MTH 001: Access Mathematics

NATIONAL OPEN UNIVERSITY OF NIGERIA

MTH 001: **Access Mathematics**

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Access mathematics: course guide

Contents

Introduction

This course is on **remedial mathematics**, which is a *non-credit course* for students having the need to update their secondary school leaving certificate examinations' results or remedy their deficiency in mathematics before their admission directly into the diploma or the degree programme of the NOUN. The course is therefore very essential for all students who need a very good foundation in mathematics to undergo any degree programme in the sciences.

The course treats a very broad combination of all the major mathematics topics contained in the syllabuses of the West African Examination Council (WAEC), National Examinations Council (NECO) and Joint Admissions and Matriculation Board (JAMB).

By the time you have successfully gone through this course, you would have mastered so many methods of solving various fundamental problems in mathematics that are essential to your subsequent studies in higher mathematics or any science programme that employs much of mathematics.

You are expected to diligently go through the course materials, do the exercises contained in the course and do the tutor-marked assignments (TMAs) at the end of each unit. Before you end this course, you would have become more grounded in your understanding of general basic mathematics essential for your intended undergraduate programme. However, to profitably go through this course and communicate with your course tutor over the internet, you will need to have access to a computer system that has the minimum configuration listed under the course materials.

What you will learn in this course

This course is aimed at taking you through all the various topics in mathematics at a pre-degree level in order to lay a good foundation for a sound mathematics background which is a prerequisite for all science and science-related courses of NOUN's undergraduate programmes.

Aims of the course

The above overall aim are to:

i) Solve problems involving bases, indices and logarithms.
ii) Identify and solve various types of equations such as linear, simultaneous and quadratic equations and those involving surds.
iii) Solve various problems of mathematics covering areas in sets, trigonometry, differentiation, integration, measuration, plane and coordinate geometry and simple statistics.
iv) Lay a good foundation in the basic concepts and applications of mathematics principles.

Course objectives

On successful completion of the course, you should be able to:

i) Solve problems in different number bases.
ii) Explain the relationship between indices and logarithms.
iii) Identify surds and solve equations involving surds.
iv) Distinguish between arithmetic progression and geometric progression.
v) State the properties of sets and represent them using Venn diagrams.
vi) Solve linear and simultaneous equations, and to also solve the latter by graphical method.
vii) Identify quadratic equations, solve them using various methods and plot their graphs.
viii) Solve inequalities and represent them graphically.
ix) Explain various types of variation and how to convert them into equations.
x) Solve problems involving matrices and calculate determinants of matrices.
xi) Employ various properties of plane geometry involving polygons, angles and circles to solve related problems.
xii) Calculate volumes of solid shapes.
xiii) Define trigonometric ratios and apply them in solving problems of triangles and bearings.
xiv) Coordinate geometry.
xv) Discuss various ways of representations of statistical data and solve simple probability problems.
xvi) Define differentiation and integration of a function and apply the definitions in solving related problems.
xvii) Solve simple problems of permutations and combinations.

Working through this course

Course materials

You will need some materials to have a successful study trip through this course. The materials are mainly the reference textbooks listed below under set textbooks.

One of your essential tools for this course is a computer system with the minimum configuration listed below:

i) Pentium Processor (preferably an Intel Processor, 233 MHz or above)
ii) 10G hard disk space
iii) 64 MB RAM
iv) Standard keyboard
v) Mouse and pad
vi) 14" SVGA display colour monitor
vii) Windows 98 or a better operating system
viii) Multimedia components

Study units

This course is made up of five (5) modules of thirty (30) units. The titles of the units are as presented below:

Units	Titles
1	Number bases
2	Indices
3	Logarithms
4	Surds
5	Venn diagrams
6	Sequences: arithmetic progressions
7	Sequences: geometric progressions
8	Sets and their properties
9	Operations of sets
10	Linear equations in one variable
11	Simultaneous linear equations
12	Solutions of simultaneous equations by graphical method
13	Quadratic equations
14	Graphs of quadratic equations
15	Inequalities and their solutions
16	Variations
17	Matrices and determinants
18	Plane geometry: polygons and angles
19	Circles and their properties
20	Volumes of solids
21	Trigonometric ratios
22	Applications trigonometry: bearings
23	Coordinate geometry
24	Representation of statistical data
25	Graphical representations in statistics
26	Probability
27	Differentiation of a function
28	Applications of differentiation
29	Simple integration
30	Permutations and combinations

Set textbooks and other resources

References and other resources for this course are as follows:

Backhouse, J. K. and S. P. T. Houldsworth (1999): *Pure Mathematics I*, Longman.
MAN (1991): *Senior Secondary Mathematics, Books 1 – 3*, University Press.
Adamu, S. O. and T. L. Johnson (1975): *Statistics for Beginners*, Onibonoje Press.
Clarke, L. H. and C. G. Lambe (1976): *Advanced Level Mathematics (Pure and Applied)*, Hodder & Stoughton.
Abbott, P. (1973): *Teach Yourself Geometry*, ELBS.
Abbott, P. (1973): *Teach Yourself Calculus*, ELBS.
Godman, A. and J. F. Talbert (1975): *Additional Mathematics (Pure and Applied)*, Longman.
Hayslett, H. T. (1981): *Statistics Made Simple*, Heinemann.

Assessment

The assessment in this course are made up of tutor-marked assignments and your final examination at the end of the course.

Tutor-marked assignments

There are tutor-marked assignments (TMAs) at the end of each unit. You are expected to do the assignments and submit them to your tutor for assessment.

Final examination and grading

The successful completion of this course depends on passing an examination at the end of the course and the satisfaction of all the conditions given by the National Open University of Nigeria.

Course marking scheme

You can have access to the solutions of all your assessment questions from your tutor for the course.

How to get the most from this course

To get the most from this course, you need to keep diligently to the study guidelines given by the National Open University of Nigeria. Such guidelines include the study hours expected to put into your study and doing all your assessments.

Summary

This course is an indispensable course you need as a trainee preparing for a degree programme in any of the sciences or science-related programmes. For example, you cannot be sufficiently prepared to pursue a successful programme in computer science if you do not have a good background in mathematics because computer programming utilises a great deal of mathematical principles and expressions.

It takes discipline and constant practice to master almost any mathematics topic. Hence you need to study closely all the examples and also do all the exercises in this course. A list of a number of references is included in this course to give you opportunity to explore the rich resources you need to build yourself up as a good student of basic mathematics.

Tutors and tutorials

You need to interact with your tutors and attempt all tutorial classes for a better understanding of this course.

Unit 1 Number bases

Contents

1.1 Introduction

Generally, number systems have bases or radixes (or scales) around which they revolve. For example, our most familiar number system, that is, the decimal system is based on number 10 and it has 10 symbols or digits. It has generally been assumed that the use of the decimal system was developed from the fact that human hands have 10 fingers which are readily available for calculations.

However, there are other number systems which use different bases. These include the duodecimal system (base 12), the binary system (base 2), the octal system (base 8), the hexadecimal system (base 16), etc.

In this unit, you will be exposed to the principles of operations in number bases. Though it would not be possible to cover all the number systems in this single unit, however, you will enjoy the unit.

1.2 Objectives

By the end of this unit you will be able to:

i) Identify different number systems operating on different number bases like the binary number bases like the binary number systems (base two); the octal number systems (base eight) etc.;
ii) Carry out arithmetic operations in some number systems.

1.3 Number bases

Some general properties

Before going into the treatment of each number bases; there are some properties or concepts that are common to all the number systems. If this is well understood in the decimal system, which you are most familiar with, it will enhance your understanding of other number systems. The properties include the following:

i) **Base**

Each of the number systems has a base, i.e. a number which indicates the number of digits used in the system. For example, base 10 system uses 10 digits; base two uses two digits, etc.

ii) **Digit value**

This the value attached to each digit.

iii) **Positional notation**

Specific weights are attached to each digit depending on its position in the number under consideration. For example, the number 245 in the decimal system means:-

$245 = 2 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$

$= 200 + 40 + 5$

Similarly, number *abc* in base two means:

$a \times 2^2 + b \times 2^1 + c \times 2^0$

What will a number *xyz* mean in base 5?

Write down your answer. Now the following general rule is a guide for re-writing a number given in any base in base 10.

Guiding rule

If B = base of a given number; d = digit value; and n = number of digits in the given number (whole number digits only), then re-writing the number in base ten, we have:

$d \times B^{n-1} + d \times B^{n-2} + d \times B^0$

when n = whole number digits only.

Example 1.1

Re-write 263_{five} in base ten.

Solution

B= 5, n= 3 (i.e. number of digits in the given number =3)

d = digit value (i.e. value of each digit in the number)

The required number is:

$2 \times 5^{(3-1)} + 6 \times 5^{(3-2)} + 3 \times 5^{(3-3)}$

$= 2 \times 5^2 + 6 \times 5^1 + 3 \times 5^0$

$= (2 \times 25) + (6 \times 5) + (3 \times 1)$, since $5^0 = 1$

$= 50 + 30 + 3$

$= 83_{ten}$

The binary number system (base two)

The binary number system has a base of 2, (or radix of 2), therefore, it uses only two symbols, these are 0 and 1. Apart from these stated facts, the rules for operating the binary number system are similar to that of decimal system (or base 10).

Converting a number from binary to decimal system

To convert a binary number into a decimal number; you need to take the following procedures:

i) Write down the binary number.

ii) Write out the binary number in the expanded form using the guiding rule stated earlier.

iii) Simplify and add up the products.
iv) The final result you obtained from the addition of the products is the required number in base 10.

Example 1.2
Rewrite the number 10110_{two} in base ten.

Solution
B = 2, n = 6 and d= value of each digit, then the required number is:

$$(1 \times 2^{6-1}) + (0 \times {}^{26-2}) + (1 \times 2^{6-3}) + (1 \times 2^{6-4}) + (0 \times 2^{6-5}) + (1 \times 2^{6-6})$$
$$= (1 \times 2^{5}) + (0 \times 2^{4}) + (1 \times 2^{3}) + (1 \times 2^{2}) + (0 \times 2^{1}) + (1 \times 2^{0})$$
$$= 32 \quad + \quad 0 \quad + \quad 8 + \quad 4 + \quad 0 + \quad 1$$
$$= 45_{ten}$$

Example 1.3
Convert 110101.1011_{two} to base 10

Note: that this binary number contains a decimal point. However the same principle is involved; after the decimal point, you will still continue to multiply the digit to the right of the decimal point by descending powers of 2 (the base) from 0. This means that the power of two to multiply with after zero is -1 (i.e. 2^{-1}) and in that order. This will be further illustrated in the following solution to this example.

Solution
B=2, n = 6 (remember that n represents whole number digits only) and d = value of each digit.
$\therefore$ The required number is given by:

$$(1 \times 2^{6-1}) + (1 \times 2^{6-2}) + (0 \times 2^{6-3}) + (1 \times 2^{6-4}) + (0 \times 2^{6-5}) + (1 \times 2^{6-6}) + (1 \times 2^{6-7}) + (0 \times 2^{6-8})$$
$$+ (1 \times 2^{6-9}) + (1 \times 2^{6-10})$$
$$= (1 \times 2^{5}) + (1 \times 2^{4}) + (0 \times 2^{3}) + (1 \times 2^{2}) + (0 \times 2^{1}) + (1 \times 2^{0}) + (1 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4})$$
$$= 32 + 16 + 0 + 4 + 0 + 1 + \tfrac{1}{2} + 0 + \tfrac{1}{8} + \tfrac{1}{16}$$
$$= 32 + 16 + 0 + 4 + 0 + 1 + 0.5 + 0 + 0.125 + 0.0625$$
$$= (53.6875)_{ten}$$
$$= 53.6875$$

Conversion from decimal to binary

Decimal numbers are converted to binary forms through the method of repeated division. The following example illustrates this:

Example 1.4
Convert 45_{ten} to base two.

Solution
Divide 45 by two repeatedly until you are left with 0 as follows:

2	45	
2	22 R 1	(i.e. remaining 1)
2	11 R 0	
2	5 R 1	
2	2 R 1	
2	1 R 0	
	0 R 1 ↑	

Now write out the remainders from the one at the bottom to the topmost (as shown by the arrow) as follows: 101101.
The result obtained is now in base two
$\therefore 45_{ten} = 101101_{two}$

To check: Write out the binary form in the expanded notation to get the following:
$(1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$
$= 32 + 0 + 8 + 4 + 0 + 1$
$= 45_{ten}$ (which is the original number)

Exercise 5.1

1. Convert the following base two numbers to base ten numbers:
   a) 101011
   b) 1110
   c) 11011.11
   d) 101111.10

2. Convert the following numbers in base ten to base two
   a) 64
   b) 91
   c) 124
   d) 320

Mathematical operations with binary numbers

The operations of binary arithmetic are guided by simple rules. These are discussed as follows:

Addition of binary numbers

To add binary numbers, the guiding rules are:

$0 + 0 = 0$	$1 + 0 = 0$
$0 + 1 = 0$	$1 + 1 = 10$

Note that when adding a set of numbers and you have $1 + 1 = 10$ in the process, the 0 is to be written down while 1 is carried over just as in the decimal system.

Example 1.5
Add the following binary numbers:
11001, 1011, 11101

Solution

$$\begin{array}{r} 11001 \\ 1011 \\ +\quad 11101 \\ \hline 1000001 \\ \hline \end{array}$$

To check: Write the decimal equivalent of each binary number and add them together in that decimal form. Then convert the binary sum obtained to decimal form; compare the two results, they should be the same, if there is no error. For example, from the above, you have

$11001 = (1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$
$= 16 + 8 + 0 + 0 + 1$
$= 25_{ten}$

$1\,0\,1\,1 = (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$

$\quad 8 + 0 + 2 + 1$

$= 11_{ten}$

$1\,1\,1\,0\,1 = (1 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$

$= 16 + 8 + 4 + 0 + 1$

$= 29_{ten}$

Adding the numbers, you have

$25 + 11 + 29 = 65_{ten}$

Now, convert the binary sum to decimal:

$1000001 = (1 \times 2^6) + (0 \times 2^5) + (0 \times 2^4) + (0 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$

$= 64 + 0 + 0 + 0 + 0 + 0 + 1$

$= 65$

Since the two sides are equal, it means the binary sum is correct.

Subtraction of binary numbers

Subtraction is done in the same way as it is carried out under the decimal system.

Example 1.6

Subtract 1100 from 11101

Solution

$$
\begin{array}{r}
11101 \\
-\ 1100 \\
\hline
10001 \\
\hline
\end{array}
$$

Multiplication of binary numbers

In multiplying binary numbers, you must take note of the following guiding rules:

i) $0 \times 0 = 0$ iii) $1 \times 0 = 0$

ii) $0 \times 1 = 0$ iv) $1 \times 1 = 1$

Apart from the above, multiplication of binary numbers obeys the same principle of long multiplication as done under the decimal system.

Example 1.7

Multiply 1110 by 111

Solution

$$
\begin{array}{r}
1110 \\
\times\ 111 \\
\hline
1110 \\
1110 \\
1110 \\
\hline
1100010 \\
\hline
\end{array}
$$

To check correctiness:
Convert the binary numbers into decimal as follows:

Binary		Decimal
1 1 1 0	$\Rightarrow$	1 4
$\times$ 1 1 1	$\Rightarrow$	$\times$ 7
1 1 0 0 0 1 0	$\Rightarrow$	98

The multiplication is therefore correct.

Division of binary numbers
Division of binary numbers is carried out by the method of long division as done in the decimal system.

Example 1.8
Divide 10101_{two} by 111_{two}

Solution

```
       1 1
111 | 1 0 1 0 1
      1 1 1
      -----
        1 1 1
        1 1 1
        -----
        0 0 0
```

$\therefore$ Answer = 11_{two}

Example 1.9
Divide 10111_{two} by 100_{two}

Solution

```
     101.11
100 | 10111
      100
      ---
       111
       100
       ---
        110
        100
        ---
         100
         100
         ---
         000
```

$\therefore$ Answer = 101.11_{two}

To check correctness:
Convert each of the binary numbers to decimal, carry out the division in decimal system and then re-convert to binary and compare your values. You proceed as follows

$$10111_{two} = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$$
$$= 16 + 0 + 4 + 2 + 1$$
$$= 23$$

$100_{two} = (1 \times 2^2) + (0 \times 2^1) + (0 \times 2^0)$

$= 4 + 0 + 0$

$= 4$

Now, $23 \div 4 = \frac{23}{4} = 5.75$ .... (A)

From the answer obtained, you have

$101.11 = (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) + (1 \times 2^{-1}) + (1 \times 2^{-2})$

$= 4 + 0 + 1 + \frac{1}{2} + \frac{1}{4}$

$= 5 + 0.5 + 0.25 = 5.75$ ...... (B)

Since A = B, it means the division is correct.

Exercise 1.2

1. Find the sum of the following binary numbers:
   a) 1101 + 110 + 10 + 1110
   b) 10110 + 111101 + 1101
   c) 11011 + 111001 + 101011 + 111000

2. Carry out the subtraction as indicated in the following binary numbers:
   a) 11011010 – 1001110
   b) 1101101.101 – 100110.11

3. Multiply the following pairs of binary numbers:
   a) 1101 and 111
   b) 10101 and 101
   c) 111011 and 110

4. Divide the following binary numbers:
   a) 11101 ÷ 100
   b) 10011 ÷ 101 to three binary points.

The octal number system (base 8)

The octal number system is also known as base 8 and it uses symbols:
0, 1, 2, 3, 4, 5, 6, 7.

Conversion of octal number to decimal

To convert a number from octal (i.e. base 8) system to decimal (base ten) system, take the following steps:

i) Write out the number in the expanded form (as in the case of binary system).
ii) Evaluate and sum up the results.
iii) The results obtained will be in the decimal system.

Example 1.10

Convert the following octal numbers (base 8) to decimal, (i.e. base ten) numbers

i) $(2435)_{eight}$
ii) $(326.25)_{eight}$

Solution

i) $(2435)_8 = (2 \times 8^3) + (4 \times 8^2) + (3 \times 8^1) + (5 \times 8^0)$

$= (2 \times 512) + (4 \times 64) + (3 \times 8) + (5 \times 1)$

$= 1{,}024 + 256 + 24 + 5 = 1{,}309$

$= (1{,}309)_{ten}$

ii) $(326.25)_8 = (3 \times 8^2) + (2 \times 8^1) + (6 \times 8^0) + (2 \times 8^{-1}) + (5 \times 8^{-2})$

$= (3 \times 64) + (2 \times 8) + (6 \times 1) + (2 \times \frac{1}{8}) + (5 \times \frac{1}{64})$

$= 192 + 16 + 6 + \frac{1}{4} + \frac{5}{64}$

$= 214 + 0.25 + 0.0078125 = (214.26)_{ten}$

Conversion of decimal to octal number systems

A decimal number is converted to an octal number through a process of repeated division as was done under binary operation.

Example 1.11

Convert the following decimal number to a base 8 number: 2432

Solution

Divide by 8 repeatedly as follows:

8	2432
8	304 R 0
8	38 R 0
8	4 R 6
	0 R 4 ↑

Now write out the remainders from below upwards as shown by the direction of the arrow to get $(4600)_8$

$\therefore 2432_{ten} = 4600_{eight}$

Conversion between octal and binary systems

To convert a given number to another number in another base other than the decimal, the usual process is to go through the decimal. Therefore, to convert a number from base eight to base two, you will first convert the number from base eight to base ten and then convert the resulting number to base two.

Example 1.12

Convert the number $(163)_{eight}$ to base two.

Solution

First convert the number to base ten:

Then $163_8 = (1 \times 8^2) + (6 \times 8^1) + (3 \times 8^0)$

$= 64 + 48 + 3 = 115_{ten}$

Then convert 115_{ten} to base two as follows:

2	115
2	57 R 1
2	28 R 1
2	14 R 0
2	7 R 0
2	3 R 1
2	1 R 1
	0 R 1 ↑

Now write the remainder upward to get $(1110011)_2$.

Exercise 1.3

1. Convert the following octal numbers to decimals:

a) 243	c) 352.26
b) 55	d) 1467.73

2. Convert the following decimal numbers to octal numbers:

a) 27	c) 1453
b) 98	d) 921

3. Convert the following octal numbers to their base two equivalents

a) 452	c) 4325
b) 14.3	d) 655

Mathematical operations of octal numbers

Addition and subtraction of octal numbers

Arithmetic operations of addition, subtraction, multiplication and division of octal numbers are quite easy. It is very much similar to that of the decimal except that the symbols 8 and 9 are absent.

Example 1.13

i) Add the following octal numbers:
132_{eight}, $(235)_8$, 473_{eight}

ii) Add 1477_{eight} and 566_{eight}

Solution

i)
$$\begin{array}{r} 132 \\ +\,235 \\ 473 \\ \hline 1062_{eight} \\ \hline \end{array}$$

Remarks

Note that when you add any figures and their sum is more than eight, divide the sum obtained by 8, carry the quotient over, and write down the remainder under the corresponding column.

ii)
$$\begin{array}{r} 1477 \\ +\,566 \\ \hline 2265_{eight} \\ \hline \end{array}$$

Example 1.14

Subtract the following numbers as stated below:

a) 575_8 from 763_8

b) 356_8 from 625_8

Solution

a)
$$\begin{array}{r} 763 \\ -\,575 \\ \hline 166_8 \\ \hline \end{array}$$

b) $\begin{array}{r} 625 \\ -\ 356 \\ \hline 247_8 \end{array}$

Note that the principles of subtraction is the same but when you 'borrow' a unit from a higher column to a lower one, that unit becomes 8 in the lower column.

Multiplication of octal numbers

Multiplication of octal numbers is like multiplication in any other base; but remember that in this case, the highest number is 7.

Example 1.15

Carry out the following multiplication

a) 145_8 by 13_8

b) 342_8 by 121_8

Solution

a) $\begin{array}{r} 145 \\ \times\ 13 \\ \hline 457 \\ 145 \\ \hline 2127_8 \end{array}$

b) $\begin{array}{r} 342 \\ \times 121 \\ \hline 342 \\ 704 \\ 342 \\ \hline 43602_8 \end{array}$

Division of octal numbers

Division of octal numbers is carried out exactly the same way as in the decimal and binary systems. You only need to always bear it in mind that the highest digit weight is 7.

Example 1.16

Carry out the following division in base 8:

a) $257 \div 7$

b) $16343 \div 21$

Solution

a) $\begin{array}{r|l} & 31 \\ \hline 7 & 257 \\ & \underline{25} \\ & 7 \\ & \underline{7} \\ & 0 \end{array}$

Answer = 31_{eight}

$$
\begin{array}{r}
663 \\
21 \overline{)16343} \\
\underline{146} \\
154 \\
\underline{146} \\
63 \\
\underline{63}
\end{array}
$$

b)

Answer = 663_8

Exercise 1.4

1. Perform the following additions in base eight:
   a) $241_8 + 365_8$
   b) $624_8 + 4502_8 + 123_8$
   c) $453_8 + 26_8 + 510_8$

2. Perform the following subtractions in base eight:
   a) $360_8 - 216_8$
   b) $1133_8 - 664_8$
   c) $25356_8 - 1166_8$

3. Multiply the following pairs of numbers in base eight:
   a) 2145 by 34
   b) 264 by 123
   c) 230.45 by 64
   d) 314.12 by 54

4. a) Divide 10624_{eight} by 137_{eight}
   b) Divide 15210.14_{eight} by 56_{eight}

1.4 Conclusion

This unit has exposed you to some other number systems apart from the decimal system. These are the binary number system (base two) and the octal number system (base eight). The binary number system, for example, has a great relevance to computer operations. This is because the computer machine understands only 0 or 1. Therefore, the study will be of immense benefit to you in this computer age when no one can afford to be a computer illiterate.

1.5 Summary

In this unit you have learnt:
i) how to identify some common number systems with their bases;
ii) how to carry out arithmetic operations in some number systems.

In your future advanced studies in number systems what you have learnt in this unit would serve as a good foundation.

1.6 Tutor-marked assignment

1. a) Convert the following binary numbers (base two) into decimal numbers (base ten):
      i) 11001011
      ii) 111010

b) Convert the following decimal numbers to base two:

i) 81

ii) 217

2. Perform the indicated operations on the following octal (base eight) numbers:

i) $642_{eight} + 225_{eight} + 420_{eight}$

ii) $33456_{eight} - 1076_{eight}$

iii) Multiply $(4265)_8$ by $(35)_8$

iv) $106245_{eight} \div 137_{eight}$

1.7 References

Clarke, H. L., (1965). *Modern Mathematics at Ordinary Level*, Heinemann, London. p. 3 – 18.

NERDC (2001). *Further Mathematics for Senior Secondary Schools*, Longman Nigeria p. 259 – 280.

Unit 2 Indices

Contents

2.1 Introduction

In your previous study of mathematics you must have learnt how to express numbers in their standard forms; you are likely able to identify bases and indices of numbers. In this unit, you will study more about number bases and indices.

2.2 Objectives

By the end of this unit, you should be able to:
i) state the laws of indices;
ii) evaluate expressions containing indices;
iii) solve equations involving indices.

2.3 Positive indices

You known from previous work that:
i) $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$
ii) $81 = 3 \times 3 \times 3 \times 3 \qquad = 3^4$
In (i), 2 is called the 'base' while 6 is defined as the 'index' and in (ii), 3 is called the base while 4 is the index (the plural form of index is indices).

Now, 2^6 and 3^4 are in index forms of 64 and 81 respectively.

In general a^m means $a \times a \times a$ ...... to m factors, where a is the **base** and m is the **index** or 'power' or 'exponent'.

Example 2.1
$a^4 = a \times a \times a \times a$
$b^5 = b \times b \times b \times b \times b.$

There are three fundamental (basic) rules for working with indices that are positive integers (i.e., numbers such as 1, 2, 3 ......). They are derived as shown below:
i) Consider the following expression:
$(a \times a \times a) \times (a \times a \times a \times a) = a \times a \times a \times a \times a \times a \times a = a^7$
$= a^3 \times a^4 = a^7$

Now, you will notice that the indices on the left hand side are summed up to obtain the index on the right hand side, i.e. 3 + 4 = 7.

Now, look at this next example.

ii) $(a \times a \times a \times a \times a) \times (a \times a \times a) = a \times a \times a \times a \times a \times a \times a \times a = a^8$

$\Rightarrow a^5 \times a^3 = a^8$

but $a^{5+3} = a^8$ (summing the indices)

$a^5 \times a^3 = a^{5+3} = a^8$

So, in general when numbers in index from but having the same base are multiplied, the result is given by the base raised to the sum of the indices expressed as a rule below:

Rule 1: $a^m \times a^n = a^{m+n}$

Example 2.2

Evaluate the following and give your answers in index form:

i) $3^2 \times 3^7$

ii) $5^4 \times 5^6$

iii) $x^3 \times x$

Solution

i) $3^2 \times 3^7 = 3^{2+7} = 3^9$

ii) $5^4 \times 5^6 = 5^{4+6} = 5^{10}$

iii) $x^3 \times x = x^{3+1} = x^4$

Consider the following divisions

i) $\frac{a^6}{a^3} = a \times a \times a \times a \times a \times a \div a \times a \times a = \frac{a \times a \times a \times a \times a \times a}{a \times a \times a} = a^3$

Therefore, you will observe that 6 – 3 = 3 (subtracting their indices)

$\Rightarrow \frac{a^6}{a^3} = a^{6-3} = a^3$

ii) $a^{10} \div a^8 = \frac{a^{10}}{a^8} = \frac{a \times a \times a \times a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a \times a} = a^2$

Note also that 10 – 8 = 2 (subtracting indices)

$\therefore \quad a^{10} \div a^8 = a^{10-8} = a^2$

You can therefore also deduce that, when two numbers having the same base are divided, the result is given by the base raised to the difference between their indices. This becomes your second rule:

Rule 2: $a^m \div a^n = a^{m-n}$

Example 2.3

Evaluate the following, giving your answers in index forms:

i) $5^{16} \div 5^9$

ii) $7^{13} \div 7^6$

iii) $y^5 \div y^2$

Solution

i) $5^{16} \div 5^9 = 5^{16-9} = 5^7$

ii) $7^{13} \div 7^6 = 7^{13-6} = 7^7$

iii) $y^5 \div y^2 = y^{5-2} = y^3$

Consider the following multiplication:

$a^3 \times a^3 \times a^3 \times a^3 = (a^3)^4$

$= (a \times a \times a)\ (a \times a \times a)\ (a \times a \times a)\ (a \times a \times a) = a^{12}$

But $(a^3)^4 = a^{3\times4} = a^{12}$ (multiplying their indices)
So, in general, you again have the next rule:

Rule 3: $(a^m)^n = a^{mn}$

Example 2.4
Evaluate the following:
i) $(5^4)^3$
ii) $(3^6)^4$
iii) $(b^3)^2$

Solution
i) $(5^4)^3 = 5^{(4\times3)} = 5^{12}$
ii) $(3^6)^4 = 3^{6\times4} = 3^{24}$
iii) $(b^3)^2 = b^{3\times2} = b^6$

2.4 Fractional indices

In this section, you will be considering cases when your indices are fractions.

Evaluating $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$ (using Rule 2)

which also means that $(5^{\frac{1}{2}})^2 = 5$

Therefore, $5^{\frac{1}{2}} = 2\sqrt{5}$ or $\sqrt{5}$ (positive root only)

Now, look at the following example:

$$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$$

Hence $(8^{\frac{1}{3}})^3 = 8$

$\therefore\ 8^{\frac{1}{3}} = 3\sqrt{8}$

Similarly, $a^{\frac{1}{3}} = 3\sqrt{a}$

$\therefore$ In general

$$a^{\frac{1}{n}} = n\sqrt{a}$$

Consider also $8^{\frac{2}{3}}$

$$8^{\frac{2}{3}} \times 8^{\frac{2}{3}} \times 8^{\frac{2}{3}} = (8^{\frac{2}{3}})^3 = 8^{\frac{2}{3}\times\frac{3}{1}} = 8^2$$

Now if $(8^{\frac{2}{3}})^3 = 8^2$, then $8^{\frac{2}{3}} = 3\sqrt{8^2}$

(by taking cube roots on both sides)
So, in general,

$$a^{\frac{1}{n}} = n\sqrt{a}$$

Therefore, you have:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

When $m = 1$, then $a^{\frac{m}{n}} = a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a}$

Example 2.5
Evaluate:

i) $81^{\frac{1}{4}}$

ii) $27^{\frac{1}{3}}$

iii) $(\frac{16}{25})^{\frac{1}{2}}$

Solution

i) $81^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3^1 = 3$

ii) $27^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^1 = 3$

iii) $(\frac{16}{25})^{\frac{1}{2}} = \frac{(4^2)^{\frac{1}{2}}}{(5^2)^{\frac{1}{2}}} = \frac{4^1}{5^1} = \frac{4}{5}$

Exercise 2.1

1 Evaluate the following indices:

i) $2^{11} \times 2^{15}$	vi) $6^2 \times 6^8$
ii) $5^{14} \div 5^7$	vii) $a^p \times a^q$
iii) $3^{19} \div 3^{12}$	viii) $a^3 \times a^5$
iv) $(7^8)^2$	ix) $a^x \times a^y \times a^z$
v) $(3^7)^3$	x) $y^2 \times y^9 \times y^2$

2.5 Zero and negative indices

a) **Meaning of a^0**

You will now see what zero index means.
Consider the following examples:
$a^5 \div a^5 = a^{5-5} = a^0$ …… (i)
Also you can express the above as follows:

$$a^5 \div a^5 = \frac{a^5}{a^5} = 1 \text{ ...... (ii)}$$

But (i) and (ii) are correct results of the same problem.
Therefore,

$a^0 = 1$, where $a \neq 0$

Example 2.6

i) $7^0 = 1$

ii) $x^0 = 1$

iii) $(a + b)^0 = 1$

In the first section of this unit, you studied positive indices. Now, you will see what negative index means.

b) **Meaning of a^{-n}**

Evaluating $a^5 \div a^7 = a^{5-7} = a^{-2}$ ...... (i)

Also, $a^5 \div a^7$

Also, $a^5 \div a^7 = \frac{a^5}{a^7} = \frac{a \times a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a}$

i.e. $\frac{\not{a} \times \not{a} \times \not{a} \times \not{a} \times \not{a}}{\not{a} \times \not{a} \times \not{a} \times \not{a} \times \not{a} \times a \times a} = \frac{1}{a^2}$ ...... (ii)

But (i) and (ii) are again correct results of the same problem.

Hence (i) = (ii)

$\therefore a^{-2} = \frac{1}{a^2}$

In general, $a^{-n} = \dfrac{1}{a^n}$

Example 2.7

Evaluate the following:

i) 2^{-2}

ii) $(\frac{1}{2})^{-3}$

iii) $(\frac{27}{8})^{-\frac{2}{3}}$

Solution

i) $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

ii) $(\frac{1}{2})^{-3} = \frac{1}{(\frac{1}{2})^3} = \frac{1}{\frac{1}{8}} = 1 \div \frac{1}{8} = 1 \times \frac{8}{1} = 8$

Now, $(\frac{27}{8})^{-\frac{2}{3}} = \frac{1}{(\frac{27}{8})^{\frac{2}{3}}} = (\frac{8}{27})^{\frac{2}{3}}$

Since $8 = 2^3$ and $27 = 3^3$, you then have

$$(\frac{8}{27})^{\frac{2}{3}} = \frac{(2^3)^{\frac{2}{3}}}{(3^3)^{\frac{2}{3}}} = \frac{2^2}{3^2} = \frac{4}{9}$$

$$\therefore (\frac{27}{8})^{-\frac{2}{3}} = \frac{4}{9}$$

Exercise 2.2

Evaluate the following:

i) $(\frac{2}{3})^{-3}$

ii) 2^{-5}

iii) $(\frac{9}{16})^{-\frac{1}{2}}$

iv) $-2(3p - 2q + 3)^0$

v) $(64)^{\frac{1}{3}}$

vi) $\frac{5s + 3t}{(5s + 3t)^0}$

vii) $81^{\frac{1}{4}}$ viii) $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$

ix $2^{81} \div (2^{56} \times 2^{25})$ x) $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$

2.6 Conclusion and summary

In this unit, you have been introduced to all the basic laws for operating indices. These laws include:

1 Laws of positive indices

i) $a^m \times a^n = a^{m+n}$

ii) $a^m \div a^n = a^{m-n}$

iii) $(a^m)^n = a^{mn}$

2 Fractional indices:

i) $a^{\frac{1}{n}} = \sqrt[n]{a}$

ii) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

3 Zero and negative indices:

i) $a^0 = 1$ where $a \neq 0$

ii) $a^{-n} = \frac{1}{a^n}$

What you have learnt in this unit is a good preparation for the next two units in particular, i.e. logarithms and surds. This is because they are very much related and you can hardly go through one without using the other.

Apart from this, you will also find indices handy as you explore other topics in mathematics. So, try to be fully conversant with the concept in order to be able to use them when needed.

2.7 Tutor-marked assignment

1 Evaluate the following

i) $(5x^3yz^2)(7x^2y^oz)$

ii) $(\frac{81}{16})^{\frac{3}{4}}$

2 i) Solve for y in the equation:

$2y^3 = 128$

ii) Solve for x in the following equation:

$2^{a^2} = 16^{a-1}$

2.8 References

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Unit 3 Logarithms

Contents

3.1 Introduction

Having gone through the study of indices, the next logical topic is Logarithms, as the two are quite interwoven. Often in an attempt to solve a problem in logarithms, you will need to use indices and vice versa. You will see the close relationship between logarithms and indices in this unit.

3.2 Objectives

By the end of this unit, you should be able to:

i) define logarithms and explain the related theorems;
ii) rewrite expressions with logarithms in index forms and vice versa;
iii) change the bases of logarithms when required;
iv) solve problems in logarithms.

3.3 Definition of logarithms

If $y = a^x$, then x can be defined as the **logarithm** of y to the base a.

If $y = a^x$, then $x = \log_a y$

Look at the following:

Since $100 = 10^2$, then $\log_{10} 100 = 2$

Since $81 = 3^4$, then $\log_3 8 = 4$

Since $\frac{1}{16} = 4^{-2}$, then $\log_4 \frac{1}{16} = -2$

The left-hand side of the expression is called the **exponential form** while the right-hand side is the **logarithm form**. Practise some conversions of one form of expression to the other until you can do it with ease.

3.4 Some basic deductions

The following three deductions can be made from the definition of logarithms.
Recall that:

1. If $y = a^x$ ...... (i)

   then, $\log_a y = x$ ...... (ii)

   If $x = 0$ in (i), then, $y = a^0 = 1$

   Therefore, in (ii) $\boxed{\log_a 1 = 0}$

   So, the **logarithm of unity** (or **one**) is **zero**.
2. By putting $x = 1$ in (i), then $y = a^1 = a$ (i.e. $y = a$)

   Therefore, in (ii) $\boxed{\log_a a = 1}$

Example 3.1

a) $\log_{10} 10 = 1$ b) $\log_5 5 = 1$ c) $\log_x x = 1$

3. Let $y = a^x$, substituting in (ii),

   then $\boxed{\log_a a^{x=y}}$

Examples 3.2
Without using tables, evaluate:

a) $\log_2 16$ b) $\log_3 \frac{1}{9}$ c) $\log_5 0.04$

Solution

a) $\log_2 16 = \log_2 2^4 = 4$

b) $\log_3 \frac{1}{9} = \log_3 3^{-2} = -2$

c) $\log_5 0.04 = \log_5 \frac{1}{25} = \log_5 5^{-2} = -2$

Exercise 3.1

1 Write the following exponential expression in logarithmic forms:

a) $4^2 = 16$ b) $6^3 = 216$

c) $3^{-2} = \frac{1}{9}$ d) $4^{-4} = \frac{1}{256}$

e) $8^{-\frac{2}{3}} = \frac{1}{4}$ f) $3^5 = 243$

2 Write the following logarithmic expressions in exponential forms and then evaluate x.

a) $x = \log_7 49$ b) $x = \log_{\frac{1}{5}} 125$

c) $x = \log_2 32$ d) $x = \log_2 64$

e) $x = \log_2 32$ f) $x = \log_{10} 0.0001$

3 Without using tables, evaluate the following:

a) $\log_2 32$ b) $\log_3 81$

c) $\log_4 \frac{1}{256}$ d) $\log_{10} 0.001$

3.5 Logarithmic rules or theorems

First theorem

$\log_a (M \times N) = \log_a M = \log_a N$

Proof

Let $M = a^x$ ...... (i)

then, $\log_a M = x$ (by definition)

Similarly,

let $N = a^y$ ...... (ii)

then, $\log_a N = y$

From (i) and (ii), $M \times N = a^x \times a^y = a^{x+y}$ (1st law of indices)

$\log_a (M \times N) = x + y$

$\Rightarrow \log_a M \times N = \log_a M + \log_a N$

since $x = \log_a M$ and $y = \log_a N$.

Second theorem

$\log_a \frac{M}{N} = \log M - \log N$

Proof

Using the same notation as in (i) and (ii), you have

$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$ (2nd law of indices)

$\therefore \log_a \frac{M}{N} = x - y$

$\Rightarrow \log_a \frac{M}{N} = \log_a M - \log_a N$

Third theorem

$\log_a M^p = P \log_a M$

Proof

Let $M = a^x$ ...... (i)

then, $\log_a M = x$ $\therefore \log_a M^P = px$ or xp

From (i), $M^p = (a^x)^p = a^{xp}$ (by the law of indices)

$\therefore \log_a M^p = px$ or xp

substituting, $\log_a M^p = P \log_a M$

Fourth theorem

$$\log_a M = \frac{\log_a M}{\log_a b}$$

Proof

Let $M = b^x$ ...... (i)

Then, $\log_b M = x$ ..... (ii)

Taking the log of both sides to base a in (i),

$\therefore\ x = \frac{\log_a M}{\log_a b}$ (making x the subject)

then, $\log_a M = \log_a b^x = x \log_a b$ (from third theorem)

$\log_b M = \frac{\log_a M}{\log_a b}$ (substituting for x in (ii)

Example 3.3

Given that $\log_a pq = \log_a P + \log_a q$

Find $\log_{10} 5 + \log_{10} 20$

Solution

$\log_{10} 5 + \log_{10} 20 = \log_{10} (5 \times 20)$

$\log_{10} (5 \times 20) = \log_{10} 100$

Now, let $\log_{10} 100 = x$

i.e. $10^x = 100 = 10^2$

and $x = 2$

Therefore, $\log_{10} 5 + \log_{20} 20 = 2$

Example 3.4

Solve the equation $\log_a (5x - 6) + \log_a (2x + 3) = \log_a (10x^2 - 3x - 5)$

Solution

Using the logarithmic addition theorem, (i.e. 1st theorem)

LHS = $\log_a (5x - 6)(2x + 3) = \log_a (10x^2 - 3x - 5)$ = RHS

Taking the antilog (or the reverse of logarithm operation) on both sides, we have

$(5x - 6)(2x + 3) = 10x^2 - 3x - 5$

$10x^2 + 3x - 18 = 10x^2 - 3x - 5$

$10x^2 + 3x - 18 - 10x^2 + 3x + 5 = 0$

This gives

$6x = 18 - 5$

$6x = 13$

$\therefore\ x = \frac{13}{6} = 2\frac{1}{6}$

Example 3.5

Evaluate (a) log 5 (b) log 0.125 (if log 2 = 0.3010)

Remember that when the base is not indicated, you assume it is base 10.

Solution

a) $\log 5 = \log\frac{10}{2} = \log 10 - \log 2$

$= 1 - \log_2$ (since $\log_2 = \log_{10} 10 = 1$)

$= 1.0000 - 0.3010 = 0.6990$

b) $\log 0.125 = \log(\frac{1}{8})$

$= \log 1 - \log 8$ (or $\log 2^3$)

$= \log 1 - 3 \log 2$

$= 0 - 3 \times 0.3010$ (since $\log 1 = 0$)

$= -0.9030 = -1 + 0.0970$

$= \bar{1}.0970$

Remarks on Example 3.5

A notation has been introduced, i.e. a bar placed upon 1 ($\bar{1}$). Generally, the logarithm of a number has two parts: a whole number, called the **characteristic** and a decimal part, called the **mantissa**.

In Example 3.5, the characteristic is –1 and the mantissa is 0.0970. To emphasise that the characteristic is negative, a bar is used.

Example 3.6

Evaluate $\log_3 81 - \log_3 9$, given that

$\log_a \frac{p}{q} = \log_a p - \log_a q$

Solution

$\log_3 81 - \log_3 9 = \log_3 (\frac{81}{9}) = \log_3 9$

Now let $\log_3 9 = x$

Then, $3^x = 9 = 3^2$

i.e. $3^x = 3^2$ or $x = 2$

$\therefore \log_3 81 - \log_3 9 = 2$

Example 3.7

Using tables, evaluate $\log_2 5$

Solution

Now $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2}$ (charging the base to 10)

From tables, $\log_{10} 5 = 0.6990$, and

$\log_{10} 2 = 0.3010$

$\therefore \frac{\log_{10} 5}{\log_{10} 2} = 2.322$ (to 4 sig. figures)

Example 3.8

Solve the equation

$(\log_3 x)^2 - 6 \log_3 x + 9 = 0$

Solution

Let $\log_3 x = p$, then the equation becomes

$p^2 - 6p + 9 = 0$

Factorising

we have $p^2 - 6p + 9 = (p-3)(p-3)$

$(p-3)(p-3) = 0$

$p - 3 = 0$ (twice)

or, simply $p = 3$

Substituting $\log_3 x$ for p, we have

$\log_3 x = 3$ or $3^3 = x$

$\therefore x = 27.$

Exercise 3.2

1 Evaluate the following and find the values of x.

a) $\log_4 64 = x$ b) $\log_3 x = 2$

c) $\log_2 x = 3$ d) $\log_x 25 = 2$

e) $\log_{\frac{1}{2}} 8 = x$

2 Express the following as a single logarithm: (Base is 10 unless otherwise indicated)

a) $\log 2 - \log 3 + \log 5$ (b) $\frac{1}{2}\log 25 - \frac{1}{3}\log 64 + \frac{2}{3}\log 27$

c) $3\log 2 - 4\log 3$

3 Evaluate log 21, given that log 3 = 0.4771 and log 7 = 0.8451.

3.6 Summary

In this unit, you have learnt that:

i) If $y = a^x$, then $x = \log_a y$

ii) $\log xy = \log x + \log y$;

iii) $\log \frac{x}{y} = \log x - \log y$;

iv) $\log x^n = n \log x$;

v) $\log 1 = 0$;

vi) $\log_a a = \log 1$;

vii) $\log_a M = \frac{\log_a M}{\log_b a}$

3.7 Tutor-marked assignment

1 If $a = \log_b c$, $b = \log_c a$, $c = \log_a b$

Prove that $abc = 1$

2 Without using tables, show that

$$\frac{\log\sqrt{27} + \log\sqrt{8} - \log\sqrt{125}}{\log 6 - \log 5} = \frac{3}{2}$$

3.8 References

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Unit 4 Surds

Contents

4.1 Introduction

Surds, indices and logarithms are very related. They are, most of the time studied together. So, having being exposed to indices and logarithms, it is good that you study a related topic known as surd.

Numbers whose square roots cannot be determined in terms of rational numbers, e.g. $\sqrt{2}, \sqrt{3}, \sqrt{5},$ etc, are called surds. Such numbers occur frequently in trigonometry when calculating ratio of angles e.g. $\cos 30 = \sqrt{\frac{3}{2}}, \tan 60 = \sqrt{3}, \tan 30° = \frac{1}{\sqrt{3}};$ and in coordinate geometry, in the calculation of distances. You will therefore find it useful to have a sound knowledge of surds.

4.2 Objectives

By the end of this unit, you should be able to

i) state the meaning of surds;
ii) carry out operations of addition, subtraction, multiplication and division of surds;
iii) rationalise denominators of surds.

4.3 Definition of surd

You should be able to recall, from your previous knowledge of numbers, that a number that is a 'square' is one that can be expressed as the square of some other rational number.

For example, $9 = 3^2, 81 = 9^2$ and $\frac{4}{9} = \left(\frac{2}{3}\right)^2$. But not all numbers are rational numbers, that is, they do not have exact square roots. Examples are $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{8}, 2\sqrt{3},$ etc. The square roots of numbers that do not have exact square roots are called *surds*. Such numbers are called *irrational* numbers.

Although approximate square roots of such numbers (i.e. irrational numbers) can be found from square root tables, it is usually simpler to work with surds themselves. Note also that whenever you deal in square roots, only positive square roots are considered.

4.4 Reduction to basic forms

Any surd which contains a square number as a factor within the radical (i.e. the square root sign), is not in the basic form. For example, $\sqrt{27}, \sqrt{50}$ and $\sqrt{108}$ are not in the basic forms as they can still be reduced further through simplification.

Example 4.1

i) $\sqrt{27} = \sqrt{9\times3} = \sqrt{9}\times\sqrt{3} = 3\sqrt{3}$

ii) $\sqrt{50} = \sqrt{25\times2} = \sqrt{25}\times\sqrt{2} = 5\sqrt{2}$

iii) $\sqrt{108} = \sqrt{36\times3} = \sqrt{36}\times\sqrt{3} = 6\sqrt{3}$

From (i), (ii) and (iii) $3\sqrt{3}$, $5\sqrt{2}$ and $6\sqrt{3}$ cannot be simplified further, they have been reduced to the basic form. Surds that cannot be further simplified are said to be in their basic forms.

4.5 Adding and subtracting similar surds

Surds that are in basic form can be added and subtracted.

Example 4.2

Simplify $\sqrt{80} + \sqrt{20} - \sqrt{45}$

Solution

First reduce all to their basic forms.

That is, $\sqrt{80} + \sqrt{20} - \sqrt{45} = \sqrt{16\times5} + \sqrt{4\times5} - \sqrt{9\times5}$

RHS (i.e. Right hand side): $= 4\sqrt{5} + 2\sqrt{5} - 3\sqrt{5}$

$= 6\sqrt{5} - 3\sqrt{5}$

$= 3\sqrt{5}$

Notice that the above surds can be added or subtracted because they are in similar form, that is, numbers under the radicals are the same and they have the same index. Mixed surds such as $3\sqrt{2} + 2\sqrt{7} - 2\sqrt{3}$ are not similar, so they cannot be added or subtracted further, that is, they cannot be simplified further.

4.5 Multiplication and division of surds

Multiplication and division of surds are carried out by operating two basic laws of surds.

i) For multiplication of surds, the rule is:

$\sqrt{a}\times\sqrt{b} = \sqrt{ab}$

Example 4.3

a) $\sqrt{5}\times\sqrt{3} = \sqrt{5\times3} = \sqrt{15}$

b) $\sqrt{2}\times\sqrt{7} = \sqrt{2\times7} = \sqrt{14}$

ii) For division of surds, the rule is:

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Example 4.4

a) $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}$

b) $\frac{\sqrt{18}}{\sqrt{3}} = \sqrt{\frac{18}{3}} = \sqrt{6}$

Exercise 4.1

1. Reduce the following surds to their basic (or simplest) forms:

i) $\sqrt[3]{80}$ ii) $a\sqrt{9b^4c^3}$ iii) $2\sqrt{27}$

2. Evaluate the following surds:

   i) $\sqrt{18} + \sqrt{50} - \sqrt{72}$

   ii) $4\sqrt{75} + \sqrt{3\frac{3}{4}} - 2\sqrt{48}$

   iii) $\sqrt{3} + \sqrt[3]{81} - \sqrt{27} + 5\sqrt[3]{3}$

3. Multiply the following surds:

   i) $(2\sqrt{7})\,(3\sqrt{5})$

   ii) $(3\sqrt{5})\,(5\sqrt{2})$

4.7 Rationalising the denominator

A surd, such as $\frac{\sqrt{3}}{5}$, cannot be simplified further; but one, such as $\frac{2}{\sqrt{3}}$, can be written in a more convenient form, as it is not normal to have the radical in the denominator.

The process of removing the radical from the denominator is called *rationalisation*. However, to carry out rationalisation, you need to know about conjugate surds.

Conjugate surds

Given a surd $(a-\sqrt{b})$, its conjugate is defined as $(a + \sqrt{b})$ and vice-versa. When a surd is multiplied by its conjugate, their product is no more a surd. You will see this below.

For example, multiply $(a + \sqrt{b})$ by its congugate $a-\sqrt{b}$ to get

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - a\sqrt{b} + a\sqrt{b} - (\sqrt{b})(\sqrt{b})$$

$$= a^2 - (\sqrt{b})(\sqrt{b}) = a^2 - b$$

You have seen that the result is not a surd. Now, the rule is: to rationalise a radical in the denominator of a surd, multiply both numerator and denominator of that surd by the conjugate of its denominator.

Example 4.5

Rationalise the denominator of the following surds:

i) $\frac{1}{2-3\sqrt{3}}$ ii) $\frac{1}{2\sqrt{2}+\sqrt{7}}$ iii) $\frac{3\sqrt{2}-1}{3\sqrt{2}+1}$

Solutions

i) $$\frac{1}{2-3\sqrt{3}} = \frac{1}{2-3\sqrt{3}} \times \frac{2+3\sqrt{3}}{2+3\sqrt{3}}$$

[i.e. multiply the numerator and denominator by the conjugate of its denominator.]

Evaluating the RHS:

$$\frac{2+3\sqrt{3}}{2+3\sqrt{3}} = \frac{2+3\sqrt{3}}{4+\cancel{6\sqrt{3}}-\cancel{6\sqrt{3}}-9(\sqrt{3})^2}$$

$$= \frac{2+3\sqrt{3}}{4-27} = -\frac{1}{23}\ (2+3\sqrt{3})$$

As you can see, the denominator is no more a surd.

ii) The conjugate of $\frac{1}{2\sqrt{2}+7}$ is $\frac{1}{2\sqrt{2}-7}$

Rationalising, we have $\frac{1}{2\sqrt{2}+7} \times \frac{2\sqrt{2}-7}{2\sqrt{2}-7}$

$$= \frac{2\sqrt{2}-7}{(2\sqrt{2})^2 - \cancel{(7)(2\sqrt{2}+7)} + \cancel{(7)(2\sqrt{2})} - (7)^2}$$

$$= \frac{2\sqrt{2}-7}{4(2)-49} = \frac{2\sqrt{(2}-7}{-41} = -\frac{1}{41}(2\sqrt{2}-7)$$

iii) The conjugate of $3\sqrt{2}+1 = 3\sqrt{2}-1$

Rationalising, we have:

$$\frac{(3\sqrt{2}-1)}{(3\sqrt{2}+1)} \times \frac{(3\sqrt{2}-1)}{(3\sqrt{2}-1)}$$

$$= \frac{(3\sqrt{2})^2 - 3\sqrt{2} - 3\sqrt{2} + 1}{(3\sqrt{2})^2 - 3\sqrt{2} + 3\sqrt{2} - 1} = \frac{9(2) - (2)(3\sqrt{20} + 1}{9(2) - 1}$$

$$= \frac{18 - 6\sqrt{2} + 1}{18 - 1} = \frac{19 - 6\sqrt{2}}{17}$$

$$= \frac{1}{17}(19 - 6\sqrt{2})$$

Example 4.6

Rationalise the denominator in

$$\frac{1}{(\sqrt{5}-\sqrt{3})^2}$$

Solution
First, simplify the denominator by multiplying out: $(\sqrt{5} - \sqrt{3})(\sqrt{5} - \sqrt{3})$

$$= 5 - 2\sqrt{15} + 3$$

$$= 8 - 2\sqrt{15}$$

Now the expression is

$$\frac{1}{8 - 2\sqrt{15}}$$

Rationalising, we have:

$$\frac{1}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}} = \frac{8+2\sqrt{15}}{8^2 - 4 \times 15}$$

$$= \frac{8+2\sqrt{15}}{64-60} = \frac{8+2\sqrt{15}}{4}$$

$$= \tfrac{1}{4}(8 + 2\sqrt{15}) = \tfrac{1}{2}(4 + \sqrt{15})$$

Exercise 4.2

1. Rationalise the denominators of the following surds:

i) $\dfrac{3}{\sqrt{3}+\sqrt{2}}$ ii) $\dfrac{5}{2-\sqrt{7}}$

iii) $\dfrac{1-\sqrt{2}}{2+\sqrt{2}}$ iv) $\dfrac{3+\sqrt{3}}{5-\sqrt{3}}$

4.8 Conclusion and summary

In this unit, you have learnt that:

1. for numbers a and b:

i) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

ii) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

iii) $\dfrac{a}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$

3. learnt that the conjugate of surd at $a+\sqrt{b} = a-\sqrt{b}$ and that of $a-\sqrt{b} = a+\sqrt{b}$.
4. learnt that you can rationalise denominators of surds using their conjugates.

Having completed the unit on surds, you are further equipped to be able to successfully explore various topics in mathematics. In particular, note that mathematical concepts are like chains connected together; and most of the time, the knowledge of one helps to understand the other.

4.9 Tutor-marked assignment

1. Evaluate the following:
   (i) $(2\sqrt{7}) \times (3\sqrt{5})$

   (ii) $\sqrt{6.75} + \sqrt{75} - \sqrt{108}$

   (iii) $\sqrt{27} + \sqrt{48} + \sqrt{8} - \sqrt{32}$

2. Rationalise the denominator in the following:

   (i) $\dfrac{3}{2+\sqrt{5}}$ (ii) $\dfrac{5+\sqrt{2}}{3-\sqrt{2}}$

4.10 References

Backhouse, J.K. and S.P.T. Houldsworth (1970), *Pure Mathematics, A First Course*. Longman Group (Ltd), pp. 165–177.

N.E.R.D.C. (2001), *Further Mathematics for Senior Secondary Schools;* Longman Plc, Nigeria, pp. 53–64.

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Unit 5 Venn diagrams

Contents

5.1 Introduction

When set theory was being developed, certain diagrams were made which usually give pictorial representations of sets. These diagrams help in the better understanding of set operations already studied in the previous units.

The diagrams were named after John Venn (1834-1923), who invented and pioneered their use. That is why they are called 'Venn diagrams'.

5.2 Objectives

By the end of this unit, you should be able to:

i) use Venn diagrams to illustrate the various set operations;
ii) solve set problems involving numbers by means of Venn diagrams;
iii) generate formulae for solving sets problems and use them in solving give set problems.

5.3 Definitions

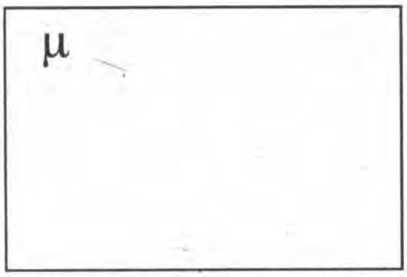
Venn diagrams are diagrammatic illustrations of sets used to show various set operations. You will recall that some of the set operational symbols that you studied in the last unit include operations for union, intersection, difference and complements of sets. Remember the laws guiding the operations of sets were studied in the last unit. In this unit, you will see those operations again and the use of Venn diagrams to illustrate the laws.

Illustrating set operations using Venn diagrams

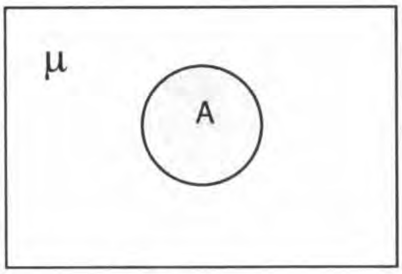
In using Venn diagrams to illustrate sets operations, the Universal set μ is normally represented by a rectangle while circles represent subsets. The diagrams in Figure 5.1 show some operations on sets.

General formulae for solving sets problems involving numbers

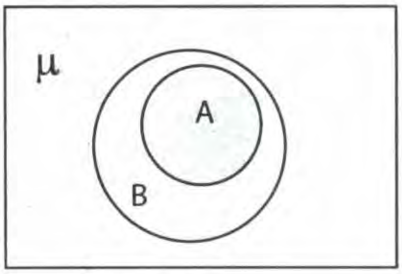
By means of Venn diagrams and set operations, general formulae for solving set problems can be derived. Consider the following examples.



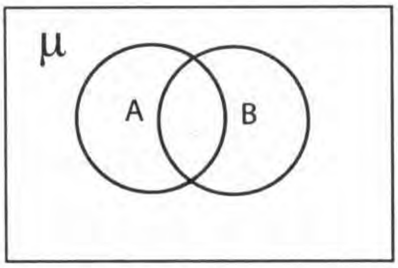
(a) Universal set is shaded



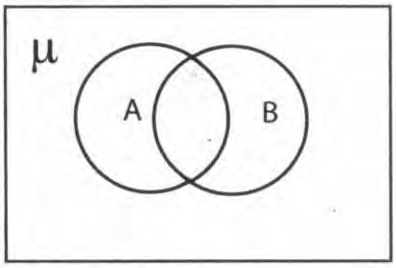
(b) A⊂U



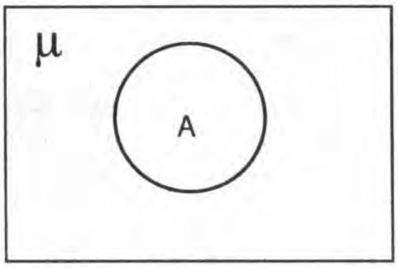
(c) A⊂B



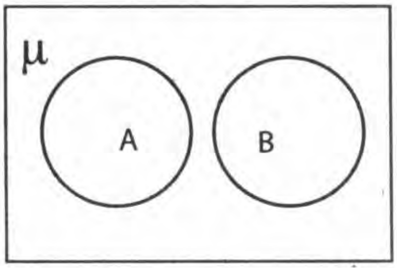
(d) A∪B is shaded



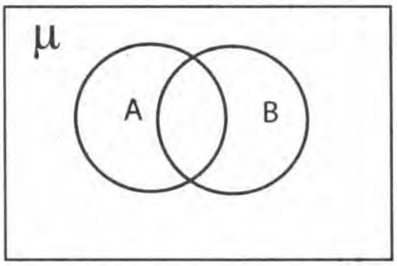
(e) A∩B is shaded



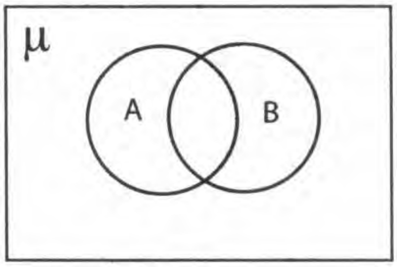
(f) A^1 is shaded



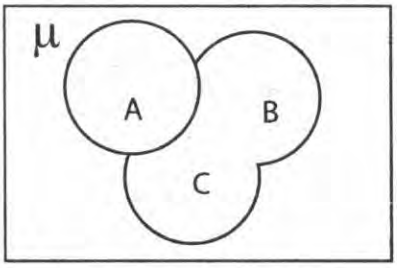
(g) A∩B = ϕ
A and B are disjoint



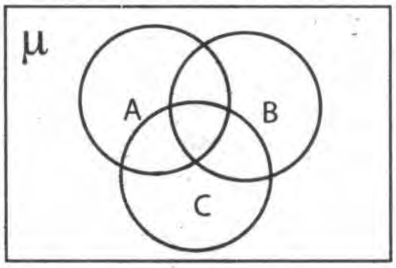
(h) A – B is shaded



(i) B – A is shaded



(j) A∪B∪C is shaded



(k) A∩B∩C is shaded

Fig. 5.1 Some operations on sets

Example 5.1
Let there be two sets A and B as shown in Fig. 5.2.

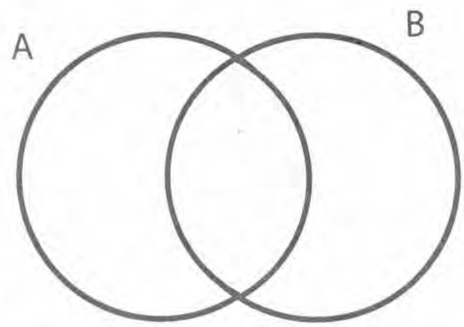


Fig. 5.2

As you can see, $(A \cup B)$ is the area surrounded by thick boundary and $(A \cap B)$ is the shaded area (since the sets intersect or overlap).

The sum of areas represented by the two circles = the sum of the whole area plus the shaded area.

Then, $nA + nB = n(A \cup B) + n(A \cap B)$.

In the case where the two sets are disjoint is then $nA + nB = n(A \cup B)$

Now use this formula to solve this problem:

From a group of students in a College of Arts and Social Sciences, 52% of the students study Geography, while 65% study Economics. If every student in the group studies either Geography or Economics. Find the percentage that study both subjects.

Solution

Let Geography students be represented by G and
Economics students by E

Then n(G) = Number of students studying Geography
n(E) = Number of students studying Economics

Therefore, $n(G) = 52\%$, $n(E) = 65\%$ and $n(G \cup E) = 100\%$

Those who study both subjects = $n(G \cap E)$

By the formula above:

$$[n(G) + n(E) = n(G \cup E) + n(G \cap E)]$$

Substituting, we have:

$52\% + 65\% = 100\% + n(G \cap E)$

$\Rightarrow (52 + 65 - 100)\% = n(G \cap E)$

$= (117 - 100)\% = n(G \cap E)$

$17\% = n(G \cap E)$

$\therefore$ 17% students study both subjects.

Go through the example again and ensure that you understand it.

Using the same principle, a formula for situations involving three sets can be derived by you. Consider three sets A, B and C. Let the thick boundary represent $A \cup B \cup C$. The shaded area is represented as $A \cap B \cap C$. (see Fig 5.3).

The letters *r, s, t,...* represent numbers in the areas where they stand.

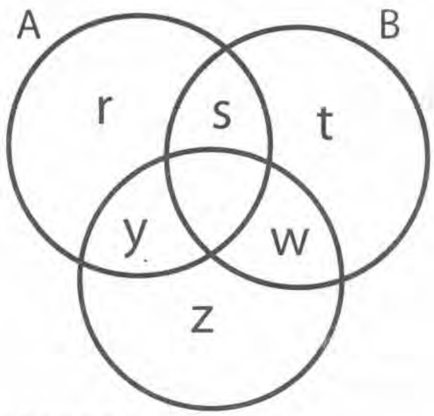


Fig. 5.3

Then n(A) = $r + s + y + x$

n(B) = $s + t + w + x$

n(C) = $w + y + z + x$

Now, n(A) + n(B) + n(C) = $(r + s + y + x) + (s + t + w + x) + (w + y + z + x)$

Re-grouping the addition, we have:

n(A)+n(B)+n(C) = $(r + s + t + w + y + z + x) + (s + x) + (w + x) + (y + x) - x$

By substituting the equivalent of what is in the bracket, on the RHS, we have:

n(A)+n(B)+n(C) = n(A∪B∪C) + n(A∩B) + n(B∩C) + n(A∩C) – n(A∩B∩C)

This is the required formula.

Now, this formula has been used to solve Example 5.2:

Example 5.2

On a particular day in a university campus, 1 000 undergraduates were sampled to study their dress patterns; the following observations were made:

(i) 271 of the students wore coats
(ii) 248 of them wore ties
(iii) 251 wore hats
(iv) 64 of them wore both coats and ties
(v) 97 wore both coats and hats
(vi) 59 wore ties and also wore hats
(vii) 434 did not wear coats, and did not wear ties or hats.

How many wore coats, ties and also put on hats?

Solution

From the above information and using set languages you can proceed as follows:

Let n(C) = {students who wore coats}
n(T) = {students wore ties}
n(H) = {students who wore hats}

Then,

n(C) = 271, n(T) = 248, n(H) = 251

n(C∩T) = 64, n(C∩H) = 97, n(T∩H) = 59

Also, n(C∪H∪T) = 1000 – 434 = 566

Now, n(C) + n(H) + n(T) = n(C∪H∪T) + n(C∩T) + n(C∩H) + n(T∩H) – n(C∩H∩T)

= 271 + 248 + 251 = 566 + 64 + 97 + 59 – n(C∩H∩T)

This becomes: 770 = 786 – n(C∩H∩T)

∴ n(C∩H∩T) = 786 – 770 = 16

∴ n(C∩H∩T) = 16

i.e. number of students who wore coats, ties and also put on hats.

Exercise 5.1

1 Using Venn diagrams, draw three sets P, Q, S, and shade the area representing $P \cup Q \cup S$.
2 Using Venn diagrams, draw three sets X, Y, Z. and shade the portion representing $X \cap Y \cap Z$.
3 In a Venn diagram, show sets E and F, and then the universal set. Shade the area for $(E \cup F)'$
4 In a Venn diagram, draw the sets A and B, and the universal set. Then shade the area representing $(A \cap B)'$

Exercise 5.2

1 In a particular state in Nigeria, everyone either speaks Hausa or English or both. It was found out that 72% speak English, while 44% speak Hausa. Find the percentage that speak both languages.
2 During a final year examination of a school, 36 students offered Biology, 32 offered Chemistry, 31 offered Physics while 4 students offered all the three subjects. How many students offered at least one of the subjects?
3 If A = {1, 2, 3}, B = {1, 2, 3 4} and C = {1, 2, 3, 4, 5}, verify that $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.
4 Let A = {a, b, c, d, e}, B = {b, d, e}, C = {d, e, f, g} and D = {b, e, f, g}. Write out the elements of sets:
   (i) $(A \cup C) \cup (A \cup D)$
   (ii) $(A \cup B) \cap (C \cup D)$

5.4 Conclusion and Summary

In this unit, you have learnt what Venn diagrams are and how they can be used to illustrate set operations, as well as solve problems on sets involving numbers. You have also learnt how to use them to derive other general formulae which offer alternative methods of solving set problems.

This unit has taken you a step further in your study of set theory. As you progress in your study of mathematics, you will find that set theory has permeated into other areas of mathematics. For example, we talk about set of integers, set of even or odd numbers. In coordinate geometry, we talk of set of ordered pairs and in algebra, we talk of a set of solutions of equations. Your knowledge of set theory will enhance your understanding of other topics in mathematics, as you will soon see even in some of the topics in the subsequent units.

5.5 Tutor-marked assignment

1 (a) Let A = {4, 5, 6}, B = {4, 5, 6, 7} and C = {4, 5, 6, 7, 8}, show that $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.
   (b) Let A = {p, q, r, s, t}, B = {$q, s, t,$}, C = {s, t, u, v} and D = {q, t, u, v}. Write down the elements of following sets.
      (i) $(A \cup C) \cup (A \cup D)$
      (ii) $(A \cup B) \cap (C \cup D)$
2 136 people were questioned as to the type of papers they read. It was found that 67 read Vanguard, 56 read New Nigerian and 40 read Punch. 11 read both Vanguard and New Nigeria 12 read New Nigerian and Punch and 9 read Vanguard and Punch. How many read all the three papers?

5.6 References

1 Clarke, L. H., (1968) *Modern Mathematics at Ordinary Level* , Heinemann Educational Books Ltd. England, p. 62-78.
2 N.E.R.D.C, (2001) *Further Mathematics for Senior Secondary Schools*, Longman Nigeria Plc.

Unit 6 Arithmetic progression

Contents

6.1 Introduction

This unit deals with ordering in number theory. Sometimes, numbers are written down in a particular order, guided by a rule. A set of numbers written down in such a pattern is called a **sequence**. When you go further to sum them up as they are, they are called a **series**. In this unit, we shall be examining a type of sequence (or series): the **arithmetic progression**.

6.2 Objectives

By the end of this unit, you should be able to:
i) identify whether or not a set of numbers forms a sequence or series;
ii) find the nth term of an arithmetic progression;
iii) sum up the series in an arithmetic progression;
iv) solve relevant everyday life problems.

6.3 Definitions

Sequence

A sequence is a set of numbers written down in a definite order, and having each of its terms formed according to a definite rule. Each member of a sequence is called a **term**.

	Sequence	Rule for forming the terms
(i)	2, 4, 6, 8, ...	add 2 to previous term
(ii)	3, 9, 27, 81, ...	multiply previous term by 3
(iii)	1, 4, 9, 16, ...	square the term number

The dots at the end of a sequence indicate that the sequence continues indefinitely. Therefore, it is possible to generate more terms of a sequence to any desired number.

Example 6.1

Generate three more terms of the series in (i) and (iii).

Solution

For (i), the next three terms are 10, 12, 14.
For (iii), the next three terms are 25, 36, 49.
Now, generate the next two terms in these two examples.

Series

When the terms of any sequence are added together, a series is formed, e.g. $2 + 4 + 6 + 8 + \ldots$ .and $1 + 3 + 5 + \ldots + 99$.

A series that has a definite number of terms, i.e. has an end is called a **finite series** while one that does not end, i.e. continues indefinitely is called an **infinite series**.

The first of the two examples given above is an infinite series while the second is a finite series.

Arithmetic progression (A.P)

Arithmetic progression is a series in which each term (after the first) is obtained by adding a constant quantity to the preceding term. The constant term that is being added is called the **common difference**. It is the difference between any two consecutive terms of the series and is denoted by d.

Example 6.2

a) $2 + 7 + 12 + 17 + \ldots, d = 5$.
b) $1 + 4 + 7 + 10 + \ldots, d = 3$.
c) $5 + 2 + (-1) + (-4) + \ldots, d = -3$.
d) $-2 + \frac{-3}{4} + \frac{1}{2} + 1\frac{3}{4} \ldots\ d = 1\frac{1}{4}$

The first 5 terms of an arithmetic progression (or arithmetic series) with 1 as the first term and 2 as the common difference are $1 + 3 + 5 + 7 + 9$. Now, obtain the first 5 terms of an arithmetic series having 2 as the first term and 3 as the common difference.

The nth term of an A.P

Let a denote the 1st term of an A.P., and d, the common difference, the formula of the nth term, i.e. last term l, can be obtained as follows:

No of terms	1st	2nd	3rd	.....	$(n\text{-}1)$th	nth
terms	a	$a+d$	$a+2d$	.....	$(a + (n\text{-}2)d$	$a+(n-1)d$

Therefore, for any A.P., the last (l or nth) term is given by $a + (n-1)d$,
i.e. $l = a+(n-1)d$ or nth term $= a + (n - 1)d$.

Example 6.3

What is the 8th term of the sequence: 10, 6, 2, -2, ...?

Solution

Here, $a = 10$, $d = 6 - 10$ or $2 - 6 = -4$ and $n = 8$.
$\therefore$ nth term $= a + (n\text{-}1)d = 10 + (8\text{-}1)(-4) = 10 + (-28) = 10 - 28 = -18$
$\therefore$ 8th term $= -18$

Example 6.4

Given an A.P. 9, 12, 15, 18, ..., generate the formula for the nth term.

Solution
Here $a = 9, d = 3$
l = nth term = $a + (n-1)d$
$= 9 + (n-1)\,3$
$= 9 + 3n-3 = 6 + 3n$
$\therefore$ l or nth term = $6 + 3n$
You can use the formula to check the terms. For example, the 2nd term, i.e. when $n = 2$, can be obtained as follows:
2nd term = $6 + (3x2) = 6 + 6 = 12$.

Example 6.5
Given that 239 is the nth term of the sequence 5, 14, 23....., find n.

Solution
Now the last term l (or nth term = 239) and $l = a + (n-1)d$; $a = 5$, $d = 9$, substitute for a and d to get $239 = 5+(n-1)9$

$$= 5 + 9n - 9$$
$$239 = 9n - 4$$
$$9n = 243$$
$$n = \frac{243}{9} = 27$$

Exercise 6.1
1 (a) Find the 10th term of the sequence 2, 6, 18, ...
(b) Find the 9th term of the sequence 8, 4, 2, ...
(c) Find the 20th term of the sequence 4, 7, 10 ....
2 (a) Given the sequence 1, 6, 11, 16, ... 66, which is an A.P. If 66 is the nth term, find the value of n.
(b) Given the A.P. series 2 + 4 + 6 + ... + 46.
If 46 is the nth term, find the value of n
3 The second term of an A.P. is 15 and the fifth term is 21. Find the first term and the common difference.

6.4 Arithmetic means

If three numbers a, b and c are consecutive numbers in an arithmetic progression, then b is called the **arithmetic mean** of a and c. The common difference of the A.P. can be determined from the following:

Common difference = $b - a$ or $c - b$

$$b - a = c - b$$
$$2b = a + c$$
$$b = \frac{a + c}{2}$$

For example, the arithmetic mean of 4 and 8 = $\frac{4+8}{2} = \frac{12}{2} = 6$.

Example 6.6
Insert five arithmetic means between 8 and 26.

Solution
What is required is an A.P., such that the 1st term = 8 and the last term = 26, i.e. 8, –, –, –, –, –, 26.
Now $l = a + (n-1)d$, where $a = 8$, $l = 26$ and $n = 7$.

Then, $26 = 8 + (7-1)d$
$26 = 8 + 6d$
$26 - 8 = 6d$
$6d = 18$
$d = \frac{18}{6} = 3$

∴ the five arithmetic means required are 11, 14, 17, 20, 23.

6.5 Sum of an arithmetic progression

Let the first term of an A.P. be a and the nth term be l. Let the sum of the 1st n terms of an A.P. be denoted by S_n, then S_n may be found as follows:

(i) $S_n = a + (a+d) + (a+2d) + ... + (l - d) + l$ (l = last term)
(ii) $S_n = l + (l - d) + (l - 2d) + ... + (a + d) + a$ (Rewritten backward.)
Adding (i) and (ii), we have:

$$2S_n = (a+l) + (a+l) + (a+l) + ... + (a+l) + (a+l)$$

Since there are n terms on the right hand side, we have
$2S_n = n(a + l)$

$\therefore S_n = \frac{n(a+l)}{2}$ or $\frac{n}{2}(a + l)$ ... (iii)

But $l = a + (n - 1)d$
Then, by substituting for l in (iii), we have

$$S_n = \frac{n}{2}\left[a + (a + (n - 1)d)\right]$$
$$\therefore S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Example 6.7
Find the sum of the first 20 terms of the series 2 + 4 + 6 + ...

Solution
Here, $a = 2, d = 2, n = 20$

Now, $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{20} = \frac{20}{2}[(2 \times 2) + (20 - 1)2]$

$= 10[4 + 38] = 10 \times 42 = 420$
$\therefore S_{20} = 420$

Example 6.8
Find the 40th term and the sum of the first 40 terms of the A.P. 10, 8, 6, ...

Solution
Here $d = 8 - 10 = -2$ or $6 - 8 = -2$
$a = 10$ and $n = 40$.
Now, $l = a + (n - 1)d = 10 + (40-1)(-2)$
$= 10 + (-78) = -68$
$l = -68$
Then $S_n = \frac{n}{2}(a + l)$
For $n = 40$,
$\therefore S_{40} = \frac{40}{2}(10 + (-68))$

$= 20(10 - 68) = 20 \times (-58)$

$= -1\ 160$

$\therefore$ 40th term $= -68$ and $S_{40} = -1\ 160$

Exercise 6.2

1 Given the following arithmetic progressions, find the sum of each:
 a) $71 + 67 + 63 + ... + 53$
 b) $1 + 3 + 5 + ... + 101$
 c) $2 + 7 + 12 + ... + 77$

2 Find the sums of the following arithmetic progressions as far as the indicated terms:
 a) $4 + 10 + ...$ (12th term)
 b) $15 + 13 + ...$ (20th term)

3 Given an arithmetical progression whose fourth term is 18, and the common difference is –5. Find the first term and the sum of the first sixteen terms.

4 An arithmetical progression has the following identity: 1st term = –12; last term 40. The sum of the progression is 196. Find:
 a) the common difference
 b) the number of terms in the series.

6.6 Summary

In this unit, you have learnt that:

i) the last term of an A.P. in general is given by $l = a + (n-1)d$ and that it is also called the nth term;

ii) the sum of an arithmetic progression is given by $S_n = \frac{n}{2}[2a + (n-1)d]$;

iii) if a, b *and* c are in an A.P., then $b = \frac{1}{2}(a+c)$ is the arithmetic mean.

6.7 Tutor-marked assignment

1 Given the first 100 positive integers exactly divisible by 7; find the sum.

2 A man owes a debt of ₦800. If he pays ₦25 in the first month, ₦27 in the second month and ₦29 in the third month. How long will he take to pay off the debt?

6.8 References

1 Backhouse, J.K. and S.P.T. Houldsworth, (1970) *Pure Mathematics, a First Course*, Longman Group Ltd, London, p. 211–225.

2 Godman, A and J. F. Talbert, (1975) *Additional Mathematics*, Longman.

3 Murray R. S. (1956) *College Algebra; Schaum Outline Series, Theory and Problems*, McGraw-Hill Book Company, p. 140–154.

4 N.E.R.D.C, (2001) *Further Mathematics for Senior Secondary Schools*, Longman Nigeria Plc. p. 65–71.

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Unit 7 Geometric progression

Contents

7.1 Introduction

This unit deals with geometric progression (GP) which is another type of sequences or series.

The exposure you had in the last unit (i.e. arithmetic progression) will help you in this unit. You should therefore, be able to go through this unit with ease.

7.2 Objectives

By the end of this unit, you should be able to:

i) find the nth term of a geometric progression;
ii) calculate geometric means;
iii) find the sum of geometric progressions;
iv) solve problems in geometric progressions.

7.3 Geometric progression

Definitions

When a sequence of numbers is such that each term is obtained by multiplying the term preceding it by a constant ratio, the sequence is said to be in a **geometrical ratio**. For example, in the following:

$$2, 4, 6, 16, \ldots$$

Can you identify the constant number that each preceding number is multiplied with to give the next? You would observe it is 2. In a geometric progression, the ratio of a term to the previous one is constant and in the example above, that constant is 2. Such a constant is called the **common ratio** denoted by r while the first term is denoted by a.

Now find the common ratio in the following series:

i) $3 + 6 + 12 + \ldots + 192$
ii) $1 + 2 + 4 + 8 + \ldots + 1024$
iii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{512}$

Compare your answers with the following:

i) the common ratio is 2

ii) the common ratio is 2

iii) the common is $\frac{1}{2}$.

The nth term of a geometric progression

Let the first term of a general G.P. be a, and r be the common ratio. Then you can compute the general or nth term of a G.P. from the following table.

Number of term	1st	2nd	3rd	4th	.....	nth
Term	a	ar	ar^2	ar^3	....	ar^{n-1}

$\therefore$ For any G.P. the n^{th} or general term is
nth term $= ar^{n-1}$

Example 7.1

Find the 10^{th} term of the series: $1 - 5 + 25 + ...$

Solution

From the series: $\frac{-5}{1} = \frac{-25}{5} = -5 = r$

$\therefore a = 1, r = -5$

Now nth term is given by ar^{n-1}

Then 10^{th} term $= 1\times(-5)^{10-1} = 1\times(-5)^9 = (-5)^9$

$\Rightarrow 10^{th}$ term $= (-5)^9$

Example 7.2

Given the G.P. series: $9 + 27 + 81 + ...$ and that the n^{th} term is 3^{19}, find n.

Solution

Here $a = 9, d = \frac{27}{9} = \frac{81}{27} = 3$

$\therefore r = 3$

And since nth term is given as ar^{n-1}:

$\Rightarrow 3^{19} = 9 \times 3^{n-1}$

$= 3^{19} = 3^2 \times 3^{n-1} = 3^{n-1+2} = 3^{n+1}$

Now, if $3^{n+1} = 3^{19}$, then $n+1 = 19$ (since the bases are equal)

Therefore $n = 18$.

Geometric mean

If three numbers a, b and c are in geometric progression, b is called the **geometric mean** of a and c; and the common ratio of the G.P. is given by $\frac{b}{a}$ or $\frac{c}{b}$

$\therefore \frac{b}{a} = \frac{c}{b}$

$\Rightarrow b^2 = ac$

$\Rightarrow b = \sqrt{ac}$

$\therefore$ the geometric mean of a and c (only positive value is accepted) $= \sqrt{ac}$

Example 7.3
Find the geometric mean of 4 and 25.

Solution
Geometric mean is given by $b = \sqrt{ac}$
i.e. $b = \sqrt{4 \times 25} = 2 \times 5 = 10$

Therefore 4, 10, 25 are in geometric progression.

Example 7.4
Find the geometric mean of:

$3\frac{3}{8}$ and $\frac{2}{3}$

Solution
The geometric mean is given by

$$\sqrt{3\tfrac{3}{8} \times \tfrac{2}{3}} = \sqrt{\tfrac{27}{8} \times \tfrac{2}{3}} = \sqrt{\tfrac{9}{4}} = \tfrac{3}{2}$$

$\therefore$ the required geometric mean = $\frac{3}{2}$.

Exercise 7.1

1. Investigate and determine which of the following sequences are geometric progressions.
   i) 16, 12, 9, ...
   ii) 3, 6, 12, ...
   iii) –1, 3, –9, ...
2. Find the common ratio of a G.P. whose first term is 1 and fourth term is 27.
3. Write down the first four terms of the G.P. whose first term is 16 and common ratio is $\frac{1}{2}$
4. What is the 12th term of the G.P. 2, –4, 8, ...?

The sum of a geometric progression

Let a = first term and r the common ratio, then the general formula for the sum of a geometric progression can be generated as follows:
Taking the sum of a general G.P. series to n terms we have:
$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$ .... (i)
Multiply both sides by r:
$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$ ... (ii)
Subtract (ii) from (i) to get:
$\therefore S_n - rS_n = a - ar^n = a(1 - r^n)$
or $S_n(1 - r) = a(1 - r^n)$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r} \text{(if } r < 1)$$

An alternative form of the formula can be obtained by multiplying the numerator and denominator by –1, which gives

$$\therefore S_n = \frac{a(r^n - 1)}{r - 1} \text{ (if } r > 1)$$

Example 7.5
Evaluate 3 + 6 + 12 + ... up to the 6^{th} term.

Solution

Now $S_n = \dfrac{a(1 - r^n)}{1 - r}$,

where $a = 3$, $r = 2$ and $n = 6$.
Substitute these into the formula to get

$$S_6 = \frac{3(2^6 - 1)}{2 - 1} = \frac{3(64 - 1)}{1} = 3 \times 63$$

$\therefore\ S_6 = 189.$

Example 7.6
Given the series:

$\frac{1}{9} + \frac{1}{3} + 1 + ...$

Evaluate the first 8 terms of the G.P.

Solution

Now, $S_n = \dfrac{a(r^n - 1)}{r - 1}$, since $r > 1$

Here $a = \dfrac{1}{9}$, $r = 3$, $n = 8$

Substitute these into the formula to get:

$$S_8 = \frac{1(3^8 - 1)}{9(3 - 1)} = \frac{(6561 - 1)}{9 \times 2} = \frac{6560}{18}$$

$= 364.44$

Example 7.7
In a particular G.P. series, the sum of the second and third terms is 6, and the sum of the third and fourth terms is – 12. Find the first term and common ratio.

Solution
Here, we are required to find two unknowns i.e. the first term and the common ratio.

But we also have two pieces of information: i.e. the sum of the second and third terms is 6; the sum of the third and fourth terms is –12. Now, write down these two equations to enable you find the two unknowns.

Now you have the following:

If a = first term and r = common ratio.

Now, second term = ar and third term = ar^2

$\therefore\ ar + ar^2 = 6$ ... (i)

Also the third term = ar^2 and the fourth term = ar^3 (for the second information)

$\therefore\ ar^2 + ar^3 = -12$ ... (ii)

Factorize L.H.S. of the two equations to get:

$ar(1 + r) = 6$ ... (iii)

$ar^2(1 + r) = -12$ (iv)

Divide (iii) by (iv) to get:

$$\frac{ar(1 + r)}{ar^2(1 + r)} = \frac{-6}{12}$$

That is,

$$\frac{1}{r} = \frac{-1}{2}$$

$\Rightarrow\ r = -2$

Substitute $r = -2$ in (iii) to get:

$-2a(1 + (-2) = 6$

i.e. $(-2a)(-1) = 6$

$2a = 6 \Rightarrow a = 3$

$\therefore$ the first term = 3 and common ratio = –2

Exercise 7.2

1 The fourth term of a G.P. is –6 and the seventh term is 48. Write down the first eight terms of the progression.
2 Find the sums of the first six terms of the G.P.: 16 + 8 + 4 + ...
3 Find the sum of the first eight terms of the G.P.: 15 + 15 + ...
4 Find the sum of the first 19 terms of the G.P. series: 1 + 7 + 49 + ...
5 In a geometric progression, the sum of the second and third terms is 9; and the seventh term is eight times the fourth. Find the first term, the common ratio and the fifth term.
6 The sum of the first two terms of a geometric progression is 3, and the sum of the second and third terms is –6. Find the first term and the common ratio.

7.4 Summary

This unit has exposed you to geometric progression which is another type of sequences and series, different from the A.P. you have already studied.

In this unit, you have learnt that:

i) a general geometric sequence is written as $a, ar, ar^2, ..., ar^{n-1}$; where a is the first term and r is the common ratio;
ii) the nth term of a geometric progression is given by ar^{n-1};
iii) for three consecutive numbers a, b, c that are in geometric progression, their geometric mean is given by $b = \sqrt{ac}$;
iv) the sum of a geometric progression with first term a, and common ratio r, is given by:

$$\frac{a(r^n - 1)}{r - 1} \text{ (if } r > 1\text{) and}$$

or

$$\frac{a(1 - r^n)}{1 - r} \text{ (if } r < 1\text{).}$$

7.5 Tutor-marked assignment

1 Given that the second term of a G.P. is 3, and the fifth term is $\frac{81}{8}$, find the eighth term.
2 The financial remuneration of a daily-paid worker is as follows: ₦10 for the first day, ₦20 for the second day, ₦40 for the third day, 80 for the fourth day, etc,. How much would he receive at the end of the 12th day?

7.6 References

1 Backhouse, J.K., and S.P.T. Houldsworth (1970). *Pure Mathematics, A First Course.* Longman Group Ltd, p. 211–225.
2 Spiegel, M. R. (1956). *College Algebra: Shaum Outline Series, Theory and Problems,* McGraw-Hill Book Company, p. 140–154.
3 N.E.R.D.C (2001), *Further Mathematics for Senior Secondary Schools,* Longman Nigeria Plc, p. 65–71.
4 Tranter C. J., and C. G. Lambe (1976), *Advanced Level Mathematics (Pure and Applied),* Hodder and Stoughton.
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Unit 8 Sets and their basic properties

Contents

8.1 Introduction

Ideas about sets are common in everyday use, and set theory as a field of mathematics is very important. A German mathematician, George Cantor (1845 – 1918) pioneered works on set theory when he used it as a basis for his consideration of 'Infinite'. At first, his work was ridiculed by many mathematicians, but by 1920, it became accepted and has led to great advances in the field of abstract thought. Later, George Boole (1815 – 1864) used the set theory to investigate theory of logic, which is now called **Boolean Algebra** after his name.

Today, set theory has percolated into the whole realm of mathematics, and it has a unifying influence on mathematics as a whole.

8.2 Objectives

By the end of this unit, you should be able to:
i) define the mathematical terms used in set theory;
ii) use set notations to make mathematical statements;
iii) find the sum of geometric progressions;
iv) solve problems related to set concepts.

8.3 Definitions

Set is the name used for any collection of distinct objects or elements having a common characteristic or property. The common characteristic makes it possible for one to precisely determine whether or not a given object or element is a member of the particular set. The objects or elements in the set can be anything, e.g. people, numbers, cars, states in a particular country, etc.
Now, take the following about sets:
i) each object of a set is called an **element** or **member** of the set;
ii) the elements of a set may be defined either by enumerating the elements or by stating a rule under which a member or element may be known;

iii) a pair of braces, i.e. { } is usually used to enclose the elements of a set;

iv) a set is usually denoted by a capital letter, such as A, B, C, X, Y, ... while the elements are denoted by lower case (i.e. small) letters a, b, c, d, x, y, ...;

v) when a set is defined by actually listing its member (or elements), e.g. A = {a, e, i, o, u}, it is called the **tabular form** of a set. However, if the set is defined by stating properties which its elements must satisfy, e.g. 'let B be the set of all odd numbers' i.e. B = {x/x is odd}, read as 'B is the set of numbers x, such that x is odd'. This is called the **set-builder form** of a set.

Example 8.1

A = {x : x is odd, $0 < x < 10$} read as 'A is the set of odd numbers greater than 0 but less than 10'.

B = {1, 3, 5, 7, 9}

You will observe that A and B describe the same set. A is written in the set-builder form while B is in tabular form. See again the following:

C = {2, 4, 6, 8, 10, 12 ....}

D = {x : x is a positive even integer}

Do you agree that C and D also describe the same set? Yes, sets C and D are the same.

E = {states in Nigeria}

F = {students of NOUN}

G = {a, e, i, o, u}, set of vowels

Note that C and G are tabular forms while D, E and F are written in set-builder forms.

When an object x, say, is an element or a member of a set B, it is indicated by $x \in B$, read as 'x belongs to B' while $x \notin B$, read as 'x does not belong to B', indicates that x is not a member or element of B. For example, if G = {a, e, i, o, u} then e $\in$ G, but c $\notin$ G.

8.4 Finite and infinite sets

When the number of elements in a set is specific and can be counted, the set is said to be **finite**, but when the number of elements is uncountable, then the set is called ***infinite***.

Example 8.2

i) Let A be the set of odd numbers less than 10, i.e. A = {1, 3, 5, 7, 9}, then A is finite.

ii) Let B be the set of all natural numbers, then B = {1, 2, 3, 4 ....}

From the above definition, B is infinite since the number of all natural numbers is countless.

8.5 Null or empty set

A set which contains no elements is referred to as **null** or **empty** set, and is denoted by ϕ or { }.

Example 8.3

Let P be the set of people with three eyes. Since it is not natural for human beings to have three eyes, then P is an empty or null set, i.e.

P = ϕ or { }.

Note that {ϕ} does not mean an empty set, but a set containing ϕ as an element.

8.6 Singleton

A set that has only one element is called a **singleton**. For example, if A is a set having 'b' as its only element, then, A = {b}.

Exercise 8.1

1 Write down the set whose elements are the first seven positive odd numbers.

2 List out members of the following sets:
   (i) {x, an even number: $5 < x < 15$}
   (ii) {x, an integer: $11 < x < 20$}
   (iii) {factors of 24}

3 State whether the following statements are true or false:
   (i) A triangle ∈ {polygons}
   (ii) A polygon ∈ {all triangles}
   (iii) 5 ∈ {factors of 36}
   (iv) 3 ∈ {prime numbers}

8.7 Equality of sets

If two sets, P and Q, have the same elements, they are said to be equal. For example,

P = {r, s, t, y, z}
Q = {t, z, r, s, y},
then P = Q. You should note that the order in which the elements are arranged does not matter.

Example 8.4

Let A = {5, 6, 7, 8, 9}
B = {3, 7, 8, 7, 6, 8, 6, 9}
C = {5, 5, 7,7, 6, 8, 9}

Here also, A = B = C, that is, a set does not change if its elements are re-arranged or repeated.

8.8 Subsets and proper subsets

If every element of a set A, say is also an element of another set B, say, then A is a subset of B and is denoted by $A \subset B$, read 'A is contained in B'.
If A is not a subset of B, we write $A \not\subset B$.
For example, if

A = {a, b, c, d, e},
B = {a, d, c} and D = {a, f, k},
then $B \subset A$, $D \not\subset A$.

Example 8.5

List all the subsets of the set P = {x, y, z}.
The subsets are {x}, {y}, {z}, {x, y}, {x, z}, {y, z} {x, y, z}, { }.

At this juncture, you should note the following:

(i) The empty set is considered to be a subset of every set.
(ii) Every set is a subset of itself.

Now, since every set is a subset of itself, another definition of equal sets can be:
Two sets A and B are equal, that is, A = B if and only if $A \subset B$ and $B \subset A$.

Proper subset

Let A and B be sets, then A is a proper subset of B if the following properties are satisfied:

(i) A is a subset of B, and
(ii) A is not equal to B, that is, $A \subset B$ and $A \neq B$.

Example 8.6

Let A = {a, b}. All subsets of A are {a,}, {b}, {a, b}, ϕ, but the proper subsets are {a}, {b}, ϕ.

8.9 Power set

Let P be a set; the **power set** of P is the family of all the subsets (that is, the totality of all subsets obtainable from a set) of P. It is denoted by 2^n where n is the number of elements in P.

Example 8.7
Let P = {1, 2}. Then the power set of P, that is, 2^P given by 2^n is $2^P = 2^2 = 4$
The list of 2^P = {1,}, {2}, {1, 2}, { }.

8.10 Universal set

If each set under consideration is a subset of another set, say μ, then μ is called the **Universal set**. For example, consider a class of boys and girls in a secondary school.

Let A = {all girls in the class}
B = {all boys in the class}
C = {all members of the class}

Then, C can be considered as the universal set. The universal set is denoted by μ or ε. Note that the universal set is not unique and so it must be defined. In this example, the set of 'all pupils in the school' can also be defined as the universal set.

Having some through the properties of sets, you should now be able to attempt the following exercises.

Exercise 8.2

1 Give 3 subsets of each of the following:
(i) A = {a, b, c, d}
(ii) B = {states of Nigeria}
(iii) C = {fruit trees}

2 Define a Universal set which will include the following pairs:
(i) {12, 14, 16 18}; {2, 4, 6, 8, 10}
(ii) {3, 5, 7, 9}; {9, 12, 15, 18}
(iii) {hens, ducks, turkeys}; {dogs, goats, sheep}
(iv) {desks}; {chairs}; {tables}.

3 Identify which of the following sets are finite:
(i) {odd numbers} (ii) {x: $2 < x < 10$}
(iii) {positive integers) (iv) {factors of 10^8}
(v) {rivers} (vi) {even numbers}.

4 Indicate which of the following pairs are equivalent:
(i) {3, 6, 9} and {e, f, g}
(ii) {The prime factors of 20} and {solutions of (y–1)(y–2)(y–3)}
(iii) {The vertices of a pentagon} and {the first four prime numbers}
(iv) {4, 6, 8} and {6, 14, 22}.

8.11 Summary

In this unit, you have learnt the various terminologies about sets and their definitions, including finite and infinite sets, null or empty set, singleton, subsets, power set, etc. In the next unit, you are going to learn about the operations on sets, similar to the operations you perform on variables.

From the foregoing, you would have noticed that the knowledge of set and its properties give you a firm foundation for further studies in set theory. We use set ideas in everyday conversation, e.g. a set of football players, a set of books, a set of dishes, a set of integers, etc. In this unit, you have studied various types of sets with some examples. What you have just done are the fundamentals of set theory and these would help you in doing further work on application of sets.

8.12 Tutor-marked assignment

1 a) Write down the subsets indicated below:
   (i) All subsets of A = {1, 2, 3}
   (ii) Two subsets of {Polygons}
   (iii) Two subsets of {Quadrilaterals}
   (iv) Two subsets of (Natural numbers}
   b) How many subsets does (a, b, c, d} have?
2 Indicate which of the following is an empty set:
   (a) Quadrilaterals whose sum of their angles is 300°.
   (b) Normal human beings having two heads.
   (c) Triangles whose sum of their angles is 180°.
   (d) Rectangles, whose diagonals are unequal in length.

8.13 References

1 Clarke, L. H., (1968) *Modern Mathematics at Ordinary level,* Heinemann Educational Books Ltd., England, p. 49-61.
2 Nigerian Educational Research and Development Council, (2001) *Further Mathematics for Senior Secondary Schools,* Longman Nigeria Plc, p. 247-249.
3 File draft.

Unit 9 Operations on sets

Contents

9.1 Introduction

In algebra or number theory, certain mathematical symbols are used for operations. Such symbols include (+), (–), (x) and (÷), all of which you are familiar with. There are corresponding mathematical symbols used to operate on sets. These include $\cup$, $\cap$ and $\subset$. This unit will treat the various operations on sets.

9.2 Objectives

By the end of this unit, you should be able to:
i) identify the different symbols used for set operations;
ii) use those symbols;
iii) state the properties of set operations.

9.3 Operations on sets

Union of sets

Let A and B be sets. The set that contains all the elements in A and B is called the **union** of A and B. It is denoted by '$A \cup B$' and read as 'A union B. For example, if on a particular day, in a school setting, you have the following sets defined:

A = { Girls that plait their hair}
B = { Girls that wear socks}
Then, $A \cup B$ = { Girls that plait hair, or that wear socks or girls that do both}
Note that if sets A and B have some elements in common, those elements will not be repeated in the union set. For example, if $A = \{5, 6, 7, 8, 9\}$ and $B = \{3, 4, 5, 6, 10\}$,
then, $A \cup B = \{3, 4, 5, 6, 7, 8, 9, 10\}$

As you can see, the elements 5, 6, 10 which are common to A and B that are not repeated in $A \cup B$. Symbolically, $A \cup B$ may also be written as

$$\mathbf{A \cup B} = \{x : x \in A \text{ or } x \in B\}$$

By the definition of union of sets, it can be seen that:
(i) $A \cup B = B \cup A$
(ii) $A \subset (A \cup B)$ and $B \subset (A \cup B)$

Example 9.1
Let $A = \{a, b, c, d\}$
$B = \{b, c, d, e, f, g\}$
Then, $A \cup B = \{a, b, c, d, e, f, g\}$
$B \cup A = \{b, c, d, e, f, g, a\}$

Intersection of sets

The intersection of sets A and B is the set of elements which are common to both A and B. It is denoted by $A \cap B$ and read as A **intersection** B. Symbolically, it can also be written as:

$$A \cap B = [x: x \in A \text{ and } x \in B\}$$

Example 9.2
Let $A = \{3, 4, 5, 6, 7\}$, $B = \{1, 2, 3, 4,\}$ and $C \{2, 3, 4, 8, 9\}$
Then, $A \cap B = \{3, 4\}$
$B \cap C = \{2,3,4\}$ and $A \cap C = \{3, 4\}$

Note from the definition that:

(i) $A \cap B = B \cap A$

(ii) $(A \cap B) \subset A$ and $(A \cap B) \subset B$

If sets A and B have no elements in common, they are said to be **disjoint.**
Then, $A \cap B = \varnothing$, e.g. if A = {odd number} and B = {even number}, then, $A \cap B = \varnothing$.

Difference of sets

Let A and B be two sets, the **difference of sets** A and B is the set of elements which belong to A but which do not belong to B. It is denoted by $A - B$ and read as A **difference** B.
Symbolically, it can also be written as:
$A - B = \{x : x \in A, x \notin B\}$
Let $A = \{a, b, c\}$ and $B = \{b, c, d, e, f\}$
Then $A - B = \{a\}$ and $B - A = \{d, e, f\}$
Note that:

(i) $(A - B) \subset A$ and $(B - A) \subset B$
(ii) $(A - B)$, $(B - A)$ and $(A \cap B)$ are mutually disjoint, i.e. the intersection of any two of them is mutually empty.

Complement of sets

Let μ be a universal set under consideration, and A, a subset of the universal set. Let there be another subset B of μ, which contains all elements of the universal set which are not in A, set B is said to be the **complement** of set A and it is denoted by A' (i.e. $B = A'$), read as <u>A</u> **complement.**

For example, let μ = {all members of a class}
and A = {all girls in the class}
Then A' = {all boys in the class}.

Let μ = {2, 3, 4, 5, 6, 7, 8, 9}
A = {3, 5, 7, 9}
B = {2, 3, 4, 5, 6, 8}
You can see that $\mu = A \cup B$
Then A' = {2, 4, 6, 8}
B' = {7, 9}

Symbolically, the complement of a set A can be written as

$A' = \{x : x \in \mu, x \in A\}$

Exercise 9.1

1 Given that $A = \{a, c, g,\}$, $B = \{b, d, g\}$ and $C = \{e, a, d\}$; write out the following sets:
a) $A \cup B$ b) $A \cap B$ c) $A \cup B \cup C$ d) $A \cap B \cap C$

2 State whether or not the following statements are always true:
a) $C \cup C = C$ b) $C \cap D = D \cap C$
c) $C \cup C = 2C$ d) $C \cup D = D \cup C$

3 Let P = { consonants} and Q = {vowels}
What is $P \cup Q$?

4 Let G = {Goats} and S = {Sheep}
What is $G \cap S$?

5 If sets A and B are disjoint, write down the value of $n(A \cap B)$ where n represents the number of elements.

9.4 Properties of set operations

Just as arithmetic operation symbols, such as (+), (x), (÷) and (–) obey various algebraic laws, e.g. cumulative, associative and distributive laws, so do symbols for set operations obey such laws as **cumulative, associative, distributive** and **idempotent** laws.

Cumulative laws

1) $A \cup B = B \cup A$
For illustration purpose, let sets A, B and C be
A = {a, b, c, d, e, f, g}
B = {d, e, f, g}
C = {a, c, e, g,}
Then, (i) $A \cup B$ = {a, b, c, d, e, f, g} $\cup$ {d, e, f, g}
= {a, b, c, d, e, f, g}
$B \cup A$ = {d, e, f, g} $\cup$ {a, b, c, d, e, f, g}
= {a, b, c, d, e, f, g}
You can therefore, see that $A \cup B = B \cup A$
2) $A \cap B = B \cap A$
$A \cap B$ = {a, b, c, d, e, f, g} {d, e, f, g)
= {d, e, f, g}

$B \cap A = \{d, e, f, g\} \cap \{a, b, c, d, e, f, g\}$
$= \{d, e, f, g\}$
$\therefore A \cap B = B \cap A.$

Associative laws

1) $A \cup (B \cup C) = (A \cup B) \cup C$
2) $A \cap (B \cap C) = (A \cap B) \cap C$

Illustrations

From sets *A*, *B* and *C* earlier defined, you have

1) $(A \cup B) \cup C = \{a, b, c, d, e, f, g\} \cup \{d, e, f, g\} \cup \{a, c, e, g\}$
$= \{a, b, c, d, e, f, g\} \cup \{a, c, e, g\}$
$= \{a, b, c, d, e, f, g\}$
$A \cup (B \cup C) = \{a, b, c, d, e, f, g\} \cup \{d, e, f, g\} \cup \{a, c, e, g\}$
$= \{a, b, c, d, e, f, g\} \cup \{a, c, d, e, f, g\}$
$= \{a, b, c, d, e, f, g\}$
$\therefore (A \cup B) \cup C = A \cup (B \cup C)$
2) $A \cap (B \cap C) = \{a, b, c, d, e, f, g\} \cap \{d, e, f, g\} \cap \{a, c, e, g\}$
$= \{a, b, c, d, e, f, g\} \cap \{e, g\}$
$= \{e, g\}$

Also, $(A \cap B) \cap C = \{a, b, c, d, e, f, g\} \cap \{d, e, f, g\} \cap \{a, c, e, g\}$
$= \{d, e, f, g\} \cap \{a, c, e, g\}$
$= \{e, g\}$

$\therefore A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws

1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

The laws are verified as follows:

1) $A \cap (B \cup C) = \{a, b, c, d, e, f, g\} \cap \{d, e, f, g\} \cup \{a, c, e, g\}$
$= \{a, b, c, d, e, f, g\} \cap \{a, c, d, e, f, g\}$
$= \{a, c, d, e, f, g\}$

$(A \cap B) \cup (A \cap C) = [\{a, b, c, d, e, f, g\} \cap \{d, e, f, g\} \cup \{a, b, c, d, e, f, g\} \cap \{a, c, e, g\}]$
$= \{d, e, f, g\} \cup \{a, c, e, g\}$
$= \{a, c, d, e, f, g\} = A \cap (B \cup C)$
$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2) $A \cup (B \cap C) = \{a, b, c, d, e, f, g\} \cup [\{d, e, f, g\} \cap \{a, c, e, g\}]$
$= \{a, b, c, d, e, f, g\} \cup \{e, g\}$
$= \{a, b, c, d, e, f, g\}$
and $(A \cup B) \cap (A \cup C) = [\{a, b, c, d, e, f, g\} \cup \{d, e, f, g\}] \cap [\{a, b, c, d, e, f, g\} \cup \{a, c, e, g\}]$
$= (\{a, b, c, d, e, f, g\} \cap \{a, b, c, d, e, f, g\})$
$= \{a, b, c, d, e, f, g\} = A \cup (B \cap C)$
$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

For arithmetic operations an elements *a*, *b* and *c* using arithmetic addition and multiplication operators, an expression obeying associative law will be
$a \times (b+c) = (a \times b) + (a \times c) = ab + ac.$

Idempotent laws

For set A:

1) $A \cup A = A$ and
2) $A \cap A = A$

Illustrations

1) $A \cup A = \{a, b, c, d, e, f, g\} \cup \{a, b, c, d, e, f, g\}$
   $= \{a, b, c, d, e, f, g\} = A$

$\therefore \quad A \cup A = A$

2) $A \cap A = \{a, b, c, d, e, f, g\} \cap \{a, b, c, d, e, f, g\}$
   $= \{a, b, c, d, e, f, g\} = A$

$\therefore \quad A \cap A = A$

Exercise 9.2

1 State whether or not this statement is always true:
$P \cup (Q \cup R) = (P \cup Q) \cup R$
Hint: You may define elements of sets P, Q, R,and use them to verify.

2 Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$ and
$C = \{1, 2, 3, 4, 5, 6\}$
Verify whether or not $(X \cap Y) \cap (X \cap Y) = A$

3 Let $= \mu\ \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3, 4\}$
$B = \{3, 4, 5\}$, $C = \{6\}$ and $D = \{4, 5\}$
Verify that $(A \cap B \cap C) \cup (A \cap B \cap C') \cup (A \cap B \cap D) = A \cap B$.

9.5 Summary

In this unit, you have learnt that:

i) the union of two sets A and B is the set of elements belonging to A or B *or* to both, and is denoted by $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
ii) the intersection of two sets A and B is the set of elements which are common to both A and B, and is denoted by $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
iii) the difference of two sets A and B is the set of elements which belong to A but which do not belong to B, and is denoted by $A - B$;
iv) if A and B are two subsets of μ, the universal set, and B contains all elements of the universal set which are not in A, then B is the complement of A, and is denoted by A^l.

9.6 Tutor marked assignment

1 Given that $A = \{a, b, c\}$, $B = \{a, b, c, d\}$ and $C = \{a, b, c, d, e\}$,
show that $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

2 a) Given that $\mu = \{\text{letters of the alphabet}\}$ and
$A = \{\text{consonants}\}$, what is A'?

b) Given that $\mu = \{1, 2, 3, -, -, -10\}$, and A, a subset of $\mu = \{4, 5, 6\}$, what is $(A^l)^l$? What can you conclude about A and $(A^l)^l$?

9.7 References

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Unit 10 Linear equations in one variable

Contents

10.1 Introduction

In your study or mathematics at the junior secondary level, you must have learnt about simple or linear equations. In this unit, you shall be further exposed to linear equations to consolidate what you learnt earlier and also to gain further knowledge of this interesting aspect of mathematics. Have an enjoyable study.

10.2 Objectives

By the end of this unit, you should be able to:

i) identity linear equations in one variable;
ii) solve linear equations in one variable;
iii) translate relevant word problems into algebraic symbols, hence into linear equations in one variable;
iv) solve number and age problems written as linear equations in one variables.

10.3 Linear equations in one variable

Definitions

An expression of the form $2x = 8$, for example, is an algebraic sentence. When an algebraic sentence contains an equality sign it is called an equation. For example, $3a = 18$ is an equation.

Alternatively, an equation is a statement in which one quantity is equal to another. Expressions such as:
$5x = 20$; $2y^2 = 6$; $r + 2 = 4$; $5 - y^3 = 10$, $2x^2 + 7x + 1 = 0$, etc.
are all examples of equations.

In these examples, the letters in the expressions represent quantities or variables whose values are not known they are therefore called **unknown quantities or variables**. Now, can you verify how many unknowns or variables occur in each of the equations above? You will notice that the number of unknowns or variables in each equation is only one. Hence they are all equations in one variables (variables will be used more frequently as from now).

Linear equations

You must have noticed that in the equations given as examples some variables have index or power of 1, others have index or power greater than one. Any equation whose variables have an index of 1 is called a **linear equation**. For example, equations $5x = 20$ and $r + 2 = 4$ are linear equations because the indices of their variables are 1.However, $2y^2 - 6$; $5 - y^3 = 10$; and $2x^2 + 7x = 1$ are not linear equations because in each case, their variables' indices are greater than one.

In this unit, you shall be studying about linear equations in one variable.

Solutions of linear equations

In any equation, the challenge is to determine the value of the unknown or variable. And whenever this is being done, that is determining the value of the variable which will make a given equation true, then you are **solving** the equation. The value, so obtained, is called the **solution** of the equation.

For example, in $2x = 8$, $x = 4$ is the solution of the equation. In this unit, you shall be exposed to techniques of solving linear equations in one variable.

Example 10.1

Solve the following equations:

i) $x + 10 = 21$

ii) $\frac{4x}{5} - 3 = 10$

iii) $2(t + 3) = 5(t - 1) - 7(t - 3)$

iv) $\frac{x-3}{2} = \frac{2x+4}{5}$

Solution

i) Given that $x + 10 = 21$

Subtract 10 from both sides:

$\Rightarrow x + 10 - 10 = 21 - 10$

$\Rightarrow x = 11$

ii) To solve $\frac{4x}{5} - 3 = 10$:

Add 3 to both sides as follows:

$\frac{4x}{5} - 3 + 3 = 10 + 3 = 13$

$\Rightarrow \frac{4x}{5} = 13$

Multiply both sides by 5:

$\frac{4x}{5} \times 5 = 13 \times 5$

$\Rightarrow 4x = 65$

Divide both sides by 4:

$\frac{4x}{4} = \frac{65}{4}$

$\therefore x = 16\frac{1}{4}$

iii) $2(t + 3) = 5(t - 1) - 7(t - 3)$

Multiply each bracket by the figures outside it to get:

$2t + 6 = 5t - 5 - 7t + 21$

Put all variable terms in t on the left hand side and constants on RHS:

$2t - 5t + 7t = 21 - 5 - 6$

$\Rightarrow 9t - 5t = 21 - 11$

$\therefore\ t = \frac{10}{4}$

$= 2\frac{1}{2}$

iv) $\frac{x-3}{2} = \frac{2x+4}{5}$

To clear the denominators, multiply by the LCM of 2 and 5, i.e. 10

$\Rightarrow \frac{(x-3)}{2} \times 10 = \frac{(2x+4)}{5} \times 10$

$= 5(x - 3) = 2(2x + 4)$

$\Rightarrow 5x - 15 = 4x + 8$

Put terms in x on one side and constants on the other side:

$5x - 4x = 8 + 15$

$\therefore\ x = 23$

Exercise 10.1

1 $y + 3(y - 4) = 4$

2 $(x - 3) -2(6 - 2x) = 2(2x - 5)$

3 $\frac{2s-9}{3} = \frac{3s+4}{2}$

4 $\frac{3}{x} - \frac{4}{5x} = \frac{1}{10}$

Translating word problems into algebraic expressions

Sometimes, there is the need to reduce word problems into mathematical or algebraic symbols. This section shall take you through examples of word problems that are reduced to linear equations in one variable.

Example 10.2

Express the following statement in terms of algebraic symbols:

i) One more than twice a number.
ii) Three less than five times a certain number
iii) Three consecutive integers (e.g. 3, 4, 5)
iv) Any odd integer.
v) Peter is twice as old as Janet and Janet is three times as old as Rufai. Express each of their ages in terms of a single variable.

Solution

i) Let x be the number
Then twice the number = $2x$
One more than twice the number = $2x + 1$

ii) Let the number be x
Then five times the number = $5x$
Three less than five times the number = $5x - 3$

iii) Let x be the smallest of the integers
Then $(x + 1)$ and $(x + 2)$ are the remaining two integers.
$\therefore$ The three consecutive integers are: $x, (x + 1), (x + 2)$

iv) Let x be any integer
Then $2x$ is always an even integer (since it is divisible by 2)
$\therefore (2x + 1)$ is an odd integer.

v) Let Rufai's age = x
Then Janet's age = $3x$
Also, Peter's age = $2(3x) = 6x$

Exercise 10.2

Express each of the following statements in terms of algebraic symbols:

i) Two more than five times a certain number.
ii) Six less than twice a certain number.
iii) The squares of three consecutive numbers.
iv) The squares of any odd integer.
v) The difference between the squares of two consecutive even integers.
vi) Bako is six years older than Tola who is half as old as Bello. Express each of their ages in terms of a single variable.
vii) The perimeter and area of a rectangle, if one side is 3m shorter than three times the other side.

Number-related problems or equations

Some word problems involve looking for some numbers, such that after the word problems have been reduced to algebraic symbols, the numbers can be found by solving the resulting equations.

Example 10.3

The sum of two numbers is 21, and one number is twice the other. Find the numbers.

Solution

Let the smaller number be x
Then the other number is $2x$
Thus $x + 2x = 21$ (since their sum = 21)
$\Rightarrow 3x = 21$, and $x = \frac{21}{3} = 7$, and $2x = 14$
$\therefore$ The numbers are 7 and 14.

Example 10.4

Ten less than four times a certain number is 14. Find the number.

Solution

Let the number be x
Four times the number = $4x$
Then $4x - 10 = 14$
$\Rightarrow 4x = 24$ (adding 10 to both sides)
$\therefore x = \frac{24}{4} = 6$
$\therefore x = 6$

Example 10.5

The sum of three consecutive integers is 24. Find the integers.

Solution

Let the three consecutive integers be x, $x + 1$ and $x + 2$
Then their sum = $x + (x + 1) + (x + 2) = 24$
$\Rightarrow x + x + 1 + x + 2 = 24$ (opening the brackets)
or $3x + 3 = 24$ (collecting like terms together)
$3x = 21$ (subtracting 3 from both sides)

$\therefore x = \frac{21}{3} = 7$

Then the three numbers are 7, 8 and 9.

Exercise 10.3

1 One half of a certain number is 10 more than one sixth of the number. Find the number.
2 Find two consecutive positive odd integers such that the difference of their squares is 64.
3 Find two consecutive even integers such that twice the smaller exceeds the larger by 18.
4 The difference between two numbers is 20, and their sum is 48. Find the numbers.

Age-related problems

Other word problems that can be solved after they are expressed in algebraic symbols include age-related problems.

Example 10.6

A man is 41 years old and his son is 9. In how many years will the father be three times as old as the son.

Solution

Let x be the required number of years.
Then the father's age in x years = 3 (son's age in x years)

$\Rightarrow 41 + x = 3(9 + x) = 27 + 3x$

Collect terms in x together as follows:

$41 - 27 = 3x - x$

$\Rightarrow 14 = 2x$

$\Rightarrow x = \frac{14}{2} = 7$

$\therefore x = 7$ years.

Exercise 10.4

1 A father is 24 years older than his son. In 8 years from now, he will be twice as old as his son. Find their present ages.
2 Paul is fifteen years older than his brother, Israel. Six years ago, Paul was six times as old as Israel. Find their present ages.

10.4 Conclusion

Having gone through this unit on linear equations in one variable, you have gone a step further towards the laying of a foundation upon which studies in higher mathematics is built. More specifically, you are now prepared to go into studying simultaneous linear equations, which is the next unit you will be going through.

10.5 Summary

In this unit, you have learnt:

i) all definitions related to linear equations in one variable;
ii) how to solve linear equations in one variable;
iii) how to translate word problems into algebraic expressions;
iv) how to solve age-related word problems having translated them to linear equations in one variable.

10.6 Tutor-marked assignment

1. Solve the following linear equations:
   i) $5x - 6 = 2x + 13$
   ii) $6(x + 11) + 2(2x - 5) = 10$
2. Express each of the following in terms of algebraic symbols using one variable only:
   i) Two numbers whose sum is 200.
   ii) Two numbers whose difference is 8.
   iii) The perimeter and area of a rectangle, if one side is 4cm longer than twice the other side.
3 i) Francis subtracted 2 from a certain number, multiplied the result by 5, and added 8. If the result is 60, find the original number.
   ii) Find two consecutive even numbers such that 5 times the smaller added to 3 times the greater, gives 262.
4. Ten years ago, Musa was four times as old as Dapo. Now, he is only twice as old as Dapo. Find their present ages.

10.7 References

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Unit 11 Simultaneous linear equations

Contents

11.1 Introduction

In your earlier studies in mathematics, you must have learnt about algebraic expressions or equations involving two unknown quantities.

This unit will expose you to more studies on simultaneous linear equations.

11.2 Objectives

By the end of this unit, you should be able to:

i) distinguish simultaneous linear equations from other types of simultaneous equations;
ii) solve simultaneous linear equations using substitution, elimination and graphical methods;
iii) formulate word problems into simultaneous linear equations and solve them.

11.3 Definitions

Two equations which have the same set of two unknown quantities are called **simultaneous equations in two variables**. For example, $y-7x-5 = 0$ and $3y-16x+9 = 0$ are simultaneous equations in two variables, x and y. The two unknown quantities, as you have already learnt are called **variables**.

When the two unknown quantities, i.e. x and y (or any other variables used) occur in the first degree only, i.e. have power of 1 (in the two equations), the equations are called **simultaneous linear equations**.

Two basic methods of solving simultaneous linear equations will be considered here, namely the substitution method and the elimination method. These are non-graphical methods.

11.4 Substitution method

In this method, one of the variables will be made the subject of one of the equations and its value will be determined and then substituted into the second equation. After the substitution, that second equation becomes a linear equation in the unknown and the rest becomes easy to solve. See Example 11.1).

Example 11.1

Solve the following simultaneous equations by the substitution method:

a) $2x - y = 7$ .... (i)
$x + y = 2$ .... (ii)

b) $3x - 2y = 12$ .... (i)
$5x + 3y = 1$ .... (ii)

Solutions

a) $2x - y = 7$ .... (i)
$x + y = 2$ .... (ii)

From equation (ii), $y = 2 - x$ .... (iii)

Substitute $y = (2 - x)$ in (i) to get:

$2x - (2 - x) = 7$ .... (iv)

Open the bracket and obtain:

$2x - 2 + x = 7$

$3x = 7 + 2 = 9$

$\therefore\ x = \frac{9}{3} = 3$

Now, put $x = 3$ in (ii) to get:

$3 + y = 2$

Subtract 3 from both sides to get: $3 + y - 3 = 2 - 3$

$\therefore y = -1$

Your solutions are: $x = 3$, $y = -1$.

b) Given that:

$3x - 2y = 12$ .... (i)
$5x + 3y = 1$ .... (ii)

From equation (ii), $5x = 1 - 3y$ or $x = \frac{1}{5}(1-3y)$

Substitute $x = \frac{1}{5}(1-3y)$ in (i) to get:

$3\,[\frac{1}{5}(1-3y)] - 2y = 12$

Multiply each term by 5 to obtain

$3(1 - 3y) - 5 \times 2y = 12 \times 5$

Simplifying, you have:

$3 - 9y - 10y = 60$

$\Rightarrow\ 3 - 19y = 60$

$-19y = 60 - 3 = 57$

or $19y = -57$

$\Rightarrow\ y = -\frac{57}{19} = -3$

$\therefore\ y = -3$

Now, put $y = -3$ in equation (i) to get:

$$3x - 2(-3) = 12$$
$$\Rightarrow \quad 3x + 6 = 12$$
$$3x = 12 - 6 = 6$$
$$\therefore \quad x = \frac{6}{3} = 2$$
$$\therefore \quad x = 2, y = -3$$

Exercise 11.1

Solve the following simultaneous equations using the substitution method.

1. $4x - y - 7 = 0$ .... (i)
   $3x - 4y - 2 = 0$ .... (ii)

2. $a - b = -1$ .... (i)
   $b = 3 - a$ .... (ii)

3. $5y = 3 - 2x$ .... (i)
   $3x = 2y + 1$ .... (ii)

11.5 Elimination method

In this method, the practice is to eliminate one of the variables from the given set of equations. This reduces the system to one-variable equation which is then solved with ease. From your previous knowledge, how do you carry out the elimination process?

The steps involved in elimination are as follows:

(i) make the coefficients of one of the unknown quantities numerically equal by (if need be) multiplying the given equations by a number that will yield this result;

(ii) if the signs of the equal coefficients are different i.e. (+) and (–); add the resulting equations;

(iii) if the signs of the equal coefficients are the same, i.e. either (–) and (–) or (+) and (+), subtract them.

Example 11.2

Solve for x and y in the following equations by the elimination method:

a) $2x - y = 7$
   $x + y = 2$

b) $3x - 2y = 12$
   $5x + 3y = 1$

Solution

a) Given that:

$$2x - y = 7 \text{ .... (i)}$$
$$x + y = 2 \text{ .... (ii)}$$

Notice that the coefficients of y in (i) and (ii) are already numerically equal, which is **unity** or 1. Also notice that the signs in the two equal coefficients are different, i.e. (+) and (–).

∴ since the two coefficients are numerically equal and the signs are different, you can add the two equations to obtain:

$$3x = 9$$

Dividing both sides by 3, we have: $x = \frac{9}{3} = 3$

Now substitute $x = 3$ in (ii) to give

$$3 + y = 2$$

Subtracting 3 from both sides we have: $y = 2 - 3$

or $y = -1$

The solutions are therefore, $x = 3, y = -1$

b) Given that

$3x - 2y = 12$ .... (i)

$5x + 3y = 1$ .... (ii)

To make coefficients of y equal in (i) and (ii), multiply (i) by 3 and (ii) by 2 to have:

$9x - 6y = 36$ .... (iii)

$10x + 6y = 2$ .... (iv)

The resulting equations (iii) and (iv) show that the y coefficients are equal, but with different signs.

Now add (iii) and (iv) to get:

$19x = 38$

Divide both sides by 19 and obtain

$$\frac{19x}{19} = \frac{38}{19}$$

$\Rightarrow \quad x = 2$

Put $x = 2$ in equation (ii) to obtain

$5(2) + 3y = 1$

$3y = 1 - 10 = -9$

Dividing both sides by 3, we have

$$\frac{3y}{3} = \frac{-9}{3} \Rightarrow y -3$$

Therefore, $x = 2$, $y = -3$.

You will observe that the same results were obtained for these equations when solved by both substitution and elimination methods.

Exercise 11.2

Solve the following simultaneous equations by the elimination method:

1. $5x + 2y = 3$
   $2x + 3y = -1$

2. $7x - 2y = 14$
   $3x + 2y = 4$

3. $2x + 3y = 3$
   $6y - 6x = 1$

11.6 Problem statements that result in simultaneous linear equations

Sometimes, some real life problems can be turned into sets of mathematical equations and then solved simultaneously. This section of the unit shall take you through examples of such problems.

Example 11.3

The sum of two numbers is 28 and their difference is 12. Find the numbers.

Solution

Let x and y be the numbers.

Then, $x + y = 28$ .... (i)

$x - y = 12$ .... (ii)

Add (i) and (ii) and get $2x = 40$ or $x = \frac{40}{2} = 20$

Subtract (i) from (ii) to get:

$+ 2y = 16$ or $y = \frac{16}{+2} = +8$

$\therefore \; x = 20, y = 8$

Example 11.4
John and Mary have their ages summed up to 48 years. Seven years ago, Mary was three times as old as John. Calculate their ages.

Solution
Let John's age be x years and Mary's age be y years.
Then, $x + y = 48$ .... (i)
7 years ago, John's age was $(x - 7)$ years and Mary's age was $(y - 7)$ years.
$\therefore 3(x - 7) = (y - 7)$
Simplify to get $3x - 21 = y - 7$
$\Rightarrow 3x - y = -7 + 21 = 14$ .... (ii)
Putting the two equations together, you have:
$x + y = 48$ .... (i)
$3x - y = 14$ .... (ii)
Adding (i) and (ii) gives:

$$4x = 62 \text{ or } x = \tfrac{62}{4} = 15\tfrac{1}{2}$$

Put $x = 15\frac{1}{2}$ in (i) to get:

$$15\tfrac{1}{2} + y = 48$$

$$\therefore \quad y = 48 - 15\tfrac{1}{2} = 32\tfrac{1}{2}$$

$\therefore$ John's age is $15\frac{1}{2}$ years and Mary's age is $32\frac{1}{2}$

Exercise 11.3
1. the sum of two numbers is 38. If their difference is 10, calculate the numbers.
2. In 12 years from now, a father will be twice as old as his daughter. Twelve years ago he was six times as old as his daughter. Find their present ages.
3. Solve each of the following pairs of simultaneous equations using any of the methods you have learnt in this unit.

   (a) $2x + y + 1 = 0$
   $3x - 2y + 5 = 0$

   (b) $\frac{2x}{3} + \frac{y}{5} = 6$

   $\frac{x}{6} - \frac{y}{5} = 6$

Hint on (b): Clear the fractions by multiplying each equation by the LCM of the denominators.

4. Given two numbers of which when the first of the two numbers is added to twice the second, the result is 21, but when the second number is added to twice the first, the result is 18. Find the two numbers.

11.7 Conclusion and summary

This unit on simultaneous linear equations has taken you through some basic definitions and two main methods of solving simultaneous linear equations. In particular, you learnt how to formulate simultaneous equations from some given problem statements. You also learnt the meaning of simultaneous linear equations. You also learnt how to solve such equations by two basic methods:

substitution method and elimination method.

You are now ready to study the third method of solving these types of equations, i.e. the **graphical method**.

11.8 Tutor-marked assignment

1. Given that:

   $2x - y = 4$

   $x + y = 5$

   Use the two methods discussed in this unit to solve these simultaneous linear equations.
2. Two years ago, a man was six times as old as his son. In 18 years from now he will be twice as old as his son. Determine their present ages.

11.9 References

1. Ezike, R.O., G.C. Obodo, H.M. Ogbu and C.J.U. Asogwa, (1992), *Basic Mathematics for Teachers and Students*, Alpha Book Company, Nsukka, Nigeria, p. 87–92.
2. Murray, R.S., (1956), *College Algebra — Schaum Outline Series: Theory and Problems*, McGraw Hill Book Company, p. 100–109.
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Unit 12 Solutions of simultaneous equations by graphical method

Contents

12.1 Introduction

In the last unit, you were tutored on simultaneous linear equations. Also some word problems that could be translated to such equations as well as the methods of solving them were treated. However, one other method that was not treated then is the graphical method of solving simultaneous equations. This unit will take you through that method. It is a method that your will also find interesting.

12.2 Objectives

By the end of this unit, you should be able to:

i) carry out the various steps involved in plotting graphs of simultaneous linear equations; such as preparing table of values and choosing suitable scales for graphs;
ii) plot graphs of simultaneous linear equations;
iii) identify various types of equations such as consistent, dependent or inconsistent simultaneous equations.

12.3 Graphical method of solving simultaneous linear equations

The process of solving simultaneous linear equations graphically is as follows:

i) Draw the graph of the two equations on the same plane. This should result in two straight lines. How can you determine the solution of the equations from the graph? See the next step.
ii) Find the point of intersection of the two graphs. The coordinates (x, y) of the point of intersection are the solutions of the two equations.

Example 12.1

Use graphical method to solve the following simultaneous linear equations.

$$y - 2x + 3 = 0$$
$$y - 5x + 6 = 0$$

Solution

First re-express the equations in the following forms by making y the subject:

$y = 2x + 3$ .... (i)

$y = 5x - 6$ .... (ii)

Now go through the following steps:

Step 1

Prepare a table of values for each of the equations. Since the equations are that of straight lines, three pairs of values are enough. Any three arbitrary values can be chosen for x. In this example, assuming you choose $x = -3$, 1 and 3.

Table of values

x	–3	1	3
$y = 2x+3$ $y = 5x-6$	–3 –21	5 –1	9 9

The values of y in the above table are obtained by substituting for the values of x in each of the equations.

Step 2

Choose suitable scales for the graphs. For this case, let 2 cm represent 1 unit on the x -axis and let 2 cm represent 5 units on the y-axis. Draw the graphs on a graph sheet to give accurate values.

Ensure you have joined the points of the first equation with a straight line before plotting the points of the second equation in order to avoid confusion.

Step 3

Locate the point of intersection of the two graphs. Read the coordinates at the point of intersection. If you have drawn the graphs well, they should intersect at the point (3, 9), at $x = 3$, $y = 9$. These are your solutions.

The graphs are as presented in Fig 12.1.

Some cases of simultaneous equations

There are instances when the graphs drawn may not intersect. Two such cases will be discussed below.

i) Cases of inconsistent equations

When the lines drawn are parallel instead of intersecting, it means their equations are inconsistent and do not have simultaneous solutions.

Example 12.2

Solve the following equations graphically.

$x + y = 2$ .... (i)

$3x + 3y = 18$ .... (ii)

Solution

For $x + y = 2$

$x = 0 \Rightarrow y = 2$ i.e. point (0, 2)

$y = 0 \Rightarrow x = 2$ i.e. point (2, 0)

For $3x + 3y = 18$

$x = 0 \Rightarrow 3y = 18 \Rightarrow y = \frac{18}{3} = 6$ i.e. point (0, 6)

$y = 0 \Rightarrow 3x = 18 \Rightarrow x = \frac{18}{3} = 6$ i.e. point (6, 0)

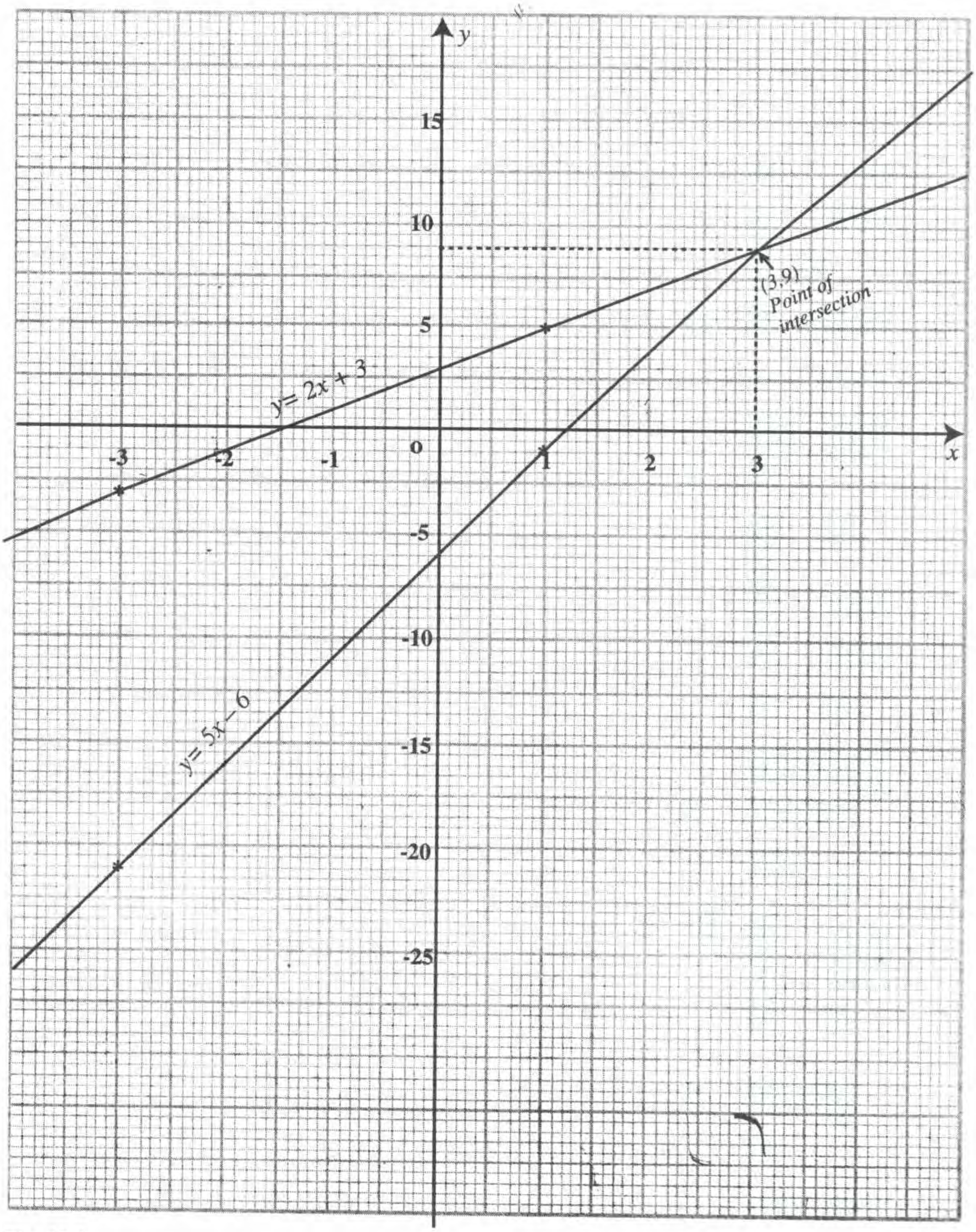
y
x
15
10
5
o
-5
-10
-15
-20
-25
-3
-2
-1
1
2
3
y= 2x + 3
y= 5x – 6
(3,9)
Point of
intersection

Fig. 12.1

The graph is sketched as shown in Fig. 12.2

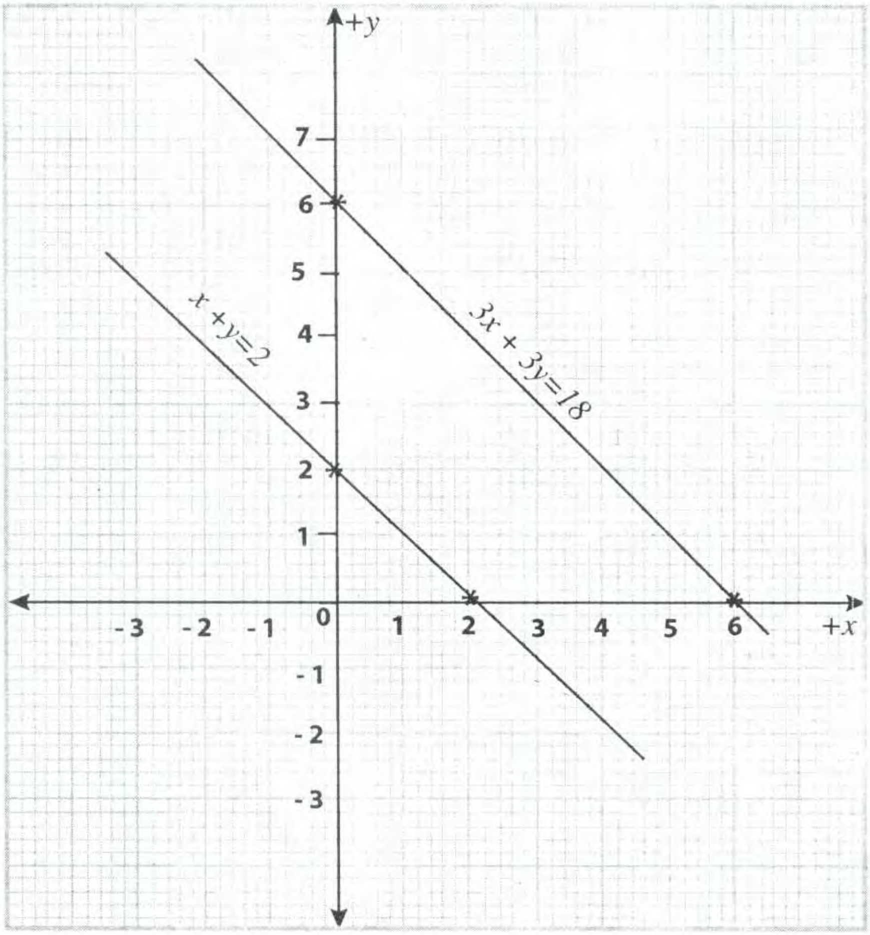

Fig. 12.2

Note that the two lines do not intersect but rather they are parallel. So the equations are inconsistent and do not have a simultaneous solution.

Note that if equation (i) is multiplied by 3 it gives $3x + 3y = 6$, which is not consistent with equation (ii)

***ii)* Cases of dependent equations**

When the graphs of the two equations result in the same line (i.e. the two equations are represented by the same line), they are said to be **dependent equations**. In such a case, every point on the line represents a solution, which means there are infinite number of simultaneous solutions.

Example 12.3

Draw the graph of the following equations

$x + y = 1$ .... (i)

$5x + 5y = 5$ .... (ii)

Solution

For $x + y = 1$,

$x = 0 \Rightarrow y = 1$ yielding point (0, 1)

$y = 0 \Rightarrow x = 1$, yielding point (1, 0)

For $5x + 5y = 5$
$x = 0 \Rightarrow 5y = 5 \Rightarrow y = 1$ i.e. point (0, 1)
$y = 0 \Rightarrow 5x = 5 \Rightarrow x = 1$ i.e. point (1, 0)

The graph is plotted as in Fig. 12.3.

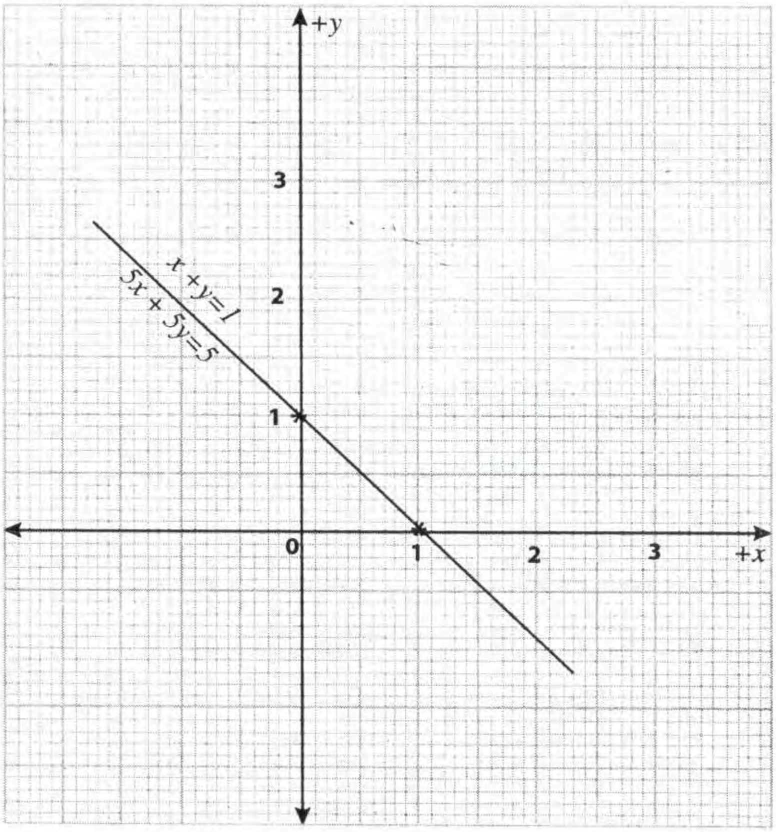

Fig. 12.3

Note that the two equations yield the same line.

You will also observe that if equation (i) is multiplied by 5, it results in equation (ii). That is, the two equations are multiples of each other, hence they are the same.

Consistent equations

When the graphs of the two lines intersect, and there is a simultaneous solution of the equations, the equations are said to be **consistent**.

Example 12.4

Solve the following graphically and indicate which of the systems are consistent, dependent or inconsistent:

a) $x + 3y = 4$
$2x - y = 1$

b) $2x - y = 5$
$2y = 7 + 4x$

c) $3x = 2y + 3$
$x - \frac{2y}{3} = 1$

Solution

a) Given $x + 3y = 4$ .... (i)

$2x - y = 1$ .... (ii)

$x + 3y = 4 \Rightarrow 3y = 4 - x$

$\Rightarrow y = \frac{4-x}{3}$

Then your table of values is as follows:

x	–4	0	4
y	$2\frac{2}{3}$	$1\frac{1}{3}$	0

For equation (ii)

$2x - y = 1 \Rightarrow - y = 1 - 2x$

$\Leftrightarrow y = 2x - 1$

Then your table of values is as follows

x	–4	0	4
y	–9	–1	7

The graph is as shown in Fig. 12.4. From the graph, the solutions are $x = 1$, $y = 1$. The equations are consistent.

b) Given that:

$2x - y = 5$ .... (i)

$2y = 7 + 4x$ .... (ii)

From equation (i):

$- y = 5 - 2x$ or $y = 2x - 5$

A table of values is as follows:

x	–3	0	3
y	–11	–5	1

From equation (ii):

$2y = 7 + 4x \Rightarrow y = \frac{7+4x}{2}$

A table of values is as follows:

x	–3	0	3
y	$-2\frac{1}{2}$	$3\frac{1}{2}$	$9\frac{1}{2}$

The graphs for the equations are as drawn in Fig. 12.5

From the graphs, the two lines are parallel, and do not meet. Therefore, the equations are inconsistent, so there is no simultaneous solution.

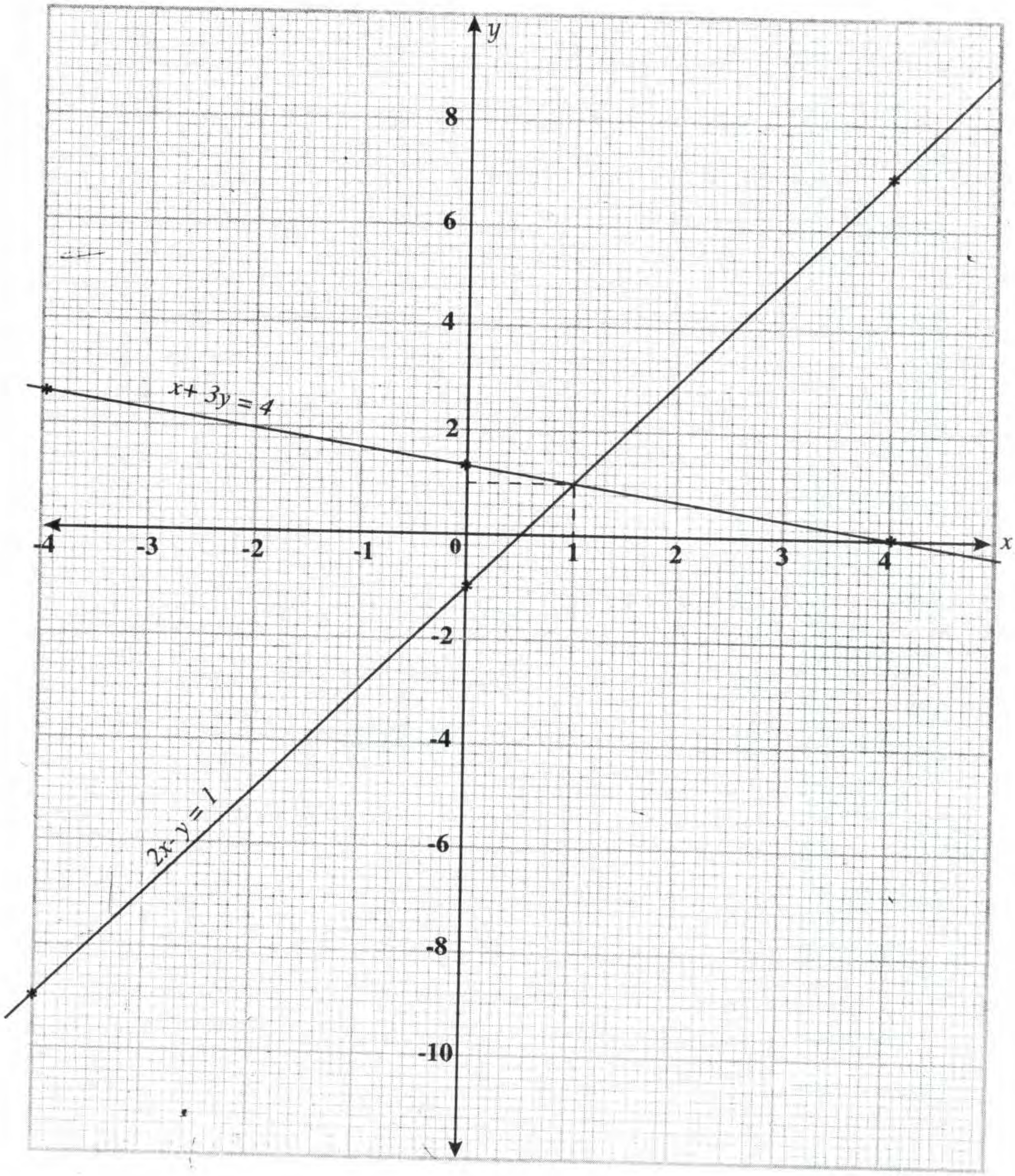

Fig. 12.4

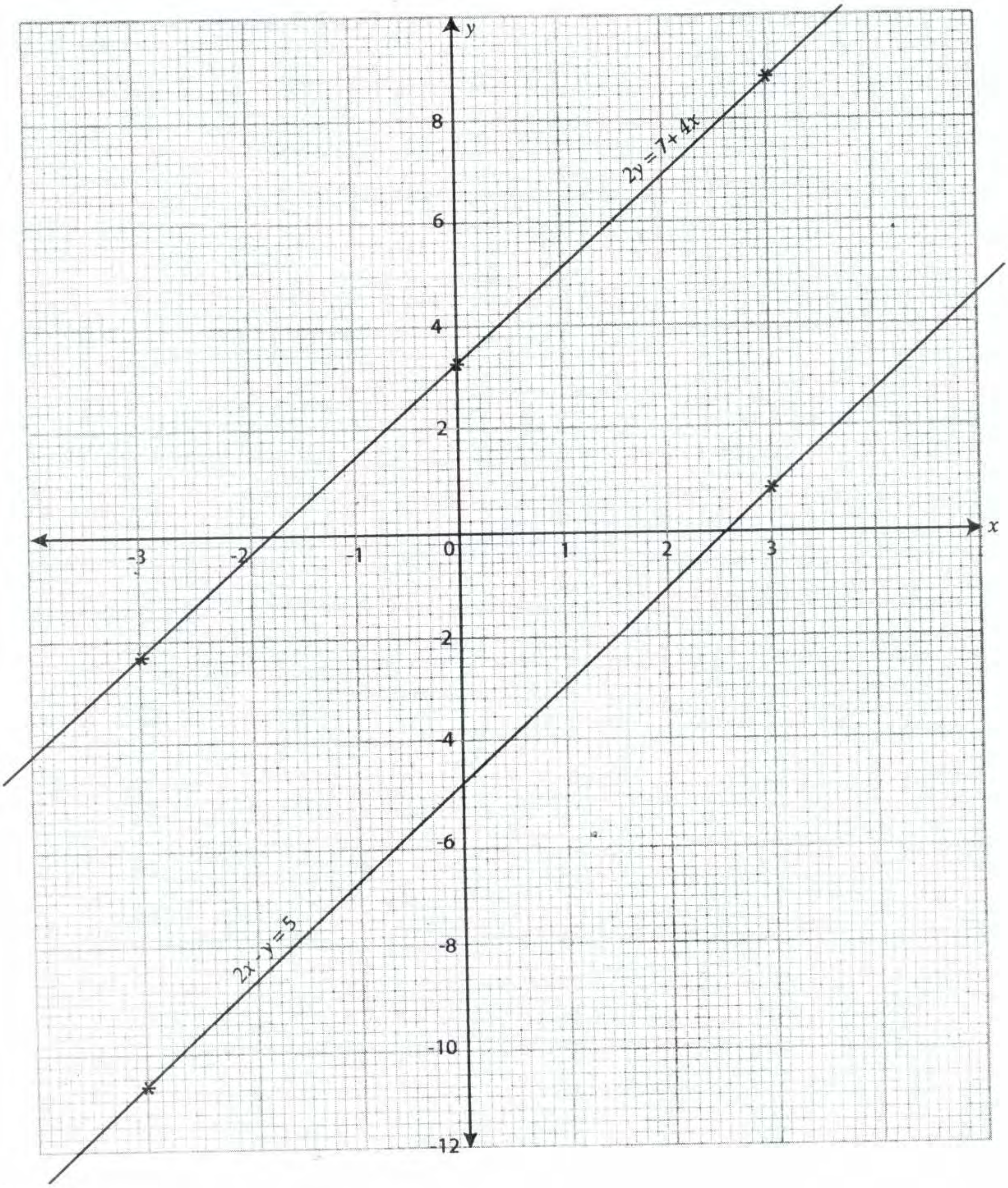

Fig. 12.5

c) Given that

$$3x = 2y + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$x - \frac{2y}{3} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From equation (i):

$$2y = 3x - 3$$

or $y = \frac{3x - 3}{2}$

A table of values is as follows:

x	–4	0	4
y	–7.5	–1.5	4.5

From equation (ii), we also have $x - \frac{2y}{3} = 1 \Rightarrow 3x - 2y = 3$

$\Rightarrow \quad 3x - 3 = 2y$

or $y = \frac{3x-3}{2}$

A table of values is as follows:

x	–4	0	4
y	–7.5	–1.5	4.5

In this case, the tables of values have shown that the graphs of the two equations will result in the same line. Therefore, the equations are dependent, and it has infinite number of simultaneous solutions. This is shown in the graph in Fig. 12.6.

Exercise 12.1

Using graphical method, solve the following pairs of simultaneous equations and indicate which pair of the equations are consistent, dependent or inconsistent:

1 a) $x + y = 1$
$x - 2y = 7$

b) $4x + 2y = 7$
$x - 3y = 7$

c) $\frac{(x+3)}{4} = 2y - 1$
$3x - 4y = 2$

2 a) $3x - y = -6$
$2x + 3y = 7$

b) $\frac{(x+2)}{4} - \frac{(y-2)}{12} = \frac{5}{4}$
$y = 3x - 7$

c) $4x + 2y = 3$
$5x - 3y = -2$

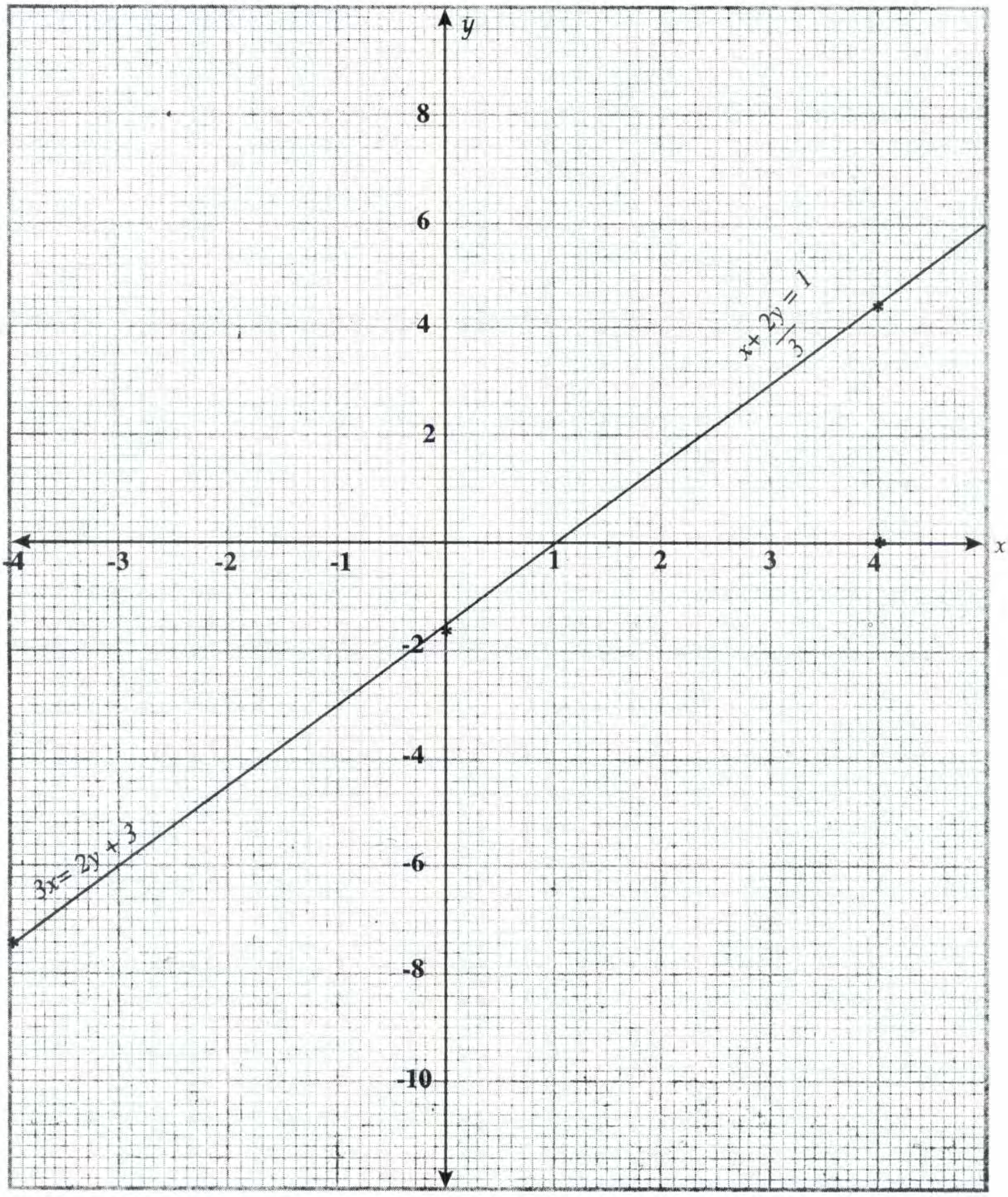

Fig. 12.6

12.4 Conclusion

This unit has taken you through the graphical method of solving simultaneous linear equations in two unknowns. This method, together with other methods which you learnt in the last unit have put you in a position to be able to handle any problems related to this topic. You are therefore climbing higher on the ladder of mathematical knowledge, a necessary foundation for scientific and technological pursuit. Well done.

12.5 Summary

In this unit, you have learnt:

i) that simultaneous linear equations in two unknowns can be solved graphically;

ii) how to follow the steps involved in plotting the graphs of simultaneous linear equations and construct table of values following the choice of suitable scales;

iii) how to identify various types of simultaneous linear equations i.e. consistent, dependent or inconsistent equations.

12.6 Tutor-marked assignment

1. Solve the following simultaneous linear equations by:
   a) Substitution method
   b) Elimination method

   $2x - y = 4$

   $x + y = 5$
2. Solve the equations given above by the graphical method.

12.7 References

Ezike, R.O., G.C. Obodo, H.M. Ogbu and C.J.U. Asogwa (1992), *Basic Mathematics for Teachers and Students*, Alpha Book Company, Nigeria, pp. 87–92.

Murray, R.S. (1956), *College Algebra — Schaum Outline Series; Theory and Problems*, McGraw Hill Book Company, pp. 100–109.

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Unit 13 Quadratic equations

Contents

13.1 Introduction

In Unit 10, you learnt how to solve equations in one variable. However, there are many real-life model problems that require products and/or quotients of physical quantities. A good example is the area of a circle given by πr^2 where r is the radius. Those kinds of problems involve second degree terms. Such equations involving second degree terms are called **quadratic equations**.

This unit will be concerned with solving quadratic equations analytically.

13.2 Objectives

By the end of this unit, you should be able to:

i) solve quadratic equations by factorisation;
ii) solve quadratic equations by completing the squares;
iii) derive the quadratic formula and apply it to solve quadratic equations;
iv) translate relevant word problems into quadratic equations and solve them;
v) identify some relationships between the roots of quadratic equations and use them to solve related problems.

13.3 Definition of quadratic equations

This is an equation of the form $ax^2 + bx + c = 0$ where x is a variable, a, b and c are constants, and $a \neq 0$. In such an equation, the highest degree (or index) of the variable is two, a is the coefficient of x^2, b is the coefficient of x and c is the constant term. Some examples of quadratic equations are:

(i) $x^2 - 3x + 5 = 0$
(ii) $5x^2 + 7x + 2 = 0$
(iii) $x^2 - 9 = 0$
(iv) $2x^2 = 6 - x$

The general quadratic equation $x^2 + bx + c = 0$ is usually referred to as the **standard form** of a quadratic equation. Equations (i)–(iii) are in the standard form, while equation (iv) is not. However, it can be rewritten in the standard form as: $2x^2 + x - 6 = 0$. Now, write $12 = 7x - x^2$ in the standard form and compare your result with $x^2 - 7x + 12 = 0$. A quadratic equation which has no x term of the linear is called a **pure quadratic equation**. Equation (iv) is an example.

Solving a given quadratic equation means finding values of x which satisfy the equation. The values obtained are called **solutions** or **roots** of the equation. For example, $x^2 + 5x - 6 = 0$, is satisfied by two values: $x = -6$ and $x = 1$. It then follows that $x = -6$ and $x = 1$ are the solutions or roots of the equation.

13.4 Methods of solving quadratic equations

The following methods are employed in solving quadratic equations: factorisation, completing the square, use of quadratic formula and graphical method.

In this unit, you will study the first three methods, while the graphical method will be treated in Unit 14.

Solution by factorisation

Recall that the product of two terms is equal to zero if any one of the two terms is zero, i.e. $A \times B = 0$ if $A = 0$ or $B = 0$. Similarly, the product of two algebraic expressions is equal to zero when one of them is zero. Factorisation method will be shown using examples.

Example 13.1

Solve the equation $x^2 - 16 = 0$.

Solution

Factorising $x^2 - 16 = 0$, you have $(x - 4)(x + 4) = 0$.

Hence, either $(x - 4) = 0$, or $(x + 4) = 0$

If $x - 4 = 0$, then $x = 4$

If $x + 4 = 0$, then $x = 4$

The solution is therefore, $x = 4$ or -4.

This is written as $x = \pm 4$

Example 13.2

Solve the equation $t^2 - 5t = 0$

Solution

Factorising $t^2 - 5t = 0$ gives $t(t-5) = 0$.

Hence, either $t = 0$, or $t-5 = 0$, i.e. $t = 5$.

The solutions: $t = 0$ or $t = 5$.

Example 13.3

Solve the equation $x^2 - 4x + 4 = 0$

Solution

Factorising, you have $(x-2)(x-2) = 0$

The solution is $x = 2$ two times. This equation has double roots: $x = 2$ (twice)

Example 13.4

Solve the equation $\dfrac{1}{t-1} + \dfrac{1}{t-4} = \dfrac{5}{4}$.

Solution
Obtain the LCM of the denominators of both sides of the equation, i.e. $4(t-1)(t-4)$.
Next, multiply both sides of the equation by this LCM, giving $4(t-4)+4(t-1)=5(t-1)(t-4)$.
You now have $8t-20=5t^2-25t+20$ or $5t^2-33t+40=0$
Factorising, we have $(5t-8)(t-5)=0$
Either, $t-5=0$, i.e. $t=5$

or $5t-8=0$ from which $t=\frac{8}{5}$.

$t=5$ or $\frac{8}{5}$.

Exercise 13.1
Now attempt the following, using factorisation method:

1 $x^2-40=9$
2 $x^2+36=9-2x^2$
3 $x^2-7x=-12$
4 $\frac{1}{4-x}-\frac{1}{2+x}=\frac{1}{4}$
5 $\frac{2x-1}{x+2}+\frac{x+2}{2x-1}=\frac{10}{3}$
6 $4x-5x^2=-12$

Solution by completing the square

It will be realised that not all quadratic equations can be solved by factorisation method. However, any quadratic equation can be solved by the process of completing the square.

Consider the expression x^2+ax. In order to turn this into a perfect square, follow these steps:

Step 1
Divide the coefficient of x by 2, i.e. $a \div 2 = \frac{a}{2}$

Step 2
Square the result got in step 1, to obtain $(\frac{a}{2})^2$ or $\frac{a^2}{4}$.

Step 3
Add the result got in step 2 above (i.e. $\frac{a^2}{4}$) to the expression x^2+ax to obtain $x^2+ax+\frac{a^2}{4}$.

Step 4
Factorise the result obtained in step 3 to obtain $a^2+ax+\frac{a^2}{4}=(x+\frac{a}{2})^2$.
Note that x^2+ax is not a perfect square, but by the processes in steps 1 to 4, the expression has been turned into a perfect square. Note also that the coefficient of the quadratic variable, i.e. x is unity (1) in the expression x^2+ax. The coefficient of the quadratic term must be reduced to unity before carrying out the processes shown in steps 1 to 4.

Example 13.5
Solve $x^2-6x-2=0$ by the method of completing the square.

Solution
Transfer the constant term in $x^2-6x-2=0$ to the R.H.S. to give $x^2-6x=2$.
Half of the coefficient of x is -3.
The square of -3 is added to both sides, giving $x^2-6x+(-3)^2=2+(-3)^2$,
i.e. $x^2-6x+9=11$.
The L.H.S. is a perfect square. Factorising, you have $(x-3)^2=11$.

Taking the square roots of both sides, you have

$$x-3=\pm\sqrt{11}$$

$$\therefore\ x=3\pm\sqrt{11}$$

Example 13.6
Using the method of completing the square, solve $x^2 = 4 - 3x$.

Solution

$x^2 = 4 - 3x$ is written as $x^2 + 3x = 4$. The square of half of the coefficient of x is $\left(\frac{3}{2}\right)^2$. This value is added to both L.H.S and R.H.S above to give

$x^2 + 3x + \left(\frac{3}{2}\right)^2 = 4 + \left(\frac{3}{2}\right)^2$, i.e. $x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{25}{4}$, or $\left(x + \frac{3}{2}\right)^2 = \frac{25}{4}$.

Taking square roots of both sides:

$$\left(x+\tfrac{3}{2}\right) = \pm\sqrt{\tfrac{25}{4}} = \pm\tfrac{5}{2}.$$

Therefore, $x = -\frac{3}{2} \pm \frac{5}{2} = \frac{1}{2}(-3 \pm 5)$

i.e. $x = 1$ or -4.

Example 13.7
Solve by completing the square: $3x^2 + 8x + 5 = 0$

Solution
Re-writing the equation to transfer the constant term to R.H.S, we have $3x^2 + 8x = -5$
Reduce coefficient of x^2 to unity by dividing through by 3:

$$x^2 + \tfrac{8}{3}x = -\tfrac{5}{3}.$$

The square of half of the coefficient of x is $\left(\frac{4}{3}\right)^2$. When this value is added to both sides of the equation above, we have $x^2 + \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = -\frac{5}{3} + \left(\frac{4}{3}\right)^2$

i.e. $x^2 + \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = \frac{1}{9}$.

The L.H.S is a perfect square. Taking the square roots of both sides gives

$$\left(x+\tfrac{4}{3}\right) = \pm\sqrt{\tfrac{1}{9}} = \pm\tfrac{1}{3}.$$

$x = -\frac{4}{3} \pm \frac{1}{3}$, from which

$x = -1$ or $-\frac{5}{3}$.

Exercise 13.2
By using the method of completing the square, solve the following quadratic equations.

1 $x^2 + 4x - 5 = 0$ **2** $x(x-3) = 4$

3 $2x^2 = x + 1$ **4** $3x^2 - 2 = 5x$

Solution by quadratic formula

Consider the general quadratic equation

$ax^2 + bx + c = 0$ .... (1)

a, b and c being constants, $a \neq 0$.

Make the coefficient of x^2 unity by dividing through by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

or $x^2 + \frac{b}{a}x = \frac{c}{a}$ .... (2)

Next, make the L.H.S a perfect square by adding the square of half of the coefficient of x to it and R.H.S, i.e. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a}$.

Since L.H.S is now a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

R.H.S can be simplified further as:

$$\frac{-c}{a} + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Thus, you have:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking the square roots of both sides:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

From which $x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$

$$= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which can be used to solve any quadratic equation.

Example 13.8

Solve this equation by using the quadratic formula:

$$3x^2 - 5x + 1 = 0$$

Solution

Comparing the general quadratic equation

$ax^2 + bx + c = 0$ with $3x^2 - 5x + 1 = 0$

You have, $a = 3$, $b = -5$, $c = 1$.

Therefore, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

$$= \frac{+5 \pm \sqrt{25 - 12}}{6}$$

$$= \frac{5 \pm \sqrt{13}}{6} \text{ or } \frac{1}{6}\left(5 \pm \sqrt{13}\right)$$

Exercise 13.3

Solve each of the following quadratic equations by the quadratic formula method:

1 $x^2 - 5x = 6$

2 $x^2 - 6 = x$

3 $16x^2 - 3x + 1 = 0$

4 $9x^2 + 6x = -4$.

13.5 Quadratic problem statements

In dealing with statements of problems involving quadratic equations, certain basic steps should be followed:

Step 1: Using pre-determined letters, define the problem in terms of mathematical symbols.
Step 2: Using the symbols and letters in step 1 above, construct a mathematical equation.
Step 3: Solve the equation constructed in step 2 above.
Step 4: Check the correctness of the solution by referring to the problem.
Step 5: Consider if the solution arrived at is reasonable (i.e. rational). For instance, if the number of people or the length of a field is required, the answer should not be negative. In such cases, negative answers should be rejected.

Example 13.9
If three times a certain number is added to twice its reciprocal, the result is 5. Find the number.

Solution

Step 1
Let that number be x.

Step 2

Three times $x = 3x$, and twice the reciprocal of $x = 2\left(\frac{1}{x}\right)$
Therefore, $3x + 2\left(\frac{1}{x}\right) = 5$.

Step 3
Solving the problem, you have

$$3x + 2\left(\frac{1}{x}\right) = 5, \text{ i.e. } 3x^2 - 5x + 2 = 0$$
$$(3x - 2)(x - 1) = 0 \text{ (by factorisation)}$$
$$x = \frac{2}{3} \text{ or } x = 1$$

Step 4
Verify the solution:

(i) If the number is $\frac{2}{3}$, then,

$$3\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}\right) = 2 + 3 = 5$$

(ii) If the number is 1, then
$3(1) + 2(1) = 5$.

Step 5

The number is $\frac{2}{3}$ or 1. Each of the solutions is correct.

Example 13.10
The perimeter of a rectangle is 50 m and it has an area of 150 m^2. Determine the dimensions of the rectangle.

Solution
Since the perimeter = 50 m, then the sum of any two adjacent sides = 25 m.
Let one side = x m

$\therefore$ The other side is $(25 - x)$m
$\therefore$ Area of the rectangle = $x(25 - x)$ m^2.

But area is also equal to 150 m^2.

$\therefore x(25 - x) = 150$

$x^2 - 25x + 150 = 0$ (the required quadratic equation)

$(x - 15)(x - 10) = 0$ (by factorisation)

$x = 15$ or $x = 10$

If $x = 15$, then $25 - x = 10$

If $x = 10$ then $25 - 10 = 15$.

The dimensions of the rectangle are 15 m by 10 m.

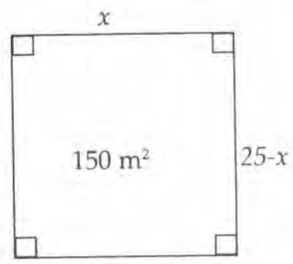


Fig. 13.1

Exercise 13.4

1 The sum of the squares of two numbers is 34. The first number is one less than twice the second number. Determine the numbers.

2 The difference between two positive numbers is 3. The sum of their reciprocals is $\frac{1}{2}$. Determine the numbers.

3 The length of a rectangle is three times its width. If the width is diminished by 1 m and the length increased by 3 m, the area will be 72 m^2. Find the dimensions of the original rectangle.

13.6 Characteristics of roots of quadratic equations

Recall that the formula for finding the roots of the general quadratic equation

$ax^2 + bx + c = 0$ is given as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

from which $x_1 = \frac{-b + \sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$

and $x_2 = \frac{-b - \sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$

where x_1 and x_2 are the two roots.

The expression $b^2 - 4ac$ is known as the **discriminant** of the roots. It is used to determine the nature of the roots of the quadratic equation as shown in the following cases. In each case, it is assumed that a, b and c are real and rational numbers.

Case 1

If $(b^2 - 4ac) = 0$, it this implies that what is inside the square root sign is zero. Then $x_1 = \frac{-b}{2a}$ or $x_2 = \frac{-b}{2a}$. The roots here are said to be **coincident**, meaning that they are **equal**. Such a quadratic expression can be factorised into a perfect square.

Consider the equations: $x^2 - 8x + 16 = 0$. Here $a = 1$, $b = -8$ and $c = 16$.

Solving by the formula, you have $x_1 = \frac{+8 + \sqrt{(-8)^2 - 4(1)(16)}}{2(1)}$

$$x_1 = \frac{+8 - \sqrt{64-64}}{2} = \frac{8+0}{2} = 4.$$

$$x_2 = \frac{+8 - \sqrt{(-8)^2 - 4(1)(16)}}{2a} = \frac{+8-0}{2} = 4.$$

The roots are said to be real and equal.

Case 2

If $b^2 - 4ac > 0$, then $b^2 > 4ac$, and whatever number is inside the square root sign is positive. The square root of such a number has two roots which are real, one positive and the other negative. The result is that the roots of such a quadratic equation are **real** and **unequal**.

For example, consider the equation $2x^2 + 3x - 20 = 0$.

Here $a = 2$, $b = 3$ and $c = -20$

$$x_1 = \frac{-3 + \sqrt{(3)^2 - 4(2)(-20)}}{2(2)} = \frac{-3 + \sqrt{169}}{4} = \frac{10}{4} = 2\tfrac{1}{2}$$

$$x_2 = \frac{-3 - \sqrt{(3)^2 - 4(2)(-20)}}{2(20)} = \frac{-3 - \sqrt{169}}{4} = \frac{-16}{4} = -4.$$

The roots are real and unequal.

Case 3

Next, we consider when $b^2 - 4ac < 0$. This implies that the number inside the square root sign is negative. But the square root of a negative number is not real. It is not within the domain of real numbers. It is said to be an **imaginary** or a **complex** number. Thus, no real roots can be obtained.

Consider $x^2 + 2x + 5 = 0$.

$$x_1 = \frac{-2 + \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 + \sqrt{-16}}{2}$$

$$\text{and } x_2 = \frac{-2 - \sqrt{2^2 - (4)(1)(5)}}{2(1)} = \frac{-2 - \sqrt{-16}}{2}$$

$$\text{i.e. } x = -1 \pm \tfrac{1}{2}\sqrt{-16}.$$

$$= -1 \pm \tfrac{1}{2}(4)\sqrt{-1}$$

In complex number theory, this is written as $x = -1 \pm 2i$, where $i = \sqrt{-1}$ is called the complex unit.

Exercise 13.5

In each case, find the discriminant ($b^2 - 4ac$) and thus determine the characteristics of the roots in the following:

1 $2x^2 - 7x + 4 = 0$

2 $3x^2 = 5x - 2$

3 $3x - x^2 = 4$

4 $2x^2 = 5 + 3x$

13.7 Relationship between roots of quadratic equations

Recall again the general quadratic equation $ax^2 + bx + c = 0$. The roots of the equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, Now, let the two roots of the equation be given as α and β.

Thus, $+\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Let $(b^2 - 4ac) = d$, then

$$\alpha = \frac{-b + \sqrt{d}}{2a} \text{ and } \beta = \frac{-b - \sqrt{d}}{2a}.$$

The expression for **sum of the roots** $(\alpha + \beta)$ is given as:

$$\alpha + \beta = \frac{-b + \sqrt{d}}{2a} + \frac{-b - \sqrt{d}}{2a}$$

$$= \frac{-b + \sqrt{d} - b\sqrt{d}}{2a} = \frac{-2b}{2a}$$

$$= -\frac{b}{a}$$

i.e. $\alpha + \beta \;=\; -\frac{b}{a}$ .... (i)

To obtain the product of the roots, we need the value of $\alpha\beta$.

$$\text{Thus, } \alpha\beta = \left[\frac{-b + \sqrt{d}}{2a}\right]\left[\frac{-b - \sqrt{d}}{2a}\right]$$

$$= \frac{b^2 - b\sqrt{d} + b\sqrt{d} - d}{4a^2} = \frac{b^2 - d}{4a^2}$$

Inserting the value of d, $\alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2}$

Therefore, $\alpha\beta = \frac{4ac}{4a^2} = \frac{c}{a}$ .... (ii)

Thus, we have:

(i) sum of roots, $\alpha + \beta = -\frac{b}{a}$

(ii) product of roots, $\alpha\beta = \frac{c}{a}$

where a = coefficient of the variable with the second degree.
b = coefficient of the variable with the first degree.
c = the constant term.

If we rewrite $ax^2 + bx + c = 0$ as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$,
then, we have $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ from (i) and (ii). This equation is seen as:
x^2 – (sum of roots) x + product of roots = 0.

This expression is very useful when studying quadratic equations.

Example 13.11
If $x^2 - 7x + 6 = 0$, find
(i) sum of the roots
(ii) product of the roots.

Solution
Since $x^2 - 7x + 6 = 0$ is similar to $ax^2 + bx + c = 0$, we see that $a = 1$, $b = -7$, $c = 6$.

(i) Sum of the roots $= \frac{-b}{a} = \frac{-(-7)}{1} = 7$

(ii) Product of the roots $= \frac{c}{a} = \frac{6}{1} = 6$.

Example 13.12

If α and β are the roots of $x + 3x^2 + 5 = 0$, find

(i) $\alpha + \beta$ (ii) $\alpha\beta$.

Solution

Re-write $x + 3x^2 + 5 = 0$ as $3x^2 + x + 5 = 0$

Here again, $a = 3, b = 1, c = 5$.

(i) $\alpha + \beta = \frac{-b}{a} = -\frac{1}{3}$

(ii) $\alpha\beta = \frac{c}{a} = \frac{5}{3}$

The next example is more demanding but interesting.

Example 13.13

If α and β are the roots of $2x^2 + 6x - 5 = 0$, find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\beta} + \frac{1}{\alpha}$.

Solution

$$\alpha + \beta = \frac{-b}{a} = -\frac{6}{2} = -3$$

$$\alpha\beta = -\frac{5}{2}$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-3)^2 - 2\left(-\frac{5}{2}\right)$

$= 9 + 5 = 14.$

(ii) $\frac{1}{\beta} + \frac{1}{\alpha} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-3}{-\frac{5}{2}} = \frac{6}{5}$.

13.8 Conclusion

Having gone through this unit of study on quadratic equations, you are now in a position to go through other topics of algebra with ease. This is because you will often come across situations involving quadratic equations in a number of algebraic problems.

In addition, understanding the principles guiding quadratic equations has given you the needed background to study the topic of the next unit.

13.9 Summary

In this unit, you have learnt:

i) the meaning of quadratic equations;

ii) how to solve quadratic equations, using such methods as factorisation, completing the square and quadratic formula.

iii) how to solve word problems involving quadratic equations, using appropriate steps.

13.10 Tutor-marked assignment

1 Solve the following quadratic equations by the factorisation method:

a) $p^2 + 3p = 28$ b) $5t - 2t^2 = 2$

2 Solve this equation by completing the square:

$x^2 - 6x + 8 = 0$

3 Solve by the quadratic formula:

$$2x^2 - 6x + 4 = 0$$

4 One positive number exceeds three times another positive number by 5. The product of the two numbers is 68. Find the numbers.

5 a) If α and β are the roots of equation $x^2 + 5x - 2 = 0$, find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\beta} + \frac{1}{\alpha}$.

b) If α and β are the roots of $8x^2 - 16x + 6 = 0$, form the equation whose roots are $\frac{1}{2}\beta$, $\frac{1}{2}\alpha$.

13.11 References

1 Backhouse, J.K. and S.P.T. Houldsworth, (1970), *Pure Mathematics, a first course*, Longman Group Limited, p. 178–180.

2 Murray, R.S., (1956), *College Algebra – Schaum Outline Series; Theory and Problems*, McGraw Hill Book Company, pp. 110–126 and pp. 122–134.

3 NERDC, (2001) *Further Mathematics for Senior Secondary Schools*, Longman Nigeria Plc, p. 99–106.

Unit 14 Graphs of quadratic equations

Contents

14.1 Introduction

In Unit 13, you learnt about three of the methods of solving quadratic equations. In this unit, you will learn about the graphical method of solving quadratic equations.

14.2 Objectives

By the end of this unit, you should be able to:

i) explain the principles of solving quadratic equations by the graphical method;
ii) plot the graphs of quadratic equations;
iii) use the graphs to solve quadratic equations.

14.3 Principles of solution by graphical method

i) Draw the graph of the given quadratic equation as accurately as possible. The graph of a quadratic equation is usually a curve, called **parabola.**
ii) After drawing the curve, notice where it crosses or touches the x-axis. Since $y = 0$ at the x-axis, it follows that point(s) where the curve cuts the axis are the solution points of the equation.

Example 14.1

Use graphical method to solve the equation $y = x^2 - 2x - 4$

Solution

Taking values of x from – 4 to 6, the table of values is given as:

x	– 4	– 3	– 2	– 1	0	1	2	3	4	5	6
$y = x^2 - 2x - 4$	20	11	4	–1	– 4	– 5	– 4	– 1	4	11	20

Table 14.1

The process of obtaining the values in Table 14.1 is as follows:

$x = -4 \Rightarrow y = (-4)^2 - (2 \times -4) - 4 = 16 + 8 - 4 = 24 - 4 = 20$
$x = -3 \Rightarrow y = (-3)^2 - (2 \times -3) - 4 = 9 + 6 - 4 = 15 - 4 = 11$
$x = -2 \Rightarrow y = (-2)^2 - (2 \times -2) - 4 = 4 + 4 - 4 = 8 - 4 = 4$
$x = -1 \Rightarrow y = (-1)^2 - (2 \times -1) - 4 = 1 + 2 - 4 = 3 - 4 = -1$
$x = 0 \Rightarrow y = (0)^2 - (2 \times -0) - 4 = 0 + 0 - 4 = 0 - 4 = -4$

$x \;=\; 1 \Rightarrow y = 1^2 - (2 \times 1) - 4 = 1 - 2 - 4 = 1 - 6 = -5$
$x \;=\; 2 \Rightarrow y = 2^2 - (2 \times 2) - 4 = 4 - 4 - 4 = 4 - 8 = -4$
$x \;=\; 3 \Rightarrow y = 3^2 - (2 \times 3) - 4 = 9 - 6 - 4 = 9 - 10 = -1$
$x \;=\; 4 \Rightarrow y = 4^2 - (2 \times 4) - 4 = 16 - 8 - 4 = 16 - 12 = 4$
$x \;=\; 5 \Rightarrow y = 5^2 - (2 \times 5) - 4 = 25 - 14 = 11$
$x \;=\; 6 \Rightarrow y = 6^2 - (2 \times 6) - 4 = 36 - 16 = 20$

The graph is shown below:

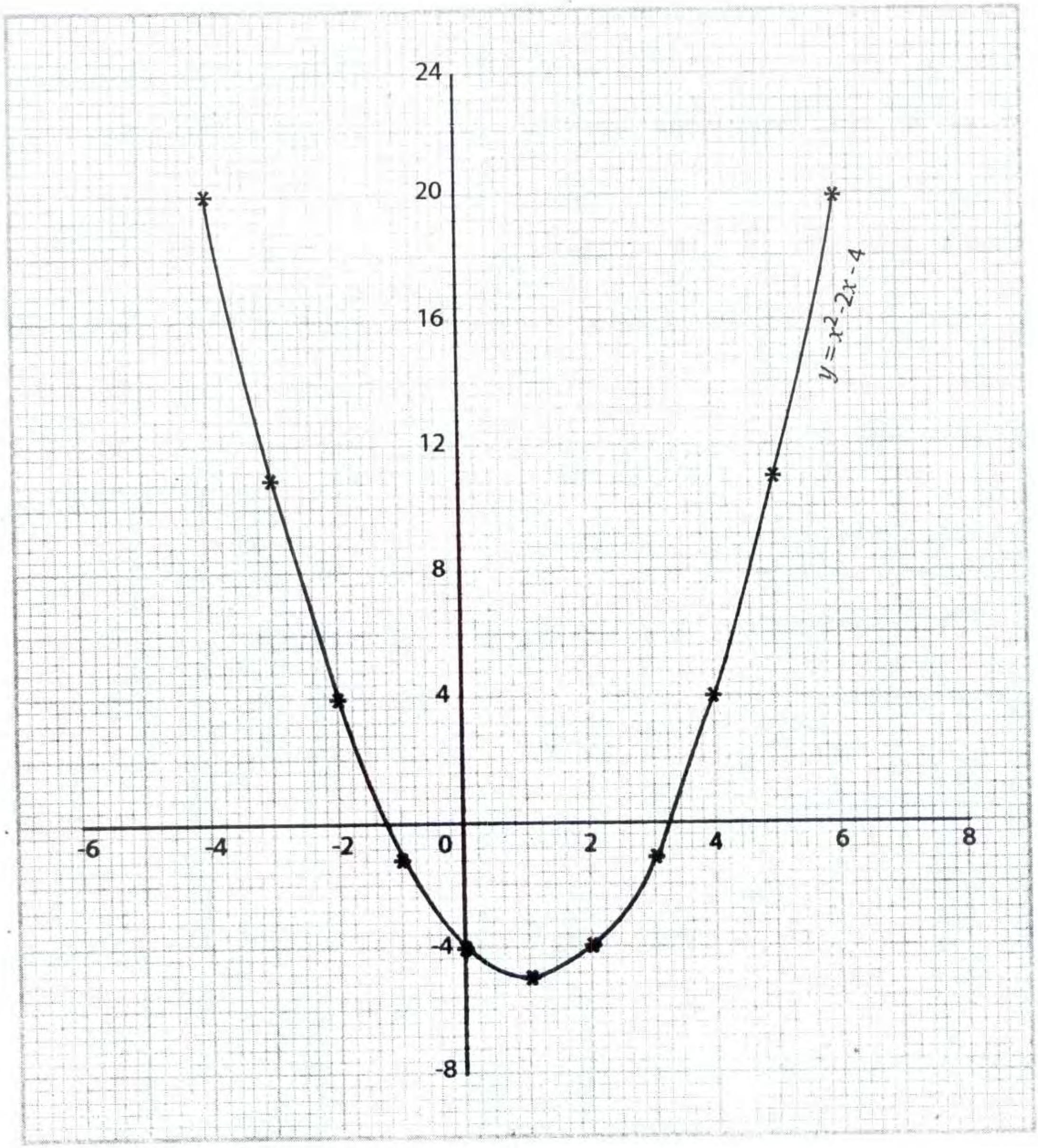


Fig. 14.1

Now, the solutions of the equation are the points of intersection of the curve $y = x^2 - 2x - 4$ with the x-axis. From the graph, the points are $x = -1.2$ and $x = 3.2$.

Example 14.2

Use graph to solve the equation $y = 5+x-x^2$, taking values of x from $x = -4$ to $x = 5$.
What is the greatest value of y?

Solution

Taking values of x from $x = -4$ to $x = 6$, the table of values is shown below. Follow the process employed in Example14.1. See Fig. 14.2.

x	-4	-3	-2	-1	0	1	2	3	4	5	6	$\frac{1}{2}$
$y = 5+x-x^2$	-15	-7	-1	3	5	5	3	-1	-7	-15	-25	$5\frac{1}{4}$

Table 14.2

(a) The intersection of the graph with the x-axis gives roots or solutions of the equation, i.e. $x = -1.7$ and $x = 2.8$

(b) Also from the graph, the greatest value of y is 5.2, i.e. the turning point of y.

Example 14.3

Draw the graph of the equation $y = x^2+x+2$ for $x = -4$ to $x = 3$.
On the same graph sheet and on the same axes, draw the graph of the straight line $y = x+4$.
From your graphs, find solutions of the following equations:

a) $x^2 + x + 2 = 0$
b) $x^2 - 2 = 0$
c) What is the least value of y?

Solution

Plot the table of values for equation $x^2 + x + 2$

x	-4	-3	-2	-1	0	1	2	3	$-\frac{1}{2}$
$y = x^2+x+2$	14	8	4	2	2	4	8	14	$1\frac{3}{4}$

Table 14.3

Plot the table of values for the straight line
$y = x+ 4$ as follows:

x	-3	0	3
$y = x + 4$	1	4	7

Table 14.4

See Fig 14.3 for the graphs.

a) The solution of equation $x^2+x+2 = 0$ should be the intersection of the curve with the x-axis. But in this case, the graph does not intersect with the x-axis. What then should the solution be? Now, in such a case as this when the graph does not intersect with the x-axis, it means that there is no real value of x for which $y = 0$.
The roots are imaginary roots. This observation is interesting, isn't it?

b) To solve $x^2 - 2 = 0$, see the steps below:
Add $x+ 4$ to both sides to get
$x^2 + (x + 4) - 2 = x + 4$

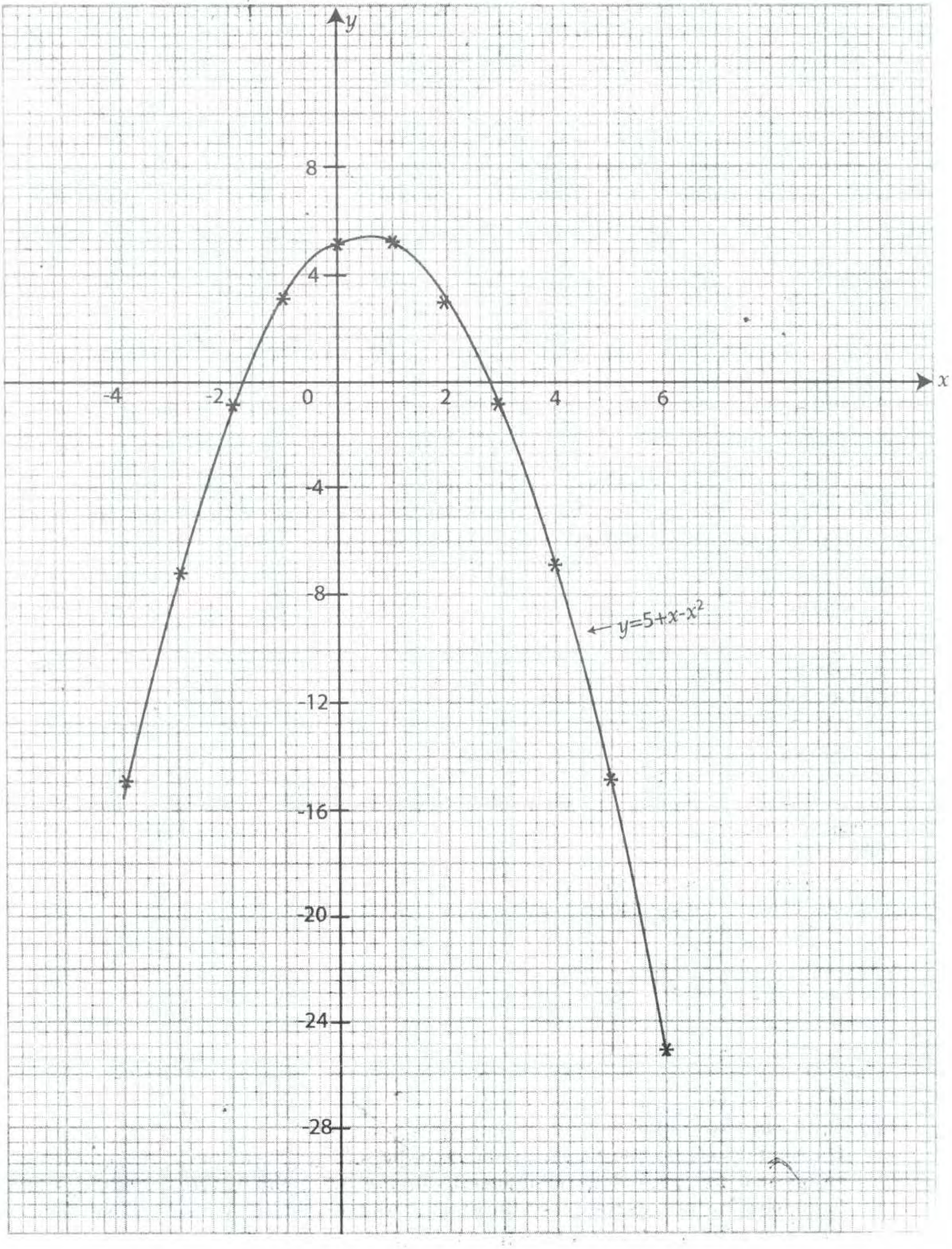


Fig. 14.2

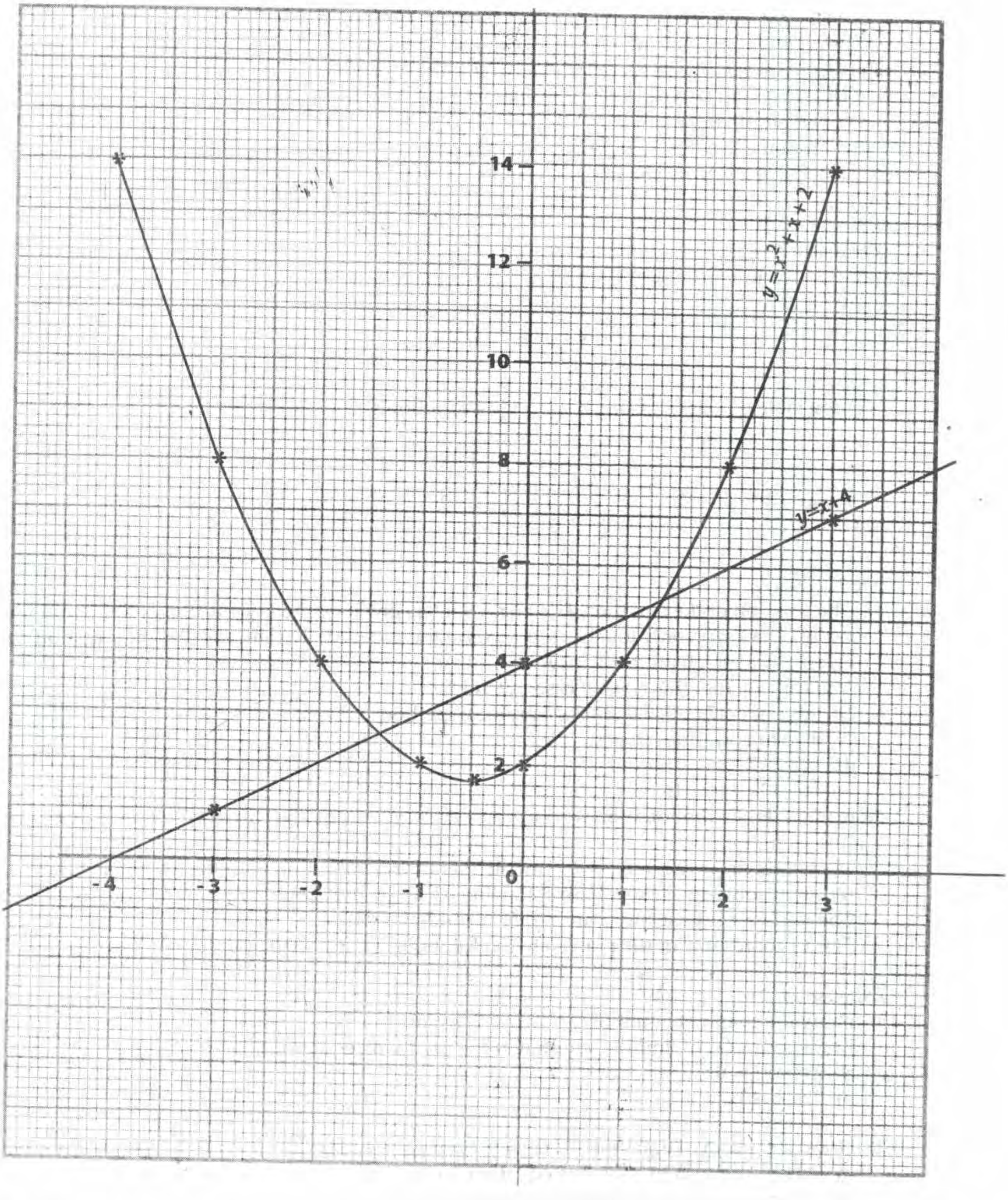


Fig. 14.3

$x^2 + x + 4 - 2 = x + 4$

$\Rightarrow$ i.e. $x^2 + x + 2 = x + 4$

You will note that the L.H.S is the equation whose graph is drawn while the R.H.S is the graph of the straight line.

$\therefore$ the intersection of the two graphs is the solution of the curve x^2-2

From the graph, the points are $x = -1.4$ and $x = 1.4$.

This solution technique is very good. Go through it again.

c) The least value of y is given by the lowest point on the graph $y = x^2 + x + 2$

From the graph, the point is at $y = 1.65$.

Now, from the last three examples, you have seen some interesting problems that could be solved by using graphs. Now, the fourth example.

Example 14.4

Draw the graph of $y = 4x^2 - 12x + 9 = 0$

from $x = -1$ to $x = 4$. Use your graph to solve the following equations.

a) $4x^2 - 12x + 9 = 0$

b) $4x^2 - 12x - 7 = 0$

Solution

Plot the table of values from $x = -1$ to $x = 4$ as follows:

x	-1	0	1	2	3	4	$1\frac{1}{2}$
$y = 4x^2 - 12x + 9$	25	9	1	1	9	25	0

Table 14.5

See Fig 14.4.

a) The solution of the equation $4x^2 - 12x + 9 = 0$ is given by the intersection of the graph with the x-axis. The graph only touches the x-axis at the point 1.5. This means that the equation has two equal roots which are $x = 1.5$ (twice)

You came across this type of case in Unit 13.

b) To solve $4x^2 - 12x - 7 = 0$ by the graphical method:

Add 16 to both sides to get:

$4x^2 - 12x - 7 + 16 = 16$,

i.e. $4x^2 - 12x + 9 = 16$

Note that L.H.S is the equation whose graph was drawn, and the constant 16 is the same as the line $y = 16$.

Therefore, draw the line $y = 16$, and its points of intersection with the graph give the required solution. Take a look at the line $y = 16$ on the graph.

The points of intersections are $(-0.5, 16)$ and $(3.5, 16)$

The x- coordinate of the points are the solutions, i.e. $x = -0.5$ and 3.5.

Exercise 14. 1

1 Draw the graph of equation $y = 2x^2 + x - 3$, from $x = -3$ to $x = +3$.

From your graph, find the solution of the following equations:

a) $2x^2 + x - 3 = 0$

b) $2x^2 + x - 13 = 0$

2 Draw the graph of equation $3 + 2x = x^2$ from $x = -4$ to $x = 6$.

a) Use your graph to solve the equation $3 + 2x - x^2$

b) When $y = -15$, find the corresponding values of x.

c) What is the greatest value of y?

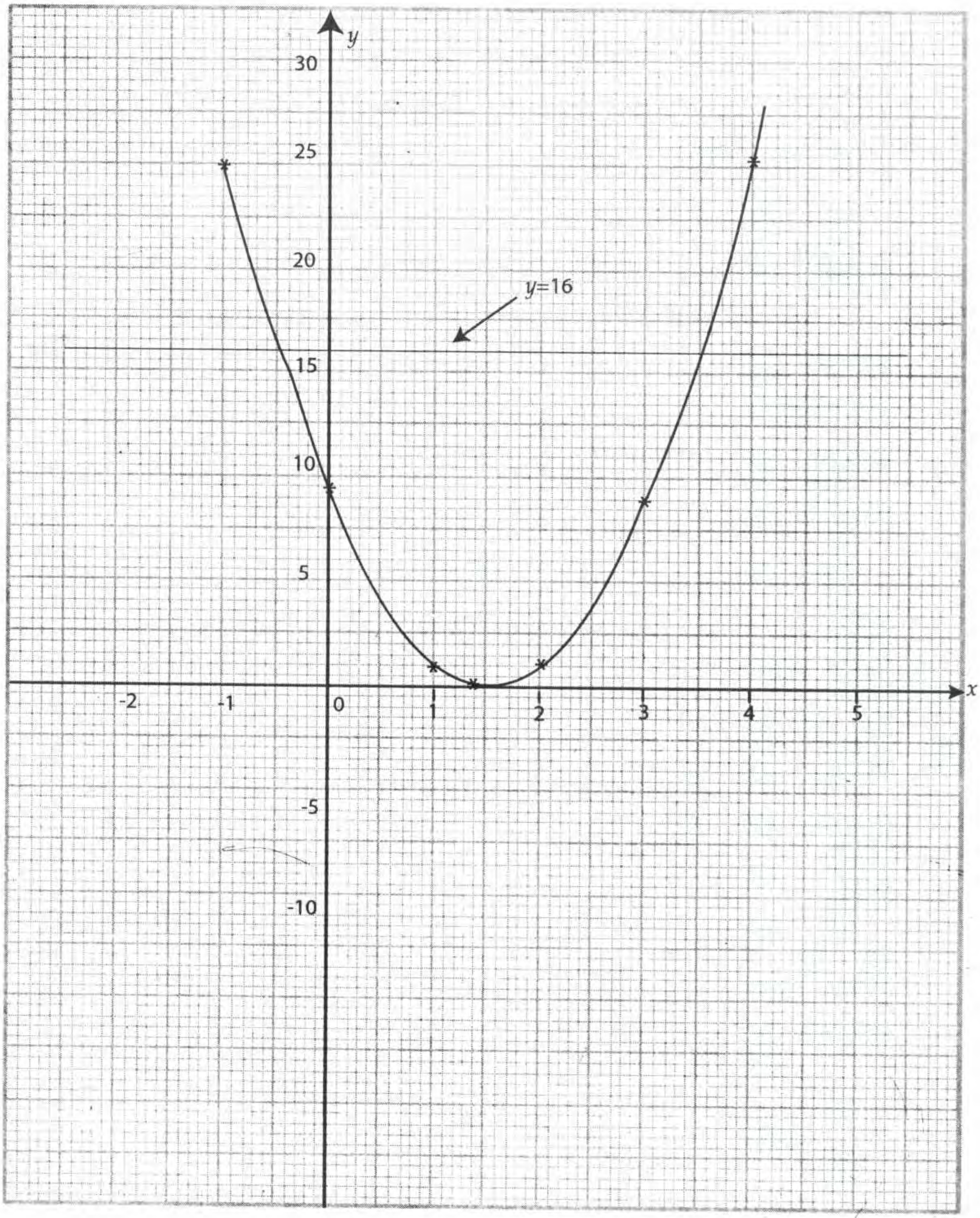


Fig. 14.4

3 Draw the graph of equation $y = x^2 - 2x - 2$ for $x = -4$ to $x = 6$.
On the same graph sheet, and on the same axes, draw the graph of the straight line $y = x + 6$.
Use your graphs to solve the following equations:
a) $x^2 - 2x - 2 = 0$
b) $x^2 - 3x - 8 = 0$
c) What is the least value of y?

14.4 Conclusion

You have been exposed to another important method of solving quadratic equations: the graphical method. With this addition, you are now well-armed to solve any problem involving quadratic equations.

Indeed, you have added more *blocks* to your algebraic foundation, upon which higher mathematics is built.

14.5 Summary

In this unit you have learnt:
i) the principles of solving quadratic equations by the graphical method;
ii) how to plot graphs of quadratic equations;
iii) how to use such graphs to solve problems in quadratic equations.

14.6 Tutor - marked assignment

1 Draw the graph of equation $y = 4 + x - x^2$ for $x = -4$ to $x = 5$.
a) Use your graph to find solutions of equation $4 + x - x^2 = 0$
b) When $y = -10$, what are the corresponding values of x?
c) What is the greatest value of y?

2 Using the same graph sheet and the same axes, draw the graphs of the equation $y = 2x^2 + 3x - 5$ for $x = -3$ to $x = 2$, and line $y = x + 5$ (using any 3 suitable points.)
From your graphs, find solutions to the following equations:
a) $2x^2 + 3x - 5 = 0$
b) $2x^2 + 2x - 10 = 0$

14.7 References

1 Murray, R. S., (1956) *College Algebra - Schaum Outline Series Theory and Problems*, McGraw Hill Book Company, pp80-81; 115.
2 N.E.R.D.C, (2001), *Further Mathematics for Senior Secondary Schools*, Longman Nigeria Plc, pp 83 - 86.

Unit 15 Inequalities and their solutions

Contents

15.1 Introduction

In our day-to-day activities, there arises the need to compare things or quantities such as 'one quantity is less or more than another'. For example, '10 is grater than 7'; '5 is less than 9'; Abu is taller than Audu, etc.

In mathematics, there are ways of expressing such comparisons; e.g. "10 is greater than 7' is written as $10 > 7$; '5 is less than 9' is written as $5 < 9$, etc. This unit is designed to expose you to such studies. You will enjoy it.

15.2 Objectives

By the end of the unit, you should be able to:

i) explain the basic principle for solving inequalities;
ii) solve linear inequalities in one variable;
iii) solve linear inequalities in two variables;
iv) solve simultaneous linear inequalities;
v) solve simultaneous linear and quadratic inequalities.

15.3 Inequalities

In everyday life, it is normal to compare things, quantities, circumstances, expressions and so on. The outcome may be that things are similar, the same, equal or different from one another. At least two things must be present in order for you to talk about comparison. In measurable attributes, two things may be equal or not equal. In the absence of equality, you have 'inequality'. In ordinary language, you deal with real quantities or expressions.

Definitions, notations and basic principles

Definitions

1. When one real quantity is greater or less than another real quantity, we have an **inequality** existing between them.

2. An **absolute inequality** is a statement that is true for all values of the expression under inequality. For example, for all values of the real numbers x and y, $(x - y)^2$ is greater than -1.
3. A **conditional inequality** is a statement that holds true only for particular values of the expression under inequality. For example, $(x - 2)$ can be greater than 3 only if x is greater than 5.

Notations

For all values of positive real numbers a and b,

1. $a > b$ means a is greater than b.
2. $a \geq b$ means a is greater than or equal to b
3. $a < b$ means a is less than b
4. $a \leq b$ means a is less than or equal to b
5. $0 < x < 5$ means x is greater than zero but less than 5.

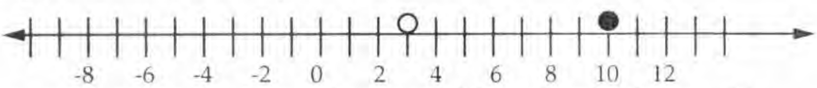
Basic principles of inequalities

1. If the whole of the R.H.S. and the L.H.S of an equality are exchanged with each other, then the symbol must change e.g.
   If $a + b > c + d$
   then $c + d < a + b$.
2. If each side of an inequality is increased or decreased by the same real number, the sense of the inequality remains unchanged. e.g.
   If $a > b$, then $a + c > b + c$.
   and $a - c > b - c$.
3. If each side of an inequality is multiplied or divided by the same positive real number, the sense of the inequality remains unchanged. e.g If $a > b$ and $m > 0$, then
   $ma > mb$ and $\frac{a}{m} > \frac{b}{m}$
4. For $m < 0$, $ma < mb$ and
   $\frac{a}{m} < \frac{b}{m}$
   (i.e. when $a > b$)
5. For positive numbers a, b, n; if $a > b$; then
   (i) $a > b^n$ (ii) $a^{-n} < b^{-n}$
6. If $a > b$ and $c > d$, then $(a + c) > (b + d)$.
7. If $a > b > c$ and $c > d > 0$, then $ac > bd$.
   The following are some examples to illustrate some of the above principles:

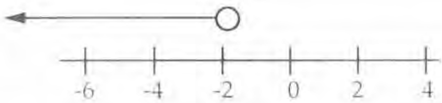
1. $4 > 3$, then $4^2 > 3^2$, but $4^{-2} < 3^{-3}$
2. $25 > 16$, then
   $25^{\frac{1}{2}} > 16^{\frac{1}{2}}$, but $25^{-\frac{1}{2}} < 16^{-\frac{1}{2}}$

Linear inequalities in one variable

A linear inequality is a first degree inequality in one unknown. It can be represented graphically on a straight line often referred to as number line:

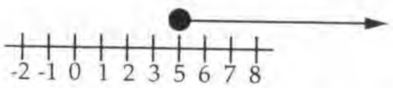


1. An arrow on the line shows the range of the values of the variable.
2. An empty circle shows that the circled number is not included. The shaded circle shows that the value circled is included. e.g. the graph of $x < -2$ is shown as



This means that x can have any value less then 2.

The graph of $x \geq 5$ in shown as follows:



That is, x can have values of 5 and above.

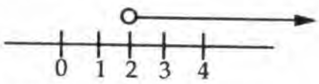
Example 5.1
Find the values of x which satisfy: $4x + 5 > 2x + 9$.

Solution
$4x + 5 > 2x + 9$.
Transpose $2x$ to LHS and 5 to RHS.
(Sign changes but sense remains unchanged)

$4x - 2x > + 9 - 5$
$2x > 4$.

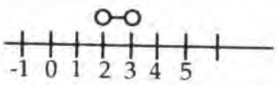
Divide both sides by 2 (sense unchanged)

$\frac{2x}{2} > \frac{4}{2}, \Rightarrow x > 2$



Example 15.2
Find the solution set of the inequalities
$x + 5 < 8$ and $x + 4 > 6$

Solution
For $x + 5 < 8$, subtract 5 from both sides:
$x + 5 - 5 < 8 - 5$, i.e. $x < 3$.
For $x + 4 > 6$, subtract 4 from the sides.
$x + 4 - 4 > 6 - 4$, i.e. $x > 2$.
The condition satisfying both $x < 3$ and $x > 2$ is the set of x such that x is greater than 2 but also less than 3. This is the set defined by $\{ x : 2 < x < 3 \}$
Graphical representation of the solution is shown as follows:



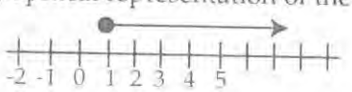
Example 15.3
Fine the range of values of x for which

$$\frac{x+2}{4x-3} \leq 3.$$

Solution

$\frac{x+2}{4x-3} \leq 3$ implies that $4x - 3 > 0$.

By cross-multiplication: $x + 2 \leq 3\,(4x - 3)$
i.e. $x + 2 \leq 12x - 9$
Transpose $12x$ to LHS and to RHS (sign changes, but sense remains): $x - 12x \leq -9 - 2 - 11x \leq -11$
Divide both sides by -11 (signs and sense change): $x \geq 1$.
The graphical representation of the solution is as follows:



Example 15.4
Find the values of x for which $2(x + 3) > 3(x - 1) + 6$

Solution
You can transpose wholly, both sides as follows:

$3(x - 1) + 6 < 2(x + 3)$

that is, $3x - 3 + 6 < 2x + 6$

$\Rightarrow 3x + 3 < 2x + 6$

Transpose $2x$ to LHS and 3 to RHS:

$3x - 2x < 6 - 3$

$x < 3.$

Another method of solving this example is:

$2(x + 3) > 3(x - 1) + 6$

$2x + 6 > 3x - 3 + 6$

$2x + 6 > 3x + 3$

Subtract $2x$ from both sides: $6 > x + 3$
Subtract 3 from both sides: $3 > x$
Transpose: $x < 3$.

You should be able to provide the graph for this solution.

Exercise 15.1

1. Represent the solution of the following inequalities graphically:
   i) $2x + 1 > 5$
   ii) $3x - 2 < 1$
   iii) $1 + 4x \geq 5$
   iv) $-1 < x$

2. Find the value of x for which each of the following holds:
   i) $2x \geq 7x + 15$
   ii) $4x \geq 14 - 3x$
   iii) $\frac{x}{2} - \frac{1}{3} < \frac{2x}{3} + \frac{1}{2}$
   iv) $\frac{1}{x} + \frac{3}{4x} > \frac{7}{8}$.

Linear inequalities in two variables

While linear inequalities in one variable involve a range, linear inequalities in two variables involve a region. The idea is to find a region in the $x - y$ plane that satisfies the inequality statement.

Example 15.5
Find the region of the $x - y$ plane which satisfies the inequality $3x - 2y > 4 + 2y - 3x$.

Solution

$3x - 2y > 4 + 2y - 3x$

Adding $3x$ to both sides: $6x - 2y > 4 + 2y$
Subtracting $2y$ from both sides: $6x - 4y > 4$
Dividing through by 2: $3x - 2y > 2$
Next, sketch the line $3x - 2y = 2$.
Then decide, from the inequality sign, on which side of the line $3x - 2y = 2$ the solution lies.

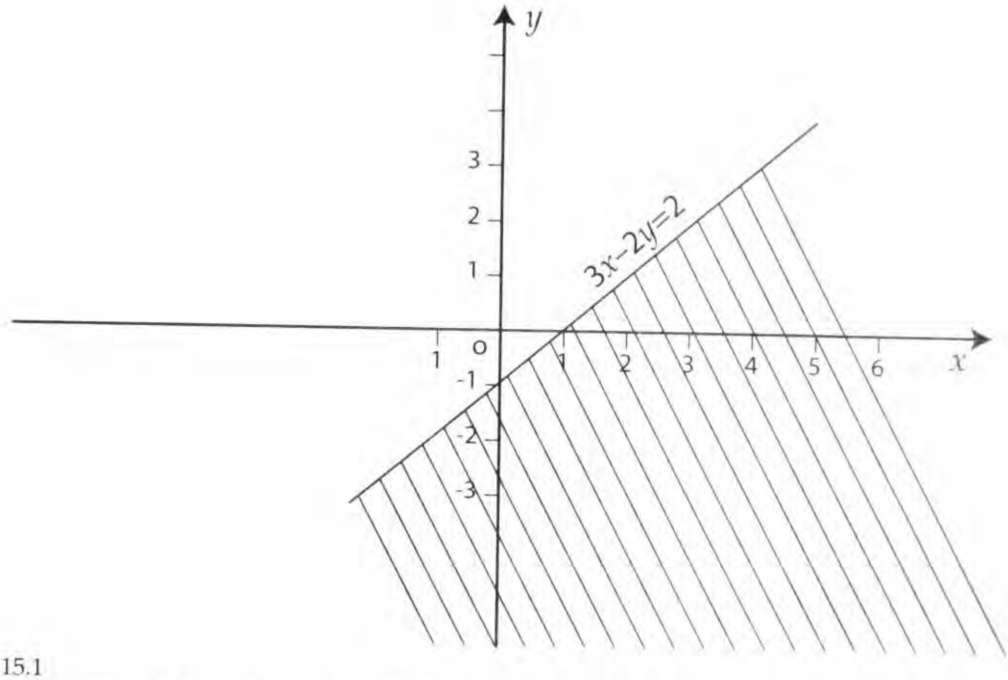


Fig. 15.1

A point (x_1, y_1) is in the solution region if when substituted in the inequality $3x - 2y > 2$, the condition holds. For instance, the point $(1, -2)$ satisfies the region since $(3(1) - 2(-2) > 2$.

The region is indicated by shading as shown in the graph and the line $3x - 2y = 2$ is not included in the region since the inequality sign (>) is a strict inequality. Note that if the inequality sign has been <, then the solution region would have been on the LHS of the line.

Exercise 15.2

Find the regions in the $x - y$ plane that satisfy the following inequalities:

1. $2x + 4y > 5x - 3y + 3$
2. $4x - 2y + 5 < 2x + 3y$

Quadratic inequalities

In solving problems involving quadratic inequalities, certain mathematical concept must be understood by you. They include the following:

For real numbers a and b,

1. The product of two numbers is positive if either both numbers are positive or both are negative, i.e. $ab > 0$ when $a > 0$ and $b > 0$ or when $a < 0$ and $b < 0$.

2. The product of two numbers is negative if only one of the two numbers is negative and the other is positive. i.e. $ab < 0$ when $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$.

3. Recall that not all quadratic expressions can be factorised. But all can be solved by the method of quadratic equation formula or completing the squares.

4. Recall also that the discriminant of the general quadratic equation has real roots if the discriminant $b^2 - 4ac \geq 0$.

Example 15.6

Find the values of x for which $x^2 - 7x + 12 > 0$

Solution:

$x^2 - 7x + 12$ can be factorised into $(x - 3)(x - 4)$.

Now, $(x-3)(x-4) > 0$ when $(x-3) > 0$ and $(x-4) > 0$, simultaneously.
i.e. $x > 3$ and $x > 4$ at the same time. Thus $x > 4$ satisfies that condition.
Also $(x-3)(x-4) > 0$ when $(x-3) < 0$ and $(x-4) < 0$, simultaneously, i.e. $x < 3$ and $x < 4$.
Thus $x < 3$ satisfies the condition.
Therefore, $x^2 - 7x + 12 > 0$ is satisfied when $x > 4$ or $x < 3$.

You should observe that the region $x > 3$ encloses the region $x > 4$ while the region $x < 4$ also encloses the region $x < 3$.

Example 15.7
Find the values of x for which $x^2 - 7x + 12 < 0$

Solution
$x^2 - 7x + 12 < 0$ implies $(x-3)(x-4) < 0$.
This is true when $(x-3) > 0$ and $(x-4) < 0$, simultaneously, or when $(x-3) < 0$ and $(x-4) > 0$, simultaneously. i.e. when

$x > 3$ and $x < 4$, which is possible or when $x < 3$ and $x > 4$, which is impossible.
Hence, $x^2 - 7x + 12 < 0$ when $3 < x < 4$.

Example 15.8
Find the minimum value of $x^2 + x - 3$ and determine where it occurs.

Solution

$$x^2 + x - 3 = x^2 + \frac{1}{4} - 3 - \frac{1}{4}$$
$$= x^2 + x + \frac{1}{4} - 3\frac{1}{4} = (x + \frac{1}{2})^2 - \frac{13}{4}$$

The least value of $(x + \frac{1}{2})^2$ is 0. Any other value assigned to x will make $(x + \frac{1}{2})^2 > 0$ and hence increase the value of

$$(x + \frac{1}{2})^2 - \frac{13}{4}.$$

Thus $(x + \frac{1}{2}) = 0$ when $x = \frac{1}{2}$

Thus, the least value of $x^2 + x - 3$ is $\frac{13}{4}$

and it occurs when $x = -\frac{1}{2}$

Graphical method
Put $x^2 + x - 3 = y$, and plot the graph of $y = x^2 + x - 3$.

x	−3	−2	−1	0	1	2	3	4
x^2	9	4	1	0	0	4	9	16
x	−3	−2	−1	0	1	2	3	4
−3	−3	−3	−3	−3	−3	−3	−3	−3
y	3	−1	−3	−3	−1	3	6	17

The lowest point on the curve where
$x = -\frac{1}{2}$ is shown as
$y = -\frac{13}{4}$ or $-3\frac{1}{4}$.

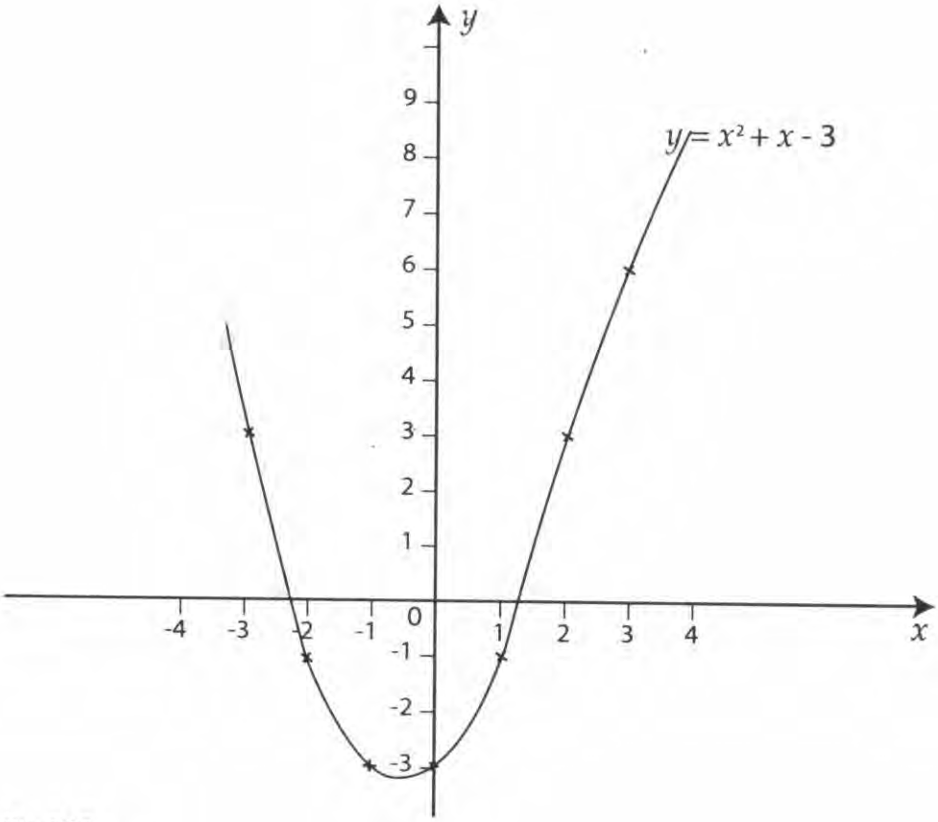


Fig 15.2

An alternative method

Put $x^2 + x - 3 = y$. from which $x^2 + x = y + 3$.

Completing the square on the LHS:

$$(x + \tfrac{1}{2})^2 = y + 3 + \tfrac{1}{4} = y + 3\tfrac{1}{4}.$$

Thus $y = (x + \tfrac{1}{2})^2 - 3\tfrac{1}{4}$

The minimum value of y occurs when

$(x + \tfrac{1}{2})^2 = 0$, as $(x + \tfrac{1}{2})^2$ can never be negative

Thus when $(x + \tfrac{1}{2})^2 = 0$, $y = -3\tfrac{1}{4}$

Also $(x + \tfrac{1}{2})^2 = 0$ when $(x + \tfrac{1}{2}) = 0$

i.e. when $x = -\tfrac{1}{2}$.

Go through this method again for a better understanding.

Exercise 15.3

1. Find the values of x for which the following inequalities hold:
   a) $(3 - x)(x + 2) > 0$
   b) $x^2 - 3x + 2 < 0$
   c) $\frac{4 - x}{2 + x} > 3$.

2. Show that the minimum value of
   $x^2 + 3x + 1$ is $-\frac{5}{4}$.

Determine where the minimum value occurs.

3. Determine, by graph method, the range of values of x satisfying $2x^2 - 5x + 2 < 0$.

Problems involving simultaneous linear and quadratic inequalities

These types of problems are best illustrated with examples. Go through the following examples:

Example 15.9

Find the values of x satisfying $x^2 - 3x + 2 < 0$

and $\frac{1}{2}x - \frac{3}{4} < 0$.

Solution

$x^2 - 3x + 2$ can be factorised as follows:

$(x - 2)(x - 1) <$

That is, either $x - 2 > 0$ and $x - 1 < 0$

(from which $x > 2$ and $x < 1$, which is not possible), *or*

$x - 2 < 0$ and $x - 1 > 0$

(from which $x < 2$ and $x > 1$, which is possible).

This gives $1 < x < 2$, as a set of solutions.

Also from $\frac{1}{2}x - \frac{3}{4} < 0$, $\Rightarrow \frac{1}{2}x < 4\frac{3}{4}$

Multiplying both sides by 2 gives

$x < \frac{3}{2}$.

If $x < \frac{3}{2}$, it holds that $x < 2$.

Combining $1 < x < 2$ and $x < \frac{3}{2}$ gives

$1 < x < \frac{3}{2}$.

Thus, $x^2 - 3x + 2 < 0$ and $\frac{1}{2}x - \frac{3}{4} < 0$ are satisfied if $1 < x < \frac{3}{2}$.

Example 15.10

Determine graphically the region of the $x - y$ plane which satisfies

$x^2 + y^2 < 9$ and $x + y < 2$.

Solution

$x^2 + y^2 < 9$ is the region inside a circle with the centre at the origin and radius 3.

$x + y < 2$ is the region below or on the LHS of line $x + y = 2$.

The shaded area in Fig. 15.3 is the solution, being the intersection of the two regions defined by the inequalities.

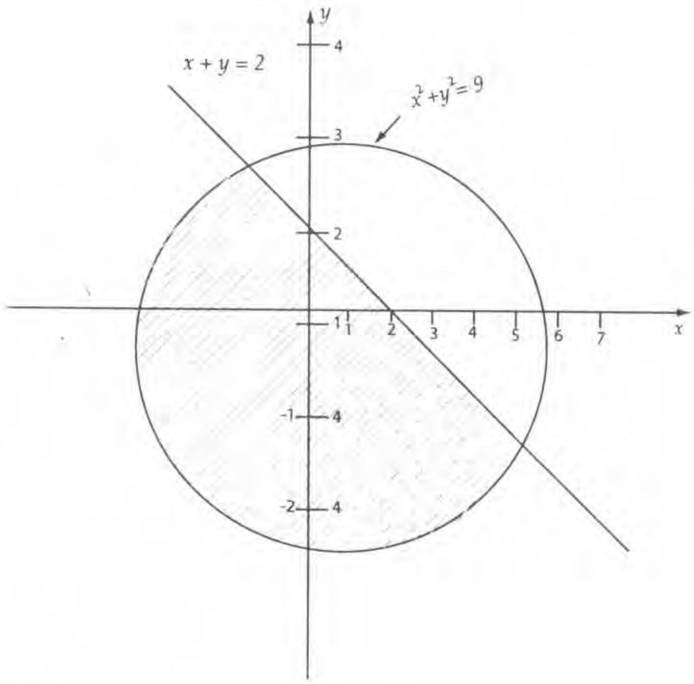


Fig. 15.3

Exercise 15.3

1. Find x for which $x^2 - 3x + 2 < 0$ and

   $x - \frac{3}{2} > 0$.

2. Solve graphically the simultaneous equation $x^2 + y^2 < 16$ and $x + 2y > 3$

15.4 Conclusion

You have been tutored on the concept of inequality which is a mathematical way of comparing things or quantities. You had earlier studied equations — both linear and quadratic equations. You now have a good foundation for exploring the world of algebra, even to higher levels.

15.5 Summary

In this unit, you have learnt that inequalities are used, mathematically, for comparing things or quantities. You have also learnt the notations and the basic principles involved in the operations associated with inequalities. This unit have also taken you through on how to solve:

i) linear inequalities in one variable;
ii) linear inequalities in two variables;
iii) simultaneous linear inequalities;
iv) quadratic inequalities.

15.6 Tutor-marked assignment

1 (a) Solve the following inequality

$$\frac{2t-3}{-3t+7} \leq 2$$

(b) Solve the following inequality and find the region of the $x - y$ plane which the inequality satisfies:

$4x - 2y \geq 8 + 2y - 4x$.

2 Find the value of p for which $p^2 + p - 2 > 0$

15.7 References

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Unit 16 Variations

Contents

16.1 Introduction

Variation in its mathematical sense is concerned with certain ways in which one variable depends on one or more others. The idea is bound up with ratio or proportion, which you have met in other areas of mathematics. It is advisable for you to revise these ideas and appreciate their importance.

Proportion arises in mathematics in a number of ways. For instance, the circumference C of a circle is proportional to its radius r. This is usually expressed in the form of an equation $C = 2\pi r$.

16.2 Objectives

By the end of this unit, you should be able to:

i) solve equations on direct variation;
ii) solve equations involving partial variation;
iii) solve joint variation problems;
iv) solve inverse variation problems.

16.3 Direct variation

To understand the notion of direct variation, consider a heavy rod of uniform section that is cut into various pieces and let the weight of each piece be proportional to its length. In addition, let the ratio of weight to length always be the same. Similarly, let a cyclist be moving at a steady speed, where the distance he covers is proportional to the time for which he has been travelling.

Both of these cases above are examples of **direct variation**. In the first case, the weight varies directly as the length, and in the second, the distance varies directly as the time. Consider also the area, A, of a circle which is given in terms of its radius r by the equation $A = \pi r^2$. Here, A is not proportional to r, but it is proportional to r^2 and you express this by saying that A varies as the square of r. The symbol used for 'varies as' is, so that the two statements just made above can be expressed as $W \propto L$ and $D \propto T$, where W = Weight, L = Length, D = Distance, T = Time. The expression, $W \propto L$ really means that the ratio $\frac{W}{L}$ is constant and it is on this ratio that all subsequent numerical work depends.

Example 16.1

If $W \propto L$ and $W = 8$ when $L = 2$, find the law connecting W and L and the value of L when $W = 20$.

Solution

If $W \propto L$, therefore $\frac{W}{L}$ is constant and let this constant be K

Then, $\frac{W}{L} = K$

$\therefore W = KL$

$\therefore 8 = K \times 2 = 2K$

$\therefore K = \frac{8}{2} = 4$

$\therefore W = 4L$, which is the law connecting W and L.

When $W = 20$, $20 = 4L$

$\therefore L = 5$

Note that $W \propto L$ means that $W = KL$ or

$L = \frac{1}{K}$ w, so that L also varies as W.

In general, if $a \propto b$, then b also $\propto a$.

Try this exercise now.

Exercise 16.1

1 In which of the following are the two quantities italicised in direct variation with one another?
   a) *Number of bricks* in a wall and *area* of wall.
   b) *Number of bricks* in a wall and *height* of wall.
   c) *Speed* and *distance* when travelling for a fixed period of time.

2 If $D \propto S$ and $D = 140$ when $S = 35$, find the law of the variation, and the value of S when $D = 176$.

3 $x \propto y$ and $x = 30$ when $y = 12$. Find x when $y = 10$ and y when $x = 14$.

($\therefore$ T = $\frac{D}{S}$). Then T α $\frac{1}{S}$, and T is said to 'vary inversely' as S.

16.4 Inverse variation

To start with, look at this illustration: If a pot of rice is shared between some family members. The greater the number of people in the family, the less rice each will receive. Similarly, if a car has to travel a certain distance D, the higher its average speed S, the less time T it will take. Therefore, $T = \frac{D}{S}$. Then $T \propto \frac{1}{S}$, and T is said to **vary inversely** as S.

These two cases are examples of inverse proportions or inverse variations. The quantity Q of rice each receives, varies inversely with the number n of the family.

This is written as $Q \propto \frac{1}{n}$.

Example 16.2

If V varies inversely with n, and $V = 330$ when $n = 10$, find V when $n = 12$.

Solution

If $v \propto \frac{1}{n}$

then $v = \frac{K}{n}$ where K is a constant

If $v = 330$ when $n = 10$, then you have:

$330 = \frac{K}{10}$ $\therefore K = 10 \times 330$.

$\therefore v = \frac{10 \times 330}{n}$

Now, when $n = 12$
then, $v = \frac{10 \times 330}{12} = 275$

Exercise 16.2

1 If y varies inversely as x, and $y = 2$ when $x = 3$, find y when $x = 6$.
2 P is inversely proportional to Q and $P = 5$ when $Q = 4$. What is the value of Q when $P = 25$?
3 If x varies inversely as the square of y, and $x = 4$ when $y = \frac{1}{2}$, what is y when x is 5?

16.5 Joint variation

So far, you have only considered examples of variation, where one variable, e.g. y; varies as some power of another variable, e.g. x. But there are many examples in science, engineering and everyday life when one variable depends on two or more others. For example, the volume V of a right circular cylinder is given in terms of its radius R and height H by the formula $V = \pi r^2 h$.

If you consider a metal rod of uniform circular cross-section which can be cut into lengths, you have a case of this law in which the radius is constant and so the volume varies as the length. Then $V \propto h$. On the other hand, if circular discs are cut out of a metal sheet or plywood, h will be constant and so the volume varies as the square of the radius, that is, $V \propto r^2$.

In summary, for a right circular cylinder:

If r is constant, $V \propto h$

If h is constant, $V \propto r^2$.

In experimental work, if one variable depends on two or more others, it is most convenient to see how the first depends on each of the rest in turns while the remainders are held constant.

The cases described above are examples of joint variation. As an illustration of this, consider Example 16.3 below.

Example 16.3

The mass of a wire varies jointly with its length and square of its diameter. If 400 m of wire of diameter 3 mm has a mass of 31.5 kg. What is the mass of 1 km of wire of diameter 2 mm?

Solution

Let M = mass in kg, d = diameter in mm and L = length in m. Then, from the problem statement: $M \propto Ld^2$ or $M = kLd^2$, where k is a constant.

Also, from the statement of the problem: $M = k \times 1\,000 \times 2^2$ (1 km = 1 000 m).

Now using $M = kLd^2$, you have

$3.15 = k \times 400 \times 3^2$

Therefore, $k = \frac{31.5}{400 \times 3^2}$

For 1 km (or 1 000 m) length of wire of diameter 2 mm, you have

$M = k \times 1000 \times 2^2$

$= (\frac{31.5}{400 \times 3^2}) \times 1\,000 \times 2^2$

$= 35$ kg

Exercise 16.3

1 If z varies directly as $\frac{x}{y^2}$ and y varies inversely as x. If $z = \frac{1}{3}$ when $x = 2$ and $y = \frac{1}{4}$ express
   a) y in terms of x.
   b) z in terms of x.
   c) If the value of x is increased by 10%, find the corresponding increase in the value of z.

2 The mass, m, of a roller varies jointly with its length, l, and the square of its diameter, d.

A roller of diameter 20 mm is 5 cm long and has a mass of 0.29 kg. Calculate the mass, in kg, of a roller 30 mm in diameter and 2 m long.

16.6 Partial variation

When a large family has to be fed, the cost of feeding depends upon two quite independent factors: first, the cost of overhead, which remains the same irrespective of how many family members are in the house, and secondly, the cost of the food itself, which is directly proportional to the number of members of the family being fed. Hence, it can be said that the cost is partly constant and it partly varies as the number of member present, or in algebra form, $C = a + kN$, C being the cost, N the number of people in the family, and a and k are both constants. This is an example of partial variation.

Example 16.4

C is partly constant and partly varies as N. If $C = 45$, when $N = 10$ and $C = 87$ when $N = 24$.

a) Find the formula connecting C and N.

b) Find C when $N = 18$.

Solution

a) Let $C = a + hN$, where a and h are constants.
Using the data from the problem, you have:
$45 = a + 10h$ ...... (i)
and $87 = a + 24h$ ...... (ii)
subtract (i) from (ii) to obtain
$42 = 14h$
$h = \frac{42}{14} = 3$

Substitute 3 for h in (i)
$45 = a + 30$
$a = 15$

Hence, $C = 15 + 3N$ is the required formula.

b) When $N = 18$
$C = 15 + 3 \times 18 = 15 + 54 = 69$

You would have noticed in this example that since there are two unknown, quantities and two equations must be formed. These are then solved simultaneously.

Exercise 16.4

1 Suppose x is partly constant and partly varies as y. When $y = 2$, $x = 0$ and when $y = 6$, $x = 20$.
   a) Find the equation connecting x and y.
   b) Find x when $y = 3$.

2 If x is partly constant and partly varies inversely as y. When $y = 3$, $x = 1$ and when $y = 6$, $x = 3$. Find x when $y = 4$.

3 A varies partly as, B and partly as the square root of B. When $B = 4$, $A = 22$ and when $B = 9$, A = 42. Find A when $B = 25$.

16.7 Conclusion and summary

In this unit, you have learnt about different types of variations: direct, inverse, joint and partial variations. They were discussed with worked examples. Variation in its mathematical sense is concerned with certain ways in which one variable depends on one or more others. The idea was built up with ratio and proportion. You should go through the examples again, since some of the topics you will meet in subsequent units will be related to what you have studied in this unit, in one way or the other. In this unit, you also studied about the characteristics of variations.

In brief, in this unit, you have learnt that:

i) a variable y is said to vary directly as another variable x (or y is proportional to x) if y is equal to some constant c times x, i.e. if $y = cx$;
ii) a variable y is said to vary inversely as another variable x if y varies directly as the reciprocal of x, i.e. if $y = \frac{c}{x}$;
iii) a variable z is said to vary jointly as x and y if z varies directly as the product xy, i.e. $z = cxy$;
iv) when two variables x and y are connected by the general linear equation $y = ax + b$, y varies as x and is partly constant, and this is known as partial variation.

16.8 Tutor-marked assignment

1 C is partly constant and partly varies as N. $C = 45$ when $N = 10$ and $C = 87$ when $N = 24$.
   a) Find the formula connecting C and N.
   b) Find C when $N = 18$.
2 The mass of a wire varies jointly with its length and the square of its diameter. 500 m of wire of diameter 3 mm has a mass of 31.5 kg. What is the mass of 1 km of wire of diameter 2 mm?

16.9 References

Channon, J. B; Smith A. M, *et al* (1985) *New General Mathematics for West Africa Book 5,* Longman Group Limited, U.K. pp. 73 – 79.

Unit 17 Matrices and determinants

Contents

17.1 Introduction

A **set** is a collection of elements in a particular order, but a **matrix** is a mathematical array of elements arranged in rows and columns. The same numbers, arranged in a different order will give a different matrix, and matrices are used to present numerical information in a compact form. In this unit, you will study matrices and determinants, and their properties.

17.2 Objectives

By the end of this unit, you should be able to:

i) define the concept and properties of matrices;
ii) solve problems relating to matrices;
iii) define the concept and properties of determinants
iv) solve problems associated with determinants.

17.3 Matrices

A matrix is a set of mn quantities arranged in a rectangular array of m rows and n columns. It is called m by n (or $m \times n$) matrix and is referred to as having order $m \times n$. In writing down a matrix, it is usual to enclose this array by large brackets and to denote the matrix by a single letter, e.g.

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{pmatrix} \text{ or } \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{bmatrix}$$

The individual quantities are called the **elements** of the matrix. A matrix of m rows and n columns is said to be a square matrix if $m = n$.

A matrix is indicated by writing the array within large square brackets, e.g.

$\begin{bmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{bmatrix}$ is a 2×3 matrix, where 5, 7, 2, 6, 3, 8 are the elements of the matrix.

When describing the matrix, the number of rows is stated before that of the columns,

e.g $\begin{bmatrix} 5 & 6 & 4 \\ 2 & -3 & 2 \\ 7 & 8 & 7 \\ 6 & 7 & 5 \end{bmatrix}$ is a matrix of order 4×3, that is, 4 rows and 3 columns.

Also, the matrix $\begin{bmatrix} 6 & 4 \\ 0 & 2 \\ 2 & 3 \end{bmatrix}$ is of order 3×2 and the matrix

$\begin{bmatrix} 2 & 5 & 3 & 4 \\ 6 & 7 & 9 & 6 \end{bmatrix}$ is of order 2×4.

Before showing the use of matrices, you must first set up an algebra defining the operations of addition, subtraction, multiplication and scalar multiple of matrices.

Equality of matrices

Two matrices are said to be equal if corresponding elements are equal. Therefore, the two matrices must also be of the same order, i.e. $a_{ij} = b_{ij}$ or $A = [a_{ij}]$ and $B = [b_{ij}]$.
For example,

$$\text{if} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 4 & 6 & 5 \\ 2 & 3 & 7 \end{bmatrix}$$

then, $a_{11} = 4$; $a_{12} = 6$; $a_{13} = 5$; $a_{21} = 2$ and so on.

Therefore, if $[a_{ij}] = [b_{ij}]$, then $a_{ij} = b_{ij}$ for all values of i and j.

Addition and subtraction of matrices

Like sets, matrices are not single numbers, so the ordinary rules of arithmetic operations cannot be used. Matrix addition and matrix subtraction are defined in a special way. Two matrices must be of the same order before they are added or subtracted by combining the corresponding elements according to an arithmetic operation.

Example 17.1

Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{bmatrix}$, then

$$C = A + B = \begin{bmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{bmatrix}$$

$$\text{and } D = A - B = \begin{bmatrix} 4-1 & 2-8 & 3-9 \\ 5-3 & 7-5 & 6-4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -6 & -6 \\ 2 & 2 & 2 \end{bmatrix}$$

In other words, addition and subtraction of matrices are both **associative** and **commutative**, that is, $A + B = B + A$ and $(A + B) + C = A + (B + C)$.

Multiplication of matrices

The result of multiplying a matrix A by *a* single number K (a scalar) is the same as multiplying each individual element of the matrix by that factor.

For example, if

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \text{ then } KA = \begin{bmatrix} K & 3K \\ 2K & 4K \end{bmatrix}$$

From this definition, it follows that the **distributive law** of elementary algebra holds also for matrices. This means that $K(A \pm B) = KA \pm KB$.

Furthermore, if you define $KA = AK$, in general, $K[a_{ij}] = [Ka_{ij}]$. It also means that you can take a common factor out of every element (not just one row or one column). For example, if

$$A = \begin{bmatrix} 10 & 25 & 45 \\ 35 & 15 & 50 \end{bmatrix}$$

Then, A can also be written as

$$A = 5 \times \begin{bmatrix} 2 & 5 & 9 \\ 7 & 3 & 10 \end{bmatrix}.$$

Multiplication of two matrices

Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second.

If $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = [b_{ij}] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

then $A \times B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$= \begin{bmatrix} a_{11} \times b_1 + a_{12} \times b_2 + a_{13} \times b_3 \\ a_{21} \times b_1 + a_{22} \times b_2 + a_{23} \times b_3 \end{bmatrix}$$

Each element in the top row of A is multiplied by the corresponding elements in the first column of B and then the products are added. Similarly, the second row of the product is found by multiplying each element in the second row of A by the corresponding elements in the first column of B.

To illustrate this procedure, study Example 17.2.

Example 17.2

$A = \begin{bmatrix} 4 & 7 & 6 \\ 3 & 3 & 1 \end{bmatrix}$ and $B \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}$

Then, $\begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \times 8 + 7 \times 5 + 6 \times 9 \\ 2 \times 8 + 3 \times 5 + 1 \times 9 \end{bmatrix}$

$$= \begin{bmatrix} 32 + 35 + 54 \\ 16 + 15 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 121 \\ 40 \end{bmatrix}$$

Remember that in this example, matrix B has only one column. The general rule for multiplication is: **the product of an ($\ell \times m$) matrix and an ($m \times n$) matrix has order ($\ell \times n$).**

If $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{bmatrix}$

then, $A \times B = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{bmatrix} \times \begin{bmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 14 - 8 + 24 & 2 + 36 + 18 \\ 21 - 18 + 20 & 3 + 81 + 15 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 56 \\ 23 & 99 \end{bmatrix}$$

Remember that multiplication of matrices is defined only when the number of rows in the first matrix is equal to the number of columns in the second matrix. It means that a product such as

$$\begin{bmatrix} 1 & 5 & 6 \\ 4 & 9 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 5 \\ 8 & 7 & 1 \end{bmatrix} \text{ has no meaning.}$$

But if A is an $(m \times n)$ matrix and B is $(n \times m)$, then products of $A \times B$ and $B \times A$ are possible.

Example 17.3

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$$

$$\text{Then, } A \times B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 7 + 16 + 27 & 10 + 22 + 36 \\ 28 + 40 + 54 & 40 + 55 + 72 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 68 \\ 122 & 167 \end{bmatrix}$$

$$\text{and } B \times A = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7+40 & 14+50 & 21+60 \\ 8+44 & 16+55 & 24+66 \\ 9+48 & 18+60 & 27+72 \end{bmatrix}$$

$$= \begin{bmatrix} 47 & 64 & 81 \\ 52 & 71 & 90 \\ 57 & 78 & 99 \end{bmatrix}$$

You may have observed from this example that in matrix multiplication, $AB \neq BA$. Therefore, multiplication is not commutative. Apart from this, matrix multiplication satisfies the associative and distributive laws, i.e. $(AB)C = A(BC)$ and $(A + B)C = AC + BC$. However, these can only be true if the products are defined.

Exercise 17.1

1 If A is a $3 \times m$ matrix and B is a 2×5 matrix. For what value of m do the matrices:
   a) AB
   b) A^2
   c) BA exist?

2 $A = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $D = [-2 \quad 13]$

   If the matrices are compatible, find the products:
   a) AC b) AB c) BC d) CD e) DC

3 If $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 5 & 7 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 & 0 \\ 6 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

   Find a) $A + B$ b) $A - B$ c) $B - A$ d) AB (e) BA

Zero matrix

A zero matrix has all its elements as zero, and the addition of a zero matrix to any matrix of the same size leaves the matrix unchanged. In other words, zero matrix is the identity matrix in the addition of matrices of the same size.

For 2×3 matrix, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Square matrix

A square matrix has the same number of rows and columns, e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{bmatrix}$$

is a square matrix of order $(n \times n)$. The diagonal containing the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is called the **leading diagonal** and the sum of these elements is called the **trace** (or **spur**) of the matrix.

Unit matrix

The unit matrix is a square matrix having the elements on the leading diagonal as 1 and all other elements as 0. The unit matrix of any size is the identity matrix for multiplication of matrices of that size, and it is usually given the symbol I_n (or In × n), where n is the matrix dimension.

For example, the (2 × 3) unit matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, if A is a square matrix of order $(m \times m)$ and I is the unit matrix of the same vector, then, $IA = AI = A$.

Likewise,

$I = I^2 = I^3 = \ldots\ldots = I^k \ldots\ldots$ where k is any positive integer.

Inverse matrix

The inverse of a square matrix A is denoted by A^{-1}, and the product of a square matrix and its inverse is the corresponding unit matrix. Thus, for 3-dimensional square matrix A,

$$AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

17.4 Determinants

The **determinant** of a matrix is an essential property that is associated with square matrices. It is defined, using a 2 × 2 matrix as follows:
Let A be defined by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then, the quantity defined by $a_{11}a_{22} - a_{12}a_{21}$ is called the **determinant** of matrix A. It is denoted by

$$det\ A = |A| = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

As you can observe, while large square brackets are used to enclose the matrix elements, two vertical lines are used to represent a determinant.

Determinants arise naturally in the solution of a set of linear equations. For example, if you have the equations

$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

where x and y are to be found, and a_1, b_1, c_1, a_2, b_2 and c_2 are given constants. Solving for x and y, using the **Cramer's rule** have the solutions as:

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and $y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

provided $a_1b_2 - a_2b_1 \neq o$. If you now define the quantity:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_1 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$$

then those solutions may be obtained from the following:

$$\frac{x}{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}} \quad \frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} \quad \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Now, study Example 17.4.

Example 17.4

Use Cramer's method to solve the simultaneous linear equations

$3x + 3y = 1$

$5x + y \ = -6$

Solution

The system becomes $Ax = B$ where $A = \begin{vmatrix} 3 & 3 \\ 5 & 1 \end{vmatrix}$, $B = \begin{vmatrix} 1 \\ -6 \end{vmatrix}$

$|A| = 3(1) - (3)(5) = 3 - 15 = -12$

Let $|A|_x = \begin{vmatrix} 1 & 3 \\ -6 & 1 \end{vmatrix} = 1 - (3)(-6) = 1 + 18 = 19$

$|A|_y = \begin{vmatrix} 3 & 1 \\ 5 & -6 \end{vmatrix} = (3)(-6) - (1)(5) = -18 - 5 = -23$

The solutions : $x = \frac{|A|x}{|A|} = \frac{19}{21} = 1\frac{7}{12}$

$$y = \frac{|A|y}{|A|} = \frac{-23}{-12} = 1\frac{11}{12}$$

Hence, the solutions are $x = 1\frac{7}{12}$, $y = 1\frac{11}{12}$

So far, you have studied examples of determinants for 2×2 matrices. The definition of the determinant of a 3×3 matrix can be obtained as follows:

Let A be defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \ldots\ldots \text{(i)}$$

Then, the determinant of A is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{12}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \ldots\ldots \text{(ii)}$$

As you can see from this expression, the terms are obtained by selecting (in this example, first row elements and then multiplying each by the determinant of a matrix formed from crossing out the row and column elements containing each selected row element. The sign of the second term is negative.

Two general definitions related to the above expression are essential at this point: the **minor** and **cofactor**. The minor, M_{ij} of any element a_{ij} in the matrix A is the determinant formed of the elements remaining after deleting the row i and the column j in which the element a_{ij} occurs.The cofactor of an element a_{ij} is equal to $(-1)^{i+j} M_{ij}$.

The determinant of any matrix may be expressed in terms of the elements of any row or column, and their respective cofactors in accordance with the rule: **Each element in the selected row (or column) is multiplied by its corresponding cofactor. Then, the sum of these products is the value of the determinant.**

You can observe that equation (ii) obeys this rule.

Example 17.5

Let a 3 × 3 matrix be given by

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

Then,

$$|B| = 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(4-2) - 2(20-0) + 3(5-0)$$

$$= 2 - 40 + 15 = -23$$

Exercise 17.2

1 Solve these pairs of simultaneous linear equations:

a) $x + y = 12$
$x - y = 6$

b) $5x + y = -1$
$2x + y = -1$

2 Find the determinants of the following matrices:

a) $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 0 \\ 9 & 2 & 1 \end{bmatrix}$

b) $B = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 6 & 10 \end{bmatrix}$

17.5 Conclusion

In this unit, you have studied concepts of matrices and determinants. You have learnt that a matrix is an array of quantities in which the elements are arranged in rows and columns. You have also learnt that the determinant of a square matrix is a single number associated with matrix, and for a 2-square matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is the numerical value of $ad - cb = |A|$ or *det A*.

You have also gone through addition, subtraction and multiplication of matrices. Now, you can define zero matrix, square matrix and unit matrix. The unit has also shown you the general rule of using cofactors to obtain the determinant of a 3 × 3 matrix. Finally, you have also studied determinants and how they are used in solving simultaneous linear equations when using Cramer's rule.

17.6 Summary

In this unit, you have learnt that:

i) a matrix is a set of mn quantities arranged in a rectangular array of m rows and n columns and denoted by a single letter, say A, such that

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots\cdots & a_{mn} \end{pmatrix}$$

ii) the determinant is defined for a 2×2 matrix by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

which is the difference of the products of the numbers in the two diagonals;

iii) the minors and cofactors of elements are needed to obtain the value of the determinant.

17.7 Tutor-marked assignment

1 Find a matrix X such that $AX = B$

when $A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 \\ 1 & 10 \end{pmatrix}$

2 Find the inverse of the matrix $\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$ and hence, the solution of the simultaneous equations

$2x - 3y = 9$

$4 - y \quad = 11$

3 Obtain the determinant of the 3×3 matrix

$A = \begin{bmatrix} 9 & 2 & 6 \\ 15 & 0 & 2 \\ 1 & 3 & 4 \end{bmatrix}$, using the first row and column.

17.8 References

1 Backhouse, J. K. and S.P.T. Houldsworth, (1999) *Pure Mathematics Book 1, 4th Edition,* Longman Addison Wesley Limited, p. 212 – 235.

2 Stephenson, G., (1981), *Mathematical Methods for Science Students,* Longman Group Limited, London, p. 276 – 323.

Unit 18 Plane geometry: polygons and angles

Contents

18.1 Introduction

In your secondary school mathematics you were expected to have studied the geometric properties of lines, angles, plane objects and solids.

In this unit, you will study proofs of some geometrical properties of plane shapes. Those properties will be shown to be true in all cases. Your study objectives for the unit are as follows:

18.2 Objectives

By the end of this unit, you should be able to:
i) state the different steps involved when giving the proof of a theorem;
ii) solve problems relating to triangles;
iii) prove theorems relating to angles;
iv) recognise the basic properties of parallelograms and polygons;
v) recognise the properties of special quadrilaterals.

18.3 Properties of triangles

In geometry, there are some basic facts which are often taken for granted and which require no proof. However, there are other more important facts, which require proof. These are called **theorems**.

A theorem is a statement which contains a given statement or hypothesis and what is to be proved. Usually in geometry, a diagram is drawn to illustrate what is given. Sometimes, a construction may be needed in giving a proof. The proof of a theorem, therefore, usually consists of the following four sections:

i) Given statement,
ii) Statement to be proved,
iii) Construction (where necessary), and
iv) Proof.

The **given statement** sets out the information contained in the theorem. It is appropriate to use symbols in stating the given facts.

What is **required to be proved** should also be stated symbolically.

The section on **construction** is a statement of the construction to be carried out in order to aid the proof of the theorem.

The **proof** is a reasoned argument in which statements are deduced from the facts given, in order to obtain what is required to be proved. The proofs of some theorems will be given in this sections.

Triangles

Theorem 1: **The sum of the angles of a triangle is 180°**

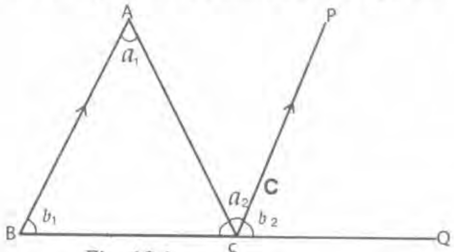


Fig. 18.1

Given Δ ABC (i.e. triangle ABC)

You are to prove that: $\hat{A} + \hat{B} + \hat{C} = 180°$

($\hat{A}$ means angle A)

Construction: Extend line $\overline{BC}$ to Q and draw $\overline{CP}$ parallel to $\overline{BA}$

Proof: In Fig. 18.1, you can see that the following angles are equal:

$a_1 = a_2$ (alternate angles BA// CP),

where BA//CP means BA is parallel to CP.

$b_1 = b_2$ (corresponding angles, BA//CP)

But $a_2 + b_2 + c = 180°$ (angles on a straight line).

$\therefore a_1 + b_1 + c = a_2 + b_2 + c = 180°$

i.e. $\hat{A} + \hat{B} + \hat{C} = 180°$.

You must have noticed that reasons for expressions are given in the brackets beside them.

Theorem 2: **The exterior angle of a triangle is equal to the sum of the opposite interior angles.**

Given: Δ PQR, QR is produced to x. PRX is an exterior angle.

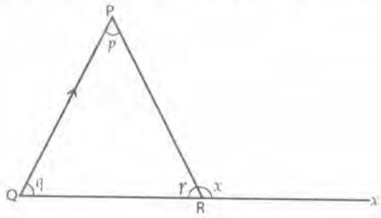


Fig. 18.2

To prove that: $P\hat{R}X = \hat{P} + \hat{Q}$.

Proof: using the letters in the diagram above,

$x + r = 180°$ (angles on a straight line)

$p + q + r = 180°$ (sum of angles in a triangle, theorem 1)

$\therefore r + x = p + q + r$

$x = p + q$

Thus $P\hat{R}X = \hat{P} + \hat{Q}$.

Angles and triangles

A triangle is a plane figure bounded by three straight lines. Triangles are classified according to their sides or according to their angles. Classification by sides is called **lateral classification**, while classification of triangles by their angles is called **angular classification.**

Lateral classification

Triangles in this class fall into the following categories based on their sides.

i) *Scalene triangle:* no two sides are equal.
ii) *Isosceles triangle:* two sides are equal.
iii) *Equilateral triangle:* all the three sides are equal.

Angular classification

Triangles in this class fall into the following categories based on their angles.
i) In an *acute-angled triangle*, all the angles are acute.
ii) In a *right-angled triangle*, one angle is a right angle.
iii) In an *obtuse-angled triangle*, one angle is obtuse.

Below are worked examples to guide you to properly understand some of the theorems as well as the proofs.

Example 18.1

a) The angles of a triangle are $a°$, $2a°$ and $3a°$. Find the value of a in degrees.

b) In triangle PQR shown in Fig. 18.3, let X be a point on $\overline{QR}$ such that $R\hat{P}X = \hat{Q}$. Prove that $P\hat{X}R = Q\hat{P}R$.

Solution

a) The sum of the angles of a triangle = 180°

$\therefore a + 2a + 3a = 180°$

$6a = 180°$

$a = \frac{180}{6}$

$\therefore \quad a = 30°$

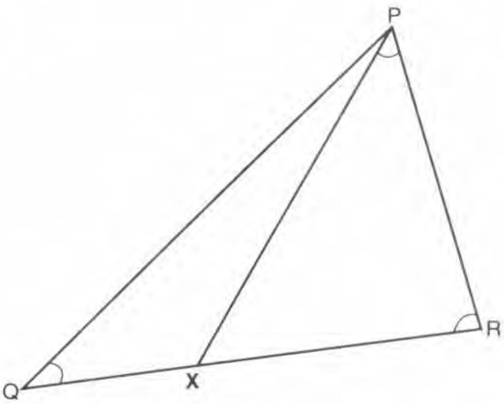


Fig. 18.3

b) $\hat{Q} + Q\hat{P}R + P\hat{R}Q = 180°$ (angle sum of aΔ)

$P\hat{X}P + P\hat{R}X + X\hat{P}R = 180°$ (angle sum of aΔ)

$\therefore P\hat{X}R + P\hat{R}X + X\hat{P}R = \hat{Q} + Q\hat{P}R + P\hat{R}Q$

but $R\hat{P}X = \hat{Q}$

Also, $P\hat{R}X = P\hat{R}Q$ (from the triangle)

$\therefore P\hat{X}R = Q\hat{P}R$

Now, you should be able to do the following exercise based on the worked examples.

Exercise 18.1

1 The angles of a triangle are in the ratio 2:3:4. What is the size of each angle?

2 Find the angles of a triangle if the first angle is thrice the second, and the third is 28° less than thrice the first.

3 In a triangle ABC, the angle at A is bisected and the bisector meets $\overline{BC}$ at D. If $B\hat{A}C = 2x$ and $A\hat{D}C = 45 + x$. Find, in term of x the angles at B and C.

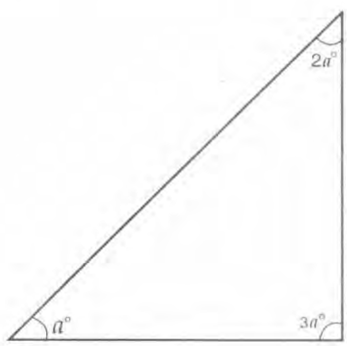


Fig.18.4

Angles and polygons

An n-sided polygon is a plane figure, bounded by n-straight lines. When the sides are equal in length, the polygon is said to be **regular**. When all the diagonals of a polygon lie inside the polygon, the polygon is called a **convex polygon**. If at least one diagonal of a polygon lies outside the polygon, then the polygon is called a **concave polygon**.

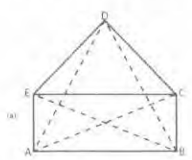
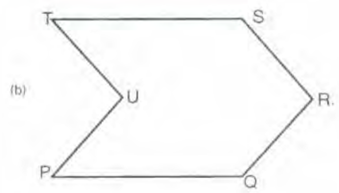


Fig. 18.5



In Fig. 18.5 (a), all the diagonals of the pentagon lie inside the polygon. Hence, ABCDE is a convex polygon. While in Fig. 18.5 (b), the diagonal PT of the hexagon lies outside the polygon. Hence, PQRSTU is a concave polygon.

Theorem 3: **The sum of the interior angles of any n-sided convex polygon is (n–2) × 180° or (2n–4) right angles.**

Given: Any convex polygon ABCDE with n-sides.

Prove that: $\hat{A} + \hat{B} + \hat{C} + \ldots = (n-2) \times 180°$

$= (2n - 4)$ right angle

Construction: Join all vertices, except two, to a common vertex, A.

Proof: By construction, there are $(n - 2)$ triangles (a polygon with n vertices).

Sum of angles in $n - 2$ triangles $= (n - 2) \times 180°$ (angle sum of a Δ).

The angles of the triangles are the same as the interior angles of the polygon of n–sides

$\therefore \hat{A} + \hat{B} + \hat{C} + \ldots = (n-2) \times 180°$

$= 2(n-2) \times 90°$

$= (2n-4)$ right angles

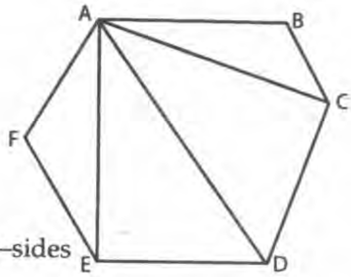


Fig. 18.6

You noticed that in a convex polygon any two of its interior points are joined by a line segment which lies in its interior. Besides, all its interior angles are less than 180°.

Theorem 4: **The sum of the exterior angles of any convex polygon is 4 right angles.**

Given: Convex polygon of n sides

To prove that: Sum of exterior angles of a polygon of n sides = 4 right angles.

Construction: Produce AB, BC, CD, ... as in Fig. 18.6.

Proof: The sum of interior and exterior angles at each vertex = 180° (angles on a straight line).

The sum of interior and exterior angles at n vertices is $n \times 180°$

The sum of interior angles of the polygon is $(n - 2) \times 180°$

$\therefore$ Sum of exterior angles

$= n \times 180° - (n-2) \times 180°$

$= 2 \times 180°$

$= 360°$

$= 4 \times 90° = 4$ right angles

The sum of the exterior angles of a polygon of n sides is 4 right angles.

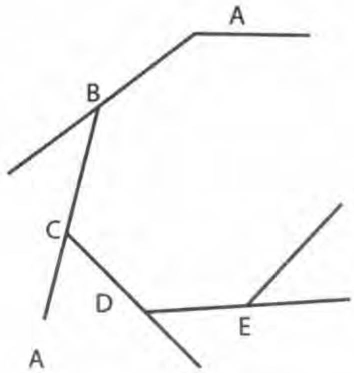


Fig. 18.7

Work through the following examples for your practice.

Example 18.2

In Fig. 18.8 ABCDEF is a regular hexagon. $\overline{AF}$ and $\overline{DE}$ produced meet at X. Prove that Δ FEX is an equilateral triangle.

Solution

Given: ABCDEF is a regular hexagon.

Prove that: Δ FEX is an equilateral triangle.

Construction: Produce AF and DE to meet at X as shown in Fig. 18.8.

Proof: Sum of angles of a hexagon $= (6 - 2) \times 180°$

$= 720°$

Each interior angle of a regular hexagon = $\frac{720°}{6}$ = 120°

$\hat{AFE} = \hat{DEF} = 120°$

$\hat{EFX} = 180° - 120°$ (angles on a straight line)

$= 60°$

Similarly, $\hat{FEX} = 60°$

$\therefore \hat{FXE} = 60°$ (angle sum of aΔ)

Δ FEX is an equilateral triangle.

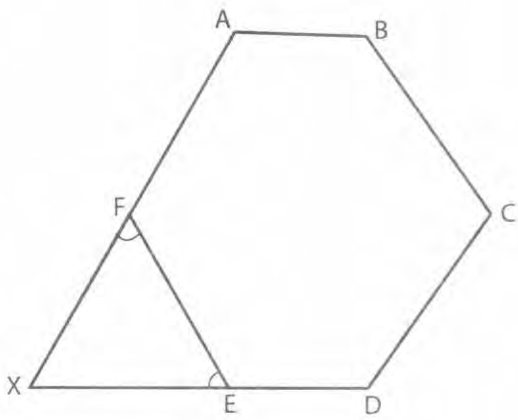


Fig. 18.8

Example 18.3

Calculate $\hat{ADB}$ in a regular pentagon ABCDE

Solution

Each interior angle is $\frac{(5-2) \times 180°}{5} = 108°$.

In the isosceles triangle AED; $\hat{EAD} = \hat{EDA} = 36°$ (angle sum of a triangle is 180°)

Similarly, in triangle BCD, $\hat{CDB} = 36°$

$\therefore \hat{ADB} = 108° - (36° + 36°)$

$= 36°$.

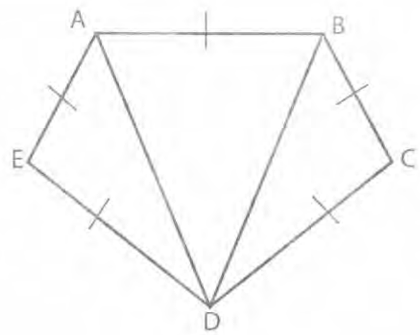


Fig. 18.9

Some special polygons

1. *Triangles:* 3 sides,
   angle sum = (3 – 2) straight angle = 180°
2. *Quadrilaterals:* 4 sides,
   angle sum = (4 – 2) straight angles = 360°
3. *Pentagon:* 5 sides,
   angle sum = (5 – 2) straight angles = 540°
4. *Hexagons:* 6 sides,
   angle sum = (6 – 2) straight angles = 720°
5. *Octagons:* 8 sides,
   angle sum = (8 – 2) straight angles = 1 080°
6. *Decagons:* 10 sides,
   angle sum = (10 – 2) straight angles = 1 440°

Now, practice by providing solutions to the following exercises.

Exercise 18.2

1 Find the interior angle of a regular
  a) octagon b) 12-sided figure c) decagon.

2 Find the number of sides of a regular polygon whose exterior angle is 40°.

3 One interior angle of a polygon is 240°, other interior angles are all equal to 120°. How many sides has the polygon?

4 The angles of a pentagon are $3x°$, $4x°$, $5x°$, $6x°$ and $7x°$. Find the value of x.

Theorems on parallelograms

In the previous sections, you have been introduced to formal proofs of theorems related to polygon and angles. In this section, you will be taken through few theorems on parallelograms which are special types of quadrilaterals.

Now, study the following definitions:

A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A **rhombus** is a parallelogram in which all the sides are equal.
A **rectangle** is a parallelogram which has all its angles as right angles.
A **square** is a rectangle in which all the sides are equal. Therefore, a square is also a rhombus.

Theorem 5: **In a parallelogram the opposite sides are equal and the opposite angles are equal**

Given: a parallelogram, ABCD, in which AB// DC and AD// BC.
Prove that: (i) $|AB| = |DC|$, $|DA| = |CB|$
(ii) $\hat{A} = \hat{C}$, $\hat{B} = \hat{D}$

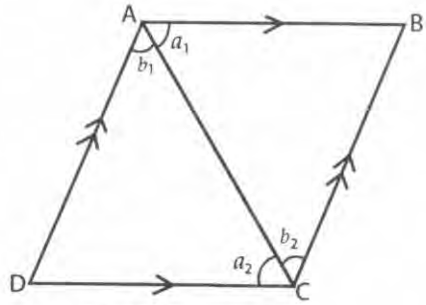


Fig. 18.10

Proof: In triangles ABC and ADC
$|AC| = |DC|$ (common sides)
$A\hat{C}B = C\hat{A}D$ (alternate angles AD //BC)
$\Delta ABC = \Delta CDA$ (ASA, i.e. 1 side, 2 angles equal)
Hence (i) $|AB| = |DC|$, $|DA| = |CB|$
(ii) $\hat{B} = \hat{D}$

Also $\hat{A} = C$ $(b_1 + a_2 = a_1 + b_2)$ from Fig. 18.10.
The fact that triangles ABC and ADC are congruent also leads to the fact that the diagonal of a parallelogram bisects the area of the parallelogram.

Theorem 6: **The diagonals of a parallelogram bisect each other**

Given: a parallelogram, ABCD, in which the diagonals, AC and BD, intersect at O.

Prove that: $|OA| = |OC|$, $|OB| = |OD|$
Proof: In triangles OAB and OCD
$|AB| = |DC|$ (opposite sides of a parallelogram)

Use letters to represent the angles in Fig. 18.11.
$a_1 = a_2$ (Alternate angles, AB//DC)
$b_1 = b_2$ (Alternate angles, AB//DC)
Therefore, $\Delta OAB \equiv \Delta OCD$ (ASA)
$\therefore$ $|OA| = |OC|$ and $|OB| = |OD|$

Some vital definitions
A line that cuts two or more straight lines is called a **transversal** of the lines. The line segment between the points where two lines meet a transversal is called **intercept** made by the lines or the transversal. In Fig. 18.12, LM is a transversal of PQ and RS, and AB is an intercept.

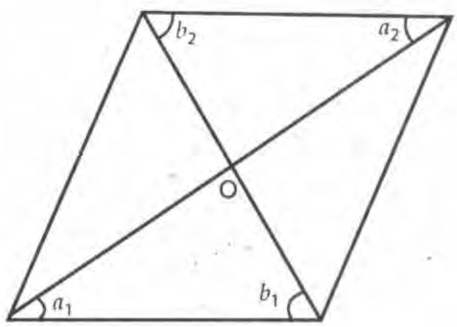


Fig. 18.11

Example 18.4
In the quadrilateral ABCD shown in Fig. 18.13, BC and EF are both parallel to AD. E is the mid point of AB and the diagonal AC meets EF at X. If $|XC| = 10$cm and $|CF| = 9$cm.
Find i) $|AX|$
ii) $|CD|$.

Solution
Given that BC, EF and AD are parallel to each other and AC and AB transversal.

If BE = EA, then, AX = XC

but XC = 10 cm

$\therefore$ AX = 10cm

Similarly, CD is a transversal of the parallel lines BC, EF and AD

$\therefore$ CF = FD

CD = 2CF

$= 2 \times 9$

= 18 cm

You should now practice with the following exercise.

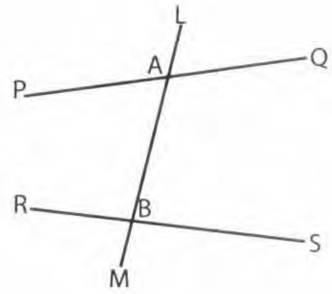


Fig. 18.12

Exercise 18.3

1 ABCDE is a pentagon with |AB| = |AE|, |BC| = |ED| and $A\hat{B}C = A\hat{E}D$. Prove that

(a) |AC| = |AD| and (b) $B\hat{C}D = E\hat{D}C$.

2 The diagonals of a parallelogram ABCD intersect at O, and P is the midpoint of $\overline{BC}$. Prove that:

i) $\overline{OP}$ is parallel to $\overline{DC}$

ii) $|OP| = \frac{1}{2}|DC|$

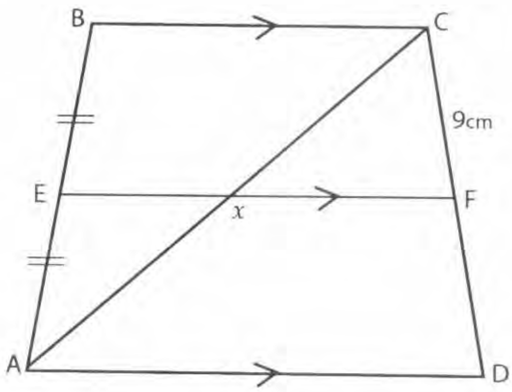


Fig. 18.13

18.4 Conclusion

In this unit, you have studied about the proofs of some geometrical properties of plane shapes. The proof of any theorem consists of the following:

i) given statement;
ii) statement to be proved;
iii) construction on (where necessary); and
iv) the main proof.

In brief, in this unit, you have learnt some of the basic theorems involving angles and sides of polygon and how to prove them.

18.5 Summary

In this unit, you have learnt a number of theorems in geometry, involving basically some geometrical shapes such as triangles, parallelograms and other polygons. For example you saw that a parallelogram is a plane figure with four straight sides and in which the opposite sides are equal and parallel. Similarly, you were taken through the proofs of some theorems or angles. You will definitely come across more theorems in your subsequent studies in higher mathematics.

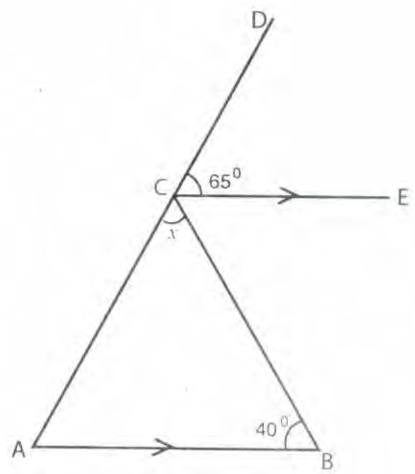


Fig. 18.14

18.6 Tutor-marked assignment

1. Find x in Fig. 18.14.
2. In the diagram shown in Fig. 18.15, ABCD is any quadrilateral and E is a point on CD such that AE//

BC. Prove that quadrilateral ABED and triangle ACD have equal areas.

18.7 References

Lassa, P. N. and S. A. Ilori (1991), *Further Mathematics for Senior Secondary Schools, (Books 1–3)*, University Press Plc. Ibadan, pp. 82–108.

MAN (1991), *Senior Secondary Mathematics*. University Press Plc., Ibadan, pp. 23–53.

Unit 19 Circles and their properties

Contents

19.1 Introduction

A **locus** in plane geometry is a line or curve joining all points in the plane which satisfy some **given conditions**. For instance, a circle is the locus of all the points in a plane which are equidistant **from a fixed point** in the plane, where the fixed point is the centre and the fixed distance is the radius. The locus also describes the circumference of the circle. An **arc,** for example, is a part of the circumference. In this unit, you will be taken through a number of the important theorems related to circles with some essential examples, to show how they are applied to problems in geometry.

19.2 Objectives

By the end of this unit, you should be able to:

i) recognise and define an arc, a sector or a segment of a circle;
ii) define angular properties of a circle;
iii) determine cyclic quadrilaterals;
iv) solve problems involving tangents to a circle.

19.3 Meanings of arc, sector and segment of a circle

A circle is the path traced out by a point that moves in such a way that it is always at a constant distance r from a fixed point O in a given plane. The fixed point O is the centre of the circle and the constant distance r is the **radius**, as shown in Fig. 19.1.

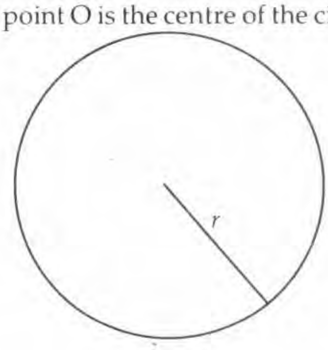


Fig. 19.1.

The **circumference** of a circle is the distance round the circle. To calculate the length of the circumference of a circle, you use the formula $2\pi r$, where r is the radius, note that $\pi = \frac{22}{7} = 3.142$, correct to 3 decimal places. **A chord** of a circle is a line segment joining any two points on a circumference. A **diameter** is a chord which passes through the centre of the circle. A chord which is not a diameter divides the circumference into two arcs of different lengths: a **major arc** and a **minor arc**, as shown Fig. 19.2. The chord also divides the circle into two segments of different sizes: a **major segment** and a **minor segment**.

From Fig. 19.3, P, Q and R are points on the circumference of a circle. $A\hat{P}B$, $A\hat{Q}B$ and $A\hat{R}B$ are angles **subtended**, at the circumference by the chord AB or by the minor arc AB, $A\hat{P}B$, $A\hat{Q}B$ and $A\hat{R}B$ are all **angles in the same major segment** APQRB.

Similarly, in Fig. 19.4, $A\hat{X}B$ and $A\hat{Y}B$ are angles subtended by the chord AB, or by the major arc AB in the minor segment AXYB.

Theorem 1

A straight line drawn from the centre of a circle to bisect a chord, which is not a diameter, is at right angles to the chord.

Given: A circle with centre O and chord AB with OM such that $|AM| = |MB|$

To prove: $A\hat{M}O = B\hat{M}O = 90°$

Construction: Join $\overline{OA}$ and $\overline{OB}$

Proof: Using the labels on Fig. 19.5, you have:

$|OA| = |OB|$ (radii)
$|AM| = |MB|$ (given)
$|OM| = |OM|$

$\therefore \Delta A\hat{M}O \equiv \Delta B\hat{M}O$ (sss)

$\therefore A\hat{M}O = B\hat{M}O$

But $A\hat{M}O + B\hat{M}O = 180°$ (angles on straight line)

$\therefore A\hat{M}O = B\hat{M}O = \frac{180°}{2} = 90°$

The following are also true:

1 A straight line drawn from the centre of the circle perpendicular to a chord bisects the chord.
2 The perpendicular bisector of a chord of a circle passes through the centre of the circle.

Now, study the worked Example 19.1.

Example 19.1

a) Find the length of a chord of the circle of radius 5 cm and whose distance from the centre is 3 cm.

Solution

Using Fig. 19.6,

AO = 5 cm
OD = 3 cm

Now, to find AB, observe that ΔAOD is a right-angled triangle.

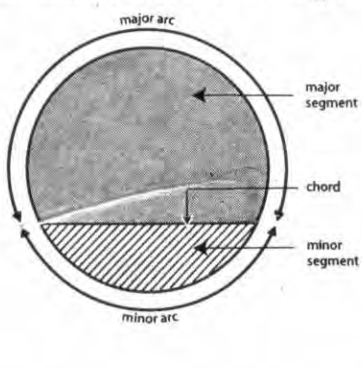


Fig. 19.2.

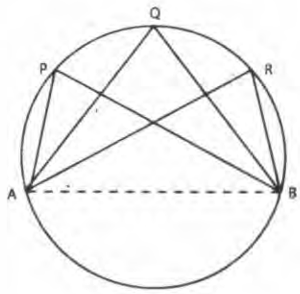


Fig. 19.3

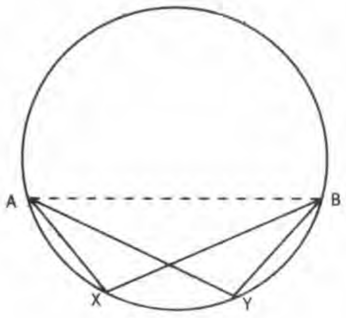


Fig. 19.4

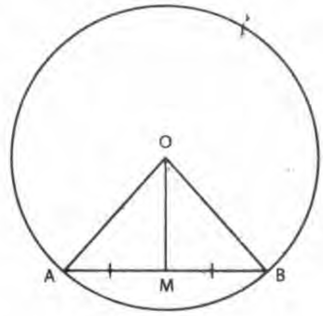


Fig. 19.5

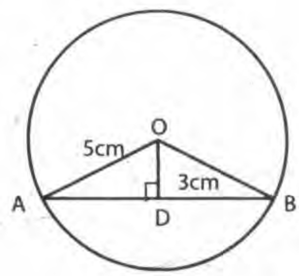


Fig. 19.6

By Pythagoras' theorem, you have

$AD^2 = AO^2 - OD^2$

$= 5^2 - 3^2 = 16$

$\therefore AD = 4$ cm

Hence $AB = 2AD = 2 \times 4$ cm $= 8$ cm

b) A chord 16 cm, is drawn in a circle of radius 10cm. Calculate the distance of the chord from the centre of such a circle.

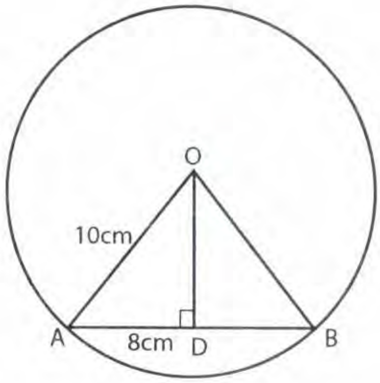


Fig. 19.7

Solutions

Given AO = 10 cm, AD = $\frac{1}{2}$AB = 8 cm

See Fig. 19.7.

You want to find OD.

$OD^2 = AO^2 - AD^2$ (by Pythagoras)

$= 10^2 - 8^2 = 36$

$OD = 6$ cm

One important fact or theorem about a chord is that the perpendicular drawn from the centre of a circle to a chord bisects that chord. This fact is used to solve these problems on the length of a chord.

A chord which passed through the centre of a circle is called the diameter while **concentric circles** are circles having the same centre but different radii. You will recall that every chord divides a circle into two segments and that the smaller segment is called the minor segment while the larger one is called the major segment.

Exercise 19.1

1 The radius of a circle is 10 cm and the length of a chord of the circle is 16 cm. Calculate the distance of the chord from the centre of the circle.

2 The distance of a chord of a circle of radius 5 cm from the centre of the circle if 4 cm. Calculate the length of the chord.

3 A chord of length 24 cm is 13 cm from the centre of the circle. Calculate the radius of the circle.

19.4 Angular properties of a circle

The minor arc AB has y as the angle subtended at the centre, and x as the angle at the circumference, as shown in Fig. 19.8.

The following properties of angles of a circle are useful in solving problems.

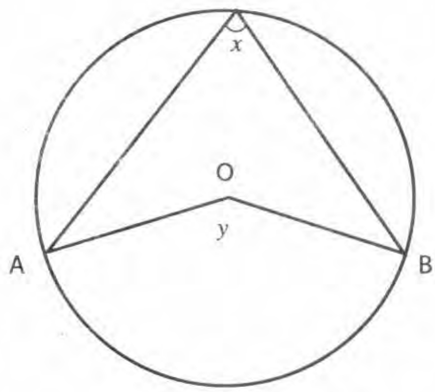


Fig. 19.8

Theorem 2

The angle that an arc or a chord of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.

Given: A circle APB with centre O

To prove: $A\hat{O}B = 2 \times A\hat{P}B$

Construction: Join $\overline{PO}$ and produce it to any point Q.

Proof: Note that there are three possible diagrams as

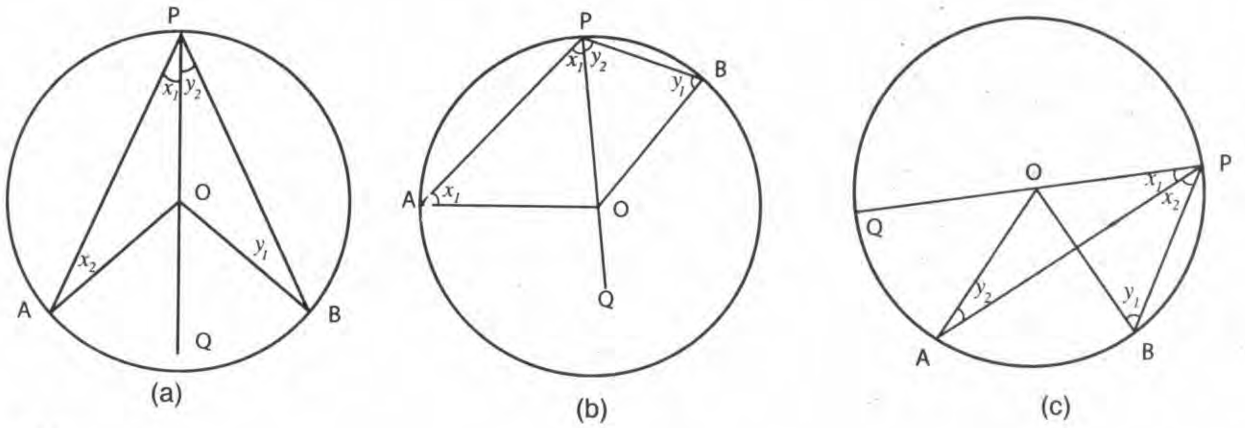


Fig. 19.9

shown in Fig. 19.9.

Using the labels you have in the diagrams (Fig. 19.9)

$|OA| = |OP|$ (radii)

$\therefore x_1 = x_2$ (base angles of Isos. Δ)

$\therefore A\hat{O}Q = x_1 + x_2$ (ext. angle of Δ AOP)

$\therefore A\hat{O}Q = 2x_2$ $(x_1 = x_2)$

Similarly, $B\hat{O}Q = 2y_2$

In Fig. 19.9 (a), $A\hat{O}B = A\hat{O}Q + B\hat{O}Q$

Also, in Fig. 19.9 (b), reflex $A\hat{O}B = A\hat{O}Q + B\hat{O}Q$

$= 2x_2 + 2y_2$

$= 2(x_2 + y_2)$

$= 2 \times (A\hat{P}B)$

In every case, $A\hat{O}B = 2 \times A\hat{P}B$

Below, is another interesting property of circles.

Theorem 3

Angles in the same segment of a circle are equal.

Given: That P and Q are any points on the major arc of circle APQB.

To prove: $A\hat{P}B = A\hat{O}B$. (See Fig. 19.10.)

Construction: Join A and B to O, the centre of the circle.

Proof: Using the letters on Fig. 19.10.

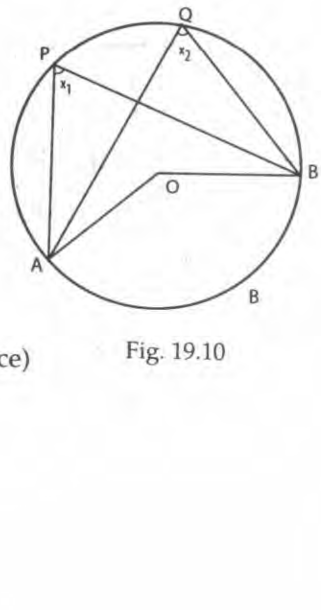


Fig. 19.10

$A\hat{O}B = 2x_1$ (angle at the centre = $2 \times$ angle on the circumference)

$A\hat{O}B = 2x_2$ (same reason as above)

$\therefore x_1 = x_2$ $(= \frac{1}{2} A\hat{O}B)$

$\therefore \hat{APB} = \hat{AQB}$

Since P and Q arc any points on the major arc, all angles in the major segment are equal to each other. The theorem is also true for angles in the minor segment.

Theorem 4

The angle in a semicircle is a right angle.

Given: That AB is a diameter of a circle, centre O and x is any point or the circumference of the circle.

To prove: $\hat{AXB} = 90°$. This is illustrated in Fig. 19.11.

Proof: From, Fig. 19.11, you have

$\hat{AOB} = 2\hat{AXB}$ (angle at centre = 2 × angle on the circumference)

But $\hat{AOB} = 180°$ (straight angle)

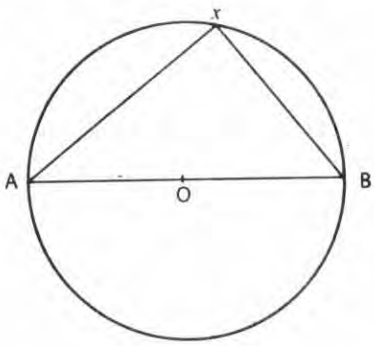


Fig. 19.11

$\therefore 2\hat{AXB} = 180°$

$\therefore \hat{AXB} = 90°.$

Now, consider the worked Example 19.2.

Example 19.2

If in Fig. 19.12 below, PQ is a diameter of circle PMQN, centre O and $\hat{PQM} = 63°$, find $\hat{QNM}$.

In Δ QPM above,

$\hat{PMQ} = 90°$ (angle in a semicircle)

$\therefore \hat{PMQ} = 180° - 90° - 63°$ (angle sum of a triangle)
$= 27°$

$\therefore \hat{QNM} = 27°$ (angle in the same segment as $\hat{QPM}$)

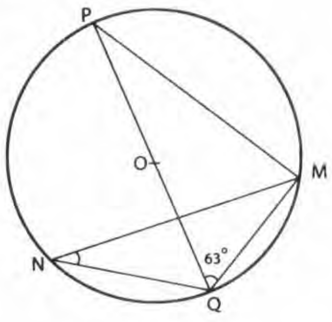
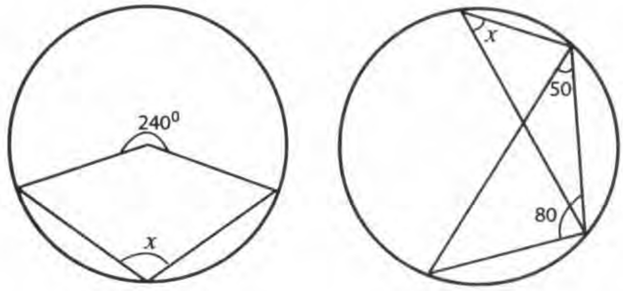


Fig. 19.12

Exercise 19.2

1 Calculate the angle marked x in the following figures.



(a) (b)

Fig. 19.13

2 ABCD is a quadrilateral circumscribed in a circle. AE = ED. AB is parallel to DC and $C\hat{B}D$ = 40°, find $B\hat{D}A$.

19.5 Tangents and circles

A straight line which touches the circumference of a circle at one point and is at right angles to the radius at this point of contact is called **tangent** to the circle.

Therefore, you should notice from this definition that the tangent at any point of a circle and the radius through the point are perpendicular to each other.

You need always to remember the following:

1 A tangent to a circle is perpendicular to radius at the point of contact.

2 The perpendicular to a tangent at its point of contact passes through the centre of the circle. Observe this graphically as shown in Fig. 19.14. You will now state and prove this next important theorem about tangents and circles.

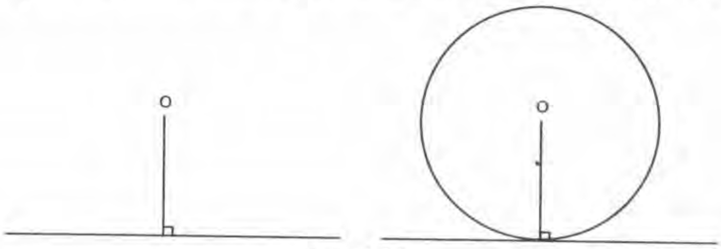


Fig. 19.14

Theorem 5

If two tangents are drawn to a circle from a point outside it, then:

i) the tangents are equal in length.

ii) the angle between the tangents is bisected by the line joining the point of intersection of the tangents to the centre.

iii) the line joining the intersection of the tangents to the centre bisects the angle between the radii drawn to the points of contact. Study the proof below.

Given: A circle with centre O and tangents DB and DA.

Construction: Draw the diagram as in Fig. 19.15.

Required: To prove the following:

i) DB = DA

ii) $\angle BDO = \angle ODA$

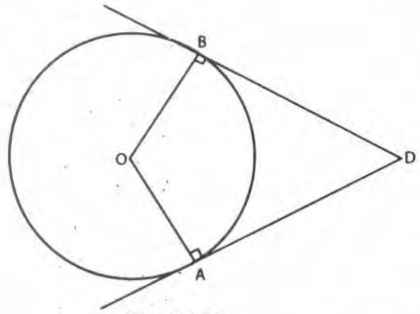


Fig. 19.15

iii) $\angle DOB = \angle DOA$

Proof: In the triangles DOB and DOA,

$OB = OA$ (radii)

$OD = OD$ (common)

$D\hat{B}O = D\hat{A}O$ (right angles)

$\therefore \Delta DOB \equiv \Delta DOA$

Hence $DB = DA$

$B\hat{D}O = O\hat{D}A$

$D\hat{O}B = D\hat{O}A$

The alternate (opposite) segment

In Fig. 19.16, ABD is a tangent to the circle at B. Considering angle DBC, you call segment BEC its **alternate** segment.

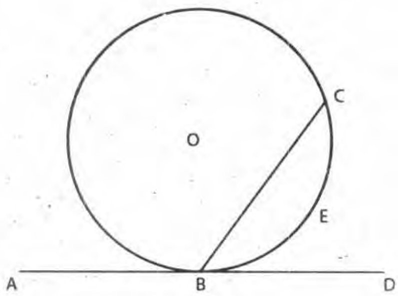


Fig. 19.16

Example 19.4

In Fig. 19.17, PT is a tangent to circle ABCT. $|BA| = |BT|$ and $A\hat{T}P = 82°$, calculate $B\hat{A}T$.

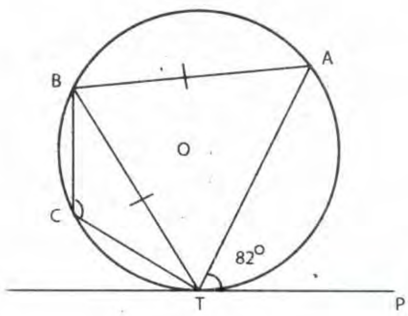


Fig. 19.17

Solution

$A\hat{B}T = 82°$ (alternate segment)

In Δ ABT, $B\hat{A}T = \frac{1}{2}(180° - 82°)$ (sum of angles of I sos D)

$= \frac{1}{2} \times 98°$

$= 49°$

$\therefore B\hat{A}T = 49°$

Exercise 19.2

1 A, B, C are three point on a circle. The tangent at C meets AB produced at T. If $A\hat{C}T = 103°$, $A\hat{T}C = 43°$. Calculate the angles of Δ ABC.

2 AB is a chord of a circle and the tangents at A and B meet at T. C is a point in the minor arc AB. If $A\hat{T}B = 54°$ and $C\hat{B}T = 23°$. Calculate $C\hat{A}T$.

19.6 Conclusion

You have learnt a number of circle theorems involving arc, chords, and about angle properties of the circle. You equally studied tangents to circles. These were discussed with worked examples and proof of each one in the unit. In geometry, there are some basic facts which are often taken for granted and which require no proof. In this unit, you have seen that important facts require proof. Go through the worked examples, which are based on the theorems. The understanding of this unit will help you in the subsequent units.

19.7 Summary

In this unit, you have learnt that:

i) a straight line from the centre of a circle that bisects a chord, is at right angles to the chord;
ii) the angle subtended at the centre of a circle is twice that subtended at the circumference;
iii) angles in the same segment are equal;
iv) the angle in a semicircle is a right angle;
v) a tangent to a circle is a straight line that touches a circle at one point and one point only.

19.8 Tutor-marked assignment

1 The angles of a pentagon are x, $2x$, $2x$, 120° and 170°. Find the value of x.
2 Find the size of one exterior angle of a regular octagon.

References

1 Aderogba, K, Kalejaiye A. O, and Ogum G. E, (1992), *Senior Secondary Mathematics* (*Books 1–3*), Longman Nigeria Plc, p. 121–134.

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3 Macrae, M. F., Kalejaiye A. O., Chima E. I., Garba G. U, and Ademosu, M. O, (2001), *New General Mathematics for Senior Secondary School Books 1–3* (3rd Edition), Longman Nigeria Plc.

Unit 20 Volumes of solids

Contents

20.1 Introduction

Historically we are told that to measure distance and direction precisely for erecting buildings and monuments stimulated the study of geometry in Greece and Egypt in the sixth century B.C. In the third century B.C., Euchid enlarged the knowledge of the subject and presented it as a continous logical development, which remains, till today, the main source of studying geometrical solids.

In this unit, you will study volumes of geometrical solids. Your study objectives for the unit are as follows:

20.2 Objectives

By the end of this unit, you should be able to:

i) determine the volumes of cubes, cuboids, cylinders and cones;
ii) calculate the surface areas of some compound solids;
iii) calculate the volume of any given frustrum of a right circular cone.

20.3 Surface areas and volumes of simple and composite shapes

You will need to do a quick revision of surface areas of simple solids presented as follows:

A cube

A cube is a solid with six square faces. In the diagram in Fig. 20.1, AB = AD = AF. To find the surface area, therefore, you will need to find the areas of the faces. If the faces are the same in dimensions, how do you find the total surface area?

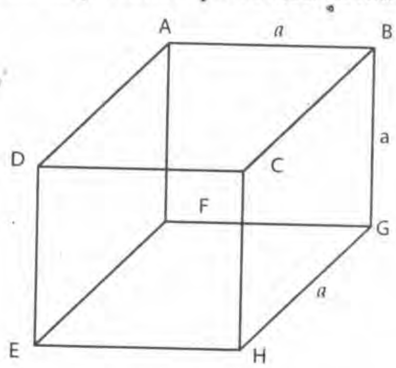


Fig. 20.1

Area of one face is $a \times a = a^2$
Total area = $a \times a \times 6 = 6a^2$
$\therefore$ A = $6a^2$ units^2

A cuboid

What makes a cuboid different from a cube? All the faces of a cuboid are not of equal dimensions. Identify the ones that are equal using the diagram in Fig. 20.2

In Fig. 20.2 figure above figure

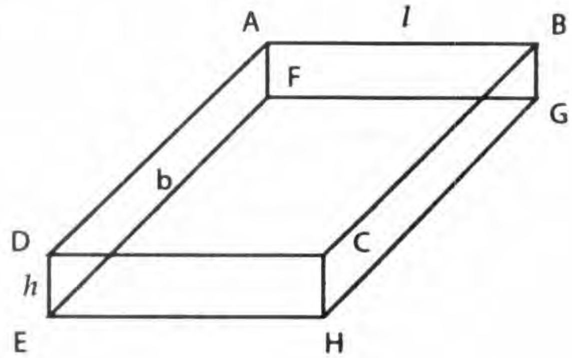


Fig 20.2

l = length of the cuboid
b = breadth of the cuboid
h = height of the cuboid
A cuboid has three different pairs of faces that add up to make its surface area.
Area of ABCD = Area of FGHE,
Area of AFGB = Area of DEHC,
Area of ADEF = Area of BGHC
$\therefore$ Area of the cuboid = 2(area of ABDC) + 2 (area of AFGB) + 2 (area of ADEF)
$= 2\,(l \times b) + 2(b \times h) + 2(l \times h)$
$= 2(lb + bh + lh)$
Below are some examples for your practice.

Example 20.1
What is the surface area of a cube of edges 12 cm?

Solution
A = $6 \times 12 \times 12 = 6 \times 144 = 864$ cm^2

Example 20.2
Given a cuboid of edge 3 cm, 5 cm, and 8 cm, calculate the surface area.

Solution
Use the formula of area of a cuboid
A = $2\,(lb + lh + lh)$
$= 2(8\text{ cm} \times 5\text{ cm} + 8\text{ cm} \times 3\text{ cm} + 5\text{ cm} \times 3\text{ cm})$
$= 2\,(40\text{ cm}^2 + 24\text{ cm}^2 + 15\text{ cm}^2)$
$= 2 \times 79\text{ cm}^2$
$= 158\text{ cm}^2$

A cylinder

You need to refresh your memory by looking briefly at the following parts of a cylinder. A cylinder can have open ends, one end open and the other closed or both ends closed.

If the curved, surface of a cylinder is cut open, it forms a rectangular shape, where the height of the cylinder forms the breadth of the rectangle and the circumference of the circular base forms the length. If the height of the cylinder is h and the length of the circumference is $2\pi r$. Area of curved surface will be $2\pi r \times h$.

However, if the cylinder is closed on one end, then the area of the circular base is added: Area of circle = πr^2

Area of cylinder with one end closed:

$$2\pi rh + \pi r^2 = \pi r(2h + r)$$

If two ends are closed:

Area = $2\pi rh + 2\pi r^2$

$= 2\pi r\,(h + r)$

where r = radius of the cylinder, h = height of the cylinder.

A cone

You will recall that the sector of a circle is a net of a cone, that is the area of the sector of a circle is the same as the surface area of the cone it forms. This is demonstrated in the diagrams in Fig. 20.3.

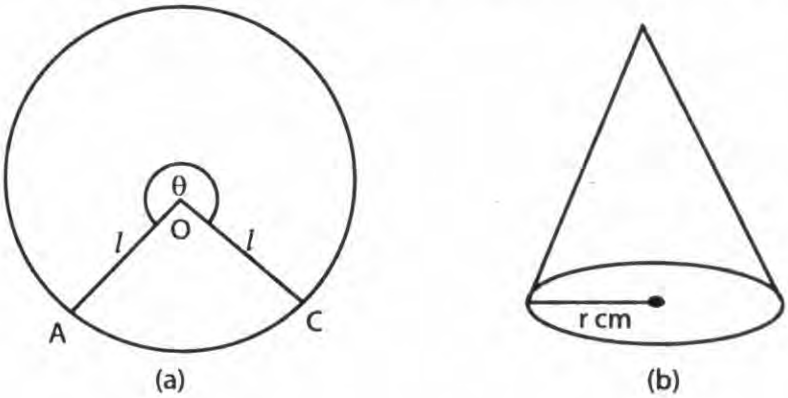


Fig. 20.3

Length of arc AC = $2\pi l \dfrac{\theta}{360^\circ}$

Where l is the slanting height of the cone when AO and CO are coincident, a cone will be formed with circular base of radius r. The circumference of the base is then $2\pi r$.

Thus $2\pi r = 2\pi l \dfrac{\theta}{360^\circ}$

$\theta = \dfrac{2\pi r \times 360^\circ}{2\pi l}$

$= \dfrac{360r}{l}$

But the area of a sector of a circle of radius l is $\pi l^2 \times \dfrac{\theta}{360^\circ}$

Since $\theta = \dfrac{360^\circ r}{l}$

$\therefore$ Area of sector = $\dfrac{\pi l^2 \times 360^\circ r}{360^\circ l}$

$= \pi rl$

This implies that area of curved side of the cone = πrl

Where l = length of slant side

r = base radius

When the cone has a base, then its surface includes the circular base

$\therefore A = \pi rl + \pi r^2 = \pi r\,(l + r)$

Consider the following worked examples

Example 20.3
Find the area of
a) curved surface;
b) circular base; and
c) total surface area of the cone in Fig. 20.4.

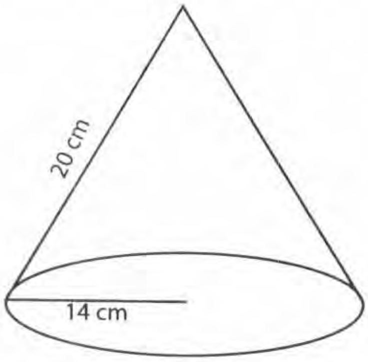


Fig. 20.4

Solution

a) $A = \pi rl$, where r = radius of the cone = 14 cm
l = slanting height of the cone = 20 cm^2
$\therefore A = \frac{22}{7} \times 14 \text{ cm} \times 20 \text{ cm} = (44 \times 20) \text{ cm}^2$
$= 880 \text{ cm}^2$

b) $A = \pi r^2 = (\frac{22}{7} \times 14 \times 14) \text{ cm}^2$
$= (308 \times 2) \text{ cm}^2 = 616 \text{ cm}^2$

c) Total area = $(880 + 616) \text{ cm}^2$
$= 1\,496 \text{ cm}^2$

Now practise with the following exercises.

Exercise 20.1

1. If the total surface area of a cube is 96 cm. What is the length of the cube?
2. What is the total surface area of a cuboid whose dimensions are 11 cm 8 cm and 5 cm?
3. What is the total surface area of a cone of radius 14 cm and slanting side of 80 cm?

20.4 Volumes of solid shapes

A geometrical solid, or polyhedrons is a three-dimensional figure consisting of intersecting polygons called **faces** in different planes. Two faces of a polyhedron meet in a line called an **edge,** two edges meet in a **corner** or **vertex**. Note that a cuboid is an example.

Units of volume

Volume is a measure of space taken up by a solid body. Since a solid has three dimensions, the units of volume are expressed in cubic units such as mm^3, cm^3, km^3 and m^3, where $1 \text{ km}^3 = 10^9 \text{m}^3$, $1 \text{ m}^3 = 10^6 \text{ cm}^3$ or 10^9 mm^3. A convenient unit for fluids for example is the litre (l) which is 1 dm^3 or 1 000 cm^3.

Boxes and cubes

A box as shown in the diagram in Fig. 20.5 has three dimensions namely length(l), breadth (b) and height (h).

The volume of the box = $l \times b \times h$ cubic units

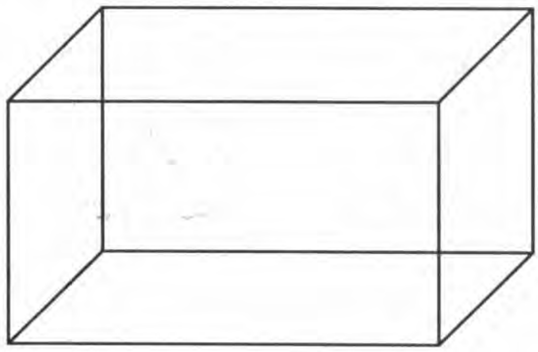


Fig. 20.5

In the case of a cube, the length, breadth and height are of equal dimensions, hence if the length is 1 cm, then the volume = $1 \times 1 \times 1 = 1^3 = 1\ cm^3$. That is a cube of size 3 cm has its volume to be $3 \times 3 \times 3 = 27$ cu cm or 27 cm^3

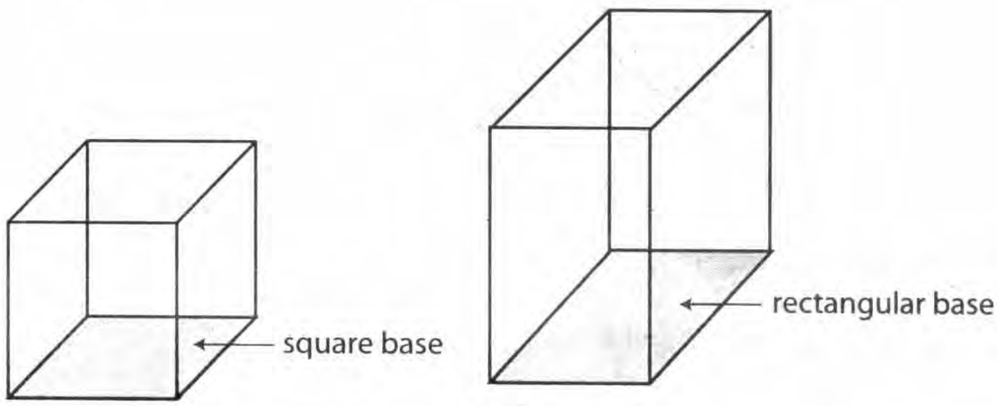
Volumes of right prisms

A prism has a polygon as base and uniform cross-section. In a right prism, the plane of the base is perpendicular (or normal) to the height of the prism. The volume of a right prism with cross-sectional area A and height h is given by the formula:

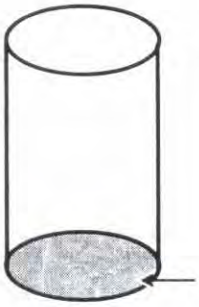
$$V = Ah \text{ cubic units}$$

The volume of a prism = (area of cross-section x lengh).

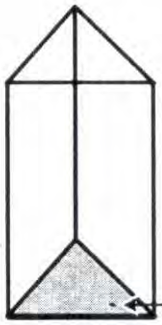
If the cross-section is triangular or square, then area of the cross-section will be the area of the triangle or square, respectively. Fig. 20.6 shows some right prisms for our study.



(a) square prism (b) rectangular prism



(c) circular prism



(d) triangular prism

Fig. 20.6

The two shapes in Fig. 20.6(a) & (b) are often called cuboids or simply box shapes.
In every case of a right prism:
Volume = area of base x height
Where all measurements must be in the same units. Now, study the following examples.

Example 20.4
Find the volume of a prism with rectangular base of 5 cm by 3 cm, and is 6 cm long.

Solution
Volume of prism = area of base x height.
First of all try to draw a diagram of what you think the shape looks like, as done.

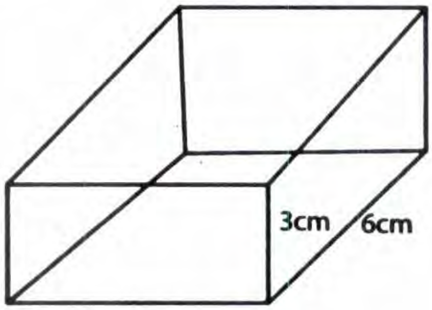


Fig. 20.7

Area of base = length × breadth
$= 5 \text{ cm} \times 6 \text{ cm}$
$= 30 \text{ cm}^2$
volume = area of base × height
$= 30 \text{ cm}^2 \times 3 \text{ cm}$
$= 90 \text{ cm}^3$ or 90 cubic cm.

Example 20.5
Find the volume of a cylinder of radius 2 cm and height 10 cm. ($\pi = 3.14$)

Solution
Volume = area of base × height
Area of base = πr^2
$= \pi \times r \times r$
$= 3.14 \times 2 \text{ cm} \times 2 \text{ cm}$
$= 12.56 \text{ cm}^2$

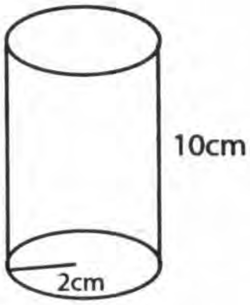


Fig. 20.8

Volume = area of base × height
$= 12.56 \text{ cm}^2 \times 10 \text{ cm}$
$= 125.6 \text{ cm}^3$

You should be able to solve the following problems having studied the last two examples.

Exercise 20.2
1. Find the volume of a circular cylinder with diameter 7 cm and height 6 cm.
2. Find the volume of a cube of scale 4.5 cm.
3. Find the height of a cuboid whose horizontal cross-sectional area is 36 cm^2 and whose volume is 144 cm^3.
4. Find the volume of a solid circular cylinder whose diameter is 8 cm and the height is 7 cm.

Right circular cones and pyramids

A right circular cone is a special pyramid with a circular base and a vertex directly above the centre of the circular base.

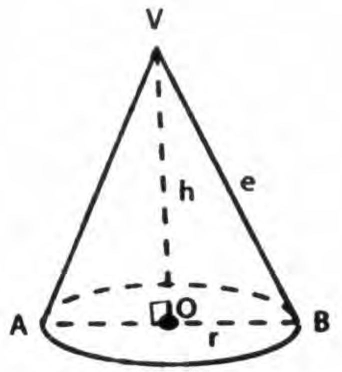


Fig. 20.9

In Fig 20.9, the line OV is called the **altitude** of the cone. The line BV is called the **slant height** of the cone while OB is the radius of the circular base. Since ΔOVB is a right-angled triangle, it follows that the slant height, *l*, altitude, h, and the radius, r ,of the circular base are related by the following expression;

$$l^2 = h^2 + r^2$$

Let the total surface area of the cone be area of the curved surface + area of base, then you have

Total surface area = $\pi r l + \pi r^2$

$= \pi r (l + r)$

A cone is a pyramid with a circular base and a curved surface. The volume of a cone with a base of radius r units and a height h is

$$V = \frac{1}{3}\pi r^2 h \text{ cubic units}$$

Right pyramids

A pyramid becomes narrower upward to a point called the **apex**, and so the area of cross-section is not constant. In a right pyramid, the normal or perpendicular height from the apex to the base passes through the centre of symmetry of the base and the length of this normal is the height of the pyramid.

The general formula for the volume of a right pyramid is

$$V = \frac{1}{3} \text{ x area of base x altitude}$$

so

$$V = \frac{1}{3} Ah \text{ cubic units}$$

Example 20.6

A right pyramid with vertex V on a square base ABCD has each of the lateral faces as an equilateral triangle of side 8 cm. If the altitude of the pyramid is 5.7 cm , find.

i) the total surface area of the pyramid;

ii) the volume of the pyramid.

Solution

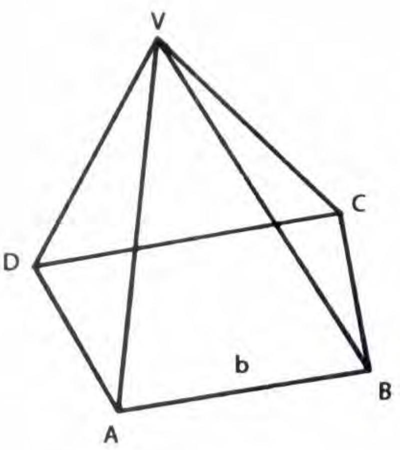


Fig. 20.10

Let the total surface area of the pyramid be equal to 4A + b, where A is the area of each of the triangular faces

$= 4A + b$

$= 4 \times 16\sqrt{3} + 64$

$= 64\sqrt{3} + 64$

$= 174.9 \text{ cm}^2$

ii) Let V be the volume of the pyramid

$V = \frac{1}{3} \times \text{area of base} \times \text{height}$

$= \frac{1}{3} \times 64 \times 5.7$

$= 121.6 \text{ cm}^3$

Example 20.7

A right-circular cone of base radius 5 cm has an altitude of 12 cm. Find, correct to 3 significant figures,

i) the total surface area of the cone;

ii) the volume of the cone (take $\pi = 3.142$).

Solution

i) Let the total surface area of the cone be $\pi r^2 + \pi rl$.

$\pi r^2 + \pi rl = 25\pi + 65\pi$

$= 90\pi$

$= 90 \times 3.142 \text{ cm}^3$

$= 282.8 \text{ cm}^2$

$= 283 \text{ cm}^3$ (3 s.f.)

(ii) Let V be the volume of the cone, then

$v = \frac{1}{3} \times \text{area of base} \times \text{altitude}$

$= \frac{1}{3} \times \pi r^2 h$

$= \frac{1}{3} \times 3.142 \times 25 \times 12$

$= 3.142 \times 100$

$= 314.2 \text{ cm}^3$

$= 314 \text{ cm}^3$ (3 s.f.)

Frustum of a cone or pyramid

When the top of a pyramid or cone is cut along a plane perpendicular to the axis joining the vertex to the centre of the base, the remaining part is called **frustum** of a pyramid or cone respectively. They are shown in Figs 20.11 and 20.12.

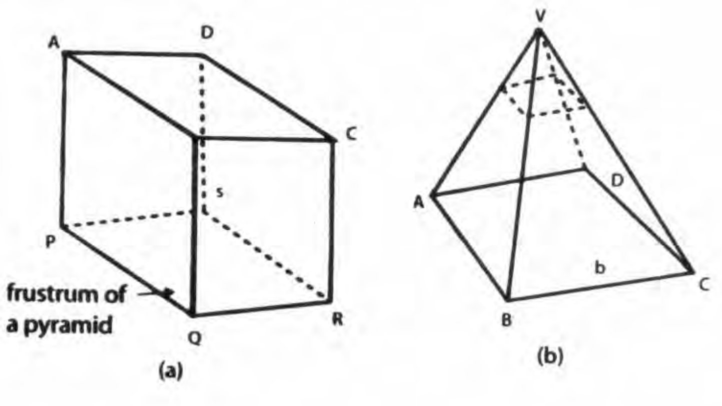


Fig 20.11

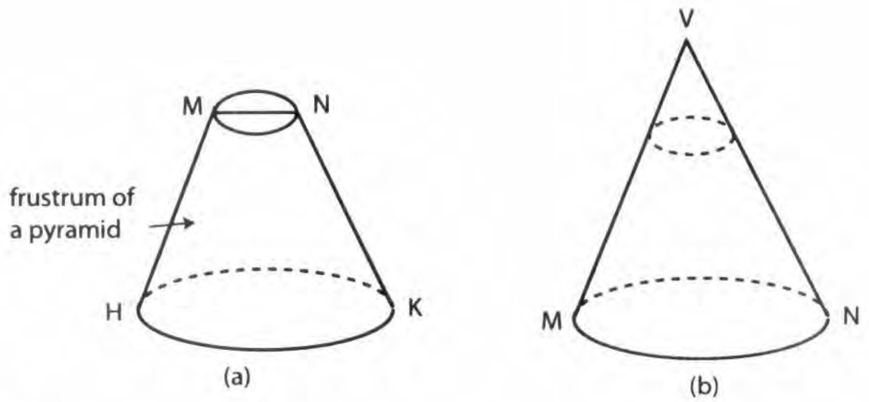


Fig. 20.12

The following examples will give you a clearer understanding of a frustum.

Example 20.8

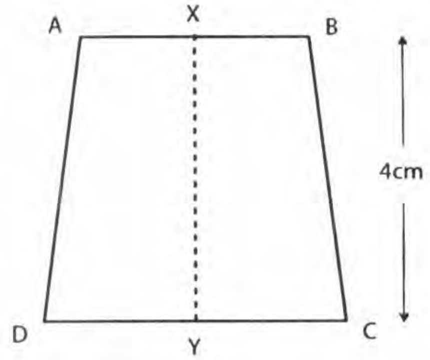


Fig. 20.13

The diagram in Fig. 20.13 is the vertical cross-section of the frustum of a right circular cone whose axis of symmetry is XY. If the top and bottom parts of the frustum have radii 3 cm and 4 cm respectively, and the height of the frustum is 4 cm; find:

i) the height of the cone that was cut to form the frustum,

ii) the volume of the entire cone of which the frustum was a part;

iii) the volume of the frustum of the cone (take $\pi = 3.142$).

Solution

i)

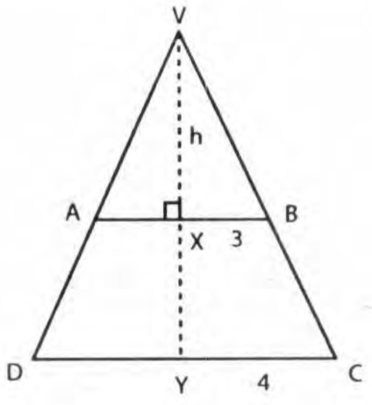


Fig. 20.14

Let h be the height of the cone that was cut to form the frustum. Let H be the height of the entire cone of which the frustum forms a part.
Then, $H = 4 + h$

ΔVXB and ΔVYC are similar,

$\therefore \frac{VX}{VY} = \frac{XB}{YC}$

$\therefore \frac{h}{h+4} = \frac{3}{4}$

$\therefore 4h = 3(4 + h)$

$= 12 + 3h$

$\therefore h = 12$ cm

ii) Let V_1 be the volume of the cone cut, then

$V_1 = \frac{1}{3} \times \pi \times 3^2 \times 12$

$= 36\pi$

$= 36 \times 3.142$

$= 113.1$ cm^3

iii) Let V_2 be the volume of the entire cone, then

$V_2 = \frac{1}{3} \times \pi \times r^2 \times (4 + 12)$

$= \frac{1}{3}\pi \times 16 \times 16$

$= \frac{256\pi}{3}$

$= \frac{256 \times 3.142}{3}$

$= 268.1$ cm^3

iv) Let V be the volume of the frustum then,

$V = V_2 - V_1$

$= (268.1 - 113.1)$ cm^3

$= 155$ cm^3

Example 20.9

Find, in cm^3, the area of material required for a lamp shade in the form of a frustum of a cone of which the top and bottom diameters are 20 cm and 30 cm respectively and the vertical height is 12 cm.

Solution

Complete the cone of which the lampshade is a frustum as follows:

With the labelling of the diagram in Fig. 20.15, the similar triangles

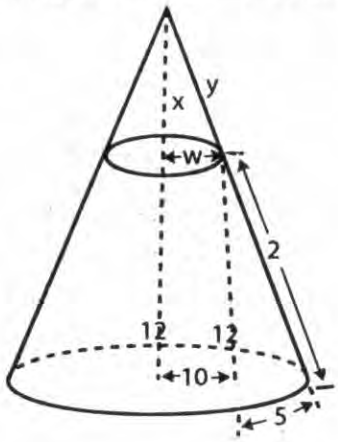


Fig. 20.15

$\frac{x}{10} = \frac{12}{5}$

$\therefore x = \frac{120}{5} = 24$

By Pythagoras' theorem, the values of y and z are found to be $y = 26$ and $z = 13$.
Therefore, surface area of frustum

$= (\pi \times 15 \times 39) - (\pi \times 10 \times 26)\ \text{cm}^2$
$= 13\pi(45 - 20)\ \text{cm}^2$
$= 13\pi \times 25\ \text{cm}^2$
$= 1\,021\ \text{cm}^2$

Area of material required = 1 021 cm^2.

Exercise 20.3

1. A frustum of a cone has top and bottom diameters of 14 cm and 10 cm respectively and a depth of 6 cm. Find the volume of the frustum in terms of π.
2. A right pyramid on a base 10 m square is 15 m high.
   a) Find the volume of the pyramid.
   b) If the top 6 cm of the pyramid is removed, what is the volume of the remaining frustum?
3. A frustum of a pyramid is 16 cm square at the bottom, 6 cm square at the top and 12 cm high. Find the volume of this frustum.

20.5 Conclusion

In this unit you have studied volumes of various solid shape and you have also learnt the following:

i) Prism: The volume of a prism is given by
Volume = area of constant cross-section × perpendicular height or area of base × height.

ii) The volume of a right pyramid is:
Volume = $\frac{1}{3}$ × base area × height

iii) A cone is a pyramid with a circular base and a curved surface.
The volume = $\frac{1}{3}\pi r^2 h$ where r = radius
Curved surface area = $\pi r l$
Total surface area = $\pi r l + \pi r^2$
$= \pi r\,(l + r)$

iv) Volume of a square – based pyramid is given by
Volume $= \frac{1}{3} b^2 h$

v) When a small cone or pyramid is cut from the top of a similar solid, the portion remaining is called a frustum. The volume of a frustum of a cone is the difference in the volumes of two similar cones.

20.6 Summary

In this unit you have learnt different types of common surface areas and volumes of solids, namely, cuboid, prism, pyramid, cylinder, triangular prism, rectangular-based pyramid, right – circular cone, square prism and frusta of cones and pyramids.

What you have learnt in this unit will help you later in your indepth study of mensuration.

20.7 Tutor-marked assignment

1. Find the volume of a right pyramid of height 6 cm and square base of side 8 cm.
2. Find the volume of a cone with perpendicular height 4 cm and base radius 3 cm.

20.8 References

1. Macrae, M. F.; Kalejaiye, A. O. *et al.* (2001), *New General Mathematics for Senior Secondary Schools*, Pearson Education Limited,pp. 176 – 200
2. M.A.N. (1991), *Senior Secondary Mathematics Book 1*, University Press Plc, Ibadan, pp. 243-269.

Unit 21 Trigonometric ratios

Contents

21.1 Introduction

Trigonometry is the branch of mathematics that is concerned with the relationship between angles of a triangle and the lengths of its sides. In particular, the ratios of the lengths of the sides of a right-angled triangle are known as trigonometric ratios.

There are basically three ratios which relate the lengths of the three sides of a triangle. These are **sine, cosine** and **tangent**. Their reciprocals are **cosecant, secant** and **cotangent** respectively. In this unit, you will study trigonometric ratios, as well as reduction to functions of positive acute angles.

21.2 Objectives

By the end of this unit, you should be able to:

i) use the tangent, sine and cosine ratios in relation to the sides of right-angled triangles to solve practical problems;
ii) identify the trigonometric ratios of the special angles 30°, 45° and 60°;
iii) derive and apply certain trigonometric identities;
iv) solve problems on similar triangles.

21.3 Basic trigonometric ratios

The basic trigonometric ratios have two basic definitions: **traditional** definition and **alternative** definition.

Traditional definition

The basic trigonometric ratios can be defined in terms of the sides of a right-angled triangle.

QPR is a right-angled triangle with $Q\hat{P}R = \theta$ and $P\hat{R}Q = 90°$

The three basic ratios can be defined as follows:

Cosine of angle $\theta = \frac{PR}{PQ} = \frac{q}{r}$

Sine of angle $\theta = \frac{QR}{PQ} = \frac{p}{r}$

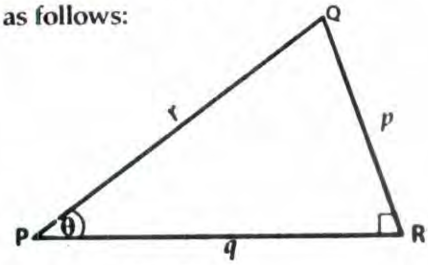


Fig. 21.1

Tangent of angle $\theta = \frac{QR}{PR} = \frac{p}{q}$

Thus, $\cos\ \theta = \frac{q}{r}$

$\text{Sine}\ \theta = \frac{p}{r}$

$\tan\ \theta = \frac{p}{q}$

Similarly, the reciprocals of the basic ratios are as follows:

$\sec\ \theta = \frac{r}{q} = \frac{1}{\cos\theta}$

$\text{cosec}\ \theta = \frac{r}{p} = \frac{1}{\sin\theta}$

$\cot\ \theta = \frac{p}{q} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

Alternative definition

An alternative definition is possible using Fig. 21.2.

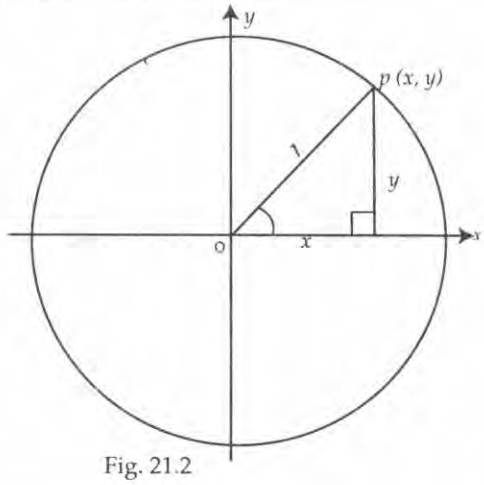


Fig. 21.2

In Fig. 21.2, if P with coordinates (x, y) is a variable point on a unit circle with centre O, OP makes angle θ with the positive x-axis. Now, x can be considered as the projection of OP of unit length in the x-axis and y can be considered as the vertical displacement of P from the x-axis. Sin θ and cos θ are defined in terms of y and x respectively as:

$\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$

The traditional approach makes it easy to find the remaining five ratios when one ratio is given. Example 21.1 will serve as a good illustration.

Example 21.1

Given that $\sin\theta = \frac{5}{13}$ and θ is acute, find:

a) cosθ b) secθ c) cosθ d) tanθ e) cosecθ

Solution

Since sine of an angle is the ratio of opposite side to the hypotenuse, you can draw a right-angled triangle PQR with QR = 5 and PQ = 13.

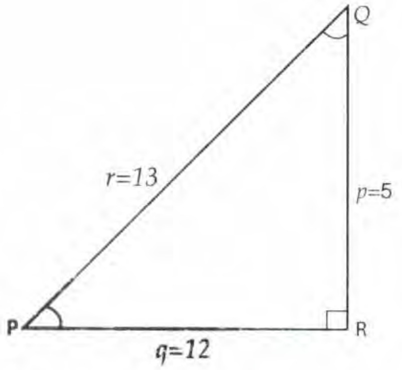


Fig. 21.3

Use Pythagoras' theorem to find PR

$PQ^2 = PR^2 + QR^2$
$13^2 = PR^2 + 5^2$
$\therefore PR^2 = 13^2 - 5^2$
$= 169 - 25$
$= 144$
$\therefore PR = 12$
Thus $q = 12, r = 13, p = 5$
So,

a) $\cos\theta = \frac{q}{r} = \frac{12}{13}$ b) $\sec\theta = \frac{r}{q} = \frac{13}{12}$ c) $\cot\theta = \frac{q}{p} = 12$

d) $\tan\theta = \frac{p}{q} = \frac{5}{12}$ e) $\operatorname{cosec}\theta = \frac{r}{p} = \frac{13}{5}$

Exercise 21. 1

1. If $\cos\theta = \frac{4}{5}$, find $\sin\theta$ and $\tan\theta$
2. If $\sin\theta = \frac{7}{25}$, find $\cos\theta$ and $\tan\theta$
3. If co $\theta = \frac{40}{41}$, find $\sin\theta$ and $\tan\theta$
4. If $\tan\theta = \frac{5}{4}$, find $\sin\theta$ and $\cos\theta$
5. If $\sec\theta = 3$, find $\cos\theta$ and $\tan\theta$

Relationships between the trigonometric ratios

You recall that trigonometry is concerned with the relationship between angles and lengths of sides of a triangles.

(sin +)	Second quadrant (90° – 180°)	First quadrant (0°–90°)	(All positive)
(Tan +)	Third quadrant (180° – 270°)	Fourth quadrant (270° – 360°)	(cos +)

Fig. 21.4

From Fig. 21.4, you will observe the following:

a) In the first quadrant, all the ratios are positive.
b) In the second quadrant, only sine ratio is positive while the rest are negative.
c) In the third quadrant, only tangent ratio is positive, while the rest are negative.
d) In the fourth quadrant, only cosine ratio is positive, while the rest are negative.

Also using Fig. 21.4, you will notice that:
sin 150° = + sin 30°; sin 210° = – sin 30°
cos 150° = – cos 30°; cos 210° = – cos 30°
tan 150° = – tan 30°; tan 210° = + tan 30°

Trigonometric ratios of special angles 30°, 45°, 60°

Consider an equilateral triangle with sides of 2 units in length and angle 60° as shown in Fig.21.5.

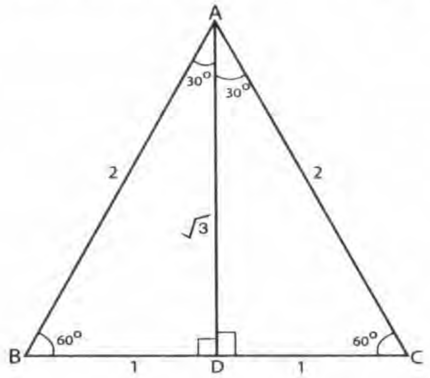


Fig. 21.5

In the triangle ABC, ABD is a right-angled triangle.
Observe that:

$\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$

$\sin 30° = \frac{BD}{AB} = \frac{1}{2}$

$\cos 60° = \frac{BD}{AB} = \frac{1}{2}$

$\tan 60° = \frac{\sin 60°}{\cos 60°} = \sqrt{3}$

$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{\sqrt{3}}$

$\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$

$\sin 30° = \cos 60°$

$\sin 60° = \cos 30°$

Now, consider a right-angled triangle XYZ in Fig. 21.6.

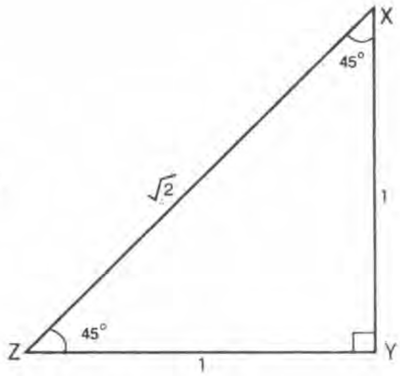


Fig. 21.6

$$\sin 45^\circ = \frac{XY}{XZ} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{ZY}{ZX} = \frac{1}{\sqrt{2}}$$

Below is a table of values for trigonometric ratio of special angles (0°, 30°, 45°, 60°, 90°):

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (infinity)
cosec	∞ (infinity)	3	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	α
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Table 21.1

$\cos^2\theta + \sin^2\theta = 1$

i.e. $\sin^2\theta + \cos^2\theta = 1$

This is called a **trigonometric** identity, from which we obtain the following:

$1 + \tan^2\theta = \sec^2\theta$

$\cot^2\theta + 1 = \text{Cosec}^2\theta$

Example 21.2

Express the following in terms of the trigonometrical ratios of acute angles:

a) cos 330°

b) sin 150°

Solution

a) $\cos 330^\circ = \cos(360^\circ - 30^\circ)$
$= \cos 30^\circ$

b) $\sin 150^\circ = \sin(180^\circ - 30^\circ)$
$= \sin 30^\circ$

Example 21.3

Solve the equation $2\sin^2\theta - \sin\theta$ for values of θ from 0° to 360° inclusive.

Solution

$2\sin^2\theta = \sin\theta$

$2\sin^2\theta - \sin\theta = 0$

$\therefore \sin\theta(2\sin\theta - 1) = 0$

$\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

If $\sin\theta = 0, \theta = 0°, 180°, 360°$

If $\sin\theta = \frac{1}{2}, \theta = 30°, 150°$

$\therefore$ the roots of the equation for $0° \le \theta \le 360°$ are
0°, 30°, 150°, 180° and 360°

Exercise 21.2

1. Solve the following equations:
   a) $1 + \cos\theta = 2\sin^2\theta, 0° \le \theta \le 180°$
   b) $3\cos^2\theta = 2\sin\theta\cos\theta, -180° \le \theta \le 180°$
2. Evaluate the following, without using tables:
   a) $\frac{\cot 25°}{\tan 65°}$
   b) $\sec^2 45° - \tan^2 45°$

Trigonometric identities and equations

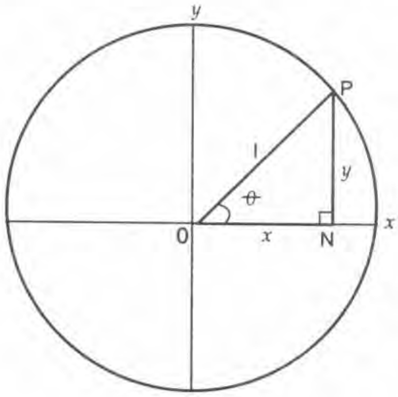


Fig. 21.7

Fig. 21.7 shows a unit circle. If Δ *OPN* is a right-angled triangle with $OP = 1$, $ON = x$ and $PN = y$, $P\hat{O}N = \theta$, then from the definition of trigonometric ratios:

$x = \cos\theta$ .... (i)

$y = \sin\theta$ .... (ii)

From (i): $x^2 = \cos^2\theta$ .....(iii)

From (ii): $y^2 = \sin^2\theta$ .....(iv)

Adding (iii) and (iv), you get

$x^2 + y^2 = \cos^2\theta + \sin^2\theta$ .... (v)

Since Δ *OPN* is a right-angled triangle:

$ON^2 + NP^2 = OP^2$

$\therefore x^2 + y^2 = 1$ .... (vi)

From equations (v) and (vi) you get

$\cos^2\theta + \sin^2\theta = 1$ .... (vii)

You got this identity before, isn't it?

Dividing both sides of (vii) by $\cos^2\theta$ gives you

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$\therefore 1 + \tan^2\theta = \sec^2\theta$ .... (viii)

Dividing (vii) through by $\sin^2\theta$

$\therefore \cot^2\theta + 1 = \mathrm{cosec}^2\theta$

$\therefore 1 + \cot^2\theta = \mathrm{cosec}^2\theta$ .... (ix)

Example 21.4
Show that $(3 - \sin^2\theta)\,\text{cosec}^2\theta = 2\,\text{cosec}^2\theta + \cot^2\theta$

Solution
Let your R.H.S. $= 3\,\text{cosec}^2\theta - \sin^2\theta + \text{cosec}^2\theta$

$= 3\,\text{cosec}^2\theta - \frac{\sin^2\theta}{\cos^2\theta}$

$= 3\,\text{cosec}^2\theta - 1$

$= 3\,\text{cosec}^2\theta - (\text{cosec}^2\theta - \cot^2\theta)$

$= 3\,\text{cosec}^2\theta - \text{cosec}^2\theta + \cot^2\theta$

$= 2\,\text{cosec}^2\theta + \cot^2\theta$

= R.H.S.

Example 21.5

Prove that $\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta} = 2\,\text{cosec}^2\theta$

Solution
Left hand side (using the LCM)

$= \frac{(1-\cos\theta)+(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$

$= \frac{2-\cos\theta+\cos\theta}{1-\cos^2\theta}$

$\therefore$ L.H.S. $= \frac{2}{1-\cos^2\theta}$

$= \frac{2}{\sin^2\theta}$ (since $\sin^2\theta + \cos^2\theta = 1$

$= 2\,\text{cosec}^2\theta\ (\frac{1}{\sin\theta} = \text{cosec}\,\theta)$

= R.H.S.

Exercise 21.3
Prove each of the following identities:

a) $\frac{\sin\theta}{1-\cos^2\theta} = \text{cosec}\,\theta$

b) $\frac{\tan\theta}{\sin\theta} = \sec\theta$

c) $\frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \tan\theta$

d) $\frac{(1-\sin\theta)(1+\sin\theta)}{\sin^2\theta} = \cot^2\theta$

Trigonometric equations
A trigonometric equation is an equation containing trigonometric ratios. Since trigonometric functions are periodic functions, there are infinite number of solutions for any trigonometric equation. It is therefore, traditional to find some solutions within a given range.

If $a\sin\theta + b = 0\ (a \neq 0)$

$a\sin\theta = -b$

$\sin\theta = -\frac{b}{a}$

$\theta = \sin^{-1}(-\frac{b}{a})$

$\sin^{-1}(-\frac{b}{a})$ means the angle whose sin θ ratio is $\frac{-b}{a}$.

It is called sine inverse of $\frac{-b}{a}$ It is also denoted as **arc sin**$\frac{-b}{a}$.

Thus, if sin θ = $\frac{-b}{a}$, then, θ = $\sin^{-1}(\frac{-b}{a})$ = arc sin $(\frac{-b}{a})$

Similarly, if $a\cos\theta + b = 0$, $(a \neq 0)$

Then, cos θ = $\frac{-b}{a}$

θ = $\cos^{-1}$ $(\frac{-b}{a})$

Example 21.6

Find the values of θ between 0° and 360° which satisfy the equation: $4\cos^2\theta - 3 = 0$.

Solution

$4\cos^2\theta - 3 = 0$

$4\cos^2\theta = 3$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

If cos θ = $\frac{\sqrt{3}}{2}$,

Then, θ = 30° and 330°

If cos θ= $-\frac{\sqrt{3}}{2}$, then

θ = 150° and 210°

Hence, the values of θ between 0° and 360° satisfying the equation are 30°, 150°, 210° and 330°.

Example 21.7

Find the values of θ between 0° and 360° which satisfy $6\sin^2\theta + \sin\theta - 1 = 0$

Solution

$6\sin^2\theta + \sin\theta - 1 = 0$

Let $p = \sin\theta$

Then, $6p^2 + p - 1 = 0$

$6p^2 + 3p - 2p - 1 = 0$

$3p(2p + 1) - 1(2p + 1) = 0$

$(3p - 1)(2p + 1) = 0$

$p = \frac{1}{3}$ or $p = -\frac{1}{2}$

∴ sin θ = $\frac{1}{3}$

θ = $\sin^{-1}(\frac{1}{3})$

= 19.47°; 160.53°

or sin θ = $-\frac{1}{2}$

θ = $\sin^{-1}(\frac{-1}{2})$

= 210°; 330°

Hence, the values of θ within the given range which satisfy the equation are 19.47°, 106.53°, 210° and 330°.

Exercise 21.4

Find the values of θ in the range $0° \le \theta \le 360°$ which satisfy:

a) $\sin\theta + \sin^2\theta = 0$

b) $(2\cos\theta + 1)(1 - \cos\theta) = 0$

c) $2\sin^2\theta - \sin\theta - 1 = 0$

d) $\frac{1 + \cos\theta}{2 - \cos\theta} = 1$

e) $4\cos^2\theta + 2\cos\theta = 1$

21.4 Conclusion

In this unit, you have studied trigonometric ratios of the general angles in relation to the sides of right-angled triangles. In particular, the ratios of the lengths of the sides of the right-angled triangles are known as trigonometric ratios.

There are also some angles whose trigonometric ratios can be calculated easily without using tables, such as 45°, 30° and 60°, as well as angles greater than 90°.

Simple trigonometric identities were also discussed with worked examples.

21.5 Summary

In this unit, you have learnt that:

i) $\sin^2\theta + \cos^2\theta = 1$;

ii) $1 + \tan^2\theta = \sec^2\theta$;

iii) $1 + \cot^2\theta = \text{cosec}^2\theta$.

21.6 Tutor-marked assignment

1. Given that $\sin\theta = \frac{5}{13}$ and θ is acute, find a) cos θ; b) tan θ; c) sec θ
2. Show that $(3 - \sin^2\theta)\,\text{cosec}^2\theta = 2\,\text{cosec}^2\theta + \cot^2\theta$.

21.7 References

1. Backhouse, J. K. and Houldsworth, S.P.T. (1999), *Pure Mathematics I*, Addison Wesley Longman Limited, England, pp 341 – 359
2. MAN, (1991), *Senior Secondary Mathematics Books 1 – 3*, University Press Plc. Ibadan, pp 187–207.
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Unit 22 Applications of trigonometry: bearings

Contents

22.1 Introduction

In the last unit, you studied trigonometric ratios: sine, cosine and tangent of angles. You will also recall that there are some special angles like 0°, 30°, 45°, 60° and 90°. Trigonometry is one of the oldest branches of mathematics and a good tool for solving practical problems. Applications of trigonometry, particular astronomy, reach far back to the history of civilisation. Modern uses of trigonometry go far beyond what you will study in this unit. However, it can be demonstrated that various things you have already learnt have many practical applications.

22.2 Objectives

By the end of this unit, you should be able to:

i) identify an angle of elevation;
ii) calculate heights and distances using angle of elevation and trigonometric ratios;
iii) identify an angle of depression;
iv) calculate heights and distances using angle of depression and trigonometric ratios;
v) solve problems involving bearings using ratios.

22.3 Angle of elevation

If you are faced with mathematical problems that requires you to use trigonometric ratios, you need to solve the problems as follows:

a) Clearly draw the diagram associated with the problem.
b) Find out a suitable right-angled triangle and label the triangle accordingly.
c) Choose the appropriate trigonometric ratio, build up your equation and solve.
d) Make use of the Pythagoras' theorem where appropriate.

What is an angle of elevation

If an observer at A is below the level of an object at B and is at a point on the same horizontal level as Y, then the angle of elevation of B from A is the angle between AB and the horizontal, that is, the angle $\angle YAB$ as shown in Fig. 22.1.

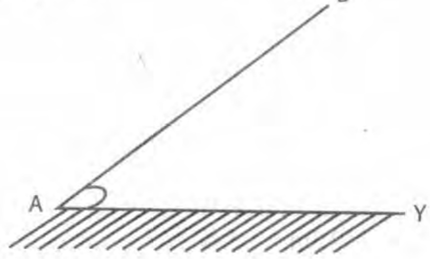


Fig. 22.1

The angle of elevation of the top of a tree B observed from a point A is the angle which the line AB makes with the horizontal at A. Fig. 22.2 gives a clearer meaning of angle of elevation.

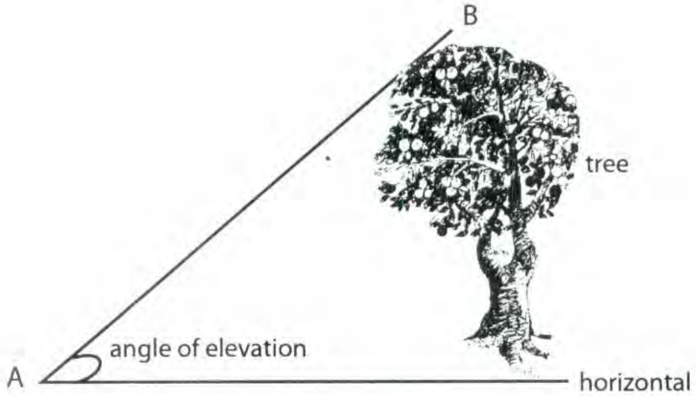


Fig. 22.2

The angle which the observer's eyes make with the horizontal is called **angle of elevation.** Now, study the following examples.

Example 22.1

A student 1.5m tall, when standing up, observes that the angle of elevation of top of a tree 20 metres away is 25°. Find the height of the tree.

Solution

You should draw the diagram to the problem as shown in Fig. 22.3.

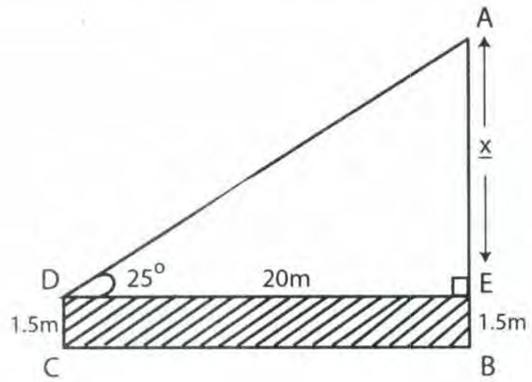


Fig. 22.3

From the diagram, note that AB represents the tree and DC the student. From triangle ADE, you obtain

$\frac{x}{20} = \tan 25°$

$x = 20 \tan 25°$

$= 20\ (0.4663)$ (From the table tan 25°)

$= 9.3\text{m}$

$\therefore$ Height of tree AB $= x + 1.5$ m

$= 9.3\text{m} + 1.5$ m

$= 10.8$ m

Example 22.2
Bala, 20 m away, observes a man working on the top of an electricity pole. If the height of the pole is 15 m, calculate the angle of elevation of the man.

Solution
A diagram to illustrate the question is shown in Fig. 22.4.

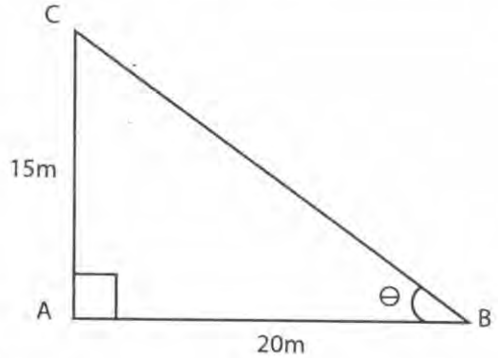


Fig. 22.4

In Fig. 22.4, AC represents the electricity pole while Bala is at point B and from the triangle ABC,
$\tan\theta = \frac{15}{20} = 0.75$
$\therefore \theta = 36.9°$
Therefore, the required angle of elevation is 36.9°
Having studied Examples 22.1 and 22.2, practise with the following exercises.

Exercise 22.1
1. A man views the angle of elevation of the top of a tower to be 30°. He is 40 metres from the foot of the tower and at the same level. Find the height of the tower.
2. From two points X and Y, 8 metres apart, and in line with the foot of a hill, the angles of elevation of the top of the hill are 30° and 60° respectively. Find the height of the hill above the ground.

22.4 Angle of depression
In this section, you will develop the idea of angle of depression as well as calculating heights and distances using angles of depression and trigonometric ratios.

Consider the diagram shown in Fig. 22.5, where a hunter, H, sitting on the top of a tree, who wants to look at an animal A on the ground, must turn his eyes from the horizontal position through the angle shown and look down. The angle which is below the horizontal is called **angle of depression.**

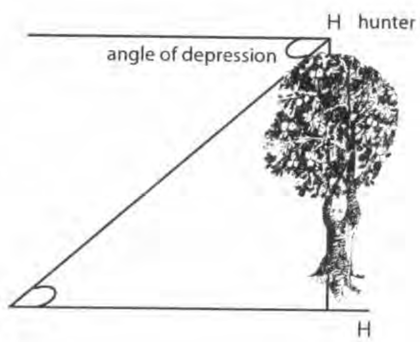


Fig. 22.5

Now, study the following examples for more understanding.

Example 22.3
A student, 1.6 metres tall, standing on the edge of a raised platform 1.4 high, views an object on a level ground at an angle of depression of 35°. Find the distance of the object from the foot of the platform.

Solution
Draw the diagram as shown in Fig. 22.6.

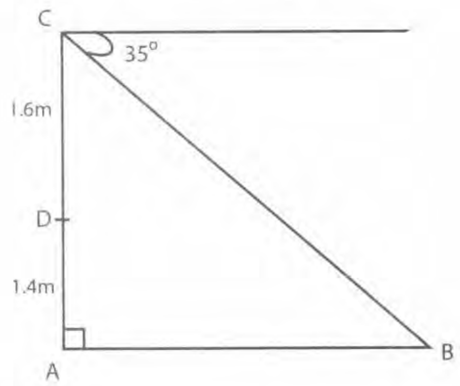


Fig. 22.6

In the diagram, CD represents the student while the object is at B. From triangle ABC, angle ACB = 55°. Hence

$\frac{AB}{AC} = \tan 55°$

AB = AC tan 55°

= 3 tan 55° = 4.3 m

The required distance is 4.3 m

Example 22.4
A man at the top of a building, 25 m high, looks down at an object 30 m away from the foot of the building. Calculate the angle of depression of the object from the man.

Solution
Draw a diagram representing the problem as shown in Fig. 22.7.

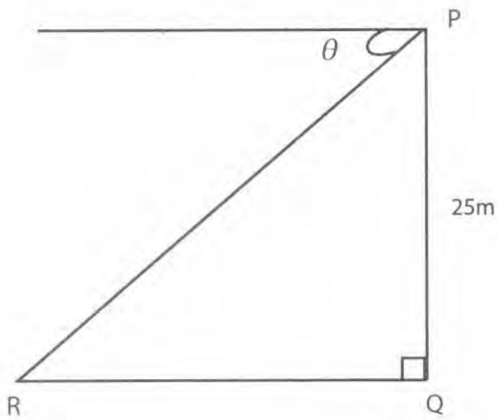


Fig. 22.7

In the diagram, the man is at P while the object is at R. From triangle PQR, angle PRQ is equal to the required angle of depression

$\therefore\ \tan \theta = \frac{PQ}{RQ} = \frac{25}{30} = 0.8333$

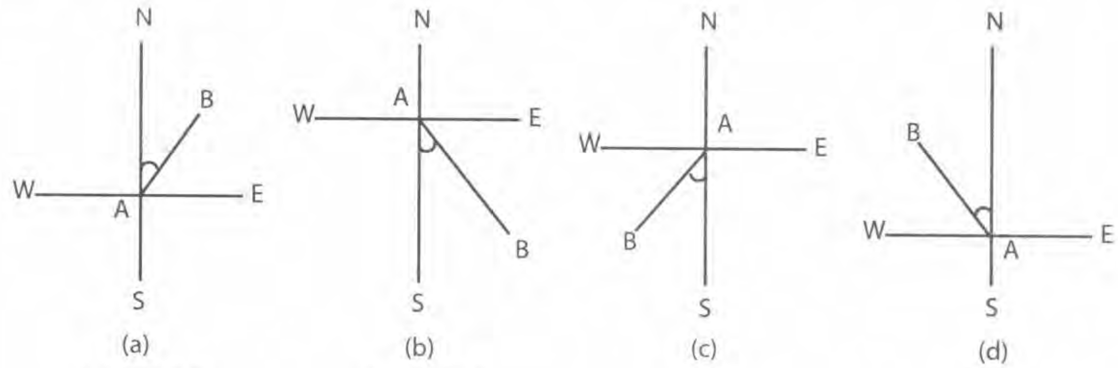
$\therefore\ \theta = 39.8°$

Exercise 22.2

1. A man standing on the edge of a cliff views a boat on the sea below at an angle of depression of 33°. If the height of the man is 15m above sea-level, find the distance of the boat from
a) the foot of the cliff,
b) the man.
2. Two points on a level ground in a valley are in the same vertical plane with an observer on top of a hill. If their angles of depression are 30° and 45° and the observer is 20 m above the valley, find the distance between the two points.

22.5 Bearings

The bearing of a point B from a point A, in a horizontal plane, is usually defined as the angle (always acute) made by the half-line drawn from A through B with the north-south line through A. The bearing is then read from the north or south line towards the east and west. For example to illustrate bearing:



(a): N35° E (b): S 35° E (c): S 35° W (b): N 35° W

Fig. 22.8

In bearing, the directions of N or S are usually stated first, followed by an acute angle then by E or W, as appropriate. The bearing of AB, N 35°E as indicated above, simply means that if you turn your head from the north through 35° towards the east you will be looking along AB.

For example, in the military, the initial direction is taken as the north from which the observer turns his head clockwise through an approximate angle to look at the desired direction. This angle is always written as a digit figure with zeros replacing empty position at the beginning. Using the military method AB is now 035°, the north is 000° or 360°. The special direction of NE, SE, SW and NW are respectively midway between the N and E, S and E, S and W and N and W, respectively.

Now, study the following examples.

Example 22.5

A man who could not trace his route was given the following instruction in order to get to his destination: walk 6 km east and them 8 km north. Find the bearing from the former position and the distance covered.

Solution

Let P be his former position and Q his destination. Let line PQ be the distance covered and θ his bearing. The diagram in Fig. 22.9 illustrates this description.

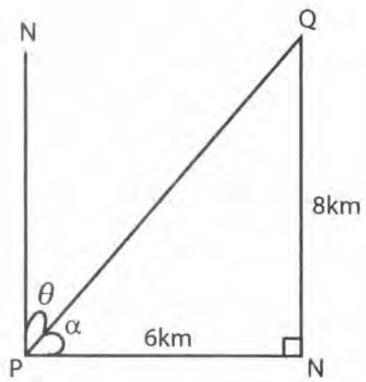


Fig. 22.9

From triangle PNQ, $\tan \alpha = \frac{QN}{PN}$

$= \frac{8}{6} = \frac{4}{3} = 1.3333$

$\tan \alpha = 1.3333$

$\therefore \alpha = 52.1°, \theta = 90° - 53.1° = 36.9°$

The distance covered,

$PQ^2 = 6^2 + 8^2$

$= 36 + 64$

$= 100$

$\therefore PQ = 10$ km

The man covered a distance of 10 km in a bearing of 36.9° or N36.9°E. While solving this problem, remember that you have applied trigonometric ratios as well as Pythagoras' theorem.

Exercise 22.3

1. A point X is 34 m due east of a point Y. The bearings of a flagpole from X and Y are N18°W and N40°E respectively. Calculate the distance of the flagpole from Y.
2. A man walks due west for 4 km. He then changes direction and walks on a bearing of 197° until he is south-west of his starting point. How far is he from his starting point?

22.6 Conclusion

In this unit you have learnt very vital concepts such as angle of elevation, angle of depression and bearing. You have also gone through a lot of related worked examples. The trigonometric ratios such as sine and cosine rules that are derived from the acute and obtuse angles are applied to the solution of triangles and problems on elevation, depression and bearings.

22.7 Summary

In this unit, you have learnt that:

i) the angle measured from the horizontal looking up to the top of an object is an angle of elevation;
ii) the angle measured from the horizontal looking down at an object is an angle of depression;
iii) the angle which a point or an object makes with the north-pointing line is the bearing;
iv) the trigonometric ratios are applied to solving problems involving angle of elevation, angle of depression and bearings.

22.8 Tutor-marked assignment

1. A simple measuring device is used at points X and Y on the same horizontal level to measure the angles of elevation of the peak P of a certain mountain. If X is known to be 5 200 m above sea level, XY = 4 000 m and the measurements of the angle of elevation of P at X and Y are 15° and 35° respectively, find the height of the mountain. (Take tan 15° = 0.3 and tan 35° = 0.7).

2. A tower and a building stand on the same horizontal level. From the point P at the bottom of the building, the angle of elevation of the top T of the tower is 65°. From the top Q of the building, the angle of elevation of the point T is 25°. If the building is 20 m high, calculate the distance PT.

   Hence or otherwise, calculate the height of the tower.

22.9 References

M.A.N. (1992), *Senior Secondary Mathematics, Book 2*, University Press Plc, Ibadan, pp. 152-190.

Unit 23 Coordinate geometry

Contents

23.1 Introduction

This unit is an aspect of coordinate geometry which focuses on Cartesian coordinates, division of a line and distance between two points.

23.2 Objectives

By the end of this unit, you should be able to:

i) understand the properties of Cartesians coordinates;
ii) divide a line segment in a given ratio;
iii) find the distance between two points.

23.3 Cartesian coordinates

You are certainly familiar with the coordinate system, but let us recall rapidly the main faults. To set up a *coordinate system* in the plane (that is, to label each point in the plane with a pair of numbers in some definite way), you draw two perpendicular lines in the plane as shown and calibrate each one using the same unit distance for both. The horizontal line is called the 'x-axis' and the vertical line the 'y-axis'. A point P in the plane is given the label (a,b), if it is at a horizontal distance, **a** from the origin and at a vertical distance **b** from the origin.

You call **a** the *x-coordinate* (or *abscissa*) of **P**, and **b** the *y-coordinate* (or *ordinate*) of **P** and you call the ordered pair of number (a, b) **coordinates** of P.

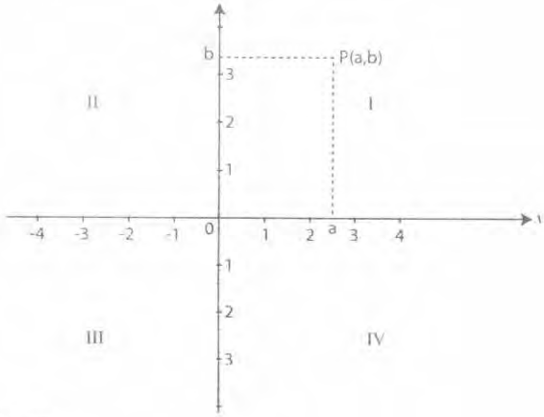


Fig. 23.1

Thus in your coordinate system to each point P there is an ordered pair of numbers. Conversely, given any ordered pair (a, b) of number, there is one and only one point P having that pair as its coordinates. How would you locate it? If P has coordinates (a, b), you will sometimes **refer to** it simply as the point (a, b). Sometime you write 'P(a, b)' to represent the point. The **coordinate** plane is called the *xy-plane*.

The point (0,0) where the two axes intersect is called the **origin.** The four quarter-plants into which the axes divide up the plane are called quadrants. They are labelled first, second, third and fourth (I, II, III, IV) as illustrated in Fig. 23.1. This way of labelling is just as arbitrary conversation but it is a common convention.

23.4 Division of a line segment in a giyen ratio

Let A and B be the points (x_1,y_1) and (x_2,y_2); let the point $P(x_p,y_p)$ divide AB in the ratio m: n as in Fig. 23.2.

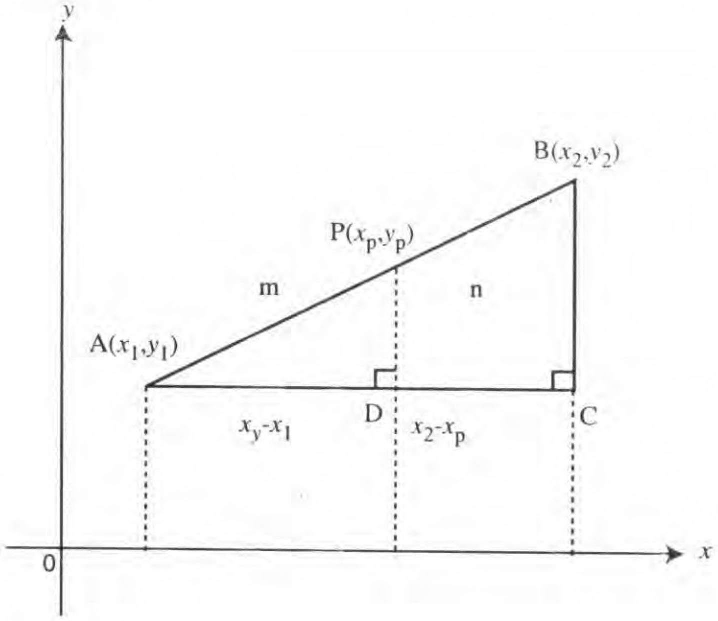


Fig. 23.2

Since $\frac{AP}{PB} = \frac{m}{n}$

then $\frac{AD}{BC} = \frac{m}{n}$

or $\frac{x_p - x_1}{x_2 - x_p} = \frac{m}{n}$

$nx_p - nx_1 = mx_2 - mx_p$

$(m + n)x_p = mx_2 + mx_1$

$x_p = \frac{mx_2 + nx_1}{m + n}$

similarly, $y_p = \frac{my_2 + ny_1}{m + n}$

So, the coordinates of the point $P(x_p, y_p)$ which divides the line joining $(x_1,y_1)$and (x_2, y_2) in the ratio m:n are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$. When m = n, the point P becomes the mid-point of AB.

Example 23.1

If A is (-5, 6) and B is (3, -7), find the point P(*x*, *y*) which divides AB in the ratio 2:1.

Solution

P coordinates = $\left(\frac{2(3) + 1(-5)}{2 + 1)}, \frac{2(-7) + 1(6)}{2 + 1)}\right)$; m = 2, n = 1

$= \left(\frac{6 - 5}{3}, \frac{-14 + 6}{3}\right)$

$= (\frac{1}{3}, \frac{-8}{3})$

Example 23.2

State the coordinates of the mid-point of the line joining the points S(2a, b) and T(a, 2b).

Solution

Here m = n, thus our formula becomes P $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= P\left(\frac{1}{2}(2a + a), \frac{1}{2}(b + 2b)\right)$

$= P\,(\frac{3a}{2}, \frac{3b}{2})$

23.5 Distance between two points

If A, B in Fig. 23.3 have coordinates (x_1, y_1), (x_2, y_2) respectively, then by Pythagoras' theorem;

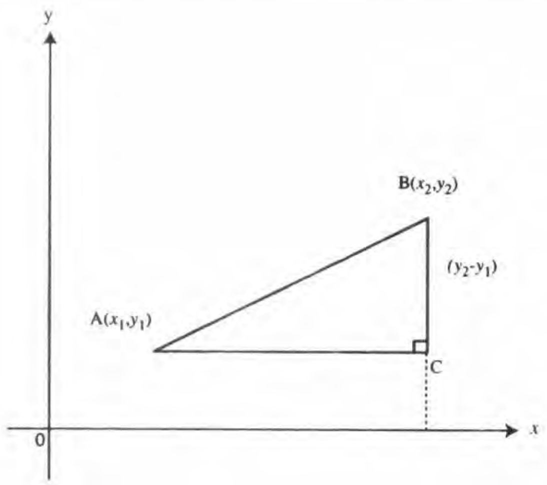


Fig. 23.3

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

So, AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Since the differences $x_2 - x_1$, $y_2 - y_1$ are squared, it does not matter which point is taken (x_1, y_1) or (x_2, y_2). However, it is important that the signs are correct.

Examples 23.3
Find the distance between the points (-5, 6) and (3, -2)

Solution

$$\text{Distance} = \sqrt{(-5-3)^2 + (6+2)^2} = \sqrt{64+64}$$

$$= \sqrt{128} = 8\sqrt{2}$$

Exercise 23.1

1. For each pair of points, find the directed length of P_1P_2.
   a) P_1 (-3, 4), P_2 (5, 4)
   b) P_1 (6, 3), P_2 (-4, 3)
   c) P_1(5, 21, P_2(5, 8)
2. Find the distance between the following pairs of points
   a) (2, 1), (8, 7), b) (-3, -2), (-5, -8), c) (0, -2), (5, -6).

It is also easy to derive the formula for the distance between two points (a_1,b_1) and (a_2,b_2) from Fig. 23.4.

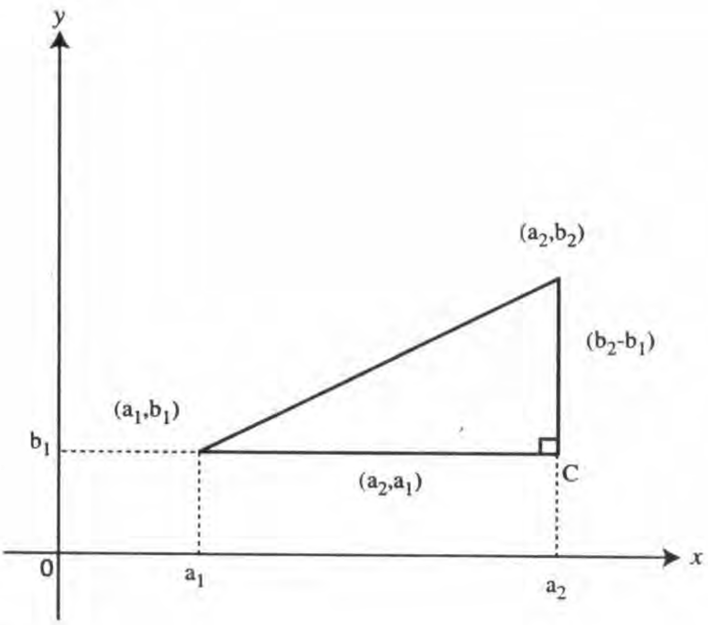


Fig. 23.4

You see that the distance you want is the length of the hypotenuse of the right-angled triangle. So by the theorem of Pythagoras, distance between (a_1,b_1) and (a_2, b_2) in Fig. 23.4

$$= \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}$$

This formula is valid, regardless of the position of the two points. The length of the horizontal length of the triangle is either $a_2 - a_1$, or $a_1 - a_2$, whichever is the positive number, but the squares of these expressions are equal. The same is true for the vertical lengths. Example 23.4 below will make this point clearer.

Example 23.4

What point of the x-axis is equidistant from (0,1) and (3, 2)?

Solution

If you call the unknown point (x, 0), the equation it satisfies is

$$\sqrt{(x-0)^2+(0-1)^2} = \sqrt{(x-3)^2+(0-2)^2}$$

Squaring both sides gives $x^2 + 1 = (x-3)^2 + 4$, from which $x = 2$. Thus, the required point is (2, 0). Try these exercises below.

Exercise 23.3

1. For each of the following pairs of points in the xy-plane, find the distance between the two points.

   a) A(1,2) and B (5, 2), b) C (3, 4) and D (7, 1), c) E (-2, 3) and F (4, 3), d) G (-6, 1) and H (6, 6).

23.6 Summary and conclusion

What you have learned in this unit centres around Cartesian coordinates, line segment division in a given ratio as well as distance between two points. It has served to introduce you to the coordinate geometry.

You also learnt how to use formula in finding the distance between two points. You need to be aware, however, that coordinate geometry is the fundamental background to other aspects of geometry. The units that follow shall build upon the introduction.

23.7 Tutor-marked assignment

1. Find the equation of a line AB, which passes through the points (2,2) and (-7, -6).
2. Find:
   i) the equation of the perpendicular from (3,7) to the line $y + 3x = 6$;
   ii) the coordinate of the foot of this perpendicular;
   iii) hence, find the distance of the point (3,7) from the line $y + 3x = 6$.

23.8 References

1. Backhouse, J. K. and S.P.T. Houldsworth (1991), *Pure Mathematics*, Longman Addison Wesley Limited England, pp, 1-18
2. Perry, O. and Perry, J. (1984), *Mastering Mathematics*, Macmillan Press Limited, pp 204-236.

Unit 24 Representation of statistical data

Contents

24.1 Introduction

When data are collected and put in numerical forms, they do not seem to be meaningful until they are summarised into tables, charts or diagrams. It is only in these forms that they can be useful in decision-making.

In this unit, you will study how to construct tables, charts and diagrams from a given set of data. This unit also covers the most elementary things you should know about collection of numerical data.

24.2 Objectives

By the end of this unit, you should be able to:

i) describe the various techniques used for presentation of data;
ii) draw bar charts, histograms and pie charts.

24.3 Presentation of statistical data

The collection of statistical data has various practical purposes. It is useful in investigating the relationship between *cause* and *effect*. It can also be used in making predictions.

Statistical data are obtained from many sources, such as questionnaires and experiments. Such data are presented in different ways. The first method of presentation with which you should be concerned in this unit is **tabulation**.

Tabulation

Tabulation is probably the easiest way of presenting data. The raw scores are simply given in form of tables, and in a more ordered manner.

Example 24.1

Raw score

67	76	81	71	61
73	74	78	79	65
78	76	80	82	85

Solution

Scores	61	65	67	71	73	74	76	76	78	78	79	80	81	82	85

Table 24.1

Tabulation makes interpretation of data easier and quicker. Here, you can quickly discover that 61 is the least score and 85 is the highest score. The method fails to be advantageous when there are many scores, particularly repeated scores.

Exercise 24.1

Present the data below in a tabular form

a)	11	8	12	9	8	7
	10	9	9	11	12	8
	7	11	10	13	10	7

b)	60	72	84	60
	73	74	83	61
	71	79	86	59
	78	80	71	42

Measurement concepts in data

The classification of a group of objects may be either in terms of **variables** or **attributes**, i.e. they may be measurable or non-measurable. Examples of measurable characteristics are height, income, I.Q and examination marks. These are called variable characteristics because they vary fairly continuously. Examples of non-measurable characteristics are sex, marital status and hair colour. These are termed attributes because they are distinguished by qualitative rather than measurable differences between individuals.

Where a group to be classified by some measurable characteristics is large, there is need to know whether the variable is **discrete** or **continuous**. This is very important because it determines the way in which the range of values which the variable can take should be divided.

Discrete data : Mark (%)	**Continuous data: Height (cm)**
1 - 25	> 60 but < 66
26 - 50	> 66 but < 72
51 - 75	> 72 but < 78
76 - 100	> 78 but < 84

Table 24.2

The values taken by some discrete variables may be too few to be classified by intervals. The appropriate method in such a case is then to determine each class by means of the actual values which individuals take.

Diagrams and charts in statistics

After tabulating a set of data, our next concern is the possibility of putting the data in some pictorial or diagrammatical form. Diagrams, like statistical tables, are designed not only to 'catch the eye', but also to convey information.

Pictorial diagrams

The simplest and most obvious way of presenting information is by means of pictorial figures or designs which are directly related to the items with which the statistical data are connected. Pictorial diagrams are popularly used by journalists (in newspapers) and advertisers to show relationship between variables. They convey broad information to be eyes, even though, the details may be absent.

When pictures are used to represent a statistical information, such a representation is called a **pictogram.** An example is shown in Fig. 24.1

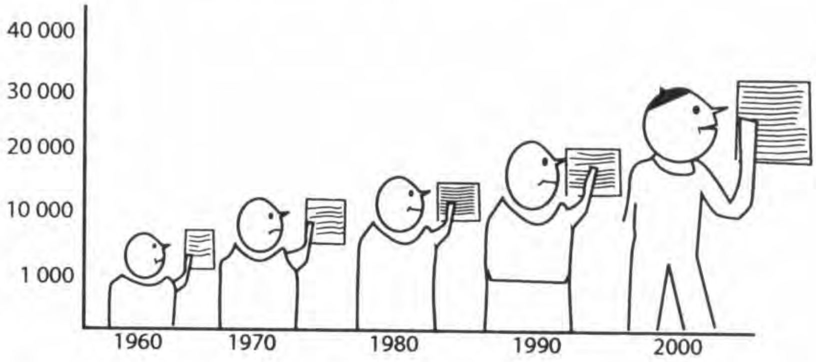


Fig. 24.1 Average lecturers' salary (₦) in the university

Bar charts

These are the most commonly used forms of diagrams. In practice, frequency diagrams are not often used as they are not particularly 'eye-catching' and it is more common to see a bar chart used where the numbers are treated in a qualitative manner. The information on bar charts is represented on bars without the axes intersecting each other. Fig. 24.2 is the bar chart of a farmer's harvest. The data are first presented in a table before the bar chart is drawn.

Year	1991	1992	1993	1994	1995
No. of harvested bags	10	20	30	40	50

Table 24.3 A farmer's harvest

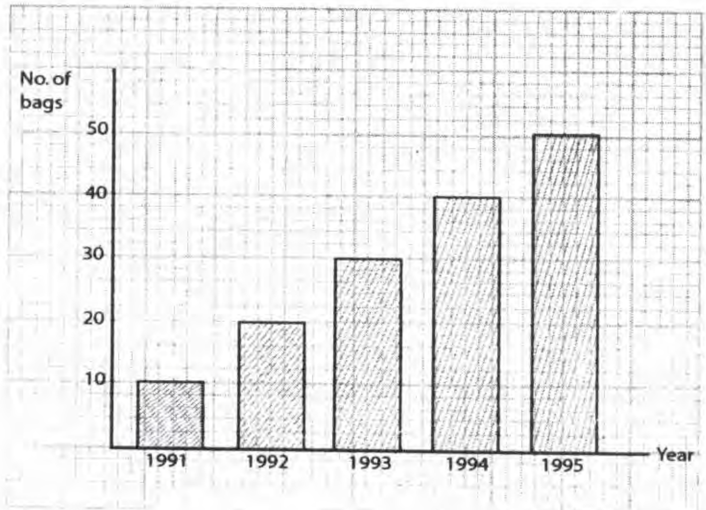


Fig. 24.2

Histograms

These are used only for quantitative continuous data. A proper continuous mathematical scale must be used along the bottom. Here there are no gaps between the bars, but they are drawn up to class boundaries.

The histogram is a diagram which looks like a graph where the data are presented in a tabular form with the axes intersecting at right angles. To illustrate this, consider the data presented in Table 24.4.

No. of children	0	1	2	3	4	5	Greater than 5	Total
No. of families	253	1 921	2 235	1 528	326	128	73	6 464

Table 24.4 Size of families

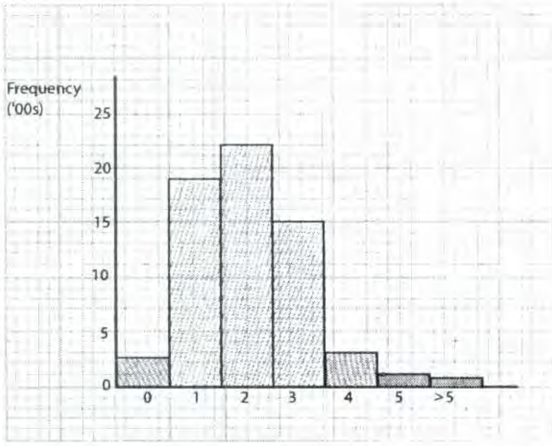


Fig. 24.3

Pie charts (Circular diagrams)

The total number of values in a set of numerical data can be represented by the area of a circle, while the size of each class is proportional to the angle of the sector representing it, i.e. the slice of the pie.

Example 24.2

A student working during the long vacation at a small factory in Abuja counted the number of staff in the various departments as shown in Table 24.5. Construct a pie chart to display this information.

Group of staff	Number
Management and sales	16
Clerical	30
Technical	24
Production	193
Others	7

Table 24.5

Solution

The circle will be drawn first and its centre marked. The size of the circle depends on the space available and the purpose for which the diagram is required, but a radius or 3 cm or 4 cm is suitable for a notebook.

Next, you calculate the angles of the sectors. The total number of staff is 16 + 24 + 30 + 193 + 7 = 270, and this is represented by the whole circle of 360° . Each person is therefore, represented by an angle of $\frac{360°}{270}$ or $\frac{4°}{3}$.

In this example, the angles can be calculated easily without finding the percentages, but in general, each 1% is equivalent to 3.6, i.e. $\frac{360°}{100}$.

The angles of the sectors are found as follows:

Management : $16 \times \frac{4^\circ}{3} = 21.3^\circ = 6\%$

Technical : $24 \times \frac{4^\circ}{3} = 32^\circ = 9\%$

Clerical : $30 \times \frac{4^\circ}{3} = 40^\circ = 11\%$

Production : $193 \times \frac{4^\circ}{3} = 257.3^\circ = 71.5\%$

Others : $7 \times \frac{4^\circ}{3} = 9.3^\circ = 2.5\%$

The angles are converted to the nearest degree, but to make the total correct and to minimise the relative error, the largest can be made up to 258°. They are measured with a protractor, and each sector is labelled as shown in Fig. 24.4. Use different kinds of shading or clearing to increase the visual impact of the diagram.

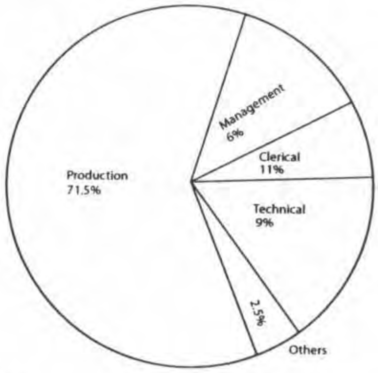


Fig. 24.4

Exercise 24.2

1 Fig. 24.5 shows the temperature of a patient recorded every hour.

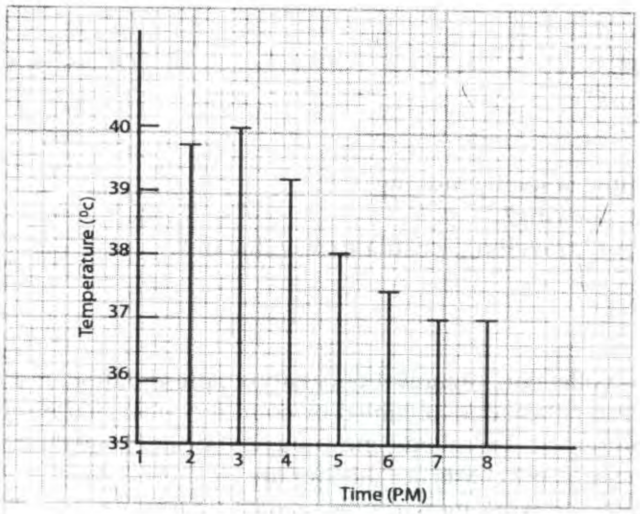


Fig. 24.5

a) At what time was the highest temperature recorded?
b) What was the patient's temperature at 6 p.m.?
c) During which hour did the temperature rise the most?
d) During which hour did the temperature fall the most?

2 The work distribution for a class test is shown in Fig. 24.6. The marks range from 3 to 9. How many students took the test?

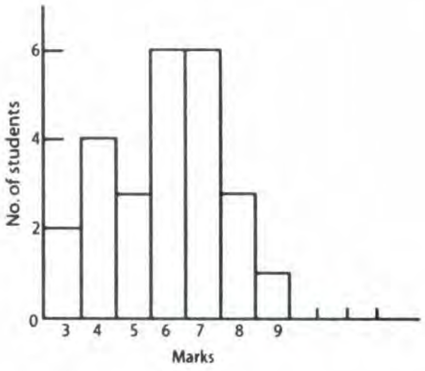


Fig. 24.6

3 The pie chart in Fig. 24.7 represents 24 hours in the life of a student.
   a) What fraction of the time is spent sleeping?

Fig. 24.7

   b) What percentage of the time is spent studying?
   c) How much time is spent studying?
   d) If 1 hour 20 minutes is spent travelling, calculate the value of x.

24.4 Conclusion

The subject of statistics embraces collection, organisation, analysis, presentation and interpretation of data for a purpose. While statistical data obtained by counting are called discrete data, statistical data obtained by measurement are called continuous data. Statistical data can be represented by pictograms, bar charts, pie charts and histograms.

Go through the examples again and work more problems relating to statistical data. This will help you in subsequent units.

24.5 Summary

In this unit, you have learnt that:

i) classification and tabulation of data form the basis for reducing and simplifying the details given in a mass of data into such a form that the main features may be brought out to make the assembled data easily understood;

ii) information collected by counting is discrete, and usually takes integer values, while continuous data are collected by measurement and include such information as the height and weight of people in a group;

iii) statistical data can be represented by pictograms, bar charts, piecharts and histograms.

24.6 Tutor - marked assignment

Fig. 24.8 shows the division of the workforce of a factory.

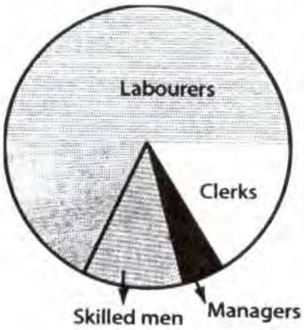


Fig. 24.8

a) In its simplest form, what fraction of the workforce are skilled men?
b) What percentage of the workforce are labourers?
c) Give the ratio of managers to clerks in its simplest terms.
d) If the factory employs 180 people, how many people are managers?

24. 7 References

1 Adamu, S. O. and Johnson, T. L. (1975), *Statistics for Beginners:* Onibonoje Press and Industries Limited, pp 1 – 40.

2 Aderogba, K. A. etal., A. O., (1991), *Senior Secondary Mathematics Book 1,* Longman Nigeria Limited. pp 63 – 71.

Unit 25 Graphical representations in statistics

Contents

25.1 Introduction

In the last unit, you began your study of descriptive statistics. You should recall that the general purpose of descriptive statistics is to summarise data. Two important elements of descriptive statistics are the arrangement and the display of numerical information. Raw numbers alone provide little insight into the underlying pattern of the data from which conclusions are to be drawn. This unit introduces various techniques that you can use in arranging, sorting and depicting relevant data. Now, having guided you through what you will meet in this unit look at the specific unit objectives below.

25.2 Objectives

By the end of this unit, you should be able to:

i) construct a frequency distribution table;
ii) draw a histogram, frequency curve and cumulative frequency curve (ogive) for grouped data;
iii) analyse data represented in histograms.

25.3 Frequency distribution

A better way of presenting data takes care of repeated scores, that is, a score and its number of occurrence, or frequency.

The importance of this method is seen more clearly when data consists of many scores with many repeated scores, such as the following:

76	79	84	79
67	84	79	67
84	76	78	67
76	80	78	67

The frequency distribution table is as follows:

Score	Frequency
67	5
76	4
78	2
79	4
80	2
81	3

It is however more convenient to use frequency tallies when the scores are very many as shown in the following table:

Score	Tally	Frequency
67	卌	5
76	||||	4
78	||	2
79	||||	4
80	||	2
84	|||	3

The presentation of data as shown in the last table has some fundamental problems. The table may be unnecessarily too long and may be very difficult to identify the sequencing of the scores before tallies are done. The latter may be compounded by the fact that many scores may be missing. The next method you will consider takes care of these weaknesses.

Grouped frequency distribution

Whenever you are constructing frequency distribution, the usual practice is to:

a) Decide on the groupings. To condense the large number of observations, all the values that occur within a particular interval are grouped. These groups or intervals are called **classes**. For example, below are two classes of a group.

40 – 44
45 – 49

b) Tally the raw data into the classes.

c) Count the number of tallies in each class.

Now see an example presented below.

Class interval	Tally	Frequency
55–59	//	2
60–64	//	2
65–69	/	1
70–74	卌 //	7
75–79	卌 卌 卌 /	16
80–84	卌 //	7
85–89	/	1

This is a more acceptable frequency distribution table.

In grouped data, the number of scores in each class is called the **width** of the class. In the example given, the width of the class is 5. Each class has a **lower** and an **upper limit**. The first class has 55 and 59 as its lower and upper limits respectively. Unless you are specifically restricted, the choice of class width is a matter of convenience. It is however to be noted that the number of classes should not be too many. If the latter is not taken care of, the purpose for using the groupings will be defeated.

You should now practice with the following exercises.

Exercise 25.1

1. Present the data below in a tabular form

11	9	12	9	8	7
10	8	9	11	12	8
7	11	10	13	10	7

2. Present the data in question (1) as a frequency distribution. What is the advantage of frequency distribution over ordinary tabulation?
3. With the use of tallies, form the frequency distribution for the following data:

14	10	11	14	13	15	12	14	12	11	14
15	14	12	15	11	14	15	15	15	16	12
12	16	13	14	10	15	12	13	12	14	11
17	14	15	16	13	11	14	10	15	12	17
17	17	12		14	15	12	16	13	11	15

Using a class of width 3, form the frequency distribution for this set of data.

25.4 Graphical representation of data

According to the Chinese proverb that 'a picture is worth a thousand words', pictures of a special variety are also used extensively to help people such as hospital administrators, business executives and consumers to get a quick grasp of statistical reports. Such pictures of course, are what you call graphs or charts. The three graphic forms commonly employed to portray a frequency distribution are the **histogram, frequency polygon**, and **cumulative frequency polygon**. You will study all the three graphical forms in this unit.

Line graph

A line graph is a chart which displays how the different components of a set of data vary with time, say.

Example 25.1

The table below shows the variation in the monthly earning of a trader.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Earning (₦)	150	120	160	180	140	200	220	200	180	160	140	170

Draw a line graph for the data.

Solution

A line graph showing variation of monthly earnings of the trader with time is shown in Fig. 25.1.

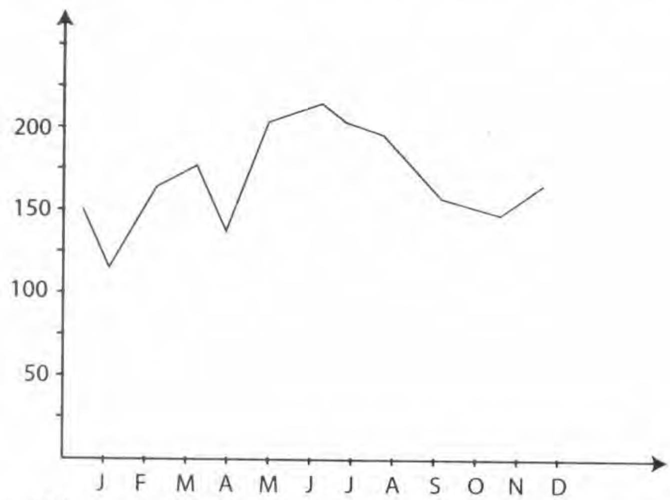


Fig. 25.1

Histogram

A histogram is a graphical display of a frequency distribution. It is one of the most easily interpreted charts. Therefore, the chart of a frequency distribution is called a **histogram**.

For this diagram the values of the variable are scaled along the x-axis and the frequencies along the y-axis. A histogram is constructed as follows:

Example 25.2

Twenty pupils of a school went to a farm to pick oranges. The number of oranges picked by each pupil was recorded as follows.

1,2, 5, 3, 0, 4, 3, 4, 1, 2, 1, 2, 1, 3, 0, 2, 1, 2, 3, 2.

and then summarised into a frequency table as follows:

No. of oranges (x)	Counts	Frequency
0	||	2
1	卌	5
2	卌 |	6
3	||||	4
4	||	2
5	|	1

Solution

Step 1: Draw a vertical line at the lower boundary of each class interval up to the height representing its frequency as shown in Fig. 25.2.

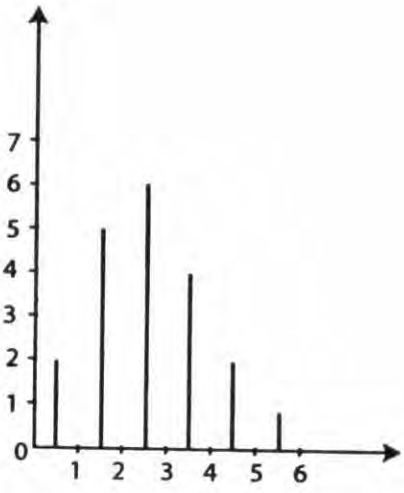


Fig. 25.2

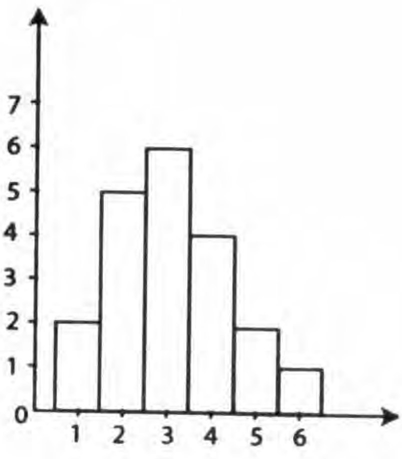


Fig. 25.3: Histogram of number of oranges

Step 2: Mark a horizontal line representing the width of each class interval on the top of each vertical line drawn in step 1, above. You should produce the vertical line for each class interval, where necessary, to get a complete rectangle.

These two steps lead to a series of connected rectangles and the whole diagram is called a histogram. Where the frequency has no class interval, any constant width is taken with each value as the mid-point. See the graph in Fig. 25.3.

You have dealt with data having constant class intervals in which case the heights of the intervals represent the frequencies. There are situations where the class intervals vary, then the height of the bar must be adjusted so that the area represents the frequency.

Frequency polygon

A frequency polygon is a line graph of a frequency distribution obtainable from a histogram by joining the mid-points of the tops of the rectangles in the histogram. The frequency polygon example is given below. The diagram you obtain is what is called **frequency polygon**.

Score	Frequency
0–9	4
10–19	7
20–29	9
30–39	12
40–49	8
50–59	6
60–69	4

The mid-point values are used for the data as in Fig. 25.4.

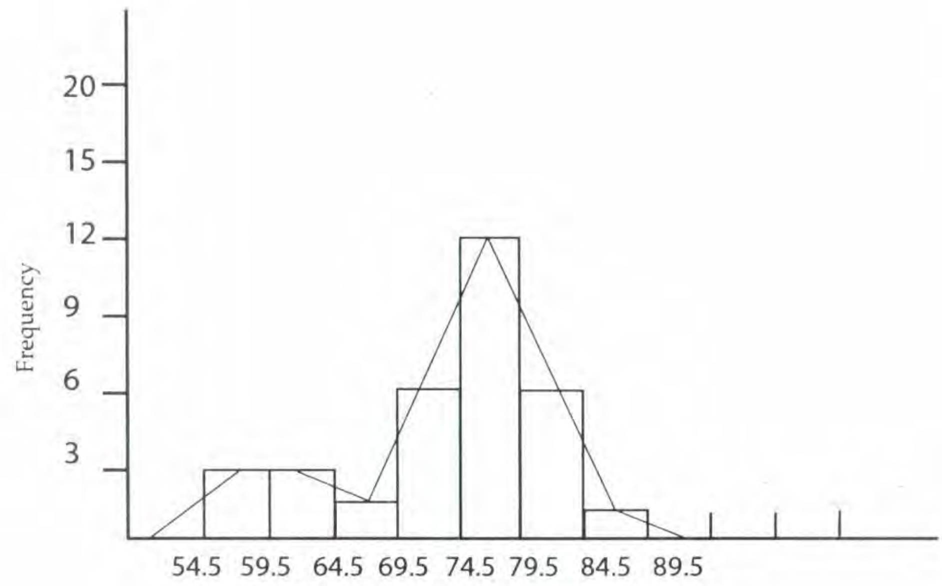


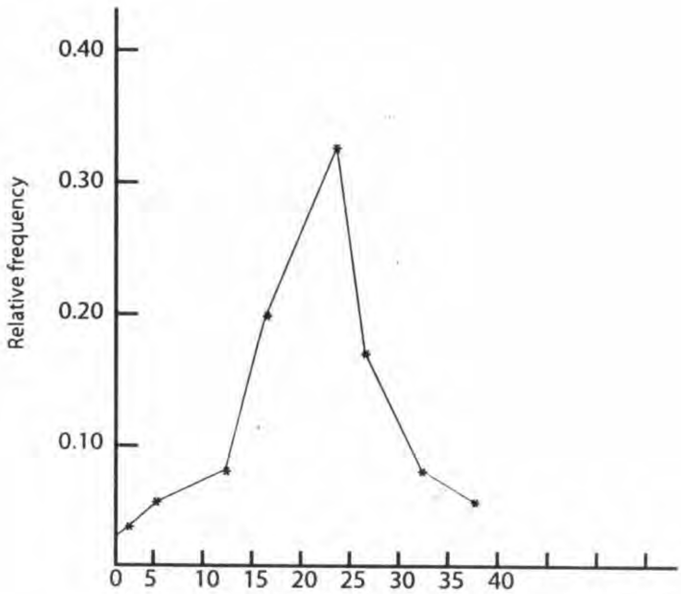
Fig. 25.4: Frequency polygon

Relative frequencies

The sum of the numbers in the second column of the following table is 50 (i.e. total frequency). In practice, it is always better to work with sets of frequencies whose sum is 1. When frequencies are so defined, they are called **relative frequencies**.

	Distance (km)			
Class interval (x)	Absolute frequency	Absolute cumulative frequency	Relative frequency	Relative cumulative frequency
0–5	2	2	0.04	0.04
5–10	3	5	0.06	0.10
10–15	4	9	0.08	0.18
15–20	10	19	0.20	0.38
20–25	17	36	0.34	0.72
25–30	8	44	0.16	0.88
30–35	4	48	0.08	0.96
35–40	2	50	0.04	1.00
	50		1.00	

The tabulation in the table above using the fourth column is represented graphically in Fig. 25.5.



Distance in km
Fig. 25.5

Therefore, you could conclude that a relative frequency for a class is the actual frequency of the class divided by the total frequency.

Tabulation of relative frequencies is very important in that it helps to bring the shapes of many frequency distributions to the same scale. This makes it easier for such frequency distributions to be compared and it gives an important introduction to the topic to be discussed in subsequent units.

Cumulative frequencies

Similarly, if the absolute frequencies or the relative frequencies are added successively, as is done in the table on page 185, the result is called **cumulative frequency distribution** in the case of absolute frequencies or called **distribution function** in the case of relative frequencies.

The graph of the cumulative frequency curve against the upper class boundary is called a **cumulative frequency** curve or **ogive**. Study the following example:

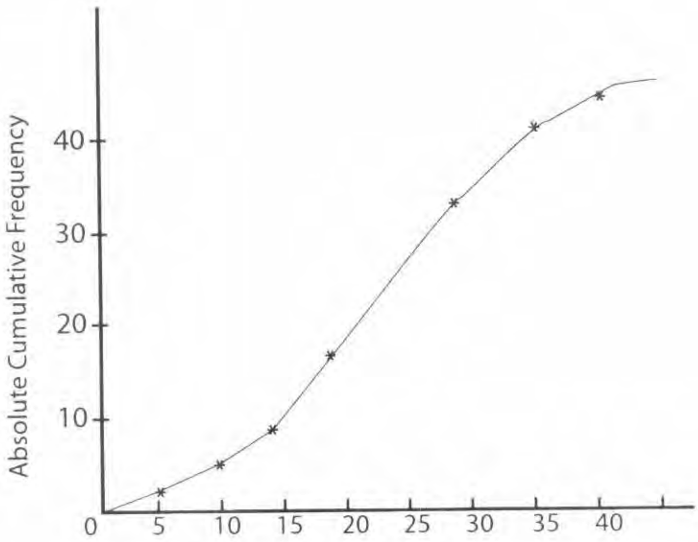


Fig. 25.6: Cumulative frequency curve of distance

You are now adequately equipped to practice with the following exercise.

Exercise 25.3

1. The following table gives the distribution of 100 families according to the number of children:

No of children	0	1	2	3	4	5	6	7
No of families	5	10	16	25	19	12	8	4

Draw the histogram and the frequency polygon.

2. The following is a record of marks obtained by a group of 40 students out of a maximum of 10 marks.

Marks	4	3	5	8	4	8	5	3	0	1	8	8	9	7	7	2	9	4	9	0
No of students	1	3	0	2	5	1	4	3	3	8	5	4	0	6	6	1	5	2	4	3

a) Tabulate these as a frequency distribution using (i) no class interval (ii) class interval of 2 marks.

b) Draw the histogram and frequency polygon.

3. T͏he number of bad oranges (x) in baskets of equal number of oranges picked in a farm for 3(·askets are:

4	3	0	3	1	3
0	1	1	4	3	2
5	2	2	1	1	0
1	0	2	0	2	1
5	3	1	0	2	4

a) Construct a frequency table for $x = 0, 1\ 2, 3, 4, 5$ and for the number of baskets having bad oranges.

b) Obtain the cumulative frequency distribution are draw the graph.

25.5 Conclusion

In this unit, you have studied how to summarise data in terms of frequency distributions and graphical presentations of data. The three graphical forms commonly employed to portray a frequency distribution are the histogram, the frequency polygon, and the cumulative frequency polygon.

In brief, in this unit you have learnt the following:

i) frequency distribution: the tabulation of a given collection of data in an order with frequency attached to each value or group of values;

ii) the chart of a frequency distribution is called a histogram;

iii) the graph of a frequency is called frequency polygon;

iv) the relative frequency for a class is the actual frequency of the class divided by the total frequency; and

v) a graph of the cumulative frequency distribution for any set of data is called cumulative frequency curve or ogive.

25.6 Summary

In this unit you have learnt different types of representing the frequency distribution graphically. These were illustrated with a number of examples for you to understand properly. You should go through the examples again since these will help you to visually represent data in various forms, that are easy to read and analyse, in subsequent topics in statistics.

25.7 Tutor-marked assignment

The following is the frequency distribution of pocket money withdrawn by 100 students during the last mid-term holidays. Pocket money, x, is measured to the nearest naira.

x	10–11	12–13	14–15	16–17	18–19	20–23	24–27	28–31
y	4	11	20	30	19	10	4	2

Draw a) the histogram
b) the frequency polygon
e) the cumulative frequency curve (ogive).

25.8 References

Adamu, S.O. and T.L. Johnson (1975), *Statistics for Beginners*, Onibonoje Press and Industries Limited, pp. 6–29.
Aderogba, K., A.O. Kalejaiye and A.O. Ogum, *Book 1*, Longman Nigeria Plc, pp. 63–71.
Mason, R.D. and D.A. Lind (1983), *Statistics: An Introduction*, Harcourt Brace Jovanovich Inc., New York, pp. 12–45.

Unit 26 Probability

Contents

26.1 Introduction

The purpose of this unit is to examine ways of calculating the probability that some events will occur. For instance, you may be interested in determining the probability that the incumbent governor in your state will be reelected the second time or in the probability that fewer than 80% of first-time traffic offenders will be fined a second time for this offence. Whatever your purpose, the ability to calculate probability will be a valuable tool in your everyday decision making. Look at the following examples of statements of probabilities:

i) 'Bello is more likely to be the head boy next year, than Bala.'
ii) 'It will probably rain today.'
iii) 'The Nigerian football team has a better chance of winning the next World Cup than a South African one.'

All these statements express an opinion that one outcome is more likely than another, but in none of them is there any attempt to say by how much.

26.2 Objectives

By the end of this unit, you should be able to:

i) define probability;
ii) explain how the probability of an event happening is measured;
iii) recognise whether or not events are selected in any way;
iv) assess the likelihood of events occurring;
v) calculate probabilities using the rules of addition and multiplication.

26.3 Definition of probability

Many intuitive ideas of chance and probability are based on the idea of symmetry. Consider the following questions:

i) If you toss a coin repeatedly, how many times will you get a head?
ii) If you roll a die how often will you get a four?
iii) If you roll two dice several times, how often will you get two fives?

For the second question, your answer should be about one in six times provided the die is a fair one. Another way of expressing this is to say that the probability of obtaining 4 is

$$\tfrac{1}{6} \Rightarrow P(4) = \tfrac{1}{6}.$$

The answer is dependent on the idea of symmetry. That is, every possible outcome (namely 1, 2, 3, 4, 5, and 6) is equally likely to occur. So the probability of any one score must be $\frac{1}{6}$.

A number that measures the likelihood that a particular event will occur is called **probability**. Three key words are used in the study of probability: **experiment, outcome** and **event**. While they commonly appear in your everyday language, in statistics they have specific meanings. These will be illustrated bellow:

Experiment

By definition an experiment is the observation of an activity or the act of taking some type of measurement. This definition is more general than the one used in the physical sciences, where we picture researchers using test tubes or microscopes in experiments. In statistics an experiment has two or more different possible results and it is uncertain which of them will occur.

Outcome

This is a particular result of an experiment. For example, the tossing of a coin is an experiment. You may observe the coin tossing, but you will be unsure whether it will come up with heads or tails. If a coin is tossed, one particular outcome might be a 'head' or, the outcome might be a 'tail', when one of the experiment's outcomes is observed, and you call it an **event**. You have noticed that an event is not always simply an outcome.

Probabilities may be expressed as fractions, decimals or percentages. A probability is always between zero and one, inclusive.

Example 26.1

If a fair coin (one that is properly minted and that cannot stand on an edge when tossed) is tossed once and then tossed twice, all the possible outcomes will be head (H) or tail (T). For simplicity, you can put these as (H,T). If tossed twice the list of outcomes will be:

{(H,H), (HT), (TH), (TT)}.

Exercise 26.1

A politician has just completed a major speech at Abuja. He receives six e-mail messages commenting on it and he is interested in the number of writers who agree with him.

(a) What is the experiment?

(b) What are the possible outcomes?

(c) Describe one possible event that might occur.

26.4 Sample and population

The set of observations taken from some sources for the purpose of obtaining information from such observations is called a sample while the entire source of those observations is called a **population**.

Statistical methods may be described as methods for drawing conclusion about population by means of sample. The words *statistics* is often used in place of statistical methods. The part of statistics concerned with collecting and summarising data is usually called **descriptive statistics** and the part concerned with reaching conclusion about the sources of the data is called **statistical inference**.

In tossing a coin the outcomes are Head (H) or Tail (T). If a coin is tossed twice, the same space $S = \{HH, HT, TH, TT\}$. That is

1st coin	**H**	**T**
2nd coin H	HH	TH
T	HT	TT

The total outcome of the experiment is called sample space, S and each of the outcome is referred to as an event. Similarly, if a die is cast once, the outcomes are 1 or 2 or 3 or 4 or 5 or 6. Each of the outcomes is called an **event** and together {1, 2, 3, 4, 5, 6} is called a **sample space**.

If a die is cast twice or two dice at a time, the outcomes are

1st die		1	2	3	4	5	6
2nd die	1	(1,1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Example 26.2

If two (2) dice are cast, what is the probability of obtaining three (that is sum of the numbers to be three)? The possible outcomes are (2, 1) and (1, 2). That is, two events out of 36 events. Hence

$\frac{2}{36} = \frac{1}{18}$.

26.5 Axioms of probabilities

A probability measure 'P' is a real valued set of function defined on a sample space S that satisfy the following:

1. $0 \leq P(A) \leq 1$ for every event A
   $P(A) \geq 0$
   $P(A) \leq 1$
   $P(0) = 0, (PA) = 1$
2. $P(S) = 1$, i.e, the probability of sample space is 1
3. $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

Mutually exclusive events

If A and B are mutually exclusive (disjoint) of experiment or events, the probability that A or B will occur is the sum of the probabilities according to Axiom (3) above, i.e.

$$P(A \text{ or } B) = P(A) + P(B)$$
$$P(A \cup B) = P(A) + P(B)$$
$$P(A + B) = P(A) + P(B)$$

On the other hand, if the two outcomes are not mutually exclusive, the probability that A or B or both will occur is the probability of A + probability of B minus the probability of A and B that is,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A + B) = P(A) + P(B) - P(AB)$$

Independent events

If the probabilities of 'n' independent events $A_1, A_2, \ldots$ An are $P_1, P_2, \ldots$, Pn respectively, then probability that all will occur is the product $P_1 \times P_2 \times \ldots \times P_n$. In particular, for two independent events A and B this is written as $P(A \cap B) = P(AB) = P(A). P(B)$ and for 'n' independent event A_1, $A_2 \ldots A_n$ we have $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_2). P(A_2) \ldots P(A_n)$

Operation on sets

Property I

If $B \subseteq A$ then

1. $P(A-B) = P(A) - P(B)$
2. $P(B) \leq P(A)$

Property II

For every event A of the sample space *S*

$0 \leq P(A) \leq 1$

Property III

$P(\phi) = 0$

Example 26.3

Let A and B be independent with $P(A) = \frac{1}{2}$ and $P(A \cup B) = \frac{2}{3}$

Find P(B) and $P(A \cap B)$

Solution

If A and B are independent events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A) . P(B)$

$\therefore \quad P(A \cup B) - P(A) = P(B) - P(A) . P(B)$

$P(A \cup B) - P(A) = P(B) [1 - P(A)]$

(i) $P(B) = \frac{P(A \cup B) - P(A)}{1 - P(A)}$

$$= \frac{\frac{2}{3} - \frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{2}}$$

$$= \frac{1}{3}$$

(ii) $P(A \cap B) = P(A) \times P(B)$

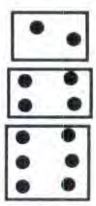
$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

Example 26.4

What is the probability that an even number will result from one roll of a single die?

Solution

The event (an even number) is composed of three outcomes, namely;

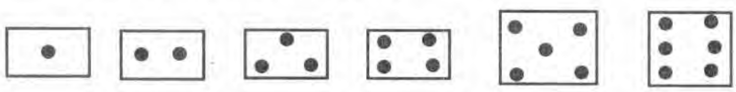


The outcome of a 2 is event A.

The outcome of a 4 is event B.

The outcome of a 6 is event C.

The six possible outcomes are



The probability of each of the outcomes (2, 4, 6) is $\frac{1}{6}$. The probability of the event 'the outcome is an even number' is found by adding the three probabilities. That is, $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$ or 0.50. In symbols, if A stands for the outcome of a 2, B represents the outcome of a 4, and C outcome of a 6, the probability of an even number appearing is computed by

$$\begin{aligned} P(\text{even}) &= P(A) + P(B) + P(C) \\ P(A \text{ or } B \text{ or } C) &= P(A) + P(B) + P(C) \\ &= \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} \\ &= \tfrac{3}{6} = 0.50 \end{aligned}$$

Exercise 26.2
Two hundred randomly selected prisoners in cell block M are surveyed and classified by type of crime committed.

Type of crime	Number
Murder	48
Armed robbery	42
Rape	101
Kidnapping	7
Other	2

(a) What is the probability that a particular prisoner selected in the sample is a convicted murderer?
(b) What is the probability that a particular prisoner selected is a convicted kidnapper?
(c) What is the probability that a particular prisoner selected is either a convicted murderer or a convicted kidnapper? What rule of probability was employed?

26.6 Tree diagrams

An interesting technique used to show probabilities, joint probabilities and conditional probabilities, is to plot them on so-called **tree diagrams**. These are sometimes called **decision trees** and may be used in other subjects such as business studies.

A simple way of calculating probability of the kind of experiments given above is to draw a tree diagram for each experiment.

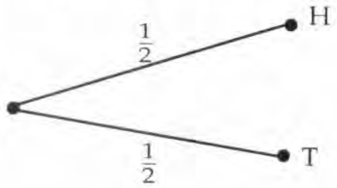
Definition

A tree diagram, as the name suggests, is a set of connected lines with each line looking like a branch of a tree. It is a device to represent the trials or stages of an experiment in a diagram. Many problems in probability can be solved with the aid of a tree diagram.

Example 26.5
Draw a tree, stating the sample points and the probabilities for each of the following experiments:
(a) Toss a fair coin once.
(b) Toss a fair coin twice.

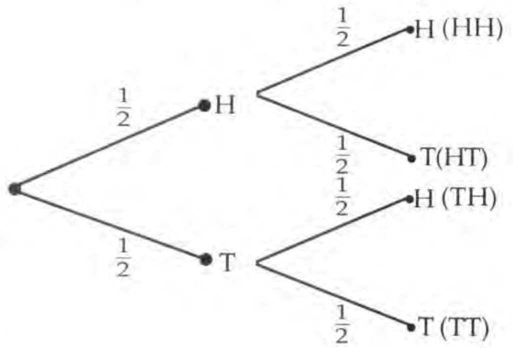
Solution

(a) Tree for tossing a fair coin once (one trial).



Sample	Probability
Point (H)	$P(H) = \frac{1}{2}$
(T)	$P(T) = \frac{1}{2}$

(b) Tree for tossing a fair coin twice (two trials).



Probability

$P(HH) = \frac{1}{4}$

$P(HT) = \frac{1}{4}$

$P(TH) = \frac{1}{4}$

$P(TT) = \frac{1}{4}$

Exercise 26.3

A committee is made up of 2 men and 4 women. Two members are selected successively at random without replacement.

(a) Draw a tree to describe the problem.

(b) What is the probability of each outcome?

26.7 Conclusion and summary

In this unit you have learned that a possible result of an experiment or game is called an outcome and each combination of outcomes is an event. The probability that an event will occur is defined as the number of outcomes favourable to the event divided by the number of possible outcomes, and the probability that event A will occur is written P(A). Other related concepts discussed include: independent events, conditional probability and mutually exclusive events.

In brief, you have in this unit, learnt the following:

(i) Concept of probability and its properties;

(ii) When the result of an experiment has no effect on the result of a second experiment, the two are said to be independent;

(iii) When two events are not independent, the probability that the second event will occur depends on whether or not the first event has occurred, and this is called conditional probability;

(iv) If only one of the possible outcomes can occur at a time, they are mutually exclusive;

(v) Special rules of addition and multiplication of probability;

(vi) Conditional probabilities are represented in a different type of diagram called a tree diagram. Every outcome of an experiment is represented by a different branch.

What you have learnt in this unit will help you later in your further study of statistics.

26.8 Tutor-marked assignment

1. Two dice are tossed six times. Find the probability (a) that 7 will show on the first four tosses and will not show on the other two; (b) that 7 will show in exactly four of the tosses.

26.9 References

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Unit 27 Differentiation of functions

Contents

27.1 Introduction

The process of finding **gradient functions** is called **differentiation**. Another name for gradient function is the **derivation** or **derived function.** The inventor of this technique is generally thought to be Sir Isaac Newton, who developed it in order to explain the movement of stars and planets. However, the German mathematician, Gottfried Wilhelm Leibuiz was the first to actually publish the idea in 1684. In this unit, you will study differentiation.

27.2 Objectives

By the end of this unit, you should be able to:

i) define such terms as constant, variable function and limit;
ii) differentiate simple functions from first principle;
iii) differentiate the product function;
iv) differentiate a function using chain rule;
v) differentiate implicit functions;
vi) differentiate some trigonometric functions;
vii) differentiate the quotient function.

27.3 General concepts associated with differentiation

You may have come across some words like constant, variable, function and limit or limiting value. Let us recall their definitions again.

Constant: This is a quantity whose value remains unaltered under a mathematical operation. For example, all numbers are constants.

Variable: A variable is a quantity, which in the cause of a mathematical operation, may assume different values. If the variable is one which may assume any value assigned to it, it is called an **independent variable.** If it assumes different values merely on account of, or as a result of changes in value of another variable, it is called a **dependent variable,** e.g. the area of a circle which depends on the radius.

Function: A function of x (or of any other quantity) is a quantity the value of which depends on the value of x, e.g. cos x, sin x and log x are all functions of x. The symbols, F(x) and Q(x) are usually employed to denote functions of x.

Limit or limiting value: Consider the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ The sum of this series is always less than 2 no matter how many terms you consider. But by taking a sufficiently large number of terms, you can make the difference between 2 and the sum of these terms as small as you please, e.g. the limit of the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots$

$= \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^n} + \ldots..$ When n approaches infinity, is 2.

Given a function $y = f(x)$, the limit of $f(x)$, as x approaches a number a is written as $\lim_{x \to a} f(x)$ Also, a limit is a number which is finite. Therefore, $\lim_{x \to a} f(x) = f(a)$.

Example 27.1

Given that $f(x) = 6x^2$. Find the limit of $f(x)$ as x approaches 2.

Solution

$f(2) = 6(2^2) = 6(4) = 24$

$\therefore \lim_{x \to 2} f(x) = f(2) = 24$

Example 27.2

Given that $f(x) = x^3 - 4x + 3$. Find the limit of $f(x)$ as x approaches -1.

$\lim_{x \to -1} f(x) = f(-1) = (-1)^3 - 4(-1) + 3$

$= -1 + 4 + 3 = 6$

There is an important rule you need to know: if $f(x)$ and $g(x)$ are polynomials in x.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$,provided $g(a) \neq 0$

Example 27.3 will illustrate this rule.

Example 27.3

If $f(x) = x^2 - x + 6x$ and $g(x) = x + 3$,

then, limit $\frac{f(x)}{g(x)} = \lim_{x \to 3} \frac{x^2 - x + 6}{x + 3} = \frac{f(3)}{g(3)} = \frac{3^2 - (3) + 6}{3 + 3}$

$= \frac{9 - 3 + 6}{6} = \frac{12}{6} = 2$

Example 27.4

Evaluate the limit of $\left[\frac{2x - 1}{8x^3}\right]$ as $x \to 1$

Solution

$\lim_{x \to 1} \frac{2x - 1}{8x^3 - 1} = \frac{2(1) - 1}{8(1)^3 - 1} = \frac{2 - 1}{8 - 1} = \frac{1}{7}$

Note that to evaluate the limit of a quotient $\frac{f(x)}{g(x)}$ as x approaches a when there is a common factor, you need to do the following:

i) divide the numerator and denominator by the common factor.
ii) evaluate the simplified expression at $x = a$.

Example 27.5

Find $\lim_{x \to 2} f(x) = \frac{x^2 - 4}{x - 2}$ $(x \neq 2)$

Solution

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x^2 - 2)(x - 2)}{x - 2}$

$= \lim_{x \to 2} x + 2 = 4$

Example 27.6

Find $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$

Solution

$$\lim_{x \to 2} \frac{(x-2)(x-3)}{x-2} = \lim_{x \to 2} x - 3$$

Exercise 27.1

Evaluate the following limits

a) $\lim_{x \to -4} x^2$ b) $\lim_{x \to 1} = \frac{x^3 - 1}{x - 1}$

c) $\lim_{x \to 2} (4x^2 - 5x)$ d) $\lim_{x \to 2} \frac{x^2 + 6x + 5}{x^2 - 2x - 3}$

Differentiation from first principle

In nature, things change with time. If the rate of change leads to a change of the original component, then you have differentiation. For example, when you heat water in a kettle, the level reduces with time due to evaporation. This is an example of differentiation.

The process of calculating the ratio of the incremental change in a function y of x to the incremental change in x, that is, of determining an expression for $\frac{\Delta y}{\Delta x}$ and then finding the limiting value of this ratio as Δx approaches zero is known as **differentiation from first principle.** The limit found in this way is generally denoted by the symbol $\frac{dy}{dx}$ and is called the differential coefficient of y with respect to x.

Example 27.7

Differentiate the function $y = 5x^2$ from first principle.

Solution

To differentiate from first principle, you have $y = 5x^2$. For elemental increment, you have:

$y + \Delta y = 5(x + \Delta x)^2$

By substitution (note that $y = 5x^2$)

$\Delta y = 5\{(x + \Delta x)^2 - x^2\} = 5(2x + \Delta x)\Delta x$

Hence, $\frac{\Delta y}{\Delta x} = 5(2x + \Delta x)$

$= 10x + 5\Delta x$

You will see that the limiting value (as $\Delta x \to 0$) of the expression on the right is $10x$.

So, in general, for a function $y = f(x)$, you have

$$\frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\}$$

Example 27.8

Differentiate the function $y = x^2$ from first principle.

Let $y = x^2$

$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\,\Delta x + (\Delta x)^2$

$y + \Delta y - y = (x + \Delta x^2) - x^2 = x^2 + 2x\Delta x + (\Delta x)^2 - x^2$

or $\Delta y = 2x\,\Delta x + (\Delta x)^2$

Then $\frac{\Delta y}{\Delta x} = \frac{2x.\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$

letting $\Delta x \to 0$, you will find that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x$$

Therefore, $\frac{dy}{dx} = 2x$

However, when you differentiate some functions from the first principle, a kind of pattern should have been observed by you in the manner the powers of the variable x reduces as seen in the last two examples.

Consider these:

If $y = x^2$, $\frac{dy}{dx} = 2x$

If $y = x^3$, $\frac{dy}{dx} = 3x^2$

If $y = x^4$, $\frac{dy}{dx} = 4x^3$

Therefore, in general, if you have $y = x^n$,

then, $\frac{dy}{dx} = nx^{n-1}$

Differentiation of product of functions

The product of two differentiable functions u and v is differentiable, and without taking you through the process of deriving this, the formula is

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

This is called the **product rule**.

Example 27.9

If $u = x^2 + 4$ and $v = x^3 + 4$,. Find the derivative of $y = (x^2 + 4)(x^3 + 4)$, using the product rule.

Solution

Recall the formula

$$\frac{dy}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

Take $u = x^2 + 4$, $\frac{du}{dx} = 2x$

While $v = x^3 + 4$, $\frac{dv}{dx} = 3x^2$

$\frac{dy}{dx} = (x^2 + 4)\, 3x^2 + (x^3 + 4)\, 2x$

$= 3x^4 + 12x^2 + 2x^4 + 8x$

$= 5x^4 + 12x^2 + 8x$

Exercise 27.2

1 Differentiate the following functions from the first principle.
 a) $5x^2$ b) $x^4 - x^2$ c) $y = 2x + 1$
 d) $7x^2$ e) $x^4 - 2x^2$

2 Differentiate the following functions:
 a) $y = 3x^4 + 2x^2 + 9$
 b) $y = 7x^5$
 c) $y = 3x^7 - 7x^3 + 21x^2$

3 Differentiate by the product rule:
 a) $y = 4(6x^2 + 1)$
 b) $y = (3x + 2)(x^2 + 5x)$
 c) $y = x^3(6x^5 + 4)$

Differentiation of quotient of functions

At a point where $v \neq 0$, the quotient $y = \frac{u}{v}$ of two differentiable functions is differential and the formula is given as follows:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$$

Example 27.10

Find the differentiation of

$$y = \frac{x + 1}{x^2 + 1}$$

Solution

Let $u = x + 1$ and $v = x^2 + 1$. So, $du = 1$ and $dv = 2x$

Therefore, $\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{y^2}$

$$\frac{(x^2 + 1) - (x + 1)\,2x}{(x^2 + 1)^2}$$

Differentiation of function of functions (chain rule)

A function like $y = (x + 3)^2$ is a function of functions, because $(x + 3)$ is a function of x and $(x + 3)^2$ is a function of $(x + 3)$. This type of relationship is termed **function of functions**, and the method of derivative is called the **chain rule.**

If u is a function of x and y is a function of u, this implies that $y = u(x)$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

See the application of this rule in Examples 27.11 and 27.12.

Example 27.11

$y = (x + 3)^2$ or $y = u^2$, where $u = x + 3$

$\frac{dy}{du} = 2u;\ \frac{du}{dx} = 1$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2u \times 1 = 2(x + 3)$

Example 27.12

$y = (x + 1)^3$ or $y = u^3$, where $u = x + 1$

$\frac{dy}{du} = 3u^2,\ \frac{du}{dx} = 1$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 + 1 = 3(x + 1)^2$

Exercise 27.3

Differentiate the following using chain rule method:

a) $y = (x^3 - 1)^3$
b) $y = 6(x - 2)^2$
c) $y = (x^3 - 3x)^3$
d) $y = (1 - 3x^2)^5$

Implicit differentiation

In this section, you will develop the technique necessary for the differentiation of implicit functions, i.e. functions where y for example, is not explicitly expressed as a function of x. Suppose, for example, y is defined as a function of x by the equation $x^2 + y^2 = 4$. This is an example of implicit functions. For this, you can do the following:

$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) = \frac{d}{dx}(4)$

Therefore, $\frac{d}{dx}(y^2) + 2x = 0$

Now y^2 is a function of y, which is itself a function of x (if you take $y = \sqrt{(4-x^2)}$.

Then, $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$

Therefore, $2y\frac{dy}{dx} + 2x = 0$

so that $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

However, you know that it is possible to express y explicitly in terms of x as

$y = \sqrt{(4-x^2)}$

Then you can set $y = u^{\frac{1}{2}}$, where $u = 4 - x^2$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2} \times \frac{1}{\sqrt{u}} \times (-2x)$

$= -\frac{x}{\sqrt{u}} = -\frac{x}{y}$

You still arrive at the same solution. However, you can also express y as a function of x as

$y = -\sqrt{(4-x^2)}$, $y = -u^{\frac{1}{2}}$ where $u = 4 - x^2$

Therefore,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{2} \times \frac{1}{\sqrt{u}} \times (-2x)$

$= \frac{x}{\sqrt{u}} = \frac{x}{-y} = -\frac{x}{y}$

In all the cases, you still have $\frac{dy}{dx} = \frac{-x}{y}$.

This is because $y = \sqrt{(4-x^2} \text{ or} -\left(\sqrt{(4-x^2)}\right)$.

Exercise 27.4

Find $\frac{dy}{dx}$ for:

a) $x^2 + y^2 = 1$

b) $xy = 1$

c) $x^2y^2 - x - y = 0$

d) $\sqrt{x} + \sqrt{y} = 1$

Differentiation of some trigonometric functions

In this section, you will study the differentiation of trigonometric functions without learning how to obtain the derivatives from first principle.

For the basic trigonometric functions, the derivatives are as follows:

If $y = \sin x$, $\frac{dy}{dx} = \cos x$

If $y = \cos x$, $\frac{dy}{dx} = \sin x$

If $y = \tan x$, $\frac{dy}{dx} = \sec^2 x$

The three other trigonometric ratios can be obtained by writing:

$$\cot x = \frac{\cos x}{\sin x};\ \sec x = \frac{1}{\cos x} \text{ and } \operatorname{cosec} x = \frac{1}{\sin x}$$

The results are:

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$; $\frac{d}{dx}(\sec\ x) = \sec\ x$

and $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Now, see example 27.13.

Example 27.13

Find $\frac{dy}{dx}$ if $y = \sin(3x + 4)$

Solution

$y = \sin(3x + 4)$

Put $u = 3x + 4$

Therefore, $y = \sin u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \cos u.\ \frac{dy}{dx} = 3$

Therefore, $\frac{dy}{dx} = 3\cos u.$

$= 3\cos(3x + 4)$

27.4 Conclusion

In this unit, you have learnt how to find the derivatives of a function from first principle. You have also gone through differentiation of product of functions, quotient of functions, function of functions, implicit functions, as well as trigonometric functions.

As you will remember, the process of finding gradient functions is called differentiation. Another name for gradient function is derivative or derived function. What you have learnt in this unit will help you later in your indepth study of calculus

27.5 Summary

In this unit, you have learnt that

i) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$: (sum);

ii) $\frac{d}{dx}(u \times v) = \frac{udv}{dx} + \frac{vdu}{dx}$: (product);

iii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$: (quotient);

iv) $\frac{d}{dx}(\sin x) = \cos x$;

v) $\frac{d}{dx}(\cos x) = \sin x$.

27.6 Tutor-marked assignment

1 Differentiate $\frac{(x-3)^2}{(x+2)^2}$, using the quotient rule.

2 Differentiate $(x^2 + 1)^3 (x^3 + 1)^2$

27.7 References

1 Lassa P. N. and S. A. Ilori, (1991) *Further Mathematics for Senior Secondary Schools Books 1-2*, University Press Plc, Ibadan, p. 408-414.

2 Tuttuh-Adegun M. R., S. Sivasubramanian and R. Adegoke, (1992) *Further Mathematics Project, Book 1*, p. 78-86.

Unit 28 Applications of differentiation

Contents

28.1 Introduction

You should have already covered the material on differentiation, and you should have used the rules of differentiation developed to solve problems. This unit extends the range of problems you can solve, including finding the greatest or least value of a function and differentiating complicated functions.

28.2 Objectives

By the end of this unit, you should be able to:

i) find maximum and minimum of a function;
ii) identify the stationary points of a function;
iii) apply the ideas of maximum and minimum to real life problems;
iv) find the second derivatives of functions.

28.3 Rate of change of the gradient

Knowing the gradients of a function at each point of the graph tells you a great deal about the function. See the following example:

Example 28.1
If $f(x) = x^2 - 2x - 2$

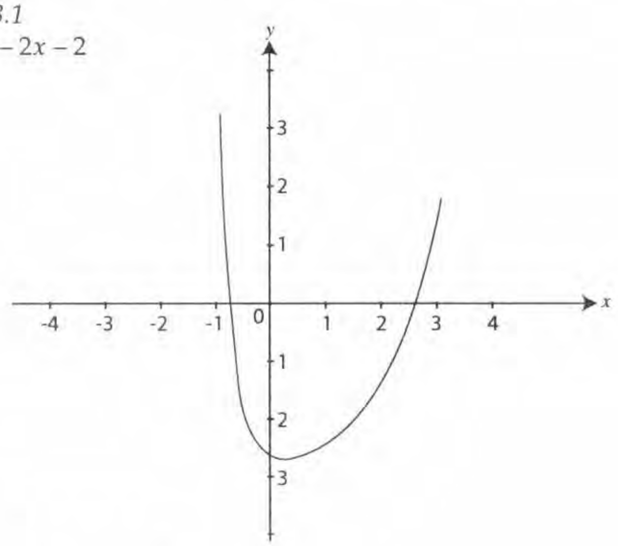


Fig. 28.1

$\frac{dy}{dx} = 2x - 2 = 2(x - 1)$

If $x < 1$, then, $\frac{dy}{dx} = 2x$

As you can see in the graph of f(x), this means that in this interval ($x < 1$) the graph of f goes down as x increases. If $x = 1$, then $\frac{dy}{dx} = 0$, which means that the line tangent to the graph at that point is horizontal, since f(1) = -3, that point will be (1, -3).

If $x > 1$, then $\frac{dy}{dx} > 0$ and the graph rises with increasing x.

Also, the sketch in Fig. 28.2 shows the graph of $y = x^3 + 3x^2 - 9x - 4$.

You have already seen that the derivative of this function is given by

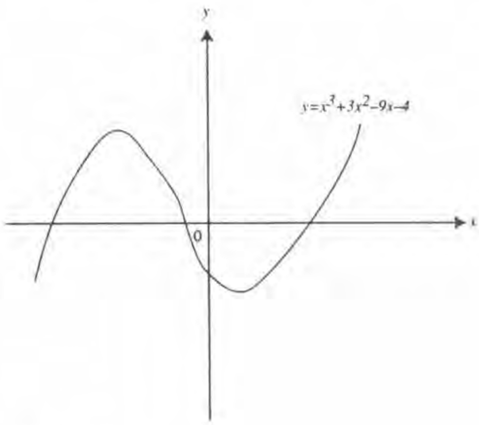


Fig. 28.2

$\frac{dy}{dx} = 3x^2 + 6x - 9$

This is also illustrated in Fig. 28.3

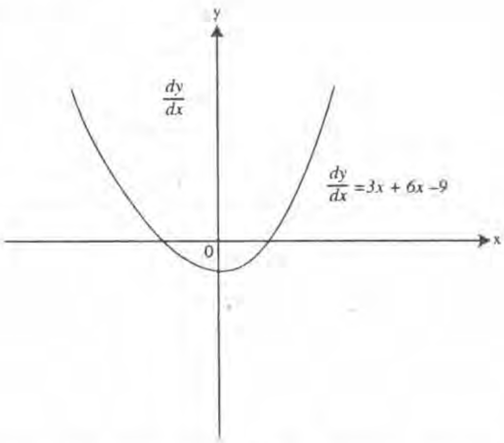


Fig. 28.3

Now as you can see, the gradient functions, $\frac{dy}{dx}$, is also a function of x, and can therefore be differentiated again to give the second differential:

$$\frac{d^2y}{dx^2} = 6x + 6$$

Again this is illustrated in Fig. 28.4.

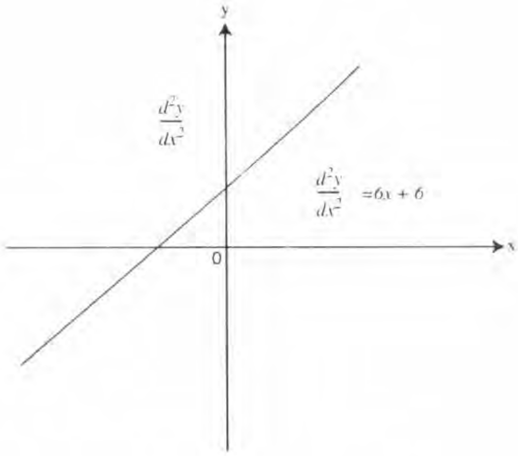


Fig. 28.4

Exercise 28.1

1. Find the second derivative of the following functions:
   (a) $y = x^3$ (b) $y = x^4$ (c) $y = x^2$
   (d) $y = 4x^3 - 12x^2 + 5$

Stationary points

In this section you will consider curves with their equations. For example,

$y = 2x^3 + 3x^2 - 12x$

This is s cubic equation and a rough sketch of its graph is shown in Fig. 28.5.

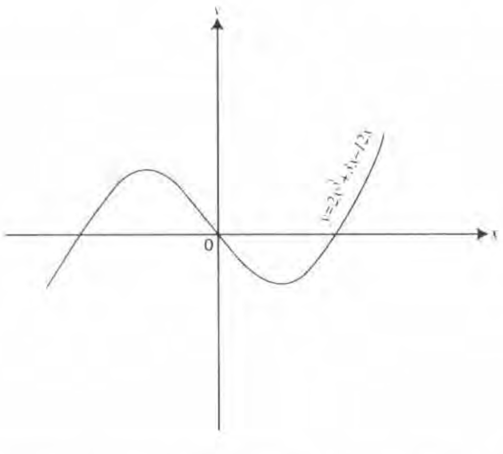


Fig. 28.5

It has two stationary points (sometimes called turning points) at which the gradient is zero. If you are to find coordinate of the stationary points for the curve with equation:

$$y = 2x^3 + 3x^2 - 12x$$

In the last unit on the differentiation of a function, you saw that the nature of the stationary points can be determined by looking at the gradient on each side of the stationary point. Here an alternative and more formal method is developed, based on using second derivatives.

Now, what you do is to draw an accurate sketch of the curve with the equation

$y = 2x^3 + 3x^2 - 12x$

between $x = -3$ and $+ 2$, for example, choose the y - axis to show values between -10 and +20. For every x value -3, -2.8, -2.6, -2.4, 0, 2, note the gradient in the diagram shown in Fig. 28.6.

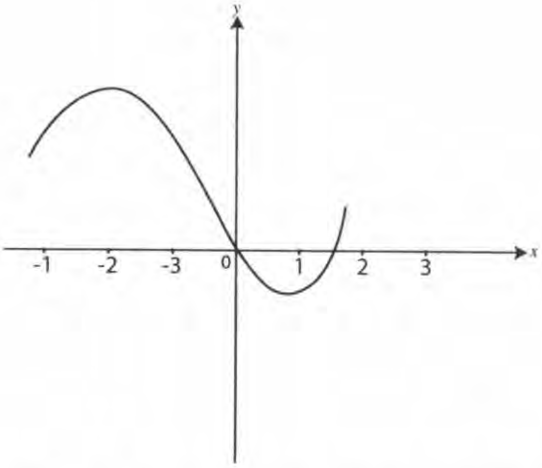


Fig. 28.6

Plot a graph of the gradient function and note how it behaves near the stationary point of the function. For a maximum value of a function, you should note that the gradient is decreasing in value as it passes through the value zero at the stationary point, whereas for a minimum value this gradient is increasing. The result can be summarised as follows:

a) for stationary points of a function $y(x)$

$$\frac{dy}{dx} = 0$$

b) If $\frac{d^2y}{dx^2} < 0$ at a stationary point, it corresponds to a maximum value of y.

c) If $\frac{d^2y}{dx^2} > 0$ at a stationary point, it corresponds to a minimum value of y.

Example 28.2

Find maximum and minimum of the curve with equation:

$$y = \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - 6x^2 + 3$$

Attempt to sketch the curve.

Solution

For stationary points, $\frac{dy}{dx} = 0$, which gives $\frac{dy}{dx} = x^3 + x^2 - 12x = 0$

So $\frac{dy}{dx} = 0 \Rightarrow x^3 + x^2 - 12x = 0$

$x(x^2 + x - 12) = 0$

$x(x + 4)(x - 3) = 0$

Hence there are stationary points at $x = 0, -4$ and 3. i.e. $x = 0, x + 4 = 0, x - 3 = 0$

To find out their nature, the second derivative with now used.

Now $\frac{d^2y}{dx^2} + 3x^2 + 2x - 12$

At $x = 0$, $\frac{d^2y}{dx^2} = -12 < 0$

That is you have a maximum at $x = 0$ and $y(0) = 3$

At $x = -4$, $\frac{d^2y}{dx^2} = 3(-4)^2 + 2(-4) - 12 = 28$

So, you have a minimum at $x = -4$

and $y(-4) = \frac{1}{4}(-4)^4 + \frac{1}{3}(-4)^3 - 6(-4)^2 + 3 = -\frac{151}{3}$

At $x = 3$, $\frac{d^2y}{dx^2} = 3(3)^2 + 2(3) - 12 = 21 > 0$

i.e. minimum exists at $x = 3$, and

$y(3) = \frac{1}{4}(x)^4 + \frac{1}{3}(3)^3 - 6(3)^2 + 3 = \frac{-87}{4}$

You should also know that as $x \to \pm \infty$

$y \to \pm \infty$

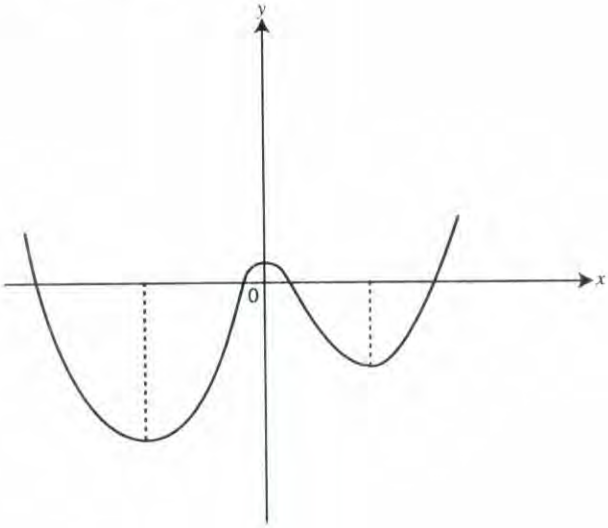


Fig. 28.7

There is another type of stationary point you should consider, and this is called a **point of inflexion**. An example is given in the following activity below:

Example 28.3

Find the stationary point of $y = x^3$.

What is the value of $\frac{d^2y}{dx^2}$ at the stationary point? Sketch the graph of $y = x^3$.

Solution

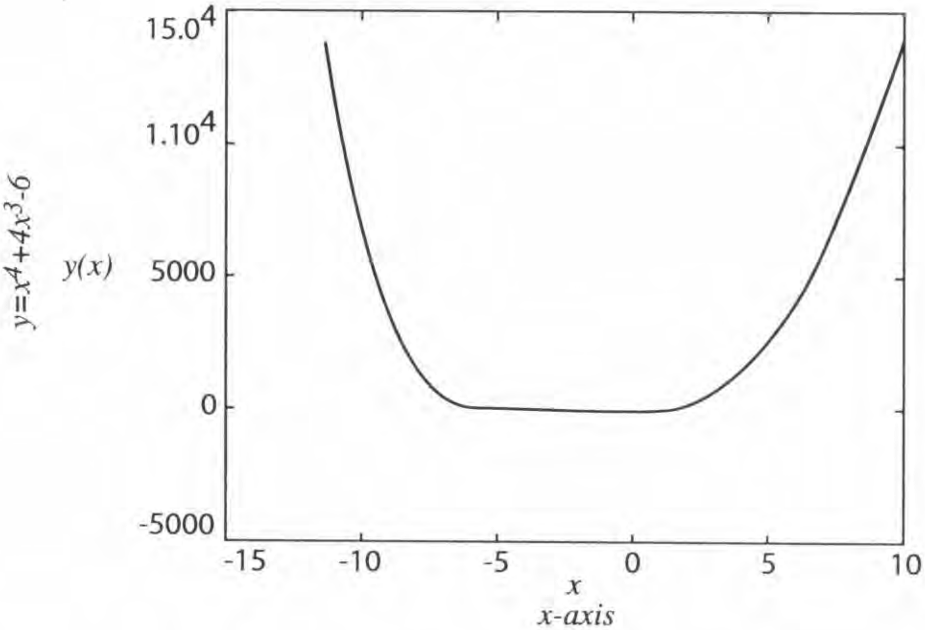
For a horizontal point of inflexion, not only does $\frac{dy}{dx} = 0$, but also $\frac{d^2y}{dx^2} \neq 0$.

$\frac{d^3y}{dx^3} \neq 0$ at the point. These are sufficient but also necessary conditions, as can be seen by considering $y = x^2$.

Fig. 28.8 (a) (b)

Example 28.4
Find t2he nature of the stationary points of the curve with the equation:
$y = x^4 + 4x^3 - 6$

Solution
Sketch a graph of the curve as follows:



Now $\frac{dy}{dx} = 4x^3 + 12x^2 = 0$ when
$4x^3 + 12x^2 = 0$
$\Rightarrow 4x^2(x + 3) = 0$
$\Rightarrow x = 0, -3$ for stationary points

But $\frac{d^2y}{dx^2} = 12x^2 + 24x$
At $x = 0$, $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} = 24x + 24$
$24 > 0$ at $x = 0$.

So there is a point of inflexion at (0, -6)
At $x = -3$, $\frac{d^2y}{dx^2} = 12(-3)^2 + 24(-3)$
$= 108 - 72 = 36 > 0$
So there is a minimum at $x = -3$ of value:
$y(-3) = (-3)^4 + 4(-3)^3 - 6$
$= 81 - 4 \times 27 - 6$
$= -33$
To sketch the curve, also note that
$y \to \infty$ as $x \to \pm \infty$

You can then deduce the form of the curve as shown in dashed form in Fig. 28.9.

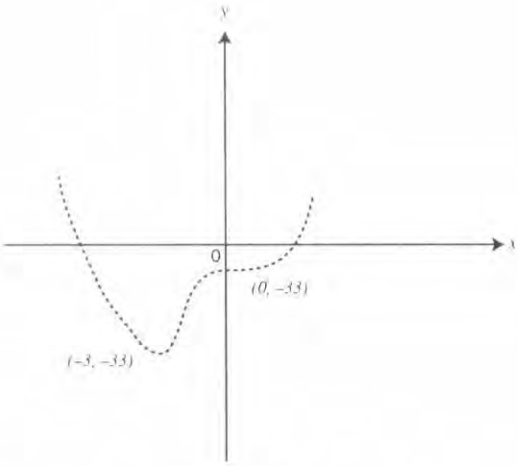


Fig. 28.9

To keep you focussed, remember the following:

a) Stationary points are always given by

$\frac{dy}{dx} = 0$, for a function $y(x)$

b) If, at the stationary point, $\frac{d^2y}{dx^2} > 0$, there is a minimum whereas if

$\frac{d^2y}{dx^2} < 0$ there is maximum.

c) If at the stationary point $\frac{d^2y}{dx^2} = 0$, then there is a point of inflexion provided

$\frac{d^3y}{dx^3} \neq 0$

Conditions b) and c) are sufficient to guarantee the nature of the stationary point, but, as you have already seen, they are not necessary.

When this analysis does not hold, that is when $\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$ at a stationary point, it is easier to consider the signs of the gradient on each side of the stationary point as illustrated in Fig 28.10.

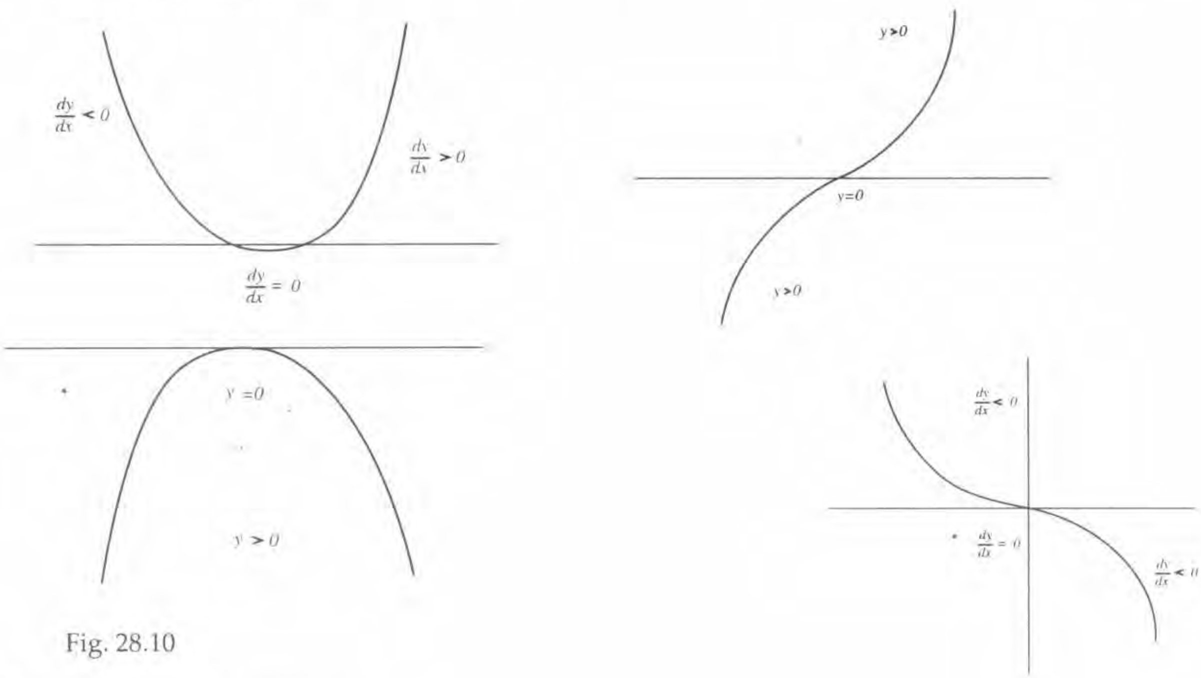


Fig. 28.10

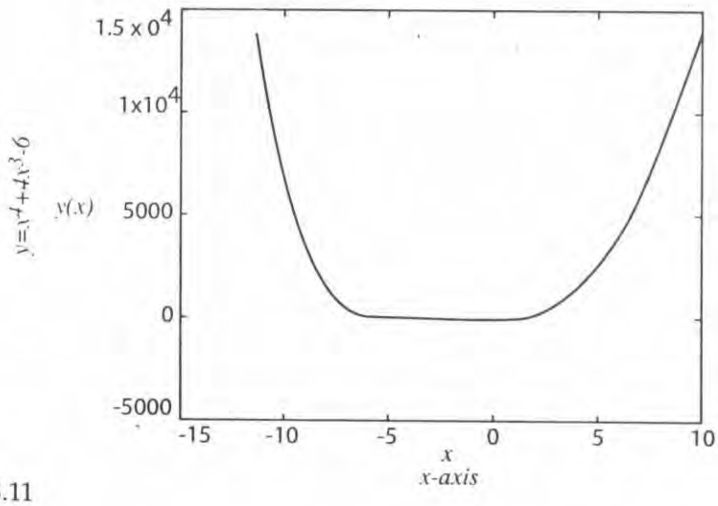


Fig. 28.11

Example 28.5
Show that $y = x^4$ has a minimum at $x = 0$

Solution
For stationary points, $\frac{dy}{dx} = 4x^3 = 0 \Rightarrow x = 0.$
But, if $x = 0$, $\frac{d^2y}{dx^2} = 12x^2 = 0$
And $\frac{d^3y}{dx^3} = 24x = 0$
For example, if you now take
$x = -0.1 \Rightarrow \frac{dy}{dx} = 4(-0.1)^3 = -0.004 < 0$
Also at:
$x = 0.1 \Rightarrow \frac{dy}{dx} = 4(0.1)^3 = 0.004 > 0$
So there is a minimum at $x = 0$

Exercise 28.2
1. Find the turning points of these curves using differentiation. In each case, find out whether the points found or maximum, minimum or point of inflexion.
   a) $y = 2x^2 + 3x + 1$
   b) $y = x^3 - 12x + 6$
   c) $y = x^3 - 2x^2 - 5x = 6.$

Application of minimum and maximum points
Example 28.6
A farmer wishes to enclose in rectangular plot for a pasture, using a wire fence on three sides and a hedge row as the fourth side. If he has 2400 m of wire, what is the greatest area he can fence off?

Solution
Let x denote the length of the equal sides to be wired, then the length of the third side is $2400 - 2x$.

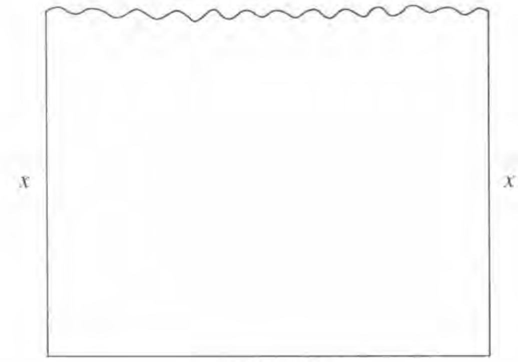


Fig. 28.12

The area is A = f(x) = x (2400 – 2x) = $2400x - 2x^2$

Now $\frac{dy}{dx} = 2400 - 4x = 4(600 - x)$ and the critical value is obtained at $\frac{dy}{dx} = 0$

i.e. $600 - x = 0$ or $x = 600$

Since $\frac{d^2y}{dx^2} = -4$, $x = 600$ yields a relative maximum.

The value becomes f(600) = 720,000 metres.

Example 28.7

A closed cylindrical can has a volume of 350 cm^2. Find the dimension of the can that minimises the surface.

Solution

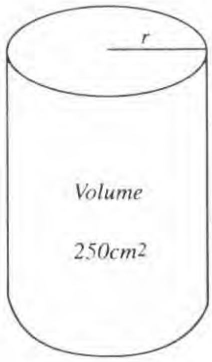


Fig. 28.13

Let the radius be *r* cm and the height h cm. Let the surface area be 5 cm^2. Then, using the surface area formula for a cylinder, you have

$S = 2\pi r^2 + 2\pi rh$ (the quantity to be minimised).

At present, S involves two variables *r* and *h*. The fact that the volume has to be 350 cm^3, means that

$\pi r^2 h = 350$ (constant)

So $h = \frac{350}{\pi r^2}$ (i.e. h the subject)

and $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \frac{350}{\pi r^2}$

$= 2\pi r^2 + \frac{700}{r}$

Now give $\frac{dS}{dr} = 4\pi r - \frac{70(}{r^2}$

At a stationary point, $\frac{dS}{dx} = 0$

Giving $4\pi r - \frac{700}{r^2} = 0$

$\Rightarrow r^3 = \frac{700}{4\pi} = 55.7$

i.e. $r = \sqrt[3]{55.7}$

on r = 3.82 cm (to 3 significant figures)

Then $h = \frac{350}{\pi r^2} = 7.64$ cm

With these examples given, you should be able to practice with the following exercises

Exercises 28.3

1. The rectangular window frame in the diagram shown in Fig 28.14 uses 20 cm window frame altogether. What is the maximum area the window can here.

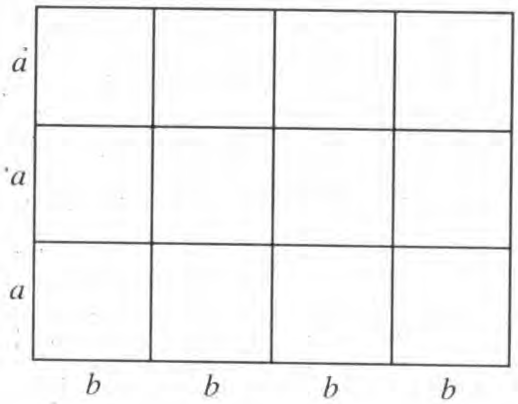


Fig. 28.14

2. A number is 10 more than another number. Find the minimum product of the two numbers.

28.4 Conclusion

In this unit you have learnt to find the maximum and minimum points of any function. This is done by setting the gradient to zero. It was also noted that stationary points can turn out to be points of inflexion.

28.5 Summary

The following are the highlights of this unit:

i) the gradient of a function;
ii) the second derivation of functions;
iii) the stationary points;
iv) the maximum and minimum points of functions;
v) solving real-life minimum and maximum problems.

What you have studied in this unit will help you to apply the knowledge of differentiation to a number of problems.

28.6 Tutor-marked assignment

1. A cylinder container with circular base is to hold 64 cubic cm. Find the dimensions so that the amount (surface area) of metal required is a minimum when
   (a) the container is an open cup; and
   (b) a closed car.

28.7 References

Backhouse, J. K. and S. P. T. Houldsworth, (1999), *Pure Mathematics, Book 1*, Addison Wesley Longman Limited. England.

Murphy, P. (1984), *Additional Mathematics Made Simple*, W. H. Allen and Co. Ltd, pp. 233-242.

Unit 29 Simple integration

Contents

29.1 Introduction

In the last unit, you studied about differentiation. **Integration** is simply the inverse operation of differentiation. Integration is also the process of finding the area under a graph. An example of an area that integration can be used to calculate is the shaded region shown in Fig. 29.1.

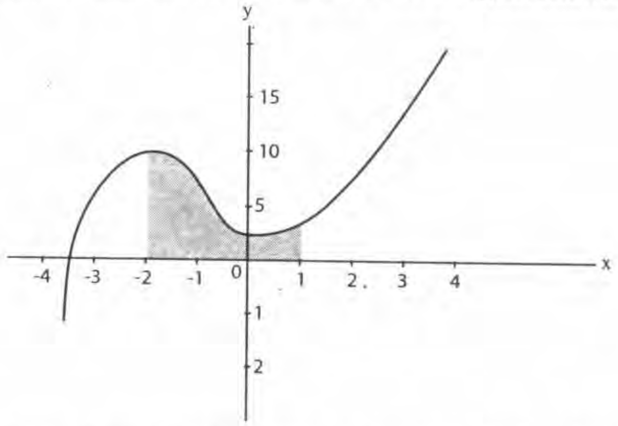


Fig. 29.1

There are several ways of estimating the area — this unit includes a brief work at such methods; but the main objective is to discover a way to find the area exactly. There are a number of problems in science and other disciplines that need the knowledge of integration to solve a problem. You will starts by looking at an example of such a problem.

You can find the derivative of many functions by now, but suppose someone gave you the derivative of a function, could you find the original function that was differentiated? Given a derivative $f(x)$, the process of finding the original function is called **antidifferentiation** and the original function is called the **antiderivative** of f. Thus, there are many antiderivatives for any one function. In general, if $F(x)$ is one antiderivative, then so is $(F(x) + c)$ for any constant c. This is because the derivative of any constant is zero. You say that the antiderivative of a function $f(x)$ is $F(x) + c$.

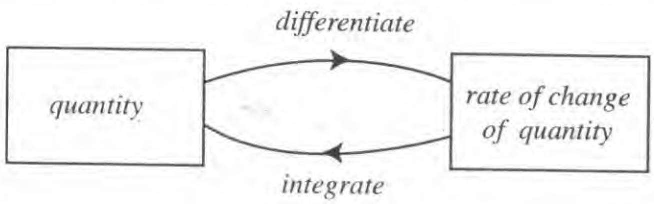


Fig. 29.2

You will recall from the last unit that the process of finding rates of change is differentation of derivative, hence integration is defined as the reverse process. See the illustration (Fig 29.2).

You should recall that finding the derivative of a composite function, a product, or a quotient, requires certain formulae. Similarly, the integration of products and quotients need some efforts.

29.2 Objectives

By the end of this unit, you should be able to:

i) describe integration as the reverse of differentation;
ii) define the concept of integration;
iii) solve and define integral problems.

29.3 Integration

If $F(x)$ is a **function** whose derivative $\frac{d(x)}{dx} = f(x) = F'(x)$, then $F(x)$ is called an integral of $f(x)$. For example, $F(x) = x^3$ is an integral of $f(x) = 3x^2$, since $F'(x) = 3x^2 = f(x)$. Also, $G(x) = x^3 + 5$ and $H(x) = x^3 - 6$ are integral of $f(x) = 3x^2$. Why? If $F(x)$ and $G(x)$ are two distant integral of $f(x)$, then $F(x) = G(x) + c$, where c is a constant.

The indefinite integral of $f(x)$, denoted by $\int f(x)dx$, is the most general integral of $f(x)$, that is $\int F(x)dx = F(x) + c$.

where $f(x)$ is any function such that $F'(x) = f(x)$ and c is an arbitrary constant. Thus, the indefinite integral of $f(x) = 3x^2$ is

$$\int 3x^2dx = x^3 + c$$

You will use the following integration formula as you go along the course.

Rules of integration

a) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq 1$.

b) $\int cf(x)dx = c\int f(x)dx$, where c is a constant

c) $\int \{(f(x) + g(x)\}dx = \int f(x)dx + \int g(x)dx$

For example, you already know that $\int 3x^2dx = x^3$, and that in reverse operation, that is equivalent to:

$$\frac{d}{dx}(x^3) = 3x^2$$

But what about $\frac{d}{dx}(x^3 + 1)$, $\frac{d}{dx}(x^3 + 4)$ and $\frac{d}{dx}(x^3 - 7)$?

They all give the answer, $3x^2$. So, in general you can write

$\int 3x^2dx = x^3 + c$; where c is any constant.

It was Leibnitz who first used the elongated 'S' (for summation) as the symbol for an integral. If y is a function of x, $\int ydx$, means 'the integral of y with respect to x'.

You must always include the arbitrary constant when working with indefinite integrals.

Example 29.1

Use the rules of integration above to verify the following:

a) $\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + c$

b) $\int(3x^2 + 5)dx = x^3 + 5x + c$

c) $\int(5x^6 + 2x^3 - 4x - 3)dx = \frac{5}{7}x^7 + \frac{1}{2}x^4 - 2x^2 + 3x + c$

d) $\int(80x^{19} - 32x^{15} - 12x^{-3})dx$

$= 4x^{20} - 2x^{16} + \frac{6}{x^2} + c$

Consider the following examples.

Example 29.2

1 $\int 5x^3 dx = 5\int x^3 dx = \frac{5x^4}{4} + c = \frac{5x^4}{4} + c$

2 $\int \frac{7}{x^3} dx = 7\int x^{-3} dx$ (because $\frac{7}{x^3} = 7x^{-3}$)

$= 7\frac{x^{-2}}{(-2)} + c = \frac{-7x^{-2}}{2} + c$

$= \frac{-7}{2} - \frac{1}{x^2} + c$

$= \frac{-7}{2x^2} + c$

3 $\int \frac{x + x^2}{3} dx = \frac{1}{3}\int(x + x^2)dx$

$= \frac{1}{3}\{\int x dx + \int x^2 dx\}$

$= \frac{1}{3}\{\frac{x^2}{2} + \frac{x^3}{3}\} + c$

$= \frac{x^2}{6} + \frac{x^3}{9} + c$

In the worked examples above, it has been assumed that integration is a linear process. In general, if $u(x)$ and $v(x)$ are any functions of x, and a and b are any constant numbers, than you have, by linearity rule:

$\int au(x) + bv(x) = a\int u(x)dx + b\int v(x)dx$

Try these exercises now.

Exercise 29.1

1 Integrate

a) with respect to x : $\frac{1}{2}, \frac{1}{2}x^2, x^2 + 3x, (2x + 3)^2, x^{-5}$ and $\frac{-2}{x^4}$

2 Integrate the following:

a) $\int \frac{3 - 2x^5}{4} dx$

b) $\int \frac{5x^2 - x + 1}{2} dx$

c) $\int \frac{3}{t^3} dt$

d) $\int \frac{3}{2y^3} dy$

29.4 Integration by substitution method

The results that have been developed in the last section are, as you will see, very useful in integration. For example, if $y = (x + 5)^4$.

then,

$\frac{dy}{dx} = 4(x + 5)^3$ using that $\frac{dy}{dy}5 = \frac{1}{y} = \frac{1}{4}a^3$

Hence $\int 4(x+5)^3dx = (x+5)^4 + c$ (c is arbitrary constant).

Similarly, $\int (x+5)^3dx = \frac{1}{4}(x+5)^4 + c' = (c' = \frac{1}{4}c)$.

But, the above can be solved by using substitution as you will see below.

Example 29.2
Solve $\int (3x-1)^9dx$

Solution

Now, let $u = (3x-1)$, $\frac{du}{dx} = 3$ and so $dx = \frac{1}{3}du$

Therefore, $\int (3x-1)^9\,dx = \int (u^9)\frac{du}{3} = \frac{1}{3}\int u^9du$

$$= \frac{1}{3}\frac{u^{10}}{10} + c$$

$$= \frac{1}{3}\frac{(3x-1)^{10}}{10} + c$$

$$= \frac{1}{30}(3x-1)^{10} + k, \quad k = \frac{c}{10}$$

Exercise 29.2
Find

1. a) $\int (2x+1)^4dx$

   b) $\int \frac{1}{(x-5)^2}dx$

   c) $\int \frac{1}{\sqrt{x+1}}dx$

   d) $\int \sqrt{(4x-1)}\,dx$

29.5 Conclusion and summary

In this unit you have learned the significance of simple integration. The process of finding rates of change is differentiation and hence integration must be the reverse process. For any function, therefore, it is said that integration is the reverse of differentiation.

In general, an arbitrary constant should always be added to the result of an integration, i.e. $\int g(x)dx = f(x) + c$. This is known as the indefinite integral of $g(x)$. A general formula common in integration as seen in this unit is:

$$\int x^n dx = \frac{x^{n-1}}{n+1} + c$$

provided $n \neq -1$, where c = constant.

The units that follow shall build upon other aspects of integration.

29.6 Tutor-marked assignment

Evaluate the following:

1 $\int \frac{3x^2 - 5x + 1}{x}dx$

2 $\int \frac{x^2 - 2x + 1}{\sqrt{x}}dx$

3 $\int\sqrt{3 - 2x}\, dx$ by substitution method

4 $\int x(3x^2 + 2)^5 dx$ by substitution method

5 $\int \frac{x^2 dx}{\sqrt{x^2 + 1}}$ by substitution method

29.8 References

Backhouse, J. K. and S. P. T. Houldsworn (1999): *Pure Mathematics Book 1*, Longman Addison Wesley Limited, London, pp. 106 – 114 and 147 – 165.

Hodson, J. D. (1996): *Elementary Pure Mathematics*, Macmillan, NY, pp. 105 – 107 and 149 – 151.

Unit 30 Permutations and combinations

Contents

30.1 Introduction

When the number of possible outcomes is small, as you have learnt in the previous unit on probability, it is not difficult to list all the possibilities. For example, to list the possible outcomes for two children in a family is quite simple. The possibilities are two boys, one boy and one girl or two girls. Suppose, however, you have to list and count the possible outcomes for a family of 15 children! Obviously, a more efficient method of counting the outcomes will be needed. In this unit, you will study the multiplication formula as well as permutations and combinations.

30.2 Objectives

By the end of this unit, you should be able to:

i) define and solve multiplication formula;
ii) define and solve problems related to permutations;
iii) definite and solve problems related to combinations;
iv) solve problems on combinations and permutations.

30.3 Multiplication formula

The multiplication formula states that if a choice consists of two steps, the first of which can be made in m ways and the second in n ways, then there are $m \times n$ possible choices.

Example 30.1

Consider a Fine Art department designing the advertisement for a new catalogue. The artist must show all possible variations on a new outfit consisting of two blouses and three skirts. The blouse options are long sleeves or short sleeves. The skirt options are solid colour, plain or long evening skirt. How many different interchangeable outfits are there?

Solution

Let m represent the number of blouse options, and n represent the number of skirt options.
Using the multiplication formula ($m \times n$).

Number of options $= m \times n$
$= 2 \times 3 = 6$

There are six options.
This counting method can be extended to more than two events.

Example 30.2
A new car buyer has a choice of five body styles, two engines and eight different colours. How many different car choices does the buyer have?

Solution
Let m = body styles, n = engines and t = different colours.
Applying multiplication formula:
$m \times n \times t = 5 \times 2 \times 8 = 80$ different choices of cars that can be bought.

Exercise 30.1

1. One of the Local Governments in Kaduna State is considering using only numbers on their automobile license plates. If only four digits are to appear on the plates (such as 8313), how many different plates are possible?
   Hint: The first digit can be any number between 0 and 9.
2. Consider the generating of telephone numbers. Within a given exchange, say 874, how many different numbers are possible? (Assume that the numbers 874.000 and 874.9999 are acceptable).

30.4 Permutations (Arrangement)

In the previous section where you learnt about the multiplication formula, one item was chosen from each of several groups, e.g. one blouse from among the blouses and one skirt from among the skirts. If however, more than one item is to be selected from the same group and the order of selection is important, the resulting arrangement is called a **permutation** of the items. Permutation is therefore, defined as an ordered arrangement of a group of objects.

The permutation formula given as

$$nP_r = \frac{n!}{(n-r)!}$$

is defined as the number of permutation of n objects, taking r at a time. For example, if there are five persons and three are to fill the offices of president, vice president and secretary, then $n = 25$ and $r = 3$.

The notation $n!$, called **n factorial** is the product: $n \times (n-1) \times (n-2) \times \ldots\ldots(1)$.
So 4! (four factorial) is $4 \times 3 \times 2 \times 1 = 24$ and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. Zero factorial, written 0! is treated as a special case, and is defined to be equal to 1.

Example 30.3
In how many ways can you form a committee of two (2) from five (5) people?

Solution

$$nP_r = 5P_2 = \frac{n!}{(n-r)!}$$

$$\frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

$$= 20.$$

There are 20 ways

Example 30.4
In how many ways can you form a committee of four (4) from ten (10)?

Solution

$$nP_r = 10P_4 = \frac{n!}{(n-r)!}$$

Now, substituting the values, you have

Lastly, combination is an arrangement where the order of selection is immaterial. This is expressed as

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

30.7 Summary

In this unit, you have learnt that:

i) multiplication formula is used to find the total number ($m \times n$) of results if there are m different results from doing one thing and n different results from doing another thing;
ii) in permutations, the order of outcomes or selections is important;
iii) the order of outcomes or selections is not important in combinations.

30.8 Tutor-marked assignment

1 In forming 5–letter words, using the letters of the word **EQUATIONS**
   a) How many consist only of vowels?
   b) How many contain all the consonants?
   c) How many begin with E and end with S?

2 Solve for n, given
   a) $nP_2 = 110$
   b) $nP_4 = 30\,nP_2$

30.9 References

1 Perry, O. and J. Perry, (1982) *Mastering Mathematics*, Macmillan Press Limited, p 248-259.
2 Stephenson, G., (1983) *Mathematical Methods for Science*, Longman Group Limited, London, p. 329-361.

There are six options.
This counting method can be extended to more than two events.

Example 30.2
A new car buyer has a choice of five body styles, two engines and eight different colours. How many different car choices does the buyer have?

Solution
Let m = body styles, n = engines and t = different colours.
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Permutation is therefore, defined as an ordered arrangement of a group of objects.

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$$nP_r = 5P_2 = \frac{n!}{(n-r)!}$$

$$\frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

$$= 20.$$

There are 20 ways

Example 30.4
In how many ways can you form a committee of four (4) from ten (10)?

Solution

$$nP_r = 10P_4 = \frac{n!}{(n-r)!}$$

Now, substituting the values, you have

$10P_4 = \frac{10!}{(10-4)!}$

$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= (10 \times 9 \times 8 \times 7) = 5\ 040$ ways.

Exercise 30.2

1 In how many ways can 6 students be assigned:
   a) to a row of 6 seats?
   b) to a row of 8 seats?

2 Using the letters of the word **MARKING** and calling any arrangement a word.
   a) How many different 7– letter words can be formed?
   b) How many different 3– letter words can be formed ?

30.5 Combinations (Selections)

In dealing with permutations, the order of the outcomes is important. For example, one slate of yellow jacket officers might be president–Bolaji, secretary–Musa, and treasurer–Peters. Another slate might have president–Peters, secretary– Bolaji and treasurer– Musa. Every time there is a change in the order, the count of possibilities is increased by one.

In another example, however, the order may not be important. If for example, Bala, Clara and Caleb were selected to serve as social committee members, it will be the same committee if the order were Clara, Caleb and Bala. In other words, the committee of Caleb, Clara and Bala is counted just once, regardless of the order of names. Technically, such a group is considered a **combination.**

The symbol $\frac{n}{r}$ or ${}^nC_r$ denotes the number of ways of selecting (or choosing) r things from n unlike things. It is alternatively stated to be the number of combination of n unlike things, taking r at a time and given by the formula

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3 \ldots\ldots r}$$

Here, $n!$ (read, factorial n) denotes the product of the positive integers from 1 to n inclusive, e.g. $1! = 1$; $3! = 1 \times 2 \times 3 = 6$; $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ and $0! = 0$.

Exercise 30.3

1 A lady organises a dinner party for six guests.
   a) In how many ways can they be selected from among ten friends?
   b) In how many ways if the two of the friends will not attend the party together?

2 In how many ways may 12 persons be divided into three groups of:
   a) 2, 4 and 6?
   b) 4 persons each?

30.6 Conclusion

In this unit, you have learnt about the multiplication formula, permutations and combinations. Multiplication is a principle used when there are m distinct results from doing one thing and n different results from doing another thing. The total possible number of results is found by $m \times n$.

Permutation is an arrangement where order is important. To count the number of permutations of n objects, taking r at a time, you have

$$nP_r = \frac{n!}{(n-r)!}$$