COURSE GUIDE

MTH 102
ELEMENTARY MATHEMATICS II

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INTRODUCTION

MTH102- Elementary Mathematics II is designed to teach you how differential and integral calculus could be used in solving problems in the contemporary business, technological and scientific world.

Therefore, the course is structured to expose you to the skills required in other to attain a level of proficiency in science, technology and engineering.

The course is a 3-credit unit and a core course in first semester. It will take you 15 weeks to complete the course. You are to spend 91 hours of study for a period of 13 weeks while the first week is for orientation and the last week is for end of semester examination. The credit earned in this course is part of the requirement for graduation. You will receive the course material which you can read online or download and read off-line.

The online course material is integrated in the Learning Management System (LMS). All activities in this course will be held in the LMS. All you need to know in this course is presented in the following sub-headings.

COURSE COMPETENCIES

By the end of this course, you will gain competency to:

- Competency in Elementary Mathematics II
- Work with calculus and integration
- Develop mathematical model from calculus and integration

COURSE OBJECTIVES

The course objectives are to:

- To Inculcate Appropriate Mathematical Skills Required in Science and Engineering.
- Educate Learners on How to Use Mathematical Techniques in Solving Real Life Problems.
- Educate The Learners on How to Integrate Mathematical Models in Sciences and Engineering.

WORKING THROUGH THIS COURSE

The course is divided into modules and units. The modules are derived from the course competencies and objectives. The competencies will guide you on the skills you will gain at the end of this course. So, as you work through the course, reflect on the competencies to ensure mastery.

The units are components of the modules. Each unit is sub-divided into introduction, intended learning outcome(s), main content, self-assessment exercise(s), conclusion, summary, and further readings. The introduction introduces you to the unit topic. The intended learning outcome(s) is the central point which help to measure your achievement or success in the course. Therefore, study the intended learning outcome(s) before going to the main content and at the end of the unit, revisit the intended learning outcome(s) to check if you have achieved the learning outcomes. Work through the unit again if you have not attained the stated learning outcomes.

The main content is the body of knowledge in the unit. Self-assessment exercises are embedded in the content which helps you to evaluate your mastery of the competencies. The conclusion gives you the takeaway while the summary is a brief of the knowledge presented in the unit. The final part is the further readings. This takes you to where you can read more on the knowledge or topic presented in the unit. The modules and units are presented as follows:

MODULE 1: FUNCTIONS

Unit 1: Function and Graphs
Unit 2: Limits
Unit 3: Idea of Continuity

MODULE 2: CALCULUS OF DIFFERENTIATION

Unit 1: The Derivative as Limit of Rate of Change
Unit 2: Differentiation Technique

MODULE 3: CALCULUS OF INTEGRATION

Unit 1: Integration
Unit 2: Definite Integrals (Application to Areas under Curve and Volumes of Solids)

There are seven units in this course. Each unit is spread over week(s) of study.

PRESENTATION SCHEDULE

The weekly activities are presented in Table 1 while the required hours of study and the activities are presented in Table 2. This will guide your study time. You may spend more time in completing each module or unit.

TABLE I: WEEKLY ACTIVITIES

Week	Activity
1	Orientation and course guide
2	Module 1 Unit 1
3	Module 1 Unit 2
4	Module 1 Unit 3
5	Module 2 Unit 1
6	Module 2 Unit 2
7	Module 2 Unit 2
8	Module 2 Unit 2
9	Module 3 Unit 1
10	Module 3 Unit 1
11	Module 3 Unit 1
12	Module 3 Unit 2
13	Module 3 Unit 2
14	Revision and response to questionnaire
15	Examination

The activities in Table I include facilitation hours (synchronous and asynchronous), assignments, mini projects, and laboratory practical. How do you know the hours to spend on each? A guide is presented in Table 2.

TABLE 2: REQUIRED MINIMUM HOURS OF STUDY

S/N	Activity	Hour per Week	Hour per Semester
1	Synchronous Facilitation (Video Conferencing)	2	26
2	Asynchronous Facilitation (Read and respond to posts including facilitator 's comment, self-study)	4	52
3	Assignments, mini-project, laboratory practical and portfolios	1	13
Total		7	91

ASSESSMENT

Table 3 presents the mode you will be assessed.

Table 3: Assessment

S/N	**Method of Assessment**	**Score (%)**
3	Tutor Mark Assignments	30
4	Final Examination	100
Total	100	

ASSIGNMENTS

Take the assignment and click on the submission button to submit. The assignment will be scored, and you will receive feedback.

EXAMINATION

Finally, the examination will help to test the cognitive domain. The test items will be mostly application, and evaluation test items that will lead to creation of new knowledge/idea.

HOW TO GET THE MOST FROM THE COURSE

To get the most in this course, you:

- Need a personal laptop. The use of mobile phone only may not give you the desirable environment to work.
- Need regular and stable internet.
- Need to install the recommended software.
- Must work through the course step by step starting with the programme orientation.
- Must not plagiarise or impersonate. These are serious offences that could terminate your studentship. Plagiarism check will be used to run all your submissions.
- Must do all the assessments following given instructions.
- Must create time daily to attend to your study.

FACILITATION

There will be two forms of facilitation–synchronous and asynchronous.

The synchronous will be held through video conferencing according to weekly schedule. During the synchronous facilitation:

- There will be two hours of online real time contact per week making a total of 26 hours for thirteen weeks of study time.
- At the end of each video conferencing, the video will be uploaded for view at your pace.
- You are to read the course material and do other assignments as may be given before video conferencing time.
- The facilitator will concentrate on main themes.
- The facilitator will take you through the course guide in the first lecture at the start date of facilitation.

For the asynchronous facilitation, your facilitator will:

- Present the theme for the week.
- Direct and summarise forum discussions.
- Coordinate activities in the platform.
- Score and grade activities when need be.
- Support you to learn. In this regard personal mails may be sent.
- Send you videos and audio lectures, and podcasts if need be.

Read all the comments and notes of your facilitator especially on your assignments, participate in forum discussions. This will give you opportunity to socialise with others in the course and build your skill for teamwork. You can raise any challenge encountered during your study.

To gain the maximum benefit from course facilitation, prepare a list of questions before the synchronous session. You will learn a lot from participating actively in the discussions.

LEARNER SUPPORT

You will receive the following support:

- Technical Support: There will be contact number(s), email address and chat bot on the Learning Management System where you can chat or send message to get assistance and guidance any time during the course.

- 24/7 communication: You can send personal mail to your facilitator and the centre at any time of the day. You will receive answer to you mails within 24 hours. There is also opportunity for personal or group chats at any time of the day with those that are online.

- You will receive guidance and feedback on your assessments, academic progress, and receive help to resolve challenges facing your studies.

COURSE INFORMATION

Course Blub: This course presents differential calculus and integration techniques for different functions. It offers diverse solution procedures in determine the derivative of function and applications in the areas of science engineering.

MAIN COURSE

CONTENTS

MODULE 1: FUNCTIONS

In this module, you will be introduced to the fundamentals of function.

Concepts related to limit and continuity and its applications will be studied. This module is made up of the following units:

Unit 1: Function and Graphs
Unit 2: Limits
Unit 3: Idea of Continuity

Unit 1: Function and Graphs

Unit Structure

1.1 Introduction
1.2 Intended Learning Outcomes (ILOs)
1.3 Main Content
- 1.3.1 Functions
- 1.3.2 Graph of a function
- 1.3.3 Bounded function
- 1.3.4 Principal values
- 1.3.5 Maxima and Minima
- 1.3.6 Types of function
- 1.3.7 Composite function
- 1.3.8 Inverse function

1.4 Summary
1.5. Conclusion
1.6 References/Further Reading
1.7 SELF Assessment Exercise(s)

1.1 Introduction

In everyday life, many quantities depend on one or more changing variable. For example:

- Speed of a moving car or object depend on distance travelled and time taken
- The voltage of electrical devices depends on current and resistance.
- The volume of given mass of gas depends on the pressure at room temperature (i.e., temperature remain constant)

A function is a phenomenon that relates how one variable or quantity depends on the other variables or quantities.

For instance, in ohm's law V I, mathematically V=IR where R is constant of proportionality.

If I increase, so does the voltage V. If I decrease, so does the voltage. Hence, from this, we can say voltage is a function of current.

2.0 Intended Learning Outcomes (Ilos)

By the end of this unit, you should be able to:

- define functions
- describe the concept of graphs
- solve problems related functions and graphs

3.0 Main Content

1.3.1 Function

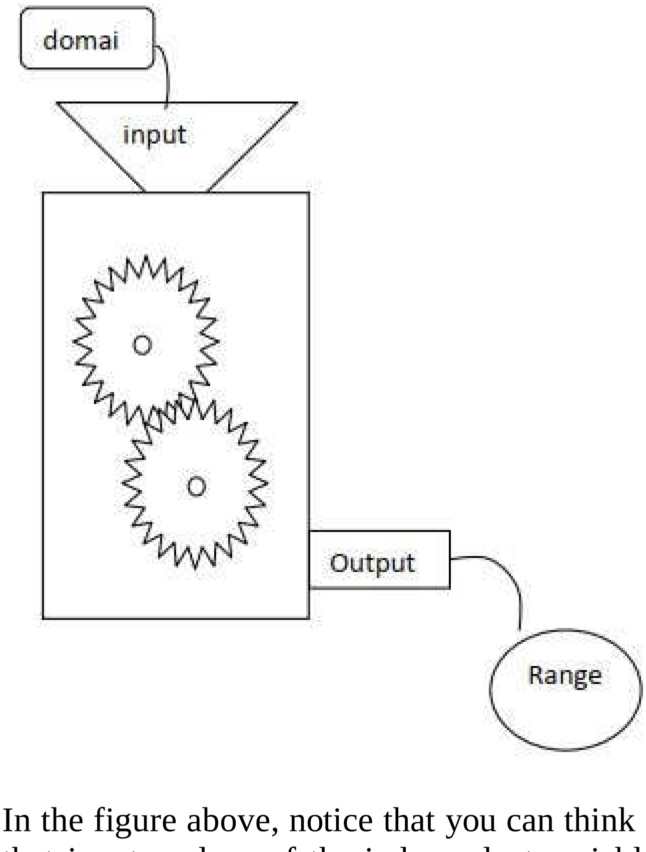
A function is composed of a domain set, a range set, and a rule of correspondence that assigns exactly one element of the range to each element of the domain.

This definition of a function place no restrictions on the nature of the elements of the two sets. If the elements of the domain and range are represented by x and y, respectively, and symbolizes the function, then the rule of correspondence takes the form $y = f(x)$.

The distinction between f and $f(x)$ should be kept in mind. f denotes the function as defined in the first paragraph. y and $f(x)$ are different symbols for the range (or image) values corresponding to domain values x. However, a ''common practice'' that provides an expediency in presentation is to read $f(x)$ as, ''the image of x with respect to the function f '' and then use it when referring to the function. (For example, it is simpler to write sin x than ''the sine function, the image value of which is $sin\ x$.''). This deviation from precise notation will appear in the text because of its value in exhibiting the ideas. The domain variable x is called the independent variable. The variable y representing the corresponding set of values in the range, is the dependent variable.

There are many ways to relate the elements of two sets. [Not all of them correspond a unique range value to a given domain value.] For example, given the equation $y^2 = x$, there are two choices of y for each positive value of x. As another example, the pairs (a, b), (a, c), (a, d) and (a, e) can be formed and again the correspondence to a domain value is not unique.

Because of such possibilities, some texts, especially older ones, distinguish between multiple-valued and single-valued functions. This viewpoint not consistent with our definition or modern presentations. In order that there be no ambiguity, the calculus and its applications require a single image associated with each domain value. A multiple valued rule of correspondence gives rise to a collection of functions (i.e., single-valued). Thus, the rule $y^2 = x$ is replaced by the pair of rules $y = x^{1/2}$ and $y = -x^{-1/2}$ and the functions they generate through the establishment of domains.



In the figure above, notice that you can think of a function as a machine that inputs values of the independent variable and inputs values of the dependent variable. Although function can be described by various means such as table, graphs and diagrams, they are most often specified by formulas or equation. For instance, the equation $y = 4x^2 + 3$ describes y as a function of x. For this function, is the independent variable and y is the dependent variable.

1.3.2 Graph of a function

A function f establishes a set of ordered pairs (x, y) of real numbers. The plot ofthese pairs $(x, f(x))$ in a coordinate system is the graph of f. The result can be thought of as a pictorial representation of the function.

Example 3.2.1

Deciding whether relation are functions.

Which of the equation below define y as a function of x?

- a) $x + y = 1$ b) $x^2 + y^2 = 1$
- c) $x^2 + y = 1$ d) $x + y^2 = 1$

Solution 3.2.1

To decide whether an equation defines a function, it is helpful to isolate the dependent variable on the left side.

For instance, to decide whether the equation $x + y = 1$ defines y as a function of x, write the equation in the form.

$$y = v - x$$

From this form, you can see that for any value of x, there is exactly one value of y. So, y is a function of x.

Original Equation Rewritten Equation Test: Is y a function of x?

- a. $x + y = 1$ $y = 1 - x$ Yes, each value of determines exactly one value of y
- b. $x^2 + y^2 = 1$ $\overline{y = \pm\sqrt{1 - x^2}}$ No, some values of x determine two values of y.
- c. $x^2 + y = 1$ $y = 1 - x^2$ Yes, each value of x determines exactly one value of y.
- d. $x + y^2 = 1$ $y = \pm\sqrt{\overline{1 - x}}$ No, some value of x determines two values of y.

Note that the equations that assign two values $(\pm)$ to the dependent variable for a given value of the independent variable do not define functions of x. For instance, in part (b), when $x = 0$, the equation $y = \pm\sqrt{1 - x^2}$ indicates that $y = \pm 1$ or $y = -1$.

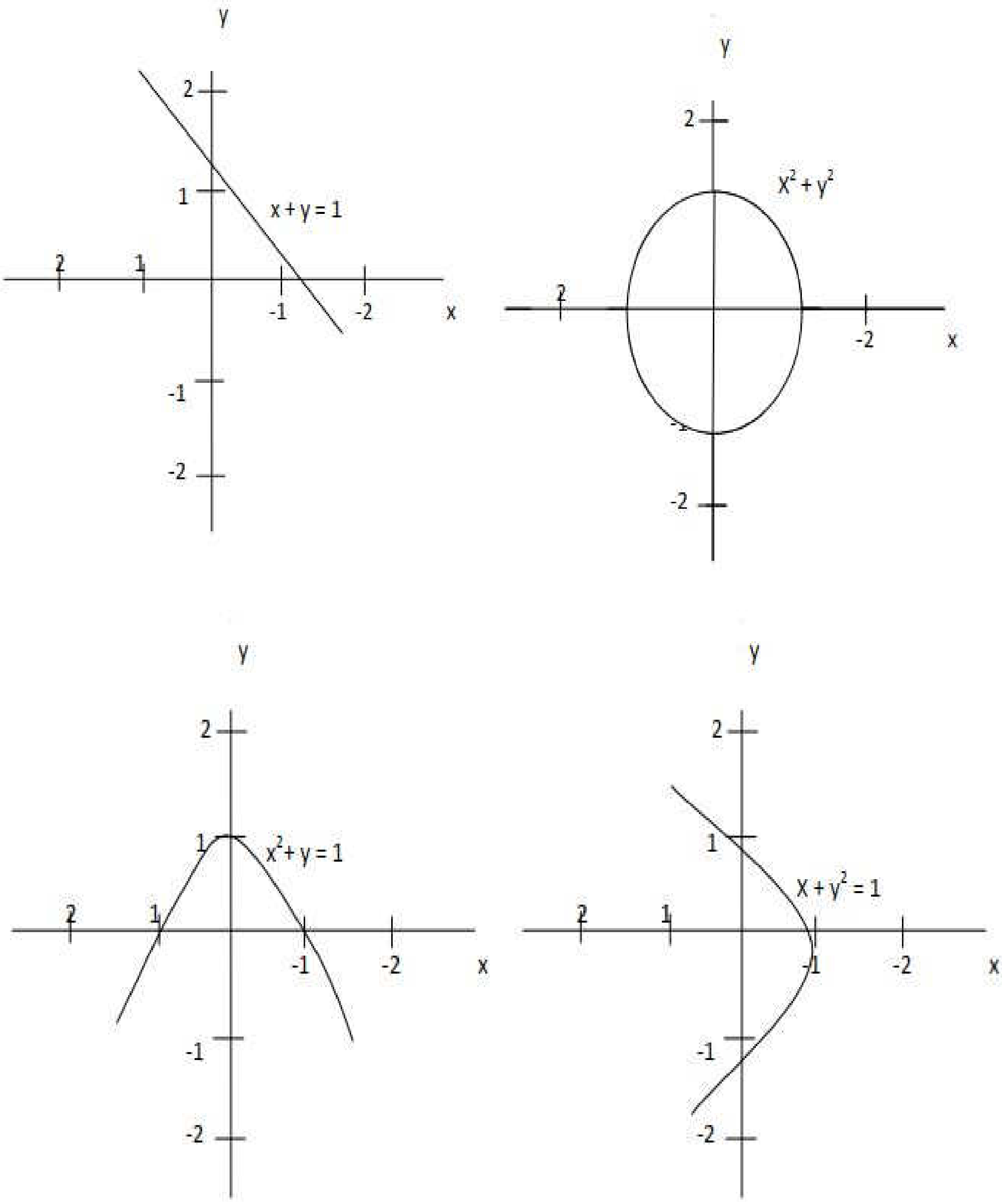


Fig 3.2.1: shows the graphs of the four equations

Checkpoint 1:

Which of the equations below define y as a function of x?

- a. $x - y = 1$ b. $x^2 + y^2 = 4$ c. $y^2 + x = 2$ d. $x^2 - y = 0$ e. $2x + y = 6$ f. $x^2 + y^2 = 1$

Example 3.2.2

Determine whether each equation defines as a function of x:

a. $x^2 + y = 4$ b. $x^2 + y^2 = 4$

Solution 3.2.2

Solve each equation for y in terms of x. If two or more values of y can be obtained for a given x, the equation is not a function.

a. $x^2 + y = 4$ **This is the given equation**.

$x^2 + y - x^2 = 4 - x^2$ **Solve for y by subtracting x^2 from** -

both sides.

$y = 4 - x^2$ **Simplify.**

From this last equation we can see that for each value of x, there is one and only one value of y. For example, if $x = 1$, then $y = 4 - 1^2 = 3$. The equation defines y as a function of x.\

b. $x^2 + y^2 = 4$ **This is the given equation**
$x^2 + y^2 - x^2 = 4 - x^2$ **Isolate y^2 by subtracting x^2 from both**

sides.

$y^2 = 4 - x^2$ **Simplify.**

$y = \pm\sqrt{4 - x^2}$ **Apply the square root properly: if $u^2 = d$ then** $u = \pm\sqrt{d}$

Then $\pm$ in this last equation shows that for certain values of x (all values between -2 and 2), there are two values of y.

For example, if $x = 1$, then $y = \pm\sqrt{4} - 1^2 = \pm\sqrt{3}$. For this reason, the equation does not define y as a function of x.

Example 3.2.3

The graphs of the functions described by $y = x^2, -1 \leq x \leq 1$, and $y^2 = x, 0 \leq x \leq 1, y \geq 0$ appear in **Fig. 3 2.2**.

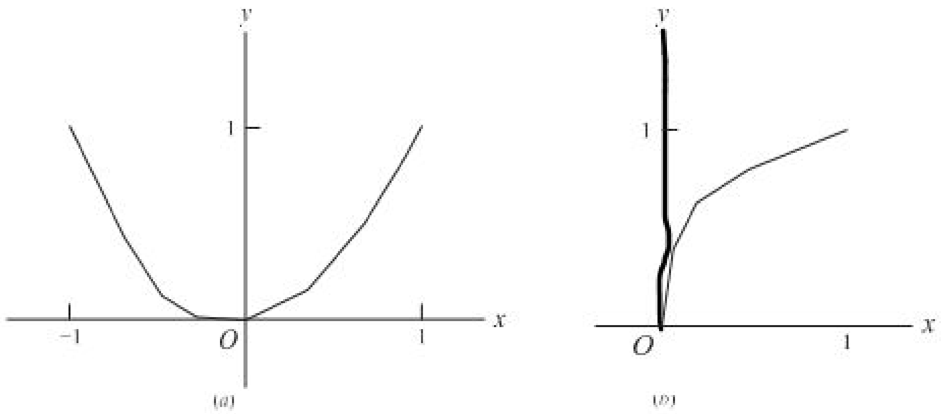


Fig. 3 2.2: The graphs of functions

- $y = x^2$
- (b) $y^2 = x, \geq 0$

1.3.3 Bounded functions

If there is a constant M such that $f(x) \leq M$ for all x in an interval (or other set of numbers), we say that f is bounded above in the interval (or the set) and call M an upper bound of the function. If a constant m exists such that $f(x) \geq m$ for all x in an interval, we say that $f(x)$ is bounded below in the interval and call m a lower bound.

If $m \leq f(x) \leq M$ in an interval, we call $f(x)$ bounded. Frequently, when we wish to indicate that a function is bounded, we shall write $|f(x)| < P$.

Example 3.3.1

(a) $f(x) = 3 + x$ is bounded in $-1 \leq x \leq 1$. An upper bound is 4 (or

any number greater than 4). A lower bound is 2 (or any number less than 2).

(b) $f(x) = \frac{1}{x}$ is not bounded in $0 < x < 4$ since by choosing x sufficiently close to zero, f(x) can be made as large as we wish, so that there is no upper bound. However, a lower bound is given by $\frac{1}{4}$ (or any number less than $\frac{1}{4}$).

If $f(x)$ has an upper bound it has a least upper bound $(l.u.b.)$; if it has a lower bound it has a greatest lower bound $(g.l.b.)$.

1.3.4 Inverse functions. Principal values

Suppose y is the range variable of a function f with domain variable x.

Furthermore, let the correspondence between the domain and range values be one-to-one. Then a new function f^{-1}, called the inverse function of f, can be created by interchanging the domain and range of f.

This information is contained in the form $x = f^{-1}(y)$.

As you work with the inverse function, it often is convenient to rename the domain variable as x and use y to symbolize the images, then the notation is $y = f^{-1}(x)$. In particular, this allows graphical expression of the inverse function with its domain on the horizontal axis.

Note: f^{-1} does not mean f to the negative one power. When used with functions the notation f^{-1} always designates the inverse function to f. If the domain and range elements of f are not in one-to-one correspondence (this would mean that distinct domain elements have the same image), then a collection of one-to-one functions may be created. Each of them is called a branch. It is often convenient to choose one of these branches, called the principal branch, and denote it as the inverse function, f^{-1}. The range values of f that compose the principal branch, and hence the domain of f^{-1}, are called the principal values.

Example 3.4.1

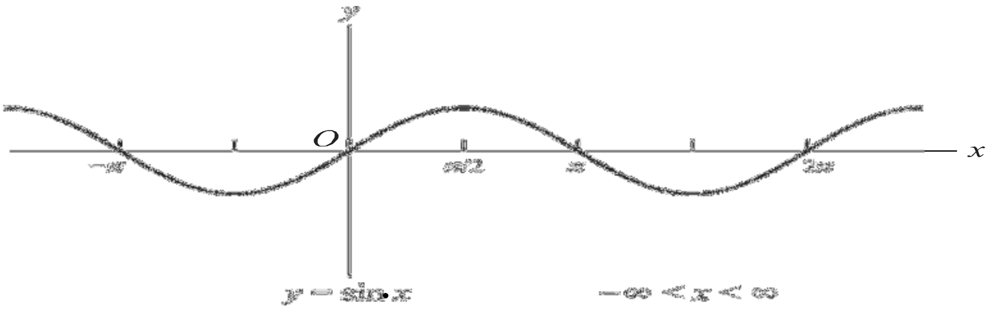
Suppose f is generated by $y = sin\, x$ and the domain is $-\infty \leq x \leq \infty$.

Then. there are an infinite number of domain values that have the same image. (A finite portion of the graph is illustrated below in **Fig.3.4.1**(a).

In **Fig.3.4.1**(b) the graph is rotated about a line at 45^0 so that the x-axis rotates into the y-axis. Then the variables are interchanged so that the x-axis is once again the horizontal one. We see that the image of an x value is not unique. Therefore, a set of principal values must be chosen to establish an inverse function. A choice of a branch is accomplished by restricting the domain of the starting function, $\sin x$. For example, choose $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Then there is a one-to-one correspondence between the elements of this domain and the images in $-1 \leq x \leq 1$.

Thus, f^{-1} may be defined with this interval as its domain. This idea is illustrated in **Fig.3.4.1** (c) and **Fig.3.4.1** (d). With the domain of f^{-1} represented on the horizontal axis and by the variable x, we write $y = sinx, -1 \leq x \leq 1$, if $x = -\frac{1}{2}$ then the corresponding range value is $y = -\frac{\pi}{6}$.

N**ote:** In algebra, b^{-1} means $\frac{1}{b}$ and the fact that bb^{-1} produces the identity element 1 is simply a rule of algebra generalized from arithmetic. Use of a similar exponential notation for inverse functions is justified in that corresponding algebraic characteristics are displayed by $f^{-1}[f(x)] = x$ and $f[f^{-1}(x)] = x$.



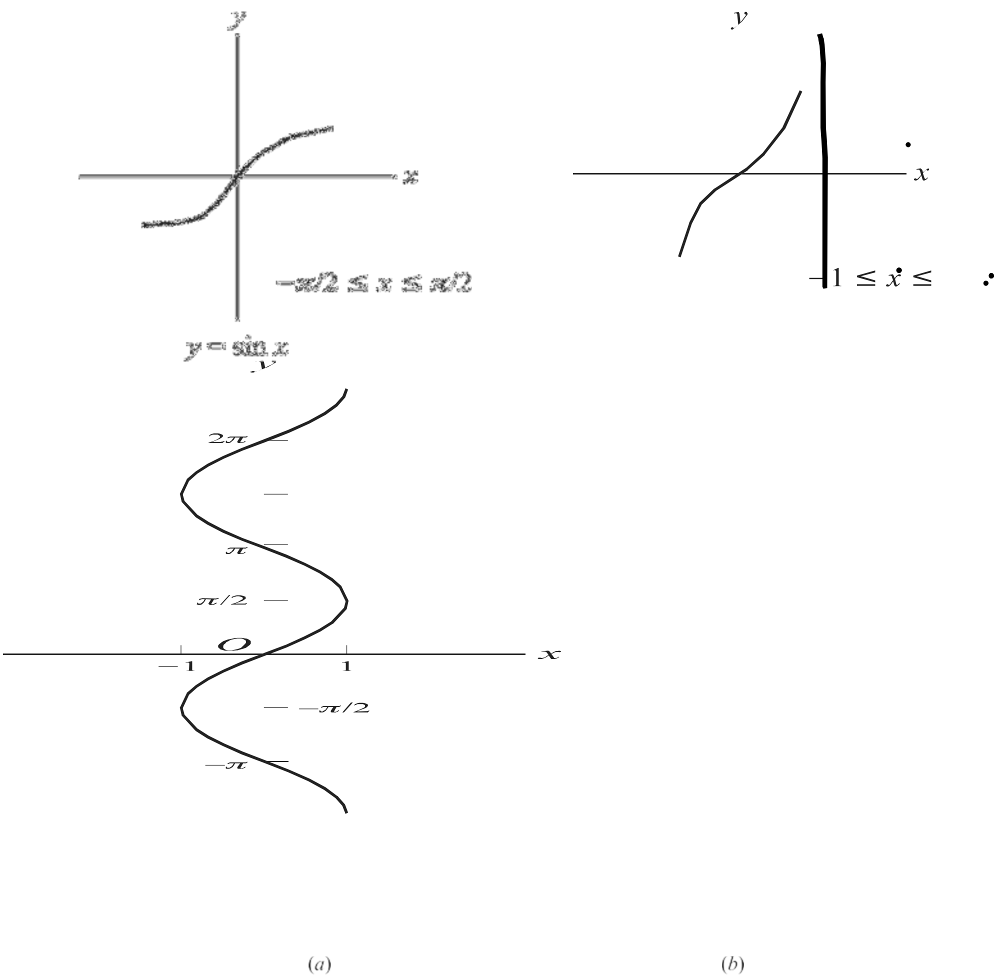


Fig.3.4.1: Graph of domain

1.3.5 Maxima and Minima

The seventeenth-century development of the calculus was strongly motivated by questions concerning extreme values of functions. Of most importance to the calculus and its applications were the notions of *local extrema*, called *relative maximums* and *relative minimums*.

If the graph of a function were compared to a path over hills and through valleys, the local extrema would be the high and low points along the way. This intuitive view is given mathematical precision by the following definition.

Definition 3.5.1

If there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all x other than c in the interval, then $f(c)$ is a relative maximum of f.

If $f(x) > f(c)$ for all x in (a, b); other than c, then f(c) is a relative minimum of f.

Functions may have any number of relative extrema. On the other hand, they may have none, as in the case of the strictly increasing and decreasing functions previously defined.

Definition 3. 5.2

If c is in the domain of f and for all x in the domain of the function $f(x) \leq f(x)$, then $f(c)$ is an absolute maximum of the function f. If for all x in the domain $f(x) \geq f(x)$ then f(c) is an absolute minimum of f. (See **Fig 3.5.1**)

Note 1: If defined on closed intervals the strictly increasing and decreasing functions possess absolute extrema.

Absolute extrema are not necessarily unique. For example, if the graph of a function is a horizontal line, then every point is an absolute maximum and an absolute minimum.

Note 2: A point of inflection also is represented in **Fig 3.5.1**. There is an overlap with relative extrema in representation of such points through derivatives.

1.3.6 Types of functions

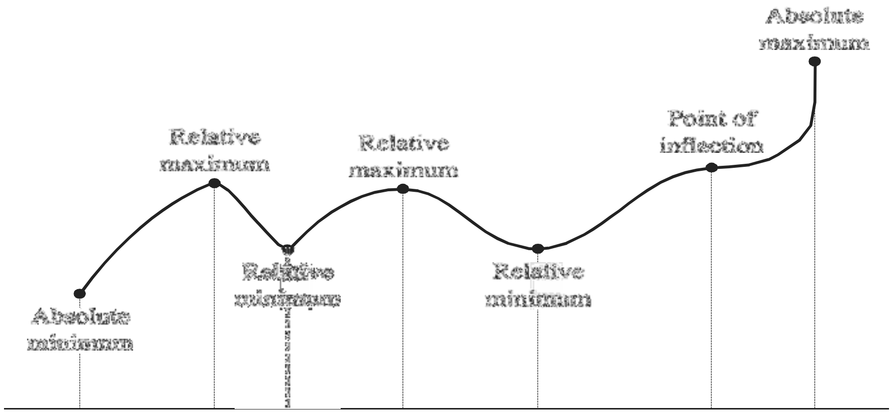


Fig 3.5.1: Maximum point

It is worth realizing that there is a fundamental pool of functions at the foundation of calculus and advanced calculus. These are called elementary functions. Either they are generated from a real variable $s$by the fundamental operations of algebra, including powers and roots, or they have relatively simple geometric interpretations. As the title ‘‘elementary functions’’ suggests, there is a more general category of functions (which, in fact, are dependent on the elementary ones). Some of these will be explored later in the book. The elementary functions are described below.

1.3.6 Polynomial functions

The polynomial functions are of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

where $a_0, \ldots a_n$ are constants and n is a positive integer called the degree of the polynomial if $a_0 \neq 0$.

The fundamental theorem of algebra states that in the field of complex numbers every

polynomial equation has at least one root. As a consequence of this theorem, it can be proved that every nth degree polynomial has n roots in the complex field. When complex numbers are admitted, the polynomial theoretically may be expressed as the product of n linear factors; with our restriction to real numbers, it is possible that 2k of the roots may be complex. In this case, the k factors generating them will be quadratic.

(The corresponding roots are in complex conjugate pairs.) The polynomial
$x^3 - 5x^2 + 11x - 15 = (x - 3)(x^2 - 2x + 5)$ Illustrates this thought.

3.6.2 Algebraic functions

Algebraic functions are functions $y = f(x)$ satisfying an equation of the form

$$P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_{n-1}(x)y + P_n(x) = 0$$

where $P_0(x), \ldots P_n(x)$ are polynomial in x.

If the function can be expressed as the quotient of two polynomials, i.e., P(x)/Q(x) where P(x) and Q(x) are polynomials, it is called a rational algebraic function; otherwise, it is an irrational algebraic function.

3.6.3. Transcendental functions are functions which are not algebraic, i.e., they do not satisfy equations of the form (2). Note the analogy with real numbers, polynomials corresponding to integers, rational functions to rational numbers, and so on.

The following are sometimes called elementary transcendental functions:

- **Exponential function:** $f(x) = a^x, a \neq 0,\ 1.$

- **Logarithmic function:** $f(x) = log_a x,\ a \neq 0, 1.$ This and the exponential function are inverse function. If a ¼ e ¼ 2:71828 . . .; called the natural base of logarithms, we write $f(x) = log_e x = lnx$, called the natural logarithm of x

- **Trigonometric functions:** (Also called circular functions because of their geometric interpretation with respect to the unit circle):

$$sinx, cosx, tanx = \frac{sinx}{cosx}, cscx = \frac{1}{sinx}, secx = \frac{1}{cosx}, cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$$

The variable x is generally expressed in radians (π radians =180^0). For real values of x, $sin\ x$ and $cos\ x$ lie between -1 and 1 inclusive.

The following are some properties of these functions:

$$\sin^2 x + \cos^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin(x \pm y) = sinxcosy \pm cosxsiny, \qquad \sin(-x) = -sinx$$

$$\cos(x \pm y) = cosxcosy \mp sinxsiny, \qquad \cos(-x) = cosx$$

$$\tan(x \pm y) = \frac{tanx \pm tany}{1 \mp tanxtany}, \qquad \tan(-x) = -tanx$$

- **Inverse trigonometric functions**: The following is a list of the inverse trigonometric functions and their principal values:

- $y = \sin^{-1} x\,,\ (-\pi/2 \le y \le \pi/2\,)$
- $y = \csc^{-1} x = \sin^{-1}\frac{1}{x},\ \left(-\frac{\pi}{2} \le y \le \frac{\pi}{2}\right)$
- $y = \cos^{-1} x\,,\ (0 \le y \le \pi/)$
- $y = \sec^{-1} x = \cos^{-1}\frac{1}{x},\ (0 \le y \le \pi\,)$
- $y = \tan^{-1} x\,,\ (-\frac{\pi}{2} < y < \pi/2\,)$
- $y = \cot^{-1} x = \pi/2 - \tan^{-1} x\,,\ (0 < y < \pi)$

- **Hyperbolic functions** are defined in terms of exponential functions as follows. These functions may be interpreted geometrically, much as the trigonometric functions but with respect to the unit hyperbola.

- $sinhx = \frac{e^x - e^{-x}}{2}$
- (ii) $cschx = \frac{1}{sinhx} = \frac{2}{e^x - e^{-x}}$
- $coshx = \frac{e^x + e^{-x}}{2}$
- (iv) $sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$
- $tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-1}}$
- (vi) $cothx = \frac{coshx}{sinhx} = \frac{e^x + e^{-1}}{e^x - e^{-x}}$

- **Inverse hyperbolic functions.** If $x = sinhy$ then, $x = \sinh^{-1} x$ is the inverse hyperbolic $sine$ of x. The following list gives the principal values of the inverse hyperbolic functions in terms of natural logarithms and the domains for which they are real.

- $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all x
- (ii) $\cosh^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right), x \ne 0$
- $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), x \ge 1$
- $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right), 0 < x \le 1$
- $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), |x| < 1$
- $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), |x| > 1$

Example 3.6.1

Find the domain and range of each function

- $y = \sqrt{x-1}$
- b. $y = \begin{cases} 1-x, & x < 1 \\ \sqrt{x-1}, & x \geq 1 \end{cases}$

Solution 3.6.1

- Because $\sqrt{x-1}$, is not defined for $X - 1 < 0$ (that is for $x < 1$), it follows that the domain of the function is the interval $x \geq 1$ or $[1, \infty]$. To find the range, we observe, $\sqrt{x-1}$ is never negative. Moreover, as x takes on the various values in the domain, y takes on all non-negative values. So, the range is the interval $y \geq 0$ or $[0, \infty]$.

- Because this function is defined for x < 1 and for $x \geq 1$, the domain is the entire set of real numbers. This function is called **piecewise – defined function** because it is defined by two or more equations over a specified domain.

When $x \geq 1$, the function behaves as in parts (a). For $x < 1$, the values of $1 - x$
is positive, and therefore the range of the function is $y \geq 0$ or $[0, \infty]$.

A function is **one to one** if to each value of the dependent variable in the range there corresponds exactly one value of the independent variable.

For instance, the function in **Example 3.6.1**(a) is one to one, whereas the function in **Example 3.6.1**(b) is not one- to -one

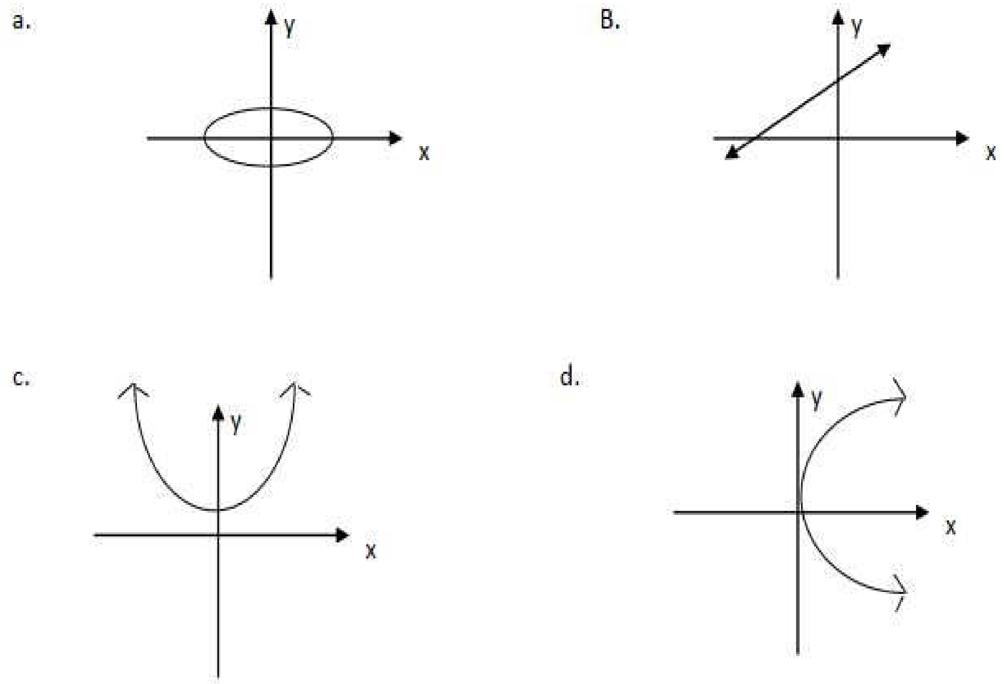
Geometrically, a function is one- to- one if every horizontal line intersects the graph of the function at most once. This geometrical interpretation is the **horizontal line test** for one-to-one functions. So, a graph that represents a one- to –one test must satisfy both the vertical line test and the horizontal line test.

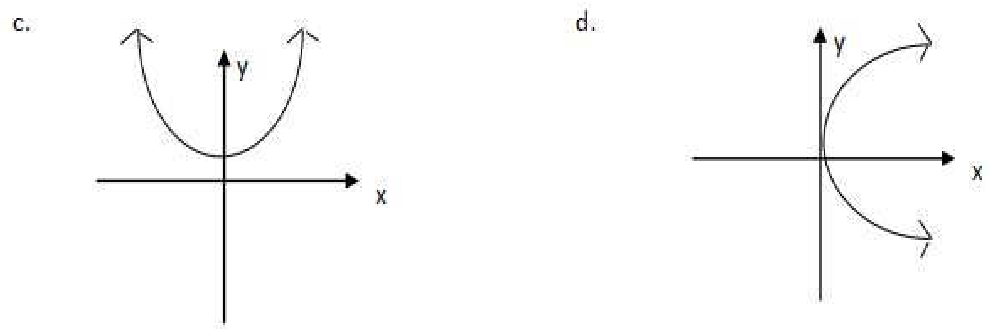
3.6.3 The vertical line test for function:

If any vertical line intersects a graph in more than one point, the graph does not define y as a function of x.

Example 3.6.3.1

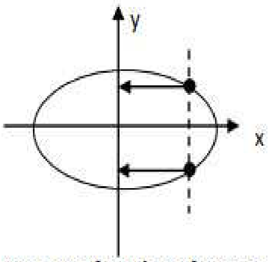
Use the vertical line test to identify graphs in which y is a function of x





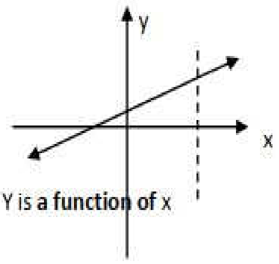
Solution 3.6.3.1

a.



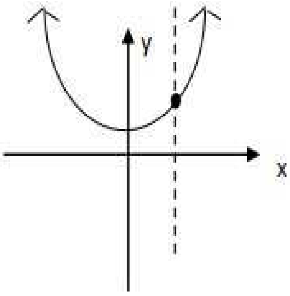
y is **not a function of** x. Two values of y corresponds to an x - value

B.

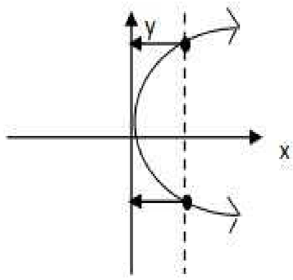


Y is **a function of** x

c.



d.



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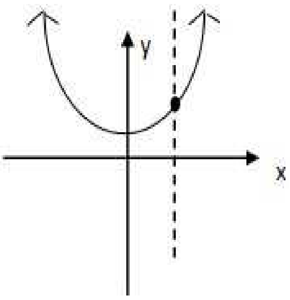

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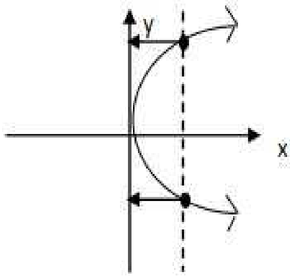

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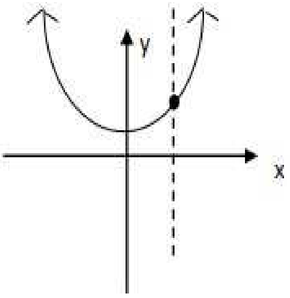

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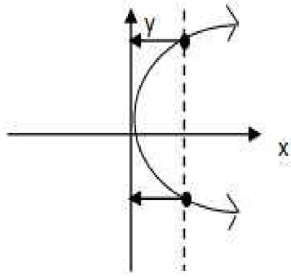
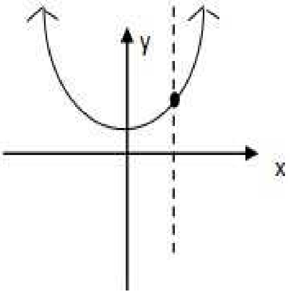
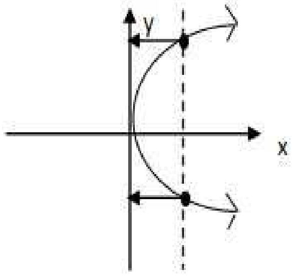


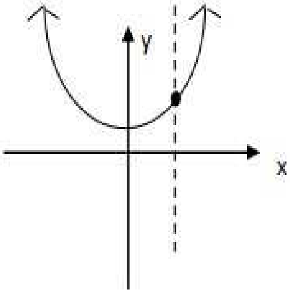
c.



d.



c.



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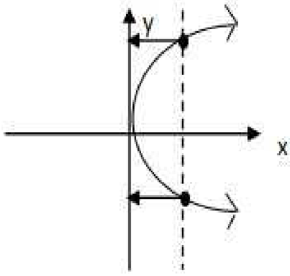

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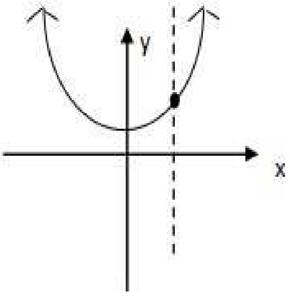

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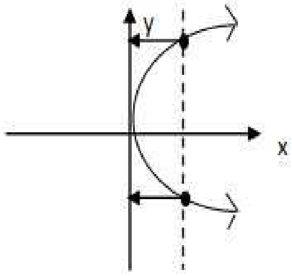

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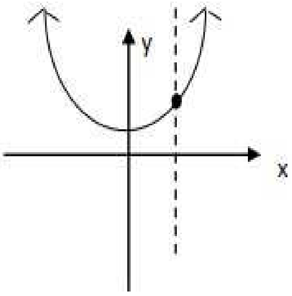

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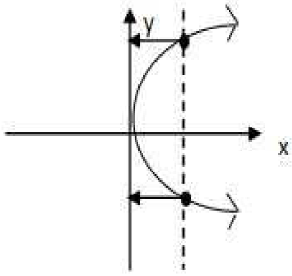

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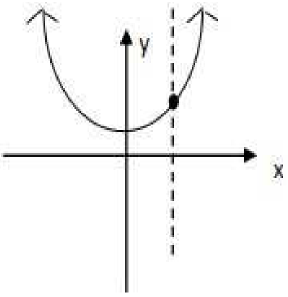

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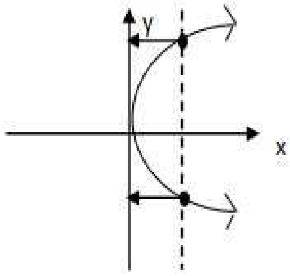

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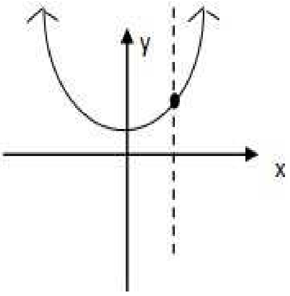

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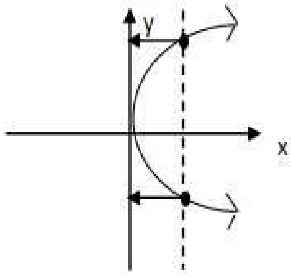

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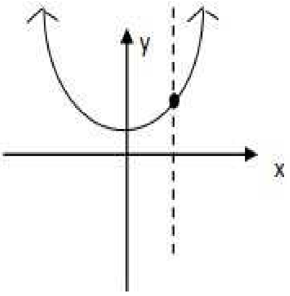

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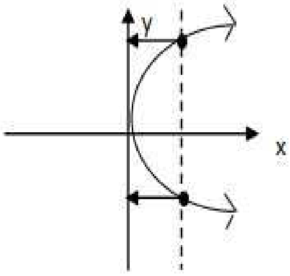

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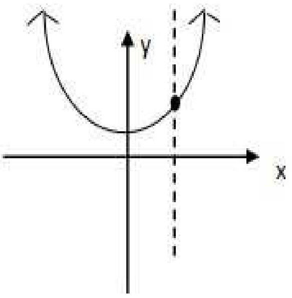

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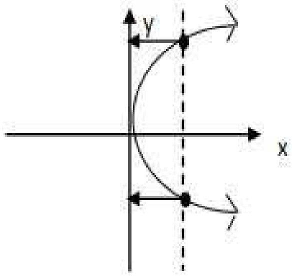

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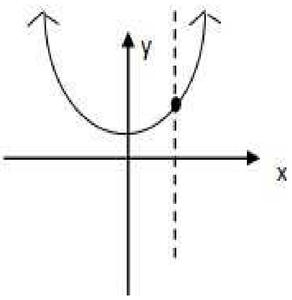

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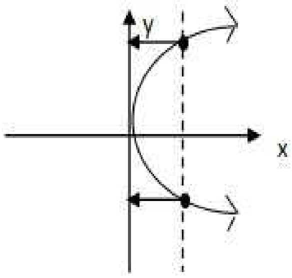

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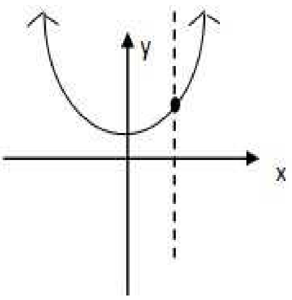

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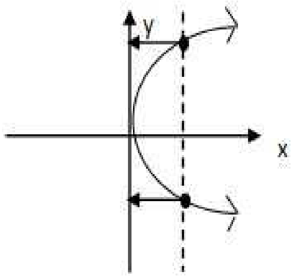


- *y* is **a function of** x
- *y* is **not a function of** x.

of y correspond to an x-value. Two values.

Example 3.6.3.2

Find the value of $f(x) = x^2 - 5x + 1$ when x is 0, 1, and 4. Is f one to one?

Solution 3.6.3.2

When x = 0, the value of
f is $f(0) = 0^2 - 5(0) + 1 = 1$
When x = 1, the value of f is $f(1) = 1^2 - 5(1) + 1$
$= 1 - 5 + 1 = -5$

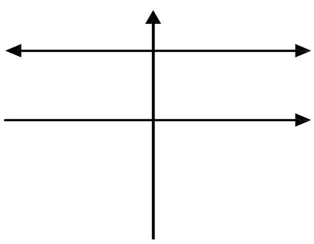
When x = 4, the value of f(x) is
$= 4^2 - 5(4) + 1$
$= 16 - 20 + 1 = -3$

Checkpoint

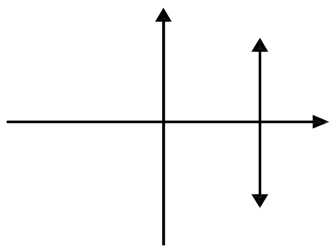
In the following exercise, use the vertical line test to identify graphs in which y is a function of x.

i.

y

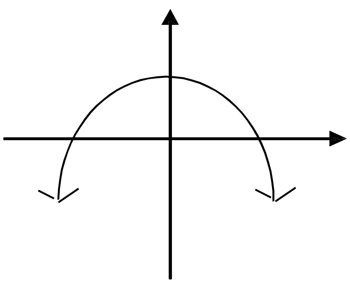


ii

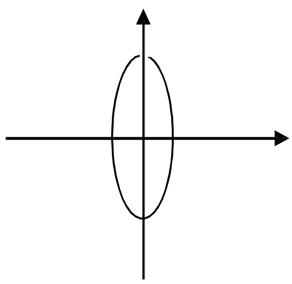


iii

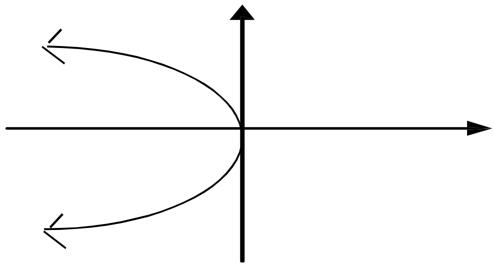
y



iv



v



1.3.7 combinations of functions (composite functions) Definition

The function given by $(f.g)\ (x) = f\ (g(x))$ is the composite of f with g. The domain of $(f.g)$ is the set of all x in the domain of g such that $g(x)$ is the domain of f.

Two functions can be combined in various ways to create new functions.

For instance, if $f\ (x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the following functions.

- $f(x) + g(x) = (2x - 3) + (x^2 + 1) = x^2 + 2x - 2$ **Sum**
- $(f(x) - g(x) = (2x - 3) - (x^2 + 1) = -x^2 + 2x - 4$ **Difference.**
- $f(x)\ g(x) = (2x - 3)\ (x^2 + 1) = 2x^3 - 3x^2 + 2x - 3$ **Product.**
- $\frac{f(x)}{g(x)} = \frac{2x-3}{x^2+1}$ **Quotient/ division**

Example 3.7.1

Forming composite functions

Let $f(x) = 2x - 4$ and $g(x) = x^2 + 3$, and find (a). $f(g(x))$ (b). $g(f(x))$

Solution 3.7.1

- The composite off with g is given by

$g(f(x)) = 2(g(x)) - 4$ **Evaluate f at g(x).**

$= 2(x^2 + 3) - 4$ **Substitute x^2+3 for g(x).**

$= 2x^2 + 6 - 4$ **Simplify**.

$= 2x^2 - 2$

$= 2(x^2 - 1)$

- The composite of g with f is given by

$g(f(x)) = (f(x))^2 + 1$ **Evaluate g at f(x).**

$= (2x - 4)^2 + 1$ **Substitute 2x – 4 for f(x).**

$= 4x^2 - 6x + 16 + 1$ **Simplify.**

$= 4x^2 - 16x + 17$

Checkpoint

Let $f(x) = 2x + 1$ and $g(x) = x^2 + 2$, and find a) $f(g(x))$. b) $g(f(x))$.

1.3.8 Inverse function

Definition of the inverse of a function. Let f and g be two functions such that: $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f.

Under these conditions, the function g is the **inverse** of the function f. The function is donated by f^{-1} which is read as f- inverse". So,

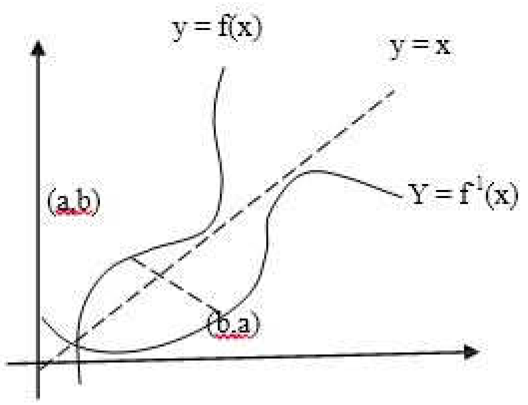
$f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$

The domain off must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Example 3.8.1

Several functions and their inverse are shown below. In each case, note that the inverse function "undoes" the original function. For instance, to undo multiplication by 2, you should divide by 2.

- $f(x) = 2x$
- $f^{-1}(x) = \frac{1x}{2}$
- b) $f(x) = 1/5x$ f^{-1}(x) = 5x
- c) $f(x) = x + 8$ f^{-1} (x) = x – 8
- d) $f(x) = 3x + 7$ f^{-1} (x) = 1/3 (x + 7)
- (e) f(x) = x^3 $f^{-1}(x) = 3\sqrt{x}$
- f) $f(x) = \frac{1}{x}$ $f^{-1}(x) = \frac{1}{x}$



The graphs of f andf^{-1} are mirror images of each other (with respect to the line $y = x$, as in **Fig 3.8.1**

Checkpoint

Informally find the inverse function of each function a. $f(x) = \frac{1}{5x}$
b $f\ (x)\ =\ 3x + 2$

Example 3.8.2

Find the inverse function of $f(x)\ =\ \sqrt{2x-3}$

Solution 3.8.1

Begin by substituting $f(x)$ with y. Then interchange x and y and solve for y

$f(x) = \sqrt{2x-3}$ Write original $y = \sqrt{2x-3}$
function. Replace $f(x)$ with y.

$x = \sqrt{2y-3}$ Interchange x and y. $x^2 = 2y-3$ square both side
$x^2 + 3 = 2y$ Add 3 to each side.

$\frac{x^2+3}{2} = y$ Divide each side by 2.

So, the inverse function has the form $f^{-1}(x) = \frac{x^2+3}{2}$
Using x as the independent variable, you can write $f^{-1}(x) = \frac{x^2+3}{2}$ as $x \geq 0$.

Note that the domain of $f^{-1}(x)$ coincides with the range of f.
After you have found the inverse of a function, you should check your result. You can check your results graphically by observing that graphs of f and $f^{-1}(x)$ are reflections of each other in the line y =x. You can check your results algebraically by evaluating $f(f^{-1}(x))$ and $f^{-1}(f(x))$ – both should be equal to x.

Check that $f(f^{-1}(x)) = x$ and that $f^{-1}\left(f(x)\right) = x$
$f(f^{-1}(x)) = f(\frac{x^2+3}{2})$ and $f^{-1}\left(f(x)\right) = f^{-1}(\sqrt{2x-3})$

$$f^{-1}\left(f(x)\right) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2}$$

$$= \sqrt{x^2} = \frac{2x}{2}$$

$$= x, x \geq \frac{3}{2}$$

Checkpoint

Find the inverse of function of $f(x) = x^2 + 2$ for $x \geq 0$
.

Note: Not every function has an inverse function. In the fact, for a function to have an inverse function, it must be one- to- one.

Example 3.8.2

A function that has no inverse function.

Show that the function $f(x) \ = \ x^2 - 1$ has no inverse function. (Assume that the domain of f is the set of all real numbers)

Solution

Begin by sketching the graph off, as shown in figure 1.25 Note that
$f(x) \ = \ x^2 - 1$
$f(2) \ = \ 2^2 - 1$
$= \ 2$

So, f does not pass the horizontal line test, which implies that if is not one- to- one and therefore has no inverse function. The same conclusion can be obtained by trying to find the inverse of f algebraically.

f (x) = x^2 – 1	**Write original function.**
y = x^2 – 1	**Replace f(x) with y**
x = y^2 – 1	**Interchange x and y**
x + 1 = y^2	**Add 1 to each side**
$\pm\sqrt{x+1} = y$	**Take square root of each side**

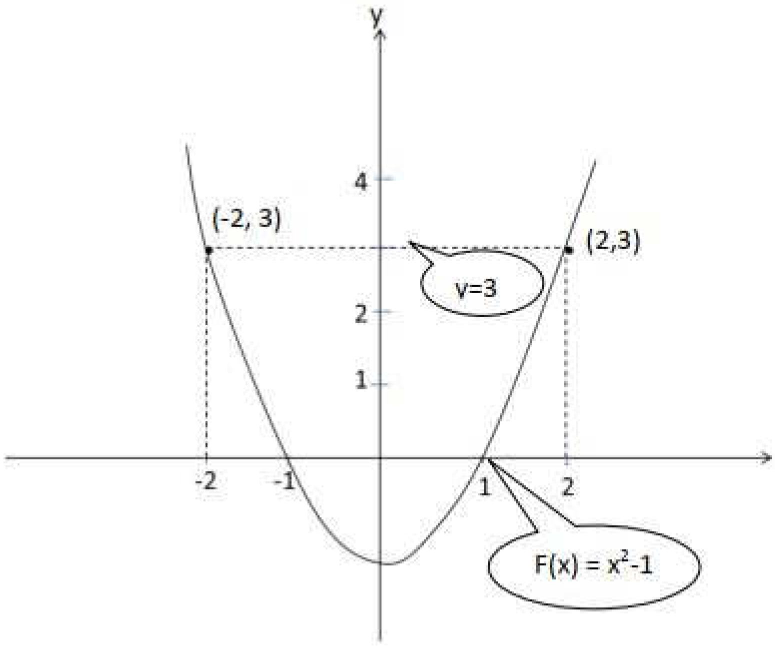
The last equation does not define y as a function of x, and so f has no inverse function y

F (x) = x^2 – 1 write original function. Y = x^2 – 1
Replace $f(x)$ with y
X = y^2 – 1 Interchange x and y

$X + 1 = y^2$ Add1 to each side

$\pm\sqrt{x+1} = y$ Take square root of each side

The last equation does not define y as a function of x, and so f has no inverse function



Checkpoint:

Show that the function $f(x) = x^2 - +4$ has no inverse function.

1.4 Summary

This unit has exposed you to the various definitions of functions and graphs, its concept and broadens your knowledge about events. The unit also exposed you to the theory of functions and graphs. This unit is structured in such a way that you will understand what lies ahead in the other units to follow.

1.5 Conclusion

- A function is a correspondence from a first set, called the domain to a second set, called the range such that each element in the domain corresponds to exactly one element in the range.

- A function is one -to -one if each value of the dependent variable in the range there corresponds exactly one value of the independent variable

1.6 References/Further Reading

Calculus an Applied Approach Larson Edwards Sixth Edition Blitzer Algebra and Trigonometry Custom 4th edition Engineering Mathematics by K.A Stroud.

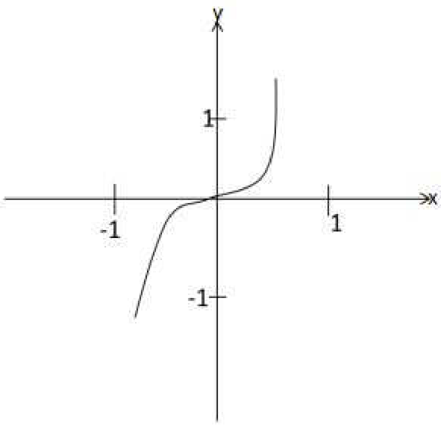
1.7 Self-Assessment Exercises

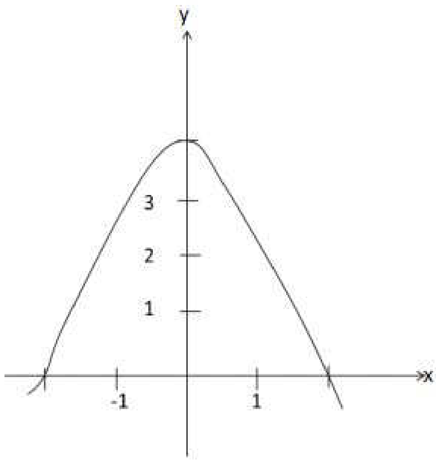
- In the following questions i – iv, decide whether the equation define y as a function of x

- $x^2 + y^2 = 4$
- $\frac{1}{2}x - 6y = -3$
- $x^2 + y = 4$
- $y^2 = x^2 - 1$

- In the following exercises i - iii, find the domain and range of the function. Use interval notation to write your result.

- $f(x) = x^3$
- $f(x) = 4 - x$





- In the following exercises i- ii, evaluate the function at the specified the values of the independent variable. Simplify the result.
- $f(x) = 2x - 3$
  (a) $f(0)$ (b) $f(-3)$ (c) $f(x-1)$ (d) $f(1+\Delta x)$
- $g(x) = \frac{1}{x}$

(a) $g(2)$ (b) $g(¼)$ (c) $g(x + \Delta x)$ (d) $g(x + \Delta x) - g(x)$

- In the following exercises i – iii, evaluate the difference quotient and simplify the result

- $f(x) = x^3 - x$

- $g(x) = \sqrt{x+3}$
- $\frac{f(x-\Delta x)-f(x)}{\Delta x} \frac{g(x+\Delta x)-g(x)}{\Delta x}$
- $f(x) = \frac{1}{x-2}$ $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

- Find (a) $f(x)+g(x)$ (b) $f(x)g(x)$ (c) $\frac{f(x)}{g(x)}$ (d) $f\big(g(x)\big)$ (e) $g(f(x))$ if defined.

- $f(x) = 2x - 5, g(x) = 5$
- $f(x) = x^2 + 1$ $g(x) = x - 1$

- Given that $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the composite functions:

a. $f(g(1))$ b. $g(f(1))$ c. $g(f(0))$ d. $f(g(-4))$ e. $f(g(x))$ f. $g(f(x))$

- In exercises 15 – 16, show that f and g are inverse functions by showing that $f(g(x)) = x$ and $g\,(f(x)) = x$. Then sketch the graphs of f and g on the same coordinate axes.
-
- $f(x) = 5x + 1$ $g(x) = \frac{x-1}{5}$
- $f(x) = 9 - x^2,\ x \geq 0$ $g(x) = \sqrt{9-x},\ x \leq 9$

- Find the inverse function of f. Then sketch the graph of f and f^{-1} on the same coordinate axis. $f(x) = \sqrt{9-x^2},\ \ 0 \leq x \leq 3$
- In the exercises i and ii, use the vertical line test to determine whether y is a function of x.

- $x^2 + y^2 = 9$
- $x^2 = xy - 1$

Unit 2: LIMITS

Unit Structure

1.1 Introduction

In everyday language, one always refers to limit one's endurance, speed limit of a car, a wrestler's weight limit or stretching a spring to its limit.

These phrases all suggest that a limit is a bound, which on some instances may not be reached but on other instance may be reached or exceeded.

Hooke's law is a perfect illustration of limit which states provided an elastic limit of a spring is not exceeded; the extension (e) is directly proportional to the tension or force acting on it. That is a spring has a limit of extension when a load is suspended on it. If it exceeds the boundary, it will reach a point of plasticity or break without returning to its initial position.

2.0 Intended Learning Outcomes (ILOs)

By the end of this unit, you should be able to:

- define limit of functions
- describe the concept of limit of functions
- solve problems related limit of functions

3.0 Main Content

2.3.1 Definition of Limit of a Function

Let $f(x)$ be defined and single-valued for all values of x near $x = x_0$ with the possible exception of $x = x_0$ itself (i.e., in a deleted neighborhood of x_0). We say that the number l is the limit of $f(x)$ as x approaches x_0 and write $\lim\limits_{x \to x_0} f(x) = l$ if for any positive number ϵ (however small) we can find some positive number δ (usually depending on) such that $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$. In such case we also say that $f(x)$ approaches l as x approaches x_0 and write $f(x) \to l$ as $x \to x_0$. In words, this means that we can make $f(x)$ arbitrarily close to l by choosing x sufficiently close to x_0.

Example 3.1.1

Let $(x) = \begin{cases} x^2 & if \;\; x \neq 2 \\ 0 & if \;\; x = 2 \end{cases}$. Then as x gets closer to 2 (i.e., x approaches 2), $f(x)$ gets closer to 4. We thus suspect that $\lim\limits_{x \to 2} f(x) = 4$. To prove this, we must see whether the above definition of limit (with $l = 4$) is satisfied.

Note that $\lim\limits_{x \to 2} f(x) \neq 2$, i.e., the limit of $f(x)$ as $x \to 2$ is not the same as the value of $f(x)$ at $x = 2$ since $f(x) = 2$ by definition. The limit would in fact be 4 even if $f(x)$ were not defined at $x = 2$. When the limit of a function exists, it is unique.

2.3.2 Right- and Left-Hand Limits

In the definition of limit no restriction was made as to how x should approach x_0. It is sometimes found convenient to restrict this approach.

Considering x and x_0 as points on the real axis where x_0 is fixed and x is moving, then x can approach x_0 from the right or from the left. We indicate these respective approaches by writing $x \to x_0^+$ and $x \to x_0^-$.

If $\lim\limits_{x \to x_0^+} f(x) = l_1$ and $\lim\limits_{x \to x_0^-} f(x) = l_2$, we call l_1 and l_2 , respectively, the right- and left-hand limits of f at x_0 and denote them by $f(x_0^+)$ or $f(x_0 + 0)$ and $f(x_0 -)$ or $f(x_0 - 0)$. The δ, ϵ definitions of limit of $f(x)$ as $x \to x_0^+$ or $x \to x_0^-$ are the same as those for $x \to x_0$ except for the fact that values of x are restricted to $x > x_0$ or $x < x_0$, respectively.

We have $\lim_{x\to x_0} f(x) = l$ if and only if $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = l$

2.3.3 Theorems on Limits

If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$, then

- $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = A + B$
- $\lim_{x\to x_0}\big(f(x)g(x)\big) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right) = AB$
- $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} f(x)/\lim_{x\to x_0} f(x) = A/B$ if $B \neq 0$

Similar results hold for right –and left-hand limits.

2.3.4 Infinity

It sometimes happens that as $x \to x_0, f(x)$ increases or decreases without bound. In such case it is customary to write $\lim_{x\to x_0} f(x) = +\infty$ or $\lim_{x\to x_0} f(x) = -\infty$, respectively. The symbols $+\infty$ (also written ∞) and $-\infty$ are read plus infinity (or infinity) and minus infinity, respectively, but it must be emphasized that they are not numbers.

In precise language, we say that $\lim_{x\to x_0} f(x) = \infty$ if for each positive number M we can find a positive number δ (depending on M in general) such that $f(x) > M$ whenever $0 < |x - x_0| < \delta$. Similarly, we say that $\lim_{x\to x_0} f(x) = -\infty$ if for each positive number M we can find a positive number δ such that $f(x) < -M$ whenever $0 < |x - x_0| < \delta$. Analogous remarks apply in case $x \to x_0^+$ or $x \to x_0^-$.

Frequently we wish to examine the behavior of a function as x increases or decreases without bound. In such cases it is customary to write $x \to +\infty$ (or ∞) or $x \to -\infty$, respectively.

We say that $\lim_{x\to x_0} f(x) = l$ or $f(x) \to l$ as $x \to +\infty$, if for any positive number ϵ we can find a positive number N (depending on ϵ in general) such that $|f(x) - l| < \epsilon$ whenever $x > N$. A similar definition can be formulated for $\lim_{x\to -\infty} f(x)$.

2.3.5 Special Limits

- $\lim_{x\to 0}\frac{sinx}{x} = 1,$ $\lim_{x\to 0}\frac{1-cosx}{x} = 0$
- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e,$ $\lim_{x\to 0^+}(1+x)^{1/x} = e$
- $\lim_{x\to 0}\frac{e^x-1}{x} = 1,$ $\lim_{x\to 1}\frac{x-1}{lnx} = 1$

Example 3.5.1

Find the limit $\lim_{x\to 1}(x^2+1)$

Solution 3.5.1

Using direct substitution by substituting 1 for x

$$\lim_{x\to 1}(x^2+1) = 1^2+1 = 2$$

Example 3.5.2

Find the limit: $\lim_{x\to 1} f(x)$.

a. $f(x) = \frac{x^2-1}{x-1}$ b. $f(x) = \frac{|x-1|}{|x-1|}$ c. $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases}$

Solution 3.5.2

- $\lim_{x\to 1} f(x) = \frac{x^2-1}{x-1} = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1}$

Factorizing the numerator by the difference of two square [$a^2 - b^2 = (a+b)(a-b)$].

$\lim_{x\to 1}(x+1) = 1+1 = 2$ Substituting 1 for x

Therefore, $\lim_{x\to 1} f(x) = \frac{x^2-1}{x-1} = 2$

- $\lim_{x\to 1}\frac{|x-1|}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} = 0$ Substituting 1 for x

Therefore, $\lim_{x\to 1}\frac{|x-1|}{x-1}$ does not exist

- $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases} = 1$

2.3.6 The limit of a polynomial function

If p is a polynomial function and c is any real number, then $\lim_{x \to c} p(x) = p(c)$

Example 3.6.1

Find the limit: $\lim_{x \to 2}(x^2 + 2x - 3)$

Solution 3.6.1

$\lim_{x \to 2} x^2 + 2x - 3 = \lim_{x \to 2} x^2 + \lim_{x \to 2} 2x - \lim_{x \to 2} 3$ Applying property II
$= 2^2 + 2(2) - 3$ Use direct substitution
$= 4 + 4 - 3 = 5$ Simplify
$= 5$

Note: Example 3.6.1 shows or states that the limit of polynomial can be evaluated by direct substitution.

Check point:

- Find the limit: $\lim_{x \to 2} f(x)$

- $f(x) = \frac{x^2-4}{x-2}$
- $f(x) = \frac{|x-2|}{x-2}$
- $f(x) = \begin{cases} x^2, & x \neq 0 \\ 0, & x = 2 \end{cases}$

- Find the limit $\lim_{x \to 2} 2x^2 - x + 4$ $\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x)$

2.3.7 Techniques for Evaluating Limits

There are several techniques for calculating limits and these are based on the following important theorem. Basically, the theorem states that "*if two functions agree at all but a single point c, then they have identical limit behavior at $x = c$*".

3.7.1 The Replacement Theorem/Technique

Let c be a real number and $f(x) = g(x)$ for all $x \neq c$. if the limit of $g(x)$ exists as $x \to c$, then the limit of f(x) also exists and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x)$$

To apply the Replacement Theorem, you can use a result from algebra which states that for a polynomial function $p, p(c) = 0$ if and only if $(x - c)$ is a factor of $p(x)$.

Example 3.7.1

Finding the limit of a function

Find the limit $\lim_{x\to 1} \frac{x^3-1}{x-1}$

Solution 3.7.1

Note that the numerator and the denominator are zero when $x = 1$

$\lim_{x\to 1} \frac{x^3-1}{x-1}$ for the numerator $\lim_{x\to 1} x^3 - 1 = 1^3 - 1 = 0$ and the denominator $\lim_{x\to 1} x - 1 = 1 - 1 = 0$.

This implies or means that x-1 is a factor of both and you can divide out this like factor using division of polynomial.

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$

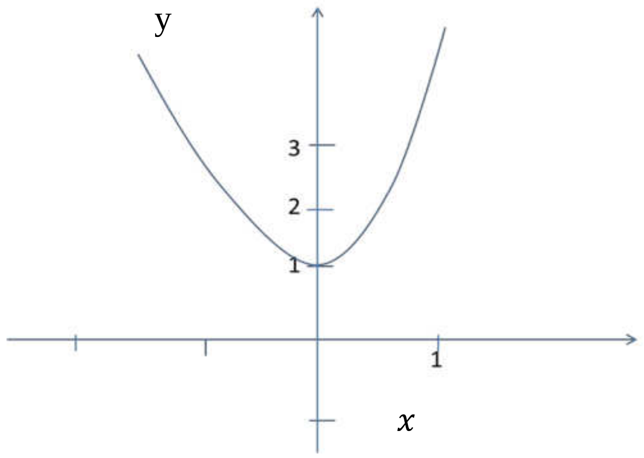
$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$ Factor numerator

$= \frac{(x-1)(x^2+x+1)}{x-1}$ Divide out factor

$= x^2 + x + 1,\ x \neq 1$ Simplify

So, the rational function $(x^3 - 1)(x - 1)$ and the polynomial function $x^2 + x + 1$ agree for all value of x other than x = 1, and you can apply the replacement theorem.

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}(x^2+x+1)=1^2+1+1=3$$



$g(x)=x^2+$

$x+1$

In figure illustrates this result graphically. Note that the two graphs are identical except that the graph of g contains the point (1, 3), whereas this point is missing on the graph of f.

Checkpoint.

Find the limit: $\lim_{x\to 2}\frac{x^3-8}{x-2}$

3.7.2 Dividing out Technique

Example 3.7.2.1

Find the limit: $\lim_{x\to -3}\frac{x^2+x+6}{x+3}$

Check:

For the numerator: $\lim_{x\to 2}(x^2+x+6)=-3^2+(-3)-6=9-3-6=0$.

Similarly for the denominator: $\lim_{x\to 2}(x+3)=-3+3=0$.

Since the limits of both numerator and denominator are zero, you know that they have a common factor of $x+3$ by factorizing the numerator.

So, for all $x\neq 3$, you can divide out this factor to obtain the following:

Solution 3.7.2.1

Using direct substitution will fails because both the numerator and the denominator are zero when $x = -3$.
$\lim_{x\to -3}\frac{x^2+x+6}{x+3} = \lim_{x\to -3}\frac{(x-2)(x+3)}{x+3}$

Factor numerator by factorization

$= \lim_{x\to -3}\frac{(x-2)(x+3)}{x+3}$ **Divide out like factor**

$\lim_{x\to -3}(x-2)$ **Simplify**

$= -3 - 2 = -5$ **Substituting -3 to be x**

Note that the graph of f coincides with the graph of $g(x) = x - 2$, except that the graph of f has a hole at $(-3, -5)$.

Checkpoint

Find the limit: $\lim_{x\to 3}\frac{x^2+x-12}{x-3}$

3.7.3 Rationalizing the Numerator Technique.

Example 3.7.3.1

Find the limit: $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

Solution 3.7.3.1

Direct substitution fails because both the numerator and the denominator are zero when $x = 0$. In this case, you can rewrite the fraction by rationalizing the numerator by taking the conjugate of the numerator and using it to both the numerator and denominator.

Taking the conjugate of the numerator of the numerator $\sqrt{x+1} - 1$ will be $\sqrt{x+1} + 1$.

Conjugate of $\sqrt{x+1} - 1$ is $\sqrt{x+1} + 1$.

i.e., $\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$

$$= \frac{x+1+\sqrt{x+1}-\sqrt{x+1}-1}{x(\sqrt{x+1}+1)}$$

$$= \frac{x+1-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0$$

Now, using the replacement theorem, you can evaluate the limit as follows

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{1+1}=\frac{1}{2}$$

Checkpoint

Find the limit: $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

One Sided Limit

One way in which a limit fails to exist is when a function approaches a different value from the left of c than it approaches from right of c. This type of behaviour can be described more concisely with the concept of a **one-sided limit**.

- $\lim_{x\to c^-} f(x)=L$ Limit from the left.
- $\lim_{x\to c^+} f(x)=L$ Limit from the right.

The first of these two limits is read as "the limit of $f(x)$ as x approaches c from the left is L". The second is read as "limit f(x) as x approaches c from the right is L".

Example 3.7.3.2

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for the function:

$$f(x)=\frac{|2x|}{x}$$

Solution 3.7.3.2

From the graph of f, you can see that $f(x)=-2$ for all $x<0$. Therefore, the

limit from the left is:

$$\lim_{x\to 0^-}\frac{|2x|}{x}=-2 \quad \textit{Limit from the left.}$$

Because f(x) = 2 for all x > 0, the limit from the right is:

$$\lim_{x\to 0^+}\frac{|2x|}{x}=2 \quad \textit{Limit from right.}$$

3.7.4 Unbounded Behaviour.

A Limit can fail to exist when $f(x)$ increases or decrease without bound as x approaches c. The equal sign in the statement $\lim_{x \to c^+} +\infty$ does not mean that the limit exists. On the contrary, it tells you how the limit fails to exist by denoting the unbounded behaviour of $f(x)$ as x approaches c.

Example 3.7.4.1

Find the limit (if possible): $\lim_{x \to 2} \frac{3}{x-2}$

Solution 3.7.4.1

$$\lim_{x \to 2^-} \frac{3}{x-2} = \infty$$

and $\lim_{x \to 2^+} \frac{3}{x-2} = \infty$

Because f is a unbounded as x approaches 2, the limit does not exist.

Example 3.7.4.2

Find the limit (if possible): $\lim_{x \to -2} \frac{5}{x+2}$

Solution 3.7.4.2

$$\lim_{x \to -2} \frac{5}{x+2} = \frac{5}{-2+2} = \frac{5}{0} = \infty$$

Because f is a unbounded as x approaches-2, the limit does not exist.

2.4 Conclusion

In conclusion, the unit discusses in comprehensive form the limit of functions under various circumstances and conditions with relevant examples and justifications.

2.5 Summary

- If f(x) becomes arbitrary close to a single number L as x approach c from either side, then $\lim_{x \to c} f(x) = L$ which is as the limit of $f(x)$ as x approaches L.

    - If p is a polynomial function and c is any real number, then, $\lim_{x \to c} p(x) = p(c)$

2.6 References/Further Reading

Engineering Mathematics by K. A Stroud.

Blitzer Algebra and Trigonometry Custom 4^{th} Edition.

Calculus An Applied Approach Larson Edwards Sixth Edition.

2.7 Self-Assessment Exercises

- In exercise i and ii, find the limit of (a) $f(x) + g(x)$ (b) $f(x).g(x)$ and(c) $\frac{f(x)}{g(x)}$ as x approaches c.

- $\lim_{x \to c} f(x) = 3$
- $\lim_{x \to c} f(x) = \frac{3}{2}$
- In the exercise i- xiii find the limit:
- $\lim_{x \to 2} x^2$
- $\lim_{x \to -2} x^3$
- $\lim_{x \to -3} (3x + 2)$
- $\text{Lim}_{x \to 1}(1 - x^2)$
- $\lim_{x \to 2}(-x^2 + x - 2)$
- $\lim_{x \to 3} \sqrt{x + 1}$
- $\lim_{x \to 4} 3\sqrt{x + 4}$
- $\lim_{x \to -3} \frac{2}{x+2}$
- $\text{Lim}_{x \to -2} \frac{3x-1}{2-x}$
- $\lim_{x \to -1} \frac{4x-5}{3-x}$
- $\lim_{x \to 7} \frac{5x}{x+2}$
- $\lim_{x \to 3} \frac{\sqrt{x+1}}{x-4}$
- $\lim_{x \to -2} \frac{x^2-1}{2x}$

- In the following exercise i – xii, find the limit (if it exists):

- $\lim_{x \to -1} \frac{x^2-1}{x+1}$
- $\lim_{x \to -1} \frac{2x^2-x-3}{x+1}$
- $\lim_{x \to 2} \frac{x-2}{x^2-4x+4}$
- $\lim_{x \to 2} \frac{2-x}{x^2-4}$
- $\lim_{t \to 5} \frac{t-5}{t^2-25}$
- $\lim_{t \to 1} \frac{t^2+t-2}{t^2-1}$
- $\lim_{x \to -2} \frac{x^3-1}{x-1}$

- $\lim_{x \to 2} \frac{x^3+8}{x+2}$
- $\lim_{x \to 0} \frac{|x|}{x}$
- $\lim_{x \to 2} \frac{|x-2|}{x-2}$
- $\operatorname*{Lim}_{x \to 3} f(x)$ where $f(x) = \begin{cases} \frac{1}{3}x, & x \leq 3 \\ -2x+5, & x > 3 \end{cases}$
- $\lim_{\Delta x \to 0} \frac{2(x+\Delta x)-2x}{\Delta x}$
- $\lim_{\Delta x \to 0} \frac{\sqrt{x+2+\Delta x}-\sqrt{x+2}}{\Delta x}$

$\lim_{\Delta t \to 0} \frac{(t+\Delta t)^2-5(t+\Delta t)-(t^2-5t)}{\Delta t}$

Unit 3: Idea Of Continuity

Unit Structure

1.0 Introduction

In mathematics, continuity means rigorous formulation of the intuitive concept of function that varies with no abrupt breaks or jumps.

Continuity of a function is expressed some times by saying if the x-values are closed together, then the y-value of the function will also be close.

2.0 Intended Learning Outcomes (ILOs)

By the end of this unit, you will be able to:

- define continuity of functions
- state continuity properties conditions
- define continuity of polynomial and rational function

3.3 Main Content

3.3.1 Idea of Continuity

3.3.1 Definition of Continuity

Let f be defined for all values of x near $x = x_0$ as well as at $x = x_0$ (i.e., in a δ neighborhood of x_0). The function f is called continuous at x = x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$. Note that this implies three conditions which must be met in order that $f(x)$ be continuous at $x = x_0$.

- $\lim_{x \to x_0} f(x) = l$ must exist.
- $f(x_0)$ must exist, i.e., $f(x)$ is defined at x_0.
- $l = f(x_0)$

In summary, $\lim_{x \to x_0} f(x)$ is the value suggested for f at $x = x_0$ by the behavior of f in arbitrarily small neighborhoods of x_0. If in fact this limit is the actual value, $f\ (x_0)$, of the function at x_0, then f is continuous there.

Equivalently, if f is continuous at x_0, we can write this in the suggestive form $\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right)$.

Points where f fails to be continuous are called discontinuities of f and f is said to be discontinuous at these points.

In constructing a graph of a continuous function, the pencil need never leave the paper, while for a discontinuous function this is not true since there is generally a jump taking place. This is of course merely a characteristic property and not a definition of continuity or discontinuity.

Alternative to the above definition of continuity, we can define f as continuous at $x = x_0$ if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Note that this is simply the definition of limit with $l = f(x_0)$ and removal of the restriction that $x \neq x_0$.

3.3.2 Right- and Left-Hand Continuity

If f is defined only for $x \geq x_0$, the above definition does not apply. In such case we call f continuous (on the right) at x = x_0 if $\lim_{x \to x_0^+} f(x) = f(x_0)$, i.e., if $f(x_0^+) = f(x_0)$. Similarly, f is continuous (on the left) at x = x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$, i.e., $f(x_0^-) = f(x_0)$.. Definitions in terms of ϵ and δ can be given

3.3.3 Continuity in an Interval

A function f is said to be continuous in an interval if it is continuous at all points of the interval. In particular, if f is defined in the closed interval $a \leq x \leq b$ or [a, b], then f is continuous in the interval if and only if $\lim_{x \to x_0} f(x) = f(x_0)$ for a < x_0 < b, $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

3.3.4 Theorems on Continuity

Theorem 1

If f and g are continuous at $x = x_0$, so also are the functions whose image values satisfy the relations $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$ and $\frac{f(x)}{g(x)}$, the last only if $g(x_0) \neq 0$. Similar results hold for continuity in an interval.

Theorem 2

Functions described as follows are continuous in every finite interval: (a) all polynomials; (b) $sin\ x$ and $cos\ x$; (c) $ax;\ a > 0$.

Theorem 3

Let the function f be continuous at the domain value $x = x_0$. Also suppose that a function g, represented by $z = g(y)$, is continuous at y_0, where $y = f(x)$ (i.e., the range value of f corresponding to x_0 is a domain value of g). Then a new function, called a composite function, $f(g)$ represented by $z = g[f(x)]$, may be created which is continuous at its domain point $x = x_0$. [One says that a continuous function of a continuous function is continuous.

Theorem 4

If $f(x)$ is continuous in a closed interval, it is bounded in the interval.

Theorem 5

If $f(x)$ is continuous at $x = x_0$ and $f(x_0) > 0$ [or $f(x_0) < 0$], there exists an interval about $x = x_0$ in which $f(x) > 0$ [or $f\ (x) < 0$].

Theorem 6

If a function $f(x)$ is continuous in an interval and either strictly increasing or strictly decreasing, the inverse function $f^{-1}(x)$ is single-valued, continuous, and either strictly increasing or strictly decreasing.

Theorem 7

If $f(x)$ is continuous in $[a,b]$ and if $f(a) = A$ and $f(b) = B$, then corresponding to any number C between A and B there exists at least one number c in ½a; b such that f$(c) = C$. This is sometimes called the intermediate value theorem.

Theorem 8

If $f(x)$ is continuous in $[a,b]$ and if $f(a)$ and $f(b)$ have opposite signs, there is at least one number c for which $f(c) = 0$ where $a < c < b$. This is related to Theorem 7.

Theorem 9

If $f(x)$ is continuous in a closed interval, then $f(x)$ has a maximum value M for at least one value of x in the interval and a minimum value m for at least one value of x in the interval. Furthermore, $f(x)$ assumes all values between m and M for one or more values of x in the interval.

Theorem 10.

If $f(x)$ is continuous in a closed interval and if M and m are respectively the least upper bound $(l.u.b.)$ and greatest lower bound $(g.l.b.)$ of $f(x)$, there exists at least one value of x in the interval for which f$(x) = M$ or $f(x) = m$. This is related to theorem 9.

3.5 Piecewise Continuity

A function is called piecewise continuous in an interval $a \leq x \leq b$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right- and lefth and limits.

Such a function has only a finite number of discontinuities. An example of a function which is piecewise continuous in $a \leq x \leq b$ is shown graphically in Fig. 3.1 below. This function has discontinuities at x_1, x_2, x_3, and x_4.

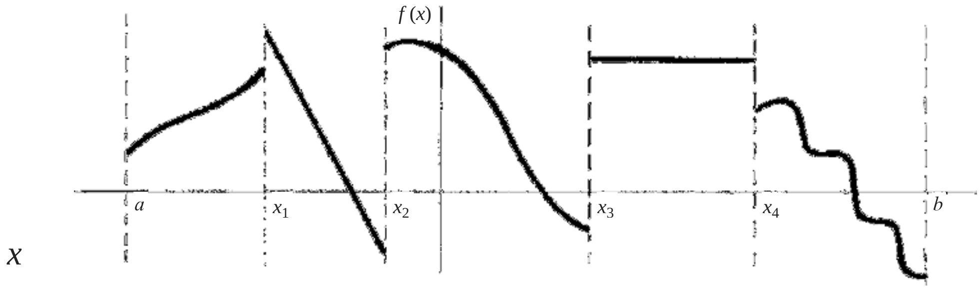


Fig. 3.1 Piecewise continuity

3.6 Uniform Continuity

Let f be continuous in an interval. Then by definition at each point x_0 of the interval and for any $\epsilon > 0$, we can find $\delta > 0$ (which will in general depend on both ϵ and the particular point x_0) such that $|f(x) - f(x_0)| < 0$ whenever $|x - x_0| < \delta$. If we can find for each δ which holds for all points of the interval (i.e., if δ depends only on ϵ and not on x_0), we say that f is uniformly continuous in the interval.

Alternatively, f is uniformly continuous in an interval if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where x_1 and x_2 are any two points in the interval.

Theorem

If f is continuous in a closed interval, it is uniformly continuous in the interval.

3.7 Continuity of Polynomial and Rational Functions

- A polynomial function is continuous at every real number
- A rational function is continuous at every number in its domain.

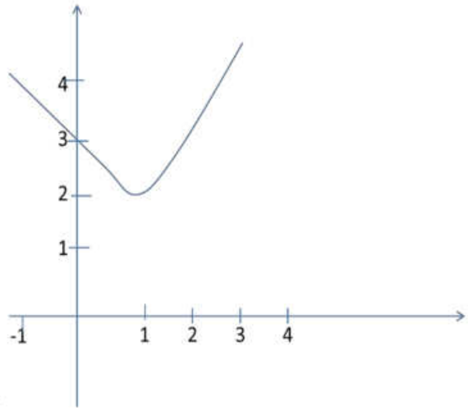
Example 3.7.1

Discuss the continuity of each function

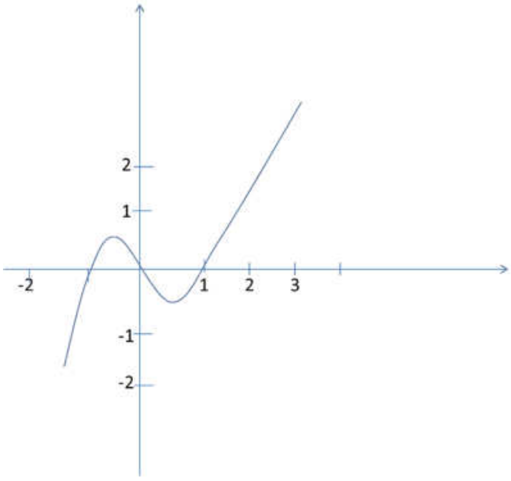
a. $f(x) = x^2 - 2x + 3$ b. $f(x) = x^3 - x$

Solution 3.7.1

Each of these functions is a polynomial function. So, each is continuous on the entire real line.



a b



Both functions are continuous on $(-\infty, \infty)$.

Example 3.7.2

Discuss the continuity of each function

a. $f(x) = \frac{1}{x}$ b. $f(x) = \frac{x^2-1}{x-1}$ c. $f(x) = \frac{1}{x^2+1}$

Solution

Each of these functions is a rational function and is therefore continuous at every number in its domain.

- The domain of $(x) = \frac{1}{x}$ consist of all real numbers except $x = 0$. So, this function is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$.
- The domain of $f(x) = \frac{x^2-1}{x-1}$ consists of all real numbers except $x = 1$. So, this function is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$.

- The domain of $f(x) = \frac{1}{x^2+1}$ consists of all real numbers. So, this function is continuous on the entire real line.

Class exercise

Discuss the continuity of each function:

- $f(x) = \frac{1}{x-1}$ (**Answer**: Continuous $(-\infty, 1)$ and $(1, \infty)$)
- $f(x) = \frac{x^2-4}{x-2}$ (**Answer**: Continuous $(-\infty, 2)$ and $(2, -\infty)$)
- $f(x) = \frac{1}{x^2+2}$ (**Answer**: Continuous on the entire real line)

Consider an open interval I that contains a real number c. If a function f is defined on I (except possibly at c), and f is not continuous at c, then f is said to have a *discontinuity* at c. Discontinuities fall into two categories: *removable* and non-*removable*.

A discontinuity is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$. For instance, the function in Example 2(b) has a removable discontinuity at $(1, 2)$. To remove the discontinuity, all you need to do is redefine the function so that $f(1) = 2$.

A discontinuity at $x = c$ is non removable if the function cannot be made continuous at $x = c$ by defining or redefining the function at $x = c$. For instance, the function in Example 2a has a non-removable discontinuity at $x = 0$.

3.8 Continuity on a Closed Interval

Definition

Let $\boldsymbol{f}$ be defined on a closed interval $[a, b]$. If f is continuous on the open interval (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

Then f is *continuous on the closed interval* $[a, b]$. Moreover, f is *continuous from the right* at a and *continuous from the left* at b.

Similar definitions can be made to cover continuity on intervals of the form *(a, b]* and $[a, b)$, or on infinite intervals. For example, the function $f(x) = \sqrt{x}$ is continuous on the infinite interval $[0, \infty)$.

Example 3.8.1

Discuss the continuity of $f(x) = \sqrt{3-x}$

Solution 3.8.1

Notice that the domain of f is the set $(-\infty, 3]$. Moreover, f is continuous from the left at $x = 3$ because:

$$\begin{aligned}\lim_{x\to 3} f(x) &= \lim_{x\to 3^-} \sqrt{3-x} \\ &= 0 \\ &= f(3)\end{aligned}$$

For all $x < 3$, the function of satisfies the three conditions foe continuity. So, you can conclude that f is continuous on the interval $(-\infty,,3]$.

Working Tip

When working with radical functions of the form $f(x) = \sqrt{g(x)}$, remember that the domain of f coincides with the solution of $g(x) \geq 0$.

Example 3.8.2

Discuss the continuity of $g(x) = \begin{cases} 5-x, -1 \leq x \leq 2 \\ x^2 - 1, 2 < x <\leq 3 \end{cases}$

Solution 3.8.2

The polynomial functions $5 - x$ and $x^2 - 1$ are continuous on the intervals $[-1,\ 2)$ and $(2,\ 3]$, respectively. So, to conclude that g is continuous on the entire interval $[-1, 3]$, you need only check the behavior of g when $x = 2$. You can do this by taking the one – sided limit when $x = 2$.

$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (5-x) = 3$ Limit from the left
$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (5-x) = 3$ Limit from the right

Because these two limits are equal

$\lim_{x\to 2} g(x) = g(x) = 3$
So, g is continuous at $x = 2$ and consequently, it is continuous on the entire interval $[-1, 3]$

3.4 Conclusion

You must have learnt continuity of function and the properties of continuity. Also, you have learnt continuity of polynomial function, dependent and independent function. Moreover, you can now solve diverse problems on continuity of functions.

3.5 Summary

- A function is said to be continuous if and only if it is continuous at every point of its domain.

- Continuity can be defined in terms of limits by saying that f(x) is continuous at x (0) of its domain if and only if, for values of x in its domain $\lim_{x \to x_0} f(x) = f(x_0)$

3.6 References/Further Reading

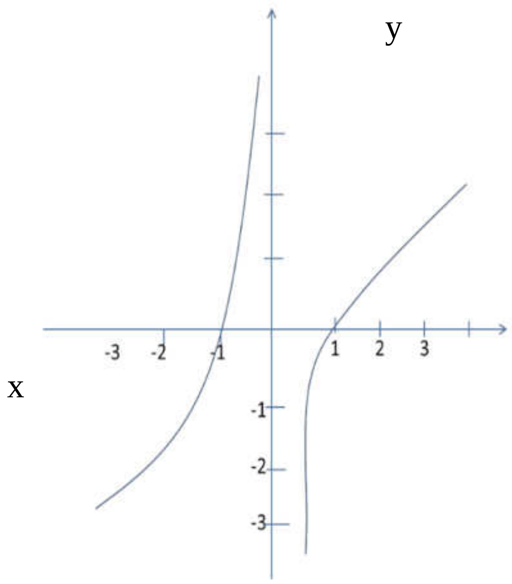
Blitzer Algebra and Trigonometry Custom 4^{th} Edition

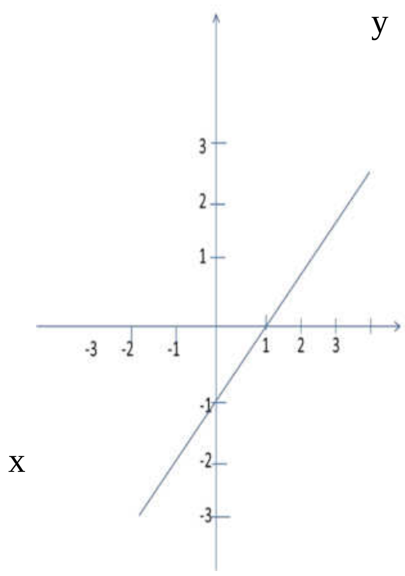
Calculus: An Applied Approach. Larson Edwards Sixth Edition Engineering Mathematics by K.A Stroud.

3.7 Self-Assessment Exercises

- In exercise i - ii, determine whether the function is continuous on the entire real line. Explain your reasoning.

i. $f(x) = 5x^3 - x^2 + 2$ ii. $f(x) = \frac{1}{x^2-4}$





- In Exercises i - xiv, describe the interval(s) on which the function is continuous.

- $f(x) = \frac{1}{9-x^2}$
- $f(x) = \frac{x^2-1}{x}$
- $f(x) = \frac{x^2-1}{x+1}$
- $f(x) = x^2 - 2x + 1$
- $f(x) = \frac{x}{x-1}$
- $f(x) = \frac{x}{x^2+1}$
- $f(x) = \frac{x-5}{x^2-9x+20}$

- $f(x) = \begin{cases} -2x+3, & x < 1 \\ x^2, & x > 1 \end{cases}$
- $f(x) = \begin{cases} 3+x, & x \leq 1 \\ x^2+1, & x > 2 \end{cases}$
- $f(x) = \frac{|x+1|}{x+1}$
- $f(x) = (x-1)$
- $h(x) = f\big(g(x)\big)$,
- $f(x) = \frac{1}{\sqrt{x}}$,
- $g(x) = x - 1, \ x > 1$.

MODULE 2 CALCULUS OF DIFFERENTIATION

In this module, you will be learning the differential calculus. Concepts related to functions will be studied and various techniques of solving differential calculus will be considered. This module is made up of the following units:

Unit 1 The Derivative as Limit of Rate of Change
Unit 2 Differentiation Technique

Unit 1 The Derivative as Limit of Rate of Change

Unit Structure

1.1 Introduction
1.2 Intended Learning Outcomes (ILOs)
1.3 Main Content
   1.3.1 The rate of change of a function
   1.3.2 Right- and left-hand derivatives
   1.3.3 Differentiability in an interval
   1.3.4 Piecewise differentiability
   1.3.5 Differentiation
   1.3.6 Derivative for power of x^n
   1.3.7 Differentiation of polynomials
   1.3.8 Standard derivative
   1.3.9 Derivatives of elementary function
   1.3.10 Higher order derivatives
   1.3.11 Mean value theorems
   1.3.12 L'hospital's rule
1.4 Conclusion
1.5 summary
1.6 References/Further Reading
1.7 Self-Assessment Exercise(s)

1.1 Introduction

Concepts that shape the course of mathematics are few and far between.

The derivative, the fundamental element of the differential calculus, is such a concept. That branch of mathematics called analysis, of which advanced calculus is a part, is the end result. There were two problems that led to the discovery of the derivative. The older one of defining and representing the tangent line to a curve at one of its points had concerned early Greek philosophers. The other problem of representing the instantaneous velocity of an object whose motion was not constant was much more a problem of the seventeenth century. At the end of that century, these problems and their relationship were resolved. As is usually the case, many mathematicians contributed, but it was Isaac Newton and Gottfried Wilhelm Leibniz who independently put together organized bodies of thought upon which others could build. The tangent problem provides a visual interpretation of the derivative and can be brought to mind no matter what the complexity of a particular application.

The derivative of a function of a real variable measures the sensitivity to change of a quantity (a function value or dependent variable) which is determined by another quantity (the independent variable).

Derivatives are fundamental tools of calculus. The derivative of a function of a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point. This means that it describes the best linear approximation of the function near that input value. For this reason, the derivative is often described as the "instantaneous rate of change", the ratio of the instantaneous change in the dependent variable to that of the independent variable. Differentiation is the action of computing a derivative.

1.2 Intended Learning Outcomes (ILOs)

By the end of this unit, you should be able to:

- define rate of change of function
- solve differentiation from first principle
- solve differentiation from second principle

1.3 Main Content

1.3.1 The Rate of Change of a Function

If y is a function of x, as x changes y will in general change. We relate the change in y to the corresponding change in x by defining the average rate of change of the function to be the change in the function divided by the corresponding change in x. If x_1 and x_2 are two values of x, and the corresponding values of y are y_1 and y_2, then the average rate of change of the function as x changes from x_1 to x_2 is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Example3.1.1

Find an expression for the average rate of change of the functions

- $y = 2x + 5$,
- $y = x^2$ in the interval x_1 to x_2.

By the above definition, the average rate of change for $y = 2x + 5$ is

$$\frac{(2x_2 + 5) - (2x_1 + 5)}{x_2 - x_1} = \frac{2(x_2 - x_1)}{x_2 - x_1} = 2$$

We notice that this is the same for each interval $x_1 - x_2$.

- The average rate of change for $y = x^2$ is

$$\frac{x_2^2 - x_1^2}{x_2 - x_1} = \frac{(x_2 - x_1)(x_2 + x_1)}{x_2 - x_1} = x_1 + x_2$$

which is different for different intervals.

If we represent the function graphically, the average rate of change of the function in the interval $x_1 - x_2$may be interpreted geometrically as being the gradient of the chord joining the points on the graph with abscissae x_1 and x_2. For the function $y = 2x + 5$, the graph is a straight line and the gradient of any chord is always 2, $\frac{QN}{PN} = \frac{Q'N'}{P'N'} = 2$. But for the graph of $y = x^2$, the gradient of the chord PQ is different from the gradient of $P'Q$', etc.

A practical application of this idea arises in connection with space-time graphs. Suppose a body moves so that the distance s travelled after time t is $s = f(r)$. Then the average rate of change of s as t changes from, $\frac{s_2 - s_1}{t_2 - t_1}$ is just the average speed of the body in the interval $t_1 - t_2$ and is the gradient of the appropriate chord on the space-time graph. The above expresses algebraically the gradient of the chord joining the points with abscissae x_1 and x_2 on the graph of $y = f(x)$.

Can the gradient of the tangent to the curve be given a similar algebraic interpretation? Geometrically, we feel no difficulty in drawing the tangent to a curve at a particular point, but in order to interpret this process algebraically we need to consider it in some detail.

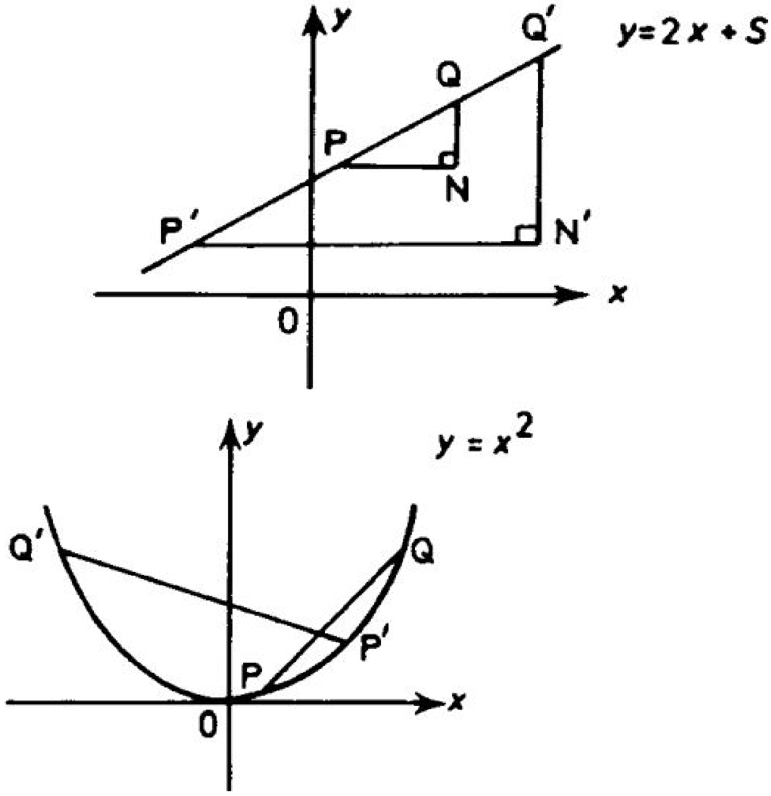
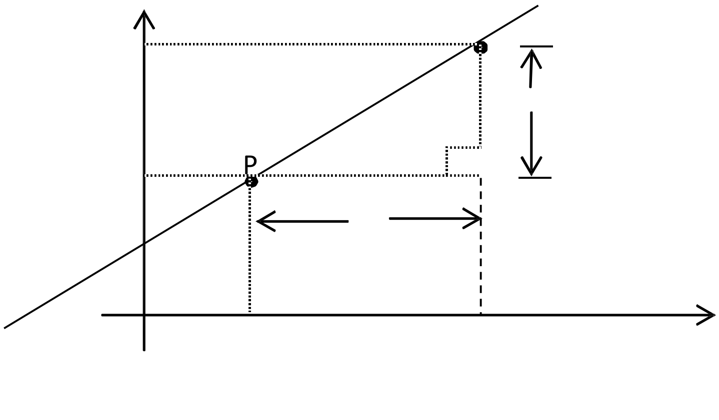


Fig 3.1.1: The graph of $y = 2x + 5$ and $y = x^2$

Gradient is defined as the ratio of the vertical distance the line rises or falls between two points P and Q to the horizontal distance between P and Q, m is the symbol used denoting gradient of a straight-line graph

i.e. $m = \frac{\delta y}{\delta x} = \frac{\Delta x}{\Delta y} = \frac{y_2 - y_1}{x_2 - x_1}$

y
Q



X

Fig 3.1.2: The Gradient of a straight-line graph

1.3.2 The Gradient of a curve at a given point (Algebraic Determination)

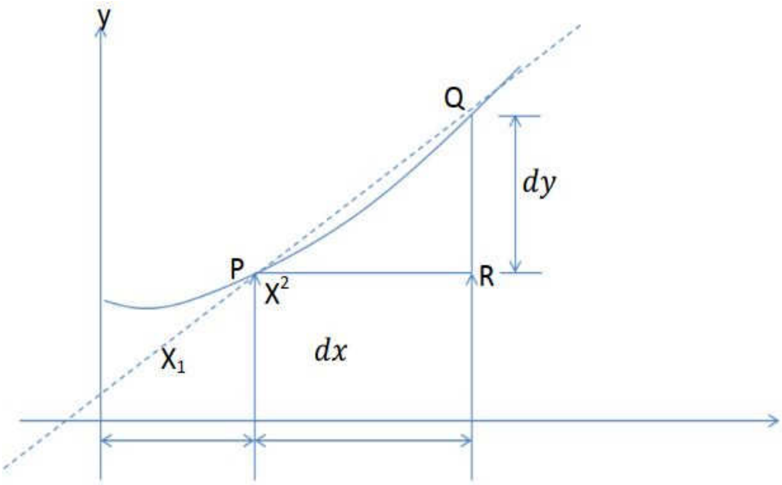


Fig 3.2.1: The gradient of the curve at a given points

Let P be a fixed point (x, y) on the curve y and Q be a neighbouring. We will notice a slight change; that we frequently use x and y to denote the respective differences in the x and y values of the points P and Q on the curve. The x and y are called the differentials. For example, we can

Fig 3.2.2 to illustrate

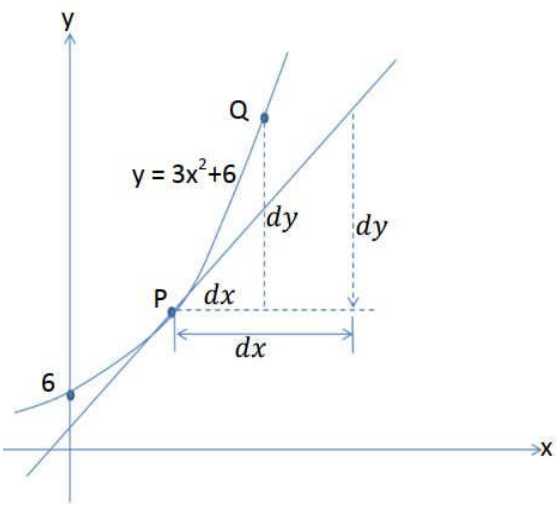


Fig 3.2.2: The graph of differentials

On the graph, P is a fixed point on the curve $y = 3x^2 + 6$ and Q a neighbouring point. There is a slight difference in x (i.e., $x + \delta x$) and in y (i.e., y+δy).

1.3.2. Right- and Left-Hand Derivatives

The status of the derivative at end points of the domain of f, and in other special circumstances, is clarified by the following definitions.

The right-hand derivative of $f(x)$ at $x = x_0$ is defined as

$$f'_+(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

if this limit exists. Note that in this case $h(=\Delta x)$ is restricted only to positive values as it approaches zero.

Similarly, the left-hand derivative of f(x) at x = x_0 is defined as

$$f'_-(x_0) = \lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

if this limit exists. In this case h is restricted to negative values as it approaches zero. A function f has a derivative at x = x_0 if and only if $f'_+(x_0) = f'_-(x_0)$.

1.3.3 Differentiability in an Interval

If a function has a derivative at all points of an interval, it is said to be differentiable in the interval. In particular if f is defined in the closed interval $a \le x \le b$, i.e., $[a;\ b]$, then f is differentiable in the interval if and only if $f'(x_0)$ exists for each x_0 such that $a < x_0 < b$ and if $f'_+(a)$ and $f'_-(b)$ both exist. If a function has a continuous derivative, it is sometimes called continuously differentiable.

1.3.4 Piecewise Differentiability

A function is called piecewise differentiable or piecewise smooth in an interval $a \le x \le b$ if $f'(x)$ is piecewise continuous. An equation for the tangent line to the curve $y = f(x)$ at the point where $x = x_0$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The slopes of these tangent lines are $f'_-(x_0)$ and $f'_+(x_0)$ respectively.

1.3.5 Differentiation

Let $\Delta x = dx$ be an increment given to x. then

$$\Delta y = f(x + \Delta x) - f(x)$$

Is called the increment in $y = f(x)$. If $f(x)$ is continuous and has a continuous first derivative in an interval, then

$$\Delta y = f'(x)\Delta x + \epsilon\Delta x = f'(x)dx + dx$$

Where $\epsilon \to 0$ and $\Delta x \to 0$. The expression

$$dy = f'(x)dx$$

is called the differential of y or f(x) or the principal part of Δy. Note that $\Delta y \neq dy$ in general.

However, if $\Delta x = dx$ is small, then dy is a close approximation of Δy.

The quantity dx, called the differential of x, and dy need not be small.

Therefore, $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

It is emphasized that dx and dy are not the limits of Δx and Δy as $\Delta y \to 0$, since these limits are zero whereas dx and dy are not necessarily zero.

Instead, given dx we determine dy, i.e., dy is a dependent variable determined from the independent variable dx for a given x.

The geometric interpretation of the derivative as the slope of the tangent line to a curve at one of its points is fundamental to its application. Also of importance is its use as representative of instantaneous velocity in the construction of physical models. In particular, this physical viewpoint may be used to introduce the notion of differentials.

Newton's Second and First Laws of Motion imply that the path of an object is determined by the forces acting on it, and that if those forces suddenly disappear, the object takes on the tangential direction of the path at the point of release. Thus, the nature of the path in a small neighborhood of the point of release becomes of interest. With this thought in mind, consider the following idea. Suppose the graph of a function f is represented by $y = f(x)$. Let $x = x_0$ be a domain value at which f_0 exists (i.e., the function is differentiable at that value). Construct a new linear function

$$dy = f'(x_0)dx$$

With dx as the (independent) domain variable and dy the range variable generated by this rule. This linear function has the graphical interpretation illustrated in **Fig 3.5.1**.

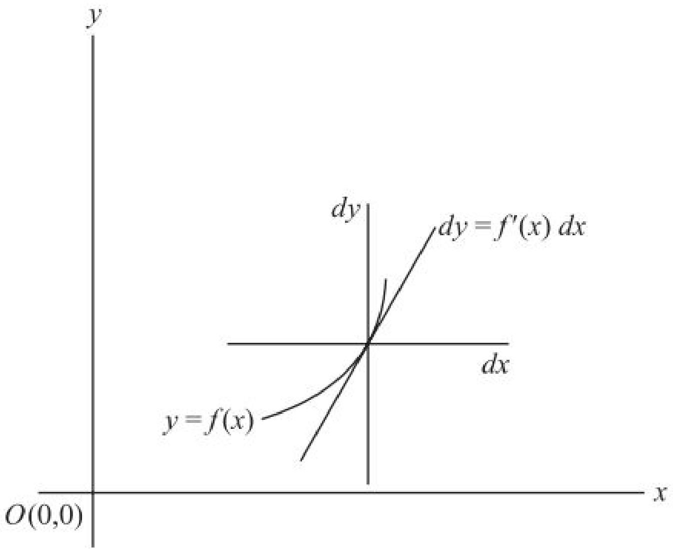


Fig 3.5.1: The graph of $f(x)$

That is, a coordinate system may be constructed with its origin at P_0 and the dx and dy axes parallel to the x and y axes, respectively. In this system our linear equation is the equation of the tangent line to the graph at P_0. It is representative of the path in a small neighborhood of the point; and if the path is that of an object, the linear equation represents its new path when all forces are released.

dx and dy are called differentials of x and y, respectively. Because the above linear equation is valid at every point in the domain of f at which the function has a derivative, the subscript may be dropped and we can write

$$dy = f'(x)dx$$

The following important observations should be made.

$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, thus $\frac{dy}{dx}$ is not the same thing as $\frac{\Delta y}{\Delta x}$.

On the other hand, dy and Δx are related. In particular, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ means that for any $\epsilon > 0$ there exists $\delta > 0$ such that $-\epsilon < \frac{\Delta y}{\Delta x} - \frac{dy}{dx} < \epsilon$ whenever $|\Delta x| < \delta$. Now dx is an independent variable and the axes of x and dx are parallel; therefore, dx may be chosen equal to Δx. With this choice

$-\epsilon\Delta x < \Delta y - dy < \epsilon\Delta x$ or $dy - \epsilon\Delta x < \Delta y < dy + \epsilon\Delta x$

From this relation we see that dy is an approximation to Δy in small neighborhoods of x. dy is called the principal part of Δy. The representation of f' by dy/dx has an algebraic suggestiveness that is very appealing and will appear in much of what follows. In fact, this notation was introduced by Leibniz (without the justification provided by knowledge of the limit idea) and was the primary reason his approach to the calculus, rather than Newton's was followed.

Example 3.5.1

Using the first principle of differentiation to evaluate $y = 3x^2 + 6$.

Solution 3.5.1

At Q: with the little increment $y + \delta y = 3(x + \delta x)^2 + 6$

Expanding the bracket in equation

$$y + \delta y = (x^2 + 2x - \delta x + |\delta x|^2 + 6$$

Subtracting y from both sides

$$y + \delta y - y = 3x^2 + 6x\delta x + 3|\delta x|^2 + 6 - y$$

Replacing the value of y in the equation above

$$y + \delta y - y = 3x^2 + 6x\delta x + 3|\delta x|^2 + 6 - (3x^2 + 6x)$$

$$\delta y = 3x^2 + 6x\delta x + 3|\delta x|^2 + 6 - 3x^2 - 6x$$

Collecting the common terms

$$\delta y = 3x^2 + 6x\delta x - 3x^2 + 3x^2 + 6x\delta x + 3|\delta x|^2 + 6 - 6$$

$$\delta y = 6x\delta x + 3|\delta x|^2$$

Dividing both side by δx

$$\frac{\delta y}{\delta x} = \frac{6x\delta x}{\delta x} + \frac{3|\delta x|^2}{\delta x}$$

Lim $\delta y \rightarrow 0$, $\delta x \rightarrow 0$

$$\frac{\delta y}{\delta x} = 6x + 3|0|$$

$$\frac{\delta y}{\delta x} = 6x$$

This is called the First Principle of Differentiation

Example 3.5.2

If $y = x^2 + 3x$, find $\frac{\delta y}{\delta x}$ using first principle

Solution 3.5.2

$$y = x^2 + 3x$$
$$y + \delta y = (x + \delta x)^2 + 3(x + \delta x)$$

Expanding the bracket in equation (2)

$$y + \delta y = x^2 + 2xdx + \delta x + |\delta x|^2 + 3x + 3\delta x$$

Subtracting y from both sides

$$y + \delta y - y = x^2 + 2xdx + \delta x + |\delta x|^2 + 3x + 3\delta x - y$$

$$\delta y = x^2 + 2xdx + \delta x + |\delta x|^2 + 3x + 3\delta x - y$$

Replacing the value of y in the above

$$\delta y = x^2 + 2xdx + \delta x + |\delta x|^2 + 3x + 3\delta x - (x^2 + 3x)$$

$$\delta y = x^2 + 2xdx + \delta x + |\delta x|^2 + 3x + 3\delta x - x^2 - 3x$$

Colleting like terms

$$\delta y = x^2 - x^2 + 2xdx + \delta x + |\delta x|^2 + 3x - 3x + 3\delta x$$

$$\delta y = 2xdx + |\delta x|^2 + 3\delta x$$

Divide throughout by δx

$$\frac{\delta y}{\delta x} = 2x\frac{\delta x}{\delta x} + \frac{|\delta x|^2}{\delta x} + \frac{3\delta x}{\delta x}$$
$$\frac{\delta y}{\delta x} = 2x + |\delta x| + 3$$

Lim $\delta y \to 0, \delta x \to 0$

$$\frac{\delta y}{\delta x} = 2x + 0 + 3$$
$$\frac{\delta y}{\delta x} = 2x + 3$$

1.3.6 Derivative of Power of x^n

If $y = x^n$ (1)

We can establish that if $y = x^n$

Using Binomial theorem to find the derivative of $y = x^n$

If $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$

If $y = x^n, y + \delta y = (x+\delta x)^n$ using first principle

$y + \delta y = x^n + nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \cdots$ (2)

Subtracting y from both side of equation (2)

$$\delta y = nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \cdots$$

Dividing throughout by δx

$$\frac{\delta y}{\delta x} = nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \cdots$$

If $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}$ and all terms on the RHS, except the first

If $\delta x \to 0, \frac{\delta y}{\delta x} = nx^{n-1} + 0 + 0 + 0 + \cdots$

If $y = x^n, \frac{\delta y}{\delta x} = nx^{n-1}$

Generally, if $y = ax^n$ then $\frac{\delta y}{\delta x} = nax^{n-1}$ where a is a constant.

If $y = k$ (where k is a constant) then $\frac{\delta y}{\delta x} = 0$

1.3.7 Differentiation of Polynomial

When differentiating a polynomial, it has to be differentiated in turn of each term.

Example 3.7.1

If $y = x^3 + 5x^2 - 4x + 2$ differentiate with respect to x

Solution 3.7.1

$$y = x^3 + 5x^2 - 4x + 2$$

$\frac{\delta y}{\delta x} = 3x^{3-1} + 2 \times 5x^{2-1} - 1 \times 4x^{1-1} + 0$ (using $y = x^n, \frac{\delta y}{\delta x} = nx^{n-1}$)

$$= 3x^2 + 10x - 4$$

Example 3.7.2

If $y = x^4 + 6x^3 - 4x^2 + 7x - 2$, find $\frac{\delta y}{\delta x}$ and the value of $\frac{\delta y}{\delta x}$ at x = 2

Solution 3.7.2

$$y = x^4 + 6x^3 - 4x^2 + 7x - 2$$

$$\frac{\delta y}{\delta x} = 4x^{4-1} + 3 \times 6x^{3-1} - 2 \times 4x^{2-1} + 1 \times 7x^{1-1} - 0$$

$$\frac{\delta y}{\delta x} = 4x^3 + 18x^2 - 8x + 7$$

At x=2, $\frac{\delta y}{\delta x} = 4(2)^3 + 18(2)^2 - 8(2) + 7$

$$= 32 + 72 - 16 + 7$$

$$= 95$$

1.3.8 Standard Derivative

Derivative of Trigonometric expression

This is established by using a number of trigonometric formulas:

- Derivative of $y \ = sinx$
- If $y = sinx$

Using first principle of differentiation

$$y + \delta y = \sin(x + \delta x)$$

Subtract y from both side of the term above

$$y + \delta y - y = \sin(x + \delta x) - y$$

$$\delta y = \sin(x + \delta x) - y$$

Replacing both the value of y in above becomes: $\delta y = \sin(x + \delta x) - sinx$

We now apply the trigonometry formulae:

$$sinA - sinB = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

Where $A = x + \delta x$ and $B = x$

$$y = 2\cos\left(\frac{x + \delta x + x}{2}\right)\sin\left(\frac{x + \delta x - x}{2}\right)$$

$$\delta y = 2\cos\left(\frac{2x + \delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\delta y = 2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

Dividing both side by δx

$$\frac{\delta y}{\delta x} = \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

$$\frac{\delta y}{\delta x} = \frac{\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\frac{\delta y}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right)\frac{\frac{sin\delta x}{2}}{\frac{\delta x}{2}}$$

When $\delta y \to 0, \frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}$

$$\frac{\delta y}{\delta x} = \cos\left(x + \frac{0}{2}\right)\sin\frac{\frac{0}{2}}{\frac{0}{2}}$$

$$\frac{\delta y}{\delta x} = cosx$$

- Derivative of $y = cosx$

If $y = cosx$

Using first principle of differentiation

$$y + \delta y = cos(x + \delta x)$$

Subtract y from both side of the term above

$$y + \delta y - y = \cos(x + \delta x) - y$$

$$\delta y = \cos(x + \delta x) - y$$

Replacing both the value of y in above
becomes: $\delta y = \cos(x + \delta x) - cosx$

We now apply the trigonometry formulae:

$$cosA - cosB = -2\sin\frac{A + B}{2}\sin\frac{A - B}{2}$$

Where $A = x + \delta x$ and $B = x$

$$y = -2\sin\left(\frac{x+\delta x + x}{2}\right)\sin\left(\frac{x+\delta x - x}{2}\right)$$

$$\delta y = -2\sin\left(\frac{2x+\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\delta y = -2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

Dividing both side by δx

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\frac{\delta y}{\delta x} = \sin\left(x+\frac{\delta x}{2}\right)\frac{\frac{sin\delta x}{2}}{\frac{\delta x}{2}}$$

When $\delta y \to 0, \frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}$

$$\frac{\delta y}{\delta x} = -\sin\left(x+\frac{0}{2}\right)\sin\frac{\frac{0}{2}}{\frac{0}{2}}$$

$$\frac{\delta y}{\delta x} = -sinx$$

- Derivative of y = Tan x

If $y = tanx$

From trigonometric expression $tanx = \frac{sinx}{cosx}$

Since the expression is $y = \frac{u}{v}$, we will use the quotient rule

$$\frac{\delta y}{\delta x} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let $u = sinx,\ v = cosx$

$$\frac{\delta u}{\delta x} = cosx\,\frac{\delta v}{\delta x} = -sinx$$

Substituting the above term in the equation above to have

$$\frac{\delta y}{dx} = \frac{cosx.cox - sinx.-sinx}{cox.cox}$$

$\frac{\delta y}{\delta x} = \frac{\cos^2 x + \sin^2 x}{cos^2 x}$ (where $\sin^2 A + \cos^2 B = 1$)

$$\frac{\delta y}{\delta x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

If $y = tanx, \frac{\delta y}{\delta x} = \sec^2 x$

1.3.9 Derivatives of Elementary Functions

In the following we assume that u is a differentiable function of x; if $u = x$, $du/dx = 1$. The inverse functions are defined according to the principal values given:

- $\frac{d}{dx}(C) = 0$
- $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
- $\frac{d}{dx}sinu = cosu\frac{du}{dx}$
- $\frac{d}{dx}cosu = -sinu\frac{du}{dx}$
- $\frac{d}{dx}tanu = \sec^2 u\frac{du}{dx}$
- $\frac{d}{dx}cotu = -\csc^2 u\frac{du}{dx}$
- $\frac{d}{dx}secu = secutanu\frac{du}{dx}$
- $\frac{d}{dx}cscu = -cscucotu\frac{du}{dx}$
- $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}$ $a > 0, a \neq 0$
- $\frac{d}{dx}\log_e u = \frac{d}{dx}lnu = \frac{1}{u}\frac{du}{dx}$,

16. $\frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$
17. $\frac{d}{dx}sec^{-1} u = \pm\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}\begin{cases} + if\ u > 1 \\ - if\ u < -1 \end{cases}$
18. $\frac{d}{dx}\csc^{-1} u = \mp\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}\begin{cases} - if\ u > 1 \\ + if\ u < -1 \end{cases}$
19. $\frac{d}{dx}sinhu = coshu\frac{du}{dx}$
20. $\frac{d}{dx}coshu = sinhu\frac{du}{dx}$
21. $\frac{d}{dx}tanhu = \text{sech}^2 u\frac{du}{dx}$
22. $\frac{d}{dx}cothu = -\text{csch}^2 u\frac{du}{dx}$
23. $\frac{d}{dx}sechu = -sechutanhu\frac{du}{dx}$
24. $\frac{d}{dx}\text{csch}\, u = -cschucothu\frac{du}{dx}$
25. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

- $\frac{d}{dx}a^u = a^u lna\frac{d}{dx}$
- $\frac{d}{dx}e^u = e^u\frac{d}{dx}$
- $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
- $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
- $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

26. $\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
27. $\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}, |u| < 1$
28. $\frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}, |u| > 1$
29. $\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
30. $\frac{d}{dx}\operatorname{csch}^{-1}u = -\frac{1}{u\sqrt{u^2+1}}\frac{du}{dx}$

1.3.10 Higher Order Derivatives

If $f(x)$ is differentiable in an interval, its derivative is given by $f'(x), y'$ or $dy = dx$, where $y = f(x)$. If $f'(x)$ is also differentiable in the interval, its derivative is denoted by f''(x), y'' or $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Similarly, the nth derivative of $f(x)$, if it exists, is denoted by $f^n(x), y^n$ or $\frac{d^ny}{dx^n}$, where n is called the order of the derivative. Thus, derivatives of the first, second, third … orders are given by $f'(x), f''(x), f'''(x)$, ...Computation of higher order derivatives follows by repeated application of the differentiation rules given above.

1.3.11 Mean Value Theorems

These theorems are fundamental to the rigorous establishment of numerous theorems and formulas.

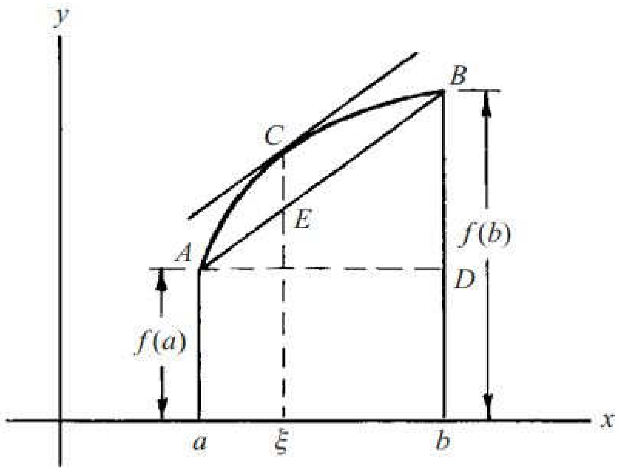


Fig 3.11.1: Mean Value theorems

Rolle's theorem.

If $f(x)$ is continuous in [a; b] and differentiable in (a, b) and if $f(a) = f(b) = 0$, then there exists a point ξ in (a, b) such that $f'(\xi) = 0$.

Rolle's theorem is employed in the proof of the mean value theorem. It then becomes a special case of that theorem.

The mean value theorem.

If $f(x)$ is continuous in (a, b) and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{b - 1} = f'(\xi), \quad a < \xi < b$$

Rolle's theorem is the special case of this where $f(a) = f(b) = 0$. The result can be written in various alternative forms; for example, if x and x_0 are in (a, b), then

$f(x) = f(x_0) + f'(\xi)(x - x_0)$ ξ between x_0 and x

The mean value theorem is also called the law of the mean.

Cauchy's generalized mean value theorem.

If $f(x)$ and $g(x)$ are continuous in $[a;\ b]$ and differentiable in $(a;\ b)$, then there exists a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}, \quad a < \xi < b$$

where we assume $g(a) \neq g(b)$ and $f'(x), g'(x)$ are not simultaneously zero.

1.3.12 L'Hospital's Rules

If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, where A and B are either both zero or both infinite, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is often called an indeterminate of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, respectively, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. The following theorems, called L'Hospital's rules, facilitate evaluation of such limits

- If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) except possibly at a point x_0 in this interval, and if $g'(x) \neq 0$ for $x \neq x_0$ then

- $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$

- whenever the limit on the right can be found. In case $f'(x)$ and $g'(x)$ satisfy the same conditions as $f(x)$ and $g(x)$ given above, the process can be repeated.

- If $\lim\limits_{x \to x_0} f(x) = \infty$ and $\lim\limits_{x \to x_0} g(x) = \infty$, the result is also valid.

- These can be extended to cases where $x \to \infty$ or $-\infty$, and to cases where $x_0 = a$ or $x_0 = b$ in which only one-sided limits, such as $x \to a^+$ or $x \to b^-$, are involved.

Limits represented by the so-called indeterminate forms 0, ∞, ∞^0, 0^0, 1^0, and $\infty - \infty$ can be evaluated on replacing them by equivalent limits for which the above rules are applicable.

1.4 Summary

- Differentiation is the action of computing a derivative. The derivate of a function $f(x)$ of a variable x is a measure of the rate at which the changes with respect to the change of the variable.

- Derivate of powers of x

  - $y = c, \frac{\delta y}{\delta x} = 0$
  - $y = x^n, \frac{\delta y}{\delta x} = nx^{n-1}$
  - $y = ax^n, \frac{\delta y}{\delta x} = anx^{n-1}$

- Gradient of a straight-line graph (m) $\frac{\delta y}{\delta x}$

- Differentiation of polynomial means differentiating each term in turn

1.5 Conclusion

You have learned the derivation of function, differentiation of elementary function and L'hospital rule. The basic theorem of governed derivation of function has been established. Rule of differentiation and properties has been discussed. Some differential problems are solved to enhance the understanding.

1.6 References/Further Reading

Additional Mathematics by Godman and J.F Talbert
Calculus An Applied Approach Larson Edwards Sixth Edition
Blitzer Algebra and Trigonometry Custom 4th Edition
Engineering Mathematics by K.A Stroud

1.7 Self-Assessment Exercises

- Differentiate using first principle.

- $y = x^3$
- $y = 5x^2 + 2$
- $y = 6x^2 - 1$
- $y = 4x^3$

  - Find the derivative of the following.

  - $y = 6x^3 + 4x^2 - 7x + 2$
  - $y = 15x^3 - 6x^2 + 10$
  - $y = 10x^5 + 7x^3 + 2x$
  - y $=6x^3 + 4x^2 - 7x + 2$ v. y$= 15x^3 - 6x^2 + 10$ iv. y$= 10x^5 + 7x^3 + 2$

Unit 2 Differentiation Techniques

Unit Structure

2.1 Introduction

This shows the useful formulas in showing that the derivative is linear. Here we will learn the quotient rule, product, function of a function and implicit function

2.2 Intended Learning Outcomes (ILOs)

By the end of this unit, you should be able:

- Construct product of function
- Solve problems on quotient of functions
- Develop and solve implicit functions
- Carryout solution on function of functions

2.3 Main Content

2.3.1 Differentiation of Products of Functions (Product Rule)

Let $y = uv$, where u and v are functions of x

If $x \to x + \delta x,\ u \to u + \delta u,\ v \to v + \delta v$ and as a result, $y \to y + \delta y$

The above expression shows that an increment ∂x in x will turn produce increments ∂u in u and also producing a change ∂v in v and a change ∂y in y.

Using first principle:

- If $y = uv$. . . . . (1)
- Therefore, $y + \partial y = (u + u)(v + v)$. . . . . .. (2)

Expanding the left-hand side in equation (2) gives

$: y + \partial y = uv + u\partial v + v\partial u + \partial y.\partial v$ . . . . (3)

Subtracting $y = uv$ in equation (3) gives

- $y + \partial y - y = uv + u\partial v + v\partial u + \partial u.\partial v - y$
- $\partial y = uv + u\partial v + v\partial u + \partial u.\partial v - y$
- $\partial y = uv + u\partial v + v\partial u + \partial u.\partial v - uv$ (where $y = uv$)
- $\partial y = u\partial v + v\partial u + \partial u.\ \partial v$

Divide throughout by ∂x

$\frac{\partial y}{\partial x} = \frac{u\partial v}{\partial x} + \frac{v\partial u}{\partial x} + \partial u.\partial v$..............................(4)

Limit if $\partial x \to 0, \frac{\partial y}{\partial x} \to \frac{\partial y}{\partial x}, \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \to \frac{\partial v}{\partial x}$

(NB tends to turns to)

Therefore, (4) now gives:

$$\frac{\partial y}{\partial x} = \frac{u\,\partial v}{\partial x} + \frac{v\,\partial u}{\partial x} + 0.\frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{u\,\partial v}{\partial x} + \frac{v\,\partial u}{\partial x}$$

Product Rule

If $y = uv$
$\frac{\partial y}{\partial x} = \frac{u\,\partial v}{\partial x} + \frac{v\,\partial u}{\partial x}$

Example 3.1.1

Differentiate with respect to x: $y = x^3 \sin x$

Solution 3.1.1

Let $u = x^3$ and $v = sin\ x$

$$\frac{\delta y}{\delta x} = cosx, \frac{\delta u}{\delta x} = 3x^2$$
$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} = x^3 cosx + 3xsinx$$

$$= x^3 cosx + 3xsinx$$
$$= x(x^2 cosx + 3sinx)$$

Example 3.1.2

Differentiate $(3x - 2)(x^2 + 3)$ with respect to x

Solution 3.1.2

Let $y = (3x - 2)(x^2 + 3)$

Let $u = 3x - 2,\ v = x^2 + 3$

$\frac{\delta v}{\delta x} = 2x, \quad \frac{\delta u}{\delta x} = 3$

$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$

$= (3x - 2)2x + (x^2 + 3)3$

$= 6x^2 - 4x + 3x^2 + 9$

$= 6x^2 + 3x^2 - 4x + 9$

$= 9x^2 - 4x + 9$

Example 3.1.3

Differentiate $y = x^5 e^x$ with respect to x

Solution 3.1.3

Let $u = x^5$ and $v = e^x$

$\frac{\delta u}{\delta x} = 5x^4, \ \frac{\delta v}{\delta x} = e^x$

$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$

$= x^5 e^x + e^x 5x^4$

$= x^4e^x(x + 5)$

Example 3.1. 4

Differentiate $y = x^2\ (2x - 5)^4$ with respect to x

Solution 3.1.4

$y = x^2\ (2x - 5)^4$
Let $u = x^2$ and $v = (2x - 5)^4$
$\frac{\delta u}{\delta x} = 2x, \quad \frac{\delta v}{\delta x} = 8(2x - 5)^3$

$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + \frac{v\delta u}{\delta x}$

$= x^2 \times 4(2x - 5)^3 \times 2 + (2x - 5)^4 \times 2x$

$\frac{\delta y}{\delta x} = 2x^2 \times 4(2x - 5)^3 + (2x - 5)^2 2x$

Simplify ~~as~~ far as possible by collecting common terms and leaving the result in factors

$= 2x(2x - 5)^3(4x + 2x - 5)$

$\frac{\delta y}{\delta x} = 2x(2x - 5)^3(6x - 5)$

Example 3.1. 5

Differentiate $y = (3x - 1)^3(x^2 + 5)$

Solution 3.1.5

$y = (3x - 1)^3(x^2 + 5)$
If u= $(3x - 1)^3, \quad v = (x^2 + 5)$
$\frac{\delta u}{\delta x} = 9(3x - 1)^2, \ \frac{\delta v}{\delta x} = 2x$

$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$
$= (3x - 1)^3 2x + (x^2 + 5)9(3x - 1)^3$

Collecting the common terms and leaving the result in factors

$= (3x - 1)^3[2x(3x - 1) + 9(x^2 + 5)$
$= (3x - 1)^3(6x^2 - 2x + 9x^2 + 45)$

$\frac{\delta y}{\delta x} = (3x-1)^3(15x^2-2x+45)$

Example 3.1.6

Differentiate the following with respect to x

1. $y = (x^2-1)(x^3+1)$

2. $y = x(x^2-1)^3$

3. $y = e^x sinx$

4. $y = 4x^3 sinx$

5. $y = 3x^3e^3$

Solution 3.1.6

1. $y = (x^2-1)(x^3+1)$

Let $u = (x^2-1),\ \ v = (x^3+1)$

Using the formula:

$$\frac{\delta u}{\delta x} = e^x,\ \ \frac{\delta v}{\delta x} = 3x^2$$
$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$
$$= 3x^4 - 3x^2 + 2x^4 + 2x$$
$$= 5x^4 - 3x^2 + 2x$$
$$= x(5x^3 - 3x + 2)$$

- $y = x(x^2-1)^3$
- $u = x,\ v = (x^2-1)^3$
- $\frac{\delta u}{\delta x} = 1,\ \ \frac{\delta v}{\delta x} = 6x(x^2-1)^2$
- $\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$
- $\frac{\delta y}{\delta x} = 6x(x^2-1)^2x + (x^2-1)^3 1$
- $\frac{\delta y}{\delta x} = 6x^2(x^2-1)^2 + (x^2-1)^3$
- $\frac{\delta y}{\delta x} = (x^2-1)^2(6x^2+x^2-1)$

- $= (x^2-1)^2(7x^2-1)$
- $y = e^x sinx$
- Let $u = e^x,\ \ v = sinx$
- $\frac{\delta u}{\delta x} = e^x,\ \ \frac{\delta v}{\delta x} = cosx$

- $\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$
- $= e^x cosx + sinx\ e^x$
- $\frac{\delta y}{\delta x} = \mathrm{e^x(cosx + sinx)}$
- $y = 4x^3 sinx$
- $u = 4x^3, \quad v = sinx$
- $\frac{\delta u}{\delta x} = 12x^2, \quad \frac{\delta v}{\delta x} = cosx$
- $\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$
- $\frac{\delta y}{\delta x} = 4x^3 cosx + sinx.12x^2$
- $\frac{\delta y}{\delta x} = 4x\text{^}2(xcosx + 3sinx)$
- $y = 3x^3e^3$
- Let $u = 3x^3, \quad v = e^x$
- $\frac{\delta u}{\delta x} = 9x^2, \quad \frac{\delta v}{\delta x} = e^x$
- $\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$
- $\frac{\delta y}{\delta x} = 3x^3e^x + e^x.9x^2$
- $\frac{\delta y}{\delta x} = 3x^2e^x + e^x(x + 3)$

2.3.2 Differentiation of a quotient of two function (Quotient Rule)

Let $y = \frac{u}{v}$ where u and v are functions of x.

An increment δx in x will turn to produce increment and a change δu in u and a change δv in v and a change δy in y

If $x \to x + \delta x, u \to u + \delta u, v \to v + \delta v$ and as a result, $y \to y + \delta y$

Using the first principle

If $y = \frac{u}{v}$
Then $y + \delta y = \frac{u+\delta u}{v+\delta v}$

Subtract y from both side of the term in equation
$y + \delta y - y = \frac{u+\delta u}{v+\delta v} - y$

$\delta y = \frac{u+\delta u-u}{v+\delta v-v}$ where $y = \frac{u}{v}$

$\delta y = \frac{v(u+\delta u)-u(v+\delta v)}{(v+\delta v)v}$

$\delta y = \frac{uv+v\delta u-uv-u\delta v}{v^2+v\delta v}$ Simplify by collecting like terms

$\delta y = \frac{v\delta u-v\delta u}{v^2+v\delta v}$

$\delta y = \frac{v\delta u+v\delta u}{v^2+v\delta v}$

Dividing both side by δx

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x}-\frac{v\delta u}{\delta x}}{v^2+v\delta v}$

If $\delta x \to 0, \delta u \to 0$ and $\delta x \to 0$ then

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x}-\frac{v\delta u}{\delta x}}{v^2+v(0)}$

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x}-\frac{v\delta u}{\delta x}}{v^2}$

Hence, the quotient rule of differentiation

Example 3.2.1

If $y = \frac{x^2}{\sqrt{x+1}}$ differentiate with respect to x

Solution 3.2.1

$y = \frac{x^2}{\sqrt{x+1}}$ Since the function above is the form $y = \frac{u}{v}$, we will use the quotient rule

Let $u = x^2, v = \sqrt{x+1} = (x = 1)^{\frac{1}{2}}$

Using the quotient rule

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x}-\frac{v\delta u}{\delta x}}{v^2}$

$\frac{\delta u}{\delta x} = 2x,\ \frac{\delta v}{\delta x} = \frac{1}{2\sqrt{x+1}}$

$\frac{\delta y}{\delta x} = \frac{\sqrt{x+1}.2x - x^2.2\frac{1}{\sqrt{x+1}}}{(x+1)^{1/2.2}},$

$$\frac{\delta y}{\delta x} = \frac{\sqrt{x+1}.2x - x^2.2\frac{1}{\sqrt{x+1}}}{x+1}$$

$$= \frac{4(x+1)-x^2}{2(x+1)^{\frac{1}{2}+1}}$$

$$\frac{\delta y}{\delta x} = \frac{4(x+1)-x^2}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{4x^2+4x-x^2}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{3x^2+4x}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{x(3x+4)}{2(x+1)^{\frac{3}{2}}}$$

Example 3.2.2

Differentiate with respect to x, if $y = \frac{sinx}{x^2}$

Solution 3.2.2

Let $u = sinx,\ v = x^2$

$$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{v\delta u}{\delta x}}{v^2} = \frac{x^2.cosx - sinx.2x}{x^{(2)2}}$$

$$= \frac{x^2 cosx - 2xsinx}{x^4}$$

$$= \frac{x^2 cosx}{x^4} - \frac{2xsinx}{x^4}$$

$$= \frac{xcosx}{x^3} - \frac{2sinx}{x^3}$$

$$= \frac{xcosx - 2sinx}{x^3}$$

Example 3.2.3

If $y = \frac{4e^x}{cosx}$, differentiate with respect to x.

Solution 3.2.3

Let $u = 4e^x,\ v = cosx$

$\frac{\delta u}{\delta x} = 4e^x, \ \frac{\delta v}{\delta x} = -sinx$

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{v\delta u}{\delta x}}{v^2} = \frac{cosx(4e^x) - 4e^x(-sinx)}{\cos^2 x}$

$\frac{\delta y}{\delta x} = \frac{cosx(4e^x) + 4e^x(sinx)}{\cos^2 x}$

$\frac{\delta y}{\delta x} = \frac{4e^x(cosx + sinx)}{\cos^2 x}$

Example 3.2.4

When $y = \frac{sinx}{cosx}$, differentiate with respect to x

Solution 3.2.4

Let $u = sinx, \ v = cosx$

$\frac{\delta u}{\delta x} = cosx, \ \frac{\delta v}{\delta x} = -sinx$

Using quotient rule

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{v\delta u}{\delta x}}{v^2} = \frac{cosx(cosx) - sinx(-sinx)}{\cos^2 x}$

In trigonometry identity $\sin^2\theta + \cos^2\theta = 1$

$\frac{\delta y}{\delta x} = \frac{1}{\cos^2 x} = \sec^2 x$

Check point

Differentiate the following with respect to x

a. $y = \frac{4x+1}{7x-4}$ (Ans. $\frac{\delta y}{\delta x} = \frac{23}{(7x-4)^2}$)

b. $y = \frac{x^2-1}{x^3}$ (Ans. $\frac{\delta y}{\delta x} = \frac{x(-x^2+3x+2)}{x^6}$)

c. $y = \frac{cosx}{sinx}$ (Ans. $\frac{\delta y}{\delta x} = -cosec^2 x$)

2.3.3 The Differentiation of Composite Functions

Many functions are a composition of simpler ones. For example, if f and g have the rules of correspondence $u = x^3$ and $y = sin\,u$, respectively, then $y = sin\,x^3$ is the rule for a composite function $F = g(f)$. The domain of F is that subset of the domain of F whose corresponding range values are in the domain of g. The rule of composite function differentiation is called the chain rule and is represented by
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}[F'(x) = g'(u)f'(x)]$.

In the example

$$\frac{dy}{dx} = \frac{d(sinx^3)}{dx} = cosx^3(3x^2dx)$$

The importance of the chain rule cannot be too greatly stressed. Its proper application is essential in the differentiation of functions, and it plays a fundamental role in changing the variable of integration, as well as in changing variables in mathematical models involving differential equations. This is the *chain rule* and is very useful in determining the derivatives of function of functions of a function.

Example 3.3.1

If $y = \frac{1}{(6x-5)^2}$, differentiate with respect to x

Solution 3.3.1

$$y = \frac{1}{(6x-5)^2} = (6x-5)^2$$

Let $u = 6x - 5$ therefore $y = u^2$

$$\frac{du}{dx} = 6, \frac{dy}{du} = -2u^{-3} = -\frac{2}{u^3}$$

Using chain rule $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$

$$\frac{dy}{dx} = -\frac{2}{u^3}\ 6 = -\frac{12}{u^3}$$

$$\frac{dy}{dx} = -\frac{12}{(6x-5)^3}$$

Example 3.3.2

If $y\ =\ (2x\ +\ 8)^2$, find the value of $\frac{dy}{dx}$

Solution 3.3.2

$y\ =\ (2x\ +\ 8)^2$ . . . . (1)

Let $u\ =\ (2x\ +\ 8)$ . . . . (2)

Therefore, $y\ = u^3$ . . . . (3) (substituting u into equation(1))

Differentiating u with respect to x and differentiating y in equation (3) with respect to u.

$\frac{du}{dx} = 2, \frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$

$\frac{dy}{dx} = 2 \times 3u^2 = 6u^2$
$\frac{dy}{dx} = 6(2x + 8)^2$

Example 3.3.3

If $y = \sin(5x - 6)$, determine $\frac{dy}{dx}$

Solution 3.3.3

$y = \sin(5x - 6)$

Let $u = 5x - 6$, therefore $y = \sin u$

$\frac{du}{dx} = 5,\ \frac{dy}{du} = cosu$

$= \frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = 5cosu$

$\frac{dy}{dx} = 5\cos(5x - 6)$

Example 3.3.4

Determine $\frac{dy}{dx}$ when $y = tan(3x + 2)$

Solution 3.3.4

$y = \tan(3x + 2)$

Let $u = 3x + 2$ theefore, $y = tanu$

$\frac{du}{dx} = 3,\ \frac{dy}{du} = \sec^2 u$

$\frac{dy}{dx} = \frac{du}{dx} * \frac{dy}{du}$

$\frac{dy}{dx} = 3\sec^2 u$

$\frac{dy}{dx} = 3\sec^2(3x + 2)$

Checkpoint

Differentiate the following with respect to x:

- $y = (2x + 3)^3$ [**Answer:** $6(2x + 5)^2$]
- $y = \sqrt{4x - 3}$ [**Answer:** $\mathbf{2/(4x - 3)^{\frac{3}{2}}}$ **]**
- $y = sin(4x + 3)$ [**Answer:** $4cos(4x + 3)$]

Exercises

Differentiate the following with respect to x:

- y = $(2x + 5)^3$
- $y = \sqrt{4x} - 3$
- y = sin (4x + 3)

2.3.4 Implicit Functions

The rule of correspondence for a function may not be explicit. For example, the rule $y = f(x)$ is implicit to the equation $x^2 + 4xy^5 +$

$7xy + 8 = 0$. Furthermore, there is no reason to believe that this equation can be solved for y in terms of x. However, assuming a common domain (described by the independent variable x) the left-hand member of the equation can be construed as a composition of functions and differentiated accordingly. (The rules of differentiation are listed below for your review.) In this example, differentiation with respect to x yields
$2x + 4\left(y^5 + 5xy^4 \frac{dy}{dx}\right) + 7\left(y + x\frac{dy}{dx}\right) = 0$

Observe that this equation can be solved for $\frac{dy}{dx}$ as a function of x and y (but not of x alone).

Steps for Differentiating Implicit function

When differentiating implicit function, it is important to determine the derivate y with respect to x and while doing so, it is the derivative of the function. It is very important to multiply the differential function of y by the derivative of the function. It is very important to also notice that derivative of constant number is zero.

Example 3.4.1

If $2x^2 + 3y^2 = 16$, find $\frac{dy}{dx}$

Solution 3.4.1

$2x^2 + 3y^2 = 16$

Differentiating the above term with respect to x

$4x + 6y\frac{dy}{dx} = 0$

Subtracting L. H. S. from R. H. S. by $4x$

$4x + 6y\frac{dy}{dx} - 4x = 0 - 4x$

$6y\frac{dy}{dx} = -4x$

Making $\frac{dy}{dx}$ subject of the formula by dividing both side by 6y

$\frac{dy}{dx} = -\frac{4x}{6y}$

$\frac{dy}{dx} = -\frac{4x}{6y}$

Example 3.4.1

If $2x^2 + 3y^2 - 4x - 3y + 6 = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 3$, and $y = 2$

Solution 3.4.1

$2x^2 + 3y^2 - 4x - 3y + 6 = 0$

Differentiating the above expression with respect to x.

$4x + 6y\frac{dy}{dx} - 4 - 3\frac{dy}{dx} = 0$

Collecting like terms

$6y\frac{dy}{dx} - 3\frac{dy}{dx} + 4x - 4 = 0$

$6y\frac{dy}{dx} - 3\frac{dy}{dx} = 4 - 4x$

$\frac{dy}{dx} = \frac{4-4x}{6y-3} = \frac{4(1-x)}{3(2y-1)}$

At $(x, y) = (3, 2)$

$\frac{dy}{dx} = \frac{4-4(3)}{6(2)-3} = \frac{4-12}{12-3} = -\frac{8}{9}$

$\frac{dy}{dx} = -\frac{8}{9}$

To calculate $\frac{d^2y}{dx^2}$, since $\frac{dy}{dx} = \frac{4-4x}{6y-3}$, we have to differentiate $\frac{dy}{dx}$ once to get $\frac{d^2y}{dx^2}$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{4-4x}{6y-3}\right)$

Using quotient rule to differentiate since the function is of the form $\frac{u}{v}$

Let $u = 4 - 4x, v = 6y - 3, \frac{du}{dx} = -4, \frac{dv}{dx} = 6\frac{dy}{dx}$

Hence, $\frac{d^2y}{dx^2} = \frac{\frac{vdu}{dx} - \frac{udu}{dx}}{v^2} = \frac{(6y-3).-4-(4-4x)6\frac{dy}{dx}}{(6y-3)^2}$

Therefore, $\frac{d^2y}{dx^2} = \frac{-4(6y-3)-(4-4x)6\frac{dy}{dx}}{(6y-3)^2}$

At $(x, y) = (3, 2)$ and $\frac{dy}{dx} = -\frac{8}{9}$

Substituting the above terms into $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{-4(6y-3)-(4-4x)6\frac{dy}{dx}}{(6y-3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-4(6(2)-3)-\left(4-4(3)\right)6\left(-\frac{8}{9}\right)}{(6(2)-3)^2} = \frac{-4(12-3)-(4-12)6\left(-\frac{8}{9}\right)}{(12-3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-4(9)-(8)6\left(-\frac{8}{9}\right)}{(9)^2} = \frac{-36-(-8)2\left(-\frac{8}{3}\right)}{81}$$

$$= \frac{-36-(-16)\left(-\frac{8}{3}\right)}{81} == \frac{-36-\left(-\frac{128}{3}\right)}{81}$$

$$= \frac{\left(\frac{-108-128}{3}\right)}{81} == \frac{-\frac{236}{3}}{81} = -\frac{236}{3}.\frac{1}{81} = -\frac{236}{243}$$

Example 3.4.2

Find $\frac{dy}{dx}$, if $x^3 + y^3 = 2xy$

Solution 3.4.2

Differentiating with respect to y, we have to treat 2xy as product of the function

$$= 3x^2 + 3y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

Collecting like terms

$$3x^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 2x) = 2y - 3x^2$$

$$\frac{dy}{dx} = \frac{2y-3x^2}{3y^2-2x}$$

Example 3.4.3

Find the equation of the tangent(s) where x= 2on the curve $x^2 + y^2 = 2x + y = 6$

Solution 3.4.3

Differentiating the expression with respect to x

$$2x + 2y\frac{dy}{dx} - 2 + \frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + \frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx}(2y + 1) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2-2x}{2y+1}$$

Substituting the value of x =2 in the original equation of the curve, we have

$x^2 + y^2 - 2x + y = 6$
$(2)^2 + y^2 - 2(2) + y = 6$
$4 + y^2 - 4 + y = 6$
$y^2 + y - 6 = 0$
$(y + 3)(y - 2) = 0$
$y = -3, 2$

There are two points on the curve where $x = (2, -3)$ and $(2, -2)$
Gradient at point $(2, -3)$

$$m = \frac{dy}{dx} = \frac{2-2(2)}{2(-3)+1} = \frac{2-4}{-6+1} = -\frac{2}{-5} = \frac{2}{5}$$

Equation of the tangent $y - y_1 = m(x - x_1)$ where $y_1 = -3, x_1 = 2, m = \frac{2}{5}$
$y - (-3) = \frac{2}{5}(x - 2)$
$y + 3 = \frac{2}{5}(x - 2),\ y + 3 = \frac{2}{5}x - \frac{4}{5}$

Multiple throughout by 5

$$5 \times y + 5 \times 3 = 5 \times \frac{2}{5}x - \frac{4}{5} \times 5$$

$5y + 15 = 2 \times -4$
$5y - 2x + 15 + 4 = 0$
$5y - 2x + 19 = 0$

Gradient at point (2, -2)

$m = \frac{dy}{dx} = \frac{2-2x}{2y+1} = \frac{2-2(2)}{2(2)+1} = \frac{2-4}{4+1} = -\frac{2}{5}$
$m = -\frac{2}{5}$

Equation of tangent $y - y_1 = m(x - x_1)$

$y - 2 = -\frac{2}{5}(x - 2)$

$y - 2 = -\frac{2}{5}x + \frac{4}{5}$

Multiplying throughout by 5

$5y - 10 = -2x + 4$
$5y + 2x - 10 - 4 = 0$
$5y + 2x - 14 = 0$

Checkpoint

- If $x^2 + 2xy + 3y^2 = 4$ find $\frac{dy}{dx}$ Ans. $\frac{dy}{dx} = \frac{-x-y}{x+3y}$
- If $x^3 + y^3 + 3xy^2 = 8$ find $\frac{dy}{dx}$ Ans. $\frac{dy}{dx} = \frac{-(x^2+y^2)}{y^2+2y}$

2.3.5 Applications

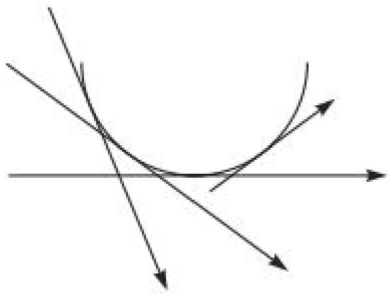
2.5.1. Relative Extrema and Points of Inflection

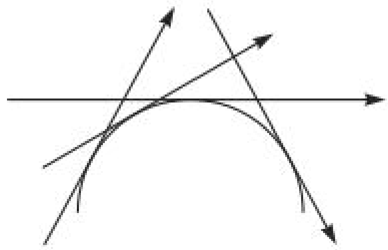
In this section, relative extrema and points of inflection shall be discussed, such points are characterized by the variation of the tangent line, and then by the derivative, which represents the slope of that line. Assume that f has a derivative at each point of an open interval and that P_1 is a point of the graph of f associated with this interval. Let a varying tangent line to the graph move from left to right through P_1. If the point is a relative minimum, then the tangent line rotates counterclockwise. The slope is negative to the left of P_1 and positive to the right. At P_1 the slope is zero.

At a relative maximum a similar analysis can be made except that the rotation is clockwise and the slope varies from positive to negative.

Because f'' designates the change of f', we can state the following theorem. As seen in the figures below.

Relative minimum Relative maximum





Counterclockwise rotating tangent clockwise rotating tangent

Theorem. Assume that x_1 is a number in an open set of the domain of f at which f' is continuous and f'' is defined. If $f'(x1) = 0$ and $f''(x1) \neq 0$, then $f(x1)$ is a relative extreme of f . Specifically:

- If $f''(x_1) > 0$, then f(x_1) is a relative minimum,
- If $f''(x_1) < 0$, then f(x_1) is a relative maximum.

(The domain value x_1 is called a critical value.)
This theorem may be generalized in the following way. Assume existence and continuity of derivatives as needed and suppose that $f'(x_1) = f^{\prime\prime(x_1)} = \cdots f^{2p-1}(x_1) = 0$ and $f^{2p}(x_1) \neq 0$ (p a positive integer). Then:

- f has a relative minimum at x_1 if $f^{2p}(x_1) > 0$,
- f has a relative maximum at x_1 if $f^{2p}(x_1) < 0$.

(Notice that the order of differentiation in each succeeding case is two greater. The nature of the intermediate possibilities is suggested in the next paragraph.)

It is possible that the slope of the tangent line to the graph of f is positive to the left of P_1, zero at the point, and again positive to the right. Then P_1 is called a point of inflection. In the simplest case this point of inflection is characterized by $f'(x_1) = 0, f''(x_1) = 0$, and $f'''(x_1) \neq 0$.

2.5.2 Particle motion

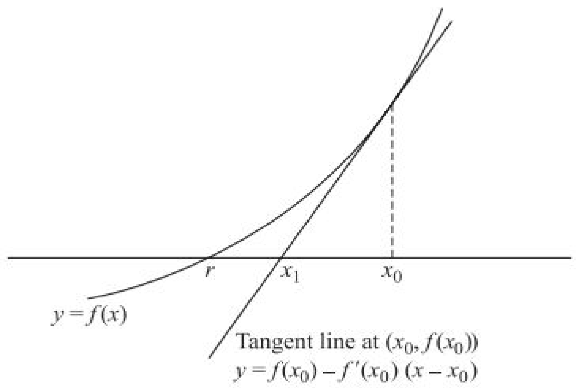
The fundamental theories of modern physics are relativity, electromagnetism, and quantum mechanics. Yet Newtonian physics must be studied because it is basic to many of the concepts in these other theories, and because it is most easily applied to many of the circumstances found in everyday life. The simplest aspect of Newtonian mechanics is called kinematics, or the geometry of motion. In this model of reality, objects are idealized as points and their paths are represented by curves. In the simplest (one-dimensional) case, the curve is a straight line, and it is the speeding up and slowing down of the object that is of importance. The calculus applies to the study in the following way.

If x represents the distance of a particle from the origin and t signifies time, then $x = f(t)$ designates the position of a particle at time t. Instantaneous velocity (or speed in the one-dimensional case) is change in distance represented by $\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)}{\Delta t}$. Furthermore, the instantaneous change in velocity is called acceleration and represented by $\frac{d^2x}{dt^2}$.

2.5.3 Newton's method

It is difficult or impossible to solve algebraic equations of higher degree than two. In fact, it has been proved that there are no general formulas representing the roots of algebraic equations of degree five and higher in terms of radicals. However, the graph $y = f(x)$ of an algebraic equation $f(x) = 0$ crosses the x-axis at each single-valued real root. Thus, by trial and error, consecutive integers can be found between which a root lies.

Newton's method is a systematic way of using tangents to obtain a better approximation of a specific real root, as demonstrated in the figure below.



Suppose that f has as many derivatives as required. Let r be a real root of $f(x) = 0, i.e., f(r) = 0$.

Let x_0 be a value of x near r. For example, the integer preceding or following r. Let $f'(x_0)$ be the slope of the graph of $y = f(x)$ at $P0[x_0, f(x_0)]$. Let $Q_1(x_1, 0)$ be the x-axis intercept of the tangent line at P_0 then

$$\frac{0 - f(x_0)}{x - x_0} = f'(x_0)$$

where the two representations of the slope of the tangent line have been equated. The solution of this relation for x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Starting with the tangent line to the graph at $p_1[x_1, f(x_1)]$ and repeating the process, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f(x_1)}{f'(x_1)}$$

and in general

$$x_n = x_0 - \sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)}$$

Under appropriate circumstances, the approximation x_n to the root r can be made as good as desired.

Note: Success with Newton’s method depends on the shape of the function’s graph in the neighborhood of the root. There are various cases which have not been explored here.

2.4 Conclusion

In this section, we have learnt differential techniques for different forms of differential functions. Product rule, quotient rule and function of function methods has been employed for solving differential calculus.

Also, implicit function as well been solved.

2.5 Summary

1. if $y = \frac{u}{v}.\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$
2. if $y = uv, \frac{dy}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

2.6 References/Further Reading

Additional Mathematics by Godman and J.F Talbert

Calculus And Applied Approach Larson Edward Sixth Edition

Blitzer algebra and trigonometry Custom 4th edition

Engineering Mathematics by K.A Stroud

2.7 Self-Assessment Exercise

- If $x^2 + y^2 - 2x + 2y = 23$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point where x =-2, y =3

- Find $\frac{dy}{dx}$ for these following implicit function

  $xy - x^3 = 6$ b. $x^2 + y^2 = 8$ c. $x^2 + 6xy = 4y$ d. $x^3 + y^3 - x - y = 3$

- Differentiate the following with respect to x and simplify as far as possible,

- leaving your answer in factor

  (a) $(2x-1)(x+4)^2$ (b) $3x^3(x^2+4)^2$ (c) $\sqrt{x}(x+3)^2$
  (d) $y = 5x^3 sinx$ (e) $y = cosxsinx$ (f) $y = e^x cosx$ (g) $y = 2x^5 cosx$

- Differentiate the following with respect to x

  (a) $y = \frac{5x^2}{cosx}$ (b) $y = \frac{6e^x}{sinx}$ (c) $y = \frac{cosx}{x^5}$ (d) $y = \frac{x^3+1}{x-1}$ (e) $y = \frac{2x^2-x+3}{2x-5}$

- Differentiate the following with respect to x

  (a) $y = (7x-2)^7$ (b) $y = (5x^3 - 2)$ (c) $y = \frac{1}{x^2+2x-3}$
  (d) $y = \cos(7x+3)$
  (e) $y = e^{2x-3}$

MODULE 3 INTEGRATION

In this module, you will be learning integration of different functions and the various basic integration techniques. This module is made up of the following units:

Unit 1: Integration
Unit 2: Volume of Solids of Revolution by Definite Integral

Unit 1 Integration

Unit Structure

1.1 Introduction
1.2 Intended Learning Outcomes (ILOs)
1.3 Main Content
  1.3.1 Integration
  1.3.2 Properties of Definite Integrals
  1.3.3 Mean Value Theorems for Integrals
  1.3.4 Connecting Integral and Differential Calculus
  1.3.5 Notations for Integration
  1.3.6 Change of Variable of Integration
  1.3.7 Standard Integral
  1.3.8 Methods of Integration
  1.3.9 Integrals of Special Function
  1.3.10 Integration by Partial Fraction
1.4 Summary
1.5 Conclusion
1.6 References/Further Reading
1.7 SELF Assessment Exercise(s)

1.1 Introduction

Integration is a way of adding slices to find the whole. Integration can be used to find areas, volumes, central points and many useful things.

Integration is like filling a tank from a tap. The input (before integration) is the flow rate from the tap. Integrating the flow (adding up all the little bits of water) gives us the volume of the water in the tank.

Hence, the processes of integration are used in many applications.

- The Petronas Tower in Kuala Lumpur, experience high forces due, to winds.

Integration was used to design the building for its strength

- It is used in finding areas under curve surfaces, centre of mass – displacement and velocity, fluid flow, modelling the behaviour of object under stress (e.g., Car engine block, piston, curvature beam of houses, bridges etc.)

1.2 Intended Learning Outcomes (ILOs)

By the end of this unit, you will be able to:

- describe the calculus integration of function
- apply basic integral theorem to functions
- evaluate change of variable in integration
- solve integration of special functions

1.3 Main Content

1.3.1 Integration

The geometric problems that motivated the development of the integral calculus (determination of lengths, areas, and volumes) arose in the ancient civilizations of Northern Africa. Where solutions were found, they related to concrete problems such as the measurement of a quantity of grain. Greek philosophers took a more abstract approach. In fact, Eudoxus (around 400 B.C.) and Archimedes (250 B.C.) formulated ideas of integration as we know it today.

Integral calculus developed independently, and without an obvious connection to differential calculus. The calculus became a ‘‘whole’’ in the last part of the seventeenth century when Isaac Barrow, Isaac Newton, and Gottfried Wilhelm Leibniz (with help from others) discovered that the integral of a function could be found by asking what was differentiated to obtain that function.

The following introduction of integration is the usual one. It displays the concept geometrically and then defines the integral in the nineteenth-century language of limits. This form of definition establishes the basis for a wide variety of applications. Consider the area of the region bound by $y = f(x)$, the x-axis, and the joining vertical segments (ordinates) $x = a$ and $x = b$. (See **Fig. 3.1.1**.)

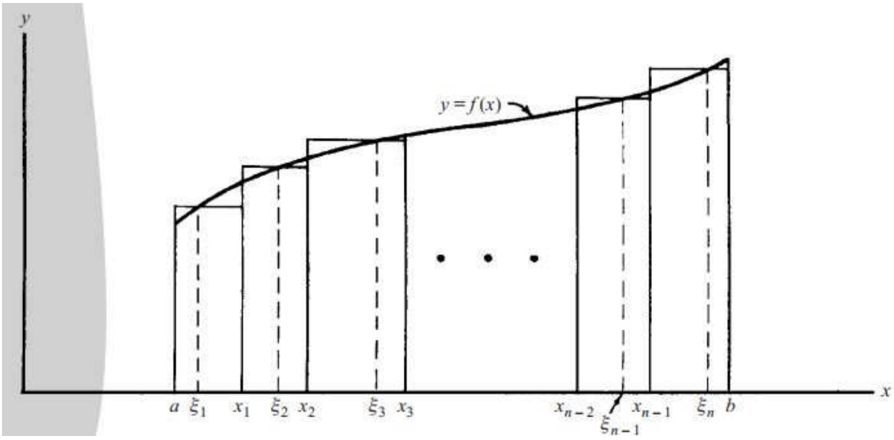


Fig 3.1.1: Region bound by $y = f(x)$

Subdivide the interval $a \leq x \leq b$ into n sub-intervals by means of the points $x_1;\ x_2;\ \ldots;\ x_{n-1}$ chosen arbitrarily. In each of the new intervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ choose points $\xi_1, \xi_2, \ldots, \xi_n$ arbitrarily. Form the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + f(\xi_3)(x_3 - x_2) + \cdots + f(\xi_n)(b - x_{n-1}) \qquad (1)$$

By writing $x_0 = a, x_n = b$; and $x_k - x_{k-1} = \Delta x_k$ this can be written

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k \qquad (2)$$

Geometrically, this sum represents the total area of all rectangles in the above figure.

We now let the number of subdivisions n increase in such a way that each $\Delta x_k \to 0$. If as a result the sum (1) or (2) approaches a limit which does not depend on the mode of subdivision, we denote this limit by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k \qquad (3)$$

This is called the definite integral of $f(x)$ between a and b. In this symbol $f(x)dx$ is called the integrand, and $[a, b]$ is called the range of integration. We call a and b the limits of integration, a being the lower limit of integration and b the upper limit.

The limit (3) exists whenever $f(x)$ is continuous (or piecewise continuous) in $a \leq x \leq b$. When this limit exists, we say that f is Riemann integrable or simply integrable in $[a, b]$. The definition of the definite integral as the limit of a sum was established by Cauchy around 1825. It was named for Riemann because he made extensive use of it in this 1850 exposition of integration.

Geometrically the value of this definite integral represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ only if $f(x) \geq 0$. If $f(x)$ is sometimes positive and sometimes negative, the definite integral represents the algebraic sum of the areas above and below the x-axis, treating areas above the x-axis as positive and areas below the x-axis as negative.

Integration simply means the inverse operation to differentiation.

Let say when $y = x^n$, the derivate i.e $\frac{dy}{dx} = nx^{n-1}$. If we differentiate $\frac{1}{n+1}x^{n+1} = \frac{x^{n+1}}{n+1}$ with respect to x.

$\frac{dy}{dx} = n + 1.\frac{1}{n+1}.x^{n+1-1}$

$\frac{dy}{dx} = x^n$

Therefore, when $\frac{dy}{dx} = x^n$, then $y = \frac{x^{n+1}}{n+1}$ that is to say the integral of x^n with respect to x is $\frac{x^{n+1}}{n+1}$ (where $n \neq 1$).

When integrating, it is very important that you show a constant term called an *arbitrary constant c*. The reason is that when we integrate $3x^2 - 1$, it might be for the following function as their differentiation e.g., $x^3 - x + 5, x^3 - x$ in each case. When integrating $3x^2 - 1$, the constant is not always recovered. In order to show there is a constant term in the integral, the arbitrary constant is added.

1.3.2 Properties of Definite Integrals

If $f(x)$ and $g(x)$ are integrable in $[a, b]$ then

- $\int_a^b \{f(x) \pm g(x)\}dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)$
- $\int_a^b Af(x)\,dx = A\int_a^b f(x)\,dx$ where A is any constant

- $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ provided $f(x)$ is integrable in $[a, c]$ and $[c, b]$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)\,dx = 0$
- If $a \le x \le b, m \le f(x) \le M$ where m and M are constants, then
- $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$
- If $a \le x \le b, f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)dx$
- $|\int_a^b f(x)dx| \le \int_a^b |f(x)|\,dx$ if a<b

1.3.3 Mean Value Theorems for Integrals

As in differential calculus the mean value theorems listed below are existence theorems. The first one generalizes the idea of finding an arithmetic mean (i.e., an average value of a given set of values) to a continuous function over an interval. The second mean value theorem is an extension of the first one that defines a weighted average of a continuous function. By analogy, consider determining the arithmetic mean (i.e., average value) of temperatures at noon for a given week. This question is resolved by recording the 7 temperatures, adding them, and dividing by 7. To generalize from the notion of arithmetic mean and ask for the average temperature for the week is much more complicated because the spectrum of temperatures is now continuous. However, it is reasonable to believe that there exists a time at which the average temperature takes place. The manner in which the integral can be employed to resolve the question is suggested by the following example.

Let f be continuous on the closed interval $a \le x \le b$. Assume the function is represented by the correspondence $y = f(x)$, with $f(x) > 0$. Insert points of equal subdivision, $a = x_0, x_1, \dots, x_n = b$. Then all $\Delta x_k = x_k - x_{k-1}$ are equal and each can be designated by Δx. Observe that $b - a = n\Delta x$. Let ξ_k be the midpoint of the interval Δx_k and $f(\xi_k)$ the value of f there. Then the average of these functional values is

$$\frac{f(\xi_1) + \cdots + f(\xi_n)}{n} = \frac{[f(\xi_1) + \cdots + f(\xi_n)]\Delta x}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(\xi_k)\Delta\xi_k$$

This sum specifies the average value of the n functions at the midpoints of the intervals.

However, we may abstract the last member of the string of equalities (dropping the special conditions) and define

$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{n}f(\xi_k)\Delta\xi_k=\frac{1}{b-a}\int_a^b f(x)dx$$

As the average value of f on $[a,b]$.

Of course, the question of for what value $x=\xi$ the average is attained is not answered; and, in fact, in general, only existence not the value can be demonstrated. To see that there is a point $x=\xi$ such that $f(\xi)$ represents the average value of f on [a; b], recall that a continuous function on a closed interval has maximum and minimum values, M and m, respectively. Thus, (think of the integral as representing the area under the curve). (See Fig. 3.3.1.)

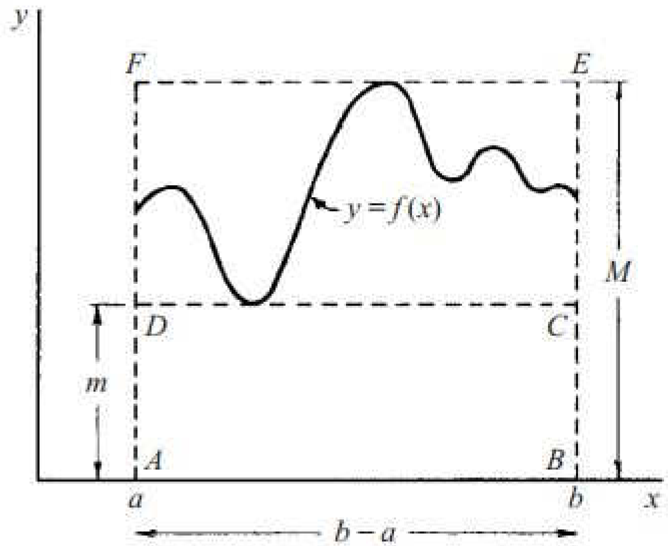


Fig. 3.3.1: The mean value theorems for integrals

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a)$$

$$Or$$

$$m\le\frac{1}{b-a}\int_a^b f(x)dx\le M$$

Since f is a continuous function on a closed interval, there exists a point $x=\xi$ in $(a;\ b)$ intermediate to m and M such that

$$f(\xi)=\frac{1}{b-a}\int_a^b f(x)dx$$

While this example is not a rigorous proof of the first mean value theorem, it motivates it and provides an interpretation.

First mean value theorem.

If $f(x)$ is continuous in $[a;\ b]$, there is a point ξ in $(a;\ b)$ such that

$\int_a^b f(x)dx = (b-a)f(\xi)$

Generalized first mean value theorem.

If $f(x)$ and $g(x)$ are continuous in $[a;\ b]$, and $g(x)$ does not change sign in the interval, then there is a point ξ in (a; b) such that

$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$

1.3.4 Connecting Integral and Differential Calculus

In the late seventeenth century, the key relationship between the derivative and the integral was established. The connection which is embodied in the fundamental theorem of calculus was responsible for the creation of a whole new branch of mathematics called analysis.

Definition

Any function F such that $F'(x) = f(x)$ is called an antiderivative, primitive, or indefinite integral of f.

The antiderivative of a function is not unique. This is clear from the observation that for any constant c

$$(F(x)+c)' = F'(x) = f(x)$$

The following theorem is an even stronger statement.

Theorem.

Any two primitives (i.e., antiderivatives), F and G of f differ at most by a constant, i.e., $F(x) - G(x) = C$.

Example 3.4.1

If $F'(x) = x^2$, then $F(x) = \int x^2 dx = \frac{x^3}{3} + c$ is an indefinite integral (antiderivative or primitive) of x^2. The indefinite integral (which is a function) may be expressed as a definite integral by writing

$$\int f(x)dx = \int_c^x f(t)dt$$

The functional character is expressed through the upper limit of the definite integral which appears on the right-hand side of the equation.

This notation also emphasizes that the definite integral of a given function only depends on the limits of integration, and thus any symbol may be used as the variable of integration. For this reason, that variable is often called a dummy variable.

The indefinite integral notation on the left depends on continuity of f on a domain that is not described. One can visualize the definite integral on the right by thinking of the dummy variable t as ranging over a subinterval $[c;\ x]$. (There is nothing unique about the letter t; any other convenient letter may represent the dummy variable.)

The previous terminology and explanation set the stage for the fundamental theorem. It is stated in two parts. The first states that the antiderivative of f is a new function, the integrand of which is the derivative of that function. Part two demonstrates how that primitive function (antiderivative) enables us to evaluate definite integrals.

1.3.5 Notation for Integration

Symbol of integration (integral sign) and both $\int$ and δx must be written.

Integrand is the function to be integrated and it is place in between the $\int$ and δx. δx is written to illustrate that the integrand is to be integrated.

If $\frac{dy}{dx} = f(x), v = \int f(x)dx + c$ where c is any constant.

Example 3.5.2

Integrate the following:

- $\int (x^3 + 3x^2 + 2x + 4)dx$
- b. $\int (s^3 + 4s)ds$ c. $\int (t^3 + 4t^2 - 2)dt$

Solution 3.5.2

Using the general formula: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

- $\int (x^3 + 3x^2 + 2x + 4)dx$
- $= \frac{x^{3+1}}{3+1} + \frac{3x^{2+1}}{2+1} + \frac{2x^{1+1}}{1+1} + \frac{4x^{0+1}}{0+1}$

- $= \frac{x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2} + \frac{4x}{1} + c$

- $= \frac{x^4}{4} + x63 + x^2 + 4x + c$

- $\int (s^3 + 4s)ds$

- $= \frac{s^{3+1}}{3+1} + \frac{4s^{1+1}}{1+1} + c$
- $= \frac{s^4}{4} + \frac{4s^2}{2} + c$
- $= \frac{s^4}{4} + 2s^2 + c$
- $\int (t^3 + 4t^2 - 2)dt$

o $= \frac{t^{3+1}}{3+1} + \frac{4t^{2+1}}{2+1} - \frac{2t^{0+1}}{0+1} + c$

- $= \frac{t^4}{4} + \frac{4t^3}{3} - 24 + c$

1.3.6 Change of Variable of Integration

If a determination of $\int f(x)dx$ is not immediately obvious in terms of elementary functions, useful results may be obtained by changing the variable from x to t according to the transformation $x = g(t)$. (This change of integrand that follows is suggested by the differential relation $dx = g'(t)dt$.) The fundamental theorem enabling us to do this is summarized in the statement

$$\int f(x)dx = \int f\{g(t)\}g'(t)dt$$

where after obtaining the indefinite integral on the right we replace t by its value in terms of x, i.e., $t = g - 1(x)$. This result is analogous to the chain rule for differentiation.

The corresponding theorem for definite integrals is

$$\int_a^b f(x)dx = \int_\alpha^\beta f\{g(t)\}g'(t)dt \qquad (7)$$

where $g(\alpha) = a$ and $g(\beta) = b$, i.e., $\alpha = g^{-1}(a), \beta = g^{-1}(b)$. This result is certainly valid if $f(x)$ is continuous in $[a; b]$ and if $g(t)$ is continuous and has a continuous derivative in $\alpha \leq t \leq \beta$.

1.3.7 Standard Integral

The following results can be demonstrated by differentiating both sides to produce an identity. In each case an arbitrary constant c (which has been omitted here) should be added.

- $\int u^n = \frac{u^{n+1}}{n+1},\ n \neq 1$
- $\int \frac{du}{u} = \ln|u|$
- $\int sinudu = -cosu$
- $\int cosudu = sinu$
- $\int tanudu = \ln|secu| = -\ln|cosu|$
- $\int\ cot\, udu = \ln|\sin u|$
- $\int\ \sec u\, du = \ln|\sec u + \tan u|$
- $\int cscu\ du = \ln|\csc u - \cot u|$
- $\int\ \sec^2 udu = \tan u$
- $\int\ \csc^2 udu = -cotu$
- $\int secu \tan u\ du = \sec u$
- $\int\ cothudu = \ln|sinhu|$
- $.\int \csc u \cot u\ du = -\csc u$
- $\int a^u du = \frac{a^u}{lna},\ a > 0, a \neq 1$
- $\int e^u = e^u$
- $\int \sinh udu = \cosh u$
- $\int\ coshu\ du = \sinh u$
- $\int\ \tan u\, du = \ln\cosh u$
- $\int sechudu = \tan^{-1}(sinhu)$
- $\int cschudu = -\coth^{-1}(coshu)$

1.3.8 Methods of Integration

In functions of a linear function, we say that an alphabet should replace by a linear function. If the alphabet stands for the linear function, the integral becomes $\int$"an alphabet" δx and before we complete the operation, we must change the variable.

Example 3.8.1

Integrate $\int (3x-2)^6 dx$

Solution 3.8.1

Let $z = 3x - 2$

$\int (3x-2)^6 dx = \int z^6 dx = \int z^6 \frac{dx}{dx}.dz$

$\int z^6 \frac{dx}{dx}.dz\left(\frac{dx}{dz}.dz = dx\right)$

Now $\frac{dx}{dz}$ can be found from the substitution of $z = 3x - 2$

For $\frac{dz}{dx} = 3$ *Differentiation with respect to x*

$\frac{dz}{dx} = \frac{1}{3}$ *Since* $\frac{dz}{dx} = \frac{3}{1} = 3dx \rightarrow \frac{dx}{dz} = \frac{1}{3}$

The integral becomes:

$\int z^6 dx = \int z^6 \frac{dx}{dz}.dz = \int z^6 \left(\frac{1}{3}\right) dz = \frac{1}{3}\int z^6 dz$ *Integrating with respect to x*

$= \frac{1}{3}\frac{z^7}{7} + c$

Finally, replacing the value of z in its terms of its original variable, x, so that

$\int (3x-2)^6 dx = \frac{1}{3}\frac{z^7}{7} + c = \frac{1}{3}\frac{(3x-2)^7}{7} + c$

$= \frac{(3x-2)^7}{7} + c$

Example 3.8.2

Integrate $\int \cos(6x+4)dx$

Solution 3.8.2

Let $z = 6x + 4$

$\int cos\ (6x+4)dx = \int coszdx$

$= \int cosz\frac{dx}{dz}dz$

$= \frac{dz}{dx} = 6$

Therefore, $\frac{dz}{dx} = 6$

$= \frac{dx}{dz} = \frac{1}{6}$

$\int cosz\frac{dx}{dz}dz = \int cosz\frac{1}{6}dz = \frac{1}{6}\int cosz\ dz$

$= \frac{1}{6}sinz + c = \frac{1}{6}\sin(6x+4) + c$

$= \frac{\sin(6x+4)}{6} + c$

Example 3.8.3

Solve the integral $\int \sec^2 8x dx$

Solution 3.8.4

Let $z = 8x$
$\int \sec^2 8x dx = \int \sec^2 z dx = \int \sec^2 z \frac{dx}{dz} dz$
$\frac{dz}{dx} = 8$ that is $\frac{dx}{dz} = \frac{1}{8}$
$\int \sec^2 z \frac{dx}{dz} dz = \int \sec^2 z \frac{1}{8} dz = \frac{1}{8} \int \sec^2 z dz$
$= \frac{1}{8tanz} + c = \frac{tanz}{8} + c = \frac{tan8x}{8} + c$

1.3.9 Integrals of the form $\int \frac{f'(x)}{f(x)} dx$ and $\int f(x).f'(x)dx$

Integral in the form $\int \frac{f'(x)}{f(x)} dx$ is any in which the numerator is the derivative of the denominator. It is of the kind $\int \frac{f'(x)}{f(x)} dx = ln\{f(x)\} + c$

Example 3.9.1

Let us consider the integral $\int \frac{2x-5}{x^2-5x+6} dx$

Solution 3.9.1

We notice that when we differentiate the denominator, we will have the expression in the numerator.

Let $z = x^2 - 5x + 6$
$\frac{dz}{dx} = 2x - 5$
$dz = (2x - 5)dx$

Making dz the subject of the formula the given integral can be rewritten in terms of z.

$\int \frac{2x-5}{x^2-5x+6} dx = \int \frac{2x-5}{z} dx$

Substituting to have

$\int \frac{2x-5}{z} dx = \int \frac{dz}{z} = \int \frac{1}{z} dz$ where $\frac{dz}{z} = \frac{1}{z} dz$
$= \int \frac{1}{z} dz = lnz = c$

Replacing back the value of z

$lnz + c = \ln(x^2 - 5x + 6) + c$

Example 3.9.2

Integrate $\int \frac{3x^2}{x^3-6} dx$

Solution 3.9.2

Let $z = x^3 - 6$
$\frac{dz}{dx} = 3x^2, \; dz = 3x^2 dx$
$\int \frac{3x^2}{x^3-6} dx = \int \frac{3x^2}{z} dx = \int \frac{dz}{z} = \int \frac{1}{z} dz = lnz + c$
$= \ln(x^3 - 6) + c$

Example 3.9.3

Integrate $\int \frac{4x-8}{x^2-4x+5} dx$

Solution 3.9.3

$\int \frac{4x-8}{x^2-4x+5} dx$
$\int 2.\frac{2x-8}{x^2-4x+5} dx$

Note: *When differentiating the denominator* $x^2 - 4x + 5$ *it gives* $2x - 4$. *Looking at the numerator, collecting common terms give* $2(2x - 4)$.

$= 2\int \frac{2x-4}{x^2-4x+5} dx$

Placing the constant number before the integral sign.

Let $z = x^2 - 4x + 5$
$\frac{dz}{dx} = 2x - 4, \; dz = (2x - 4)dx$

$= 2\int \frac{2x-4}{x^2-4x+5} dx = 2\int \frac{dz}{z} = 2\int \frac{1}{z} dz = 2lnz + c$

$= 2\ln(x^2 - 4x + 5) + c$

Example 3.9.4

Integrate $\int \cot xdx$

Solution 3.9.4

From trigonometric function $cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$ where $tanx = \frac{sinx}{cosx}$
Substituting $cotx = \frac{cosx}{sinx}$
$\int cotx\ dx = \int \frac{1}{tanx} dx = \int \frac{cosx}{sinx} dx$

Let $z = sinx$

$\frac{dz}{dx} = cosx,\ dz = cosx\ dx$

$\int \frac{cosx}{sinx} dx = \int \frac{cosx}{z} dx = \int \frac{dz}{z} = \int \frac{1}{z} dz$

$= lnz + c$
$= lnsinx + c$

1.3.10 Integration of Product (Integration by parts)

Let u and v be differentiable functions. According to the product rule for differentials

$$d(uv) = udv + vdu$$

Upon taking the antiderivative of both sides of the equation, we obtain

$$uv = \int udv + \int vdu$$

This is the formula for integration by parts when written in the form

$\int udv = uv - \int vdu$ or $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

where $u = f(x)$ and $v = g(x)$.

The corresponding result for definite integrals over the interval $[a;\ b]$ is certainly valid if $f(x)$ and $g(x)$ are continuous and have continuous derivatives in $[a;\ b]$.

Example 3.10.1

Integrate $\int x^2 lnx dx$

Solution 3.10.1

Let $u = lnx,\ v = x^2, \frac{du}{dx} = \frac{1}{x},\ du = \frac{1}{x}dx, \frac{dv}{dx} = x^2$

Integrating $\int \frac{dv}{dx} = \int x^2 dx\ ,\ \ v = \frac{x^3}{3}$

Using the integral by part definition

$$\int udv = uv - \int vdu$$
$$= lnx\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}.\frac{1}{x}dx$$
$$= lnx\left(\frac{x^3}{3}\right) - \int \frac{1}{3}.x^3.\frac{1}{x}dx$$
$$= lnx\left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^3.\frac{1}{x}dx$$
$$= \frac{x^3}{3}lnx - \frac{1}{3}\int x^2 dx$$
$$= \frac{x^3}{3}lnx - \frac{1}{3}.\frac{x^3}{3} + c$$
$$= \frac{x^3}{3}\left(lnx - \frac{1}{3}\right) + c$$

Example 3.10.2

Integrate $\int x^2 e^{3x} dx$

Solution 3.10.2

Using $\int udv = uv - \int vdu$

Let $u = x^2,\ v = \frac{e^{3x}}{3}$

$$\int udv = x^2.\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2xdx$$
$$= x^2\frac{e^{3x}}{3} - \int \frac{1}{3}.e^{3x}.2xdx$$
$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int e^{3x}.xdx$$

The integral $\int e^{3x}x\ dx$ will also be integrated by part. That is $\int e^{3x}.xdx$

Let $u = x, dv = e^{3x}dx, \frac{du}{dx} = 1,\ du = dx,\ v = \frac{e^{3x}}{3}$

Using $\int udv = uv - \int vdu$

$= x.\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} dx$

$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int e^{3x}xdx$ becomes

$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3} dx\right)$

$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x} dx\right)$

$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{3} + \frac{2}{9}\int e^{3x} dx$

$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{3} + \frac{2}{9}\frac{e^{3x}}{3} + c$

$= \frac{e^{3x}}{3}\left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) + c$

1.3.11 Integration by Partial Fractions

Any rational function $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, with the degree of P(x) less than that of Q(x), can be written as the sum of rational functions having the form

$\frac{A}{(ax+b)^r}, \frac{Ax+B}{(ax^2+bx+c)^r}$ where $r = 1, 2, 3, \ldots$

which can always be integrated in terms of elementary functions

1.3.11 Integration by Partial Fraction for an Irreducible Denominator

Example 3.11.1

Integrate $\int \frac{x+1}{x^2-3x+2} dx$

Solution 3.11.1

When we clearly look at this, the numerator is not the derivative of the denominator. This is not an example of standard integral.

$$\int \frac{x+1}{x^2-3x+2} dx$$

Since the denominator is irreducible, we will have to factorize it

$$\int \frac{x+1}{x^2-3x+2} dx = \int \frac{x+1}{(x-2)(x-1)} dx$$

Resolving into partial fraction

$$\frac{x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$
$$x+1 = A(x-1) + B(x-2)$$

To get A, put $x = 2$, then $A = 3$

Similarly, to get B, put $x = 1, B = -2$

$$\int \frac{x+1}{(x-2)(x-1)} dx = \int \left(\frac{A}{x-2} + \frac{B}{x-1}\right) dx$$

Replacing back the values A and B respectively into the term above.

$$\int \frac{x+1}{(x-2)(x-1)} dx = \int \left(\frac{3}{x-2} - \frac{2}{x-1}\right) dx$$
$$= \int \frac{3}{x-2} dx - \int \frac{2}{x-1} dx$$
$$= 3\int \frac{1}{x-2} dx - 2\int \frac{1}{x-1} dx$$
$$= 3\ln(x-2) - 2\ln(x-1) + c$$

1.3.12 Integration of Partial Fraction by repeated rule.

Example 3.11.2

Determine $\int \frac{x^2}{(x+1)(x-1)^2} dx$

Solution 3.11.2

$$\int \frac{x^2}{(x+1)(x-1)^2} dx = \int \left(\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right) dx$$

Resolving into partial fraction

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiplying LHS and RHS throughout by the LCM of the denominator

$$(x+1)(x-1)^2$$
$$x^2 = A(x-1)^2 + B(x-)(x+1) + C(x+1)$$

To get C, put x = 1

$$1^2 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$$
$$C = \frac{1}{2}$$

To get A, put x = -1

$$(-1)^2 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$$
$$A = \frac{1}{4}$$

Also, by substituting the values of A and C, $B = \frac{3}{4}$

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{1/4}{x+1} + \frac{3/4}{x-1} + \frac{1/2}{(x-1)^2}$$

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{1}{4(x+1)} + \frac{3}{4(x-1)} + \frac{1}{2(x-1)^2}$$

$$\int \frac{x^2}{(x+1)(x-1)^2}dx = \frac{1}{4}\int \frac{1}{x+1}dx + \frac{3}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{(x+1)^2}$$
$$= \frac{1}{4}\ln(x+1) + \frac{3}{4}\ln(x-1) + \frac{1}{2(x-1)} + c$$

Example 3.11.3

Determine $\int \frac{x^2+1}{(x+2)^3}dx$

Solution 3.11.3

Resolving $\frac{x^2+1}{(x+2)^3}$ into partial fraction

$$\frac{x^2+1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

Multiplying throughout by the LCM of the denominator.

$$x^2 + 1 = A(x+2)^2 + B(x+2) + C$$

To get C, put x = -2.

$$(-2)^2 + 1 = A(-2+2)^2 + B(-2+2) + C$$

Hence, C = 5

Solving and simplifying, A =1 and B = -4

$$\frac{x^2+1}{(x+2)^3}=\frac{1}{x+2}-\frac{4}{(x+2)^2}+\frac{5}{(x+2)^3}$$

Therefore:

$$\int\frac{x^2+1}{(x+2)^3}dx=\int\frac{1}{x+2}dx-\int\frac{4}{(x+2)^2}dx+\int\frac{5}{(x+2)^3}dx$$

$$=\ln(x+2)-4\int(x+2)^{-2}dx+5\int(x+2)^{-3}dx$$

$$=\ln(x+2)+\frac{4}{x+2}-\frac{5}{2(x+3)^2}+c$$

Example3.11.4

Determine $\int\frac{x^2}{(x-2)(x^2+1)}dx$

Solution 3.11.4

Resolving $\frac{x^2}{(x-2)(x^2+1)}$ into partial fraction

$$\frac{x^2}{(x-2)(x^2+1)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}$$

Multiplying throughout by the LCM of the denominator.

$$x^2=A(x^2+1)+(Bx+C)(x-2)$$

To get A, put x =2

$$2^2=A(2^2+1)+(B(2)+C)(2-2)$$

$$A=\frac{4}{5}$$

To get B and C, expanding equation and taking the coefficient

$$B=\frac{1}{5},C=\frac{2}{5}$$

$$\frac{x^2}{(x-2)(x^2+1)}=\frac{\frac{4}{5}}{x-2}+\frac{\frac{1}{5}x+\frac{2}{5}}{x^2+1}$$

$$\frac{x^2}{(x-2)(x^2+1)}=\frac{4}{5(x-2)}+\frac{x+2}{5(x^2+1)}$$

$$\int \frac{x^2}{(x-2)(x^2+1)} dx$$
$$= \int \frac{4}{5(x-2)} dx + \int \frac{1}{5(x^2+1)} dx + \int \frac{2}{5(x^2+1)} dx$$
$$= \frac{4}{5}\int \frac{1}{(x-2)} dx + \frac{1}{5}\int \frac{1}{(x^2+1)} dx + \frac{2}{5}\int \frac{1}{(x^2+1)} dx$$
$$= \frac{4}{5}\ln(x-2) + \frac{1}{5}\int \frac{\frac{1}{2}(2x)}{x^2+1} dx + \frac{2}{5}\tan^{-1} x$$
$$= \frac{4}{5}\ln(x-2) + \frac{1}{5}.\frac{1}{2}\int \frac{2x}{x^2+1} dx + \frac{2}{5}\tan^{-1} x$$
$$= \frac{4}{5}\ln(x-2) + \frac{1}{10}\ln(x^2+1) + \frac{2}{5}\tan^{-1} x + c$$

1.4 Summary

- $\int$ is the symbol of integration (integral sign)
- Integration simply means the inverse operation to differentiation
- $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (provided $n \neq 1$)
- $\int udv = uv - \int vdu$
- $\int \frac{f'(x)}{f(x)} dx = \ln\{f(x)\} + c$

1.5 Conclusion

From the unit, you must have learnt the basic integral theorem and their applications, calculus of integration of function, standard integral (integration of special function). Now you can identify an integral and with appropriate and applicable integration techniques. You can solve integration using by part method and partial fraction method.

1.6 References/Further Reading

Calculus An Applied Approach Larson Edwards Sixth Edition

Blitzer Algebra and Trigonometry Custom 4th Edition

Engineering Mathematics by K.A Stroud 5th Edition

Additional Mathematics by Godman, and J.F Talbert.

4.0 Self-Assessment Exercise(s)

- **Find the following:**
- i. $\int (3x-1)dx$ ii. $\int (3t+4t^2)dt$
- iii. $\int (x+1)(x-2)dx$ iv. $\frac{\int 2(x+3)}{x^3}dx$
- Integrate the following
- i. $\int (2x-7)^3dx$ ii. $\int \cos(7x+2)dx$
- iii. $\int \sec^2(3x+1)dx$ iv. $\int (5x-4)^6dx$
- v. $\sin\int (3x+8)\, dx$
- **Integrate the following**
- i. $\int \frac{3x^2}{x^3-4}dx$ ii. $\int \frac{2x+3}{x^2+3x-5}dx$
- iii. $\int \frac{(2x+4)}{(x^2+4x-1)}dx$ iv $\int \frac{x-3}{x^2-6x+2}dx$
- v. $\int \frac{\sec^2 x}{tanx}dx$ vi. $\int tanxdx$
- vii $\int \frac{2x-3}{x^2+3x-7}dx$ viii. $\int \frac{9x^2}{x^3-7}dx$
- **Integrate the following by part**
- i. $\int e^{3x}sinxdx$ ii. $\int xlnxdx$
- iii. $e^{5x}sin3xdx$ iv. $\int x^3e^{2x}dx$

Unit 2: Volume Of Solids of Revolution by Definite Integral

Unit Structure

2.1 Introduction

The use of the integral as a limit of a sum enables us to solve many physical or geometrical problems such as determination of areas, volumes, arc lengths, moments of inertia, centroids, etc.

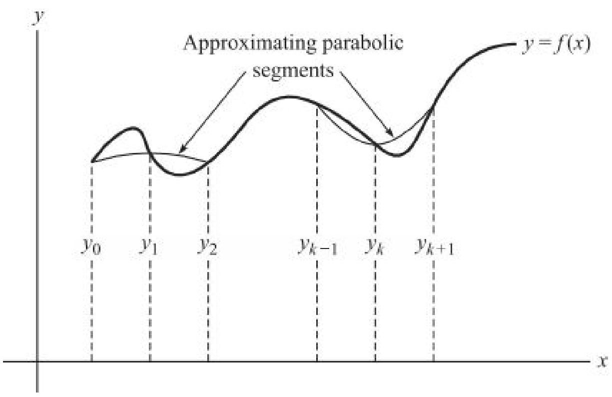


Fig 1.1: approximating parabolic segments

2.2 Intended Learning Outcomes (ILOs)

By the end of this unit, you will be able to:

- find the arc length under integration
- determine the volume of solid revolution
- evaluate the volume of a sphere, spherical and cone.

2.3 Main Content

1.3.1 Arc Length

As you walk a twisting mountain trail, it is possible to determine the distance covered by using a pedometer. To create a geometric model of this event, it is necessary to describe the trail and a method of measuring distance along it. The trail might be referred to as a path, but in more exacting geometric terminology the word, curve is appropriate. That segment to be measured is an arc of the curve. The arc is subject to the following restrictions:

- It does not intersect itself (i.e., it is a simple arc).
- There is a tangent line at each point.
- The tangent line varies continuously over the arc.

These conditions are satisfied with a parametric representation $x = f(t)$; $y = g(t)$; $z = h(t)$; $a \leq t \leq b$,where the functions f , g, and h have continuous derivatives that do not simultaneously vanish at any point. In this introduction to curves and their arc length, we let $z = 0$, thereby restricting the discussion to the plane.

A careful examination of your walk would reveal movement on a sequence of straight segments, each changed in direction from the previous one. This suggests that the length of the arc of a curve is obtained as the limit of a sequence of lengths of polygonal approximations. (The polygonal approximations are characterized by the number of divisions $n \to \infty$ and no subdivision is bound from zero.

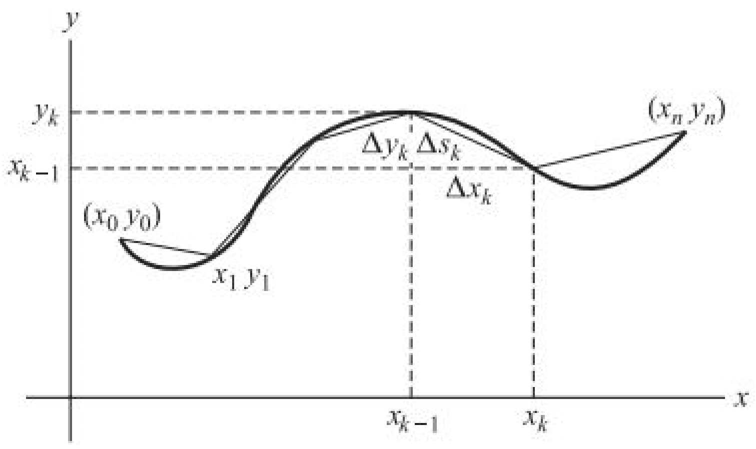


Fig 3.1: Polygon approximation

Geometrically, the measurement of the kth segment of the arc, $0 \leq t \leq s$, is accomplished by employing the Pythagorean Theorem, and thus, the measure is defined by

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left\{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right\}^{\frac{1}{2}}(\Delta x_k)$$

Where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$

Thus, the length of the arc of a curve in rectangular Cartesian coordinates is

$$L = \int_a^b \{[f'(t)^2 + [g'(t)]^2\}^{\frac{1}{2}} dt = \int \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{1/2} dt$$

(This form may be generalized to any number of dimensions.) Upon changing the variable of integration from t to x we obtain the planar form

$$L = \int_{f(a)}^{f(b)} \left\{1 + \left[\frac{dy}{dx}\right]^2\right\}^{1/2}$$

(This form is only appropriate in the plane.)
The generic differential formula $ds^2 = dx^2 + dy^2$ is useful, in that various representations algebraically arise from it. For example, expresses instantaneous speed

$$\frac{ds}{dt}$$

.

2.3.2 Area

Area was a motivating concept in introducing the integral. Since many applications of the integral are geometrically interpretable in the context of area, an extended formula is listed and illustrated below. Let f and g be continuous functions whose graphs intersect at the graphical points corresponding to $x = a$ and $x = b, a < b$. If $g(x) \geq f(x)$ on [a; b], then the area bounded by $f(x)$ and $g(x)$ is

$$A = \int_a^b \{g(x) - f(x)\} dx$$

If the functions intersect in (a; b), then the integral yields an algebraic sum. For example, if $g(x) = sin\, x$ and $f(x) = 0$ then:

$$\int_0^{2\pi} sinx dx = 0$$

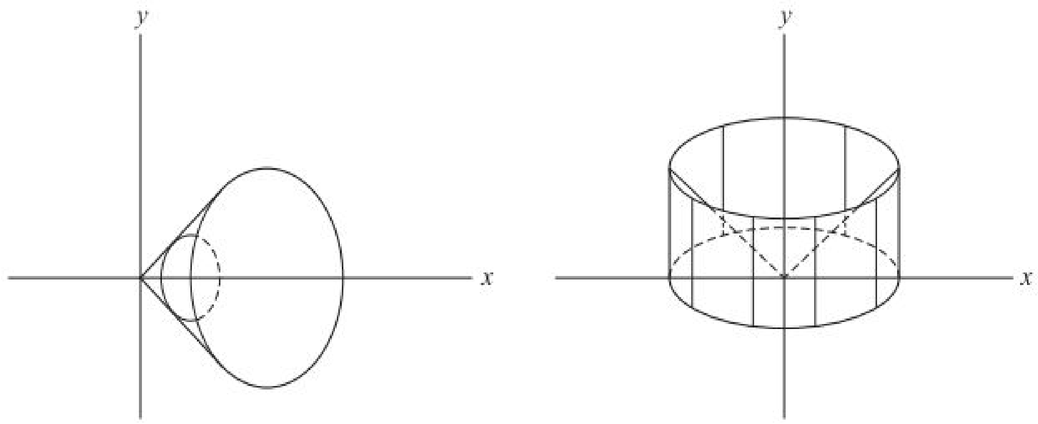
2.3.3 Volumes of Revolution

2.3.1 Disk Method

Assume that f is continuous on a closed interval $a \leq x \leq b$ and that $f(x) \geq 0$. Then the solid realized through the revolution of a plane region R (bound by $f(x)$, the x-axis, and $x = a$ and $x = b$) about the x-axis has the volume

$$V = \pi \int_a^b [f(x)]^2 dx$$

This method of generating a volume is called the disk method because the cross sections of revolution are circular disks.



- **Fig 3.3.1. Disk method**

Example 3.3.1

A solid cone is generated by revolving the graph of y = kx, k > 0 and $0 \leq x \leq b$, about the x-axis. Its volume is

$$V = \pi \int_0^b k^2 x^2 dx = \left[\pi \frac{k^3 x^3}{3}\right]_0^b = \frac{k^3 b^3}{3}$$

2.3.2 Shell Method

Suppose f is a continuous function on $[a; b]$, $a \geq 0$, satisfying the condition $f(x) \geq 0$. Let R be a plane region bound by $f(x), x = a, x = b$, and the x-axis. The volume obtained by orbiting R about the y-axis is

$$V = \int_a^b 2\pi x f(x) dx$$

This method of generating a volume is called the shell method because of the cylindrical nature of the vertical lines of revolution.

Example 3.3.2

If the region bounded by y = kx, $0 \leq x \leq b$ and x = b (with the same conditions as in the previous example) is orbited about the y-axis the volume obtained is

$$V = 2\pi \int_0^b x(kx) dx = \left[2\pi k \frac{x^3}{3}\right]_0^b = 2\pi k \frac{b^3}{3}$$

By comparing this example with that in the section on the disk method, it is clear that for the same plane region the disk method and the shell method produce different solids and hence different volumes.

3.3.3 Moment of Inertia

Moment of inertia is an important physical concept that can be studied through its idealized geometric form. This form is abstracted in the following way from the physical notions of kinetic energy, $K = \frac{1}{2}mv^2$, and angular velocity, $v = \omega r$. (m represents mass and v signifies linear velocity). Upon substituting for v

$$K = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}(mr^2)\omega^2$$

When this form is compared to the original representation of kinetic energy, it is reasonable to identify mr^2 as rotational mass. It is this quantity, $l = mr^2$ that we call the moment of inertia. Then in a purely geometric sense, we denote a plane region R described through continuous functions f and g on [a; b], where a > 0 and f(x) and g(x) intersect at a and b only. For simplicity, assume $g(x) \geq f(x) > 0$. Then

$$l = \int_a^b x^2[g(x) - f(x)]dx$$

By idealizing the plane region, R, as a volume with uniform density one, the expression $[g(x) - f(x)]dx$ stands in for mass and r^2 has the coordinate representation x^2.

2.3.4 The Volume of a Sphere

Find the volume of a sphere generated by a semicircle $y = \sqrt{r^2 - x^2}$ revolving around the x- axis.

Solution.

Since the end points of the diameter lying on the x-axis and – r and r as shown below, the

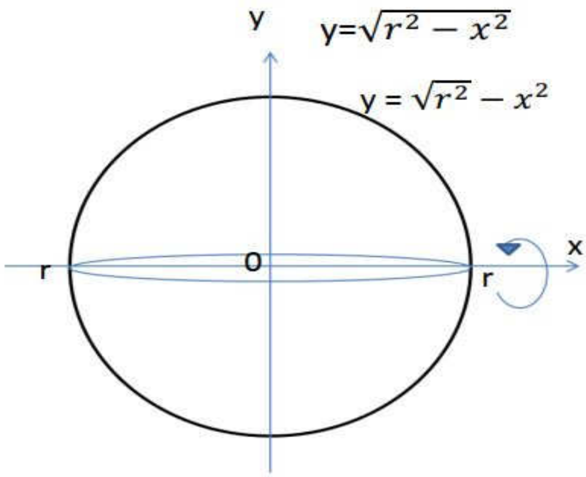


Fig 3.4.1

Since $y = \sqrt{r^2 - x^2}$
Squaring both side

$$y^2 = (r^2 - x^2)$$

$$V = \pi\int_a^b y^2dx = \pi\int_{-r}^{r}(r^2 - x^2)dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^2r - \frac{r^3}{3}\right) - \left(r^2.-r - \frac{(-r)^3}{3}\right)\right]$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{(r)^3}{3}\right)\right] = \pi\left[\frac{3r^3 - r^3}{3} - \frac{-3r^3 + r^3}{3}\right]$$

$$= \pi\left[\left(\frac{2r^3}{3}\right) - \left(\frac{-2r^3}{3}\right)\right] = \pi\left[\frac{2r^3}{3} + \frac{2r^3}{3}\right] = \pi\left[\frac{2r^3 + 2r^2}{3}\right]$$

$$V = \frac{\pi 4r^3}{3}$$

2.3.5 The Volume of a Spherical segment

Example 3.5.1

Find the volume of a spherical segment generated by the portion of the right semi-circle between $y = a$ and $y = a + h$, revolving around the y-axis, is as shown in the below figure:

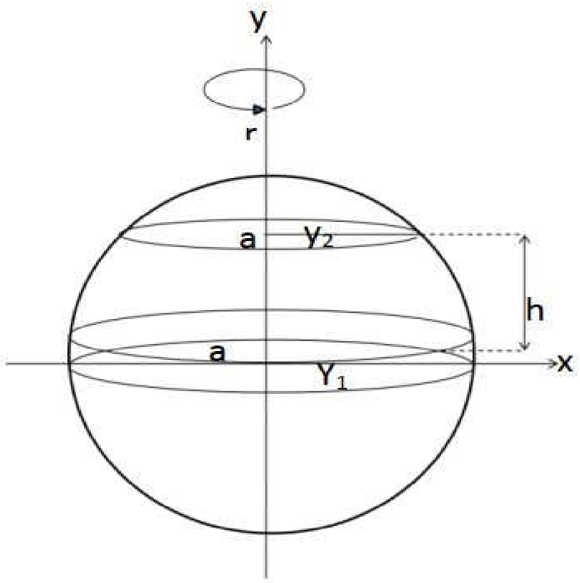


Fig. 3.5.1

Solution 3.5.1

Since the right semi-circle equation

$$x = \sqrt{r^2 + y^2}$$
$$V = \int_c^d x^2 dy = \int_a^{a+h} (r^2 - y^2) dy$$
$$= \pi \left[r^{2y} - \frac{y^3}{3} \right]_a^{a+h}$$

$$V = \pi[r^2(a + h) - \frac{(a+h)^3}{3} - \left(r^2 . a - \frac{1}{3}(a^3) \right)]$$
$$= \pi \left(r^2 a + r^2 h - \frac{1}{3}(a^3 + 3a^{2h} + 2ah^2 + h^2 a + h^3) - \left(r^2 a - \frac{1}{3} a^3 \right) \right]$$
$$= \pi \left[r^2 a + r^2 h - \frac{1}{3} a^3 - a^2 h - \frac{2}{3} ah^2 - \frac{1}{3} h^2 a - \frac{1}{3} h^3 - r^2 a + \frac{1}{3} a^2 \right]$$
$$= \pi \left[r^2 a + r^2 h - \frac{1}{3} a^3 - a^2 h - \frac{2}{3} ah^2 - \frac{1}{3} h^2 a - \frac{1}{3} h^3 - r^2 a + \frac{1}{3} a^2 \right]$$

Collecting like or common terms

$$= \pi \left[r^2 a - r^2 h + r^2 h - \frac{1}{3} a^3 + \frac{1}{3} a^3 - a^2 h - \frac{2}{3} ah^2 - \frac{1}{3} ah^2 - \frac{1}{3} h^3 \right]$$
$$= \pi \left[r^2 h - a^2 h - ah^2 - \frac{1}{3} h^3 \right]$$
$$V = \pi h \left[r^2 - a^2 - ah - \frac{1}{3} h^2 \right]$$

2.3.6 The volume of a cone

Example 3.6.1

Find the volume of a right circular cone generated by the line (segment) passing through the origin and the point (h, r), where h denotes the height of the cone and r is the radius of its base, revolving around the x-axis, as shown in the figure below

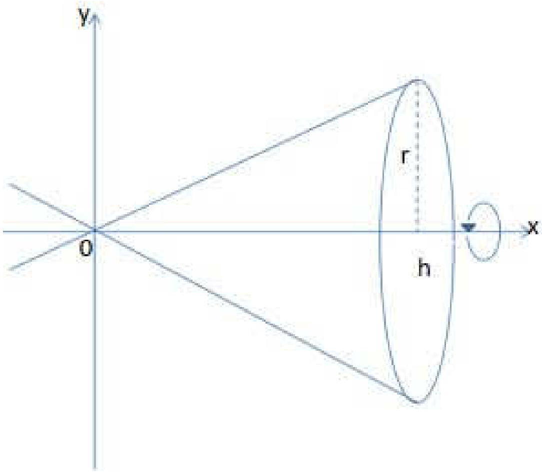


Fig 3.6.1 Circular cone

Solution 3.6.1

The equation of the generating line

$$y = mx$$
$$m = \frac{\Delta x}{\Delta y} = \frac{r}{h}$$

By substitution

$$y = \frac{r}{h}x$$
$$V = \pi \int_a^b y^2 dx$$

On substituting the above in each other

$$= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2x^2}{h^2} dx$$
$$= \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h$$

$= \frac{\pi r^2}{h^2}\left[\frac{h^3}{3}\right] = \frac{\pi r^2 h^3}{3h} = \frac{\pi r^2 h}{3}$
$V_x = \frac{\pi r^2 h}{3}$
$V_c = \frac{1}{3}\pi r^2 h$

Example 3.6.2

Find the volume of a solid of revolution generated by a plane bounded by the segment of a curve y = + 3x and the x–axis, revolving around the x–axis, as shown in the figure below:

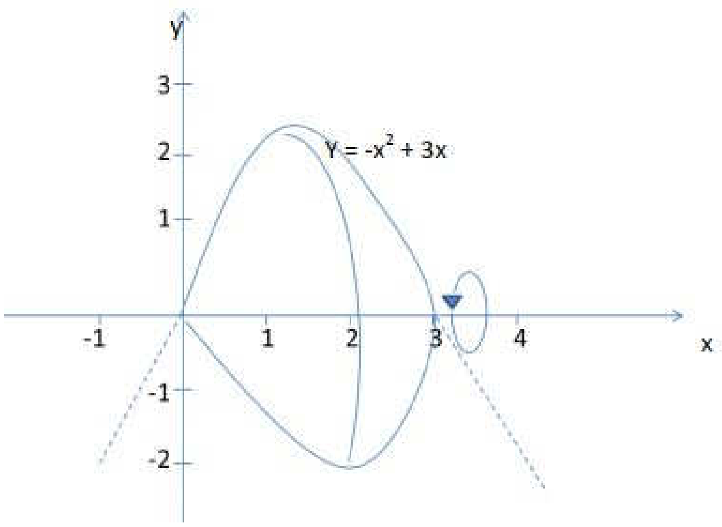


Fig 3.6.2: Revolution Generation

Solution 3.6.2

When $y = -x^2 + 3x$, let $y = 0$

The limits of the integration: $-x^2 + 3x = 0$

Factorizing the term above

$x(-x + 3) = 0$
$x = 0$, or $-x + 3 = 0$
$x = 0, x = 3$
$V_x = \pi \int_a^b y^2 dx = \pi \int_0^3 (-x^2 + 3x)^2 dx$

Expanding $(-x + 3x)^2 = (-x^2 + 3x)(-x^2 + 3x)$
$= x^4 - 3x^3 - 3x^3 + 9x^2$
$= x^4 - 6x^3 + 9x^2$

$$V_x = \pi \int_0^3 (-x^2 + 3x)^2 dx$$
$$= \pi \int_0^3 (x^4 - 6x^3 + 9x^2) dx$$
$$= \pi \left[\frac{x^5}{5} - \frac{3x^4}{2} + 3x^3\right]_0^3$$
$$= \pi \left[\frac{3^5}{5} - \frac{3(3)^4}{2} + 3(3)^3\right]$$
$$= \pi \left[\frac{243}{5} - \frac{243}{2} + \frac{81}{1}\right] = \pi \left[\frac{486-1215+810}{10}\right]$$
$$= \pi \left[\frac{81}{10}\right] = \frac{81\pi}{10}$$

2.4 Summary

In this unit, you have a good of the basic applications of calculus of integration. You can discuss and evaluate different applications of integration in arc length, volume of sphere, volume of cone and volume of spherical.

2.5 Conclusion

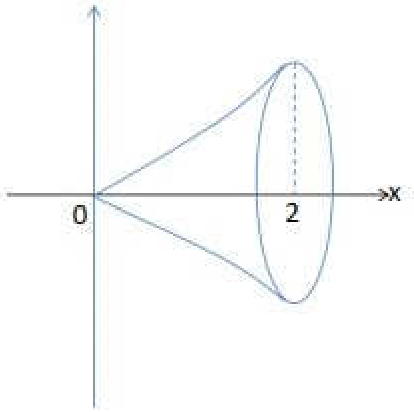
You have learned the integral of arc length and volume of revolution the basic assumptions. You as well been taught how to evaluate the volume integral for a sphere, spherical and cone.

2.6 References/Further Reading

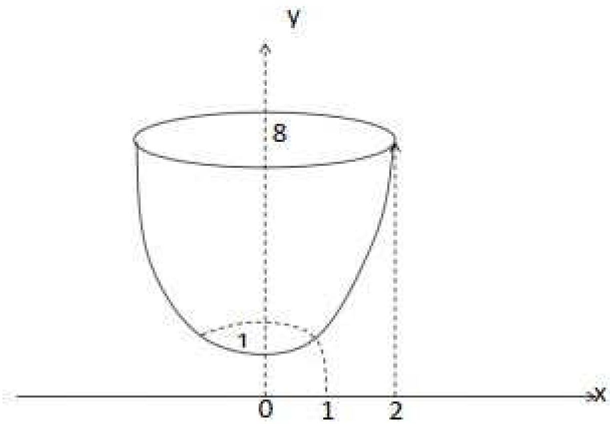
Engineering Mathematics by K. A Stroud 5th Edition.
Additional Mathematics by Godman& Y. F Talbert.
Calculus An Applied Approach Larson Edwards Sixth Edition.
Blitzer Algebra and Trigonometry Custom 4th Edition

2.7 Self-Assessment Exercise

- The portion of the curve $y = x^2$ between $x = 0$ and $x = 2$ is rotated completely round the x-axis. Find the volume of the solid created.



- The part of the curve $y = x^3$ from $x = 1$ to $x = 2$ is rotated completely round the y – axis. Find the volume of the solid generated.



- The finite area enclosed by the line $2y = x$ and the curve $y^2 = 2x$ is rotated completely about x-axis. Calculate the volume of the solid produced.

- Find the volume generated by rotating the curve $y = 3x^2 + 1$ from $x = 1$ to $x = 2$ completely round the x – axis.

- If the area enclosed between the curves $y = x^2$ and the line $y = 2x$ is rotated around the x–axis through four right angles, find the volume of the solid generated.

- Find the volume of a solid generated by rotating a curve $y = 2sinx$ between $x = 0$ and $x = 2$ around the x-axis.