rejected.

(ii) The Internal Rate of Return (IRR) Technique

This rather equates the net present value to zero. Hence, the IRR is defined as the discount rate which equates the gross present value of a project to its initial outlay (cost). It is also referred to as the yield.

$$IRR = \sum_{t=1}^{n} \frac{Bt}{(1+r)t} - C = 0$$

That is:= $$\frac{B_1}{(1+r)^1} + \frac{B_2}{(1+r)^2} + \frac{B_3}{(1+r)^3} + \ldots\ldots + \frac{B_n}{(1+r)^n} - C = 0$$

Where r* = Internal Rate of Return (yield)

For decision to be taken, the internal rate of return is compared with the investor's opportunity cost of capital (r).

If the yield is greater than the cost of capital, the investment is worth undertaken, if not, it is rejected. That is

$r^* > r$; it is profitable and accepted

$r^* < r$; it is not profitable and thus rejected this can be graphically illustrated in figure 3.1 below.

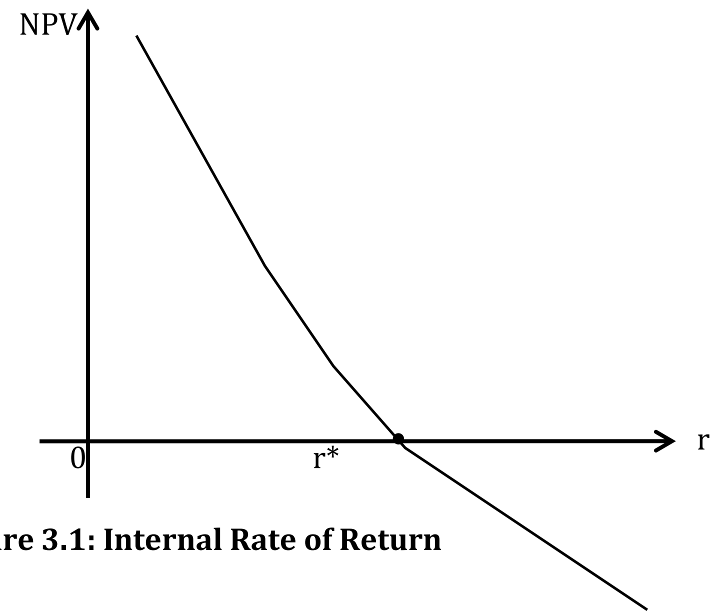

Figure 3.1: Internal Rate of Return

The yield from figure 3.1 is the intercept of the horizontal axis when the NPV is graphed.

Example 8:

Find the internal rates of return (yields) of the following cash flows:

(a) -100, 125

(b) -100, 80, 50

Answer

$$-100 + \frac{125}{(1+r^*)} = 0$$

Therefore: $$\frac{125}{1+r^*} = 100$$

$100\,(1+r^*) = 125$

$1 + r^* \quad = \frac{125}{100}$

$1 + r^* = 1.25$

Hence: $r^* = 1.25 – 1$

$r^* = 0.25$ or 25%

So, if the r = 10% or 20%, and with r*= 25%, the investment is worthwhile and it should be accepted, if not, it should be rejected.

(b) $-100 + \frac{80}{1+r^*} + \frac{50}{(1+r^*)^2} = 0$

Let X = 1+r*, so

$-100 + \frac{80}{x} + \frac{50}{x^2} = 0$

Multiply through by x^2

$-100\, x^2 + 80 + 50 = 0$

Divide through by 10, that is

- $10\, x^2 + 8x + 5 = 0$
- This gives rise to quadratic equation, of which

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus a = -10, b=8, c=5

By substitution: =

$x = \frac{-8 \pm \sqrt{8-4(-10)(5)}}{2(-10)} = \; -8\, \frac{\pm\sqrt{64+200}}{-20}$

$X = \frac{-8 \pm 16.25}{-20}$

Therefore: $x = -\frac{24.25}{-20}$ or $\frac{8.25}{-20}$, that is

x= 1.2124 or -0.4125

Since the discount rate cannot assume any negative value, we discard away with the second root. Hence,

x = 1 + r* = 1.2124

r* = 1.2124 – 1

r* = 0.2124 0r 21.24%

In conclusion, since the yield is positive and also greater than 10 or 20%, the investment is worthwhile, and should be accepted. If not, it should be rejected.

It should be noted however that whenever the economic life of a project is more than two years, the yield can only be obtained by interpolation.

SELF ASSESSMENT EXERCISE 4

You should always remember that whenever the net present value is greater than zero, the project is profitable and should be accepted. Otherwise, it should be rejected. But where the projects are mutually exclusive, the one with the highest net present value should be accepted. Furthermore, whenever the internal rate of return is higher than the existing cost of capital, the project should be accepted. Otherwise it should be rejected.

4.0 CONCLUSION

At the end of this unit we will be summarizing all that we have covered so far.

5.0 SUMMARY

Following the unit covered in this block, the following points should be noted:

1. The present value is more focusing on the time value of money. Hence the elements of compound interest and discounting are significant to its computation.

2. The discounting techniques are more realistic than the non discounted techniques.

3. Though the discounted cash flow techniques are cumbersome whenever the cash flows are more than two years especially when using the internal rate of return. Therefore in the computation process, you must be careful.

6.0 TUTOR – MARKED ASSIGNMENT

1. Noun Limited has two projects, Book publication and Journal Publication with the following cash flows.

YEAR	**CASH FLOWS**	
	Book Project	**Journal Project**
0	-₦20,000	-₦20,000
1	13,000	12,000
2	14,000	6,000
3	1,000	16,000

(a) Calculate the net present values of each project with the cost of capital at 20 percent

(b) Using the NPV rules, which project(s) should Noun Limited undertake if they are independent projects and why?

(c) Which project should be accepted if they are mutually exclusive?

2. Find the yields of the following cash flows:
   (a) – 200, 300
   (b) -150, 120, 100

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MODULE 4

Unit 1 Linear Programming

UNIT 1 LINEAR PROGRAMMING

CONTENTS

1.0 INTRODUCTION

Recall that in all the modules already covered, we have been able to discuss virtually all that are necessary to take business decisions especially the aspect of funding such businesses. In the following unit we are going to concentrate on the approach to optimize an investor's return by investing in the project that the yields the highest return

2.0 OBJECTIVES

At the end of this unit you should be able to:

- Understand the types of linear programming and the various elements that are involved in each case.
- Identify the components of linear programming.
- To formulate linear programming
- Determine the optimum activity mix that yields the highest return.

3.0 MAIN CONTENT

3.1 Meaning and Types of Linear Programming

Linear Programming has been necessitated by the fact that resources are scarce. For every investor, he seeks to either maximize the use of these resources or minimize the cost of these resources in other to maximize profit.

The resources can be time, materials, space and money to mention but few. Since all these are not available at will, they are regarded as constraints to the realization of the investor's objectives.

Linear Programming can thus be regarded as a useful technique for allocating resources among competing uses. It is therefore a mathematical technique designed to help investors find an optimal decision (or an optimal plan) chosen from a large number of possible decision.

A Linear Programming can be of two types. These are primal and dual programming. Primal Programming involves decisions to maximize profit, output, returns to mention but few. Dual programming on the other hand involves decisions to minimize risk, space, time and cost to mention but few.

Either way, a linear programming model consists of two major elements. These are objective function that must define specific aim to be achieved. Secondly the set of constraints must be defined. These are in form of reflections on the availability of resources or meeting minimum requirements.

For any general Linear Programming problem, five basic assumptions are prima facie significant to the formulation of a linear programming. These are:

(a) Linearity: The objective function and the constraint equations are assumed to be linear (straight line) in other words there is one-one mapping among the variable.

(b) Additivity: All variables and their coefficients can be summed altogether.

(c) Certainty: The method prohibits probability. The stated conditions are assumed to follow precisely the given mathematical expressions.

(d) Divisibility: The values of the decision variable may take on fractional or integer values.

(e) Non-negativity: The values of the variables that the method selects (decision variables) must be greater than or equal to zero.

3.2 Stages of Formulation and Computation Methods of Linear Programming

Formulation of Linear Programming follows the following stages:

1. **Decision Variables**: This involves deciding the volume of investment for example in stocks.
2. **Statement of objectives**: This includes deciding whether the investor wishes to maximize return or minimize risk. Where information is given on both cost and yield, it is generally acceptable to work with return figures.
3. **Statement of objective function**: This is stated as a sum of the variables multiplied by their respective unit return or cost.
4. **Set of constraints**: The constraints are all expressed as functions of the variables chosen.
5. **Conversion of all inequalities**: All the inequalities ≤ or ≥ signs in the set of constraints must be in conformity with your objective. For instance, a prima programming carries ≤ sign while the dual programming carries ≥ sign.
6. **Statement of Non negativity**: It must always be stated that all variables are non negativity that is ≥ 0.

Example 1

Focus loans and Mortgage finance house is considering investing directly in housing construction and purchasing stocks. Both investments require time in two processing activities, valuation and listing on the stock exchange market. Following are the information in respect of the activity.

Activities

Processing	Housing	Stock	Available Hours
Valuation	2	4	100
Listing	3	2	90
Return per unit	₦25	₦40	

The investor wishes to find the better of the two activities that yields highest return.

Answer:

Decision variables: $x_1 =$ Housing; x_2= Stocks.

Statement of objective function: Maximize total return (R)

$$R = 25x_1 + 40x_2$$

Sets of constraints: $2x_1 + 4x_2 \leq 100$ (Valuation)

$3x_1 + 2x_2 \leq 90$ (Listing)

Non negativity Constraints: $\sum_{i=1}^{2} x_i \geq 0 \; or \; x_1 x_2 \geq 0$

Therefore, the LP models is:

Maximize: $R = 25x_1 + 40x_2$

Subject to: $2x_1 + 4x_2 \leq 100$

$3x_1 + 2x_2 \leq 90$

Non negativity: $\sum_{i=1}^{2} x_i \geq 0$

Notice that this is a primal programming and can be converted into dual programming. This is as follows:

Decision Variable: Z_1 =Valuation time

Z_2=Listing time

Statement of objective function: Minimize total available hours (T)

$$T = 100\, z_1 + 90z_2$$

Set of constraints: $2z_1+3z_2 \geq 25$

$4z_1+2z_2 \geq 40$

Non negativity Constraint: $\sum_{i=1}^{2} Z_i \geq 0; Z_1Z_2 \geq 0$

Therefore, the dual programming model is:

Minimize: $T = 100\, z_1 + 90z_2$

Subject to: $2z_1+3z_2 \geq 25$

$4z_1+2z_2 \geq 40$

Non negativity: $\sum_{i=1}^{2} Z_i \geq 0$

Having formulated our linear programming model through the types involved, the LP can be computed through two basic methods. These are graphical and the simplex method.

The graphical method is easier to use but it is strictly restricted to the LP problems involving two (or at most three) decision variables. In using the graphical method, the following steps are involved.

1. Change inequalities to equalities
2. Graph the equalities
3. Identify the correct side for the original inequalities by shading.
4. Identify the feasible region, the area of feasible solution
5. Solve the constraints

6. Determine the return or contribution margin at all corners in the feasible region.

Example 2:

Using the information and the LP model from example 1, follow the steps 1-6. The following feasible region (shaded area) would be obtained.

1. $2x_1$+$4x_2$ = 100 -------------------- (1)

   $3x_1$+$2x_2$ = 90 --------------------(2)

2. For Equation 1,

   If $x_1 = 0, x_2$ = 25; if x_2=0, x_1=50; connect x_2= 25 and x_1=50

   For equation 2

   If $x_1 = 0, x_2$ = 45, if x_2 =0, x_1=30; connect x_2= 45 to x_1=30

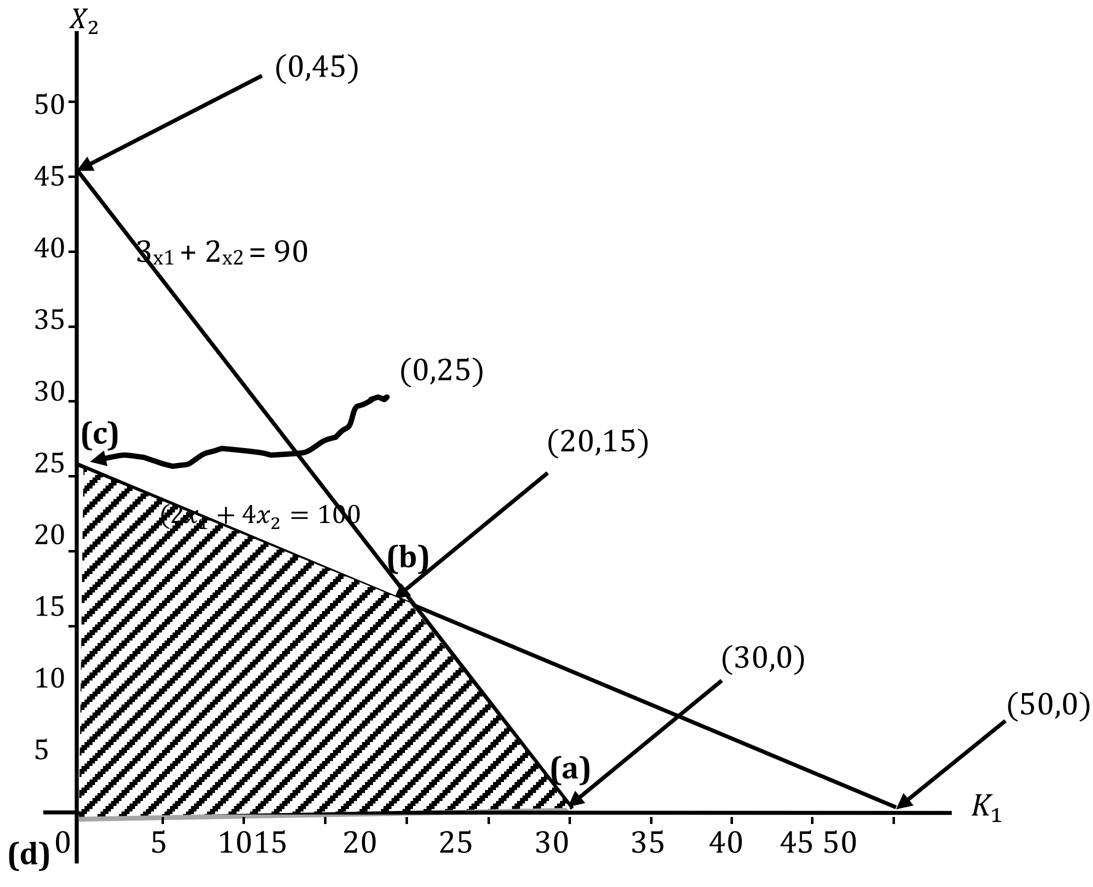

Figure 1.1: The feasible region and corner points.

Note that points (a) – (c) fall within the feasible region while point (b) had been solved simultaneously using equations (1) and (2).

Using the corner points, the optimum activity is the one that yield the highest return. This is shown below:

Corner points	x_1	x_2	Return **₦25x_1 + ₦40x_2**
A	30	0	₦25 (30) + (0) = ₦750
B	20	15	₦25 (20) +₦40 (15) = ₦1,100
C	0	25	₦25 (0) +₦40 (25) = ₦1000
D	0	0	₦25 (0) +₦40 (0) = ₦0

In conclusion, the corner point (b) (x_1=20, x_2= 15) generates the highest return of R* = ₦1,1000

SELF ASSESSMENT EXERCISE 1

Linear Programming has been a veritable tool of analyzing an investor's performance by identifying the true position of investment. This is however subject to availability of accurate information. It is very easy to compute especially when it involves two decision variables. But where it has more than two decision variables, the simplex method would be used. It should also be noted that a primal linear programming can equally be converted into dual and vice versa.

4.0 CONCLUSION

We will now be ending our discussion on unit 1, Module 4 by summarizing all that we have covered.

5.0 SUMMARY

By now you should be familiar with the following points that have been covered in this unit:

1. Linear programming is made up of dual and primal programming
2. Linear programming has been based on assumptions of linearity, certainty, divisibility, additivity and non negativity constarints.
3. The linear programming problem can be solved through graphical and simplex methods.
4. In using the graphical method the feasible region must first be identified.
5. The simplex method basically makes use of iteration technique.
6. Linear Programming follows systematic stages of formulation.

6.0 TUTOR – MARKED ASSIGNMENT

1. Explain the concept of linear programming. Of what use is it to an investor?
2. What are the assumptions of a linear programming? Are these realistic?
3. Johnson Investment Incorporation invested in long term and short term securities, X and Y. Their contribution margins are ₦50 and ₦90, respectively. Each investment passes through three phases: Advertising, Valuation and listing. The number of hours required by each phase for each investment and capabilities are as follow:

Hours required in each phase.

Securities	Advertisement	Valuation	Listing
X	2	4	3
Y	1	6	2
Capacities (Hours)	300	500	250

(a) Formulate the objective function and constraints to determine the optimal security mix.

(b) Complete the dual programming of this problem.

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