MTH 112
DIFFERENTIAL CALCULUS

NATIONAL OPEN UNIVERSITY OF NIGERIA

COURSE GUIDE

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DIFFERENTIAL CALCULUS

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TABLE OF CONTENTS PAGE

Introduction

This course calculus I, which is also known as differential calculus, is the first of the two calculus courses that will be offered by all 100 level students wishing to have a Bachelor's degree in mathematics, chemistry) physics, mathematics education, physics education, chemistry education, social sciences, computer science and computer science education. This course could serve as a reference course for all who wish to model real life situations that will involve optimization or predication of resource or commodities.

A vary good starting point is to ask you to attempt to provide an answer to this question "what is calculus" calculus is the word used in the Roman empire to describe a little pebble that was used in counting. But today the word is used to describe that branch of mathematics that extends the basic concept of elementary algebra and geometry into a new tool used in solving a variety of real-life problems. Calculus is now one of the important mathematical tools in solving most work of life. For example you could use it in the prediction of the orbit of earth satellites, in study of inertial navigation systems and radar systems.

In the modeling of economy, social and psychological behavior calculus is widely used. Calculus is useful in solving problems in the field of business, biology, medicine, animal husbandry and political science. The history of discovery of differential calculus could be traced back to 2nd half of the seventeenth century, when Sir Isaac Newton (1642 - 1727) was able to explain the motion of the planets about the sun as a consequent of the result known today as the Law of gravitational attraction. In 1675 a German Mathematician Gottfried Whilhem Leibniz (1646 - 1716) introduced the famous and basic notation dx andds which you will use in this course and the second course respectively. Other mathematicians since then have contributed to the growth and development of differential calculus among such are Bernoulli of Busel, Taylor, Lagrange and Maclaurin.

This course will consist of ten units divided into 2 modules. In the first module you shall be introduced to basic mathematical concepts such as real numbers functions, units and continuity. Differentiating a function from first principles is shield in the first module. In the second module you will acquire techniques of differentiation of functions such Sin x, Cos x, Log x, e^x, Cosh x, Sinh x, arc sinh x, arc cosh x, and arcsinh x. you will also apply these techniques in sketching the graph of curves, solving problems of minimization and maximization of values of functions, finding rates at which quantity vary deriving the formula for the equation of tangent and Normal to curves at a given point and evaluation of the velocity and acceleration of a moving body in a give interval of time.

In this course you would be required to do correctly every exercise given within and end of the units. The course has been designed for you in such a way that your proper understanding of unit 1 will make it very easy for you to study unit 2 and so on. Do not forget to study all definition properly and explain the definition. This is because it will make you overcome unnecessary difficulties in the study of calculus as a course. Finally in this course emphasis is placed on the technique application and problem solving rather than theorems and proofs this course on your own with minimum fusses through continuous practice of solved examples and exercises.

What You Will Learn In This Course

In this course you will learn how to use the limiting process to find the derivation of a function. You will combine the properties of numbers and functions to find the derivatives of special functions. You will extend these laws to finding derivatives of sum, difference, product, quotient and composite of differentiation to find the derivatives of transcendental functions such as sin x, cos x, e^x, cu x, sin hx, cos hx etc.

You will also be required to apply the skills acquired in techniques of differential to the following (i) Sketching graphs of curves (ii) Solving minimum and maximum problem (iii) finding approximate value of a quantity (iv) finding the equation of tangent and normal lines (v) finding velocity and acceleration of a moving body.

Course Aims

This course aims to position you to a level of competence in the techniques or skills in solving any type of problem on differentiation.

This you will be able to achieve by aiming to:

i) Review properties of real numbers
ii) Define properties of limit of a function
iii) Define basic properties of a continuous function both at a point and within an interval of points
iv) Define basic concept of derivatives of a function
v) Acquire special skills in techniques of differentiation
vi) Relate differentiation of functions to real-life problems.
vii) Know rules of differentiation
viii) Apply the rules of differentiation to solving physical problems.

Course Objectives

On successful completion of this course you should be able to:

i) List all the properties of Real Numbers
ii) Define and identify a function
iii) Identify the type and characteristic of a given function
iv) Determine if the limit of a given function exists
v) Define and give example of a continuous function
vi) Calculate the derivatives of any given function
vii) Recall the rules of differentiation
viii) Sketch the graph of a function using differentiation as a tool
ix) Calculate the maximum and minimum values of any given function within an interval
x) Determine the rate at which a given quantity changes
xi) Calculate the velocity and acceleration of a moving body within an interval of time
xii) Derive the equation of a tangent and normal of a curve at a given point
xiii) Find an approximate value of a quantity or function

Working through this Course

To successfully complete this course you need to read this course guide and study all the units sequentially. You need not jump any unit. The course has been prepared in such a way that knowledge gained in a previous unit will be needed to understand any unit under study. Each study unit is divided into sections. All sections of any unit should be studied. Each section has self assessment questions with answers. Each unit have tutor marked assignments. These are assignment that will be submitted to your tutors at your study center. The course should take you about 34 weeks to complete. How you will spend your time in each section of each unit will be given to you below in the course materials.

Course Materials

You shall now be given list of materials; you will need to successfully complete the course. They are as follows:

i) Course guide for MTH 114
ii) Study units for MTH 114
iii) Recommended Textbook

iv) Assignment file
v) Dates of Tutorials, Assessment and examination.

Study Units

There are ten study units in this course. It is given to you as follows:

Module 1

Module 2

Textbooks and References

The following are recommended textbooks you could borrow or purchase them:

Godwin Odili (Ed) (1997): Calculus with Coordinate Geometry and Trigonometry, Anachuma Educational Books, Nigeria.

Thomas G.B and FINNEY R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

Satrmino L.S. & Einar H. (1974) Calculus "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

Osisiogu U.A (1998) An introduction to Real Analysis with Special Topic on Functions of Several Variables and Method of Lagranges Multipliers, Bestsoft Educational Books Nigeria Flanders H, Korfhage R.R, Price J.J (1970) Calculus academic press New York and London. Osisioga U.A (Ed)(2001) fundamentals of Mathematical analysis, best soft Educational Books, Nigeria

Assignment File

Assignment File and Tutor Marked Assignment (TMA) are assignments. There are at least 10 assignments at the end of each unit. You are to make sure you do at least six of them in each unit and submit to your tutor at the study center attached to your answer. You will find these assignments in your assignment files.

Assessment

Final Examination & Grading

The final examination for these courses MAT 114 will be between 2 to 2 % hours durations.

Course Marking Scheme

Assessment	Marks
Assignment	**25 out of which the best 3 from unit will be chosen so the mark expected is15 x 3 = 45**

BONUS

Five-bonus mark will be given for attempting all the assignment in the assignment file.
Final exam: 50% of overall course mark
Total = 45 + 5 + 50 = 100.

Strategies for Studying the Course

The course has been presented with less theory and more practice. Therefore self - test exercise have been provided at the end of most sections in the unit. A careful study of the solved examples will be a useful guide to the exercises provided at the end of each section of a unit. Also working through the exercises at end of each section will help you to solve the assignment files. With several worked examples you will not find it difficult to solve and achieve the objective of the course.

While reading through this course makes sure that you check up any topic you are referred to in any previous unit you have studies. These references were given so that you use them to understand the topic under study.

Tutors and Tutorials

To be supplied by NOUN.

Summary

In this course you have studied

i) The basic properties of real numbers
ii) Types and characteristics of functions
iii) Properties of limits and Algebra of limits of functions
iv) Continuous and discontinuous function
v) How to apply differentiation of functions to solve real life problems and evaluate target and normal to a curve at a given part.

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TABLE OF CONTENTS PAGE

MODULE 1

UNIT 1 BASIC PROPERTIES OF REAL NUMBERS

CONTENTS

1.0 INTRODUCTION

In this unit you will be introduced to the basic concepts of real numbers. Basics properties of real numbers is the first topic or concept you are required to study in this course. There are reasons among others why real numbers should be the first topic to study in this course.

Firstly numbers are very important in all calculations, a fact you are already familiar with

Secondly, the properties of numbers is very essential to the development of calculus.

Lastly, all the topics in mathematics that you will be required to study during your programme will involve the use of some properties of real numbers.

In view of the above you should endeavor to carefully study all the topics covered in this unit and as well as complete all assignment Materials learnt in this unit will help you in understanding all other topics you will learn throughout this course.

2.0 OBJECTIVES

After reading through this unit you should be able to:

1. List correctly all types of numbers
2. Recall 3 basic axioms of a real number system
3. Identify all types of intervals
4. Define 4 properties of absolute value of a real number
5. Define a bounded set.

3.0 MAIN CONTENT

3.1 Sets

When you collect items of similar characteristics or functions together, you could say that you have a 'set of such items'. For example, you can have a set of books, a set of furniture, a set of dissecting instruments etc.

Therefore you could define a set as a collection of distinct (definite distinguishable) objects which are selected by means of certain rules or description. Hence a set is a mathematical concept used to describe a list, collection or a class of objects, figures etc.

The objects in the list or collections are called elements or members of the set

Example of Sets

1) The students of national Open University
2) {1,2,3,4...... }Set of Natural Numbers
3) The set of stalls in a market

Sets are denoted by single capital letters A, B, C etc. or by the use of braces, for example, {a, b, c} denotes the set having a, b, and c as members or elements.

You will now be introduced to some specific sets, and symbols associated with them which you will likely use throughout this course

Specific Sets	**Symbols**	**Statements**
1.Null or Empty set	ø	It is a Set which has no member.
2. An element of Set	α∈ A	a is an element of A
3. Not an element of	b∉ A	b is not an element of A
4. Universal Set	น	largest Set containing all elements under considerations (largest Set containing all Sets).
5. Subset	A⊂B or BcA	A is a subset of B (each element of A also belong to B).

Specific Sets	**Symbols**	**Statements**
6. Proper Subset	A⊂B	A is a proper subset of B (A is subset of B)
	A\B	B has at least one element Which is not in A.
7. Union of Sets	A∪B	This is the set of all elements which belong to A or B i.e: AVB = {x : x E A or x(=-BI
8. Intersection of Sets	A∩B	This is the set of all element which belong to A and B both i.e; A(V= {x:xcA and xEB}

SELF ASSESSMENT EXERCISE 1

Look around where you are right now identify 10 different objects. Group each one that could be used for:

l. Eating
2. Sleeping
3. Cooking
4. Sitting
5. Decoration etc.

Classify each of the identified objects.

3.2 Real Numbers

You will continue the introduction to the course for differential calculus with the study of real numbers. You are already familiar with the following types of real numbers.

i. Natural Numbers

The set of positive whole number/1,2,3,4,....., are called natural numbers. The letter N is used to denote the set of natural numbers. These numbers are used extensively for counting processes. For example, they are used in counting elements of a set. You can represent this numbers as N= {X:X =1,2,3,...) set of.

ii. Integers

Next in line to the set of Natural Numbers is another set that makes subtraction possible i.e, it allows you to find the solution to a simple equation as x + 2 = 6.

This set is derived by adding the set of negative numbers and zero to the set of Natural numbers.

It is called the set of integers and it is denoted by the letter Ƶ or I. Hence the set of integers is given as

Ƶ = {x:x = ..., -3 -2, -1, 0, 1, 2, 3, ---}

iii. Rational Numbers

You can see a gradual building up process in the various stages of development of numbers. That is, in order for you to be able to carry out division and multiplication correctly, you need to enrich or add a new set of numbers to the set of integers. So that you will be able to find a solution to equations like 2x = 3.

Therefore if we add the set of Negative and positive fractions to the set of integers we get a new set of numbers called the set of Rational Numbers.

The word `rational' is from the word ratio. Since 2:1 = 2/1 and 1:2 = ½. Set of rational numbers could be given as that number that can be expressed as the ratio of two integers of the form p/q, where p and q are relatively prime integers, i.e; p and q have no common division other than 1. The set of rational number is denoted by the letter Q.

iv. Irrational Numbers

A number which is not rational is irrational. Irrational numbers are not expressible as p/q. The set of irrational is denoted by the letter IQ

Examples of such numbers are $\sqrt{2}, \sqrt{3}, \sqrt{7}$ log Π, etc. The above number can be written as infinite decimal i.e; or decimal that does not repeat itself.

Rules Governing Addition of Number

Given that a,b, and c belong to the set R of real numbers then;

A1. R is closed under addition

$a+b \in R$ (This implies that the sum of any two real number must be a real number)

A2. Addition is commutative
$a+b = b+a$

A3. Addition is associative
$(a+b\}+c= a+(b+c)$

A4. Existence of additive identity
$0+a= a+0 = a$ (i.e; 0 is the additive identity)

A5. Existence of additive inverse
$a+b = b+a = 0$ (i.e; corresponding to each $a\in R$ there is $b\in R$ such that $a+b = 0 \rightarrow b = -a$)

similar to the rules for addition you have those for multiplication, still assuming that a, b and c R the.

MI. R is closed under multiplication.
If as $\in$ R (i.e the product of any two real numbers is a real numbers)

M2. Multiplication is commutative
$cb = bc$

M3. Multiplication is associative
$(bc) = (ab)\ c$

M4. Existence of multiplicative identity
-a. 1 = 1.a = a (1 is the multiplication identity)

M5. Existence of multiplication inverse
ab = ba = 1 (i.e; corresponding to each $a \in R$ a 0, there is $b \in R$
b=a-' such that $b=a^{-1}$

D1 Multiplication is distributive over addition
a(b+c) = a.b + a.c

The set of real numbers combined by means of the two band operations namely addition (+) and multiplication (.) as expressed above forms a field. The above rule Al-A5, M1M5 and D1 are known as the field axioms. Because of the field axioms satisfied by elements of the set of real number, the set R is a field.

Question: Is the set of rational numbers a field?

The third axiom possessed by the set of real numbers is the axiom of order. Thus there exist an ordering relation between any two elements of the set of real number. The relation is denoted by the symbol $>$ or $<$ which is read as `greater than' or `less than'.

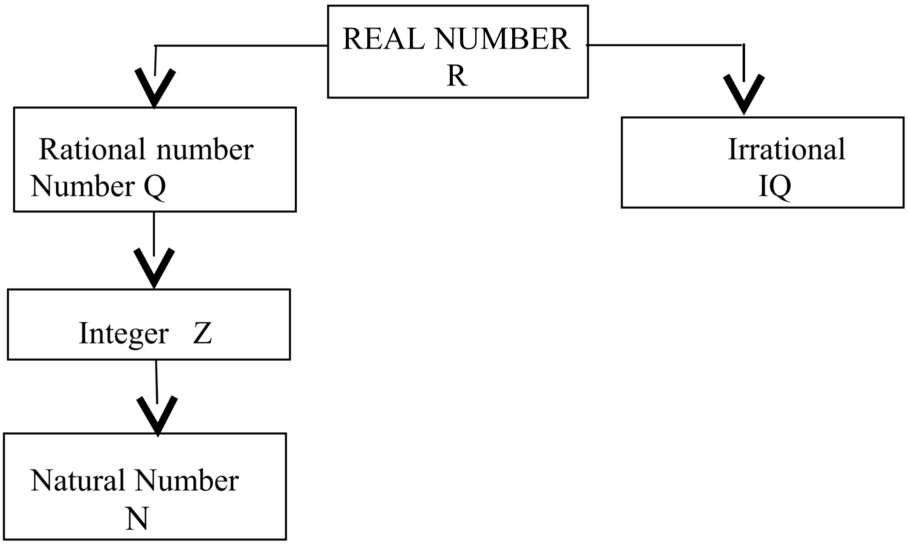
If a-b =0 then a=b or b=a. If a-b then $a>b$ or $b<a$.

The properties of the order axiom will be stated based on `>' (the ones based on $<$ are implied)

v. Real Numbers

The union of the set of rational (Q) and irrational (IQ) form the set of real number. It is denoted by the letter R.

You can visualize the development of real numbers system in a flow chart below:



In symbols we have this relationship

$NCƵ$ - $CQ, Q \cap I Q = O$ and $Q \cup I Q = R$

$N \subset Ƶ \subset Q, Q \cap I Q = \varnothing, Q \text{ U } I Q = R$

From the above relationship what can you say about the following statements?

1. All integers are natural numbers
2. All rational numbers are real numbers
3. Some rational numbers are natural numbers
4. Not all real numbers are rational numbers
5. All natural numbers are irrational numbers.

3.3 Basic Axioms of Real Numbers

You are already familiar with the four arithmetic operations of addition, subtraction, multiplication and division of real numbers. From the last section you noticed that each arithmetic operation is directly or indirectly involved in the stages of the built-up of the structure of real numbers. This built up is derived from a set of fundamental axioms or truths which is turn are used to deduce other mathematical results or formulation. Such axioms are categorized into the following.

For example; The extend axiom says that the set of real numbers has at least two distinct elements

Next is how any two or more elements of the set of real numbers could be added. You must be familiar with addition. You will now see that addition of two or more real numbers is carried out under some specific rules.

i. If a, b and c belong to R then the law of trichotomy holds. Either a>b, a=b or b>a

ii. If a>b and b>c then a>c (i.e.; `>' is transctive)

iii. If a>b then a+c > b+c (i.e; addition is monotone)

iv. If a>b and c>o then ac>bc (i.e.; multiplication is monotone)

Remark: If a∈R and a>o then a is said to be positive. If on the other hand a<o then a is said to be negative. If a = 0 then a is to be non-negative.

**** So far you have studied.***

Open Interval If a < b

3.4 Interval and Absolute Value

In this section you will continue the study of properties of real number by reviewing the concept of the real number line. After which you will be introduced to what an interval is and how a solution set of an inequality could be represented as a set of point in an interval.

Real numbers can be represented as points on a line called the real axis or number line. There is one-to-one correspondence between the members of the set of real numbers and the set of points on the number line. Commonly known to you is the fact that the set of real numbers to the right of 0 is called the set of positive numbers, while the set of real numbers to the left of 0 is called the set of negative numbers. 0 is neither positive nor negative.

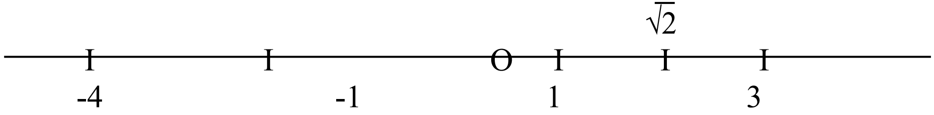


Figure 1: Showing the number line.

Remark: The one-to-one correspondence between the real numbers and the points of the number line makes it possible for us to use point and members interchangeably.

Definition of an Interval

Let ab∈ R and a<b then the set of all real numbers contained between a and b is called an interval, these two real numbers a and b, Bare, referred to as the end points of the interval .

Open Interval

If a<b then the set of real numbers specified by the inequalities{x: a<x <b} is called an open interval and is denoted by (a, b) have a and b are not member of this set of real number.

Closed Interval

If a<b, then the set of real numbers specified by the inequalities (x: a≤x≤b} is called a closed interval and is denoted by [a,b]. All points between a and b as well as a and b belong to this set [â;b].

Half Open or Half Closed Interval

The set specified by the inequalities.

> {x:a≤x<b} or {x:a<x≤b} is called half open or half closed interval and is denoted by [a,b) or (a,b].

Infinite Interval

The set of all numbers less than or equal to a given number c or the set of numbers greater than or equal to a given number c is called an infinite interval

i.e the set {x:a≥c} = [c, α)

{x:x≤c}= [-α, c)

{x:x∈R}= [-α, α)

{x:x>c}= [c, α)

{x:x<c}= [-α, c)

3.5 Bounded Sets

You will now learn about bounded sets and use it to identify intervals that are bounded or unbounded Upper Bounds.

Upper Bound

Lets S be a non-empty subset of R. If there exist a number $K \in R$ such that $x \leq k$, for all $x \in S$ then the set S is said to be bounded above. And K is known as an Upper bound.

Supremum

If there is a least member among the set of Upper bounds of the set S, this member is called Least Upper Bound (LUB) or Supremum of the set S and is denoted as Sup. S.

Example: Given that, S={1, 3, -1, -2, 4, 10}

i) List 4 Upper bounds for S
ii) Identify the Supremum for S

Solution

From the definition above any number K such that xc-s and $x \leq k$, is an Upper bound i.e. K= {10,11,12, 13,)

The least among the k's is 10 therefore the Sup S = 10

Example let (i) S={x:x = 2} Then Sup. S = 2 why?

Lower Bounds

Let S be non-empty subset R. if length of Interval

The number (b-a) is called the length of the interval (a,b), and [a,b]

You are familiar with inequalities and you could recall that to solve an inequality is to find the set of numbers that satisfy it. Inequalities play such an important role in calculus, that is imperative that you know how to use the concept of interval to represent the set of solutions that satisfy a given inequalities.

Example:
Solve the inequality $2-2x \geq 4$

Solution:
$2(1-x) \geq 4$ (divide by 2)

1- ≥2 (subtracted 1)

$x \geq - 1$ (multiplied by - 1)

Solution in set is given in the interval [-l, α)

SELF ASSESSMENT EXERCISE 2

Solve the following inequalities:

i. $3x - 3 \geq 9$

ii. $4x - 8 \geq 10$

iii. $4x - 7 \geq -10$

Absolute Value

You are familiar with the distance between zero and a point on the number line. You are equally aware that length or distance cannot have a negative value.

Let the distance between 0 and a be denoted by the symbol. $|a|$

$|x|$ = distance between 0 and x

$|a-b|$ = distance between a and b or b and a

therefore $|a| > 0$. You can define the absolute value of a number x as the distance between the point x and zero which satisfy the following conditions:

i. $|x| = x$ if $x > 0$.

ii. $|x| = x$ if $x < 0$

iii. $|x| = 0$ if $x = 0$

Example:

$|-3| = 3$

$|3| = 3$

$|0| = 0$

The relation

$|k| = x, x>0$

is equivalent to the relation

$-x \leq k \leq x$

$\Rightarrow k \in [-x, x]$

Some properties of Absolute value:

1. $|a| = a = |-a|$
2. $|ab| = |a| \ |b|$
3. $|a|^2 = a^2$
4. $|a+b| \leq |a| + |b|$ triangle inequality
5. $|a-b| \geq |\,|a| - |b|\,|$

*Remark Properties 3 and 4 imply that the sum of the length of two sides of a triangle is always greater than the length off the third side.

Example: Show that:

i. $|a+b| \leq |a| + |b|$
ii. $|a+b| \geq |\,|a| - |b|\,|$

Solution:

1. $(|a| + |b|)^2 = |a|^2 + 2|a|\,|b| + |b|^2$

$a^2 + b^2 \ 2ab$ (since $|a| > a$)

$\Rightarrow$ $(|a| + |b|)^2 \ (a+b)$ (taking square root of both sides)

$|a| + |b| \ \ |a+b|$ (sine $|a|^2 = a^2$)

Hence the required result

ii. $|a-b| \ |\,|a| - |b|\,|$

Let c = a-b then a = c+b

$|a| = |c+b| \ |c| + |b|$

$= |a-b| + |b|$

$|a| - |b| \ |a-b|$ which is the required proof.

SELF ASSESSMENT EXERCISE 3

Show that $|a-b| \ c \ \left|\,|a| - |b|\,\right|$

There exists a number $k \in R$ such that $x \ k$ for all $x \in s$ then the set S is said to be bounded below and k is known as a lower bound of S.

Infimum

If there is a greatest member among the lower bounds of the set S, then that member is called Greatest Lower Bound (GLB) or infimum of the set.

***Remark:** The supremum of a set if it exists is unique. The same applies to infimum of a set in other wards there cannot two distinct elements called the Sup S.

Examples: given the set s = (-4, -3, -1, 0, 2)

i. list 5 lower bounds of set S
ii. identify the infimum

Solution:

i. let k be the set of lower bound of s then k = (-10, -6, -8, -5, -4)
ii. the greatest member of set k is -4. Therefore inf S =-4.

Bounded Set

Let S be a non-empty subset of R. if there exist a number $k \in R$ such that $|x| = x$ for all $x \in s$ then the set S is said to be bounded.

In other words a set S is said to be bounded if it is bounded below and above

Example:

Given the following sets of number identify

i. bounded sets
ii. unbounded sets
iii. infimum
iv. supremum

Given the following sets of numbers

$A = \{-1,-2,0,1,2,3,4,)$
$B = \{x:-2 < x\ 5\}$
$C = \{x:x > -1\}$
$D = \{x : x \in (-\alpha,\alpha)\}$
$E = \{x : x \in (-\alpha, 0\}$

Identify (i) bounded sets

Determine which of the sets that are bounded or unbounded, for the bounded set, identify the supremum and infimum

Solution:

Set A is bounded
Sup A=4, and Inf A = -2
Set B is bounded
Sup B = 5 and Inf B = -2
Set C is unbounded
Set D is unbounded
Set E is unbounded

4.0 CONCLUSION

In this unit you have been able to learn about properties of real numbers and the development of real number system. You have observed that using the axioms you have studied you see a gradual and logical build up of the set of real numbers starting from the set of natural numbers.

You have studied how a set of real numbers could be represented using:

i) the concept of interval
ii) the concept of absolute value
iii) Inequalities

You have studied that a set of numbers represented by an interval can be bounded or unbounded.

5.0 SUMMARY

In this unit you have studied fundamental concepts of a set:

1. Extend axiom, Field axiom and order axiom of a set of real numbers

2. The gradual extension of the set of natural numbers to the Real number
3. The definition of absolute value of a real number as:
   $|x| = \{x$ if $x \quad 0$
   $\{-x$ if $x < 0$
4. Types of intervals of set of real numbers namely
   a. Open interval $(a,b) = \{x : a < x < b\}$
   b. Close interval $[ajb] = \{x : a x \quad b\}$
   c. Half-closed or half open interval
   $[a,b)=\{x:a\ x\ b\}$

(a,b] _ {x : a < x b }
where $a_1 b \in R$

5. That a bounded set is that set that is bounded below and above i.e. there is a number of K∈ R such that
$|x|$ k for x∈ s then set S is said to be bounded

6.0 TUTOR-MARKED ASSIGNMENTS

1. Define a bounded set.

2. Determine if the following sets are bounded:
   i. S=(x : x∈ = [0,I]}
   ii. S=(x : x∈ (0,1) }
   iii. S=(x : x∊ (0,1]j
   iv. S=(x : x∈ [0,1)}
   v. S=(x : x∈ (-α,1]}
   vi. S=(x : x∈ [-2, α)}

3. Given examples to illustrate the following:
   a. A set of real numbers having a supremum
   b. A set of real numbers having an infimum
   c. A set of real numbers that is bounded

4. Determine if the set of Natural numbers is bounded below. What is the infimum if any

5. List elements of the following sets of integer:
   i. S=(x : x∈ - (-4,I)}
   ii. S=(x : x∈ [-2,4]}
   iii. S=(x : x∈ (-1,3]}

6. State whether the following are true or false in the set of real numbers:
   a. 2 ∈ (- 2, 2)
   b. -1 ∈ (-00,0)
   c. 4 ∈ [4, 0o)

7. Show that $\left| |x| - |y| \right| \leq |x-y|$

8. Give a precise definition of:
   i. The supremum of a bounded set
   ii. The infimum of a bounded set

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 BASIC PROPERTIES OF REAL NUMBERS

CONTENTS

1.0 INTRODUCTION

The concept of functions and its corresponding definition as well as its properties are very crucial to the study of calculus. Simple observation of any physical phenomena has made it imperative for us to be interested in how variable objects are related. For example, you are familiar with how distance traveled by a body freely falling in a vacuum is related to the time of the fall or how the concentration of a medicine in the blood stream is related to the length of time between doses, or how the area of a circle is related to the to the radius of the circle.

The types of relationship between two variables in this unit will be considered, also the study of the concept of a function is very important since the properties of functions -are what you will use whenever you want to find the derivative of a function. It is important you study carefully and diligently all the various types of functions and their characterization.

2.0 OBJECTIVES

After studying this unit you should be able to

1. Define a function
2. Identity all types of functions
3. State the domain and range of a function
4. Combine functions to form a new function.

3.0 MAIN CONTENT

3.1 Definitions of a Function

You will start the study of this unit with the definitions of a function and its various forms of representation.

Definition 2.1

A function is a rule which establishes a relationship between twosets. Suppose X and Y are two sets, a function f from X to Y is a rule which attributes to every member $x \in X$ a unique member $y \in Y$ and it is written as

$$f : X \rightarrow Y$$ (which reads `f is a function from X to Y)

The set X is called the domain of the function, while the set Y is called the co-domain of the function.

Another definition of a function is given below as:

Definition 2.2

A variable y= f (x) (in words f (x) reads f' of x) is to function of a variable x in the domain X of the function if to each value of $x \in X$ there corresponds a definite value of the variable $y \in Y$.

Basically every function is determine by two things:

(1) the domain of the first variable x and
(2) the rule or condition the set of ordered pairs (x,y) must satisfy to belong to the function.

You will have a better understanding of the definitions above after going through the following examples.

Example 1

Let the domain of x be the set X={-2, -1, 0, 1, 2}
Assign to each value of X the number;

Y= 2x The function so defined is the set of pairs (-2, -4), (-1,-2), (0,0), (1,2) and (2,4)

Example 2

f : N→Z, defined by f (x) =1-x is a function since the rule f (x)=1-x assigns a every member x∈ N to a unique member of the set Z.
Z is a set of integers.

Example 3

If to each number in the set x∈ (-1, 2) we associates a number $y = x^2 +1$ then the correspondences between x and $x^2 +1$ defines a function.

3.2 Representations of Function

In the above definition of a function you were introduced to the concept of a domain .From the definition of a function ,the domain of a function could be defined as the set of value for which a function is defined. The independent variable x is a member of the domain. The dependant variable y that corresponds to a particular x-value is called the image of the x-value. The set of value taken by the independent variable y is called the range of the function. The range is the image of the domain.

Any method of representation of function must indicate the domain of the function and the rule that the ordered pairs (x,y) must satisfy in order to belong to the function> In this unit you will study two basic methods of representing a function namely:

1. Analytical method (i.e.; representation by means of a mathematics formula)
2. Graphical method

1. Analytical Representation

This is given by a formula which shows you how the value of the function corresponding to any given value of the independent value can be determined.

Example refer to example 5.5. in Unit 5.

The formulas $y=x^2 +1$ and y = 3-x specify y as a function of x. In the above example the domain of the function is assumed to be a subset of R for which the formula, representing the function makes sense.

2. Graphical Representation

A function is easily sketched by studying the graph of the function. In unit .... you would be required to plot the graphs of certain function so materials of this section will be useful to you then. Let us define what a graph of a function is

Definition 4. The graph of the function define by y= f (x) is the set of points in a rectangular plane whose co-ordinate pairs are also the ordered pairs (x,y) or [x, f(x)] of the function.

Another way you can view the above definition is to look at the steps of describing or drawing the graph of the function y = f (x) . To do this you choose a system of coordinate axes in the x-y plane. For each x∈ X, the ordered pair [x, f (x)] determines a point in the plane (see fig. 1)

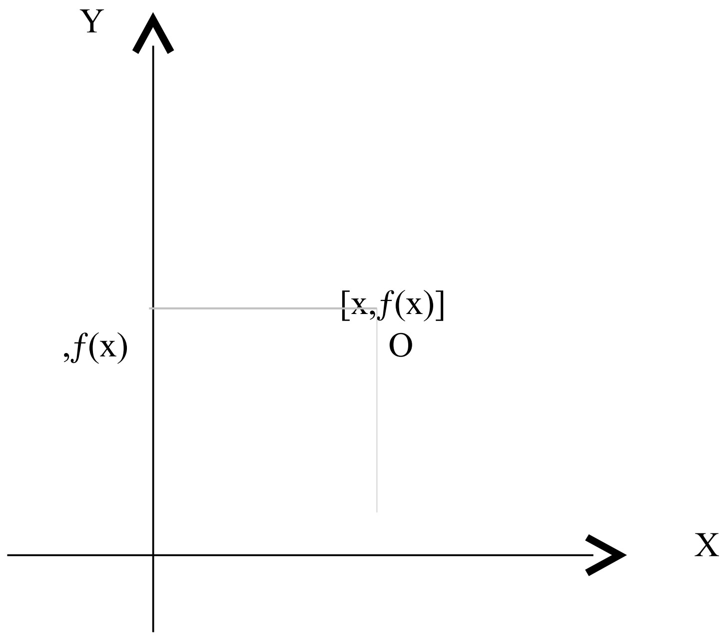


Fig. 1.

You will come across graphs of each type function that will be considered in this unit, the role each graph plays in understanding their respective functions will then become clearer to you.

3.3 Basic Elementary Functions

You will continue the study of function by considering the various types of functions and their graphs

1. Constant Functions

The simplest function to study is the constant functions. A constant function have only one constant value y for all values of belonging to the domain. i.e. ,$f(x) = a$ for all $x \in X$ where X is the domain of the function (see fig 2)

You noticed that the graph of a constant function is a straight line parallel to the x- axis.

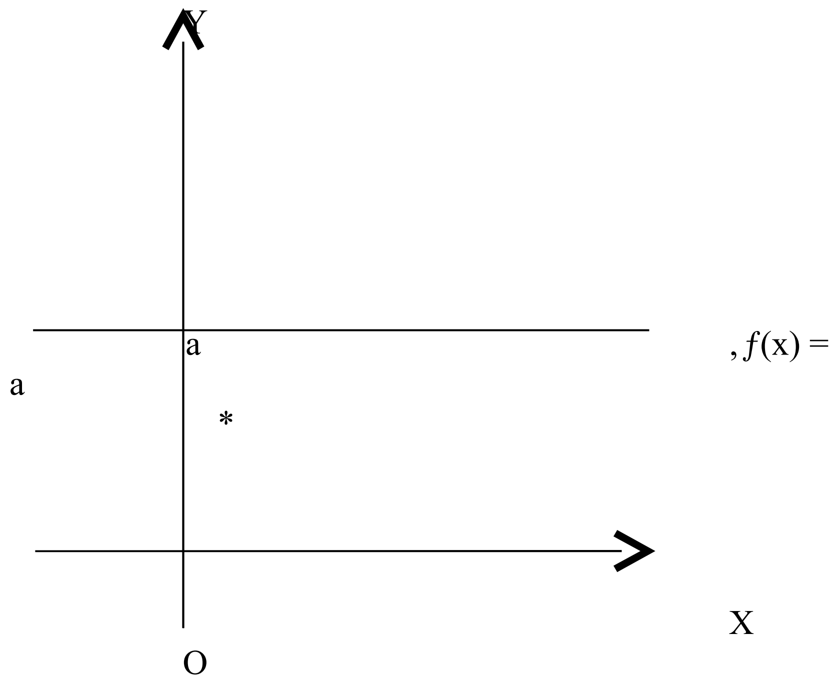


Fig. 2

In fig.2. ,$f(x) = a$ is a graph parallel to the x-axis at a distance $|a|$ units from the x-axis.

2. Polynomial Function

1. Any function that can be expressed as

   . ,$f(x)) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} +$ -----+

   $x_{n-1} + a_n$ ———————————— (A)

   where a_1, ,,, a_1........ a_n, are constant coefficients is called a polynomial function of n degree.

   You can derive various forms of functions with different graphs by varying the value of n.

Example 1 - If you substitute n =1 into expression (A) above you get a linear function i.e., . $f(x) = a_o x + a_1$
The graph is in Fig. 3a.

2. If you put n=2 in expression a you get a quadratic functions.

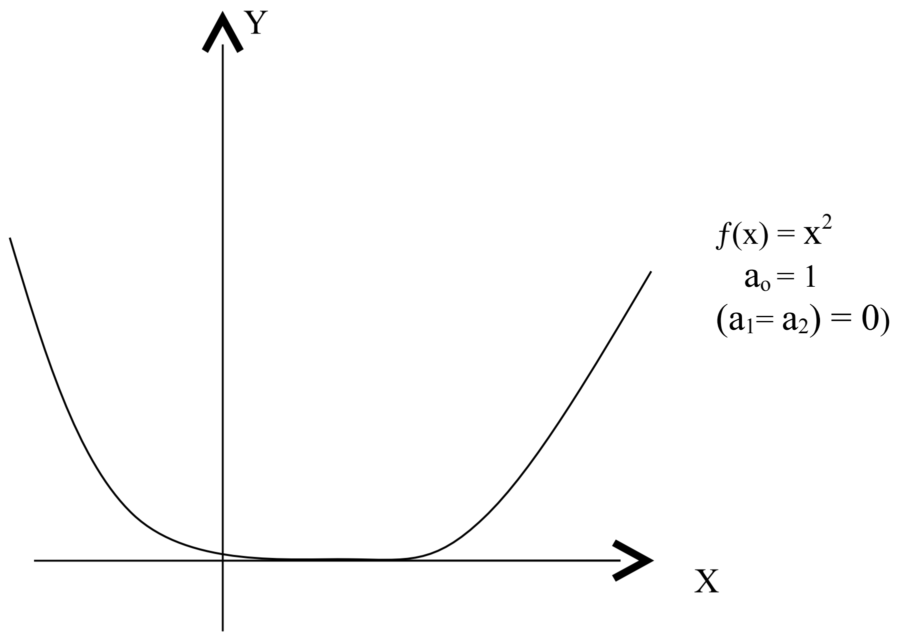
i.e. $f(x) = a_o x^2 + a_1 x + a_2$

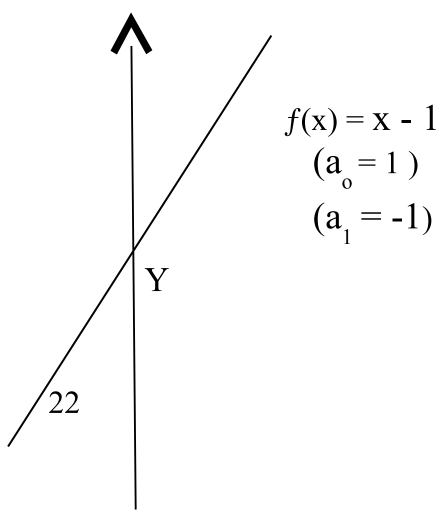
See Fig 3b.

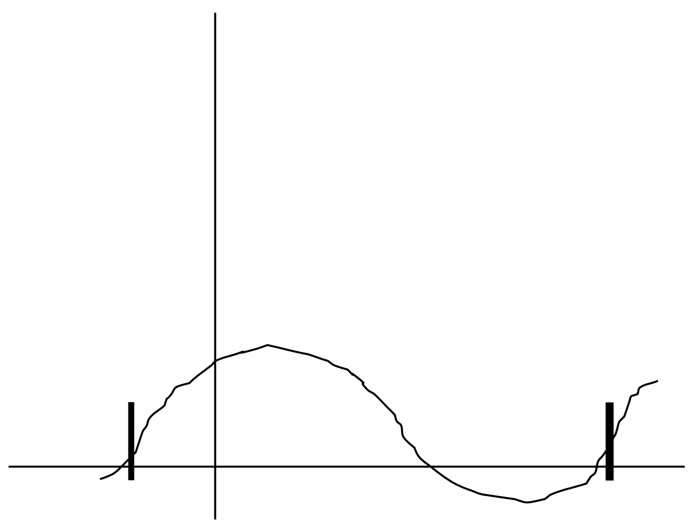
3. If you substitute n=3 into expression (A) you get a cubic function

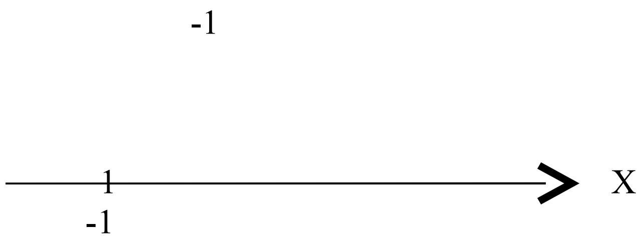
i.e. $f(x) = a_o x^3 + a_1 x^2 + a_2 x + a_3$

You could continue this process as much as you want.









get equate for MathCAD

3. Identify Function

There is a function that assigns every member of the set domain to itself.

Let X be domain of the function then ,f(x) = x for all x$\in$X. In some other textbooks identify function are denoted as Ix. The graph of an identity function is a straight line passing through the origin
(see Fig 4)

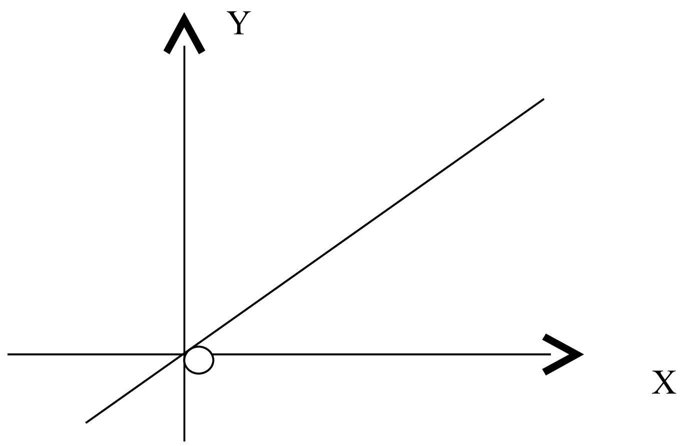


Fig 4

4. Algebraic Function

When two polynomial are combined together to construct a function of this form.

$$\frac{P(x)}{Q(x)} = \frac{a_o x^n + ... + a_n}{b_o x^m + ... + b_m}$$

The above function is called a rational algebraic function.

Example $,f(x) = \frac{2x}{x - 1}$

3.4 Transcendental Function

This is a. class of functions that do not belong to the class of algebraic functions discussed above. They are very useful in describing or modeling physical phonemic. Therefore you need to study them because they will be needed in the subsequent units.

1. The Exponential Function

A function $,f(x) = a^x$ where $a > 0$ and $a \pm 1$ is called an exponential function.

A special case of an exponential function is where a = e i.e. $,f(x) = e^x$ this function is know as the natural exponential function. Its graph is shown in Fig. 5

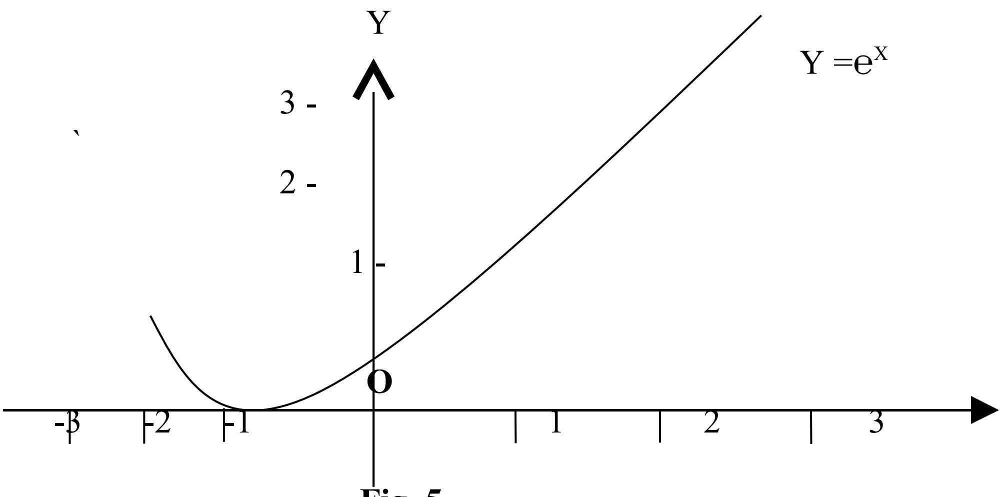


Fig. 5.

2. Logarithm Function

Any function ,f(x) which has the property that:

ff(x, y) =,f(x) + ,f(y) for all x, y > 0

is called a logarithm function

Example: Let ,f(x) be a logarithm function then

$f(1) = f(1 \,.\, 1) = f(I) + f(1) = 2f(1)$

$\Rightarrow f(1) = 2f(I)$

$\Rightarrow f(I) = 0$

for x > 0 $f(l) = f(x\ .1/x) = f(x) + f(1/x) = 0$

$\Rightarrow f(1/x) = -f(x)$.

Let x > 0 and y > 0 then

,f(y/x)= ,f(y-1/x)= f (y) + f (1/x)

Since ,f(1/x) = - ,f(x) then

,f(y/x) = ,f(y) - f(x)

Let ,f(x) = log x.

Then log (x/y) = log x - log y.

And log (y/x) = log y - log x.

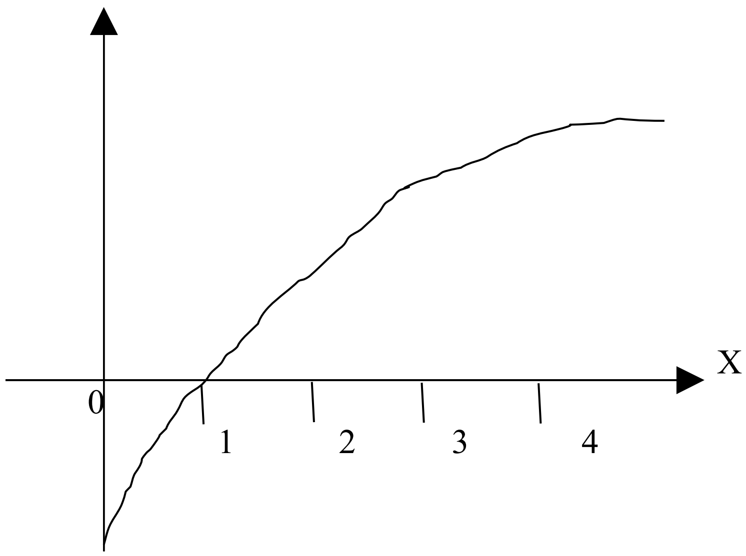
Log (1/x) = - log x.

2. The Natural Logarithmic Function

The ,f(x) = Ln (x) where x ± 0

Is called the natural logarithmic function.

Its graph is shown in fig. 6. (this function derived its definition from calculus see unit...)



3. The Trigonometric Function

The function define as:

,f(x) = Sinx, ,f(x) = Cos x,,f(x) = tanx.
,f(x) =Cot x, ,f(x) = Cosec x, and f(x) =Sec x.

are called trigonometric functions.

4. Inverse Trigonometric Functions

The function define as
f(x) Sin $^{-1-}$x, ,f(x) =Cos^{-1}x, ,f(x) = tan^{-1}x.

are called inverse trigonometric functions.

5. Hyperbolic Functions

There are classes of function that can be form by combing the exponential function.

For example:

$$,f(x) = \frac{e^{x} + e^{-x}}{2} = \text{Cosh } x$$

$$,f(x) = \frac{e^{x} - e^{-x}}{2} = \text{Sinh } x$$

These functions are very useful in computing the tension at any point in high-tension cables you see in some of the highways across the country. They are also important in solving some classes of problems in calculus. The rules governing them are like that of the trigonometric functions.

The functions: Cosh x = $1/2(e^{X} + e^{-X})$ (cash reads gosh x) and Sinh x = $1/2 (e^{X} + e^{-X})$ (sinh reads cinch x)

may be identified with coordinates of point (x,y) on the unit hyperbola $x^2 - y^2 = A$

Recalled that the functions sin x and cos x with the point (x, y) on the unit circle $x^3 + y^2 = 1$ in some text trigonometric functions are called circular functions. So the name hyperbolic is formed from the word hyperbola. Other hyperbolic functions like tanh x, Coth x, Sech x, and Cosech x can be derived from cosh x and sinh x.

6. Inverse Hyperbolic Functions

The functions $Y = \sinh^{-1} x$, $Y = \text{Cosh}^{-1} x$, are called inverse hyperbolic functions.

Others are $Y = \tanh^{-1} x$, $Y = \text{Cech}^{-1} x$, $Y = \text{Cosech}^{-1} x$,

4.0 CONCLUSION

In this unit you have studied the definitions of a function. You have studied two ways a function can be represented. You have studied types of functions - elementary and transcendental functions.

5.0 SUMMARY

You have studied to:

i. State the definition of a function of one independent variable.
ii. Use graph to describe types of functions, quadratic, sin etc.
iii. Recall various types of function.

6.0 TUTOR MARKED ASSIGNMENT

Give precise definitions of the following:

1. Domain of a function
2. Function of an independent variable.
3. Exponential functions.
4. Logarithmic functions
5. Give two ways a functions can be represented.

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 CHARACTERISTIC OF FUNCTIONS

CONTENTS

1.0 INTRODUCTION

Investigation of function are carried out by observing the graph of the function or the value of the function as the independent variable changes within a given intervals. In other words a function is investigated by characterization of its variation (or its behaviour) as the independent variable changes. The classification of the variety of function is very vast. The following types defined in this unit is by no means this unit you continue the study of functions by considering special features that` characterize a function.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

i. Identify basic characteristics of functions such as monotonic property boundedness etc.
ii. Define an inverse function.
iii. Define a composite function
iv. Combine functions, to form a new function.
v. Determine whether a given function has an inverse or not.

3.0 MAIN CONTENT

3.1 Types of Functions

Zero of a function: The value of x for which a function vanishes, that is for which $f(x) = 0$ is called The Zero (or root) of the function.

Example la.

The function ‚f(x) = x^2-3x+2 has two roots i.e.; x=2 or 1.

One of the roots of the function.

‚$f(x) = x^3-2x^2 - 5x + 6$ is 1.
i.e., ‚$f(1) = 1^3 - 2.1^2 - 5.1 + 6 = 0$

SELF ASSESSMENT EXERCISE 1

Find other roots of the above function.

1. Even and Odd Functions

A function y =‚f(x) is said to be even, if the changes of the sign of any value of the independent variable does not affect the value of the function.

F(-x) _ F(x) .
i. e, $f(-x) =$ ‚$f(x)\ \forall x \in X$

A function y = ‚f(x) is said to be odd if the change of sign of any value of the independent variable results in the change of the sign of the function

i.e. ; $f(-x) = -$‚$f(x)$

Example

The function y = x^2 is an even function while the function y = sin x and y = x^3 are odd functions.

Remark: Arbitrary functions such as y = x +1, y = 2 sin x + 3 Cos x can of course be neither even nor odd.

2. Periodic Function

A function y = ‚f(x) is said to be periodic if there exists a number n $\neq$ 0 such that for any x belonging to the domain of the function the values x + n of the independent variable also belonging to the domain of the function and the identity.

‚$f(x + n) =$ ‚$f(x)$ holds where n is called
the period of the function. 1

Example

If ,f(x) is a periodic function with
period n the f(x + n) =,f(x + 2n) = ,f(x)

Generally for any periodic function f (x) with period n.
f (x + nk) = f (x)for any x∈ Z, k∈ N,

A simple example of a periodic function is the function f(x) =s i nx or f(x) Cosx.

See Fig. 9.

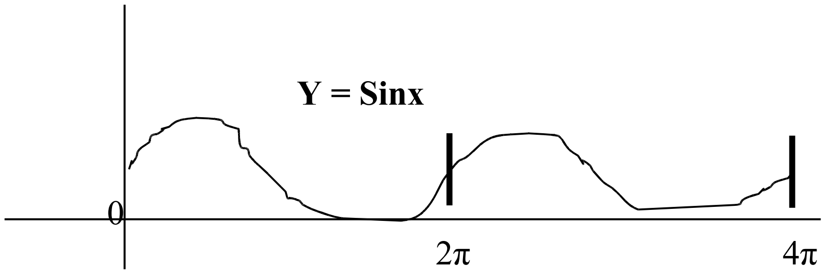


Fig. 9a.

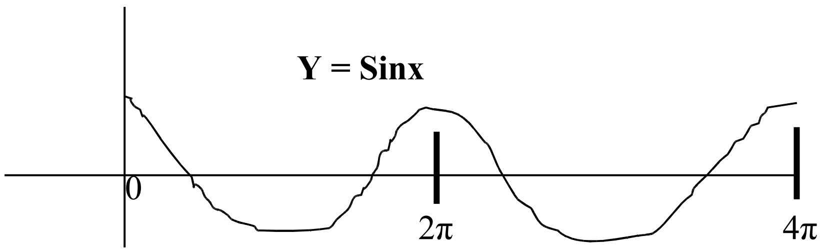


Fig. 9b

3. Monotonic Functions

A function is said to be monotonic if it is either increasing or decreasing within a given interval.

The study of monotonic function is an important concept in the application of calculus, this will be treated in the last two units of this course.

You will now consider explicit definitions of a monotonic increasing function and monotonic decreasing function within a given interval.

Definition l: A function $f(x)$ is said to be monotonic increasing in an interval.

If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
for any two points x1, $x_2 \in I$,
If $f(x_1) < f(x_2)$ then the function $f(x)$ is said to be strictly increasing.

Definition 2: A function $f(x)$ is said to be monotonic decreasing in an interval I

If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
for any two points x1, $x_2 \in I$

If $f(x_1) > f(x_2)$ then the function $f(x)$ is said to be strictly increasing.

Example

The function $y = x^2$ is monotonic decreasing in the interval $(-\infty, 0]$ and monotonic increasing in the interval $[0, \infty)$. See fig. 10

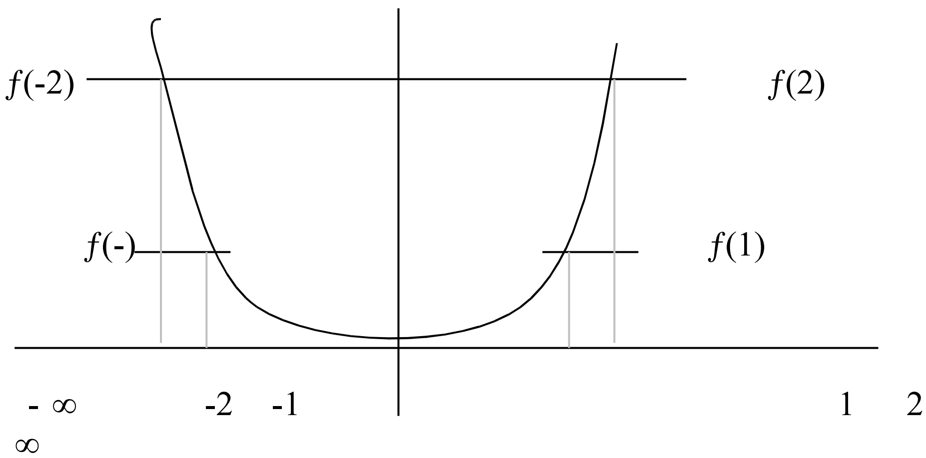


Fig. 10.
$-1,-2 \in (-\infty, o]$ and $-2 < -1$ but $f(-2) > f(-1)$
$1,2 \in\in [o, \infty)$, $1<2$ and $f(1) < f(2)$

SELF ASSESSMENT EXERCISE 2

Determine whether the function $f(x) = 2^x$ is monotonic increasing or decreasing in the interval $I = (-\infty, \infty)$.

Determine whether the following functions are monotonic increasing or decreasing in the interval (o, ∞):

i. f (x) = 2^x
ii. f (x) = 2^{-x}
iii. f (x) = 2^3
iv. f (x) = 2

4. Bounded Functions

Recall the definition of a bounded set defined in Unit 2. You will now use the same concept to define a bounded function. If a function f(x) assumed on a given interval I a value M which is greater than all other values (i.e.; f(x) < M for all x∈I) then the function f (x) is said to be bounded above. The M is called the greatest value of the function f(x) at that interval I. Similarly, if there is a constant M such that all other values of the functions is greater than (i.e.; f (x) >M for all x∈1) then we say that f (x) is bounded below and the value M is called the least value of the function f (x) in I.

Definition of a Bounded Function: A function f (x) is said to be bounded in an interval 1. If there exists a number k∈R such that

f(x) K for all x∈I y Lvle

alternatively, if given M, f (x) > M in the interval I we say f (x) in bounded, below

Example 1

The function f (x) = 2x+1 is bounded in the interval [-2,2]

i.e f(-2) f(2) is bounded in the interval (-2, 2).

2. The function $f(x) = 2x^2-3x+2$ is bounded in the interval x∈ [o,2]

SELF ASSESSMENT EXERCISE 3

Determine whether the following function are bounded in the given intervals.

i. $f(x) = x^2 - 4x + 4 \quad x \in (-\infty, \infty)$.
ii. $f(x) = x^2 - 4x + 4 \quad x \in (2, 10)$.
iii. $f(x) = 2 + x + x^2 \quad x \in (-1, 2)$.

3.2 Inverse Function

Domain and Range: since the domain and range will be useful in the study of inverse of a function you have to briefly review the concept as you have studied the fact that one of the ways a function can be determined is through the domain of the function i.e. the set containing the first variable for which a function makes sense. You shall consider some few examples of domain of a given function.

Example

i. Given the function
$f(x) = X^2$, x is a real number.

Here the domain of f is the set of all real numbers. The range is therefore $R^+ = [o, \infty)$. In symbols you write.

$D = \{x: x \in R\}$ and $R = [y; y \in R^+\}$.

ii. Given the function

$f(x) = x - 1$, x is a real number.

Here the domain of f is the set of all real numbers greater than

1. i.e.; $D = \{x: x > 1\}$ Since any other value of x will result to the square root of a negative number which does not make sense in the set of real numbers. The range $R = \{y : y \in R^+\}$

iii. Given the function

$$f(x) = \frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)}$$ the domain

$D = \{ x : x \in R, \quad x \neq -1 \text{ or } 1\}$. If x = -1 or 1 the value of the function will be meaningless.

SELF ASSESSMENT EXERCISE 4

1. Find the range of example (iii) above.
2. Let the function f assign to each state in Nigeria its capital city. State the domain of f and its range.

You will continue the study in this section by giving definitions of certain features of functions. (there have been kept purposely for this moment.)

1. Onto Functions

Let the function y = f (x) with domain of definition X {i.e. the admissible set of values of x) and the range Y (the set of the corresponding values of y). Then a the function y = f(x) is an Onto function if to each point or element of set there corresponds a uniquely determined point (or element) of the set Y, i.e., if every point in set Y is the image of at least one point in set X.

Example: consider the function shown in fig 11

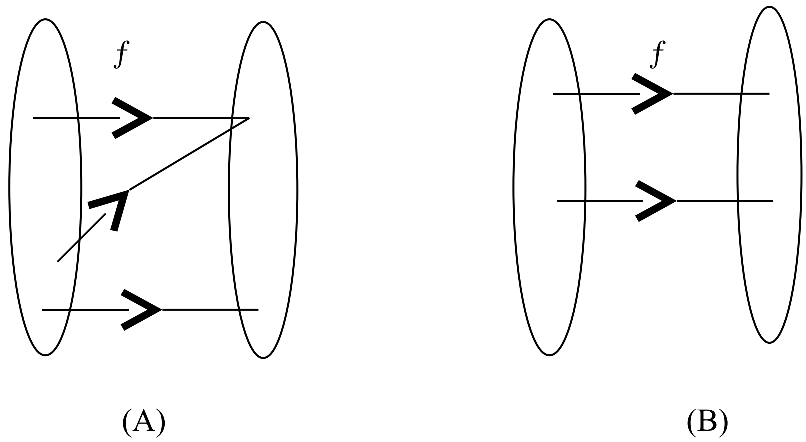


Fig 11.

The function Fig. (A) is an Onto function. The function in Fig. (B) is not an onto function

Example: The function f (x) = x^2 is an Onto function

SELF ASSESSMENT EXERCISE 5

Give reason why the function in the Fig. (a) above is an onto function and the other one in Fig(b) is not.

2. One-to-One Function

Let the function y = f(x) be an onto function. If in addition each point (or element) of set X corresponds to one and only one point (or element) of set Y then the function y = f(x) is said to be one to one function.

Example

The function $y = x^2$ is an onto function and not a one to one function. Whereas the function $y = x^3$ is an onto function as well as a one to one function
(see fig 12)

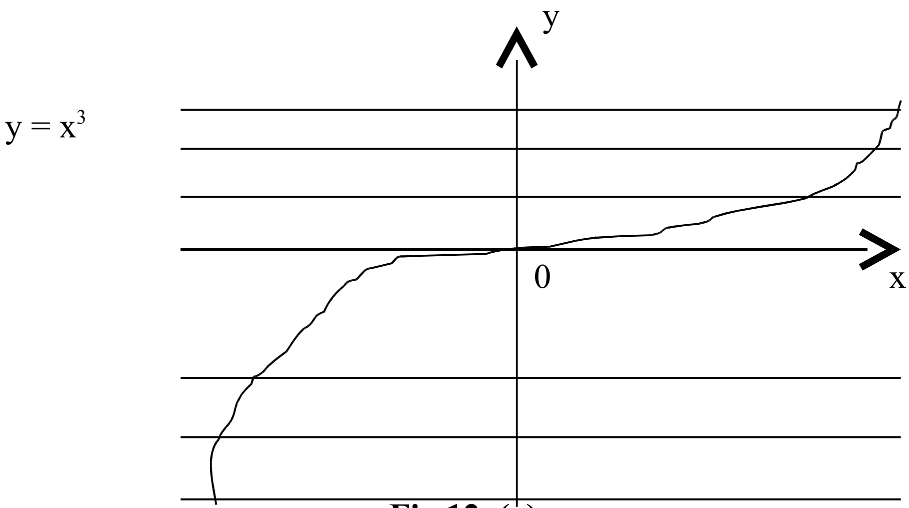
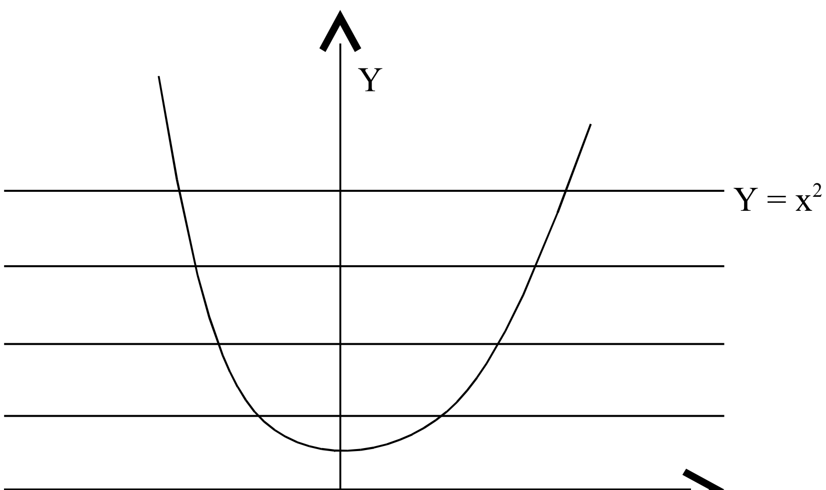


Fig 12. (a)



X

O

Fig. 12. (b)

In fig. 12 (a) no horizontal line intersect to the graph more than once thus the function.

$Y = x^3$ is one to one function.

In Fig. 12. (b) the horizontal lines intersects the graph in more than one point thus the

$f(x) = x^2$ is not a one to one function.

3.3 Composite Functions

Generally functions with a common domain can be added and subtracted. That is, if the functions f (x) and g(x) have the same domain. Then:

$$(f \pm g)(x) = f(x) \pm g(x)$$

Example:

Let $f(x) = x^2$ and $g(x) = 3x - 2$
Then $f(x) + g(x) = x^2 + 3x - 2$

The above concept can be extended to the case of multiplication. i.e.; given that f (x) and g(x) have the same domain then

$$fg(x) = f(x)\, g(x).$$

Using the above example we have that:
$f(x)\, g(x) = x^2 (3x - 2) = 3x^3 - 2x^2$

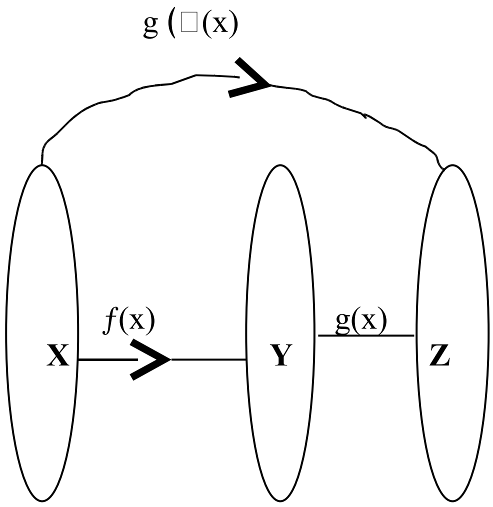
Division is also allowed between functions having the same domain.

Let $f(x) = 2x$ and $g(x) = x - 1$

Then; $\dfrac{f\,(x)}{g(x)} = \dfrac{2x}{x-1}$

There is another way function can be combined which is quite different from the ones described above. In this case two function f (x) and g(x) are combined by first finding the range of f (x) and making it the domain of g(x).

This idea is shown in fig 13.



The function you get by first applying f to x and then applying g to f (x) is given as g (□(x) and called the composition of g and f and is denoted by the symbol

g o f (which reads g circle f)
i.e.; (go f) (x) = g (□(x))

Example

1. Given that f (x) = 1/x and g(x) = $x^2 + 1$
   $f$og= f (g(x)) = $\dfrac{x}{x^2 + 1}$

   go$f$= g o (□ (x)) = $1/x^2 + 1.$ = $\dfrac{x^2 + 1}{x^2}$

2. Given that f(x) = x^2 and g(x) = x + 1

   go$f$ = g f(□(x)) = $x^2 + 1$

$f\text{og} = f(g(x)) = (x+1)^2 = x^2 + 2x + 1$

In the two examples above you can easily conclude that go$f \neq f$og
The composition of functions can be extended to three or more functions.

Example

Let $f(x) = x - 1$, $g(x) = x^2 + 1$, $h(x) = 2x$.

Then hogo f = h (go f) = h (g (□(x)))
=2 (x- 1)^2 + 1) = 2x^2 – 4x +4

SELF ASSESSMENT EXERCISE 6

Give that $f(x) = x$, g(x)=x-1, h(x) = √x -1
Find the following composite functions.

1. f og
2. go f
3. ho f
4. hog
5. f ogoh

You will now use materials discussed above in this section to study and define the inverse for any given function. A function that will have an inverse must fulfill the function, since the inverse function is a unique function in respect of the original function.

Definition of Inverse. of. a Function: If a function y= f (x) is a one to one function, then there is one and only one function x = g(y) whose domain of definition is the range of the function y = f (x). such that;

fg (f (x)) = x and g(x) = f^{-1} (x)

Examples

1. If given that $f(x) = x^3$ then $f^{-1}(x) = 3\sqrt{x}$
2. Use the above and illustrate the fact that f^{-1} o f = f^{-1} o f

Given that f^{-1} (x) = g (x) =3√ x = x ⅓
And f (x) = x^3

f^{-1} of =go$f$=g (□(x)) = (x^3) ⅓ = x
And fof^{-1-} = $f$og = f (g(x)) = $((x^{⅓})^3$ = x

Find the inverse of the following function.

1. 2x-4
2. 6x -5
3. f (x) = x^5
4. $2x^3$ -1

Solutions:

1. Let y =2x-4
   Then y+4=2x
   $\Rightarrow x = \frac{y + 4}{2}$ (solving for x)
   then $f^{-1}(x)$ $\frac{x + 4}{2}$ (interchanging x and y)
2. Let y = 6x -5
   Then y + 5 = 6x (solving for x)
   $x = \frac{y+5}{6}$
   $f^{-1}(x)$ $\frac{x+5}{6}$ (interchanging x and y)

3. Let y = x^5

   then x 5 $\sqrt{y}$ (solving for x)

   $f^{-1}(x)$ = 5 $\sqrt{x}$ (interchanging x and y)

4. Let y = $2x^3$ -1
   y + 1= $2x^3$
   $\frac{y+1}{2} = x^3$

   $x = 3\sqrt{\frac{y+1}{2}}$ (solving for x)

   f^{-1} (x) = $\sqrt[3]{x+1}$ (interchanging x and y)

SELF ASSESSMENT EXERCISE 7

1. Show that $f^{-1} o f = f o f^{-1} = x$ in example 1 to 4 above.
2. Given the following functions
   a. $f(x) = 6x - 3$
   b. $f(x) = x^7$
   c. $f(x) = mx = b$
   d. $f(x) = 1/1 - x$
   e. $f(x) = \frac{1}{x^3 - 1}$
   d. $f(x) = \frac{1}{1+x}$

   i. State the domain of each function.
   ii. Derive the inverse of each function.

4.0 CONCLUSION

In this unit you have studied characteristics of functions You have used graphs to represent functions and identity some characteristics exhibited by these functions. You have studied how to form a new function by combining two or more functions.

Furthermore, you have studied how to determine whether a function has an inverse or not.

5.0 SUMMARY

In this unit you have:

a. Defined a function
b. Discussed various types of functions
c. Use graphs to describe the characteristics of functions such as periodic, monotonic, one to one onto and transcendental functions.
d Defined domain and range of a function
f Formed new functions by combining two or more functions - composition of functions.
g Discussed the inverse of a one to one function.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Give a precise definition of the following unit examples
   a. domain of a function
   b. inverse of a function

c. composition of functions
d. bounced function
e. an even function
f. a periodic function
g. a monotonic decreasing function in an interval
h. maximum value of a function is an interval.

2. Given the following functions.

a. $f(x) = \frac{2x}{x - 5}$

b. $f(x) = \frac{1}{x^3 - 1}$

c. $f(x) = 27x^3 - 2$

d. $+f(x) = \frac{x}{(x-1)(x+2)}$

1. State the domain of definition for each function

2. Find the inverse of each function if it exists.

3 Given the following function $f(x) = x^2$, $g(x) = 2x-1$, $h(x) = x+1$
Find the:

a. fg
b. f/g
c. fog
d. $fogoh$
e. $(g-h) of$

7.0 REFERENCES/FURTHER READINGS

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UNIT 4 LIMITS

CONTENTS

1.0 INTRODUCTION

In the last units, you have been adequately introduced to the concept of a function In this unit you will be indorsed to the concept of the limit of a function. This is one of the most important concept in the study of this course calculus. Generally, it is believed that calculus begins with the idea of a limitations process. The history behind the study of limits of function is an interesting one and it will be nice if you hear some of the story.

> A French mathematician by name Joseph Lious Lag range (1736 - 1813) was among the first mathematicians that initiated the concept of limits. This was followed by another French mathematician Augustine-Lious Cauchy (1789-1837) and a Czech priest by name Bernhard Bolzono (1781-1848).

However the present day definition of limit is largely due to the work of Heinrich Edward Heine and Karl Weiestrass. In this unit an attempt to be a bit expansive in the study of the limit of functions will be made. Therefore you should be more patient when studying the materials of this unit. Bear in mind that discussions on the concept of limit of a function will easily be re-introduced into the concept of continuity of function in unit 4 and 5.

2.0 OBJECTIVES

After studying this unit you should be able to correctly;

1. define a limit of a function

2. show that the limit of a function is unique
3. evaluate the limit of a function
4. to evaluate the right and left hand of a function.
5. use the "∈, δ" method to prove that a number ℓ is the limit of a function at a given point.

3.0 MAIN CONTENT

3.1 Definition of Limit of a Function

In this Section; you will begin the; concept of limit shall be studied by first presenting it in an informal and intuitive manner. You are familiar with the word "limit". It gives you the picture of a restriction or boundary. For example consider a regular polygon with n sides inscribed in a circle. As you increase the sides of the regular polygon then each side of the n-side regular polygon gets closer to the circumference of the circle. Here if you consider sides of the polygon as the independent variable denoted by n and the shape of the regular polygon as the dependent variable then the shape of the n-sided regular polygon approaches the shape of the circle as n approaches infinity (see fig 13)

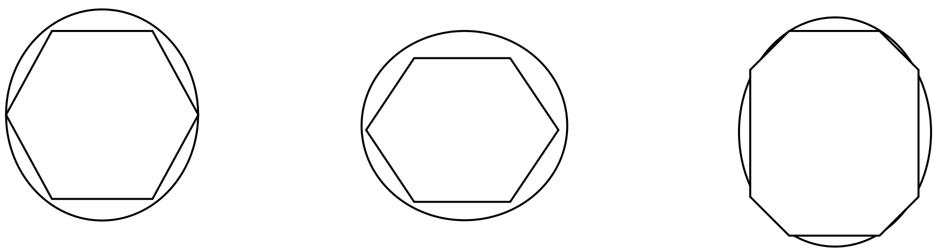


Fig. 13.

In this case we say that the limit of inscribed n-sided regular polygon is the circle as n tends to ∞.

Now consider the function $f(x) = x^2 - 1$ what is the value of $f(x)$ when x is near 1 ? In table 2.

X	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$	1.	.99	.96	.91	.84	.75	.64	.51	.36	0.19	0

Table 1

X	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
f (x)	0	.21	.44	.69	.96	1.25	1.56	1.89	2.24	2.61	3

Table 2

You can see that as x gets closer and closer to 1 f (x) approaches 0. so the value of f (x) can be made to get closer to 0 by making x get closer to 1. This is expressed by saying that as x tends to l, the limit of f (x) is 0.

Another way is to start by noting that a function f (x) could be observed to approach a given number L as x approaches a known value x_o. That is once a number ℓ is identified as x approaches x, without insisting that f (x) must be defined at xo then we can write that

$$\text{Lim } f(x) = \ell$$

In other words we say that the limit of the function f (x) as x approaches or tends to x,, is the number ℓ or as x tends to xo , f (x) tends to ℓ or for x approximately equal to xo f (x) is approximately equal to ℓ

In the above definition you will observed that there are two important things to note namely.

1. the existence of the unique number ℓ , and
2. the fact that the function need not be defined at the point x_o.

What is more important is that the function is defined near the number x_o

Consider the following example that will further explain the concept of limit.

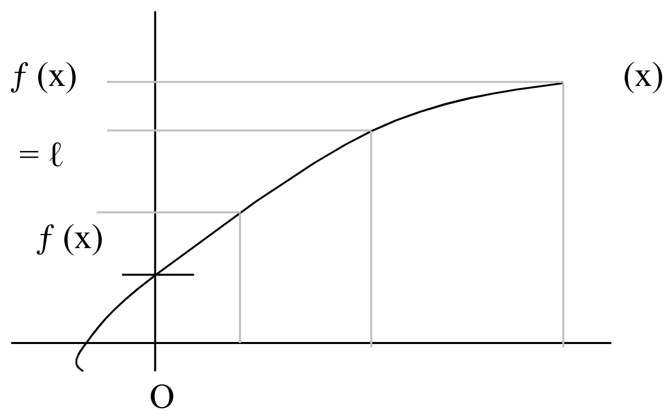


Fig. 14.

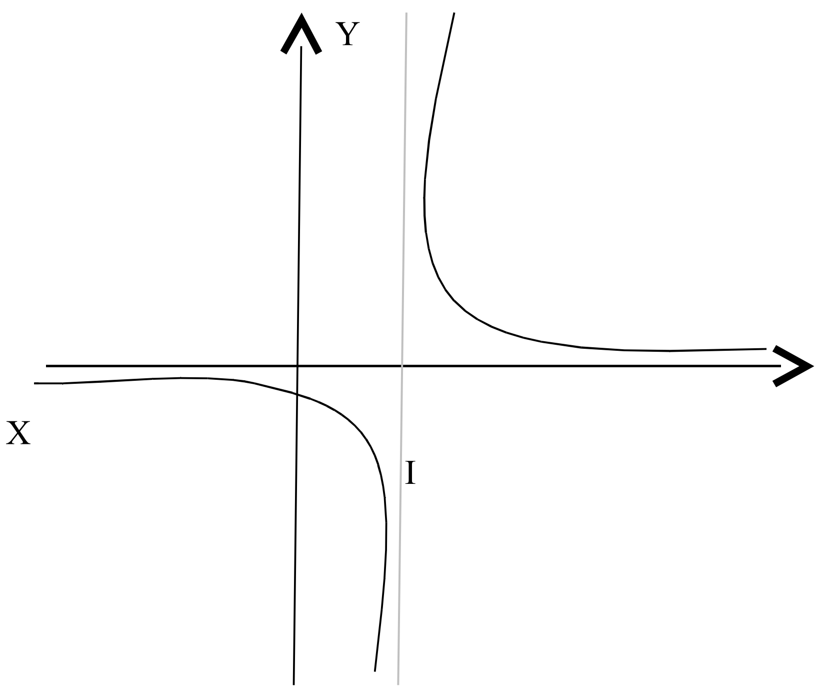
In fig(14), the curve represent the graph of f (x) . The number xo appears in the x-axis, the limit ℓ appears in the y-axis. As x approaches x_o from either side (i.e.; along the x-axis). f (x) approached ℓ along the y-axis.

Examples: Find the limit of the functions as x→1

1. $f(x) = \frac{1}{x - 1}$
2. $f(x) = \frac{1}{|x - 1|}$
3. $f(x) = \frac{x^2 - 1}{x - 1}$

Solutions:

1. $f(x) = \frac{1}{x - 1}$ in the graph of f (x) as x approaches 1. (see graph below



From the right f (x) becomes arbitrary large. Larger than any pre-assigned positive number. As x approaches curve from the left f (x) becomes arbitrarily large negative-less than any pre-assigned negative number. In this case f (x) cannot be said to approach any fixed number. The above gives a clear picture where the limit of function does not exist as x approaches a given point for a fuller understanding, you will consider two more examples.

1. $f(x) = \frac{1}{|x-1|}$ (see fig. 16)

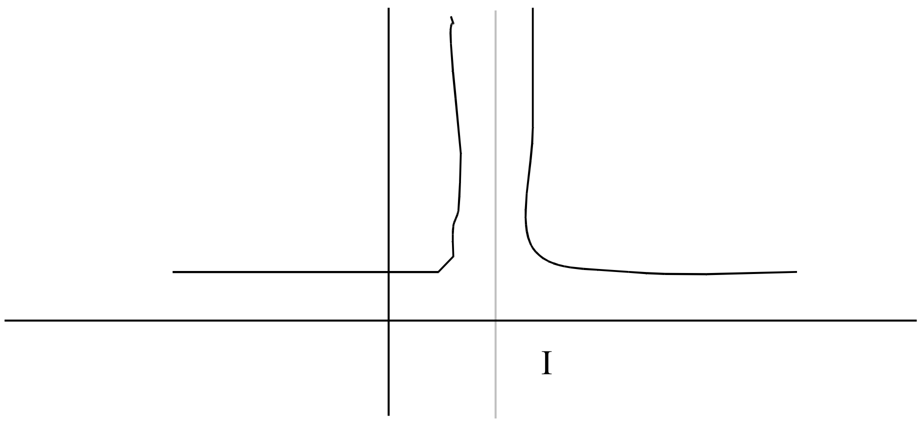


Fig. 16.

In fig. 16, as x approaches 1 from the left and the right 2. f (x) becomes arbitrary large. In this case 2. f (x) becoming arbitrary large cannot approach any fixed number ℓ .

Therefore $f(x) = \frac{1}{|x-1|}$ does not have a limit as x tends to 1.

2. $f(x) = \frac{x^3 - 1}{x - 1}$ (see table A & B below)

X	0	.1	.2	.5	.4	.5	.6	.7	.8	.9	1
f (x)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2

Table A

X	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
f (x)	0	.21	.44	.69	.96	1.25	1.56	1.89	2.24	2.61	3

Table B

At a first glance f (x) is not defined at the point x = 1, since division by zero is impossible. Recall that in finding the limit of function at a given point xo it is not required that f (x) must be defined at xo. The above is a clear example of functions having limits at points where they are not defined. You will meet other examples of such function as you progress in this course.

In tables A & B the limit of the function as x tends to 1 is 2.

By direct evaluation you can simplify:

$$f(x) = \frac{x^3 - 1}{x - 1} \quad \text{as}$$

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$$

Therefore lim. as x → 1 of f (x) = x + 1 is 1 + 1 = 2.

SELF ASSESSMENT EXERCISE 1

Determine whether the function

1. $f(x) = \dfrac{x^2 - 4}{x - 1}$ 2. $f(x) = \dfrac{1}{x - 2}$

have limits as x approaches 2. If so, find the limits.

3.3 Properties of a Limits of a Function

A formal definition of limit is hereby given.

Definition : The number ℓ is said to be the limit of the function y = f(x) as x tends to x_o if for any positive number ∈ > o (however small) we can find some positive number δ such that:

$$|fx - \ell| < \in \quad \text{whenever} \quad o < |x - x_o| < \delta$$

Using the above definition it can be shown that the limit of the function f(x) = 3x -1 is equal to 2 as x tends to 1.

To prove the above insufficient to show that for ∈ > o you can find δ > o such that the inequalities:

$$|(3x - 1) - 2| < \in \Rightarrow o < |x - 1| < \delta \quad \text{is satisfied equivalently.}$$

$$|3x - 1 - 2| = |3x - 3| = 3|x - 1| < \in$$

$$\Rightarrow \quad |x-1| < \in/3$$

Since x must be near 1 as much as possible we chose $\delta = \in/3$. Hence:
$x - 1 < \delta = \in/3$, which is the required proof.

Remark: The definition above implies that the distance between f (x) and L must be small as much as the distance between x and x_o is. Recall that the absolute value of a number 1.1 is a distance function (see unit 1) The method of the proof used above is called the "$\in, \delta$ proof" in this course you will get a better view about the definition if you go through another example when the graph of the function is shown with the limit indicated

Examples: show that the limit of $f(x) = x^2$ as x approaches (tends to) 2, is $\ell=4$ (Use the $\in, \delta$ proof)
Solution:
In finding the solution to the above you have to show this: For any positive number $\in > o$ you look for another positive number $\delta > o$ such that the inequalities

$$|x^2 - 4| < \in \quad \text{whenever} \quad o < |x-2| < \delta \quad \text{is satisfied}$$

Note that:

$$|x^2 - 4| = |(x-2)(x+2)| = |x-2| \; |x+2|$$

is the product of a factor $|x+2|$ that is near 4 and a factor $|x-2|$ that is near 0 when x is near 2. If x is required to stay within, say, 0.2 of 2 then you will have a situation like this:

$$2-0.2 < x < 0.2+2$$

$$= 1.8 \; x < 2.2 \text{ and}$$

$$3.8 < x + 2 < 4.2.$$

As a result

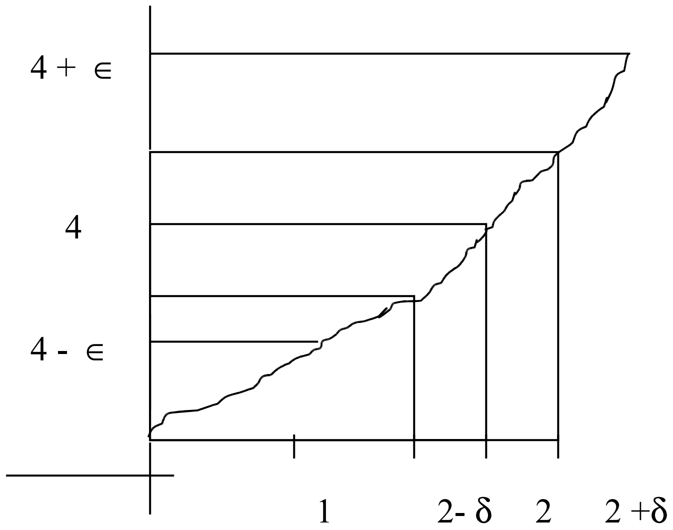
$$|x^2 - 4| \quad 4.2|x-2|$$

Now $4.2|x-2| < \in$ proved $|x-2| < \in/4.2$

Therefore you could choose δ, as the min $\{0.2 \in/4.2\}$ if you do then you will have that:

$$|x^2 - 4| < \in \text{ when } 0 < |x - 2| < \delta$$

See Fig. 17.



3.4 Right And Left Hand Limits

A function $f(x)$: could have one limit as x approaches x_o from the right and another limit as x approaches xo from the left. Recall that in the above definitions of limits of function the word "arbitrarily close" was loosely used, to describe the approach of x to xo without indicating how x should approach xo how x should approach x_o. If x approaches xo from the right-hand side:

i.e., for values of

$x > x_o$ you write that: $\to$x-3 x_o

and for values of

$x < x_o$ you write $x \to x_o$
and say that x approached x_o from the left hand side.

Definition: If lim $f(x) = \ell$ and

$x \to x_o$

where $\ell+$ is called the right hand limit of the function $f(x)$ and L^- is the left hand limit of the same function $f(x)$.

Remark: If the limit of a function exists as x $\to$ x_o then

$$\lim f(x) \quad = \quad \lim f(x)$$

$$x \quad x_o^{+} \qquad\qquad x \;\; x_o^{+}$$

Example: Investigate the limit of the function defined by

$$f(x) = \begin{cases} -1. x < 0 \\ 1. x > 0 \end{cases}$$

as x approaches 0.

Solution

From the above $f(x) = -1$ for $x < 0$

$\Rightarrow$ x 0^{o} $f(x) = -I$ and

x 0^{∞} $f(x) = 1$

Thus lim $f(x)$ does not exit.

Hence lim $f(x)$ does not exist x $\to$ x_o.

An interesting function you would not like to miss when dealing with one-sided limits is the greatest - integer function defined as;

[x] = greatest integer x_0

Example: Investigate the limit of the function F defined by:

$f(x)$ = [x] as x approaches

(see Fig. 18)

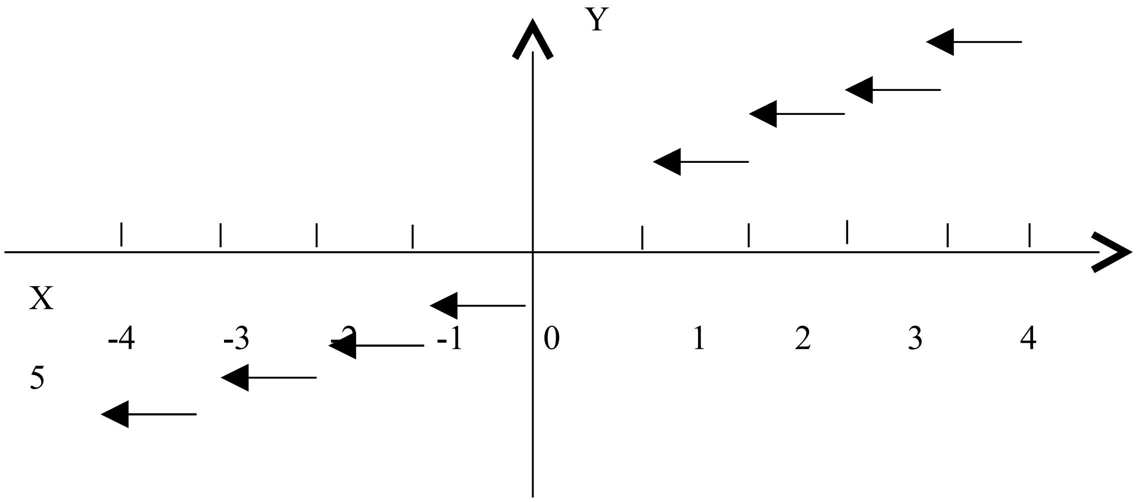


Fig. 16.

In fig 16, the function is 0 at 0 and remain 0 to 1 jumps to 1 remain 4 throughout the interval [1,2). At 2 the function jumps to 2 remain 2 in the interval [2,3) at 3 jumps to 3 and remains 3 in the interval [3,4) and so on.

To investigate the limit we take values less than 3 and values greater 3.

$$f(x) = [x] = 3 \quad \forall x \in [3, 4)$$

$$f(x) = [x] = 2 \quad \forall x \in [2, 3)$$

Therefore:

$$\lim_{x \to 3.} f(x) = 2$$

$$\lim_{x \to 3^+.} f(x) = 3$$

Since $\lim_{x \to 3^-} f(x) \neq \lim_{x \quad 3^+} f(x)$

Then the $\lim_{x \to 3} f(x)$ does not exist

Generally for the greatest -integer function:

$\lim_{x \to x_o^+} [x] = x_o$ and $\lim_{x \to x_0^+} [x] = x_o - 1$

Example:

$$g(x) \begin{cases} x2 & x > 0 \\ \sqrt{x} & x < 0 \end{cases}$$ investigate the limit as $x \to 0$

Solution

$\lim_{x \to x_0^+} g(x) = 0$ and $\lim_{x \to x_0^+} g(x) = 0$, hence $\lim_{x \to x_0} g(x) = 0$

You shall now look at one of the most important properties of the limit of a function. This is the uniqueness property.
Uniqueness: If the limit of a function $f(x)$ exists as x approaches x_o it is unique.

The above property is a theorem which you will be required to give the proof.

Example: Proof that if the limit of a function f (x) as x approaches x_0 exists it is unique.

Proof:
Let $\lim_{x \to x_0} f(x) = \ell_1$

Another one for the same function $f(x)$ be given as:

$\lim f(x) = L_2$

You will be required to show that:
$L_2 = \ell_1$ by providing that the assumption $L_2 \neq L_1$ leads to absurd result that:

$| L_2 - L_1 | \quad | L_2 - L_2|$

By definition of limit: for any positive

$\in_1 > 0$ there is $\delta_1 > 0$

$| f(x) - L_1 | < \in_1$

when $0 < | x - x_o | < \delta_1$

and for $\in_2 > 0$ there is $\delta_2 > 0$

such that $| f(x) - L_1 | < \in_2 |$

when $0 < | x - x_o | < \delta_2$

Let $0 < | \ell_1 - \ell_o | = | \ell_2 - f(x) + f(x) - \ell L_2$

$| \, | \ell_2 - f(x) | + | f(x) - \ell L_2 |$ (By triangle inequalities)

$< \in_1 + \in_2$) by definition above

$\in_{1} = \frac{1}{2} \mid \ell_1 - \ell_2 \mid$ and $\in_{2} = \frac{1}{2} \mid \ell_1 - \ell_2 \mid$

Then $\frac{1}{2} \mid \ell_1 - \ell_2 \mid < \in_1 + \in_2$

$= \frac{1}{2} \mid \ell_1 - \ell_2 \mid + \frac{1}{2} \mid \ell_1 - \ell_2 \mid = \mid L_1 - L_2 \mid$

$\mid L_1 - L_2 \mid < \mid \ell_1 - \ell_2 \mid$ which is absurd or contradictory. Hence he assumption that $\ell_1 \neq \ell_2$ is false. Therefore $\ell_1 = \ell_2$ which is the required result.

4.0 CONCLUSION

You have studied the informal and formal definitions of the limit of a function, which is a major starting point for the study of the subject called calculus. You have studied the important properties like uniqueness of the limit of a function. You have used the δ and ∈ method to prove that a given number ℓ is the limit of a function as $x \to x_o$ for a function $f(x)$.

5.0 SUMMARY

In this unit you have studied how to

1. State an informal definition of the limit of a function $f(x)$ as x tends to x_o

2. State the formal definition of the limit ℓ of a function $f(x)$. as $x \to x_o$ using the δ and ∈ symbols. i.e.; If $\mid x - x\ \text{-}1 \mid < \delta > 0$ then $\mid f(x), \ell \mid < \in > 0$
3. To show that if $\lim_{x \to x_o} f(x) = \ell$ exists then ℓ is unique.

4. To determine whether the number ℓ is the limit of a function $f(x)$ as $\to x_o$

5. The left hand and right hand limits and thus

The $\lim f(x)$ If $\lim f(x) =$ if $f(x)$ $x \to x_o$

6.0 TUTOR-MARKED ASSIGNMENT

1) Define a limit of a function
2) Show that the limit of a function is unique
3) Evaluate the limit of the following:
   (a) limit $f(x)=x^2+3x-6$, as $x \to 2$
   (b) limit $f(x)=3x^5+7xe^x-56$, as $x \to -3$

7.0 REFERENCES/FURTHER READINGS

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UNIT 5 ALGEBRA OF LIMITS

CONTENTS

1.0 INTRODUCTION

You have studied properties of a limit of a function in the previous unit. In this unit you will conclude the study of limit of a function with the following; Algebra of Limits i.e.; Sum and Difference of Limits as well as Products and Quotient of Limits. This has a direct link to the rules of differentiation that will be studied in unit 7 and 8.

2.0 OBJECTIVES

After studying this write, you should be able to correctly:

1. State the theorem on limits sum, product and quotients theorem
2. Evaluate limits of functions using the Sum, Product and Quotient theorem s on limits of a function.
3. Evaluate limits of function as $x \to \infty$ and $x \to -\infty$

3.0 MAIN CONTENT

3.1 Sum of Limits

In the last section we applied the "δ proof" to prove a more general cases involving the algebra of limits.

You will begin by considering the following theorems on limits of functions

Theorem 1: If $\lim_{x \to x_o} f(x) = f$ and $\lim_{x \to x_o} g(x) = g$

then

1. $\lim_{x \to x_o} [f(x) + g(x)l = f + g$

2. $\lim_{x \to x_o} [x\, f(x) = x\, f$

The proof of the (1) of the theorem will follow the pattern used in proving the uniqueness property.

<u>Proof</u>. Let $\in > 0$. To prove (1) above you must show that you can find $\delta > 0$ such that:

If $0 < |x - x_o| < \delta$ then $|[f(x) + g(x) – ([f(+ g)|$

Note that:

$|f(x) + g(x) – (f + g)| = |\{f(x) - f) + (g(x) – g)| \leq$

$|f(x) - f + (g(x) – g)|$ (by triangle inequality)

You will make $|f(x) - g(x) – (f + g)|$ less than $\in$ by making

$|f(x)- f|$ and $|g(x)- g|$ each less than $\frac{1}{2} \in > 0$

Since $\in > 0$ this implies that $\frac{1}{2} \in > 0$

Since by the statement of the theorem

$\lim_{x \to x_o} f(x)- f$ and $\lim_{x \to x_o} g(x)- g$

therefore there will exist two number $\delta_1 > 0$ and $\delta_2 > 0$ such that

If $0 < |x - x_o| < \delta_1$ then $|f(x)- f| < \frac{1}{2} \in$

and If $0 < |x - x_o| < \delta_2$ then $|f(x) - f| < \frac{1}{2} \in$ and $|g(x)- g| > \frac{1}{2} \in$

Now set δf = minimum of δ_1 and δ_2

Therefore $|f(x) - g(x) - (f + g)| = |f(x) - f| + |g(x) - g|$
$< > \frac{1}{2}\in + > \frac{1}{2}\in$

$\in$

Thus it is shown that if:
If $0 < |x - x_o| < \delta_2$ $|f(x) + g(x) - (f+g)| < \in$

Which is the required proof.

2. To prove that
$\lim_{x \to x_o} k f(x) = k f$

Let $\in > 0$. You must find $\delta > 0$ such that:

If $0<1$ $|x-xo| < s$ then $|k f(x) = k f|$ $< s$

There are two cases to consider:

1. when $k = 0$
2. $k \neq 0$

If $k = 0$ then $|0 - 0| < \in$ when $0 < |x - x_o| < \delta$

From the above any value for $\delta > 0$ will do.

To prove the case $k \neq 0$.
Since $\lim_{x \to x_o} f(x) = f$
then there is $\delta > 0$ such that
If $0 < |x - x_o| < \delta$ then $|f(x) - f| < \frac{\in}{|k|}$

From the last inequalities you have that

$|k| \, |f(x) - f| < \in$
$\Rightarrow$ $|k f(x) - fk| < \in$ which is the required proof.

SELF ASSESSMENT EXERCISE 1

If $\lim_{x \to x_o} f(x) = f$ and $\lim_{x \quad x_o \to} g(x) = g$

Show that:

$$\lim_{x \to x_o} f(x) - g(x) = f - g$$

The result of the last two theorems can be extended to any finite number of function.

Example: If $\lim_{x \to x_o} f_1(x) = f_1$, $\lim_{x \to x_o} f_2(X) = f_2$ ...

$\lim_{x \to x_o} f_o(x) = f_n$ then

$$\lim_{x \to x_o} (k_1, f_1(x) + k_2 f_2(x) + \ldots\ldots + k_n f_n(x)) = k_l f_1 + k_2 f_2 + \ldots\ldots k_3 f_3$$

3.3 Products and Quotients of Limits

You shall now consider further theorems on limits (the t proof are beyond the scope of this course).

Theorem2: If $\lim_{x \to x_o} f(x) = f$ and $\lim_{x \to x_o} g(x) = g$

Then (I) $\lim_{x \to x_o} f(x) = f$ and $g(x) = f g$

(II) $\lim_{x \to x_o} \frac{1}{g(x)} = \frac{1}{g}$ $g(x) \neq 0$, $g \neq 0$

(III) $\lim_{x \to x_o} f(x) = \frac{f}{g(x)}$ $f(x), \neq 0$, $g \neq 0$

Theorem3. Let $f(x)$, g(x) and h(x) be functions defined on an interval I containing a o, except possible the functions are not necessary defined at xo, such that:

$f(x)$ g(x) h(x) for all $x \in I$

and lim $f(x)$ = lim h(x) = ℓ

$x \to x_o \qquad x \to x_o$

then = $\lim_{x \to x_o} g(x) = \ell$

From the above theorems it easy to conclude that every polynomial

$f(x) = a_n x^n + a_1 x^n + a_1 x^{n-1} + ... + a_1 + a_o$

satisfies $\lim_{x \to x_o} f(x) = f(x_o)$.

Examples

Evaluate the following, limits:

i. $\lim_{x \to 2} (2x^2 - 5x + 1) = 2(2)^2 - 5(2) + 1 = 2.4 - 10 + 1 =$

ii. $\lim_{x \to 0} (3x^5 - 6x^4 - 3x^2 + x + 10) =$

iii. $\lim_{x \to x_o} (x^5 - x^3 - 4x^2 + x + 1) = (1)^5 - (1)^3 - 4(1)^{22} + |1| = -2$

As consequence of theorem 2 you can see that if P and Q are two polynomials and $Q(x_o) \neq 0$ then

$$\lim_{x \to x_o} \frac{p(x)}{Q(x)} = \frac{P(x_o)}{Q(x_o)}$$

if $Q(x_o) = 0$

then $\lim \frac{p(x)}{Q(x)}$ = does not exist.

Examples

Find the limits of the following functions.

i. $\frac{2x-1}{x^2-3} = \frac{4-1}{4-3} = \frac{3}{1} = 3$
$x \to 2$

ii. $\lim_{x \to 1} \frac{x^2+x+1}{x^2+2x} = \frac{1+1}{1+2} = \frac{3}{3} = 1$

iii. $\frac{2-x}{} = \frac{2-4}{} = \frac{-2}{} = \frac{-1}{}$

4x 8 8 4

iv. $\lim_{x \to 2} (3x – 2x + 1) = 3 – 6 + 1 = -2$

3.5 Infinite Limit

In this section you will be studying about functions whose limits tends to infinity as x approaches a given number.

If a function $f(x)$ increases or decreases without bound as x tends to certain point xo we say that $f(x)$ diverges. That is for a function $f(x)$ if corresponding to every number $K \in R$ there is $\delta > 0$ such that

if $0 < |x - x_o| < \delta$ then $f(x) > k$

Then $f(x)$ is said to approach $+\infty$ as x tend x_o in symbols you write it as

$$\lim_{x \to x_o} f(x) = +\infty$$

It is possible to have a situation whereby point $\ell < \infty$ as x increases or decreases without bound. In other words a function $f(x)$ is said to tend to ℓ as $x \to +\infty$ as if to each there is a number $k \in R$ such that:

If $x > k$ then $|f(x) - \ell| < \in$

symbolically this could be represented as:

$$\lim_{x \to x_0} f(x) = \ell$$

In a similar manner $f(x)$ is said to tend to L as $x \to -\infty$, if to each $\in > 0$ there $K \in R$ such that

$x < k \Rightarrow |f(x) - L| < \in$

In symbols you write

$$\lim_{x \to x_o} f(x) = \ell$$

Example

Take a look at the graph of the function $f(x) = \frac{1}{x}$, $x > 0$ **(see fig.(3.3))**

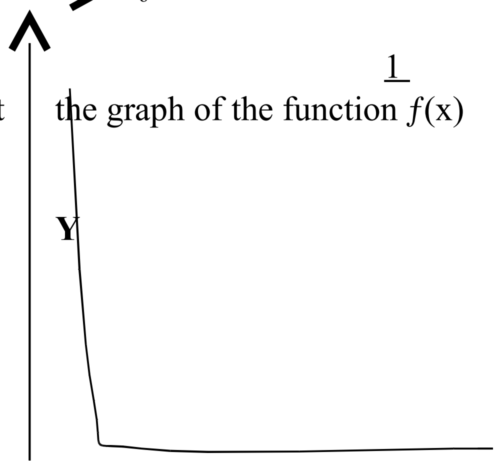


Fig. 17.

The function is a decreasing function. As x gets longer and larger $f(x)$ gets smaller and smaller. This suggest that

$$\lim_{x \to a} \frac{1}{x} = 0$$

also as gets smaller and smaller the function $f(x)$ gets bigger and bigger the value 0 ~(x) takes arbitrary large value. In this case

$$\lim_{x \to a} \frac{1}{x} = +\infty$$

SELF ASSESSMENT EXERCISE 2

Draw the graph of $f(x) = \frac{1}{x}$ $x > 0$ investigate the limit as

1. x tends to 0.
2. x tends to $-\infty$

Finally it could be possible that $f(x)$ increases or decreases without bound just as x also increases or decreases i.e. given an arbitrary number k, there exists $k_2 \in R$ such that $x > k, \Rightarrow f(x) > k_2$. In that case you write symbolically:

$$\lim_{x \to a - \infty} f(x) = \infty$$

For the case $x<k_l$ and $f(x) > k_2$ you write

Lim $f(x) = \infty$

$x \to a - \infty$

and for the case of $x < k_1$

and $f(x) < k_2$ you have $\lim_{x \to -\infty} f(x) = \infty$

For each of the following function.

Given the following function.

1 $f(x) = \dfrac{1}{x - 1}$

2 $f(x) = \dfrac{1}{x^2 - 1}$

3 $f(x) = \dfrac{1}{x^2 - 4}$

Find the limits as (1) $x \to 1^+$, (ii) $x \to -\infty$ (iii) $x \to +\infty$

Sketch the graph in each case. Consider the following functions and their graphs and

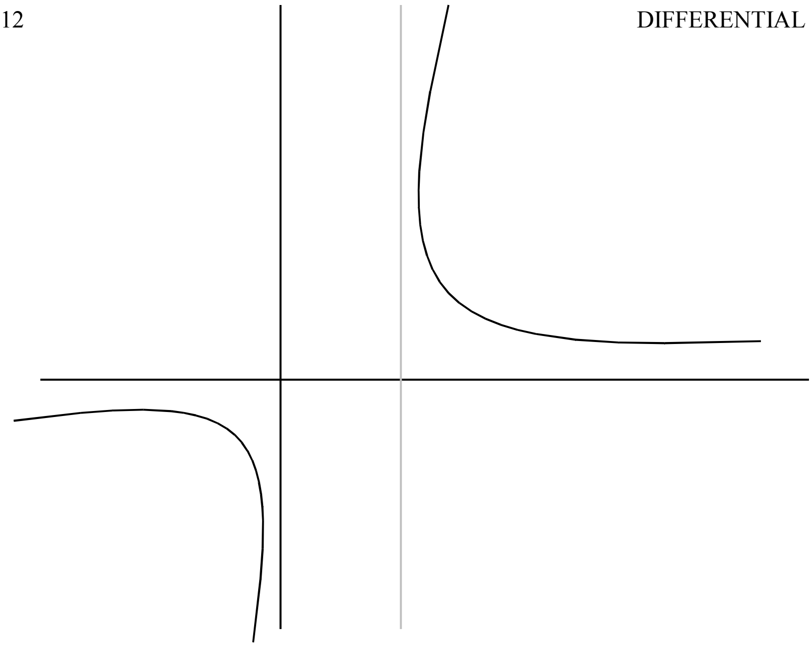
Limits of each:

1 $\lim_{x \to 1^+} = \dfrac{1}{x - 1} = +\infty$

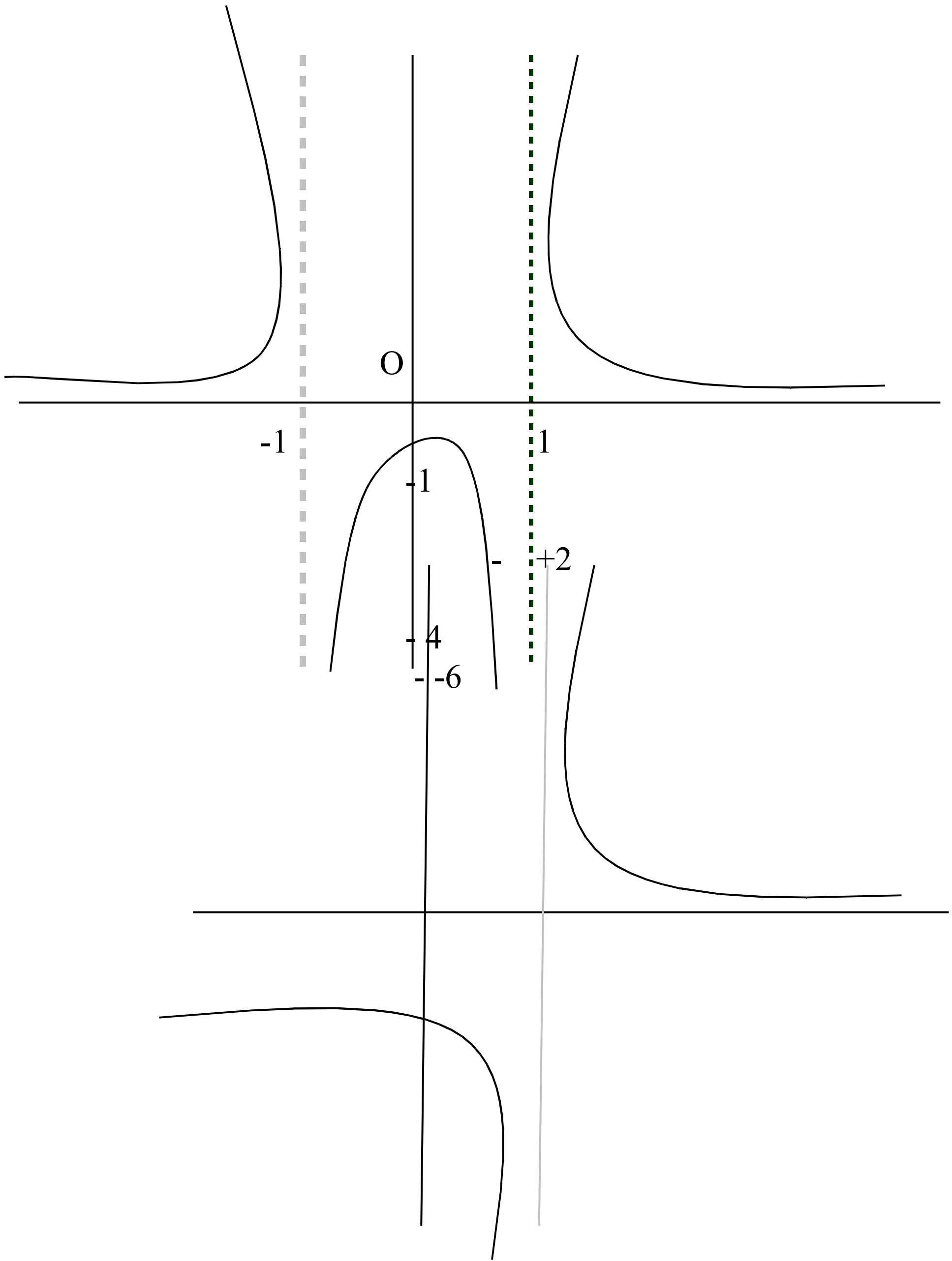
2 $\lim_{x \to 1^-} = \dfrac{1}{x - 1} = -\infty$

3 $\lim_{x \to \infty} = \dfrac{1}{x - 1} = 0$

4 $\lim_{x \to \infty} = \dfrac{1}{x - 1}$



$$f(x) = \frac{1}{x^2 - 1}$$



0 1

– -1

– -2

a (I) $\lim_{x \to 1^+} f(x) = +\infty$ (II) $\lim_{x \to \infty} f(x) = 0$

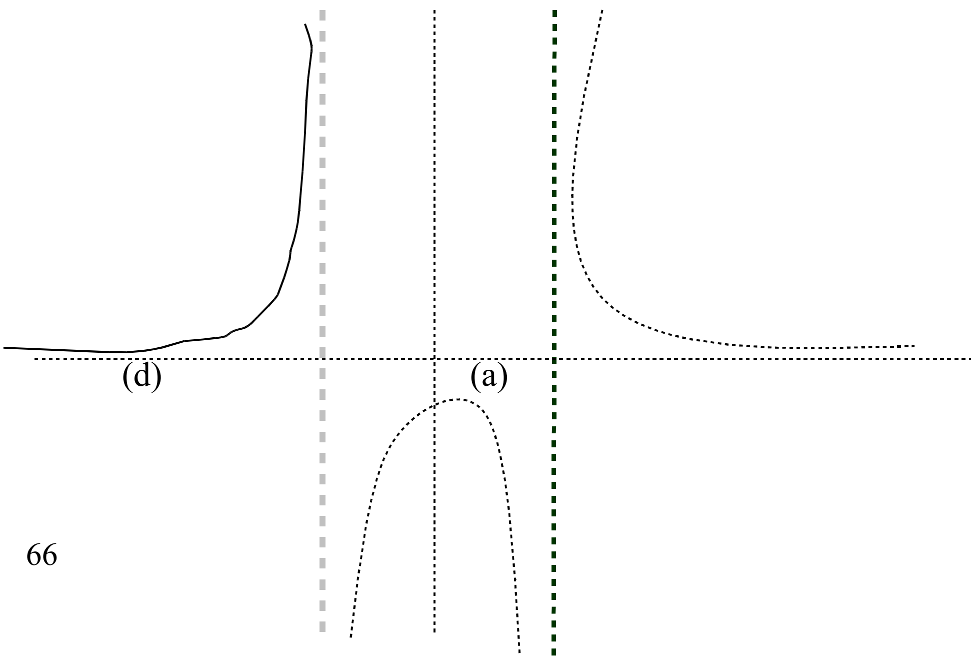
(III) $\lim_{x \to 1^2} f(x) = -\infty$ (IV) $\lim_{x \to \infty} f(x) = 0$

(ii) Investigate the limits of the function

$$f(x) = \frac{1}{x^2 - 1}$$

(I) $x \to 1^+$ (II) $x \to 1^-$ (III) $x \to -1^+$ (IV) $x \to -1^-$

(V) $x \to -\infty$ (VI) $x \to +\infty$



(f)
(e)

O

-1 1

-1

(c) (b)

a (a) $\lim_{x \to 1^+} f(x) = +\infty$ (b) $\lim_{x \to 1^-} f(x) = -\infty$

(c) $\lim_{x \to -1^+} f(x) = +\infty$ (d) $\lim_{x \to -1^-} f(x) = 0$

(e) $\lim_{x \to \infty} f(x) = +\infty$ (d) $\lim_{x \to -\infty} f(x) = 0$

From the above you can easily see that: $\lim_{x \to 1} f(x)$ does not exist.

Since the left hand limit $\neq$ Right hand limit

i.e.: $\lim_{x \to 1^+} f(x) = +\infty \neq -\infty = \lim_{x \to 1^-} f(x)$

Also , $\lim_{x \to -1} f(x)$ does not exist

Because $\lim_{x \to 1^+} f(x) = -, \infty \neq -\infty = \lim_{x \to -1^-} f(x)$

4.0 CONCLUSION

You have seen how arithmetic operation on limits is used in evaluating limits of various functions especially polynomials. You have seen how the graph of a rational function could aid in evaluating infinite limits. You will see how the limiting process that we have studied in this unit will continue to serve as a reference point in subsequent units of this course

5.0 SUMMARY

In this unit you have studied:

1. That the limit of the sum of a finite number of functions is equal to the sum of their limits.

   i.e.; $\lim_{x \to x_o} d_1, f_1, (x) + ...+ \lim_{x \to x_o} d_n f_n(x) = \lim_{x \to x_o} ef_1(x) +.....+ d_n f,)$

2. The limit of the product of a finite number of the product of their limits.

   i.e.; $\lim f_1 (x) ....\lim_{x \to x_o} f_n (x) = \lim_{x \to x_o} (f_1 (x) f_2 (x) .... f_n (x)$ $x \to x_o$

3. The limit of the quotient of two functions is equal to the quotient of their limits

$$\lim \frac{f(x)\ x \to x_o}{\lim g(x)\ x \to x_o} \qquad \lim \frac{f(x)\ x \to x_o}{\lim_{x \to x_o} g(x)}$$

6.0 TUTOR-MARKED ASSIGNMENTS

Give a precise definition of the following with suitable example where necessary:

1. The limit of a function $f(x)$ as x tends to x_o.
2. The right-hand limit of a function as x tends x_o
3. The limit of a function $f(x)$ as (a) $x \to -\infty$ (b) $x \to +\infty$
4. State the definition of the left and right hand limits. Hence give examples of functions y- $f(x)$ possessing limits as $x \to x_o$ and $x \to x^{o-}$ and having no limits as $x \to x_o$
5. Find the limits that exists:

Limits of each:

(i) $\lim_{x \to 1^+} = \frac{x^2}{x + 1}$ (ii) $\lim_{x \to 1^-} \frac{x^2}{x + 1}$

(iii) $\lim_{x \to 2} = \frac{[x] = x}{x}$ (iv) $\lim_{x \to 1} = \frac{x^2 + 2x + -1}{3 x 3 + x + 1}$

(v) $\lim_{x \to -1} = \frac{x + x^2}{x + 1}$ (vi) $\lim_{x \to 5} = \frac{2x - 10}{x^2 - 8x^2 + 17x - 10}$

6. Sketch the graph
$f(x) = \frac{x}{x^2 - 9}$

Hence find the limits that exist as

(i) $x \to -3^+$ (ii) $x \to -3^-$

(iii) $x \to -3$ (iv) $x \to 3^-$

(v) $x \to 3^+$ (vi) $x \to -\infty$

7.0 REFERENCES/FURTHER READINGS

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MODULE 2

UNIT 1 ALGEBRA OF LIMITS

CONTENTS

1.0 INTRODUCTION

You would have been familiar with the word continuous ordinarily to say that a process is continuous is to say that the process goes on without changes or interruptions. In this section the word continuous has almost the same meaning. That is a function is continuous in the sense that you plot the graph continuously without lifting your pencil from the graph paper. In calculus it is demanded that functions must be continuous at points or interval of investigations that is why you must study this unit with some care.

2.0 OBJECTIVES

After studying this you should be able to correctly:

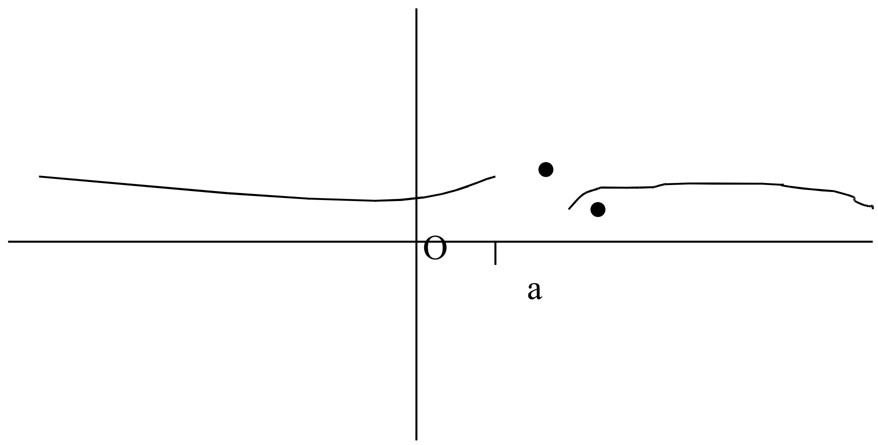
1. define a continuous function at the part $x = x_o$
2. recall properties of continuous function
3. state theorems on continuous function.
4. state the 3 conditions for continuity of a function at a given part.
5. identify parts of continuity and discontinuity of a function.

3.0 MAIN CONTENT

3.1 Definitions of a Continuous Function

Consider The Graph shown in Fig. (21 a) and Fig (21b

.



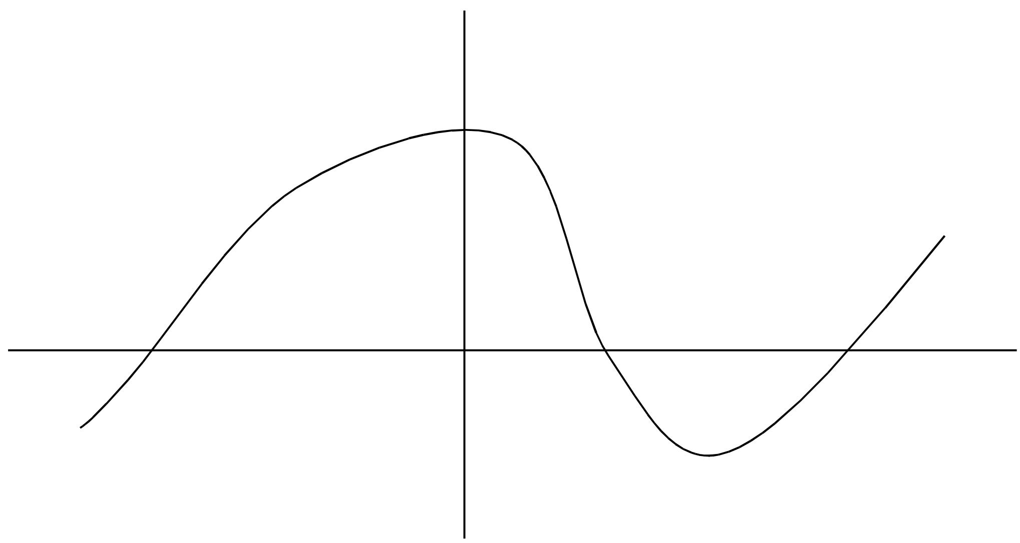


Fig. 21 b.

In Fig 21a the function changes abruptly at the part a: whereas the graph in Fig.21b is continuous.

Definition: A function $f(x)$ is said to be continuous at the point x = xo if and only if:

$$\lim_{x \to x_o} f(x) = f(x_o)$$

The above definition can be broken down into 3 main conditions a function must satisfy for it to be continuous at the point $x = x_o$

Definition: A function $f(x)$ is said to be continuous at the point $x = x_o$ if the following 3 conditions are satisfied.

1. $f(x)$ must be defined at $x = x_o$
2. $\lim_{x \to x_o} f(x) = L$ must exist
3. $f(x_o) - L$

Examples:

a. Is the function $f(x) = x^2 - 4$ continuous at the point x

Solution:

Checking for the 3 conditions.

1. Let $x = 2, f(x) = 2^2 - 4 = 0$
2. $\lim_{x \to 2} x^2 - 4 = 0$
3. $\lim_{x \to 2} x^2 - 4 = f(2) = 0$

b. Is the function $f(x) = \frac{1}{x - 1}$ continuous at point $x = 1$

Solution:

i. let $x = 1$ $f(x)$ $\frac{1}{1-1} = \frac{1}{0}$

Since division by zero is not possible then f (x) is not defined at the point x = 1. Needless to check for the remaining conditions you can conclude by saying that

$f(x) = \frac{1}{x - 1}$ is not continuous at the point $x = 1$

Definition: A function $f(x)$ which is not continuous at the point $x = x_o$ is said to be discontinuous at that point.

Example:

Determine the points of discontinuities of the function

$$f(x) = \frac{x}{x^2 - 9}$$

Solution:

The same manner a function is said to be continuous (on the left) at $x = x_o$ if. $\lim_{x \to x_o} f(x) = f(x_o)$

Just as the case of limit, a function is said to be continuous at $x = x_o$ if it is both continuous from left and from right at x_o

1. Function that are Continuous on R

Every polynomial function is continuous on R.

i.e.; $f(x) = a_n x^n + a_n, x^{n-1} + ...+ a_{n-1}, x + a_n - 1$

In unit I section 3.3 it was shown that
i. e. ; $\lim f(x) = f(x_o)$ for all polynomial

Thus $f(x)$ is continuous.

2. The Trigonometric Functions

For examples $f(x) = \sin x$ and $f(x) = \cos x$ are continuous on R

$|\sin x| < |x|$ and $|\cos x| < 1$ for all $x \in R$
see the graph of sin x and cos x at Fig. 22a and 22b

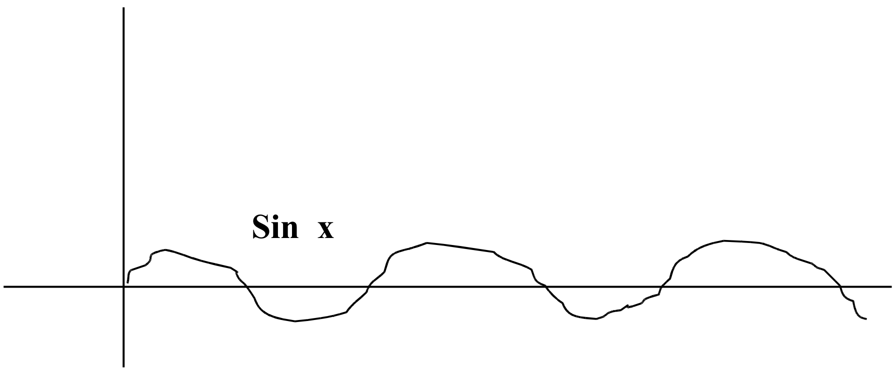


Fig. 22a

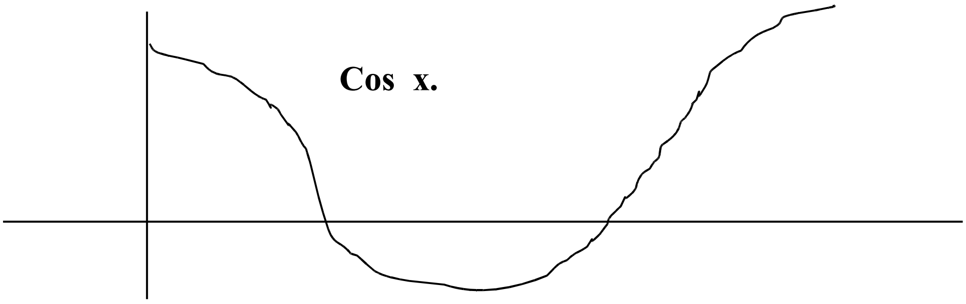


Fig. 22b.

3. Removable Discontinuity

It has earlier been defined that a function that is not continuous at a point $x = x_o$ is discontinuous at that point. However there are basically two types of discontinuities.

If for any function $f(x)$, the $\lim_{x \to x_o} f(x) \neq f(x_o)$ initially, but by redefining the function $f(x)$ is done in such a way that $f(x_o) = \lim_{x \to x_o} f(x)$

then the point x_o is said to be a point of removable discontinuity of $f(x)$.

Example:

Show that $f(x) = \frac{x - 4}{x - 2}$

Has a removable discontinuity at point x = 2.

Solution:

Since $f(x)$ is not defined at x = 2. But by appropriate factorization the function.

$$f(x) = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

then $\lim_{x \to 2} x=2 = f(2) = 4$

Hence at x = 2 is a point of removable discontinuity.

A very simple way to solve this is to define the domain of the function. You can easily see that the domain D is given as

$$D = \{ x; x \in R, x \neq -3 \text{ or } 3)$$

Therefore the points of discontinuity are -3 and 3.

Finally, another definition of limit will now be given using the familiar $\in$, δ^+ symbol.

Definition: A function $f(x)$ is said to be continuous at the point x = x_o if for $\in > 0$ there is $\delta > 0$ such that

If $|x - x_o| < \delta$ then $|f(x) - f(x_o)| < \in$

Remark:

The above definition is an extension of the definition of a limit. In the above if we replace $f(x_o)$ with L and remove the restriction that $f(x_o)$ must be defined you get back definition of a limit.

Example:

Show that the function $f(x) = x^2$ is continuous at the point x = 2.

Solution;

Let $\in > 0$ if you can find a $\delta > 0$

Such that

If $|x - 2| < \delta$ then $|f(x) - f(x_o)| < \in$

Note that $f(2) = 0$ and

$$|x^2 - 4| = |(x - 2(x + 2)| = |x + 2|\,|x - 2|$$

By keeping x close to 2 we make the factor x - 2 as small as we please and the second factor x + 2 gets close to 4.

If the domain D = [-2,]say then we can be sure that the factor (x + 2) 4.

If $x \in [-2, 2]$ then

$X + 2 \; 10$ and $| f(x) - f(2) | < \in$

When $| x - 2 | < \delta\mu$

Provided that $\delta \le \in/10$

If $\in > 0$ then

$| f(x) - f(2) | \; 10 \; | (x - 2 | < \in$ if $| x - 2 | < \in /10 = \delta$

In fact the function in the above example is continuous for all points in the interval 1 = [- 2, 2]. When such happens the function f(x) is said to be uniformly continuous on the interval 1.

3.3 Properties of Continuous Functions

1. Uniform Continuity

A function f(x) is said to be continuous in an interval I. If for $\in >$ 0 there is a $\delta > 0$ (depending on alone) such that:

If $| x_1 - x_2 | < \delta$ then $| f(x_1) - f(x_2) | < \varepsilon$

2. Continuity of Function From Left and Right of a Point

A function f(x) is said to be continuous (on the right) at a point $x = x_o$

If $\lim f(x) = f(x_o)$ $x \to x_o$

With f (x) defined at x = 2. You can say that f (x) is now continuous at the point x =2.

Type II: Non-Removable Discontinuity

If for a given function f(x) the right hand and left hand limits as $x \to x_o$ exist but are unequal i.e.;

$$\lim_{x \to x_o} f(x) \neq \lim_{x \to x_o} f(x)$$

or if either the $\lim_{x \to x_o} f(x)$ or $\lim_{x \to x_o} f(x)$

does not exist then the function $f(x)$ is said to have a non-removable discontinuity $x \to x_o$

Example

The function $f(x) = \sin(1/x)$ is continuous except for x = 0. The function has non-removable discontinuity at x = 0. Both right and left hand limits do not exist.

Example

Determine whether the function $f(x) = [x]$ (the greatest integer function) is continuous at the point x = 3

Solution:

$$\lim_{x \to x_o} [x] = 3$$

and $$\lim_{x \to 3} [x] = 2$$

Since $\lim_{x \to 3^+} f(x) \neq \lim_{x \to 3^-} f(x)$

Then the function $f(x)$ has a non-removable discontinuity at x = 3.

See Fig 23.

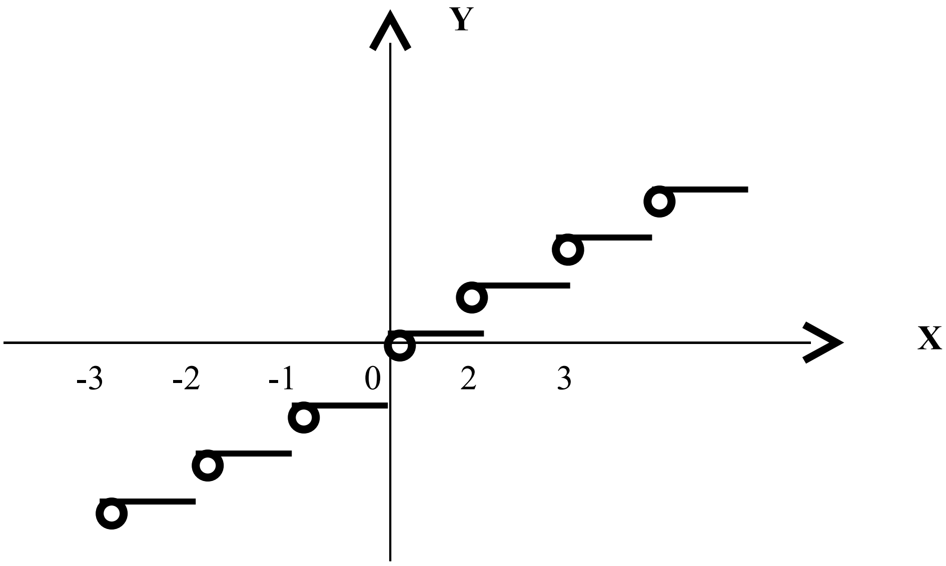


Fig. 23.

The above gives us a picture of a function that is one sided continuous. In fig. 23. the function is continuous from the right and discontinuous from the left.

5. Continuity on [a,b]

If a function *f*(x) is defined on closed interval [a, b) the most continuity we can possibly expect is:

1. Continuity at each point xo of the open interval (a, b).
2. Continuity from the right at a, and
3. Continuity from the left at b.

Therefore any function that satisfies conditions 1 to 3 above is said to be continuous on [a, b]

Functions that are continuous on a closed interval are of special interest to mathematicians, because they possess certain special properties which discontinuous function do not have.

SELF ASSESSMENT EXERCISE 1

Draw the graph of the above function.

1. Define any function that satisfies conditions 1 to 3 above is said to be continuous on [a, b]

2. Fractions that are continuous on a closed interval are of special interest to mathematicians, because they possess certain special properties which discontinuous functions do not have.

Example

The function $f(x) = \sqrt{1 - x^2}$ is on the in [-1, 1]

SELF ASSESSMENT EXERCISE 2

Draw the graph of the above function.

3.4 Algebra of Continuous Functions

Recall the theorem, on limits you studied in the last Unit. You will now do the same for continuous function. Using the following theorems it can be shown that continuity is preserved through algebraic operations on functions.

That is:

Theorem 1:

If the functions $f(x)$ and $g(x)$ are continuous at the point $x = x_o$ then the sum $f(x) + g(x)$ is continuous at $x = x_o$

Fortunately enough the proof of the above theorem is not complicated in the sense that all that is required for a function to be continuous at a point $x = x_o$ is that $\lim f(x) = f(x_o)$

From the theorems of limits (see Unit 3).

The $\lim_{x \to x_o} f(x) + g(x) = \lim_{x \to x_o} f(x) + \lim g(x)$

Therefore the function $f(x) + g(x)$ is continuous since

$$\lim_{x \to x_o} f(x) + g(x) = \lim f(x_o) + g(x_o)$$

Theorem 2:

If the functions $f(x)$ and $g(x)$ are continuous at the point $x_o = x$ then the sum $f(x) + g(x)$ is continuous at $x = x_o$

Proof:

Let $\lim_{x \to x_o} f(x) = f(x_o)$ and

Let $\lim_{x \to x_o} g(x) = g(x_o)$ since

they are continuous at $x = x_o$

Therefore $\lim_{x \to x_o} f(x) \cdot g(x) = \lim_{x \to x_o} f(x) \cdot \lim g(x) = f(x) \cdot g(x)$

hence the function $f(x)\, g(x)$ is continuous at $x = x_o$

Theorem 3:

If the functions $f(x)$ and $g(x)$ are continuous at the point $x = x_o$ then the function $f(x) / g(x)$, $g(x) \neq 0$ is continuous at $x_1 = x_o$

Proof:

The proof is similar to the one above if left as an exercise for you (good proving).

Theorem 4:

If the functions $y=f(x)$ is continuous at the point $x = x_o$ and $z = g(y)$ is continuous at $y = y_o$ where $y_o = f(x_o)$

Then the function $z = g(f(x_o))$ is continuous at point $x = x_o$.

The Proof follows from the fact that

$$\lim_{x \to x_o} f(x) = f(x_o)$$

and $$\lim_{y \to y_o} g(y) = g(y_o) = g(f(x_o))$$

hence $gf(x)$ is continuous at $x = x_o$

Example:

Use the theorems on continuous function to determine whether the following functions are continuous at the given points.

(i) $f(x) = 6x^2 - 2$ at $x = 2$

(ii) $f(x) = \frac{x^2 - 1}{x + 1}$ at $x = 1$

(iii) $f(x) = \begin{cases} x^2 + 9 & x < 3 \\ 6x & x \geq 3 \end{cases}$ at $x = 3$

(iv) $f(x) = \cos^2 x\,(x^3 + 2x - 1)$ at $x = x_o$

(v) $f(x) = \frac{\cos x}{e^x + \sin x}$ at $x = x_o$

(vi) $f(x) = \sin(x^2 - 1)$ at $x = x_o$

(vii) $f(x) = \sqrt{\frac{x^3 - 1 - x^4}{x}}$ at $x = 2$

(viii) $f(x) = \frac{x^2 - 2x - 1}{x - 2}$ at $x = 1$

(viii) Is continuous at $x = x_o$ since $\cos x$ is continuous

. . (Cos x) (Cos x) and x3 + 2x + 1 is continuous

.. by theorem 2 (Cos x)(Cosx) $(x^3 + 2x + 1)$ is continuous at $x = x_o$

Solution:

(I) $f(x) = 6x^2 - 2$ then

$$\lim_{x \to 2} 6x^2 - 2 = 6(2)^2 - 2 = 22$$

$$f(2) = 6(2)^2 - 2 = 2$$

Since $\lim_{x \to 2} f(x) = f(2)$ it is continuous

(II) $f(x) = \dfrac{x^2 - 1}{x + 1}$

then $\lim_{x \to -1} \dfrac{x^2 - 1}{x + 1} = \lim x - 1 = x - 1 = -2$
$-1 \; -1 = -2$

$$\lim_{x \to 2} f(x) = f(-1) = -2$$

(III) $f(x) = x^2 - 9 \quad x < 3$
$6x \quad x \geq 3$

$\lim_{x \to 3^-} f(x) = 18$ and $\lim_{x \to 3^+} f(x) = 18$

$\lim_{x \to 3^-} f(x) = 18$ and $\lim f(3) = 18$

$\lim_{x \to 3} f(x) = 18$ and $\lim f(3) = 18$

Since $\lim_{x \to 3} f(x) = f(3)$

It is continuous at x = 3

(IV) $f(x) = \text{Cos}^2 x \; (x^3 + 2x + 1)$ at $x = x_o$

Cos x is continuant at x = x = x_o
So is (Cos x) (Cos x) by Theorem 2.

x3 + 2x + 1 is continuous since it is a polynorminal therefore

(Cos x) (Cos x) (x^3 + 2x + 1) is product of continuous function which is continuous.

(V) $\frac{\text{Cos } x}{e^x + \text{Sin } x}$ at = x = x_o

Cos x, $\in^x$ and sin x are all continuous of x_o = x
$\in^x$ + Sin x is continuous by Theorem I

and $\frac{\text{Cos } x}{\in^x + \text{Sin } x}$ is continuous by Theorem II

(VI) $f(x)$ = Sin ($x^2 - 1$) at x = x_o
since f(g(x) is a continuous function.

If $f(x)$ and g(x) are both continuous

Therefore sin ($x^2 + 1$) is continuant at x = x_o

(VII) $f(x) = \frac{x^3 + 1 - x^4}{x}$ at x = 2

$$\lim_{x \to 2} = \frac{x^3 + 1 - x^4}{x} = \frac{\sqrt{8 + 1} - 16}{2} = \frac{3 - 16}{2} = \frac{-13}{2}$$

$f(2) = \frac{-13}{2}$

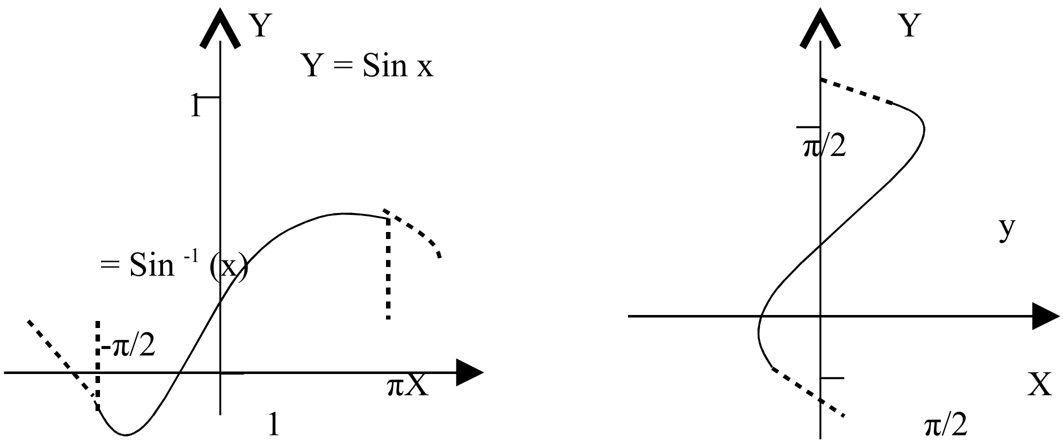
It is continuous since $\lim_{x \to 2} f(x) = f(2)$

The Theorem of continuity of inverse function will now be stated. This theorem is important in that once a continuous function is defined and it is a one to one function then it becomes easy for you to determine whether the inverse function is continuous. The concept of continuity of functions brings out the hidden beauty in the study of both differential and integral calculus.

Theorem 5: Continuity of Inverse Function

If $f(x)$ is continuous on an interval I and either strictly increasing or strictly decreasing and one to one in the interval I, then there exists an inverse function $f^{1}(x)$ which is continuous and one to one and either strictly increasing or strictly decreasing.

See Fig (24)a and Fig (24)b



In the interval [-π/2, π/2] the function $f(x)$ = Sin x is continuous and one to one so is $f(x) = \text{Sin}^{-1}$ x is also a continuous on [-π/2, π/2] and one to one.

4.0 CONCLUSION

In this Unit, you have defined a continuous function. You have used the concept of limit of a function to identify points of continuous and discontinuity of a function in a given interval of points. You have studied theorem on continuous function and used the theorem to examine points of continuities or discontinuities of a function. You are now aware that a function that is continuous at a point $x = x_o$ is defined at that point and that the limit of the function must exist as x approach the point x_o. You are also aware that the a converse is not necessary true i.e. the Limit of a function might exist at a point x_o and not continuous at that point. This logical reasoning will be extended in the next Unit.

5.0 SUMMARY

In this unit, you have studied the following:

(1) the definition of a function.

(2) how to determine points of continuity and discontinuities of a given function.

i.e. $\lim_{x \to x_o} f(x) = f(x_o)$ then x_o is a point of continuity for the function

(3) how to use the following theorems:
If f and g are continuous functions

Then
(i) $f \pm g$, (ii) fg and f/g are continuous function to determine a function. That is continuous or not.

(4) that all polynomials, cos x and sin x are continuous in R. In the unit that follows this Unit, you will see that all the results on this unit will be used.

6.0 TOTAL-MARK ASSIGNMENTS

(1) Give a precise definition of the limit of a function $f(x)$ at the point $x = x_o$

(2) State 3 condition a function must satisfy for it to be continuous at the point $x = x_o$

(3) State two properties possessed by a function $f(x)$ which is continuous in a closed interval [a, b].

(4) using the "$\in$, δ" symbols explain what is meant by saying that a function $f(x)$ is discontinuous at a point x_o

(5) give examples of two types of point of discontinuities of functions. Hence for what values is each of the following functions discontinuous.

$$f(x) = \frac{x}{x+1} \qquad f(x) = \frac{2x+1}{x^2 - 3x + 2}$$

(6) Show that the function $f(x) = \text{Sin } x$ is continuous for $x = x_o$

(7) Determine whether the following functions are continuous at the given points.

(i) $f(x) = \dfrac{x}{x^2 - 2}$ $\qquad x = 1$

(ii) $f(x) = \dfrac{1 - \text{Sin } x}{2 - \text{Cos } x}$ $\qquad x = 0$

(iii) $f(x) = \dfrac{1}{x^3 - 1}$ $\quad x = 1$

7.0 REFERENCES/FURTHER READINGS

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Satrmino L.S. & Einar H. (1974) **Calculus** "2nd Edition", John Wiley & Sons New York. London, Sydney. Toronto.

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UNIT 2 DIFFERENTIATION

CONTENT

1.0 INTRODUCTION

In this unit you will learn how to differentiate a function or find the derivative of a function at a given point. This will be done by looking at the slope of a line, which you will extend to general case of slope of a curve. Then you will apply the concept studied in unit 2 to study the limiting process of a function along the given line.

After which you use the concept of a slope of a curve and a target at a given point on the curve to solve two type of problems among others namely:

1. given a function $f(x)$, determine those value of x (in the domain of $f(x)$) at which the function is differentiable

2. given a function $f(x)$ and a point $x = x_o$ at which the function is differentiable find the derived function you will finally extend this to differentiation of a polynomials functions. A section on solved problems has been included to sharpen your skills in differentiation.

This unit is a formal bridge between concept studied so far in units 1 to 4 and those you will be studying in units 6 to 10. Therefore carefully read and understand all definitions and solved examples given in this unit – wishing you a successful completion of this unit.

Below are the objectives of this unit.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

1. define the slope of a point on a curve.
2. define the derivative of a function at a given point $x = x_o$
3. evaluate the derivative of a function using the limiting process (i.e. Δ - process or from first principle).
4. derive standard formula for differentiation of polynomials
5. find the derivative of polynomials functions using the Δ - process or a standard formula.

3.0 MAIN CONTENT

3.1 Slope of a Curve

You will start the study this unit by reviewing the following:

i. the coordinate system
ii slope of a line

1. The Coordinate System

This is the system that contains

i) a horizontal line in a plane extending indefinitely to the left and to the right and which is known as x axis or axis of abscussas

ii) a vertical line in the same plane extending indefinitely up and down this is known as the y- axis or aixs of ordinates. A unit is the chosen for both axis. See fig. 25.

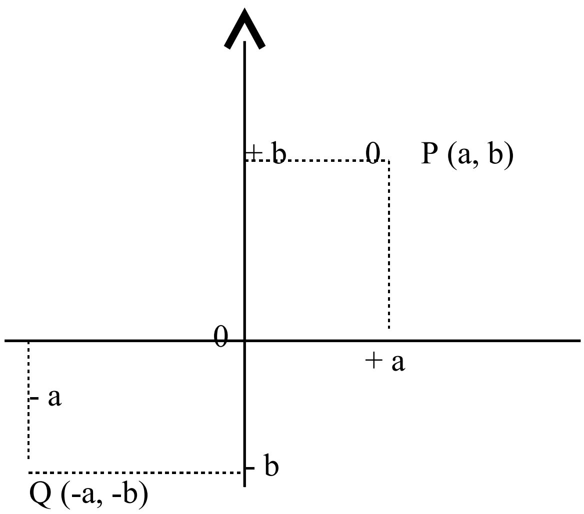


Fig. 25. Showing the coordinate system.

2. The Slope of a Line

You are also familiar with the concept of increment.

For example. If a body starts at a point $Q_1(x_1, y_1)$ and goes to a new position $Q_1(x_2, y_2)$ you say that its coordinates have changed by an increment Δx (i.e. delta x) and Δy(i.e delta y).

Let a body move from point p(2, 4) to Q(4, 6) as shown in Fig (25). Find Δx and Δy.

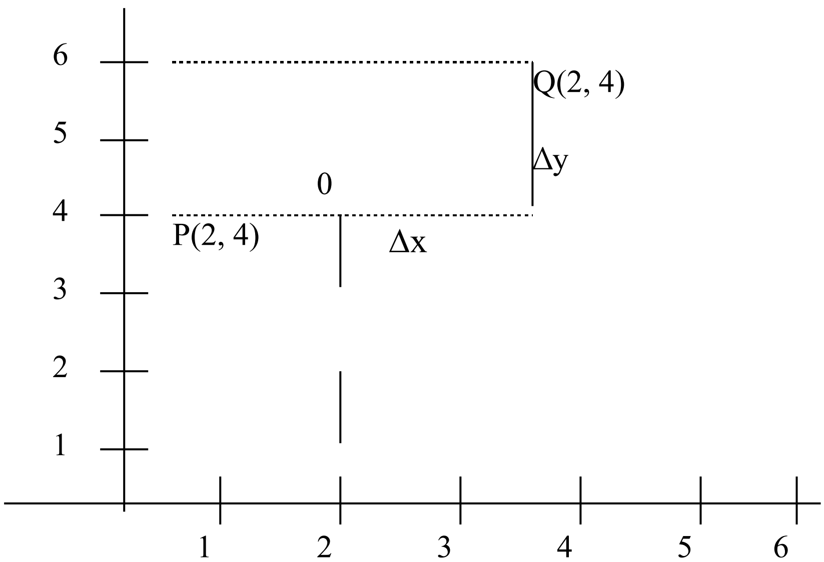


Fig 25.

$\Delta \quad x = 4 - 2 = 2$
$\Delta \quad y = 6 – 4 = 2.$

Generally if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two given points then

$\Delta x = x^2 - x_1$ if	P	Q
$\Delta x = x_1 - x_2$ if	Q	P
$\Delta y = y_2 - y_1$ if	P	Q
$\Delta y = y_1 - y_2$ if	Q	P

Using the above you can now turn your attention to finding the slope of a line the idea here is that lines in any coordinate plane rise or fall at a constant rate as we move along them from left to right unless, of course they are horizontal or vertical.

You can define the slope or gradient of line as the rate of rise or fall as you move from left to right along the given line.

Example

Describe the slope of the line L in fig 26.

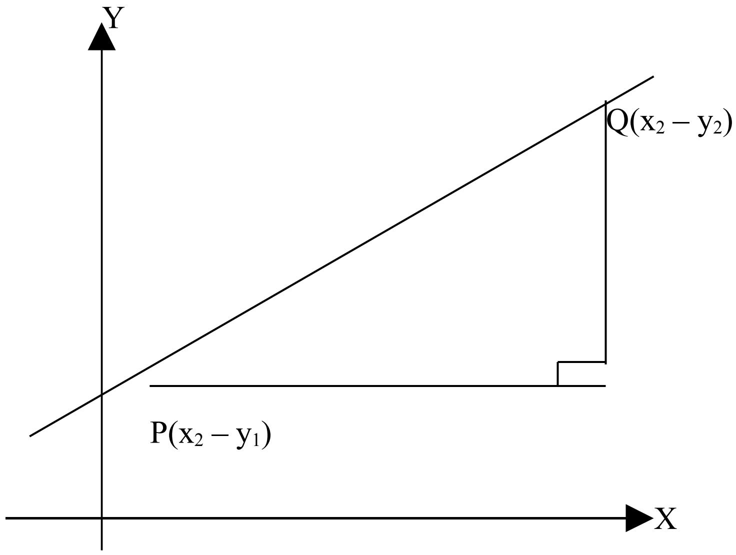


Fig. 26.

As you move from P to Q along line L, the increment $\Delta y = y_2$ is called the rise from P to Q. The increment $\Delta x = x_2 - x_1$ is called the run from P to Q. Since the line L is not vertical line then $\Delta x \neq 0$. The slope of the line L can now be defined as

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_2}{x_2 - x_1} = m$$

Remark

$$m = \frac{y_2 - y_1}{x_2 - x_1}, = \frac{y_1 - y_2}{x_1 - x_2}$$

Example

Let P (4, -2) and Q (-3 , 2) then the slope of the line joining P and Q is given as:

$$m = \frac{2-(-2)}{-3-4} = \frac{4}{-7} = \frac{-2-2}{4-(-3)} = \frac{-4}{7}$$

Slope of a Curve

You will now extend method of finding the slope of line to finding the slope or gradient of curve. To do this you start by finding the slope of a secant line through P and Q by two points on the curve C. See Fig. 27.

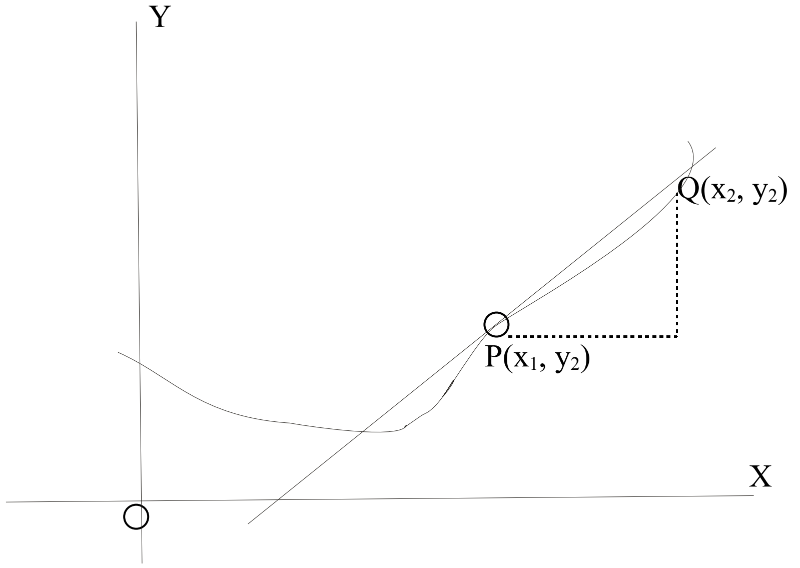


Fig. 27

In fig 27. the slope of the secant line PQ is given as

$$\text{Slope of PQ} \quad \frac{y_2 - y_{11}}{x_2 - x_{11}} = \frac{\Delta y}{\Delta x}$$

$$\Delta x, \ \Delta x \neq 0$$

The goal is to find the slope of the curve at point P to achieve this goal you hold P fixed and move Q along the curve towards P as you do so, the slope of the secant line PQ will vary. As Q moves closer and closer to P along the curve the slope of the secant line varies by slope of the secant line varies until it approaches a constant limiting value. This limiting value is what is called the slope of the curve at point P. See Fig 28.

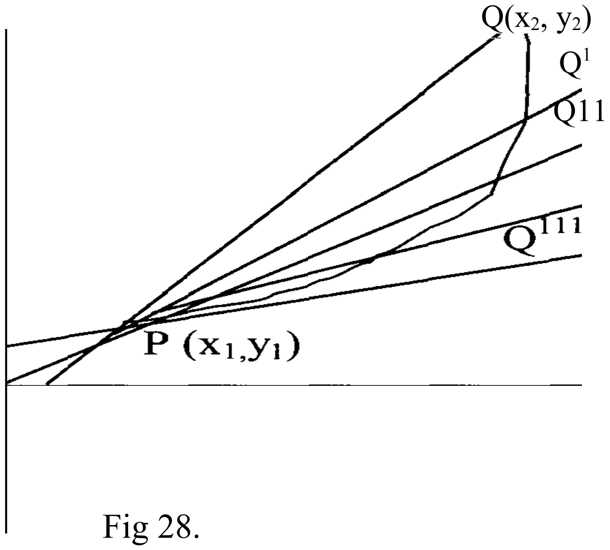


Fig 28.

In fig. 27 you will notice that as the secant line PQ moves along the curve towards the point P the slope of the secant line approaches or tends to the slope of the target line at the point P. Interesting the increment Δx tends 0 as Q --->P. Thus it will right to say that the limiting position to the tangent line TP. Therefore the slope of the secant line has a limiting value approximately equal to the slope of the tangent line.

Example

Find the slope of the curve

$Y = x^2$ at the point (x_1, y_l)

Solution

Since the point $P(x_l, y_l)$ lies on the curve then its co-ordinate must satisfy the equation.

$y= x^2$ i.e. $y_l = x_i^2$

Let $Q(x_2, x_2)$ be a second point on the curve $y = x^2$.

If $\Delta x = x_2 - x_l$ =:> $x = \Delta x_2 + x_l$
And $\Delta y = y_2 - y_l$ => $y_2 = \Delta y + y_l$

Since the point Q is on the curve limits coordinates must satisfy the equation

$y_2 = x^2$ that is $y_2 = x_2^2$

Hence $y + \Delta y = (x + \Delta x)^2$

$$= x_1^2 + 2x_1 \Delta x + (\Delta x)^2$$
$$= x\ 1^2 + 2x_1 \Delta x + (\Delta x)^2 - y_l$$
$$\therefore \quad \Delta y = x_1^2 + 2x_1 \Delta x + (\Delta x)^2 - x_1^2$$
$$= 2x^1 \Delta x + (\Delta x)^2 \; . \; . \; ... \; (A)$$

To find the slope of the line PQ you divide both sides of equation (A) by $\Delta x \neq 0$ and equation A becomes:

$$\frac{\Delta y}{\Delta x} = Qx + \neq x = \text{slope secant PQ}$$

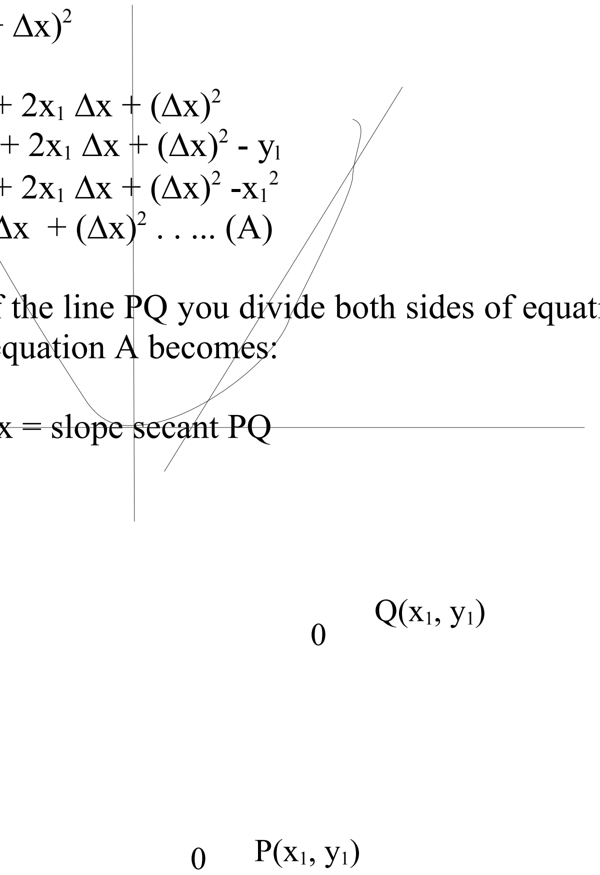


Fig. 29.

As Q gets closer to P along the curve Δx approaches zero and the slope of PQ gets closer to $2x_1$

i. e. $\frac{\Delta y}{\Delta x}$ 2x, as $\Delta x \text{ ---> } 0$

By definition this means that

$$\lim_{\Delta x \text{--}\sim 0} \frac{\Delta y}{\Delta x} = 2x,$$

The value of this limit is the slope of the tangent to the curve or slope of the curve at the point (x_l, y_l). Since the point $(x_{l,} \, y_1)$ is chosen arbitrarily (i.e. can be any point on the curve $y = x^2$) you could remove the subscript 1 from x_l and replace x_l by x. Then the slope of the tangent will be given as

$$m = 2x.$$

This is the value of the slope of the curve at any point P(x, y) on the curve.

SELF ASSESSMENT EXERCISE 1

a. Plot the points and find the slope of the line joining them:
(i) (1, 2) and (-3, - ½) (ii) (- 2, 3) (1, -2).

b. Find a formula for the slope of the line y = mx + b at any point on the line.

c. Use the example 2 above to find the slope of the curve $y = x^2$ at the following points (i) (-2, 2) (ii) (3, -2).

3.2 Definition of Derivative of a Function

In this section you will use the concept of the slope of a curve at a given point to defined the derivative of function at a given point

Let P (x_1, y_1) be a point on the curve where;

$x_1 = x_1 + \Delta x$

And $y_2 = \Delta y + y,$

In Fig 29 below $\Delta y = f(x + \Delta x) - f(x)$

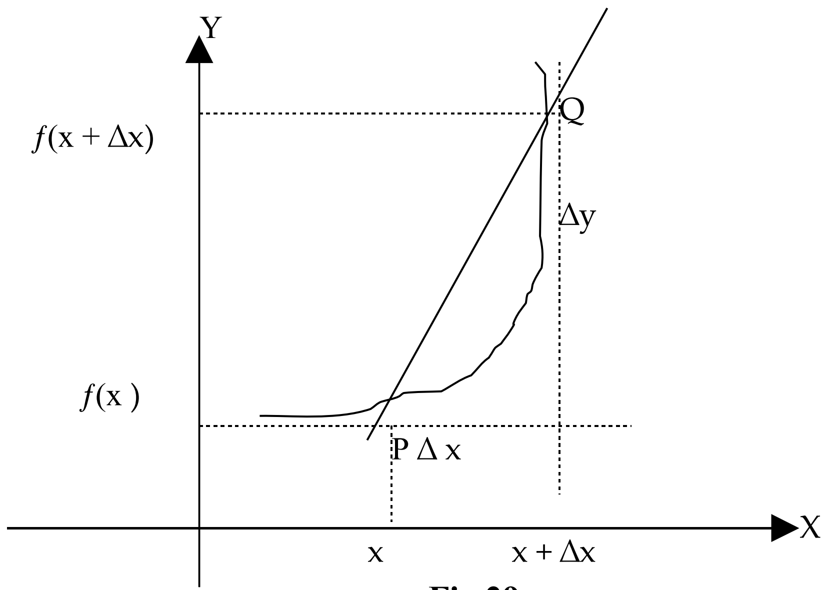


Fig 29.

Then the slope the PQ (i.e. secant) is:

$$M_S = \frac{\Delta y}{\Delta xx} = \frac{f(x + \Delta) - f(x)}{\Delta x} \; . \ldots \Delta$$

If the slope of the secant m_s approaches a constant value when Δx gets smaller and smaller, then this constant value is the limit of m_s Δx tends to zero. This limit is defined to be the slope of the tangent (m_l) to the curve at point p (x_1 y). The mathematics to describe the above could be symbolized as follows.

$$M_l = \lim_{Q \to P} m_s = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If the slope of the secant ms approaches a constant value when Δx gets smaller and smaller, then this constant value is the limit of m_s as Δ x tends to zero. This limit is defined to be the slope of the tangent (m_t) to the curve at point P(x, y) . The mathematics to describe the above could be symbolized as follows.

$$M_t = \lim_{Q \to P} m_s = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limit above if it exists is related to $f(x)$ at the point x by writing it as f (x) (read f prime of x). which is define as

$$f(x) = \lim_{\Delta x \to 30} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The above limit may exist only for some or all values of x in the domain of definition of the function $f(x)$. That is the limit may fail to exist for other values of x belonging to the domain whenever the limit exist for any point x, belonging to the domain, then the function $f(x)$ is said to be differentiable at the point x.

Remark

All the curves used for or mentioned so far are assumed to be smooth (i.e. continuous) In the previous unit it was mentioned

(and it is by definition) that a function that is continuous at a function that is a limit at that point. Since continuity implies existence of lim f(x). It will be shown later that a function that is differentiable at the point $x = x_o$, $x \to x_o$
is continuous at the point $x = x_o$.

Example

In the previous section 3.1 it was shown that the slope of the curve $y = x^2$ at a given point x is 2x. This implies that the function $y = x^2$ possess a derivative whose value at a point x is given to you that the function $f^1(x) = 2x$.

Example

Let us find $f^1(x)$ for the function given as $f(x) = 3x + 2$.

Solution:

This will be curried out in the steps/stages as follows:

Step 1:

For the given function $f(x) = 3x + 2$ you start by defining $f(x + \Delta x)$. This is given as $f(x + \Delta x) = 3(x + \Delta x) + 2$ (i.e. by direct substituting $x + \Delta x$ for x).

$\therefore \quad f(x + \Delta x) = 3x + \Delta x + 2.$

And $\quad f(x) = 3x + 2$

Step 2:

Subtract $f(x)$ from $f(x + \Delta x)$ i.e.

$$f(x + \Delta x) - f(x) - 3x + 3\Delta x + 2 - (3x - 2) = 3\Delta x$$

Step 3:

Divide result of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{3\Delta x}{\Delta x} = 3$$

Step 4;

Evaluate the limit as Δx ----> 0

$$f^1(x) = \lim \frac{f(x + \Delta x) - f(x)}{\Delta x} - s$$

The above process from step 1 to step 4 is referred to as "differentiating from first principle"

Example

Find the $f^1(x)$ if $f(x) = x^2 + 3x$

Solution:

Proceed as before.

Step 1: Write out $f(x + \Delta x)$ and $f(x)$

$$f(x + \Delta x) = (x + \Delta x)^2 + 3(x + \Delta x)$$
$$= x^2 + 2x\,\Delta x + (\Delta x) + 3x + \Delta x$$
$$f(x) = x + 3x$$

Step 2: Subtract $f(x)$ from $f(x + \Delta x)$ i.e.

$$f(x + \Delta x) - f(x) = x^2 + 2x\Delta x + (\Delta x)^2 + 3x + \Delta x$$

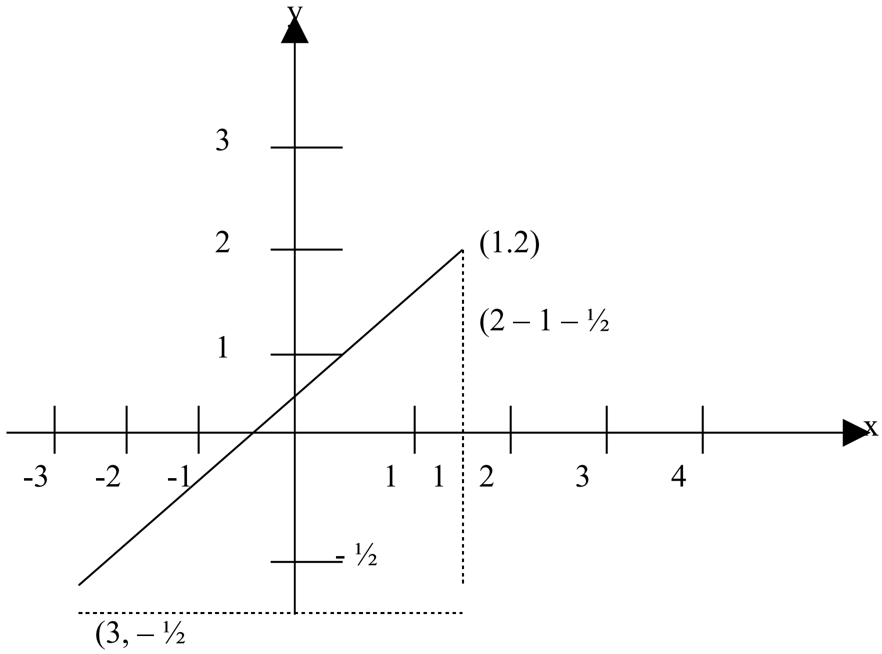
$$-x^2 - 3x$$
$$= 2x\,\Delta x + (\Delta x)^2 + 3\,\Delta x$$

Step 3: Divide the result in step 2 i. e

$$\frac{fx + \Delta x) - f(x)}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2 + 30x}{\Delta x}$$
$$=2x + 3 + \Delta x$$

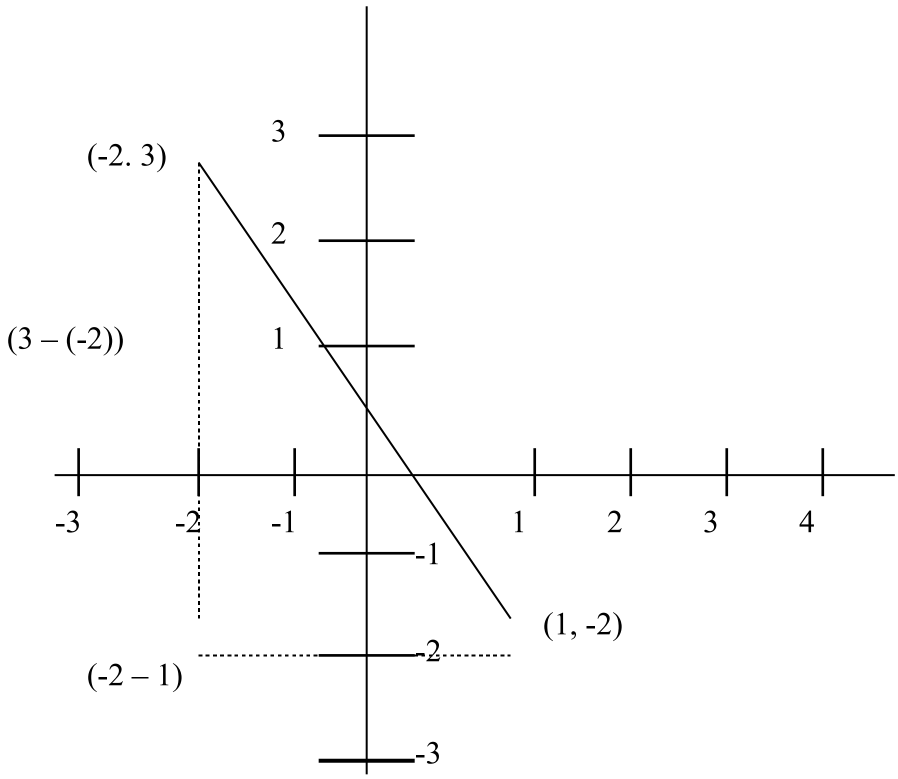
Step 4: Evaluate the limit of result of step 3 as Δx ---->0

$$f^1(x) = \lim_{\Delta x \to 0} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} 2x + 3 + \Delta x = 2x + 3$$



(1 – (-7))

(i) slope= $\frac{21/2}{4}$



(ii) slope = 2/-3

(b) $Y = mx + b.$
let $P(x_1, y_1)$ be a point on the line i.e.

$Y = mx + b.$

$Q(x_2, y_2)$ be another point. If

$\Delta X = x_2 - x_1$ and $\Delta y = y_2 - y_1$

$\Longrightarrow$ $x_2 = \Delta x + x_1$ and $y_2 = \Delta y + y,$

but $y_2 = mx_2 + b \Rightarrow \Delta y + y,=(\Delta x + x_1)\, m + b.$

$\Longrightarrow$ $\Delta y + y_1 = mx_1 + m\Delta\, x + b.$

$\Delta y = mx_1 + m\Delta\, x + b - y_1$

$= mx_1 + m\Delta x + b - (mx_1 + b)$ since $(y_1 = mx_1 + b.)$
$= m\Delta x.$

(to get the slope of the line you divide by $\Delta x \neq 0$) this gives.)

$\frac{\Delta y}{\Delta x} = m$ = slope of lin $y = mx + b.$

(c.) from solved example slope of the curve $y = x$ is gin as:

$m = 2x$ therefore slope at

(i) point (-2, 2) is $2. -2 = -4$
(ii) point (3, -2 is $2.3 = 6$

Example

If $f(x) = x^3 + x$ find $f^1\,(x0)$

Solution

Step 1: write out $f\,(x + \Delta x\,)$ and $f(x)$ i.e.

$f\,(x + \Delta x) = (x + \Delta x)^3 = (x + \Delta x)$

$= x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)$
$f(x\,) = x^3 + x$

Step 2: subtract $f(x)$ from $f\,(x + \Delta x\,)$ i.e.

$f(x + \Delta x) - f(x) = x^3 + 3x^2 \Delta x + 3x (\Delta x)^2 + (\Delta x)^3 + x + \Delta x - x^3 - x$

$f(x) = x^3 + \Delta x + 3x (\Delta x)^2 + (\Delta x)^3 + \Delta x$

Step 3: Divide result of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^2 + \Delta x}{\Delta xx}$$

$$= 3x^3 + 3x\Delta x + (\Delta x)^2 + 1$$

Step 4: Evaluate the limit of result in step 3 as Δx 0

$$f^1 (x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)\, \Delta x}{\Delta x}$$

Based on the discussion so far a formal definition of the derivative of function $f(x)$ at a point x can now be given

Definition: A function $f(x)$ is said to be differentiable (i.e. to have a derivation) at the point x If the limit:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x} = f^1 (x) \text{ exists}$$

Definition: A function $f(x)$ is said to be differentiable at the point x if for $\varepsilon > 0$ there is $\delta > 0$

Such that:

$$\text{If} \quad 0 < / \Delta x/ < \delta \text{ then } \left| \frac{f(x + \Delta x) - f(x)}{\Delta x -} - f^1 (x) \right| < \varepsilon .$$

Remark: The limit of the quotient $\frac{f(x + \Delta) - f(x)}{\Delta x}$ or $\frac{\Delta y}{\Delta x}$ as Δx-->0 have various notation.

Such as:

$f^1(x)$ (read f prime of x)

y^1 (read y prime)

(read y clot)

$\frac{dy}{dx}$ (read dee y dee x)

In this course the notation $f^{1}(x)$ and dy/dx will frequently be used.

3.3 Differentiation of Polynomial Function

In unit 2 you have studied function of this type

$f(x) = a_n x^n + a_{n-1}, \quad x^{n-1} +..... + a_1 x + a_0$

as a polynomial function. You will now investigate the derivation of such function.

1. Differentiation of a constant function

Let $y = k$

Then $f(x) = k$

$f(x + \Delta) = k$

$f(x + \Delta) - f(x) = k - k$

$\frac{f(x + \Delta) - f(x)}{\Delta x} = 0$

$\lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x} = 0$

2. Find the derivative of the function $f(x) = x^n$ at any point x.

Solution

Let $f(x) = x^n$

Step 1: $f(x + \Delta x) = (x = \Delta x)^n$

By binomial expansion:

$(x + \Delta x)^n = x^n + \binom{\prime}{I} mx^{n-1} \Delta x + \binom{\prime\prime}{I} x^{n-2} (\Delta x)^2 + ...+$

$\binom{n}{1}(\Delta x)^n$

(If you are not very familiar with binomial expansion you can read it up in any course in algebra suggested in the unit).

Step 2 Subtract $f(x)$ from $f(x = \Delta x)$

$f(x + \Delta x) - f(x) = x + nx^{n-I} \Delta x + (n - 1)^n x^{n-1} (\Delta x)^2$

$+. . .. +(\Delta x)^n - x^n$

$= nx^{n-I} \Delta x + (n-I)nx^{n-1} (\Delta x)^2 +...+ (\Delta x)^n$

Step 3: Divide result of step 2 by Δx

$\frac{f(x + \Delta) - f(x)}{\Delta x} = (n - 1)nx^{n-1} + (n - 1)nx^{n-1} \Delta x \ldots + (\Delta x)^{n-1}$

Step 4: Evaluate limit of result of step 3 as

$\lim_{\Delta x \to 0} \frac{f(x + \Delta) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} nx^{n-1} + (n-1)nx^{n-1} \Delta x +...+ (\Delta x)^{n-1}$

$= nx^{n-1}$

Example

$f(x) = k$ implies that $n = 1$ in that case $f(x) = x$ and $f^1(x) = 1$

Example

$\frac{dy}{dx}$ if (1) $y = x^8$ (ii) $y = x^5$

Solution

(i) $\frac{dy}{dx} = 8x^{8-1} = 8x^7$

(ii) $\frac{dy}{dx} = 5x^{5-1} = 5^4$

3. Differentiation of the function y = ku where a is a function of x

Solution

Let $f(x) = u$

Then $f(x + \Delta x) = u + \Delta u$
Since $y = ku$

Then $y + \Delta y = k(u + \Delta u)$

$\therefore \quad 4y = k(u + \Delta u) - y$
$= k\Delta u$
divide by Δx you obtain

$$\frac{\Delta y}{\Delta x} = k \frac{\Delta u}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta X} = \lim_{\Delta x \to 0} k \frac{\Delta u}{\Delta x} \; K \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \quad \frac{du}{dx} \quad \frac{dy}{dx}$$

$$\frac{dy}{dx} = k \frac{du}{dx}$$

Example: Find $\frac{dy}{dx}$ if $y = 5x^4$

Solution

$$\frac{dy}{dx} = \frac{d(5x^4)}{dx} = 5 \frac{d(x^u)}{dx} = 5.4x^3 = 20x^3$$

Generally If $y = kx^n$

Then $\frac{dy}{dx} = knx^{n-1}$

3.4 Solved Problems

1. Given the function (i) $y = 1/x$ (ü) $y = \sqrt{x}$.

Find the derivative by the limiting process i.e. from a suitable difference quotient $\frac{\Delta y}{\Delta x}$ and evaluate the limit as Δx tends to zero.

Solution 1

Step 1. write out $f(x + \Delta x)$ and $f(x)$ i.e.

$$f(x + \Delta x) = \frac{i}{x + \Delta x}$$

$$f(x) = 1/x$$

Step 2. Subtract $f(x)$ from $f(x + \Delta x)$

$$\text{i.e. } f(x + \Delta x) - f(x) = \frac{1}{x + \Delta x} - \frac{1}{x}$$

$$= \frac{x - (x + \Delta x)}{X (x + \Delta x)}$$

$$= \frac{-\Delta x}{x(x + \Delta x)}$$

Step 3: divide results of step 2 by Δx

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{- \Delta x}{x + \Delta x} \Big/ \Delta x = \frac{-1}{x(x + \Delta x)}$$

Step 4: Evaluate limit of result of step 3 as Δx--->0

$$\text{i.e. } \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{x(x + \Delta x)} = \frac{1}{x^2}$$

Solution 2

Step 1: write out $f(x + \Delta x)$ and $f(x)$

$$f(x + \Delta x) = \sqrt{x + \Delta x}$$

$$f(x) = \sqrt{x}$$

Step 2: Subtract $f(x)$ from $f(x + \Delta x)$

$$\text{i.e. } f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}$$

Step 3 : Divide the result of step 2 by Δx

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta xx}$$

let $\Delta x = h$

then $$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(\sqrt{x+h} - \sqrt{x}\ \ (x+h+\sqrt{x})}{(h\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{x+h-x}{h\sqrt{x+h} + \sqrt{x}} = \frac{h}{h(\sqrt{x+h} = \sqrt{x}}$$

$$= \frac{1}{h\left(\sqrt{x+h} = \sqrt{x}\right)}$$

Step 4: Evaluate the limits of result of step 3 as h ->0

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h(\sqrt{x+h} = \sqrt{x})} - \frac{1}{2\sqrt{x}}$$

2. show that a function $f(x)$ that is differentiable at the point $x = x_o$ is continuous at that point.

Solution

Since the function $f(x)$ is differentiable at $x = x_o$ then $f^1(x_o)$ exists.

i.e. $f^1(x_o) = \lim_{h \to 0} f(x_o + h - f(x_o)$ (note $h = \Delta x$

but $$f(x_o + h) - f(x_o) = \frac{f(x_o + h) - f(x_o)}{h} .h \quad - I$$

$\therefore$ $$\lim_{h \to 0} f(x_o + h) - f(x_o) = \lim_{h \to 0} \frac{f(x_o + h) - f(x_o)}{h} .h$$

$$= \lim \left(\frac{f(x_o + h) - f(x_o)}{h}\right) .h \quad \lim h$$

h --->0 h -->0

$= 0$ (since $\lim_{h \to 0} h = 0$) -II

$$\lim_{h \to 0} \left(f(x_o + h) - f(x_o) \right) = \lim_{h \to 0} f(x_o + h) - \lim_{h \to 0} f(x_o)$$

$$= \lim_{h \to 0} f(x_o + h) - f(x_o) \qquad \text{-III}$$

==> then from equations II and III you get $\lim_{h \to 0} f(x_o + h) - f(x_o) = 0$

$$\Longrightarrow \quad \lim_{h \to 0} f(x_o + h) - f(x_o) \qquad \text{-IV}$$

Note that $x - x_0 = h$ then $h \to 0 \Longrightarrow x \to 0$

$$\therefore \quad \lim_{h \to 0} f(x_o + h) = \lim_{x \to 0} f(x_o + h) \ \lim_{x \to 0} f(x_o) = 0$$

Therefore equation IV becomes

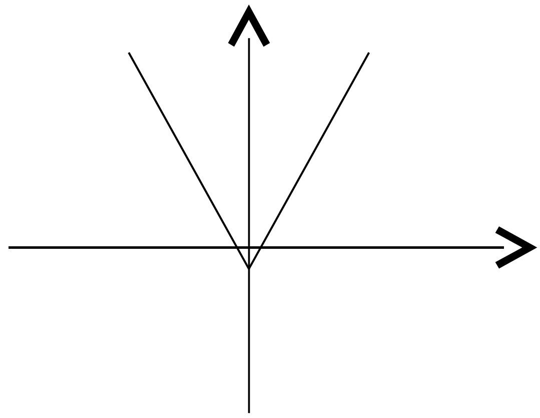
$$\lim f(x) = f(x_0)$$

Which shows that $f(x)$ is continuous at the point $x = x >$ The converse is not true. There exist functions that are continuous at given points but not differentiable at those points.

Example

Is the absolute value function, which you are very familiar with (see unit 1)

This function $f(x) = /x/$ is continuous at $x = 0$ but not continuous at $x = 0$ see fig 5.6.



The limit $f^1(0)$ does not exist

i.e. $f^1(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = -1$

$f^1(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$

Since $\lim_{h \to 0^+} f(x) \neq \lim_{h \to 0^-} f(x)$

then $f^1(x)$ at x = 0 does not exist

4.0 CONCLUSION

You have studied how to find the slope a line you have extended this to finding the slope of a curve by evaluating the limit of the slope tangent line at a given point on the curve. You have related the value of this limit with the slope of the curve at any point. You have defined this limit whenever it exist as the derivative of the function at the point x. You have used these definitions to find the derivative of certain functions such as:

$y = k, y = x, y = x^2, y = 1/x, y = \sqrt{x}$, and $y = kx^n$

by the method of limiting process. You are now posed to use materials studied here to find derivatives of sum, product quotients of functions in the next unit. Make sure you do all your assignment correctly. You will definitely need them, because you will refer to them directly or indirectly in this remaining part of this course.

5.0 SUMMARY

The principle of using limiting processes to derive the derivatives of a function is called the first principle. The principle must be well understood because it is very useful in Advanced Analysis which you will study as you progress in the study of Mathematics.

6.0 TUTOR-MARKED ASSIGNMENT

1) Differentiate from th the derivative e first principle the following functions:
(a) $f(x) = 3x^4 + 56x^3$
(b) $f(x) = e^{2x} + 5x + 6$

2) If $y = x^n$, show that the derivative of y is nx^{n-1} using the first principle.

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 RULES FOR DIFFERENTIATION I

CONTENT

1.0 INTRODUCTION

In this unit you will learn a few simple rules for differentiation of all sorts of functions constructed from the ones you are already familiar with so far you have observed that the study of differential calculus started with the study of behaviour of functions and their limits at different points. In the previous unit you studied that functions that are differentiable are continuous. This implies that rules governing the results on continuous functions in respect of algebra of continuous functions can easily be extended to differentiable functions.

In this unit you will formulate rules based on theorems on limits and theorems on continuous functions you will be expected to apply the rules formulated in this unit to differentiate sum and product of polynomials function. You will then use the same rules throughout the remaining part of this course. Until the following study style will be adopted; first the rules will be stated with example. Then the rules will be justified.

2.0 OBJECTIVES

After studying this unit you should be able to correctly.

i) derive the following rules for differentiation
   a. Sum rule
   b. Difference rule
   c. Product rule

ii) differentiate all types of polynomial functions.

3.0 MAIN CONTENT

3.1 Differentiation of Sum of Functions

Sum Rule

The derivative of the sum of finite number of functions is the sum of their individual derivatives.

i.e.

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Solution

The proof for the case of the derivative of the sum of two functions u and v will be given first after which the prove of a finite sum will then follow.

Example 1

Let $u = x^2$, $v = 2/x$ find $d/dx\ (u+v)$

$$\frac{d}{dx}(x^2 + 2/x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2/x) = 2x - \frac{2}{x^2}$$

Now let $y = u + v$ where $u = u(x)$ and $v = v(x)$ both $u(x)$ and $v(x)$ are differentiable

Let Δx be an increment in x which will give a corresponding increments in y, and v given as Δy, Δu and Δv respectively.

Then $\Delta y = (u + \Delta u) + (v + \Delta v)$

$$\Delta y = (u + \Delta u) + (v + \Delta v) - (u + v)$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \quad \text{(divide by } \Delta x\text{)}$$

$$\lim \frac{\Delta y}{\Delta x} = \lim \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right) = \lim \left(\frac{\Delta u}{\Delta x} \right) + \lim \frac{\Delta v}{\Delta x}$$

$$\frac{\Delta y}{} = \frac{\Delta u}{} + \frac{\Delta v}{}$$

$$\Delta x \quad \Delta x \quad \Delta x$$

$$\therefore \frac{d(u+v)}{dx} = \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}$$ which is the required result.

If $y = u + v + w$

Then $y = z$, where $z = v + w$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{dx} \quad \text{but} \quad \frac{dz}{dx} = \frac{\Delta v}{\Delta x} + \frac{dw}{dx}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{dw}{dx}$$

This can be extended to a finite sum of differentiable functions i.e.

if $y = u_1 + u_2 + \dots\dots + u_n$ then

$$\frac{d(u_1 + u_2 + .. + u)}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \dots\dots + \frac{du_2}{dx}$$

Example 2

Find $\frac{dy}{dx}$ if $y = 3x^3 + 2x^2 + x + 1$

Solution

First find the derivative of each of the term and then add the result.

$$\frac{dy}{dx} = \frac{d(3x^3)}{dx} + \frac{d(2x^2)}{dx} + \frac{d(x)}{dx} + \frac{d(1)}{dx}$$

$= 3.3x^2 + 2.2x + 1 + 0$
$= ax^2 + 4x + 1$

Example 3

Find the dy/dx if.

(i) $y = x^3 + 1/x$ (ü) $y = 1/3\ x^3 + ½\ x^2 + 2x$

(iii) $y = 5x^7 + 4x^5 + 10x$ (iv) $y = x^4 + 5x^2 + 1$

Solutions

(i) $y = x^3 + 1/x$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^3) + \frac{d}{dy}(1/x) = 3x^2 - \frac{1}{x^2}$$

(ii) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x$

$$\frac{dy}{dx} = \frac{d}{dx}(1/3x^3) + \frac{d}{dx}(½\,x^2) + \frac{d}{dx}(2x)$$

$= 3.\ 1/3x^2 + 2.\ ½x + 2$

$= x^2 + x + 2$

(iii) $y = 5x^7 + 4x^5 + 10x$

$$\frac{dy}{dx} - \frac{d(4x)}{dx} + \frac{d(5x^2)}{dx} + \frac{dy(1)}{dx}$$

$= 4x^3 + 2.5x + 0$

$= 4x^3 + 10x$

In unit 5, it was shown that

$\frac{d(ku)}{dx} = k\frac{du}{dx}$ where k is a constant and u is a differentiable function.

3.2 Differentiation of Difference of Functions

The above will be used to establish the difference rule.

Difference Rule: The derivative of the difference of a finite number of functions is the difference of the individual derivatives.

$$\boxed{\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx.}}$$

Example 4

Find dy/dx if $y = x^2 (x - 1)$

Let $u = x^2$ and $v = x - 1$

Then $uv = x^2(x - 1) = x^2 - x$

$$\frac{d(uv)}{dx} = \frac{d(x^3 - x)}{dx} = \frac{d(x^3)}{dx} = \frac{d(x)}{dx}$$

In unit 2 you studied that the product of two functions will result to another function. For some complicated functions it might not be very easy to carry out a straight forward multiplication or find the product before differentiable. Therefore arises the need to find a rule that will side track finding of the product of functions before differentiation. That rule is what you will derive now.

Product Rule

Suppose y = uv is the product of two differentiable functions of x then the derivative is given as:

$$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

Proof: Let y = uv where u and v are differentiable functions of x. Let Δx be an increment in x which will result in a corresponding increments in y, a and v given as Δy, Δu and Δx respectively.

Then $\Delta y + y = (u + \Delta u)(v + \Delta v)$

$= uv + v\,\Delta u + u\,\Delta v + \Delta u\,\Delta v$

Subtract y from Δy + y you get

$\Delta y = u\,v + v\Delta u = u\,\Delta v + \Delta u\,\Delta v - u\,v$
$= v\Delta u + u\Delta v + \Delta u\Delta v$

Next
divide by $\Delta x \neq 0$

$$\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\quad}{\Delta x}\frac{\Delta v}{\Delta x}\ \Delta v\ \Delta v$$

Evaluate the limit of dy/dx as $\Delta x \rightarrow 0$ you get

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left(v\frac{\Delta u}{\Delta x} + u\frac{\Delta x}{\Delta x}\ \Delta v\ \frac{\Delta v}{\Delta x} \right)$$

$= v\ \text{llim}\ \underline{\Delta u} + u \lim \underline{\Delta v} + \lim \Delta u \lim \underline{\Delta v}$

$\Delta x \qquad \Delta x \qquad \Delta x$

$\Delta x \to 0 \qquad \Delta x \to 0 \qquad \Delta x \to 0$

The $\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \Delta x = \frac{du}{dx} . 0 = 0$

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} + 0 . \frac{dv}{dx}$$

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

This can now be extended to the case of product of a finite number of functions:

i.e. if $y = u_1 u_2 u_3 ... u_n$ then

$$\frac{d(u_1 u_2 u_3 .... u_n)}{dx} = \frac{du_1}{dx}(u_2 .... u_n) + u_1 \frac{du_2}{dx}(u_3 ..... u_n)$$

$$+ .... + (u_1 ..... u_{n-1}) \frac{du_n}{dx}$$

Example 5

Differentiate the following functions

(i) $y = (x^2 - 1)(x + 3)$

(ii) $y = (x^3 - 2x)(x^2 - 1)$

(iii) $y = (x^2 + 1)(2x^2 - 1)(x - 1)$

(iv) $y = (x^2 - 1)(3x - 1)(x^4 - 1)(3x^2 - 1)$

Solutions

(i) $y = (x^2 - 1)(x + 3)$

let $u = (x^2 - 1)$ and $v = (x + 3)$

then $\frac{dy}{dx} = u \frac{du}{dx} + v \frac{dv}{dx} = (x^2 - 1) \frac{d(x + 3)}{dx} + (x + 3) \frac{d(x^2 - 1)}{dx}$

$= (x^2 - 1)\,1 + (x + 3)\,(2x)$
$= x^2 - 1 + 2x^2 + 6x$
$= 3x^2 + 6x - 1$

(ii) $y = (x^3 - 2x)(x^2 - 1)$

Let $u = (x^3 - 2x)$ $v + (x^2 - 1)$

$$\frac{dy}{dx} = u\frac{du}{dx} + v\frac{dv}{dx} = (x^3 - 2x)\frac{d(x^2 - 1)}{dx} + (x^2 - 1)\frac{d(x - 2x)}{dx}$$

$= (x - 2x)(2x) + (x^2 - 1)(3x^3 - 2x)$

$= 2x^4 - 4x^2 + 3x^4 - 3x^2 - 2x^2 + 2$

$= 5x^4 - 9x^2 + 2.$

(iii) $y = (x^2 + 1)(2x^2 - 1)(x-1)$

$y = u\,v\,w$

$$\frac{dy}{dx} = u\,v\frac{dw}{dx} + u\,w\frac{dv}{dx} + v\,w\frac{du}{dx}$$

let $u = (x^2 + 1)$ $v = 2x^2 - 1$ $w = x - 1$

$$\frac{dy}{dx} \quad \frac{d\,(uvw)}{dx} = \frac{du}{dx}(v\,w) + u\frac{dv}{dx}w + (u\,v)\frac{dw}{dx}$$

$= 2x(2x^2 - 1)(x^2 - 1) + (x + 1)\,4x\,(x - 1) + (x^2 + 1)(2x^2 - 1)\,1$

$= 10x^4 - 8x^3 + 3x^2 - 2x - 1$

(iv) $y = (x^2 - 1)(3x - 1)(x^4 - 1)(3x^2 - 1)$

Let $u = (x^2 - 1)$ $v = (3x - 1)$, $w = (x^4 - 1)$ $z = (3x^2 –$

1) $$\frac{dy}{dx} \quad \frac{d(uvwz)}{dx} = \frac{du}{dx}v\,w\,z + u\frac{dv}{dx}w\,z = uv\frac{dw}{dx}\,z\,u\,v\,w\frac{dz}{dx}$$

$= 2x\,(3x – 1)(x^4 – 1)(3x^2 – 1)$

$+ (x^2 – 1)(3)(x^4 – 1)(3x^2 – 1)$

$+ (x^2 – 1)(3x - 1)(4x^3)(3x^2 – 1)$

$+ (x^2 - 1)(3x - 1)(x^4 - 1)\,6x$

Example 6

More solved samples find dy/dx if

(i) $y = \left(\frac{1}{x}\right)\left(\frac{1}{x-1}\right)(x^2 - 1)$

(ii) $y = (2x - 1)\left(\frac{1}{x-1}\right)(\sqrt{x})$

(iii) $y = \left(\frac{1}{x}\right)(\sqrt{x})(x^2 - 1)$

Solutions

(i) $y = \left(\frac{1}{x}\right)\left(\frac{1}{x-1}\right)(x^2 - 1)$

let $u = \left(\frac{1}{x}\right)$, $v = \left(\frac{1}{x-1}\right)$ $w = (x^2 - 1)$

$$\therefore \frac{dy}{dx} = \frac{-1}{x^2} \cdot \frac{1}{(x-1)}\left(x^2 - 1\right) - \frac{1}{\left(x(x-1)^2\right)} \cdot \left(x^2 - 1\right) + \frac{2}{(x-1)}$$

(ii) $y = (2x - 1)\left(\frac{1}{x-1}\right)(\sqrt{x})$

$u = (2x - 1)$, $v = \left(\frac{1}{x-1}\right)$, $w = (\sqrt{x})$

$$\therefore \frac{dy}{dx} = 2.\ \frac{1}{x+1}\sqrt{x} - \frac{2x -}{(x+1)}\sqrt{x} + \frac{1}{2}\frac{(2x-1)}{(x+1)\sqrt{x}}$$

$$= \frac{x\sqrt{x}}{(x+1)} - \frac{2x^{3/2}}{(x+1)^2} + \frac{1}{(x+1)^2}\sqrt{x} - \frac{1}{2(x+1)\sqrt{2}}$$

(iii) $y = \left(\frac{1}{x}\right)(\sqrt{x})(x^2 - 1)$

$y = \frac{1}{\sqrt{x}}(x^2 + 1)$

Let $u = \frac{1}{\sqrt{x}} = x^{-½}$ $v = (x^2 + 1)$

$$\frac{dy}{dx} = \frac{-1}{2x^{3/2}}(x^2 + 1) + 2(\sqrt{x})$$

$$= \frac{3}{2}\sqrt{x} \quad \frac{1}{2x^{3/2}}$$

4.0 CONCLUSION

In these unit you have acquired necessary techniques or methods of differentiation. These techniques are governed by rules which you have just studied. You will be required to apply these rules when dealing with differentiation of transcendental functions. The rules of differentiation you have studied as follows: sum rule, Difference rule, Product rule, Quotient rule and Composite rule. You have used the rules in the examples given in the unit some of these examples with little modification or changes. Endeavor to solve all your exercises. It builds confidence in you.

5.0 SUMMARY

In this unit you have studied the following rules for differentiation.

1) Sum rule: $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

2) Difference rule: $\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

3) Product rule: $\frac{d(uv)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

In this unit you have studied the following

(i) the slope of a line i.e. $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

(ii) the slope of a curve at a given point i.e. $\lim_{\Delta x \to 0} \Delta y / \Delta x$

(iii) the definition of derivative of a function f (x) at a point x i.e.
$f(x) =$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(iv) how to differentiate a polynomial function using the limiting process i.e. evaluating the limit of a suitable quotient such as $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$

(v) that not all continuous functions are differentiable e.g. $f(x) = /x/$ is continuous at x = 0 but not differentiable at x = 0.

6.0 TUTOR-MARKED ASSIGNMENTS

Differentiate with respect to x

(1) $4x^{10}$

(2) $3x^7$

(3) $\frac{1}{4} x^{7/2}$

(4) $2x^{-2}$

(5) $4x^4 - 8x^3 + 2x$

(6) $3\sqrt{7}x$

(7) $(x + 1) (x^2 - 1)$

(8) $(x +1) (x+2) (x +3)$

(9) $(x^2 + 1)(4x^2 - 1)$

(10) $(x^3 - 1 (x)$

(11) drive the product rule.

$$\frac{d}{dx} (uv) = v \frac{du}{dx} + u \frac{dv}{dx}$$

(12) show that $\frac{d}{dx}(u+v) = v \frac{du}{dx} + u \frac{dv}{dx}$

(13) $\frac{d}{dx}(x^2 - 1) (2x + 1) =$

(14) $\frac{d}{dx}((3x)^2 - 1) (x - 1) =$

(15) $\frac{d}{dx}(3x^{½} - 1)(x^2 - 1) =$

7.0 REFERENCES/FURTHER READINGS

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UNIT 4 RULES FOR DIFFERENTIATION II

CONTENTS

1.0 INTRODUCTION

In the previous unit you have studied rules of differentiation namely

1) Sum rule
2) Difference rule
3) Product rule

In this unit you will also extend the properties of differentiability to include quotient rule and differentiation of a composite function. This unit will follow the same concept used in the previous unit. The introductions in unit 7 will fit perfectly as an introduction to this unit.

2.0 OBJECTIVES

After studying this unit you should be able to:

(1) Derive the formula for quotient rule from first principle using the limiting process.
(2) Derive the Chain rule.
(3) Differentiates all types of rational functions with denomination of this type
 $a\,x^n + a_l\,xn^{-1} + ... + \quad c/n$

3.0 MAIN CONTENT

3.1 Differentiation of Quotient of Functions

Quotient Rule. The derivation of the quotient of two functions is given as:

i.e.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad v \neq 0$$

Example 1

Find dy/dx if $y = \frac{x^2 - 2}{x - 1}$

Solution

Let $u = x^2 - 2, \quad v = x - 1$

$$\frac{d}{dx} = \frac{(x-1)2x - (x^2 - 2)}{(x-1)^2}$$

$$= \frac{2x^2}{(x-1)} - \frac{x^2 - 2}{(x-1)^2}$$

Quotient Rule

Suppose y = u/v $\qquad v \neq 0$

Is the quotient of two differentiable functions of x then the derivative is given as

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad v \neq 0$$

Proof: Let $y = {}^{u}\!/_{v}$, where u(x) and v(x) are both differentiable at a domain where

$v \neq 0$. Let Δx be an increment in x in the given domain. Then Δy, Δu and Δv are corresponding increments in y, u and v respectively.

Then

1. $$\Delta y + y = \frac{(u + \Delta u)}{(v + \Delta u)} \text{ where } \Delta v + v \neq$$

subtracting y from $\Delta y + y$ you get

$$\Delta y = \frac{u + \Delta u}{v + \Delta u} - \frac{u}{v}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)}$$

2. $$\frac{v\Delta u - u\Delta v}{v(v + \Delta V)}$$

division by $\Delta x \neq 0$ yields

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad v \neq 0$$

Taking the limit as Δx tends to zero you obtain

3. $$\lim d = \lim_{\Delta x \to 0} \left(\frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta x)} \right)$$

$$\frac{\lim\limits_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} - \lim\limits_{\Delta x \to 0} u\frac{\Delta v}{\Delta x}}{\lim\limits_{\Delta x \to 0} v(v + \Delta v)}$$

As $\Delta x \to 0$ $\lim (v + \Delta v) = 1\text{im } v \lim (v + \Delta v)$

$= \lim v (\lim v + \Delta v)$

$= v(v+0)$

Recall that $\lim\limits_{\Delta x \to 0} \Delta v = \text{Iim}_{\Delta x \to 0} \Delta v \quad \Delta x = dv \quad .0$

$$\lim_{\Delta x \to 0} v(v + \Delta v) = V^2$$

Hence from (3) you get:

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{V^2}$$

$$\frac{dy}{dx} = \lim \frac{\Delta y}{\Delta x}$$

Example 2

If $y = \frac{x^2 + 2}{x^3}$ Find $\frac{dy}{dx}$

Solution

Let $u=x^2 + 2,\ v = x^3$

Then $\frac{dy}{dx} = \frac{(x^3)(2x) - (x^2 + 2)(3x^2)}{(x^3)^2}$

$$\frac{2}{x^2} - \frac{3(x^2 + 2)}{x^4}$$

Example 3

Differentiate the following functions if

(i) $y = \frac{x3 + 1}{x - 1}$

(ii) $y = \frac{x + 1}{\sqrt{x}}$

(üi) $y = \frac{x^2 + 1}{\sqrt{x + 1}}$

(iv) $y = \frac{3x^2 + 2x + 1}{x^2 - 1}$

Solution

(1) $y = \frac{x^3 + 1}{x - 1}$, $y = x^3 + 1$, $v = x - 1$

$$\frac{dy}{dx} \quad \frac{(x-1)3x^2 - x^3 + 1}{(x-1)^2} = \frac{3x^2}{x-1} - \frac{(x^3+1)}{(x-1)^2}$$

(ii) $y = \frac{x+1}{\sqrt{x}} \qquad u = x + 1, \; v = \sqrt{x}$

$$\frac{dy}{dx} = \frac{x - \frac{1}{2}(x+1)(x^{-1/2})}{x}$$

$$= \frac{1}{\sqrt{x}} - \frac{1}{2}$$

$$= \frac{x-1}{2x^{3/2}}.$$

(üi) $y = \frac{x^2+1}{x+1} \quad u = 3x^2 + 2x + 1, \qquad v = x^2 - 1$

$$\frac{dy}{dx} = \frac{(6x+2)(x^2-1) - 2x(3x^{22} + 2x + 1)}{(x^2-1)^2}$$

$$\frac{6x+2}{x^2-1} - \frac{2x(3x^2+2x+1}{(x^2-1)^2}$$

Let $u = g(x)$ and $y = u^n$ then

$$\frac{dy}{dx} = \frac{du''}{dx} = nu^{n-1}\frac{du}{dx}$$

Recall that in unit 5 you proved that $\frac{dy}{dx} = mx^{n-1}$ when u = x. You will use the same principle to show the above, i.e. Let $y = u^n$ where u is a differentiable function of x and $n \in N$.

Therefore Δy and Δu are corresponding increment in y and u respectively.

Then $y + \Delta y = (u + \Delta u)^n$
By trinomial equation you get that:

$y + \Delta y = u^n + nu^{n-1}\,\Delta u$ + (terms in u and higher powers of (Δu) .

From the above subtract y from $y + \Delta y$ You get:

$$\frac{dy}{dx} = nu^{n-I}\frac{du}{dx} + 0$$

(iv) $y = (x+1)^4 (x - 1)^3 (x^2 + 1)^2$

Let $u = (x + 1)^4$, $v = (x- 1)^3$, $w = (x^2 + 1)^2$

Then $\frac{du}{dx} = 4(x+1)^3$ $\frac{dv}{dx} = 3(x- 1)^2$ $\frac{dw}{dx} = 4x(x^2 + 1)$

$\frac{dy}{dx} = 4(x\ \ 1)^3 (x - 1)^3 (x^2 + 1)^2\ 3(x -1)^4 (x + 1)^2 (x - 1)^2$

$+ 4(x + 1)^4 (x - 1)^3 (x^2 + 1)$

If n is negative instead of $y = u^n$ you have

$$Y = u^{-n} = \frac{1}{u^n}$$

Using the quotient rule you will get that

$$\frac{dy}{dx} = \frac{d\left(\frac{1}{U^n}\right)}{dx} \quad \frac{u^n \frac{d(l)}{dx} - 1\frac{d(u^n)}{dx}}{(u^n)^2}$$

$$= \frac{0 - nu^{n-1}\frac{du}{dx}}{u^{2n}}$$

$$\frac{dy}{dx} = (-nu^{n-1}\ u^{-2n})\frac{du}{dx}$$

Given that:

(1) $y = x^4 + \frac{1}{x^4}$ (2) $Y = x^3(x - 1)^{-2}$

(3) $y = 2x(4x^2 - 3)^{-3}$ Find dy/dx in each case.

Solutions

(1) $y = x^4 + \frac{1}{x^4} = x^4+X^{-4}$

$\frac{dy}{dx} = 4x - 4/x^5$

(2) $Y = x^3(x - 1)^{-2}$

let $u = x^3$ $v = (x - 1)^{-2}$

$\frac{dy}{dx} = 3x^2(x-1)^{-2} – 2x^3 (x - 1)^3$

$\frac{3x^2}{(x - 2)^2} - \frac{2x^3}{(x - 2)^3}$

(3) $y = 2x(4x^2 - 3)^{-3}$

let $u = 2x$ $v = (4x^2 - 3)^{-3}$

$\frac{dy}{dx} = \frac{2}{(4x^2 - 3)^3} - \frac{48x^2}{(4x^2 - 3)^4}$

3.2 The Chain Rule for Differentiation

You will now learn how to differentiate composite functions, which you studied in unit 2. i.e. if f (x) and g(x) are functions defined in the same domain then f og =f(g (x))

Remark: Now suppose that u(x) is a differentiable functions of x, then u (x) changes du times as fast as x does. If f changes n times as fast as g and g changes n times as dx fast as w, then f changes mn times as fast as w.

Suppose a function y = f (g(x)) where f and g are both differentiable functions of x.

Then $\frac{dy}{dx} = \frac{d}{dx}[f(g(x))]$

This implies that by the above remark f (g(x)) changes $f^1(g(x))$ times as fast as g(x) And g(x) changes $g^1(x)$ x $g^1(x)$ times as fast as x.

$f^1 (g(x) g^1 (x)$ is the derivative of f (g(x) with respect to x . which means that

$d [f (g(x))] = f^1 (g(x) g^1 (x)$.

The above is called the chain rule for the differentiation of a composite function (i.e, function of function). In the previous section you have studied that if u is a differentiable function of x and $y = u^n$ then $\frac{dy}{dx} = nun^{-1} \frac{du}{dx}$ for n εQ

The coefficient of the term $\frac{dy}{dx}$ of the above equation can be written as

$$nu^{n-1} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du}, \quad \frac{du}{dx}$$

To prove the chain rule you assume that $y = f(u)$ is differentiable at point $u = u_0$, then an increment Δu will produce a corresponding increment Δy is y such that:

$$\Delta y = f^1(u_0)\, \Delta u + \varepsilon 1\, \Delta u \quad \_\_\_\_\_\_\_ \quad (A)$$

If $u = g(x)$ is a differentiable at a point $x = x_0$, then an increment Δx produces a corresponding increment Δy such that

$$\Delta u = g^1(x_0)\, \Delta x + \varepsilon_2\, \Delta x \quad \_\_\_\_\_\_\_\_ \quad (B)$$

If $\varepsilon_1 \to 0$ then $\Delta u \to 0$

And if $\varepsilon_2 \to 0$ then $\Delta x \to 40$

Combining equations (A) and (B) you get:

$$\Delta y = (f^1(u_0)\, \varepsilon_1)\, \Delta x\, (g^1(x_0) + \varepsilon 2) \quad \_\_\_\_\_\_\_\_\_\_ \quad (C)$$

Dividing equation (C) by $\Delta x \neq 0$ you get

$$f^1(u_0)\, g^1(x_0) + f^1(u_0)\, \varepsilon_2 + g^1(x_o)\, \varepsilon_1 + \varepsilon_1\, \varepsilon_2 \quad \_\_\_\_\_\_\_ \quad (D)$$

Taking limits on both side D you get

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to O} f^1(u_0)\, g^1(x_0) + f^1(u_0)\, \varepsilon_2 + g^1(x_o)\, \varepsilon_1 + \varepsilon_1\, \varepsilon_2 \quad (E)$$

(E)

But as $\Delta x \to 0 \Rightarrow \varepsilon_2 \to 0 \Rightarrow \Delta u \to 0 \Rightarrow \varepsilon_1 \to 0$

Then Equation (E) becomes

$$\left(\frac{dy}{dx}\right)_{x^0} = f^1(u_0)\, g^1(x_0)$$

which is $\left(\frac{dy}{du}\right)_{u_o} \left(\frac{du}{dx}\right)_{x_o}$

since x_0 and u_0 are chosen arbitrarily then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The above could be written as

$$(fog)^1\,(x) = f^1(g(x))\, g^1\,(x)$$

or $(fog)^1 = f^1,(g(x))\, g^1\,(x).$

Solution

Let $u = \frac{1}{x - 1}$

$$y = u^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = 4u^3, \qquad \frac{du}{dx} = \frac{-1}{(x - 1)^2}$$

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\; 4\left(\frac{1}{x-1}\right)^3 \frac{-1}{(x - 1)2}$

Example 4

Differentiate the following function with respect to x

(1) $y = (1- 3x)^{-1}$ (2) $y = \left(\frac{2x}{x^2 - 1}\right)^3$

(3) $y = \left(\frac{1}{x - 1} - \frac{1}{x + 1}\right)^3$ (4) $y = \left(\frac{x^2 + 2}{x^2 - 1}\right)^4$

(5) $y = (2x^4 + x^2 - 1)^6$ (6) $y = \left(\frac{x^3}{2} - \frac{x^{-2}}{2} - \frac{x^{-1}}{1}\right)$

Solutions

(1) $Y = (1-3x)^{-1}$

$y = u^{-1}$ $u = 1 - 3x$

$\frac{dy}{du} = -u^{-2}$ $\frac{du}{dx} = -3.$

$\frac{dy}{dx} = -u^{-2}$ $-3 = 3u^{-2} = \frac{3}{(1-3x)^2}$

(2) $y = \left(\frac{2x}{x^2 - 1}\right)^3$

Let $y = u^3$, $u = \frac{2x}{x^2 - 1}$, $\frac{dy}{du} = 3u^2$

$\frac{du}{dx} = \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3\left(\frac{2(x^2 + 1)}{(x^2 - 1)}\right)^2 \cdot \left(\frac{-2(x2 + 1)}{(x^2 - 1)^2}\right)$

$\frac{-24x^2(x^2 + 1)}{(x^2 - 1)^4}$

(3) $y = \left(\frac{1}{x - 1} - \frac{1}{x - 1}\right)$

$u = \left(\frac{1}{x - 1} - \frac{1}{x - 1}\right)$ $y = u^3$

$\frac{du}{dx} = \frac{-1}{(x - 1)^2} + \frac{1}{(x + 1)^2}$ $\frac{dy}{du} = 3u^2$

$dy = 3\left(\frac{1}{x - 1} - \frac{1}{x + 1}\right)^2 \left(\frac{1}{(x - 1)^2} - \frac{1}{(x + 1)^2}\right)$

$\frac{-48x}{(x - 1)^4 (x + 1)^4}$

(4) $y = \left(\frac{x^2 + 2}{x^2 - 1}\right)^4$

Let $u = \left(\frac{x^2 + 2}{x^2 - 1}\right)^4$ $y=u^4$

$\frac{du}{dx} \quad \frac{2x}{(x^2 -1)} - \frac{2x(x^2 + 2)}{(x^2 -1)^2} = \frac{- 6x}{(x^2 -1)^2}$

$\frac{dy}{} = 4u^3 \, du$

$dy = 4 \left(\frac{(x^2 + 2)}{(x^2 -1)^2}\right)^3 = \left(\frac{- 6x}{(x^2 -1)^2}\right)$

$\frac{- 24x(x^2 + 1)^3}{(x^2 - 1)^5}$ -

5) $y = (2x^4 + x^2 - l)^6$

$u = 2x^4 + x^2 - 1,$ $y = u^6$

$\frac{du}{dx} = 4x^3 + 2x,$ $\frac{dy}{du} = 6u^5$

$\frac{dy}{dx} = 6(2x^4 + x^2 - 1)^5(4x^3 = 2x)$

(6) $y = \left(\frac{x^3}{3} - \frac{x^2}{2} - \frac{x}{1}\right)$-

Let $u = \frac{x^3}{3} - \frac{x}{2} - \frac{x}{1}$ $y = u^{-1}$

$\frac{dy}{du}$ u-2, $\frac{du}{dx} = x2 - x - 1$

$$\frac{dy}{dx} = \frac{-(x^2 - x - 1}{\left(\frac{x^3}{3} - \frac{x^2}{2} - x\right)^2}$$

4.0 CONCLUSION

In this unit you have derived rules for differentiating quotient of functions. You have also derived the important chain rule. As mentioned in the previous unit all these rules are very important because you will use them throughout the remaining part of this course.

5.0 SUMMARY

In this unit you have studied the following rules for differentiation.

(i) Quotient rule

$$\text{i.e. } \frac{d}{x}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(ii) Chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

You have used these rules differentiate polynomials and rational functions.

6.0 TUTOR-MARKED ASSIGNMENTS

Differentiate the following functions with respect to x.

(1) $4x^3 + 2x^2 + \frac{1}{x}$

(2) $\frac{x + 1}{2x}$

(3) $x (x - 4)^3$

(4) $\frac{x^2}{x - 1}$

(5) $\frac{x + 1}{x\sqrt{x}}$

(6) $\frac{x3 - 2x + 1}{x^2 - 1}$

(7) $\left(x - \frac{2}{x}\right)^5$

(8) $\frac{6x}{(x^2 - 1)^2}$

(9) $\frac{2(x^2 + 1)}{(x^2 - 1)^2}$

(10) $\frac{4}{(x - 1)^5}$

(11) $x^3 (x - 2)^{-1}$

(12) $x^3 - \frac{1}{x^3}$

(13) $(x^2 + x)^3 (3x + 1)^3$

(14) $(x + 1)(2x_2 - 1)(2x - 1)$

(15) $(x - 1)^4 (x + 1)^2$

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MODULE 3

UNIT 1 FURTHER DIFFERENTIATION

CONTENTS

1.0 INTRODUCTION

By now you would have known that the subject differential calculus has a lot to offer to mankind. In order to be able to solve a large number of problems it is important to study the derivative of certain class of functions. You already know that if a function f is a one to one function then f has an inverse f^{-1}. The question now is suppose f is a differentiable and one to one function. Will the inverse function f^{-1} be differentiable? And under what conditions will (f^{-1}) exist. This is one question among others that you will be able to answer in this unit. In addition problems of relating to motion of a body a along a curve can only be fully described if the derivative known. In this unit higher derivations of function will be discussed so that you and others may be able to solve completely the problem of motion rte: a body along a curve. Optimization of scarce resources can easily be solved with the knowledge of higher derivatives of function most of the functions that have been treated so far are expressed explicitly in terms of one independent variable x. There are certain functions that might not be expressed explicitly, such function fall into the class of functions known as implicit function. They are so called in the sense that dependant and independent variables are expressed implicitly. Finding derivatives of such functions will be discussed; it will save you the time of trying to express the dependent variable in terms of the independent variable before differentiating y.

2.0 OBJECTIVES

Therefore after studying this unit you should be able to

1) Differentiate the inverse of a function
2) Evaluate higher derivatives of any given function.
3) Differentiate an implicit function

3.0 MAIN CONTENT

3.0 Differentiation of Inverse Functions

You could recall that the inverse of the function $y = x^3$ is given as $y = x^{1/3}$ (see Unit 2)

If $y = x^{1/3}$

$$\frac{dy}{dx} - \frac{1}{3} x^{1/3 -1} \quad x^{-2/3} \text{ and } x \neq 0$$

In the above example the function $y = x^3$ is a one to one function and also a differentiable function. Also the function $y = x^{1/3}$ is a one to one and also differentiable at a specified domain provided $x \neq 0$.

You recall that in unit 2 you studied that the composite function of $f(x)$ and its inverse $f^{-1}(x)$ in any order yields the identity function.

i.e. $f(f^{-1}(x)) = f(f^{-1}(X)) \quad = x$

Using the function $f(x) = x^3$ you have that $f^{-1}(x) = x^{1/3}$

Then $f(f^{-1}(x)) \quad = \quad (x^{1/3})^3 = x$

And $f(f^{-1}(x)) \quad = \quad (x^3)^{1/3} = x$

Using the above illustration, you can now differentiate the composite function given as

$$f(f^{-1}(x)) \quad = x$$

i.e. $\frac{d}{dx}[f(f^{-1}(x))] = \frac{dx}{dx}$

Using the chain rule studied in Unit 6.

Let $f^{-1}(x) = g(x)$

Then $f(g(x)) = x$.

But $\frac{d}{dx}[f(g(x))] = f^{'1}(g(x)) = \frac{dx}{dx} = 1$

$f^1(g(x)).\ g^1(x) = 1.$

$\Rightarrow \quad g^1(x) = \frac{1}{f^1(g(x))}$

This gives the derivative of inverse of a function

i.e. $(f^{-1}(x)) = \frac{1}{f^1(g(x))}$

Example

Let $f(x) = x^3$ Find $(f^{-1}(x))$

Solution

$(f^{-1}(x))^1 = \frac{1}{f^1(f^1(x))} = \frac{1}{3(f^{-1}(x))^2} = \frac{1}{3(x^{1/3})^2}$

$\frac{1}{3x^{2/3}}$

by direct differentiation of y = x you get:

$\frac{dy}{dx} = x.$

Example

Let $f(x) = x^3 + 1$ find the derivation of the inverse

Solution

$f(x) = x^3 + 1;\ f^1(x) = 3x^2$

inverse $f^{-1}(x) = (x - 1)^{1/3}$

$$(f^{-1}(x))^1 = \frac{1}{(f^{-1}(x))^1} = \frac{1}{3(f^{-1}(X))^2} = \frac{1}{3(\overline{(x-1)^{1/3})^2}}$$

The derivative of inverse of the function $f(x) = x^n$. , $x > 0$

Given that $f(x) = x^n$ and $f^1(x) = x\ 1/n$

$f^1(X) = nx^{n-1}$

therefore $(f^{-1}(x))^1 = \frac{1}{f^1(f^{-1}(x))} = \frac{1}{n(f^{-1}(x))^{\overline{n-1}}}$

$$\frac{d}{dx}[x\ 1/n] = \frac{1}{n}x^{1/n-1} \qquad \frac{1}{n(x1/n)n-1} = \frac{1}{n\,x^{1-1/n}}$$

$$\frac{1}{n}x^{1/n-1}$$

Thus for $x > 0$ and $f^1(x)=x^{1/n}$

$$\boxed{\frac{d}{dx}[x^{1/n}] = \frac{1}{n}x^{1/n-1}} \qquad \text{(A)}$$

and for $x \neq 0$ and n odd

$$\frac{d}{dx}[x^{1/n}] = \frac{1}{n}x^{1/n-1}$$

Examples

Find the derivative of the following functions.

(I) $y = x^{½}$

(II) $y = x^{I/7}$

(III) $y = x^{p/E}$

(IV) $y = x^{2/5}$

(V) $y = x^{4/3}$

(VI) $y = x^{-5/3}$

Solutions

(I) $y = x^{1/2} = \frac{dy}{dx} = \frac{1}{2} x^{-1/2}$

(II) $y = x^{1/7} = \frac{dy}{dx} = \frac{1}{7} x^{-6/7}$

(III) $y = x^{p/E}$

Let $x^{p/q} = (x^{1/q})^p$

$\frac{d}{dx} [x^{p/q}] \quad \frac{d}{dx} ([x^{1/q}])^p$

Let $u = x^{I/q}$ then $y = u^p$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\frac{dy}{du} = p u^{p-1}$ and $\frac{du}{dx} = \frac{1}{q} x^{1/q - 1}$

$\frac{dy}{dx} = p u^{P-1} \frac{1}{q} x^{1/q-1} = p(x^{1/q})^{p-1} \cdot \frac{1}{q} x^{1/q-1}$

$= \frac{p}{q} x(\frac{p}{q} - 1)$

$$\boxed{\frac{d}{dx} [x^{p/q}] = \frac{p}{q} x(\frac{p}{q} - 1)} \quad \text{———} \quad (B)$$

(IV) $y = x^{2/5}$

$\frac{dy}{dx} = \frac{2}{5} x^{2/5-1} = \frac{2}{5} x^{-3/5}$

(V) $y = x^{4/3}$

$\frac{dy}{dx} = \frac{4}{3} x^{4/3-1} = \frac{4}{3} x^{1/3}$

(VI) $y = x^{-5/3}$

$$\frac{dy}{dx} = \frac{-5}{3} x^{-5/3-1} = \frac{-5}{3} x^{-8/3}$$

The above equation (B) could be extended to the case f (x) = u where u is to the case *f* (x) = u; where u is a differentiable functions of x.

i.e.
$$\boxed{\frac{d}{dx}(u(x))^{p/q} = \frac{P}{q}[u(x)]^{p/q-1}\frac{d}{dx}[u(x)]}$$

For the above to make sense, then u (x) ≠ 0 when q is odd and u (x) > 0 when q is even.

Example

(1) If $y = [(x^2 - 1)^{1/5}]$ then

$$\frac{dy}{dx} = \frac{1}{5}(x^2-1)^{1/5-1}\,2x$$

$$\frac{2x}{5}(x^2-1)^{-4/5}$$

(2) Evaluate $\frac{d}{dx}[2x^2 - 7]^{1/3}] = \frac{1}{3}(2x^2-7)^{-2/3} \cdot 4x$

$$\frac{4x}{3(2x^2-7)^{2/3}}$$

(3) Evaluate

$$\frac{d}{dx}\left[\sqrt{\frac{x^2+1}{x^2-1}}\right] = \frac{d}{x}\left[\left(\frac{x^2+1}{x^2-1}\right)^{½}\right]$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right) \quad -½ \;.\; \frac{d}{x} = \left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-½} \quad \frac{2x(x^2-1) - 2(x^2+1)}{(x^2-1)^2}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-½} \quad \left(\frac{-4x}{(x^2-1)^2}\right)$$

$$= \frac{-2x}{(\sqrt{x^2+1})(x^2-1)^{3/2}}$$

3.2 Implicit Differentiation

So far you have been finding the derivatives of functions of the class of functions whose right side of the equality sign is an expression of one variable (i.e. x). Such functions are said to be explicit functions. However, there are functions such as

$$x^2y = 2xy^2 + 6$$

This type of such is expressed implicitly. To obtain an explicit expression of an implicit expression you resolve in transpose (make subject of formula) the equation in the dependant variable or one variable

Example

$2x^2 + 3y = 6$ transposing for y

yields $y = \frac{2x^2 - 6}{-3}$

However there are implicit functions where it will not be possible to solve for y. Example of such functions are

(1) $x^2 + xy^4 + y^3x + x^3 = 2$

(2) $x^2 + xy^2 + 5x^3 + y^2 = 1$

In the above although it is not possible to solve for y, they can be differentiated by the method of known as implicit differentiation. Appropriate applications of the rules for differentiations u, u v, $u^{1/n}$, U, etc. which you have studied in unit 6. You should be able to carry out implicit differentiation. The next question that comes to your mind should be "what is implicit differentiation" this question is best answered by finding the derivative of the functions.

$$x^2 + xy^4 + y^3x + x^3 = 2$$

Solution: Differentiating both sides of the equation you get

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy^4) + \frac{d}{dx}(y^3x) + \frac{dx^3}{dx} = \frac{d(2)}{dx}$$

$$=2x + \frac{dx}{dx}, y^4 + x\frac{d(y)^4}{dx} + y^3\frac{dx}{dx} + x\frac{d(y)^3}{dx} + 3x^2 = 0$$

$$= 2x + I.\ y^4 = 4xy^3\frac{dy}{dx} + y^3 + 3x\,y^2\frac{dy}{dx} + 3x^2 = 0$$

$$= (2x + y^4 + 3x^2 + y^3) + (4xy^3 + 3x\,y^2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2x + y^4 + 3x^2 + y^3)}{4xy^3 + 3xy^2}$$

Implicit differentiation is useful in finding the slope of a tangent to curves.

Example: Find $\frac{dy}{dx}$ of the following functions.

1. $x^2 + y^2$ (2) $y^2 = \frac{x^2 - 2}{x - 1}$

3. $x^2\,y^2 + y + 2 = 0$ (4) $x^3 - xy + yl = 0$

5. $2xy - y^2 = x - y$ 6. $(x + y)^2 + (x-y)^2 = x^3 + y^3$

Solutions

1. $\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(4)}{dx}$

$$=2x^2 + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y},\ y \neq 0$$

2. $\frac{d(x^2)}{dx} = \frac{d}{dx}\left(\frac{x^2 - 2}{x - 1}\right)$

$-2y\frac{dy}{dx} = \frac{2x(x-1)-(x^2-2)}{(x-1)^2} = \frac{x^2 - 2x + 2}{(x-1)^2}$

$2y\frac{dy}{dx} = \frac{x^2 - 2x + 2}{(x-1)^2}$

$\frac{dy}{dx} = \frac{x^2 - 2x + 2}{2y(x-1)^2}$

3. $\frac{d(x^2y^2)}{dx} + \frac{d(y)}{dx} + \frac{dy}{dx} = 0$

$x^2\frac{d(y^2)}{dx} + y^2\frac{d(x^2)}{dx} + \frac{dy}{dx} = 0$

$x^2 2y\frac{dy}{dx} + y^2 2x + \frac{dy}{dx} = 0$

$2xy^2 + (2yx + 1)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2yx^2}{2yx-1}$

4. $\frac{d(x^3)}{dx} + \frac{d(xy)}{dx} = \frac{d(y^2)}{dx} \quad = 0$

$3x^2 - y\frac{d(x)}{dx} - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$3x^2 - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$3x^2 - y + (2y - x)\frac{dy}{dx} = 0$

$\frac{dy}{dx} \quad \frac{y - 3x^2}{2y-x}$

5. $\frac{d(2xy)}{dx} - \frac{d(y^2)}{dx} = \frac{d(x)}{dx} - \frac{dy}{dx}$

$$2x \frac{dy}{dx} + 2y - 2y \frac{dy}{dx} = 1 - \frac{dy}{dx}.$$

$$2y - 1 = (2y - 2x - 1) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2y - 1}{2y - 2x - 1}$$

6. $$\frac{d(x + y)^2}{dx} + \frac{d(x - y)^2}{dx} = x^3 + y^3$$

$$\frac{d(x + y)^2}{dx} + \frac{d(x - y)^2}{dx} = \frac{d\,(x^3 + y^3)}{dx}$$

$$2x(x + y)(1 + \frac{dy}{dx}) + 2(x - y) (1 - \frac{dy}{dx}) = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$\Longrightarrow 2(x + y) = 2(x - y) + [(2(x + y) - (2x - y)] \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$\Longrightarrow 4x + 4y \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$\Longrightarrow (4x+3y^2) \frac{dy}{dx} = 3x^2 + 4x$$

$$\frac{dy}{dx} \quad \frac{3x^2 - 4x}{4y - 3y^2}$$

Example

By differentiating the equation

$x^2 y^2 = x^2 + y^2$ implicitly show that

$c = \frac{k(1 - y^2)}{x - 1}$, where $k y = x$.

Solution

Given that $x^2 y^2 = x^2 + y^2$

Then $d \frac{(x^2 y^2)}{x^2 - 1}$, where $k y = x$.

Solution: Given that $x^2 y^2 = \quad x^2 + y^2$

Then $\frac{d(x^2 y^2)}{x^2 - 1}$, $\quad \frac{d}{d}(x^2 + y^2)$

$$2xy^2 + 2yx^2 \frac{dy}{dx} = 2x + 2y \frac{dy}{dx} \qquad \_\_\_\_\_\_\_\_\_ \quad (1)$$

Collecting like terms you equation (1) becomes

$$(2x + 2y)\, dy = 2x - 2xy^2 \qquad \_\_\_\_\_\_\_\_\_\_ \quad (II)$$

Dividing equation II by $(2yx^2 - 2y)$

$$\frac{dy}{dx} = \frac{x - xy^2}{yx^2 - y} = \frac{x(1 - y^2)}{y(x^2 - 1)}$$

since $k = x/y$ then.

Example

By differentiating the equation

$x^2 + y - y^2$ implicitly show that

$$\frac{dy}{dx} = \frac{2y - x}{2x + y}$$

Solution: Differentiating with respect with y you have:

$$\frac{d(x^2)}{dy} \quad \frac{d(xy)}{dy} - \frac{d(y^2)}{dy} = 0$$

$$2x \quad \frac{dx}{dy} + x + y \frac{d}{d} - 2y = 0$$

$$\frac{dx}{dy}(2x + y) = 2y - x$$

$$\frac{dx}{dy} = \frac{2y - x}{2x + y}$$

3.3 Higher Order Differentiation

You will start this section with the study of second derivative of a function where it exist and then extend it to higher order.

Let y = f (x) be a differentiable function of x. Then it has a derivative given as

$$\frac{dy}{dx} = f^1(x)$$

f^1 (x) is a function let:

f^1 (x) = g(x)

Then $g^1(x) = \frac{d^2y}{dx^2}$ which is the second derivative of function y = f (x) .

Which is written as $\frac{d^2y}{dx^2}$ or f^{11} (x)

Example

Let $y = 4x^3$ Find $\frac{d^2y}{dx^2}$

Solution

$$\frac{dx}{dy} = \frac{d}{d}(4x^3) = 12x^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d}(12x^2) = 24x$$

Example

$y = x^3 - 2x^2 + x.$ find $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \frac{d}{d}(x^3 - 2x^2 + x) = 3x^2 - 4x + 1$$

$$\frac{d^2y}{dx^2} = (3x^2 - 4x + 1)$$

$$= 6x - 4$$

Since you now know what a second derivation of higher order. The idea here is that so long as you have differentiability, you can continue to differentiate y = f (x) from $\frac{dy}{dx}$

$f^{1}(x)$ and $\frac{dy}{dx} = f^{1}(x)$ to form $\frac{dy}{dx^2} = f^{11}(x)$ and $\frac{dy}{dx^2} = f^{11}(x)$ to form $\frac{dy}{dx^3} = f^{111}(x)$

And so on until you get to an n^{th} order

i.e. $\frac{d(y)}{dx} = \frac{dy}{dx}$, $\frac{d}{d} \frac{dy}{dx} = \frac{d^2y}{dx^2}$

$$\frac{d}{d} \quad \frac{d^2y}{dx^2} \quad \frac{d^3y}{dx^3} \quad \frac{d}{d} \quad \frac{d^3y}{dx^3} = \frac{d^4y}{dx^4}$$

$$\frac{d}{d}\left(\frac{d^{n-1} g}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$$

Example

Let $y = x^5 + x^4 + x^3 + 1$

Then $\frac{dy}{dx} = 5x^4 + 4x^3 + 3x^2$

$$\frac{d^2y}{dx^2} = 20x^3 + 12x^2 + 6x$$

$$\frac{d^3y}{dx^3} = 60x^3 + 24x^2 + 6x$$

$$\frac{d^4y}{dx^4} = 120x + 24$$

$$\frac{d^{555}y}{dx^5} = 120$$

$$\frac{d^6y}{dx^6} = 0$$

$$\frac{d^{>}1y}{dx^1} = \frac{d^8y}{dx^8} = \dots = \frac{d\,''y}{dx^n} = 0$$

In the above example all derivatives of order higher than 5 are identically zero. You can see that derivative of a polynomial function is again a polynomial function. This implies that polynomial functions have derivatives of all order so also, is all rational functions.

Example

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-1}{x^2}, \qquad \frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$$\frac{d^3y}{dx^3} = \frac{-6}{x^4} \qquad \frac{d^4y}{dx^4} = \frac{24}{x^5}$$

Example

Find $\frac{d^4y}{dx^4}$ If $y = \frac{2x}{x - 1}$

Solution

$$y = \frac{2x}{x - 1}$$

$$\frac{dy}{dx} = \frac{-2x}{(x - 1)^4}, \qquad \frac{d^2y}{dx^2} = \frac{4x}{(x \_ 1)^3}$$

$$\frac{d^3y}{dx^3} = \frac{-12x}{(x - 1)^4}, \qquad \frac{d^4y}{dx^4} = \frac{48x}{(x - 1)^5}$$

Example

By differentiating implicitly find d^2y if $x^2 = 1 + y^2$, leave your answer in terms of x and y only.

Solution

Given that $x^2 - 1 = y^2$

Then $2x = 2y\frac{dy}{dx}$

$$X = y \frac{dy}{dx}$$

$$1 = \frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2}$$

$$\Rightarrow \qquad I = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}$$

$$\Rightarrow \qquad 1 - \left(\frac{dx}{dy}\right)^2 = y\frac{d^2y}{dx^2}, \quad \text{where } {}^{x}\!/_{y} = {}^{dy}\!/_{dx}.$$

$$\frac{d^2y}{dx^2} = \frac{1 - (x/y)^2}{y} = \frac{y^2 - x^2}{y^3}$$

Example

By differentiating implicitly find: (1) dy/dx. (2) d^2y/dx^2 in the following equations.

(1) $x^2 - y^2 = 4x$

(2) $x^3 - y^3 = 27$

(3) $x^2 + y + y^2 = 1$

(4) $x^2 y^2 = 16$

Solutions

(1) $x^2 - y^2 = 4x$

$$\frac{d(x^2)}{dx} - \frac{d(y^2)}{dx} = \frac{d(4x)}{dx}$$

$$2x - 2y\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{2x - 4}{2y} = \frac{x - 3}{y}.$$

Given that $y\frac{dy}{dx} = x - 2$

$$\frac{d}{d}\left(y\frac{dy}{dx}\right) = 1 .$$

$$\frac{dy}{dx}\frac{dy}{dx} + y\frac{d^2y}{dx^2} = 1$$

$$\frac{d^2y}{dx^2} = \frac{1 - \left(\frac{dy}{dx}\right)}{y} = \frac{1 - \left(\frac{x-2}{y}\right)^2}{y} = \frac{-(y^2 + x^2 - 4x + 4)}{y^3}$$

(2) $x^3 - y^3 = 27$

$\frac{d(x^3)}{dx} - \frac{(dy^n)}{dx} = 0$

$3x^2 – 3y^2 \frac{dy}{dx} = 0$

$====> \frac{dy}{dx} = \frac{x^2}{y^2}$

$\frac{d}{dx}\left(y \frac{dy}{dx}\right) = \frac{d}{dx} (x)$

$y^2 \frac{d^2y}{dx^2} + 2y \frac{d}{dx} . \frac{d}{dx} = 2x$

$y^2 \frac{d^2y}{dx^2} = 2(x – y \left(\frac{dy}{dx}\right)^2)$

$\frac{d^2y}{dx^2} = \frac{2 (x – y \left(\frac{x}{y}\right)^4)}{y2} = \frac{2x \ 1 \left(- \frac{x^3}{y^3}\right)}{y^2}$

$= \frac{2x(y^3 - x^3)}{y^5}$

(3) $x^2 y + y^2 = 1$

$\frac{d}{dx}(x^2y) - \frac{d(y^2)}{dx} = 0$

$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$

$-2xy = (x^2 + 2y) \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}$

Differentiating implicitly the equation

$$\frac{d}{dx}(2x\,y)\ \frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) + \frac{d}{dx}\left(2y \frac{dy}{dx}\right) = 0$$

you get:

$$2y + 2x \frac{dy}{dx} + 2x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \cdot \frac{dy}{dx} + 2y \frac{d^2y}{dx^2} = 0$$

collecting like terms you get:

$$2y + 4x \frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x^2 + 2y) \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = 2y + 4x \quad \frac{\left(\frac{-2xy}{x^2+2y}\right) + 2\left(\frac{-2xy}{x^2+2y}\right)^2}{x^2+2y}$$

(4) $x^2 y^2 = 16$

$$\frac{d}{dx}(x^2 y^2) = 16$$

$$2x\,y^2 + x^2\,2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2xy^2}{2yx^2} = \frac{-y}{x}$$

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) = \frac{d}{dx}(-y) = \frac{d}{dx}\left(x \frac{dy}{dx}\right)\ \frac{d}{dx}(-y)$$

$$\frac{dy}{dx} = -2 \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-2\,dy/dx}{x} = \frac{-2(-y/x)}{x} = \frac{2y}{x^2}$$

4.0 CONCLUSION

In this unit you have applied rules of differentiation to find derivatives of inverse of a function which is turn lead to differentiation of function such as $y = x^n$ where nEQ. You have studied implicit differentiation will be useful when finding the normal or tangent of curve at a given point. Higher order derivatives of functions, which you studied in this unit, is a very useful tool for studying applications of differentiation. The various solved examples in this unit is given to enable you acquire the necessary tools for further differentiation.

5.0 SUMMARY

In this unit, you have studied how to

(1) Fine the derivatives of inverse of a given function
i.e.

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f^{1}(f^{-1}(x))}$$

(II) Differentiate a given equation implicitly

(III) Fine higher order derivatives of functions

i.e. $\frac{dy}{dx} \quad \frac{d}{dx}\frac{(dy)}{dx} \quad \frac{d}{dx} \quad \frac{d^2y}{dx^2} = \frac{d^3y}{dx^2}, \ \ldots,$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \quad \frac{d^n y}{dx^n}$$

(IV) Differentiate functions with fractional powers i.e. $\frac{d}{dx}(x^{p/q}) = P/q\ x^{(p/q)}$

6.0 TUTOR-MARKED ASSIGNMENT

For exercise (1) - (2) find the derivatives of the inverse of the following functions:

(1) $y = x^2 - 1$

(2) $y = 4x^5 - 2$

(3) $y = \frac{2x}{x - 1}$

(4) $y = \frac{1}{x^3 + 1}$

(5) Find the derivatives of the following functions

(I) $y = x^{1/5}$ (II) $y = x^{1/9}$

(III) $y = x^{3/5}$ (1V) $y = x^{-2/3}$

(6) Find the derivative of the following functions

(1) $y = (x-1)^{1/5}$ (II) $y = (2x^2 - x)^{1/3}$
(III) $y = (x^3 - 1)^{2/3}$

(7) Evaluate $\frac{d}{dx}\sqrt{\left(\frac{x2 - 1}{x + 2}\right)}$

(8) Evaluate $\frac{d}{dx}\left(\frac{x + 1}{x^2 - 2}\right)^{2/3}$

(9) If $y \sqrt{l+x^2} = 2$ find $\frac{dy}{dx}$

(10) If $y = \frac{x - 1}{x + 2}$ show that

$(x+2)^2 \frac{dy}{dx} = 3$

(11) If $y = \frac{1}{\sqrt{x^2 + 1}}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

(12) Find the value of $\frac{dx}{dx}$ and $\frac{d^2y}{dx^2}$
at the point p(2, 3) if $x^2 + xy = y$.

(13) Find $\frac{dx}{dy}$ if $(x - y) + (x + y)2 = x^2 + y^2$

(14) By differentiating the equation

$2x - y^2 = x^2 - 2y$ show that

$(1- y) \frac{d^2y}{dx^2} = 1 + \left(\frac{x - 1}{1 - y}\right)^2$

(15) What is the value of $\frac{d^2y}{dx^2}$ if $y = x^6$

(16) Find $\frac{d^4y}{dx^4}$ if $y = \frac{x + 1}{x^2}$

(17) Find $\frac{d^3y}{dx^3}$ if $y = \sqrt{\frac{1 + x}{1 - x}}$

(18) Find $\frac{d^4y}{dx^4}$ if $y = 3x^4 + \frac{1}{x}$

(19) show that $\frac{dx}{dy} = \frac{-x}{y}$ if $x^2 y^2 = 16$.

(20) Find $\frac{d^2}{dx^2}\left(\sqrt{\frac{x^2 + 1}{x - 1}}\right)$

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UNIT 2 DIFFERENTIATION OF LOGARITHMIC FUNCTIONS AND EXPONENTIAL FUNCTION

CONTENTS

1.0 INTRODUCTION

So far you have studied differentiation of functions such as polynomials and rational functions. In this unit you will be studying differentiation of two special functions namely natural log and exponential which have practical applications in real life problems such as computation of compound interest accruing from money deposited or borrowed from financial institutions. Another application where the differentiation of these two special functions could be applied is in the prediction of growth or decay of a radioactive substance. The two functions natural logarithm and exponential functions that will be subject of study in this unit, are related to one another because one is the inverse functions of the others. That is f^{-1}(natural logarithm) = exponential function and the f^{-1} (exponential function) is the natural logarithm.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

1) Differentiate logarithmic functions
2) Carry out logarithmic differentiation
3) Differentiate exponential functions
4) Find the derivative of the function a^u
5) Find the derivative of the function $\log_a u$.

3.0 MAIN CONTENT

3.1 Differentiation of the Logarithm Functions

You will review some properties of logarithm functions you are already familiar with.

(1) $x = \log_a y$ if $a^x = y$

(2) the log of a product = the sum of the logs. Keeping the above in mind you should be able to recall the following:

let $f(x) = \log x$

then $f(x, y) = f(x) + f(y)$ …….. (I)

suppose $f(x) = 1$

then $f(1) = f(1 . 1) = f(1) + f(1) = 2 f(1)$

$\Rightarrow$ $f(1) = 2 f(1)$

$\Rightarrow$ $0 = f(1)$. .... (II)

if this is true for $x > 0$

Then $0 = f(1) = f(x . 1/x) = f(x) + f(1/x)$

$\Rightarrow$ = $f(1/x) = f(x)$ ………….. (III)

Taking $x > 0$ and $y > 0$ then

$f(y/x) = f(y . 1/x) = f(y) + f(1/x) = f(y) – f(x)$

(Using equation II)

$f(y/x)$ $f(y) – f(x)$ .... (IV)

Still keeping $x > 0$ and using equation I you get

$f(x . x . x.... x) = f(x^n) = f(x) + ...+ f(x)$

$= n f(x)$

$f(x^n) = n f(x)$.

Replacing f (x) by log x you get the 3 basic properties.

(1) log (x y) = log x +log y

(2) log(y/x) = log x – log y

(3) log (x^p) = p log x.

You will now attempt to derive a formula for the derivative of a logarithm function.

Let log x = f (x) and x > 0 where f is assumed to be a non-constant differentiable function of x which has all the properties of a logarithm stated above let Δx be an increment resulting in a corresponding increment in f(x). Then the difference quotient can be formed as

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \_\_\_\_\_\_\_\_\_\_ \quad (A)$$

Since f (x) is a logarithmic function.

You can re-write the above equations (A) as

$$f(x + \Delta) - f(x) = f\frac{(x+\Delta x)}{x} = f(1 + \frac{\Delta x}{x})$$

Hence $$\frac{f(x + \Delta) - f(x)}{\Delta x} = f\frac{(x+\Delta x)}{x}$$

Multiplying equation (B) x/x and noting that f(l) = 0 you get

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{1}{x}\left[\frac{f(1 + \Delta x/x) - f(1))}{\Delta x/x}\right]$$

Taking limits of equation (C) as Δx -----> 0 you get

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \lim_{\Delta x \to 0} \frac{1}{x} \quad \lim_{\Delta x \to 0}\left[\frac{f\ (1 + \Delta x/x) - f(1)}{\Delta x/x}\right] = I$$

$$= 1 \lim_{\Delta x \to 0} \left[\frac{f(1 + \Delta x/x) - f(1))}{\Delta x/x} \right]$$

$$f^1(x) = 1/x\, f^1(1)$$

$$\Rightarrow \frac{d}{dx}[f(x)] = \frac{d}{dx}[\log x] = \frac{1}{x} \text{ where } f^1(x) = 1$$

(Since f is a non-constant function $f^1(1) \neq 0$).

If u is a differentiable positive function of x (i.e. $u(x) > 0$ and $u^1(x)$ exist)

$$\text{Then } \frac{d}{dx}(l\,nu) = \frac{d}{du} \ln u \frac{du}{dx} = \frac{1}{u} \frac{du}{dx}$$

Example: Find $\frac{d}{dx} \log(x^2 + 1)$.

Solution

Let $u = x^2 + 1$ here $u > 0 \ \forall\ \varepsilon$ R.

$$\text{Then } \frac{d}{dx}(x^2 + 1). = \frac{1}{x^2 + 1}\ 2x = \frac{2x}{x^2 + 1}$$

Example: Find $\frac{d}{dx} \log \frac{1}{x^2 + 1}$

Solution

Let $u = \frac{1}{x^2 + 1}$ here $u > 0$ V x ε R.

$$\text{Then } \frac{d}{dx} \log\left(\frac{1}{(x^2 + 1)}\right) \frac{d}{dx} (\log u) = \frac{1}{u} \frac{du}{dx}$$

$$= x^2 + 1 \frac{-2x}{(x^2 + 1)^2} = \frac{-2x}{(x^2 + 1)}$$

Example: Show that $\frac{d}{dx}(\log /x/) = \frac{1}{x}$

Solution: For x > 0

$\frac{d}{dx}(\log /x/) = \frac{d}{dx}(\log x) = \frac{1}{x}$ /x/ = x for x > 0

For x < 0, / x/ = x .

Therefore $\frac{d}{dx}(\log /x/) = \frac{d}{dx}(\log -x)$

Here let u = - x $\frac{du}{dx} = -1$ $\frac{d}{dx}(u) = \frac{1}{u}$

$\frac{d}{dx}(\log /x/) = \frac{1}{u}.(-1) = \frac{1}{x}.(-1) = \frac{1}{x}$

Example: Find $\frac{d}{dx}\log$ $/1 - x^2/$

Solution

Let $u = 1 - x^2$, then $\frac{d}{dx}\log /u/$

$= \frac{1}{1-x}. -2x = \frac{-2x}{1-x^2} = \frac{2x}{x^2-1}$

Example: Find $\frac{d}{dx}\log\left(\frac{x^4}{x-1}\right)$, $x \neq 1$.

Solution

Let u = $\frac{x\Delta}{x+1}$ $\frac{du}{dx} = \frac{(3x-4)x^3}{(x-1)^2}$,

Then $\frac{d}{dx}\log = \frac{1}{u}.\frac{du}{dx} = \frac{x-1}{x^4}.\frac{(3x-4)x^3}{(x-1)^2}$

$= \frac{(3x-4)}{x(x-1)}$

3.2 Logarithmic Differentiation

The Natural Logarithm: In previous section you have differentiated a general logarithm function. That is the base to which the logarithm is taken was not mentioned. Every logarithm studied so far are mainly of two types $\log_{10} x$ or $\log_e x$. The latter is the one you will study in this section.

Remark: The natural logarithm is that function $f(x) = \log_e x$ that is the logarithm to base e (the number e is taken after Leonard Euler (1707 -1783)(There are logarithm to base other than e or 10.) The interesting thing about the study of differentiation of the natural logarithm is that its definition depends so much on calculus. You will consider the definition after you have studied the second course on calculus i.e. integral calculus. You have to make do with the fact that

$$\text{Ln } x = \log_e x = \text{the natural logarithm.}$$

The above satisfies all the basic properties of a logarithm function reviewed in the previous section.

In practice it has been observed that finding the derivatives of certain functions could be a difficult task.

But with appropriate application of the natural logarithm, derivatives of such functions could easily be found. The method involves taking the natural logarithm Ln of both sides of the given equation before differentiation. This method is called logarithmic differentiation.

Example

Suppose $y = f(x)$ Find dy/dx

Step 1. take natural logarithm both sides

$$\text{Ln } y = \text{In } f(x) \qquad \text{\_\_\_\_\_\_\_\_ (I)}$$

Step 2. Differentiate both sides with x

$$\frac{d}{dx}(\text{In } y) = \frac{d}{dx}(\text{In } f(x)) \qquad \text{\_\_\_\_\_\_\_\_\_ (II)}$$

$$\frac{dy}{dx} = y \frac{d}{dx}(\text{In } f(x))$$

Let $y = u^n$ $u(x)$ is a differentiable function of x.

Taking log of both sides you get:

$Lny = 1n\ U^n$

$Lny = n\ln U$

$\frac{d}{dx}(Iny) = \frac{d}{dx}(nlnu)$

$\frac{1}{y}\frac{dy}{dx} = n\left(\frac{1}{u}\frac{du}{dx}\right)$

$\frac{dy}{dx} = n\ y\ \left(\frac{1}{u}\right)\frac{du}{dx}$

$= \frac{nu^n}{u}\frac{du}{dx}$ (since $y = u^n$)

$= nu^{n-1}\frac{du}{dx}$

Which is the same result derived in unit 7.

Example: Find dy/dx If $y = x^{x+1}$, $x > 0$

Solution

$Y = x^{x+1}$ (taking natural log of both sides)

$Ln\ y = \ln(x^{x+1})$

$Lny = x + 1\ In\ x$ (differentiate with x)

$\frac{1}{y}\frac{dy}{dx} = In\ x + \frac{(x+1)}{x}$

$\frac{dy}{dx} = \left(\ln x + \frac{(x+1)}{x}\right) y = \left(\ln x + \frac{(x+1)}{x}\right)x^{x+1}$

$=(x\ln x + x + 1)x^x$

Example: Find $\frac{dy}{dx}$ if $y = \frac{(x^2+1)^3(2x-1)^2}{(x^2+I)}$

Solution:

$$Y = \frac{(x^2+1)^3(2x-1)^2}{(x^2+1)} \quad \text{(taking In of both sides)}$$

$$\text{Lny} = 1\text{n}\left(\frac{(x^2+1)^3(2x-1)^2}{(x^2+1)}\right)$$

$$\text{Lny} = \text{In}(x^2+1)^3 + \text{In}(2x-1)^2 - \text{In}(x^2+1)$$

$$\frac{d}{dx}(\text{lny}) = \frac{d}{dx}\left(31\text{n}(x^2+1) + 2\ 1\text{n}(2x-1) - \text{In}(x^2+1)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x^2+1}.2x + \frac{2}{2x-1}.2 - \frac{2x}{x^2+1}.$$

$$\frac{dy}{dx} = \left(\frac{6x}{x^2+} + \frac{4}{2x-1} - \frac{2x}{x^2+1}\right) \cdot \frac{(x+1)^3(2x-1)^2}{(x^2+1)}$$

$$= 4(3x^2 - x + 1)(x^2+1)(2x-1).$$

Example: Find $\frac{dy}{dx}$ if $y\ 3/2 = \frac{(x^2-1)(3x-4)^{1/3}}{(2x-3)5\,(x+1)^2}$

Solution

$$y^{3/2} = \frac{(x^2-1)(3x-4)^{1/3}}{(2x-3)^{1/5}(x+1)^2}$$

$$\text{In}\ y\ 3/2 = \text{In}\left(\frac{(x^2-1)(3x-4)^{1/3}}{2x-3)^{1/5}(x+1)^2}\right) \quad \text{(taking in of both sides)}$$

$$= 1\text{n}(x-1) + \text{In}(3x-4)^{1/3} - \text{In}(2x-3)^{1/5} - \ln(x+1)^2$$

$$\frac{d}{dx}\left(\frac{3\text{In}\ y}{2}\right)\frac{d}{dx}\left(\text{In}(x2-1) + \frac{1}{3}\text{In}(3x-4) - \frac{1}{5}\text{In}(2x-3) - 2\text{In}(x+1)\right)$$

$$\frac{3}{2y}\frac{dy}{dx} = \frac{2x}{x2-1} + \frac{3}{3(3x-4)} - \frac{2}{5}\left(\frac{1}{2x-3}\right) - \frac{2}{x+1}$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2-1} + \frac{1}{3x-4} - \frac{2}{5}\left(\frac{1}{2x-3}\right) - \frac{2}{x+1}\right)\frac{2}{3}\left(\frac{(x^2-1)(3x-4)^{1/3}}{(2x-3)^{1/5}(x+1)^2}\right)$$

$$\frac{2}{}\left(\frac{4x^3+53x^2-1774x+127}{}\right)$$

15 $(x = 1)^{5/3}\,(2x - 3)^{17/15}\,[3x - 4]^{7/9}\,(x - 1)^{1/3}$

Example: $y^{\frac{1}{5}} = \dfrac{x^6}{\sqrt{x+1}}$

Solution

$$\text{Ln } y^{1/5} = 1\text{n}\left(\frac{x^6}{(x+1)\ ½}\right)$$

$$\frac{1}{5}\text{ In } y = 6\text{ Inx} - \frac{1}{2}\text{ ln }(x = 1)$$

$$\frac{1}{5}\ \frac{1}{y}\ \frac{dy}{dx}\ \frac{6}{x} - \frac{1}{2(x+l)}$$

$$\frac{dy}{dx} = 5y\left(\frac{6}{x} - \frac{1}{2(x+l)}\right)$$

$$5.\left(\frac{x^6}{(x+1)^{1/5}}\right)^{1/5}\left(\frac{6}{x} - \frac{1}{2\,(x+1)}\right)$$

Example: $y = x\,(x - 1)(x^2 + 1)(x - 2)(x^2 - 3)$ find dy/dx

Solution

$y = x\,(x - 1)(x^2 + 1)(x - 2)(x^2 - 3)$

In y = In $(x\,(x - 1)(x^2 + 1)(x - 2)(x^2 - 3)$ (taking In of both sides.)

$= \text{lnx} + \text{ln}\,(x-1) + \text{ln}(x^2 + 1) + \text{In}\,(x - 2) + \text{In}\,(x^2 - 3)$

d (Iny) = d [In x + In (x - 1) + In $(x^2 + 1)$ + In (x - 2) + In $(x^2 - 3)$]

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x} + \frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{1}{x-2} + \frac{2x}{x^2-3}\right).y$$

$$\frac{dy}{} = \frac{1}{} + \left(\frac{1}{} + \frac{2x}{} + \frac{1}{} + \frac{2x}{}\right)(x\,(x - 1)(x^2 + 1)(x - 2)(x2 - 3)$$

dx x x-1 $x^2 + 1$ x – 2 x^2 - 3

3.3 Differentiation of Exponential Function

You will now be introduced to the function that cannot be changed by any differentiation.

The function f (x) = e^x for all real number x is called the exponential function. At this stage you will review some properties of the exponential function which you are already familiar with.

(I) log e^x = x for all real number x

(II) $e^x > 0$ for all real number x

(III) $e^{\log x} = x$ for all x > 0

(IV) $e^{x+y} = e^x e^y$ for all real x and y

(V) $e^{x-y} = e^x/e^y$ for all real x and y

The derivative of the exponential function is the exponential function. This singular property distinguishes it as the only indestructible function.

i.e. $\frac{d}{dx}(e^x) = e^x$ ______________ (I)

To prove the above you start by noting that

$\text{Log } e^x = x$ ___________ (2)

Taking the derivative of both sides of equation(2) you get:

$\frac{d}{d\,.}(\log e^x) = \frac{d}{dx}(x)$ ________________ (3)

You can write equation (4) above in a general form. By letting y = e° where u is a real and differentiable function of x.

i.e. $\frac{dy}{dx} = \frac{d}{dx}(e^u)$ by applying

the chain rule for differentiation you get that

$$\frac{dy}{dx} = e^u \frac{du}{dx}$$

Example: find $\frac{dy}{dx}$ if

(I) $y = e^{\sqrt{x}}$ (II) $y = e^{x2}$

(III) $y = e^{(x+1)2}$ (IV) $y = e^{\sqrt{x}+I}$

(V) $y = e^{(x+1)3}$

Solutions

(I) $y = e^{\sqrt{x}} = e^u$ where $u = x^2$

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e\sqrt{x} . \tfrac{1}{2} \quad x^{-1/2}$$

(II) $y = ex^2 \quad e^u$, where $u = x^2$

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e^{x2} . 2x$$

$$=2x\, e^{x2}.$$

(III) $y = e^{(x+1)2} \quad e^u$, where $u = (x+ 1)^2$

$$\frac{du}{dx} = 2(x + 1)$$

$$\frac{dy}{dx} e^u \frac{dy}{dx} = e^{(x+ 1)2}, \quad 2(x + 1)$$

$$=2(x+ 1)\, e^{(x+1)2}$$

(IV) $y = e^{\sqrt{x+1}} \quad e^u$, wherein $u = (x + 1)^{½}$

$$\frac{du}{dx} \quad \frac{1}{2} (x+12)^{-1/2}$$

$$\frac{dy}{dx} = e^u . \frac{du}{dx} = e^{\sqrt{x+1}} . ½ (x+12)^{-1/2}$$

$$= \frac{1e\sqrt{x+1}}{2\sqrt{x+1}}$$

(V) $y = e^{(x+1)3}\, e^u$, where $u = (x^2 - 1)^2$

$$\frac{du}{dx} = 2(x^2 - 1).2x = 4x(x^2 - 1)$$

$$\frac{dy}{dx} = e^u . \frac{du}{dx} = e(x^2 -1)^2.. 4x = (x^2 - 1)$$

$$= 4x = (x^2 - 1).\ e^{(x2 -1)2}$$

Example : Find dy/dx if $y = e^x - \ln x$

Solution

$$Y = e^{x - \ln x} \Rightarrow \frac{e^x}{e^{\ln x}} = \frac{e^x}{x}$$

$$\frac{dy}{dx} = e^x\ x^{-1} + (-1)\ (x)^{-2}\ ex$$

$$\frac{e^x}{x} - \frac{x}{x^2} = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

Example: find dy if $y = e^{\sqrt{x}}\ \text{In}^{\sqrt{x}}$

Solution: $Y = uv$, where $v = e^{\sqrt{x}}$ $u = \ln(x)^{½}$

$$\frac{dv}{dx} = \frac{e^{\sqrt{x}}}{2^{\sqrt{x}}},\ \frac{du}{dx} = \frac{1}{2x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{\sqrt{x}}\ \frac{1}{2x} + \frac{e^{\sqrt{x}}}{2}.\ \text{In}(x)^{1/2}$$

$$= e^{\sqrt{x}}\left(\frac{1}{2x} + \frac{\text{Inx}}{4\sqrt{x}}\right)$$

Example: if $y = \frac{1}{2}(ex + c^x)$ fine $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

If $y = x^2e^{-x}$ find $\frac{d^3y}{dx^2}$

Solution

Let $y = uv$, where $vv = x^2$, $u = e^{-x}$

$\frac{dy}{dx} = 2\,x\,e^{-x} - x^2\,e^{-x}$

$\frac{d^2y}{dx^2} = 2\,e^{-x} - 4x\,e^{-x} + x^2\,e^{-x}$

$\frac{d^2y}{dx^2} = -\,6e^{-x} + 6x\,e^{-x} - x^2\,e^{-x}$

3.4 Differentiation of the Function A^u

You will use the method above to differentiate the function y = au where u is a real differentiable function of x.

if $a > 0$ and

$b = 1n\,a$ ——— I

Then $e^b = e^{lna} = a$

Given that u is a differentiable function of x and

$au = e^{lna} = a$

then $In\,a^u = uIna$

To find the derivative of $y = a^u$.

Given that $y = a^u$

Then $\frac{dy}{dx} = \frac{d}{dx}\,a^u = \frac{d}{dx}(e^{uIna})$

$= e^{uIna}\,\frac{d}{dx}e.\,(uIna)$

$= e^{uIna}.\,Ina\,\frac{du}{dx}.$

$= a^u \ln a \frac{du}{dx}$.

$\frac{d}{dx}(a^u) = a^{ulna}$ du

Example: find $\frac{dy}{dx}$ if

(I) $y = 4^{\ln x}$

(II) $y = 2^{-(x2+1)}$

(III) $y = 5^{\sqrt{x}}$

Solutions

(1) $y = 4^{\ln x}$

let a = 4, u = Inx

$y=a^u$

$\frac{dy}{dx} = a^u \text{In } a \frac{du}{dx} = 4^{\text{Inx}} . \text{In } 4 . \frac{1}{x}$

$\frac{4^{\text{In } x} \text{ In}4}{x}$

(II) $y = 2^{-(x2+1)}$

$y= a^u$, a=2, $u=-(x^2+ 1)$

$\frac{dy}{dx} = a^u \text{In } a \frac{du}{dx} = 2^{-(x2+1)}. \text{ In } 2(-(x^2+1))$

$= 2^{-(x2+1)}. \text{ In } 2. -2x$

$= 2x \ln 2(2^{-(x2+1)})$.

(III) $y=5^{\sqrt{x}}$

$y = a^u$, a = 5, $u = \sqrt{x}$

$\frac{du}{dx} = 5^{\sqrt{x}} . \text{ Ln } 5. \frac{1}{2} x^{-½}$

Further Examples

Find $\frac{dy}{dx}$ if

(I) $y = e^x \text{In } x^3$

(II) $y^2 = e^{-x}$

(III) $y^{2/3} = \left(\frac{x + 1}{(x - 1)}\right)^{1/5}$ $x > 1$

(IV) $y = x^{1/x}$

(V) $y = 1n\,(1nx^2)$

(VI) $x = \text{In } y.$

Solution

(I) $y = e^x \text{In } x^3$

$= e^x\, 31nx.$

Let $u = e^x$, $v = 31nx.$

$\frac{dy}{du} = u\frac{dv}{dx} + v\frac{du}{dx} = \frac{3}{x}e^x + e^x\, 31nx$

$= 3e^x\,(\frac{1}{x} + \ln x).$

(II) $y^2 = e^{-x}$

$21ny = 1ne^{-x} = -x$

$\frac{d}{dx}(2\ln y) = -1$

$\frac{2}{y}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{y}{2} = \frac{\sqrt{e\text{-}x}}{2}$

(III) $y^{2/3} = \left(\frac{x+1}{x-1}\right)^{1/5} \quad x > 1$

$\text{In } (y)^{2/3} = \text{In}\left(\frac{x+1}{x-1}\right)^{1/5} \quad x > 1$

$\frac{2}{3}\frac{dy}{dx} = \frac{1}{5}\text{ Inu} \quad \text{where } u = \left(\frac{x+1}{x-1}\right)$

$\frac{2}{3y}\frac{dy}{dx} = \frac{1}{5} \;.. \quad \frac{(x-1)-(x+1)}{(x-1)^2}$

$\frac{2}{3y}\frac{dy}{dx} = \frac{1}{5}\left(\frac{x+1}{x-1}\right) . \frac{-2}{(x-1)^2}$

$\frac{dy}{dx} = \frac{3}{10}\left(\frac{-2}{x^2-1}\right)\left(\frac{x+1}{x-1}\right)$

(IV) $y = x^{1/x}$

$\text{In } y = \text{Inx}^{1/x} = \frac{1}{x}\text{ Inx.}$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} . \frac{1}{x} + \frac{-1}{x^2}\text{ Inx}$

$\frac{dy}{dx} = y\ \frac{1}{x^2} - \frac{1}{x^2}\text{ In x} = x^{1/x} . \frac{1}{x}(1 - \text{Inx})$

$= x^{1/x-1}(1 - \ln x)$

$X^{1-x/x}(1 - \text{Inx})$

(V) $y = \text{In}(\ln x^2)$

$y = 1n\ u, \quad \text{where } u = 1nx^2$

$\frac{du}{dx} \quad \frac{1}{x^2} . 2x = \frac{2}{x}$

$= \frac{2}{x\text{Inx}^2}$

(VI) $x = \text{In } y.$

x = In y.

$1 = \frac{1}{y}\frac{dy}{dx}$ ====> $y = \frac{dy}{dx}$

but y = e^x ==> $\frac{dy}{dx}$ ex ($e^x = e^{lng}$ g = y.)

4.0 CONCLUSION

In this unit you have studied how to differentiate logarithmic and exponential functions. You have studied additional methods of finding the derivative of functions, by application of logarithmic differentiation. Differentiations of certain function that are rigorous have been made easy by the method of logarithmic differentiation. The differentiation of exponential function which is very useful in solving problems of growth or decay and computing compound interest on invested money has been studied by you in this unit. you will use the knowledge gained in this unit to solve problems involving differentiation of trigonometric and hyperbolic functions in the next unit. Make sure you do all your assignments. Endeavour to go through all the solved examples.

5.0 SUMMARY

In this unit you have studied how to

(1) differentiate the function f (x) =1nu
i.e. $\frac{d}{dx}$ (lnu) = $\frac{1}{u}\frac{du}{dx}$

(II) differentiate the function f (x) =$10g_a$ u

(III) differentiate the function f (x) = e^u

i.e $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

(IV) to final derivative of complicated functions by applying logarithmic differentiation (i.e taking natural logarithmic of both sides of the equation before differentiating).

(V) to find the derivative of the function $f(x) = a^u$

i.e. $\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$

6.0 TUTOR-MARKED ASSIGNMENT

1. Find $\frac{d}{dx} \log, \sqrt{\frac{x+1}{x^2}}$

2. Find $\frac{d}{dx} \log\ / x^4 - 1 /$

3. Differentiate $y = \log \frac{x^2}{x^3 + 1}$

4. Differentiate $y = x^{2x}$

5. If $y = \frac{(x - 1)(x^2 - 1)^{1/5}}{x^2 + 1}$ find $\frac{dy}{dx}$

6. find $\frac{dy}{dx}$ if $\sqrt{y} \left(\frac{x^2 + x}{x^2 - 1}\right)^{1/7}$

7. If $y = \frac{(x + 1)(x - 2)(x^2 + 1)(x^2 - 1)}{x^3}$ find $\frac{dy}{dx}$

8. If $7\sqrt[3]{y} = e^x$ find $\frac{dy}{dx}$

9. find dy/dx if $y = e^{\sqrt{x2+1}}$

10. Given that $y = x e^x$ find d^3y/dx^3

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 DIFFERENTIATION OF TRIGONOMETRIC 41 FUNCTIONS

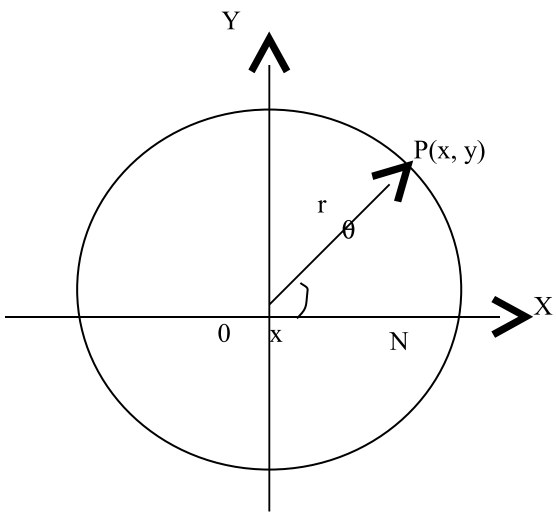
CONTENTS

1.0 INTRODUCTION

So far you have studied how to differentiate various types of functions such as polynomial, rational, fractional, exponential and logarithm functions. You have applied rules of differentiation to differentiate the sums, products, quotients and roots of these functions. In this unit you will be differentiating the class of functions which are periodic. Such periodic function are best studied using trigonometric ratios such as sine and cosines. You are already familiar with trigonometric ratios of cosines and sines in your SSCE/GCE mathematics. Their properties are briefly studied here as (see Fig 9.1)

(i) $\sin \theta = y$ (ii) $\cos \theta = x/r$ (iii) $\tan \theta =$ (iv) $\text{cosec}\, \theta = r/y$

(v) $\sec \theta = r/x$ and (vi) $\cot = x/y$



The trigonometric ratios given above are structured by placing an angle of measure θ in standard position at the center of a circle of radius r and finding the ratios of the sides of the triangle O PN.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

i) derive the derivation of the function y = sin x from first principle.
ii) derive the derivatives of trigonometric function such as cos x, tan x, cosec x and sec x.
iii) differentiate combination of various types of trigonometric functions.

3.0 MAIN CONTENT

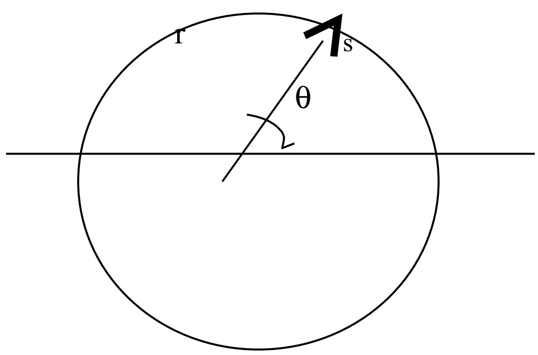
3.1 Differentiation of Sines

A good starting point for the differentiation of the trigonometric ratio of sine is imbedded in the concept of evaluating the limit.

$$\lim_{\theta \to \theta} \frac{\sin \theta}{\theta}$$ where θ is measured

in radian (a radian measure is uniteless)

Fig 9.2



θ (in radian)

= s/r

Prove that $\lim_{\theta \to \theta} \frac{\sin \theta}{\theta}$ θ is 1

From the above a direct calculation will not be possible because division by zero is not possible. Therefore, you have to go through a formal proof of the above since you will need to find the derivative of the function $f(x) = \sin u$.

Proof

Let θ > O and also measured in radian

Let θ be a small angle at the center of the circle (see Fig 9.3) or radian radius r = 1

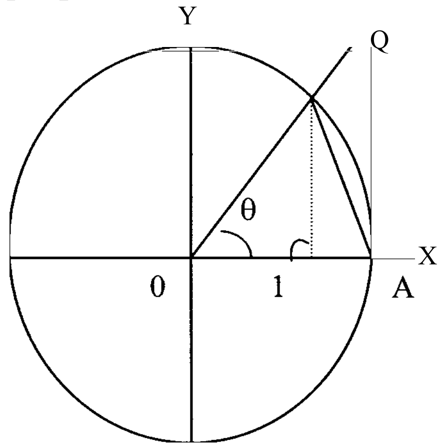


Fig 9.3.

In fig 9.3 OP and OA are side of the angle 0. OA is the targent to the cicle at point A and meets side OP at Q.

Note that

Area of AOPA = ½ base x height
= ½ (OA) (h) = ½ (1) (OP sineθ)
= ½ (1) (1)sinθ = ½ sinθ

Area of sector OPA = $\frac{1}{2} r^2 \theta = \frac{1}{2} (1)^2 \theta$.

Area of ΔOQA = ½ (OA) (QA) = ½ (1) (tan θ)
= ½ tan θ .

Fig. 9.3

Area of ΔOPA < Area of sector OPA < Area of ΔOQA

$$\Rightarrow \quad \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta \qquad \text{\_\_\_\_\_\_} \qquad 1$$

Since θ> 0 and small than sin) > 0
dividing the inequalities in (12) by ½ sin θ you get.

$1 \leq \theta < 1$ _________ II

taking the reciprocal (II) you get _________ III

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

taking limits in (III) as θ -->0

$$\lim_{\theta \to 0} 1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > \lim_{\theta \to 0} \cos\theta$$

$$1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > 1$$

the above hold for θ < 0. Since cos θ is an even function (see unit 2) i.e.

cos (-θ) - cos θ and sin θ is add i.e.

$$\sin(-\theta) = -\sin\theta \Rightarrow \frac{\sin(-\theta)}{-\vartheta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

Using the above fact you can now derive a formula for $\frac{d}{dx}(\sin u)$

Let Δ u as usual be an increment in u with a corresponding increment

Δy is y. if y = sin U
then y + Δy = sin (u + Δu) _________ I

subtracting y from y + Δy you get

$$\Delta y = \sin(u + \Delta u) - y$$
$$= \sin(u + \Delta u) - \sin U \quad \text{\_\_\_\_\_\_\_\_\_\_ II}$$

applying the factor formula i.e. $\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$
to the right side of equation II you get

$$\Delta y = 2\cos\frac{((u+\Delta u).- u)}{2}\sin\frac{((u+\Delta u) - u)}{2}$$

$$= 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_ III}$$

dividing equation III through by Δu you have

$$\frac{\Delta y}{\Delta u} = 2\cos u + \frac{\Delta u}{2}\left(\frac{\sin \Delta u/2}{\Delta u}\right)$$

$$= \cos\left(u + \frac{\Delta u}{2}\right)\frac{\sin \Delta u/2}{\Delta u} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_IV}$$

setting $\theta = \Delta u/2$ equation IV becomes

$$\frac{\Delta y}{\Delta u} = \cos(u + \theta)\frac{\sin \vartheta}{\theta}$$

taking limits in equation (V) as $\Delta u \to 0$

$$\lim_{\Delta u \to 0}\frac{\Delta y}{\Delta u} = \lim\left[\cos(u+\theta)\frac{\sin\theta}{\theta}\right] \qquad \text{\_\_\_\_\_\_\_\_\_\_\_ VI}$$

$$\lim_{\theta \to 0}\left[\cos(u+\theta)\frac{\sin\theta}{\theta}\right]$$

(since $\theta = \frac{\theta u}{2}$, so as $\Delta u \to 0, \theta \to 0$)

Equation VI becomes

$$\frac{dy}{du} = \cos U.\ 1 = \cos U.$$

since U is a differentiable function of x by the chain rule you get

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

i.e. if y = sin u, $\frac{-dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{du} = \cos u,$

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

The above process is known as differentiation of sin u from first principle or limiting process.

Example find $\frac{dy}{dx}$ if

(i) y = sin 5x

(ii) y = sin x^2

(iii) y = sin √x

(iv) y = sin (In x)

Solution:

(i) y = sin 5x

Let sin U where U = 5x

Then $\frac{dy}{dx} = \cos U$ $\frac{dy}{dx} = \cos 5x \cdot 5$

= 5 cos 5x

(ii) $y = \cos e^x + \sin x^2$

let $y = \cos u + \sin v$, where $u = e^x$, $v = x^2$.

$\therefore \frac{dy}{dx} = -\sin u \frac{du}{dx} + \cos v \frac{dv}{dx}$

$= -\sin e^x \cdot e^x + \cos x^2 \cdot 2x$
$= 2x \cos x^2 - e^x \sin e^x.$

(iii) $y = \frac{\sin x}{\cos x}$

let $y = \frac{u}{v}$, $u = \sin x$, $v = \cos x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

(iii) $y = \frac{\cos x}{\sin x}$

let $y = \frac{u}{v}$, where $u = \cos x$, $v = \sin x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\sin x(-\sin xx) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

(iv) $y = (\sin x)^{-1}$

let $y = u^{-1}$ and $u = \sin x$

$$\frac{dy}{dx} = \frac{-1}{u^2} \cdot \frac{du}{dx}, \quad \frac{du}{dx} = \cos x.$$

(v) $y = (\cos x)^{-1}$.

Let $y = u^{-1}$ and $u = \cos x$

$$\frac{dy}{dx} = \frac{-1}{u^2} \cdot \quad \frac{du}{dx} = \frac{-1}{\cos^2 x} - \sin x$$

$$\frac{\sin x}{\cos^2 x}.$$

(vi) $y = \cos(\sin^2 x)$

let $y = \cos(u)$
let $u = v^2$ where $v = \sin x$

AQ

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

$$\frac{dy}{du} = -\sin u, \quad \frac{du}{dv} = 2v, \quad \frac{dv}{dx} = \cos x.$$

$$\frac{dy}{dx} = -\sin(\sin^2 x).\ 2\sin x . \cos x.$$

$$= 2\sin x \cos x \sin(\sin^2 x)$$

DIFFERENTIATION OF tan u.

Since $\tan u = \frac{\sin u}{\cos u}$

Let $y = \tan u = \frac{\sin u}{\cos u}$

Using example (III) above we get:

$$\frac{dy}{dx} - \frac{d}{du}(\tan u) = \frac{\cos u(c \cos u)-(-\sin u)(\sin u)}{\cos^2 u}$$

$$= \frac{1}{\cos^2 u} = \sec^2 u.$$

$$\boxed{\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}}$$ ~

Exercise: Derive the formula for the derivative of cot u, where u is a
differentiable function of x. (see example (IV) above).

$$\boxed{\frac{d}{d}(\cot u) = \operatorname{cosec}^2 u \frac{du}{dx}}$$

Differentiation of sec u

Let $y = \sec u = \frac{1}{\cos u}, = (\cos u)^{-1}$

$$\frac{d}{dx}(\sec u) = -1\,(\cos u)^{-2} . (-\sin u)\ \frac{du}{dx}$$

$$= \frac{\sin u}{\cos^2 u}\ \frac{du}{dx}$$

$$\frac{\sin u}{\cos u} . \frac{1}{\cos u}\ \frac{du}{dx}$$

$$= \tan u \sec u\ \frac{du}{dx}$$

$$\boxed{\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx}}$$

SELF ASSESSEMENT EXERCISE 1

Derive the formula for the derivative of cosec u, where u is a differentiable function of x (see example (IV)above)

$$\boxed{\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx}}$$

3.1 Differentiation Of Other Trigonometric Functions

Example: Find $\frac{dy}{dx}$ if

(i) $y = \cot \sqrt{x}$ (ii) $y = \sqrt{x}\tan(\sqrt{1-x})$

(iii) $y = \sec^2 2x$ (iv) $y = \tan x \sec x.$

(v) $y = \tan(x^2 + \sec x)$ (vi) $x \cos 2y = y \sin x$

(vii) $y = \frac{2x}{\cos 3x}$ (viii) $y = \frac{x + \cos^2 x}{\sin x}$

(vx) $y = \cot^2 x \tan x$ (x) $y = e^x \cos x^2\ y = \sin y.$

Solution

(i) $y = \cot\sqrt{x}$

$y = \cot u$, $u = x^{½}$

$\frac{dy}{dx} = -\text{cosec}^2 u \frac{du}{dx}$, $\frac{du}{dx} = \frac{1}{2} x^{-½}$

$\frac{dy}{dx} = -\text{cosec}^2 u\,(x) \frac{1}{2\sqrt{x}} = \frac{-\text{cosec}2(\sqrt{x})}{2\sqrt{x}}$

(ii) $y = \sqrt{x} - \tan(\sqrt{1 - x})$

let $y = uv$, $u = \sqrt{x}$, $v = \tan z$, $z = (I - x)^{½}$

$\frac{dy}{dx} = \frac{du}{dx}.v + u \frac{dv}{dx}$

$\frac{\tan z.}{2\sqrt{x}} + \sqrt{x} . \left(\frac{dv}{dz} . \frac{dz}{dx}\right)$

$= \frac{\tan (1 - x) ½}{2\sqrt{x}} + \sqrt{x} \left(\sec2\,(I - x)\,½ . \frac{1}{2} (1 - x) - ½\right)$

$= \frac{\tan(1 - x)^{½}}{2\sqrt{x}} + \frac{\sqrt{x}}{4} \left(\frac{\sec^2 \left[\frac{1}{\sqrt{1 - x}}\right]}{\sqrt{(1 - 4)}}\right)$

(iii) $y = \sec^2 2x$

$y = \sec^2 (u)$, $u = 2x$

$y = (\sec u)^2$

$\frac{dy}{dx} = 2\sec u. \frac{d}{d} (\sec u)$

$2 \sec u. - (\tan u \sec u) . 2$

$\frac{dy}{dx} = 4 \sec^2 (2x) \tan 2x.$

(iv) $y = \tan x \sec x.$

let $y = uv$ $u = \tan x$ $v = \sec x$

$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$, $\frac{du}{dx} = \sec^2 x \frac{dv}{dx}$ tan x sec x

$\frac{dy}{dx} = \tan x(\tan x \sec x) = \sec 2$ xtnx.

(v) $y = \tan (x^2 + \text{sce } x)$

$y = \tan u$, $u = x^2 + \sin x$.

$\frac{dy}{dx} \sec^2 u \frac{du}{dx}$, $\frac{du}{dx} = 2x = \cos x$

$\frac{dy}{dx} = \sec^2 (x^2 + \sin x)(2x + \cos x)$

(vi) x cos 2y = y sin x

using implicit differentiation you

if u = cos 2y.

$\frac{du}{dx} = 2 \frac{dy}{dx}(- \sin 2y)$,

.'. $- 2x \sin 2y \frac{dy}{dx} + \cos y = \sin x \frac{dy}{dx} + y \cos x$.

∴ $\cos y – y \cos x = (\sin x + 2x\sin 2 y)\frac{dy}{dx}$

∴ $\frac{dy}{dx} = \frac{\cos y - y \cos x}{\sin x + 2x \sin 2y}$.

(vii) $y = \frac{2x}{\cos 3x}$

let $y = \frac{u}{v}$

$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$u = 2x, \quad v = \cos 3x, \quad \frac{du}{dx} = 2, \quad \frac{dv}{dx} = -3 \sin 3x.$

$$\frac{dy}{dx} = \frac{2\cos 3x - 2x.(-3\sin 3x)}{\cos^2 3x.}$$

$$= \frac{2\cos 3x + 6x \sin 3x}{\cos^2 3x.}$$

(viii) $y = \frac{x + \cos^2 x}{\sin x}$

$y = \frac{u}{v,} = x + \cos^2 x, \; v = \sin x.$

$\frac{du}{dx} = 1 - 2\cos x.\sin x \quad \frac{dv}{dx} = \cos x.$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dsx}}{v^2.}$$

$$= \frac{\sin x\,(1 - 2\cos x \sin x) - (x + \cos^2 x)\cos x}{\sin^2 x}$$

(ix) $y = \cot^2 x \tan x$

let $y = uv, \quad u = \cot^2 x, \quad v = \tan x$

$\frac{du}{dx} = 2\cot x\,(-\text{cosec}^2 x)$

$\frac{dv}{dx} = \sec^2 x.$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \cot^2 x.\sec^2 x + \tan x.(-2\cot x\,\text{cosec}^2 x)$

$= \cot x(\cot x\sec^2 x - \tan x\,\text{cosec}^2 x)$

$$= \frac{-(1 - \text{cosec}^2 x + \text{cosec}^2 x\,.\,\cos^2 x)}{(\cos^2 x - 1).}$$

(x) $y = e^x \cos x^2\, y = \sin y.$
let $u = e^x, \quad v = \cos z, \quad z = x^2 y$

$$\frac{du}{dx} = ex\ ,\quad \frac{dv}{dz} = -\sin z,\quad \frac{dz}{dx} = 2xy + x^2\frac{dy}{dx}$$

$$\frac{duv}{dx} = u\frac{dv}{dx} + \frac{du}{dx}$$

$$\cos y\, dy + e^x (=(\sin x^2 y) = (2xy + x^2 dy) + \cos x^2 y.e^x$$

collecting like terms

$$\cos y \frac{dy}{dx} + e^x \sin x^2 yx^2 dy = -e^x \sin x^2 y2xy + \cos x^2 ye^x$$

$$\Rightarrow \frac{dy}{dx}(\cos y + x^2 e^x \sin x^2 y) = e^x (\cos x^2 y - 2xy \sin x^2 y)$$

$$\frac{dy}{dx} = \frac{ex (\cos x^2 y - 2xy \sin x^2 y}{\cos y + x^2 e^x \sin x^2 y.}$$

4.0 CONCLUSION

In this unit you have studied now to derive the derivative of f (x) = sinx from first principle i.e. using the limiting process. You have extended it to finding basic formula for the derivative of cos x, tan x, cosec x, and sec x., You have used rules for differentiation studied unit 8 to find the derivatives of functions involving trigonometric functions.

5.0 SUMMARY

In this unit you have studied how to;

(i) Derive the formula $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ from first principle.

(ii) Use $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ to derive the formula $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$.

(iii) Differentiate functions involving various combination of trigonometric functions. Such as cos(sec^2 x) x sin^2 3x^2 etc.

(iv) How to differentiate functions involving inverse hyperbolic functions such as arc sin h u, arc cos h u and arc tan h u.

6.0 TUTOR-MARKED ASSIGNMENTS

(1) find $\lim_{x \to 1} \frac{(x - 3)}{\sin(x - 3)}$

(2) $\lim_{x \to 3} \frac{\sin(x^2 - 1)}{\sin(x^2 - 1)}$

(3) find the $\frac{dy}{dx}$ if

(i) $\cos y = \sin x^2$
(ii) $y = \cos(\ln x)$
(iii) $y = \tan(x^2 - 1)$

(4) If $2\sin y = \cos(\tan x)$ find $\frac{dy}{dx}$

(5) Find dx if $y = \sqrt{x} \cos(\sqrt{x})$

(6) Find $\frac{dy}{dx}$ if $y^2 = \tan x \operatorname{cosec} x$.

(7) Given that $y = e^x \cos(e^x)$

(8) Derive the formula $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

(9) Derive the formula $\frac{d}{dx}(\cos ec u) = - \operatorname{cosec} u \cot u \frac{du}{dx}$

(10) Derive the formula $\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u \frac{du}{dx}$

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UNIT 4 DIFFERENTIATION INVERSE TRIGONOMETRIC FUNCTIONS AND HYPERBOLIC FUNCTIONS

CONTENTS

1.0 INTRODUCTION

You have already studied how to differentiate trigonometric functions of sins, cosines, tangent, secant and cosecant. In this unit you will study how to differentiate their respective inverses. The derivatives of inverse trigonometric functions are very useful in evaluating integral, of a certain trigonometric functions. Therefore your understanding of this unit will help you tremendously in the course on integral calculus i.e. calculusis.

In this unit you shall also differentiate a special class of function that is derived as a combination of exponential e^x and e^{-x} which you are already familiar with in previous units. These combination produce functions that are called hyperbolic functions. They are engineering problems.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

1) differentiate the inverse trigonometric functions such as arc(sin u) , arc (cos u) arc (tan u), arc (sec u) and arc (cosec u).
2) Find the derivative of the inverse hyperbolic function of arc (sin hu) and arc (cos hu) etc.

3.0 MAIN CONTENT

3.1 Differentiation of Inverse Sine and Cosines Functions

In this unit you will use the knowledge you acquired when you studied unit 2 and unit 9 to study the inverse of a trigonometric function. This section is important because the concept you will study here will be useful in the second course of calculus. Recall that the inverse of a function $f(x)$ is that function. $f^{-1}(x)$ for which its composite with $f(x)$ yields the identical function:

$$\text{i.e.} \quad f(f^{-1}(x)) = f^{1}(f(x)) = x.$$

You could begin the study of differentiation of trigonometric functions by examining the inverse of the sine function. Consider the equation

$$x = \sin y$$

In this equation you can show that infinitely many values of y corresponds to each x in the interval [-1,1] i.e. only one of these values y lies in the interval

$$\left(\frac{-\pi}{2}, \frac{\pi}{2} \right).$$

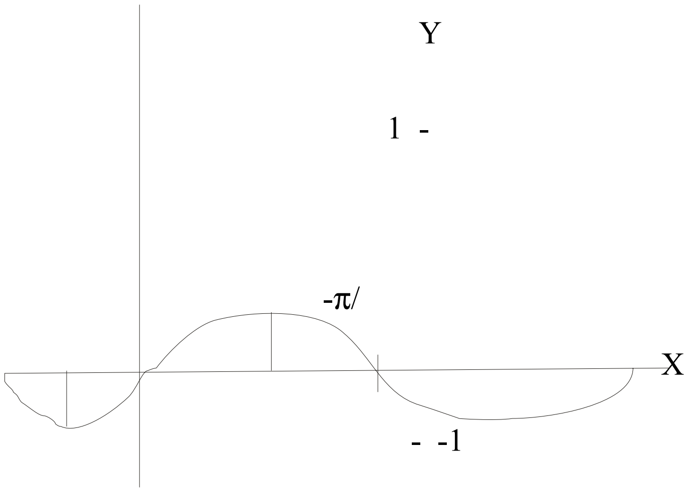
For example if x = ½ then you might wish to know the values of all angles y such that sin y = ½ . These two angles y = 30° and y = 150° will come readily to your mind. Multiples of these two angles will give the sine value to be ½ .

$$\text{i.e. } \sin 30° = ½ , \sin 150° = ½ . \sin k\, y = ½$$

for k = 1, 2, ..., and y = π/6, note that

$$150^0 = 5(30°).$$

Consider the graph of y = sin x as shown in fig. 9.3.



If you interchange the letters(variables) x and y in the original equation y = sin x you will clearly see the what is being discussed so far in this section .

That is x = sin y, x ε [-l, 1] and $\left[-\pi/2, \pi/2\right]$

In the interval [- π/2, - π/2] the function f (x) = sin x is a one to one function (see Fig 9.3 no horizontal line cuts the graphs only once).

Therefore within the interval [- π/2, - π/2] the inverse exist and it called the inverse sine function and it is written as y = arc sin x (or sin-' (x)) (see Fig 9.4 and 9.5.)

Remark: You will use the arc sin x frequently to represent the inverse sine function. The notation $\sin^{-1}(x)$ could be used if you are sure you will not confuse it with the function

$$\frac{1}{\sin x}.$$

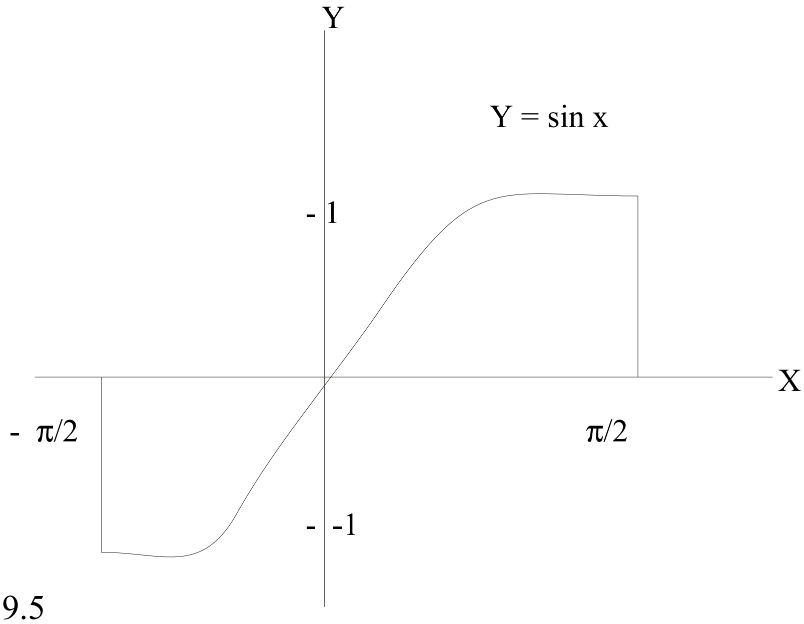


Fig: 9.5

In Fig 9.4 the function y = sin x is a continuous so also is y = arc (sin x).

$$\text{The function } \frac{dy}{dx} = \frac{d}{d2}(\sin x) = \cos x$$

is defined in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. and there is no x ε $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

such that cos x = 0. So also the derivative of the inverse sine function does not take any value zero in the open interval (-1, 1) i.e.

$$d(\text{arc} \sin x) \neq 0 \; \forall \; x \; \varepsilon \; (-1, 1)$$

With the above information you can now proceed to derive a formula for the derivative of arc sin x.

Let $f(x) = \sin x$

Then $f^{-1}(x) = \text{arc} \sin x$

Note that $f(f^{-1}(x)) = x$.

Therefore $\frac{d}{dx} - (f(f^{-1}(x)) = \frac{d}{dx}(x) = 1$

$$\Rightarrow \quad \frac{d}{dx} \sin(\arcsin x) = 1.$$

$$\Rightarrow \quad \cos(\arcsin x) \frac{d}{dx}(a \arcsin x) = 1.$$

Hence $\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(a \arcsin x)}$

$$= \frac{1}{\sqrt{1 - \sin^2 (\arcsin x)}}$$

note that : $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos x = \sqrt{1 - \sin^2 x)}$

thus $\frac{d}{dx}(\arc \sin x) = \frac{1}{\sqrt{1 - x^2}}$

note also that;

$\sin(\arc \sin x) = x$ then $(\sin(\arc \sin x)^2 = x^2)$

you could also derive the above formula by applying implicit differentiation Given that:

$y = \arc \sin x$

$\Rightarrow \quad \sin y = x$

then $\Rightarrow \quad \frac{d}{d}(\sin y) = \frac{dx}{dx}$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

putting $y = \arc \sin x$ and $\sin y = x$ you get :

$$\frac{d}{dx}(\arc \sin x) = \frac{1}{\sqrt{1 - x^2}}$$

let $y = \arc \sin u$ where u is a differentiable function of x.

Then;

$$\frac{d}{dx}(\text{arc sin } u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx.}$$

Differentiation of Inverse Cosine

Given that y = arc cos u

Let cos y = u

Then $\frac{d}{dx}(\cos y) = \frac{dy}{dx}$

$-\sin y \frac{dy}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-1}{\sin y} \frac{dy}{dx}$

$\sin y = \sqrt{1 - \cos^2 y}$. where cos y = u

Therefore

$$\frac{d}{dx}(\text{arc cos } u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx.}$$

Differentiation of Inverse Tangent

Given that y = arc tan u

Let tan y = u

then $\frac{d}{dx}(\tan y) = \frac{du}{dx}$

$\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{\text{Sec}^2 y} \frac{du}{dx.}$

(note that if $\tan^2 y = \sec^2 y$ and tan y == u) therefore $\frac{dy}{dx} = \frac{1}{1 + \tan^7 y} \frac{du}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$

hence

$$\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

Differentiation of arc sec u.

Given that y = arc sec u

Let sec y = u

$\frac{d}{dx}(\sec y) = \frac{du}{dx}$

$\sec y^t \text{ an } y \frac{dy}{dx} = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \frac{du}{dx}$

(note that tan y = ± √sec² y – 1 and sec y = u)

therefore $\frac{dy}{dx} = \frac{1}{u \pm \sqrt{u^2 - 1}} \frac{du}{dx}.$

hence

$$\frac{d}{dx}(\text{arc tan } u) = \left(\frac{1}{/u/\sqrt{u^2 - 1}}\right) \frac{du}{dx}.$$

DIFFERENTIATION OF y = arc cot u

Give y = arc cot u

Let cot y = u

$\frac{d}{dx}(\cot y) = \frac{du}{dx}$

$$- \text{cossec}^2 y \frac{dy}{dx} = \frac{dx}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y} \frac{du}{dx}$$

(but $\text{cosec}^2 y = 1 + \cot^2 y$, $\cot y = u$)

then

$$\frac{d}{dx}(\text{arc cot } u) = \frac{1}{1 + u^2} \frac{du}{dx.}$$

DIFFERENTIATION OF y = arc cosec u.

Given that y = arc cosec u.

Then cosec y = u

$$- \cot y \text{ cosecy} \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\cot y \text{ cosecy}} \frac{du}{dx.}$$

but $\cot y = \pm \sqrt{\text{cosec}^2 y - 1}$ cosec y = u

then $$\frac{dy}{dx} = \frac{-1}{u\sqrt{u^2 - 1}} \frac{du}{dx.}$$

Therefore

$$\frac{d}{dx}(\text{arc tan } u) = \frac{1}{/u/\sqrt{u^2 - 1}} \frac{du}{dx.}$$

Examples

Find dy/dx if;

1. $y = \text{arc sin } x^2$
2. $y = \text{arc cos } 2x^3$
3. $y = \text{arc tan } (x + 1)^2$
4. $y = \text{arc cot} \left(\frac{x + 1}{x - 1}\right)$
5. $y = x^2$ (arc sec 2x).

Solutions

(1) $y = \text{arc sin } x^2$

let $y = \text{arc sin } u, u = x^2$

$$\frac{dy}{dx} = \frac{1}{1 - u^2} \quad \frac{dy}{dx} = \frac{1}{1 - (x)^2} 2x$$

$$= \frac{2x}{\sqrt{1 - x^4}}$$

(2) $y = \text{arc cos } 2x^3$

let $y = \text{arc cos } u, u = 2x^3$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - u^2}} \frac{du}{dx} = \frac{-1}{\sqrt{1 - (2 \text{ x } 3)^2}} .6x^2$$

$$=- \frac{6x^2}{1 - 4x^6}$$

(3) $y = \text{arc tan } (x + 1)^2$

let $y = \text{arc tan } u, u = (x +1)^2$

$$\frac{dy}{dx} = \frac{1}{1 + u^2} \frac{du}{dx} = \frac{1}{1 + (x + 1)^4} .2(x + 1)$$

$$= \frac{2(x + 1)}{1+(x+1)^4}$$

(4) $y = \text{arc cot } \left(\frac{x + 1}{x - 1}\right)$

let $y = \text{arc cot } u, u = \left(\sim \frac{x + 1}{x - 1}\right)$

$$\frac{dy}{dx} = \frac{-1}{1 + u^2} \frac{du}{dx} = \frac{-1}{\left(\frac{x + 1}{x - 1}\right)^2} \cdot \frac{-2}{(x - 1)^2}$$

$$= \frac{1}{x^2 + 1}$$

5. $y = (x^2 \text{ arc sec } 2x)$

let y = u v and u = x^2, v = arc sec 2x

let v = arc sec z, z = 2x.

$$\frac{dv}{dx} = \frac{1}{|2|\sqrt{2^2 - 1}} . \frac{d2}{dx} = \frac{1}{2x\sqrt{4x^2 - 1}} .2x$$

but

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \frac{x^2}{\sqrt{4x2 - 1}} + (arc\ \sec\ 2x).2x.$$

$$\Longrightarrow \frac{dy}{dx} - \frac{x^2}{4x^2 - 1} + 2x(\text{arc sec } 2x)$$

3.3 Differentiation of Hyperbolic Functions

You are already familiar with the differentiation of exponential function e^x and e^{-x}. These combinations occur in two basic forms ½ (e^x + e-') and ½ (e^x - e^{-x}). They occur so frequently that they have to be given a special attention. The types of function described above are known as hyperbolic functions (see unit 2 sec 3.2 for more details).

Definition: The hyperbolic sine and cosine are functions written as

Sin h x = ½ (e^x + e^{-x})and

Cos h x = ½ (e^x - e^{-x})

(Recall that the word hyperbolic is formed from the word hyperbola see unit 2 sec. 3.2).

Given sin h x and cos h x defined above you can easily form other hyperbolic function of tangent cotangent, secant and cosecant by noting that

Sin h x = ½ (e^x + e^{-x})and

Cos h x = ½ (e^x – é$^{-x}$)

Then

1. $\tan h\, x = \frac{\sin\ h\, x}{\cos h\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

2. $\cot h\, x = \frac{\cos h\, x}{\sin h\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

3. $\text{cosec}\, h\, x = \frac{1}{\sin h\, x} = \frac{2}{e^x + e^{-x}}$

4. $\sec hx = \frac{1}{\cos h\, x} = \frac{2}{e^x + e^{-x}}$

You will briefly review some of the identities associated with hyperbolic functions they follow the same pattern with those derived for trigonometric functions.

Note that the equation of a unit hyperbola is given as

$$x^2 - y^2 = 1$$

if you put x =cos h θ and sin h θ = y

then $x^2 = \cosh^2 \theta$ and $y^2 = \sin h^2\, \theta$

$\Rightarrow \quad \cos h^2 - \sin h^2\, \theta = 1$ __________ (1)

Then by substituting appropriately you get the following identities

$1 - \tan h^2\, \theta = \text{cosec}\, h^2\, \theta$ __________ (2)

$\cot h^2\, \theta - 1 = \text{cosec}\, h^2\, \theta$ ___________(3)

The identities will be useful in finding the derivative of inverse hyperbolic functions.

DIFFERENTIATION OF Sin h u.

Let y = sin h u., where u is a differentiable function of x.

then:

$$\frac{dy}{dx} = \frac{d}{dx}(\sin h\, u = \frac{d}{dx}\left(\frac{(e^u - e^{-u}}{2}\right)$$

$$\frac{\frac{d}{dx} e^u - \frac{d}{dx} e^{-u}}{2}$$

$$\frac{e^u \frac{d}{dx} + e^{-u} \frac{d}{dx} e^{-u}}{2}$$

$$= \frac{1}{2}\left(e^u + e^{-u}\right)\frac{du}{dx}$$

$$= \cos h \frac{du}{dx.}$$

$$\frac{d}{dx}(\sin h\ u) = \cos h\ u \frac{du}{dx}$$

DIFFERENTIATION OF cos h u

Let y = cos h u

Then:

Therefore

$$\frac{d}{dx}(\cos h\ u) = \sin\ h\ u \frac{du}{dx}$$

DIFFERENTIATION OF tan h u.

Let y = tan h u.

Then:

$$\frac{dy}{dx} = \frac{d}{dx}(\tan h) = \frac{\sin h\ u}{\cos h\ u}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos h\ U \frac{d}{dx}(\sin h\ U) - \sin h\ U \frac{d}{dx}(\cos h\ U)}{\cos h^2 u}$$

$$= \frac{1}{\cos h^2 u} = \sec h^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\tan h\ u) = \sec h\ u \frac{du}{dx}$$

SELF ASSESSMENT EXERCISE 1

Using the method above:

(1) show that $\frac{d}{dx}(\cot h\ u) = -\operatorname{cosec} h^2 u\ \frac{du}{dx}$.

(2) show that $\frac{d}{dx}(\sec h\ u) = -\sec h\ u \tan h\ u\ \frac{du}{dx}$

(3) show that $\frac{d}{dx}(\operatorname{cosec} h\ u) = -\operatorname{cosec} h\ u \cot h\ u\ \frac{du}{dx}$

Examples

Find $\frac{dy}{dx}$ if

(i) $y = \tan h\ 3x$ (ii) $y = \cos h^2\ 5\ x$

(iii) $y = \sin h\ 3x^2$ (iv) $y = \sec h^3\ 2x^2$

(v) $\sin h\ x = \tan y$.

Solution

(i) $y = \tan h\ 3x$

let $y = \tan h\ u$, $u = 3x$

$\frac{dy}{dx} = \sec h^2\ u\ \frac{du}{dx} = \sec h^2\ (3x).\ 3$

$= 3 \sec h^2\ 3x$.

(ii) $y = \cos h^2\ 5\ x$

let $y = \cos h^2\ 5x$, $u = 5x$

$\frac{dy}{dx} = 2 \cosh u . \sin h\ u\ \frac{du}{dx}$

$= 2 \cos h\ 5x \sin h\ 5x.\ 5.$

$= 10 \cos h\ 5x \sin h\ 5x.$

(iii) $y = \sin h\ 3x^2$

let $y = \sin h\, u, \quad u = 3x^2$

$$\frac{dy}{dx} = \cos h\, u \frac{du}{dx} = \cos h\, 3x^2 \,.\, 6x$$

$= 6x \cos h\, 3x^2$.

(iv) $y = \sec h^2\, 2x^2$

let $y = \sec h^3\, u, \quad u = 2x^2$.

$$\frac{dy}{dx} = 3\sec h\, u\, (-\sec h\, u \tan h\,).4x$$

$= - 3 \sec h^2\, 2x^2 \,.\, \sec h\, 2x^2 \tan h\, 2x^2 \,.\, 4x$

$= 12x \sec h^3\, 2x^2 \tan h\, 2x^2$

(v) $\sin h\, x = \tan y$.

$$\cos h\, x = \sec h^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cos h\, u}{\sec h^2\, y}.$$

3.4 Differentiation of Inverse Hyperbolic Functions

In this section you will adopt the same pattern used in studying the differentiation of inverse trigonometric function to finding the derivative of the derivative of the inverse hyperbolic functions. In this course only the following hyperbolic inverse will be treated.

(I) Inverse hyperbolic sine i.e. $y = \text{arc} \sin h\, x$

(II) Inverse hyperbolic cosine i.e. $y = \text{arc} \cos h\, x$

(III) Inverse hyperbolic tangent i.e. $y = \text{arc} \tan h\, x$.

DIFFERENTIATION OF arc (sin h u)

Let $y = \text{arc} \sin h\, x$

$\Rightarrow \quad \sin hy=x$

$\Rightarrow \quad \frac{1}{2} (e^y - e^{-y}) = x$.

$e^y - e^{-y} = 2x$ / multiplying through by e^y we get.

$e^{2y} - 1 = 2x\,e^y$

$\Rightarrow \quad e^{2y} - 2xe^y = 0$

let $\quad e^y = P$

then $\quad P^2 - 2xp - I = 0$

solving for P you get

$P = \quad ½\,(2x \pm \sqrt{4x^2 + 4}\,)$

$e^y = x \pm \sqrt{x^2 + 1}$

Now find $\frac{dy}{dx}$ by Logarithmic differentiation

i.e take In of both sides

In ey = In (x + $\sqrt{x2 + 1}$) (note e^y . 0 hence you drop the minus sign.)

$y = \ln(x + x^2 + 1)$ y = In u

$\frac{dy}{dx} = \frac{1}{u} \frac{du}{dx},$ $\quad$ u (x + $\sqrt{x2 + l}$

$\frac{du}{dx} = 1 + \frac{x}{\sqrt{x^2 + 1}}$

therefore $\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{x}{x\sqrt{\ } + 1}\right)$

$= \frac{1}{\sqrt{x^2} + 1}$

$$\frac{d\,(\text{arcsin } hu)}{dx} = \frac{1}{\sqrt{x^2 + 1}} \frac{du}{dx}$$

DIFFEREENTIATION OF arc cos h u

Let $y = \text{arc cos h } x$

.'. $\cos h\ y = x$

$\frac{1}{2}(e^y + e^{-y}) = x$

$e^y + e^{-y} = 2x$/ multiplying through by e^y

$e^{2y} + 1 - 2x\ e^y = 0$

let $P = e^y$ you get a quadratic equation of the form.

$P^2 - 2x P + 1 = 0.$

Solving for P you get;

$P = \frac{1}{2}(2x \pm \sqrt{4x^2 - 4})$

$\therefore$ $ey = x \pm \sqrt{x^2 - 1}$

for $e^y > 0$ then

$e^y = x + \sqrt{x^2 - 1}$

To find $\frac{dy}{dx}$ by logarithmic differentiation you take natural logarithm of both sides and get;

$\text{Ln } e^y = \text{In }(x + \sqrt{x^2 - 1})$

$\frac{dy}{dx} = \frac{d}{d^2} \text{In } u,$ $u = x + \sqrt{x^2 - 1}$

$= \frac{1}{u} \frac{du}{dx},$ $\frac{du}{dx} = 1 + \frac{x}{\sqrt{x^2 - 1}}$

$\frac{1}{x + x^2 - 1} \quad 1 + \left(\frac{x}{\sqrt{x^2 - 1}}\right)$

$= \frac{1}{\sqrt{x^2 - 1}}$

$$\frac{d}{dx}(\text{arcos hu}) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}$$

DIFFERENTIATION OF arc tan h u.

Let $y = \text{arc tan h } x$

$\Rightarrow$ $\tan h\, y = x$

$$\frac{\sin h\, y}{\cos h\, y} = x.$$

$$\Rightarrow \quad \frac{e^y - e^{-y}}{ey + e^{-y}} = x$$

Multiplying through by e^{-y}

$$\frac{e^{2y} - 1}{e^{2y} + 1} = x$$

$\Rightarrow$ $e^{2y} - 1 = (e^{2y} + 1)\, x$

collecting like terms

$e^{2y} - x\, e^{2y} = x + 1$

$(1 - x)\, e^{2y} = x + 1$

$$e^{2x} = \frac{x + 1}{1 - x}$$

Differentiating by taking natural logarithm of both sides you get:

$$\text{Ln } e^{2y} = 1\text{n } \frac{x + 1}{1 - x}$$

$$2y = \text{In} \left(\frac{x + 1}{1 - x} \right)$$

$$2 \frac{dy}{dx} = \frac{d}{dx} \text{In } u = \frac{1}{u} \frac{du}{dx}, \quad u = \left(\frac{x + 1}{1 - x} \right)$$

$$\frac{dy}{dx} = \frac{1}{u\,.\,dx,}\,.\frac{du}{dx} \quad \frac{du}{(x - 1)^2} \quad \frac{-2}{}$$

$$2\frac{dy}{dx} = \left(\frac{1-x}{x+1}\right) . = \frac{1}{1-x^2}$$

$$\therefore \quad \boxed{\frac{d}{dx}(\text{arcos hu}) = \frac{1}{1-u^2}\frac{du}{dx}}$$

Example: Given that $\cos h^2 y - \sin h^2 y = 1$

Show that $\frac{d}{dx}(\text{arc h u}) = \pm \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

Solution

Let y= arc cos h u

Then cos h y = u

$$\frac{d}{dx}(\cos h\ y) = \frac{du}{dx}$$

$$\sin h\ y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sin h\ y}\frac{du}{dx}$$

but $\cos h^2 y - \sin h^2 y = 1$

$\therefore \quad \cos h^2 y - 1 = \sin h^2 y$

$\Rightarrow \quad \sin h\ y = \pm \sqrt{\cosh^2 y - 1}$

but cos h y = u

then $\sin h\ y = \sqrt{u^2 - 1}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\pm\sqrt{u - 1}}\frac{du}{dx}$$

SELF ASSESSEMENT EXERCISE 2

Use the above exercise to show that

$$\frac{d}{dx}(\text{arc sin h } u) = \frac{1}{1+u^2}\frac{du}{dx}$$

Example

Find $\frac{dy}{dx}$ If

(1) y = arc sin h (4x) (II) y = arc tan h (sin x)

(III) y= arc cos h (In x) (IV) y = arc cos h (cosx)

Solution

(1) let y =arc sin h u, u = 4x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$
$$= \frac{1}{\sqrt{1+16x^4}} = \frac{4}{1+16x^4}$$

(II) let y = arc tan u, u = sin x.

$$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx} = \frac{1}{1-(\sin x)^2} . \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{1-\sin^2 x} = \frac{1}{\cos x}$$

(III) let y = arc cos h u, u = (In x)

$$\frac{dy}{dx} \quad \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(\ln x)^2 - 1}} . \frac{1}{x} = \frac{1}{x\sqrt{(\ln x)^2 - 1}}$$

(IV) let y = arc cos h u, u = cos x

$$\frac{dy}{dx} = \quad 1 \quad du \quad \frac{du}{} = -\sin x$$

dx $\sqrt{u^2 - 1}$ dx dx

$\frac{dx}{dx}$ $\frac{- \sin x}{\sqrt{\cos^2 x - 1}}$

4.0 CONCLUSION

In this unit you have studied three types of functions and their respective derivative that is inverse trigonometry, hyperbolic and inverse rules for differentiation to differentiate functions involving inverse trigonometric and hyperbolic functions. You have been exposed to numerous examples involving the differentiation of these function discussed. Some of the examples were repeated in another format for example some of the examples used in unit 8 were used to explain the concept of differentiation of trigonometric and hyperbolic functions. This is a deliberate attempt so that you will master the technique studied in this unit. The differentiation of inverse function of trigonometric and hyperbolic will be very useful when studying the next course on calculus that is integral calculus. Make sure you go through the example thoroughly because you will need them in the second course in calculus.

5.0 SUMMARY

In this unit you have studied how to:

(1) Derive the formula for inverse trigonometric function such as

$\frac{d}{dx}(\text{arc} \sin u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}$ $\quad d(\text{arc} \cos u) = - \frac{I}{\sqrt{u^2 - 1}} \frac{du}{dx}$ $\frac{du}{dx}$ etc.

(II) Derive the formula

(a) $\frac{d}{dx}(\sin h\, u) = \cos h\, u \frac{du}{dx}$ etc

(b) $\frac{d}{dx}(\cos h\, u) = \sin h\, u \frac{du}{dx}$

(III) Differentiate functions involving inverse hyperbolic functions such as arc sin h u, arc cos h u, arc tan h u etc.

6.0 TUTOR-MARKED ASSIGNMENTS

(1) Find $\frac{dy}{dx}$ if $y = e^x$ arc sin (In x)

(2) Find $\frac{dy}{dx}$ if $y = \frac{\text{arc cot}(\sqrt{1-x})}{\sin h\, x^2}$

(3) Find $\frac{dy}{dx}$ if $\cos h^2\, 2(\sin x)$

(4) Find $\frac{dy}{dx}$ if $\sin h^2\, y = \tan h\,(x)$

(5) Derive the formula $\frac{d}{d}(\tan h\, u) = \sec h^2\, u\, \frac{du}{dx}$

(6) Derive the formula $\frac{d}{d}(\text{arc sin } h\, u) = \frac{1}{1+u^2} \quad \frac{du}{dx}$

(7) Find $\frac{dy}{dx}$ if $y = \frac{\sinh^3(e^{2x})}{\ln(\sin x)}$

(8) Derive the formula $\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1+u^2} \quad \frac{du}{dx}$

(9) Derive the formula $\frac{d}{dx}(\text{arc sec } u) = \frac{du}{|u|\sqrt{u^2-1}}$

(10) Derive the formula d (arc cos u) = $\frac{-\,du}{1-u^2}$

7.0 REFERENCES/FURTHER READINGS

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MODULE 4

UNIT 1 CURVE SKETCHING

CONTENTS

1.0 INTRODUCTION

Most polynomial and some rational functions could be sketch with the knowledge of the signs of the first derivative dx/dy and the second derivative d^2y/d^2xThe signs of the first derivative can give an idea of the behaviour of the curve within a given interval. The second derivative is used to determine points at which the curve is concave upward or concave downwards or information could then be used to sketch the curve of a given functions. In this unit you will study how to use both the first and second derivative to sketch the graph of a function at every points of the graph.

2.0 OBJECTIVES

After studying this unit you should be able to:

1) Use the first to determine

(i) points at which the given curves is increasing i.e. $\frac{dy}{dx} > 0$.

(ii) Points at which the a given curve is decreasing i.e. $\frac{dy}{dx} < 0$ and

(iii) Points at which a given curve is stationary i.e. $\frac{dy}{dx} = 0$

(2) Use the second derivative to determine points at which a graph is concave upwards or concave downwards.

3.1 Significance of The First Derivative to Curve Sketching

You will now consider the application of differentiation to curve sketching some curve could easily be sketched with the knowledge of the first and second derivatives of the function and the points where the first derivatives vanish i.e. equal to zero. You have already studied functions that are monotonic increasing or decreasing within an interval (see unit 2, section 3.2). You could determine whether a function is monotonic decreasing or increasing in a given interval. This is done by checking if the value of the first derivative within the given interval is positive or negative. This is stated as follows:

A function $y = f(x)$ is monotonic

(i) increasing in $x \varepsilon [a, b]$ if $\frac{dy}{dx} > 0 \ \forall \ x \varepsilon \ [a, b]$

(ii) decreasing in $x \varepsilon [a, b]$ if $\frac{dy}{dx} < 0 \ \forall \ x \varepsilon [a, b]$

(iii) constant (stationary) in $x \varepsilon [a, b]$ if $\frac{dy}{dx} = 0$

Example

Use the above stated facts to sketch the curve $y = x^2$ for $x \varepsilon [-10, 10]$.

Solution

$y = x^2$

$\frac{dy}{dx} = 2x$ when $x = 0$ $\frac{dy}{dx} = 0$

Start by first considering the values of $dy/dx = 2x$ at points on the left of $x = 0$.

For $x \, \varepsilon \, [-10, 10]$. $\frac{dy}{dx} = 2x \leq 0$ (decreasing)

For the values of $\frac{dy}{dx} = 2x$ at points on the right of $x = 0$.

i.e. $x \; [-10,10]$; $\frac{dy}{dx} = 2x \geq 0$ (increasing)

(see fig: 10. 1)

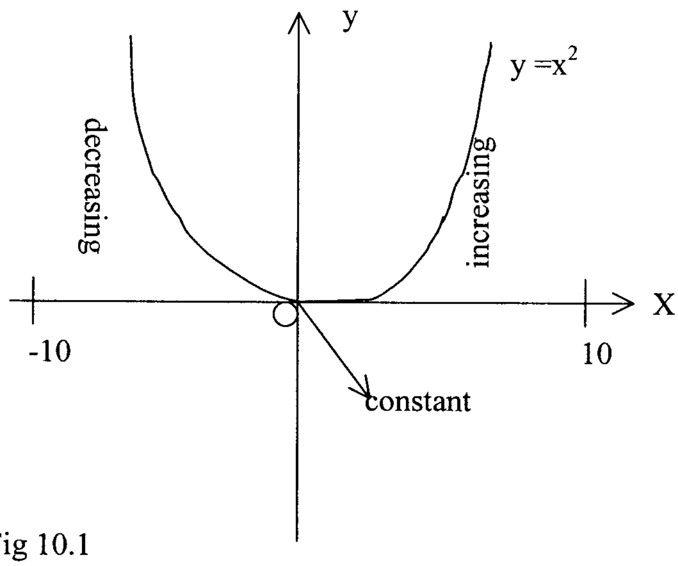


Fig 10.1

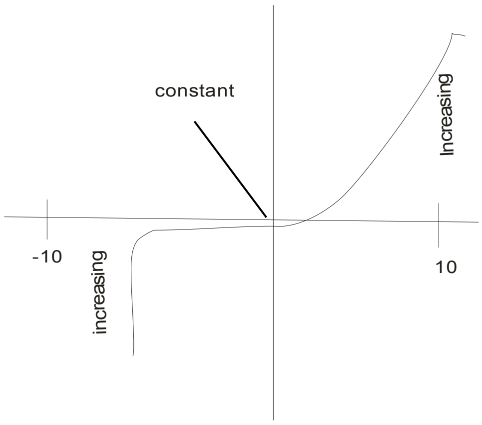


Fig 10.2

Example

Let $y = x^3$

$\frac{dy}{dx} = 3x^2$

when $x = 0$, $\frac{dy}{dx} = 0$

The values of dx/dy at points to the left and right of the point x = 0 is given as

(i) for. x [-10, 0]; $dy = 3x^2 \geq o$ (increasing)

(ii) for. x [0, 10]; $\frac{dy}{dx} = 3x^2 >- o$ (increasing)

see fig 10.2.

Example

Given $y = \frac{1}{3}x^3 + x^2 - 8x + 1$

$\frac{dy}{dx} = x^2 + 2x - 8$

$= (x - 2) x (x + 4) = 0$

x = 2 or - 4.

As before you will consider values of $\frac{dy}{dx}$ respectively. i.e. at points to the left and right of 2 and – 4 respectively. i.e.

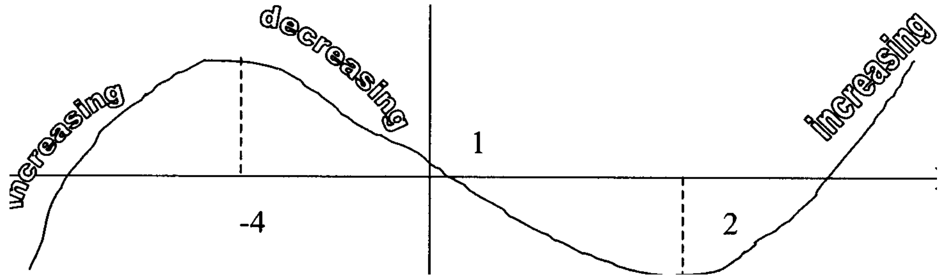
At x = 2

2. $\frac{dy}{dx} > 0$ for all x ε (2, α) i.e. increasing at the right side of

$\frac{dy}{dx} < 0$ for all x ε (-4, 2) between - 4 and 2

$\frac{dy}{dx} > 0$ for all x ε (-α , - 4)

(see Fig 10.3)



3.2 Significance of the Sign of Seemed Derivative Curve Sketching

The above sketch could be improved if you apply the information you get by taking the second derivative of the function under investigation. A quick look at the graph shown in Fig. 10 3. shows that within the interval x ε (- 4, 1o) the graph is concave upward. Within the interval x x ε (-10, - 4) the curve is concave downward. Once you find the point at which dy/dx = 0 (i.e. the turning points along the curve y = *f*(x). Then by finding the second derivative d^2y/dx^2 you can determine which of the turning points (i.e. points at dx/dy = 0) is the concave upwards or downwards.

Definition:

If $\frac{d^2y}{dx^2}$ exists and $\frac{d^2y}{dx^2} > 0$,

for all x in a specified interval 1, then dy/dx is said to be increasing in I and the graph of *f*(x). is said to be concave upwards. If $\frac{d^2y}{dx^2} < 0$ for all x ε I, the $\frac{dy}{dx}$ is decreasing in I.

So that the graph of y = f(x). is said to be concave downwards.

3.3 Curve Sketching

Definition of Points of Inflection

A point where the curve changes its concavity from downwards to upwards or vice versa is called a point of inflection This occurs where

$$\frac{d^2y}{dx^2} = 0 \text{ or where } \frac{d^2y}{dx^2} \text{ does not exists.}$$

Example

$$y = 3x^{1/3}$$

$$\frac{dy}{dx} = x^{-2/3}$$

at x= 0 $\frac{dy}{dx}$ is not defined.

So the point x = 0 is a point of inflection for the $y = 3x^{1/3}$

Example

Given that $y = f(x)$.

Let the following explain the behaviour of the curve.

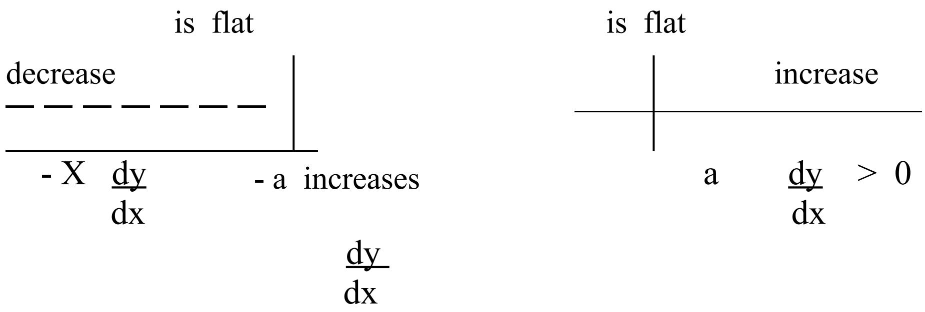
(1) At point x = a

$$\frac{dy^1}{dx} = 0 \text{ if } x = \pm a$$

$$\frac{dy}{dx} > 0 \quad if \quad x^2 > a$$

$$\frac{dy}{dx} < 0 \quad if \quad x^2 > a$$

The curve increases or decreases as indicated below.



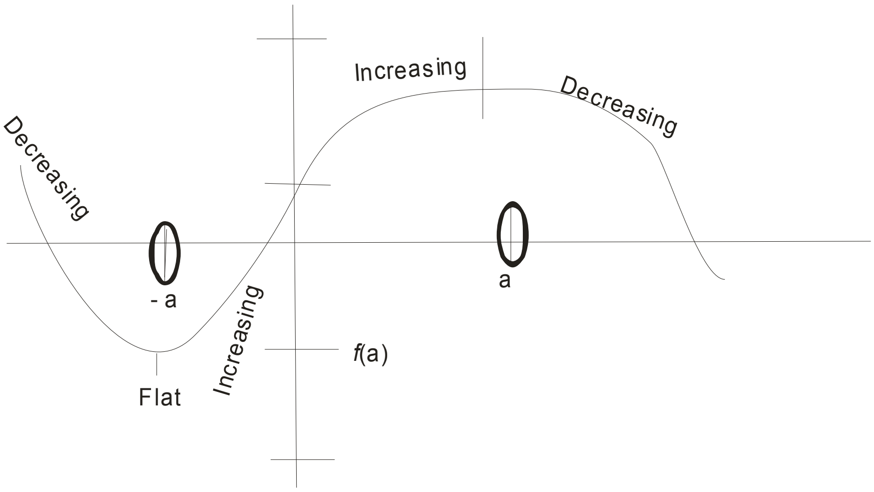
Let the point x = - a be the lowest point of the curve and the point x = a be the highest point of the curve.

Let the curve cut the y - axis at the point y = b, x = 0.

Let the curve cut the x- axis at the points $-x_1$ and x_2 sketch the given curve.

Step 1. $\frac{dy}{dx} = 0$ at $x \pm a$

Locate point $x = \pm a$



Step 2. Find the values of $y = f$ (a) and $y = f$ (-a)

Step 3. Find the point $y = f$ (o)

Step 4. Find $\frac{d^2y}{dx^2}$ at $x = a^o$, $x = - a$

For the above $\frac{d^2y}{dx^2} > o$ at x = - a and $\frac{d^2y}{dx^2} < 0$ at x = a..

Step 5: Use the above sketch the curve.

On step 5 use the information about the curve increasing and decreasing

i.e. left of x = - a it is decreasing. and right it is increasing etc. see fig. 10.3.

SELF ASSESSMENT EXERCISE 1

Given the function

(a) $y = -1 + 3x - x^3$ (b) $x^3 - 3x + 1$

Find the following:

(i) Find $\frac{dy}{dx}$ (ii) solve the equation $\frac{dy}{dx} = 0$

(iii) Find $\frac{d^2y}{dx^2}$ (iv) solve $\frac{d^2y}{dx^2} = 0$

(v) Find y for which x = 0

Solutions

(a) (i) $3(1 - x^2)$ (ii) ± 1

(iii) -6x. (iv) -1

(b) (i) $3(x^2 - 1)$ (ii) ± 1

(iii) 6x (iv) 1

Hints For Sketching Curves

Below are seven useful hints for curve sketching

Hint 1 Find dy/dx

Hint 2 Find the turning points by solving the equation $\frac{dy}{dx} = 0$

Hint 3 Evaluate $y = f(x)$ at the turning points

Hint 4 Evaluate d^2y/dx^2 at the turning points to determine which of them is the maximum or minimum points.

Hint 5 Investigate the behaviour of the curve as x-> turning points either from left or right.

Hint 6 Investigate the behaviour of the curves as
(1) $x \to \infty$ (II) $x \to -\infty$ (III) $x = 0$

Hint 7 Sketch the graph with information gathered from hint 1 to hint 6 above.

Example: sketch the curve

$y = \frac{1}{3}x^3 - 4x + 2$ using

Hint 1; $y = \frac{1}{3}x^3 - 4x + 2$

$\frac{dy}{dx} = x^2 - 4$

Hint 2: $\frac{dy}{dx} = 0 \ (x^2 - 4) = 0$

$X = \pm 2.$

Hint 3: $x = 2$ $y = \frac{1}{3}(2)^2 - 4.\ 2 + 2 = \frac{-10}{3}$

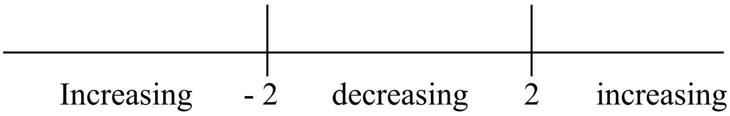
$x = -2$ $y = \frac{1}{3}(-2)^2 - 4(-2) + \frac{22}{3}$

Hint 4: $y = \frac{d^2y}{dx^2} = 2x$

at $x = -2$, $\frac{d^2y}{dx^2} = 2(-2) = -4 < 0$

at $x = 2$ $\frac{d^2y}{dx^2} = 2.2 = 4 > 0$

Hint 5:



e.g.: $x = -6$
$y = -46$
$x = -3$
$y = 5$

Hint 6: x --> ∞, y-> ∞
x--> ∞, y-->- ∞
x = 0, y = 2

Hint 7: (see graph below.)

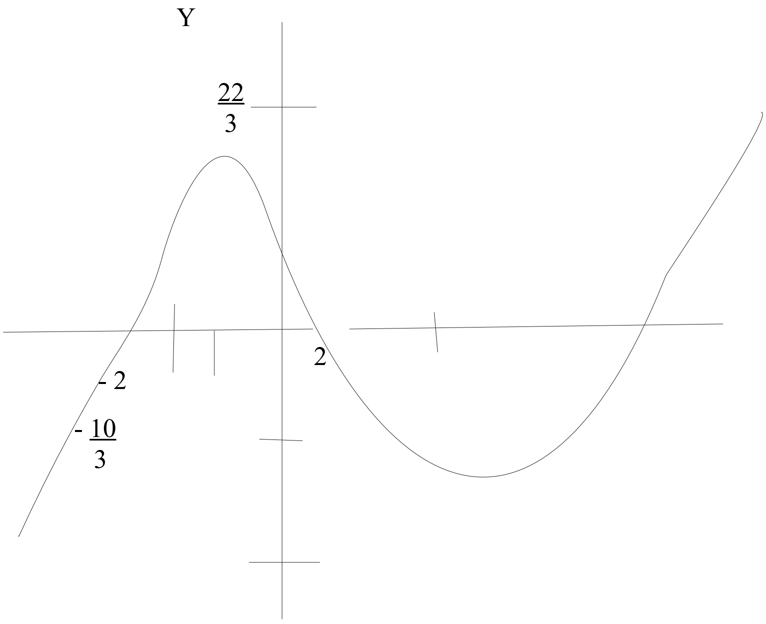


Fig 10.4

Example: $y = x + 9/x$

Hint 1: $\frac{dy}{dx} = 1 - \frac{9}{x^2} = \frac{x - 9}{x^2}$

Hint 2: $\frac{dy}{dx} = 0, \Longrightarrow \frac{x^2 - 9}{x^2} = 0$

$x = \pm 3.$

Hint 3 $x = 3,\ y = 3 + \frac{9}{3} = 6$

$X = -3\ \ y = -3 + \frac{9}{-3} = -3 -3 = -6$

Hint 4:

Increasing	- 3	decreasing	3	increasing

Y = -10, x =

Hint 5:

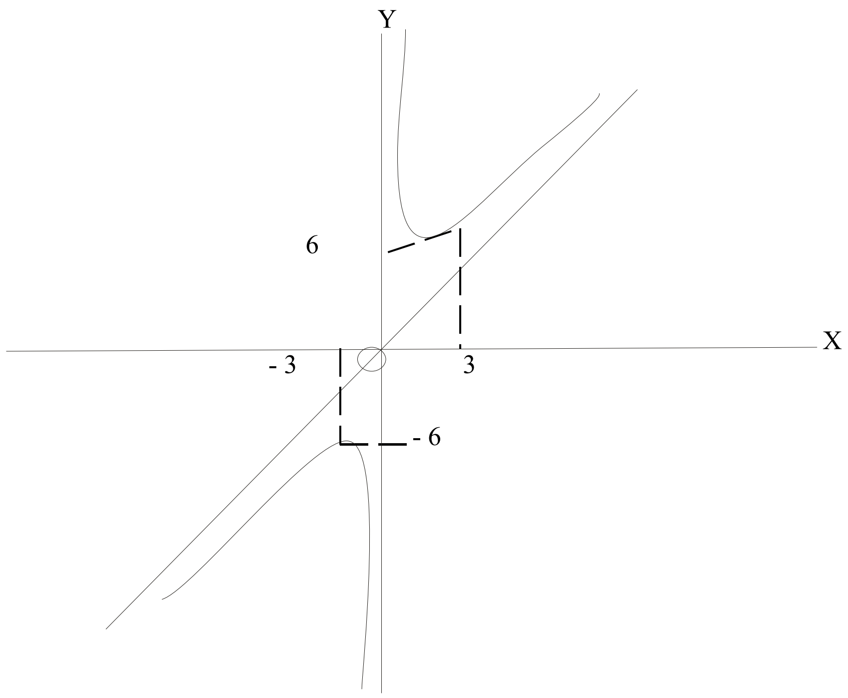
x ----> -3, y ---> 6, x ----> 3^+, y ---> -6

x ----> -∞, y ---> - ∞,

x ---> 3x y ----> 6, ----> 3^+, y ---> -6

x ----> ∞ y ---> ∞

Hint 6: Sketch



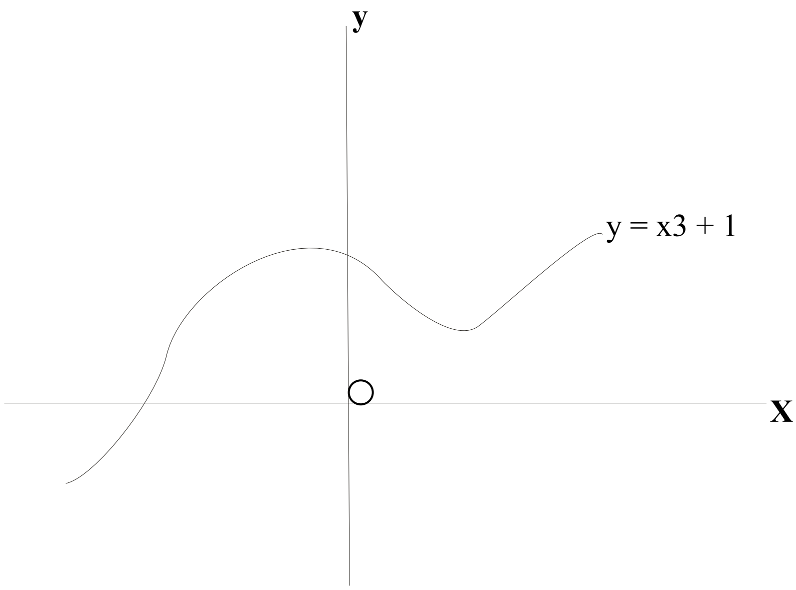
SELF ASSESSEMENT EXERCISES 2

Use the hints given above to sketch the graph of

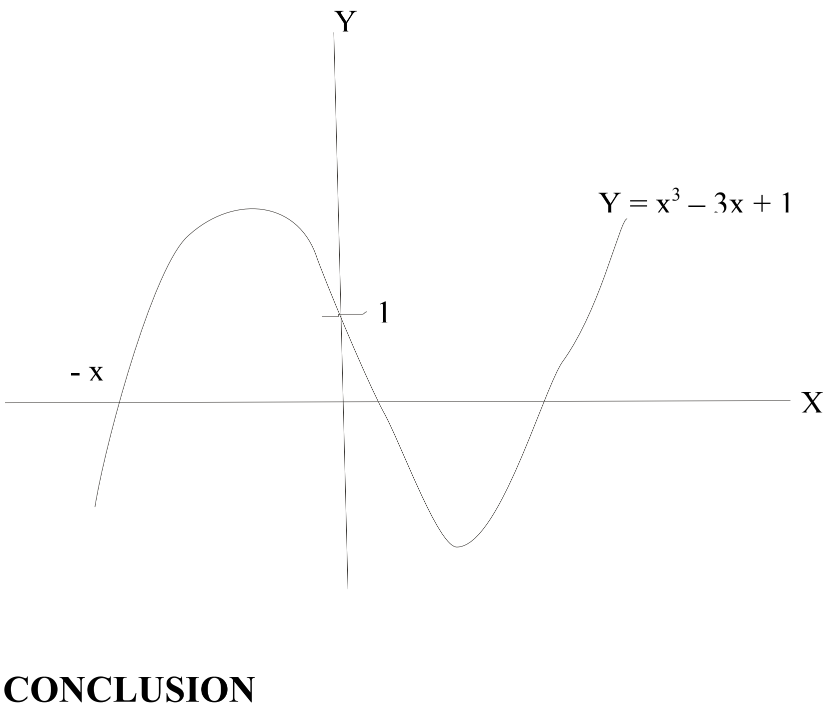
(1) $y = x^3 + 1$ (II) $y = x^3 - 3x + 1$

Solutions:

(i)



(ii)



4.0 CONCLUSION

In this unit you have studied how to use the sign of the first derivative of a function to determine a function that is monotonic increasing and monotonic decreasing. You also studied how to use the first derivative to determine stationary point. You have used the signs of the second derivative to determine a curve that is concave upwards or concave downward. You have studied how to use the information above with other information to sketch the graph of a function within the interval [a, b] or $(-\infty,\infty)$

5.0 SUMMARY

In this unit you have studied how to:

(I) Investigate the behaviour of a function $y = f(x)$ when

(a) $\frac{dy}{dx} < 0$ (b) $\frac{dy}{dx} > 0$

(c) $\frac{dy}{dx} = 0$ (d) $\frac{d^2y}{dx^2} < 0$

(e) $\frac{d^2y}{dx^2} > 0$

(II) Use the information in (10 above with other relevant ones such as the behaviour of y as $x \to \infty$ or ∞ to sketch the graph of $y = f(x)$

6.0 TUTOR-MARKED ASSIGNMENT

If $y= \frac{1}{3}x^3 - 2x^2 + 3x + 2$

(1) Find $\frac{dy}{dx}$ (2) solve the equation $\frac{dy}{dx} = 0$

(3) Find $\frac{d^2y}{dx^2}$ (4) solve $\frac{d^2y}{dx^2} = 0$

(5) Find y for which $x = 0$

(6) Find y when (i) $x \to \infty$ (ii) $x \to -\infty$

(7) Sketch the curve of $y = \frac{1}{3}x^3 - 2x^2 + 3x + 2$

(8) Locate the global minimum and maximum points on the graph in (7) above and the point of inflection.

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 MAXIMUM – MINIMUM AND RATE PROBLEMS

CONTENTS

1.0 INTRODUCTION

In this unit you will study how to use first and second derivative of a function to solve optimization problems in social sciences, physics, chemistry, engineering etc. That is any problem where the information on how small or how big a given quantity should be is needed. It assumed that such problem should be able to be modelled by any mathematical formula. By differentiating such a function you can determine it minimum or maximum value. Differentiation could be applied to problems where it necessary to determine the rate at which a quantity is changing with respect to another quantity. There are various classes of problems that could be solved by appropriate application of differential calculus the ones enumerated in this unit and some where else is the course is by no mean exhausture.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

1) use first and second differentiation of a function to solve problems where the minimum amount of resources or material is required.
2) Use first and second differentiation of a function to solve problems where maximum value of a resources or material needed or should be attained
3) Use differentiation of a function to determine the rate at which a given quantity is changing with respect to another quantity.

3.0 MAIN CONTENT

3.1 Definition of Global and Local Minimum ad Maximum Value

In this section you will be able to use both first derivative and where necessary the second derivative to solve problems of classical optimization in economics, engineering, medicine and physics etc. The words minimum and maximum give the impression of a problem where you may wish to determine how small (minimum) or how large (maximum) a variable quantity may attain. You will start by considering the following important definitions.

Definition: A function $f(x)$ is said to have a local (or relative) maximum at point $x = x_o$ if

$$f(x) \geq f(x_o + h)$$

For all positive and negative value of h however small It is said to have a local (or relative minimum at point $x = x_o$ if

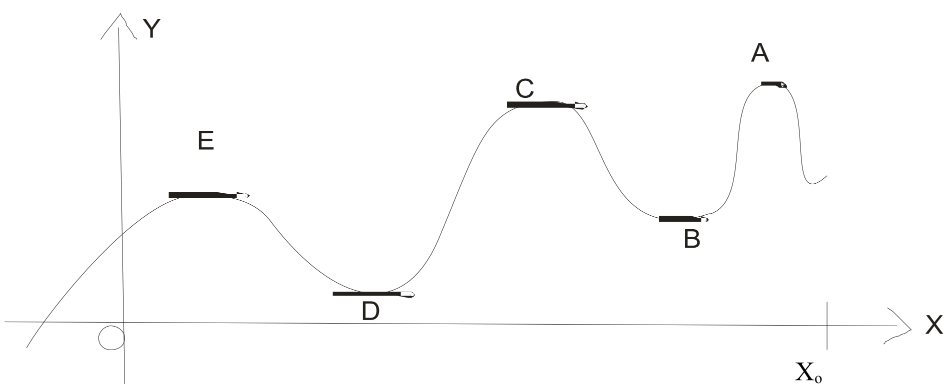
$$f(x) \leq f(x_o + h) \text{ for all values of h however small.}$$

Definition: A function $f(x)$ is said to have an absolute (global) maximum at $x = x_o$ if $f(x) \geq f(x)$ for all values of x in the domain of definition. It is said to have an absolute (global minimum at x = xo if $f(x_o) \leq f(x)$ for all values of x in the domain of definition.

Example

In Fig 10.5, you will notice that there are five turning points i.e. stationary

points the points at which $\frac{dy}{dx} = 0$



Points E and C are points where the curve attains maximum or local maximum, while at point A the curve attains an absolute maximum or global maximum within the interval $x \varepsilon [0, x_o]$ point B is a local maximum point while point D is a global minimum point.

Example: Given that $y = x + 9/x$

$$\frac{dy}{dx} = \left(1 - \frac{9}{x^2}\right) = (x^2 - 9) = 0, x = \pm 3$$

The point x = -3 is a relative maximum and not global maximum (see Fig 10.4) The point x = -3 is a relative minimum point.

SELF ASSESSMENT EXERCISE 1

Explain why the point x = 3 is not a global minimum for the curve given as

$$y = x + \frac{9}{x}$$

(Hint consider points near x = 3)

Example if $y - 3x^4 - 16x^3 + 24x^2 + 1$

Determine whether the function has a maximum or minimum points.

Solution

$$Y = 3x^4 - 16x^3 - 24x^2 - 1$$

$$\frac{dy}{dx} = 4.3x^3 - 3.16x^2 - 2.24x$$

$$= 12x^3 - 48x^2 - 48x$$

$$= 12x(x^2 - 4x + 4)$$

$$= 12x(x - 2)(x - 2) = 0$$

$$x = 0, \text{or } 2.$$

To determine which of the two points is a maximum or minimum the second derivative of these points will have to be evaluated

i. e. $\frac{d^2y}{dx^2} = 3.\ 12x^2 - 2.48x + 48$

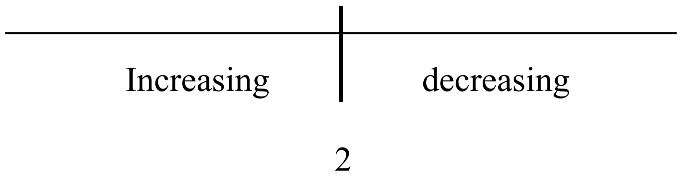
$$= 36x^2 - 96x + 48$$

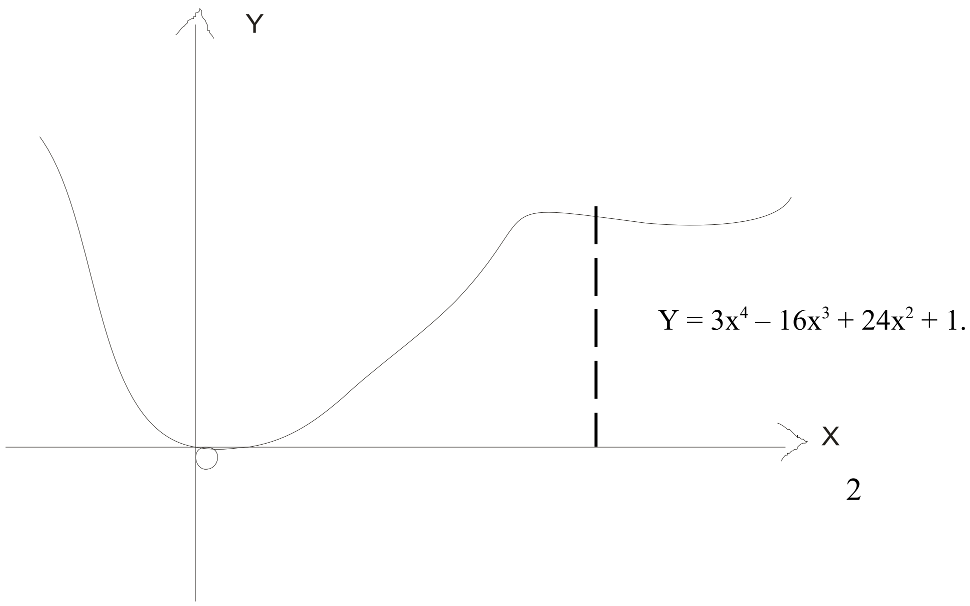
At $x = 0$, $\frac{d^2y}{dx^2} = 48$

$\frac{d^2y}{dx^2} > 0$ is a point of global minimum.

At $x = 2$ $\frac{d^2y}{dx^2} = 36(2)^2 - 96(6) + 48 = 0$

Look at points near 2.





Therefore the point $x = 2$ is a point of inflection.

3.2 Application to Minimum and Maximum Problems

It time to use the theory explained above to solve practical problems that call for the minimization or maximization of values of a function.

Example: Find two possible positive numbers whose product is 36 and whose sum could be made relatively small

Solution

You could start by making a guess based on the fact that the factors of 36 that are positive are:

(6, 6), (9, 4), (36, 1), (12, 3), (2, 18).

Taking their sum you see that

$$12 < 13 < 15 < 20 < 37.$$

So the number could is 6. It could be cumbersome to solve such a problem if the number is very large, find factors of a possible short cut is to apply differentiation. We need to note, that x and 36/x are two numbers whose product is 36.
Their sum is given as a function y = f(x)

$$\text{i.e.} \quad y = x + \frac{36}{x}$$

The issue at stake is to see the positive values of x that will give the problem is now reduced to the problem of minimization of the value of the function
$y = x + \frac{36}{x}$

Since $y = x + \frac{36}{x}$

$\frac{d}{dx} = 1 + \frac{36}{x^2} = 0$

$\Longrightarrow \quad x^2 - 36 = 0$

$\Rightarrow \quad x = \pm\ 6$

$\frac{d^2y}{dx^2} = +\frac{72}{x^3}$

if x = - 6 $\quad \frac{d^2y}{dx^2} < 0$ maximum point

if x = 6 $\quad \frac{d^2y}{dx^2} > 0$ minimum point.

Therefore the answer is x = 6.

Example

The management of a manufacturing company found out that their profit was not enough as that wish it should be. A mathematical formulation gives the cost of production and distribution as ₦ a. The selling price is ₦ x. The number sold at a given period is put at $n = K/(x^2 - a) + C(100 - x^2)$ where K and C are certain constants. What selling price will bring a maximum profit to the company.

Solution

The first thing to do is to find or formulate an equation that gives the total profit for the company.

Profit = Selling Price - Total Cost

i.e. $P = N(nx - na)$.

$= (x - a)(K/(x - a) + C(100 - x)$

$= K + C(100-x)(x-a)$

$\frac{dp}{dx} = C[(100-x)1 + (x-a)-1]$

$\frac{dp}{dx} = 0 =:> 100 - x = x - a$

$2x = 100 - a$

$x = \frac{100 + a}{2}$

₦ x is the selling price therefore $x > 0$. Having this in mind you find

$\frac{d^2P}{dx^2} = -23C$ hence $C > 0$.

To guarantee that $\frac{d^2P}{dx^2} < 0$.

The selling price that give the maximum profit is given as $x = \frac{100+a}{2}$.

Hence the above shows that the selling price is determined by the cost of production and distribution.

Example

An open storage tank with square base and vertical sides is to be constructed from a given amount of plastic material calculate the dimensions that will produce a maximum volume.

Solution

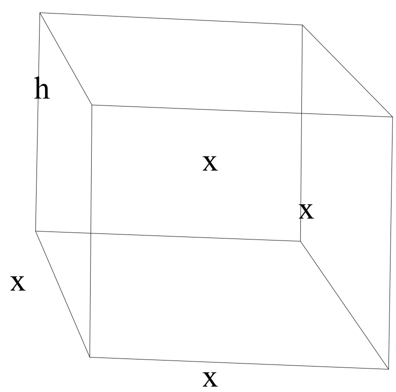
The formula for the volume of a rectangular box is given as

V = base x height (see Fig. 10.6)

Let base = x^2 (square base of side = x)

$\therefore \quad V = x^2h$

Total surface area of tank
= Total amount of material



Therefore

$A = 4\,x\,h + x^2$

Solving for h you get

$\Rightarrow \quad h = \frac{a - x^2}{4^2}$

but $V = x^2h = \frac{x^2\,(A - x^2)}{4x} = \frac{x}{4}(A - x^2$

$\therefore \quad \frac{dv}{dx} = \frac{1}{4}\,(A - x^2) + \frac{x}{4}\,(-2x)$

$= \frac{1}{4} \, [(A - x^2 - 2x^2)]$

Let $\frac{dv}{dx} = 0 \Rightarrow [(A - x^2 - 2x^2)] = 0$

$\Rightarrow \quad A = 3x^2 \quad \Rightarrow x = \pm \sqrt{A/3}$

Since x represent sides of the tank definitely $x > 0$ therefore $x = \pm \sqrt{A/3}$
To check if this value of V you evaluate second derivative at

$$x = \sqrt{A/3} \text{ i.e. } \frac{d^2y}{dx^2} = \frac{1}{4}(-6x) < 0$$

$$\text{for } x > 0 \ \frac{d^2v}{dx^2} < 0. \text{ Thus } x = \sqrt{A/3}$$

is a maximum point for the function V = x2 h.

Therefore the dimension that will yield maximum volume is given as $x = \sqrt{A/3}$and

$$h = \frac{A - x^2}{4x} = \frac{A - A/3}{4\sqrt{A/3}} = \frac{1}{2}\sqrt{A/3}$$

$$\text{Thus } x = x = \sqrt{A/3}, \ h = x = \tfrac{1}{2}\sqrt{A/3}$$

This is dependent on the surface area of tank chosen.

In the next example you will consider the case where you will seek to minimize the amount of material to be used.

Example

Given a storage rectangular tank with a square base can contain $32m^2$ of water.

Find the dimensions that require the least amount of material to construct such a tank. (Neglect the thickness of the material and the waste in construction.

Solution

Let $V = x^2 h$ be the volume of the rectangular tank (see Fig 10. 6)

But $\quad V = -32m^3$ therefore $32 = x^2 h$

$$\Rightarrow \quad \frac{32}{x^2} = h$$

Since the amount required to build the tank is equal to surface area of the tank which is given as

$$A = 4\,x\,h + x$$

$$= 4x\,(\frac{32}{x^2}) + x^2 = \frac{128}{x} + x^2$$

$$\therefore \quad \frac{dA}{dx} = \frac{-128}{x^2} + 2x$$

equating to zero you get

$$\frac{-128}{x^2} + 2x = 0$$

$$\therefore \quad -128 + 2x^3 = 0$$

$$x^3 = 64.$$

$$x = 4$$

For $x = 4$ $\frac{dA}{dx} = 0$.

To check if $x = 4$ is a minimum point you evaluate d^2A/dx^2 at $x = 4$

$$\text{i.e.} \quad \frac{d^2A}{dx^2} = \frac{256}{x^3} + 2$$

Since $x > 0$ =:> $\frac{d^2A}{dx^2} = 4 + 2 = 6 > 0$

Thus at $x = 4$ the function has a minimum value.

Therefore the dimension of the tank that will minimize amount of material is given as 4, 4, 2.

Example

A milk can is to be made in the form of a right circular cylinder from a fixed amount of plastic sheet Find the radios and height that will produce the maximum volume.

Solution (see fig, 10.7)

$$V = \pi r^2 h \qquad \_\_\_\_\_\_\_\_\_\_\_\_ \qquad (1)$$

The idea is find the values of r and h that will Maximize V.

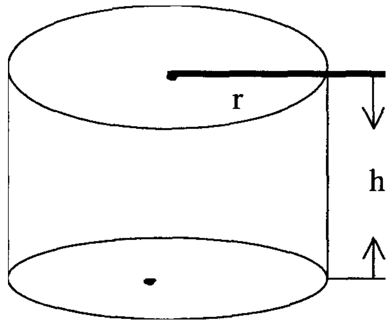


Fig 10.7

Let A be the surface area of a right circular cylinder

$$A = 2\pi r^2 + 2\pi r h \qquad \_\_\_\_\_\_\_\_\_\_\_\_ \qquad (ii)$$

Thus $h = \frac{A - 2\pi r^2}{2\pi r}$

Substituting h into equation (i) you have that:

$$V = \pi r^2 \left(\frac{a - 2\pi r^2}{2\pi r} \right)$$

$$= \quad ½ (A - 2\pi r^2)$$

Taking Differentiations

$$\frac{dv}{dr} \quad \frac{A}{2} - 3\pi r^2$$

Equating to 0 and solving for r you get

$$3\pi r^2 = A/2$$

$$r^2 = A/6\pi$$

$$\therefore \; r = \frac{\sqrt{a}}{6\pi}$$

$$h = \frac{A - 2\pi.\frac{A}{6\pi}}{2\pi\sqrt{A/6\pi}} = 2\sqrt{A/6\pi}$$

Now $\frac{d^2N}{dr^2} = -6\pi r . < 0 \; r = +\sqrt{A/6\pi}$

The radius $r = \sqrt{A/6\pi}$ $h = 2\pi$ that will produce maximum volume

Example: In Fig 10 .7 let $V = \pi r^2 h = 8$ and $h = 8\pi r^2$

Find the value of r that fail use the teart amount to produce a volume of 8 cm^3

In other words minimize the surface area. Given that the surface area

$$A=2\pi r + 2\pi r.\frac{8}{m_2} = 2\pi r^2 + \frac{16}{r}$$

$$=2\pi r^2 + \frac{16}{r^2}$$

Differentiations

$$\frac{dA}{dr} = 4\pi r - \frac{16}{r^2}$$

equating to zero and solving for r you get

$$4\pi r - \frac{16}{r^2} = 0$$

$$\pi r^3 - 4 = 0$$

$$\Longrightarrow \quad r^2 = 4/\pi$$

$$r = \left(\frac{4}{\pi}\right)^{1/3}$$

$$\frac{d^2A}{dx^2} = 4\pi + \frac{32}{r^3} > 0 \text{ for } r = \frac{4}{\pi}^{1/3}$$

Therefore $r = \left(\frac{4}{\pi}\right)^{1/3}$

SELF ASSESSMENT EXERCISE 2

Given the graph of the $f(x)$ in Fig 10.8.

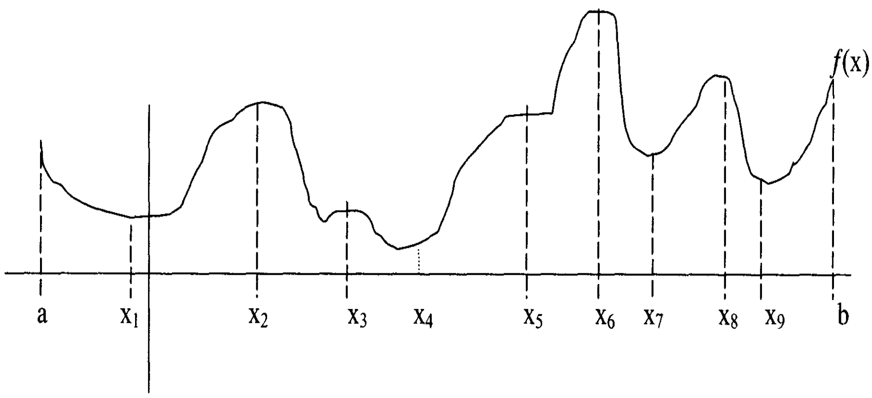


Fig 10.8.

(i) Define a global maximum of $f(x)$ and indication at which point in Fig(10.8) is this maximum attained if $x \varepsilon [a, b]$.

(ü) Define a relative minimum of a function. Indicate at which points did the function $f(x)$ shown in Fig (10.8) is relatively minimized.

(iii) Define point of inflection, Indicate them.

Remark: To find the global a maximum and minimum of f (x) in the interval
$a \leq x \leq b$. Locate all points where $f^{1}(x) = 0$. Call these points x_l, x_L, x_3,..., x_7 The global maximum is the largest of the numbers $f(x^2)$ =, $f(x^6)$, $f(x_8)$

Solution:

(i) $f(x_6)$ is the largest value therefore at $x = x_6$ = global maximum is attained.

(ii) Points of relative minimum are at $x = x_l$, $x = x_4$, $x = x_7$ and $x = x_9$

(iii) points of inflection are at $x = x_3$ and $x = x_5$.

Remark: Definitions are given in the beginning of this section

SELF ASSESSMENT EXERCISES 3

1. Find the minimum value of the function f (x) = ½ x - sin x in the interval x ε [0, 4 7π]

   Solution

   X = π/3, X = 7π/3.

2. Find the largest value of f(x) 108 x - x^4

   Solution

   f (3) 243.

3.3 Application to Rate Problem

You have already study that the derivative of a function gives its rates of change. In other words suppose you have a qualities y which varies with another quality x, them the rate of change of y with respect to x is given as dy/dx.

You would have study this type of problem under the topic variation during your course of study in your preparation for the GCE 0 level or SSCE examination. That is y is increasing or decreasing with respect to x as according as dy/dx is positive or negative. This situation was described under the section on application to curve sketching, i.e. dy/dx > 0 implies that the function y = f (x) is increasing while dy/dx == f (x) is decreasing

Example: Suppose the amount of petrol y litres in a car tank after traveling x kilometers is given as

$$y = 30 – 0.02x$$

Then $\frac{dy}{dx} = -0.02$

The negative sign means that y decreases as x increases. Hence the amount of petrol in the car's tank is decreasing at the rate of 0.02 It. I km.

Example: A circular sheet of material has a radius of 4cm. At what rate is the area increasing with respect to the radius? If the radius increases to 4.2cm, what is the approximate increase in the area.

Solution

Let radius of circular sheet be r cm and the area be represented as A cm 2

Therefore $A = \pi r^2$ (area of a circle with radius r)

The rate of increase of A with respect to r is given as

$$\frac{dA}{dr} = 2\pi r$$

Thus if r = 4cm them $\frac{dA}{dr} = 2\pi (4) = 8\pi$ cm

But $\frac{dA}{dr} = 2\pi r$

Let ΔA and Δr be small changes in area and radius respectively

The $\Delta A = 2\pi r \ \Delta r$

Since r = 4 and $\Delta r = 0.2 = (4.2 - 4)$ cm

Then $\Delta A = 2\pi (4)\, 0.2 = 1.4\pi$

Therefore the approximate increase in area is 1.4π cm^2.

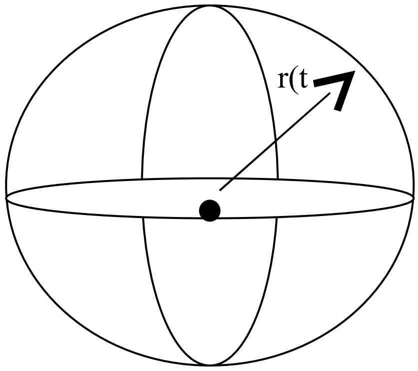
Example: A spherical balloon is inflated with gas. If its radius is increasing at the rate of 2cm per second, how fast is the volume increasing when the radius is 8cm (see Fig 10.9)

Solution

The volume at time t is expressed in terms of the radius at time t by the formula

$$v(t) = \frac{4}{3} \pi (r(t))^3$$

$$\frac{dv(t)}{dt} = 4\pi (r(t))^2 \frac{dr}{dt}$$



If at time t the radius is 8 cm

Then $\frac{dv(t)}{dt} = 4\pi (8)^2 .2 = 512\pi.$

This implies that the volume is increasing at the rate of 512π cm^3 per second.

Example: water ns into a large concrete conical storage tank of radius 5 and its height is 1om (see Fig 10.10) at a constant rate of $3m^3$ per minute. How fast is the water level rising when the water is 5m deep

Solution:

In this case the list of variable quantities will be given

Quantities that are changing is

V = the volume (m) of water in the tank at time (t)

x = the radius (m3) of the Surface of the water at time t.

y = the depth (m) of water in the tank at time t.

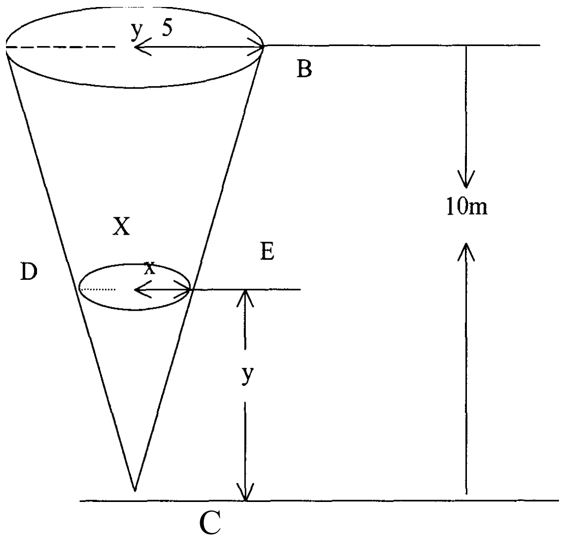


Fig. 10.10

The rate at water flow into the tank is constant and is given as dv/ dt = $3m^3$ 1 min.

The function establishing a relation between the variable quantities is given as

1. $v = \frac{1}{3}\pi x^2 y$ (this is volume of cone $= \frac{1}{3}\pi r^2 h$.)

The function is better expressed in terms of only one variable x or y . You can expressed it in terms of y alone. By noting that in Fig 10.10 Δ ABC and ΔDEC are similar triangles.;

Therefore $\frac{DE}{Ab} = \frac{CX}{CY} = \frac{X}{5} = \frac{y}{10}$

$\Rightarrow \quad x = ½ y$

Hence equation (1) becomes

$$V \frac{1}{3} = \pi (½ y)^2 y = \frac{\pi y^3}{12}$$

Therefore $\frac{dv}{dt} = \frac{\pi y^2}{4} \frac{dy}{dt}$

Given that $\frac{dv}{dt}$ =3 and y = 5.

Therefore. $\frac{dy}{dt} = \frac{4}{\pi y^2} . \frac{dv}{dt} = \frac{4}{\pi r} \left(\frac{1}{5}\right)^2 , \qquad 3$

$\Rightarrow \qquad \frac{12}{25}\ m^3\ 1min$

Example

A boat is pulling into a dockyard, for repairs by means of a rope with one and attached to the tip of the boat the other end passing through a ring attached to the dockyard at a pint 5m higher than the tip in at the rate of 3m/sec, how fast is the boat approaching the dockyard when 13m of the rope are out? (see Fig 10.11)

Solution

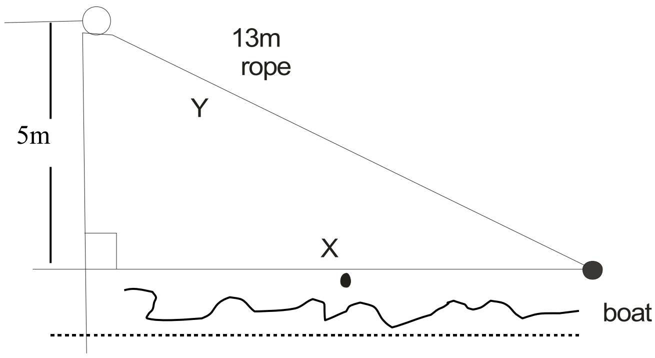


Fig 10.11.

There are two quantities dockyard that are changing. They are as follows:

(i) x = distance of boat to dockyard

(ii) y = the length of the rope.

The quantities that are fixed are:

(i) height of the dockyard = 5m

(ii) rate of change of rope to time t, $\frac{dy}{du}$

the formula connecting these quantities is given as (see Fig 11.11)

$$x^2 + (5)^2 = y^2$$

$$\Rightarrow \qquad x^2\ 25 = y^2$$

Differentiating you get;

$$2x \frac{dx}{dt} = 2y \frac{dy}{dt}.$$

$$\therefore \quad \frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt}.$$

given that y = 13 and h = 5 (constant)

and x =

$$\sqrt{13^2 - 5^2} = 12$$

$$\frac{dx}{dt} = \frac{13}{12} .3 = \frac{13}{4} \text{m/sc.}$$

That is the boat is approaching the dockyard at the rate of 13/4 m/sec.

SELF ASSESSMENT EXERCISE 4

(1) A small spherical balloon is inflated by a young boy that injects air into 10 mm^3/sec. At the instant the balloon contains 288mm^3 of air, how fast is its radius moving.

Solution:

$$\frac{5\pi}{72\pi} \text{ mm/sec.}$$

(2) A company that sells vegetable oil has a conical distribution tank of radius 3m and height of 6m. If vegetable oil is poured into the tank at a constant rate of 0.05m^3 per second. How fast is the oil level rising when the oil is 2m deep.

Solution:

$$\frac{0.05m^3}{\pi} \text{/sec.}$$

4.0 CONCLUSION

In this unit you have studied the applications of differentiations to minimum and maximum problem. You have also applied differentiation to determine the rate a given quantity changes with respect to another.

You have specifically studied how to use the first and second derivatives of a to solve problem where a minimum amount of material or resource is needed and also where certain maximum value is to be attained. You have applied rules for differentiation to solve these problems

5.0 SUMMARY

In this unit you have studied how to:

(1) calculate the minimum amount of material or resources needed in a project by using first and second derivatives of a function.

(2) Calculate the maximum value of a quantity or commodity using the first and second derivatives of a function.

(3) Determine the rate at which a quantity changes with respect to another quantity i.e. rate problems.

6.0 TUTOR-MARKED ASSIGNMENT

l. Given x = sin h t and y = -cos h t

(a) find dx (b) dy (c) $\frac{dx}{dy}$

2. An auto catalyst reaction R is defined as

R(x) = 2x (a - x), a = the amount of substance
X = product of reaction

(i) At what values of a and x will the reaction attain its maximum?

(ii) Find the maximum value of R.

(Remark: An auto catalyst reaction is a chemical reaction where the product of the reaction acts as a catalyst).

3. A closed rectangular dish with a square base can contain a maximum of $20cm^3$ of liquid. Find the dimensions that require the least amount of materials to construct such a tank (Neglect the thickness of the material and waste in construction).

4. A state owned water corporation department has a large plastic conical storage tank of radius r = 6m and height h = 9m. Because of water scarcity water is allowed to run into the tank at a cons tank rate of $4m^3$ per minute;

(i) How fast is the water level rising when the water is 6m deep?

(ii) How deep will water when the water level is rising at the rate of lm^3 /min.

7.0 REFERENCES/FURTHER READINGS

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UNIT 3 APPROXIMATION, VELOCITY AND ACCELERATION

CONTENTS

1.0 INTRODUCTION

One of the areas of application of differentiation is the approximation of a function $y = f(x)$. In this unit you will study how to estimate the changes produced in a function $y = f(x)$. when x changes by a small amount Δx. In other words if there is a change in x by a very small amount Δx, then there will be a corresponding changes Δy in y. The approximate estimate. An important rate of change of distance with respect to time. Which corresponds to the speed if a body in motion. Of moving object at a given instant of time could be computed using the derivative of the distance function. The second derivative of the distance function will produce the acceleration of the moving body. The objective to achieve in the study of this unit is hereby studied.

2.0 OBJECTIVES

After studying this unit you should be able to

(i) Find the approximate value of the change in y = (x) if there is a small change in x.
(ii) To compute the velocity of a moving body at a given instant of time.
(iii) Compute the acceleration of a moving body at a given instant of time.

3.0 MAIN CONTENT

3.1 Application to Approximation

3.1.1 Differential

You will start the study of the application of differentiation by examining the concept of "differential" of y and x where f (x). = y.

Definition: If x is the independent variable and y = f (x). has a derivative at xo, say, define dx to be an independent variable with domain = R and define dy to be $\frac{dy}{dx.} = f^1(x).$

You will now consider two types of changes that can take place within a specified domain of a function.

Let y = f (x). then an increment Δx in x produces a corresponding increment Δy in y .

Given that:

$$\lim_{\Delta x \to 0} f(x + \Delta x) - f(x) = f^1(x)$$

$$\text{but} \quad f^1(x) \lim_{\Delta x \to 0} \frac{f^1(x)\,\Delta x}{\Delta}$$

$$\text{then} \quad \lim \frac{f(x + \Delta x) - f(x) - f^1(x)\,\Delta x}{\Delta x} = 0$$

The difference $f(x + \Delta x) - f(x)$ is called the increment of f from x it x + Δx and is denoted by the symbol:

$$\Delta f = f(x + \Delta x) - f(x)$$

The product $f^1(x)\,\Delta x$ is called the differential at x with increment Δx and is usually denoted by d f:

$$\text{i.e.} \quad d f = f^1(x)\,\Delta x.$$

Given that:

$$f(x + \Delta x) - f(x) = f^1(x)\,\Delta x. + y(x)$$

for a very small Δx.

$$\Delta f \approx d f .$$

where $\lim\limits_{\Delta x \to 0} \frac{g(x)}{\Delta x} = 0$

since $y = f(x)$ you could re-write the above as:

$$\Delta y \Rightarrow dy$$

then $dy = f^1(x)\, dx$

let $x = f(t)$ and $y = g(t)$

then $dx = f^1(t)dt$ and $dy = g^1(t)dt$,

if $dt \neq 0$ and $f^1(t) \neq 0$

then $dy = \frac{g^1(t)}{f^1(t)}$

from the above you can say that a function $y = f(x)$ is the product of the derivative of the function and dx

Example:

The dimension of a square is x cm.

(i) how do small changes is x affect the area given as $A = x^2$

Solution

$$\Delta f = f(x + \Delta x) - f(x)$$

$$= f(x + \Delta x)^2 - x^2 = 2x\, \Delta x + (\Delta x)^2$$

$$\Rightarrow \quad f\, d = f^1(x)\, \Delta x = 2x\, \Delta x$$

The error of the estimate is the difference between the actual change and the estimated change that is

$$\Delta f - d f = (\Delta x)^2$$

If $\Delta x \to 0$ then this error $\to 0$.

Example: Use differentials to determine how much the function. $Y = x^{1/3}$ changes when

(1) x is increased from 8 to 11

(2) x is decreased from 1 to 0.5

Solution

(1) Let $x_o = 8$ and $\Delta x = 11 - 8 = 3$

Given that; $y = x^{1/3}$

$$\frac{dy}{dx} = \frac{1}{3} x^{-1/3} \Rightarrow \Delta y \cong \frac{1}{3} (x^{-1/3})dx$$

$$\Delta y = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}\,\frac{1}{8^{2/3}} = \frac{1}{3}(\sqrt[3]{8})^{-2} = \frac{1}{3}(2)^{-2}$$

$$\Delta y = \frac{1}{3}\cdot\frac{1}{4}\cdot\Delta x = \frac{1}{3}\cdot\frac{1}{4}\cdot 3 = \tfrac{1}{4} = 0.25$$

A change in the value of x from 8 to 11 increased the value of y by 0.25.

(2) let $x_o = 1$ and $\Delta x = (0.5 - 1) = -0.5$.

$$\Delta y = 1/3(1)^{-2/3} \cdot (-0.5) = -1/6 = -0.167$$

Example

If $Q = \frac{9}{x}$ and x is decreased from 1 to 0.85, what is the approximate change in

the value of Q.

Solution:

Let $x_o = 1$

Then $x - x_o = \Delta x = 1 - 0.85 = -.15$.

$$\frac{dQ}{dx} = \frac{-9}{x^2}$$

when x = 1, $\frac{dQ}{dx} = -9$

$\Delta Q = -9.(-.15) = 1.35.$

Example

If x = sin t and y = cost

$0 < t < \pi$

Find (1) dx and dy (II) dy/dx

Solution

(1) dx = cos t and dy = sin t

(2) $\frac{dy}{dx} = \frac{-\sin t}{\cos t} = \frac{-x}{y.}$

Example

Use differential to approximate the value of $\sqrt{125}$

Solution

Let $y = \sqrt{x}$

You are aware that $\sqrt{121} = 11$

So the problem reduces to finding an approximate change in y as x increases from 121 to 125.

$x_o = 121, \quad \Delta x = 125 - 121 = 4$

$\Delta y \frac{dy}{dx} \Delta x .$

since $y = x^{1/2}$ ==> $\frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{x}}$

if x = 121 $\frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{121}} = \frac{1}{22}$

$\Delta y = \frac{dy}{dx}\Delta x \frac{1}{22} .4 = \frac{2}{11}$

then $\sqrt{125} \cong \sqrt{121} + 2 = 11 + 0.1182$

$\sqrt{125} \cong 11.182$

Example

Given that $P = (3q^2 - 2)^2$

When q = 3 it is increased by 0.8%. Find the approximate percentage change in P

Solution

Let $q_o - 3$, $\Delta q = .9\%$ of q_o

Given $\Delta P = \frac{dP}{dq} \Delta q$

$P = (3q^2 - 2)^2, \frac{dP}{dq} = 2(3q^2 - 2) . 6q.$

$\Delta q = 0.008 \times 3 = 0.024$

$\Delta P = (2(3(3)^2 - 2)63) \times 0.024.$

$= (50 \times 6 = 3) \times 0.024 = 21.6$

% change in p $\cong \frac{\Delta P}{P} \times 100$

$P = (3q^2 - 2)^2 = (3(3)^2 - 2)^2 = (25)^2 = 125.$

$\therefore$ change in P $= \frac{21.6}{125} \times 100 = 17.28\%$

SELF ASSESSMENT EXERCISE 1

Use differential to estimate the following:

(1) $\sqrt{l04}$ (2) $(1020)^{1/3}$ (3) $(24)^{-1/4}$

(4) If $y = \sqrt{x}$, find the approximate increase in the value of y if x is increased from 2 to 2.05.

(5) if $y = (x^3 + 1)^2$ when x = 2 it is increasing by 0.5% find the approximate percentage change in y.

Solutions

(1) $\sqrt{104} \approx 10.2$

(2) $(1020)^{1/3} \approx 10 + 2/30 = 10.067$

(3) $(24)^{-\frac{1}{2}} = 1/24 \approx 21 - (0.063)$

(4) $y \approx 1.432$

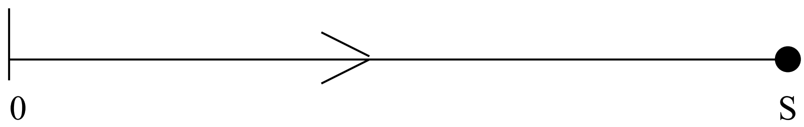
(5) 2.7%

3.2 Application of Differentiation to Velocity

You are familiar with the word speed. This is defined as the distance traveled by a body divided by the time it takes the body to arrive the distance.

i.e $$\text{speed} = \frac{\text{distance}}{\text{time}}$$

In the above the motion must occur along a straight line. Such as



(motion of car along a straight road)

However it might be a tricky question if you want to know the speed of a body in at a given instant along a curve.

For example: The motion of a rocket fired into the air to be able to describe how fast the rocket is rising after say a few second it is fired might not be very clear as finding the speed of the car along a straight road. To be able to know the speed of the rocket one must know the function that described the motion of the rocket in the air. Again one must be able to know how fast the rocket is rising after it is fired for every single point in the curve describing the motion. Un other words to be able to know how fast such a body is traveling one must know the velocity of the body at a given instant. You are aware that to find the velocity of a body all you need to do is to find the distance between the staring or initial point and point at which you want to find the velocity.

i.e. $$\text{Average velocity} = \frac{\text{Dist}\,(t_o) - \text{Dist}(t_1)}{t_1 - t_o}$$

where t_o = initial time and t_1 = time traveled The above formula cannot give you the value of instantaneous velocity which is the object of study in the section.

Let a body travel along a curve $f(t) = t^2$, if you sub divide the entire time it took to cover this curve in a given interval of time t [t_o, t_1 in a subinterval of length h . Then the average velocity between to and to th will be given as;

$$\frac{f\,(toth) - f\,(to)}{h}$$

The smaller h is the closer thus average velocity is to ($2t_o$+ h). so at the time t = t_o this average velocity is $2t_o$. Take a closer took again at equation (1) you will agree that the average velocity at a given instant in the same process used in deriving the slope of a curve y = f (t) at the point t = to thus the velocity of the body at time to is the same numerically with the slope of y = f (t) = t2 at t = t_o.

Definition: If f (t) is the position of a moving body at time t, its velocity is defined as ;

$$\frac{d\,f\,(t)}{dt} = \lim_{h \to 0} \frac{f\,(tth) - f\,(t)}{h}$$

its speed is defined as $\left(\frac{d\,f\,(t)}{dt}\right)$

the above implies that you might have a negative velocity. This always happen in the case of falling bodies. In most textbooks the velocity of a body is represented as y.

i.e. if y = f (t) then $\frac{d\,f\,(t)}{dt}$ = y

Example

A shot is fired from a top of Abia Tower at Umuahia which is at a height $100 + 24t - 12t^2$ After t seconds. Find

(i) its velocity after 2 seconds

(ii) its maximum height

(iii) its velocity as the bullet hits the ground.

Solution

Let $y(t) = 100 + 24 = - 2t^2$

Then $\frac{d f(t)}{dt} = y(t) = 24 - 4t$

(i) After 2 saec t = 2.

$\therefore \quad y(2) = 24 - 4.2 = 16 \ m^2/sec.$

Since y(2) > 0 it implies that the bullet is rising.

Ans. = $16m^2/sec.$

(ii) The maximum height can only be attained at the maximum value of y(t) i.e.

$y(t) = 0, \Rightarrow 24 - 4t = 0$

t = 6.

$Y(12) = 100 + 24.6 - 2(6)^2 = 172$

Observe that when the velocity = 0.

Then the bullet will start to fall down. This occurs at the instant the bullet attain it highest point (see Fig 10.12)

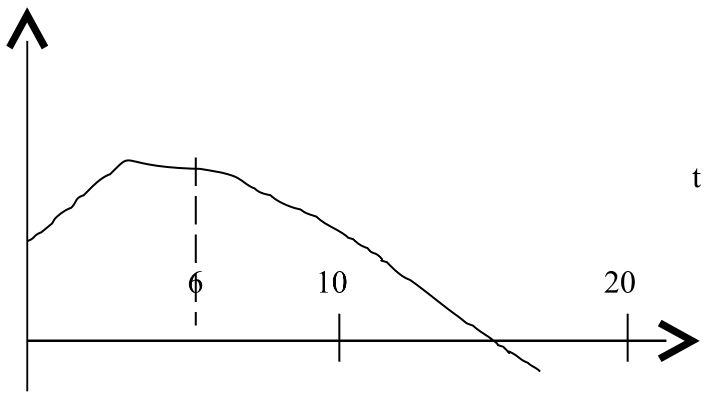


Fig 10.12.

At time t = 6 y(6) = 0

Y(16) = 172m. is the maximum height

Ans. =172m.

(iii) When the bullet hits ground y(t) = 0

.'. $100 + 24t - 2t^2 = 0$

==> $t^2 - 12t - 50 = 0$

$$\frac{-12 \pm \sqrt{144 + 4.50}}{2}$$

$$\frac{12 \pm \sqrt{344}}{2} = \frac{12 \pm 18.547}{2}$$

= 15.274 (see Fig 10.12)

at = t = 15 .274.

y(15.274) = 24 - 4 (15.274)

= -37.096.

Ans. -37.096 m/s. This should be expected

Since the bullet is falling down in practical terms the speed = 1 -37.096 m/s/ = 37.01

Compare to the rising speed of 16m/s at t = 2.

SELF ASSESSMENT EXERCISE 2

(a) A ball thrown straight up has a height $f(t) = -16t^2 + 160t$ after t seconds Find its

(i) maximum height

(ii) the velocity when it hits the ground

Solution

(i) 400m. (ii) -160.m/s.

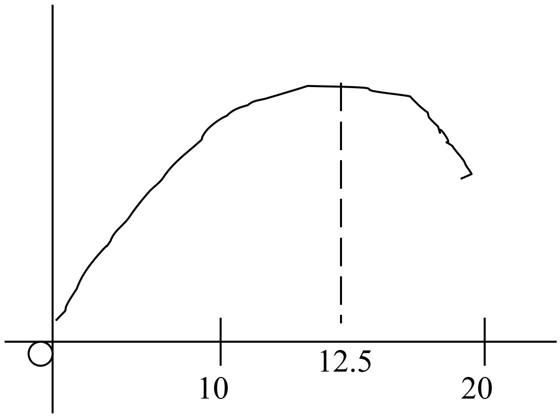
(b) a ball thrown upwards from a building attains a height of $f(t) = (-16t^2 + 400t + 8000)$ m after t seconds. Find

(i) the time it attains its maximum height
(ii) the maximum height
(iii) the velocity after 15 seconds.
(iv) Sketch the curve between t = 0 and t = 20

Solution

(i) 12.5 secs (ii) 10.500m (ii) - 80m/s.

(iv)



3.3 Application of Differentiation to Acceleration

If you applied brakes on a moving car it moves slower and slower it velocity decreases. This implies that velocity, of a moving body can either be increasing decreasing or constant.

Definition : If the velocity of a moving body at t is given as v(t) the acceleration of the body is given as

$$\frac{dv}{dt}(t) - v(t)$$

Put simply acceleration is the derivative of the velocity, i.e. if measures the rate of change of velocity during motion.

If f(t) represent the distance covered by a moving body after t second.

Then $\frac{d^2 \int(t)}{dt^2}$ = acceleration.

Simply put acceleration is the second derivative of the equation of the distance covered after t seconds.

Example

A stone thrown above the ground attains a height of 100 + 24t -8t after t seconds.

Find the acceleration at time t.

f (t) = 100+ 241 - 8t^2

f (t) = 24 - 16t

f (t) = -16 m/s.

Ans. = -16m/s.

Example

The distance covered by a moving body after t seconds is given as

$$f\ (t) = t^3 - 3t^3 + 2$$

Find the

(i) acceleration of the body at t = 2

(ii) At what time will the acceleration equal to zero.

Solution

f (t) = t^3 - 3t^2 +2.
f (t) = 3t^2 - 6t

f (t) = 6t - 6.

At t = 2 f (2) = 2.6-6 = 6m^2 /see.

f (t) = 0 => t = 1.

Ans: (i) 6m^2/sec. (ii) 1.

SELF ASSESSMENT EXERCISE 3

A small body is made to travel in a straight line so that at time t sec after a start in distance f (t) from a fixed point 0 on the straight line is given by

$$f\ (t) = t^3 - 4t^2 + 12.$$

Find;

(1) how far has the body traveled starting from the point 0.

(ii) Evaluate the velocity after t = 2.

(iii) A what time is the acceleration zero.

(iv) What is the acceleration after 4 seconds.

Solution:

(1) 12m (ii) -4m/s. (ii) 1.33 secs. (iv) $16m^2/S$.

4.0 CONCLUSION

In this unit you have studied the applications of differentiation to problem solving. You have applied the first derivative at a function to

(i) Approximate. Values of a variable quantity
(ii) Find the rate at which a quantity is changing with respect. To another.
(iii) Find the equation of a tangent and normal to a curve.

You have also studied how to use the first and second derivatives of a function to find the approximate value of the change in value of a given quantity with respects to a small change in another value. You have studied how to compute the instantaneous velocity and acceleration of moving body.

5.0 SUMMARY

In this unit you have studied how to

(1) Approximate a value of a function f by its differential d f i.e. df $\approx f^1$ (x)Δx.

(2) calculate the velocity and acceleration of a moving body i.e. v(t) $\frac{df}{dt}$ (t) and

$$\text{ace }/^n = v(t0 = \frac{d^2 f(t)}{dt^2}$$

where f (t) is the distance covered in any measurable unit after t second.

6.0 TUTOR-MARKED ASSIGNMENT

1. If P = 4/x and x is decreased from 0.5 to 0.1 what is the approximate change in the value of P.

2. Given that f(x) x ½ estimate the change in f if
   (i) x is increased from 32 to 34.
   (ii) x is decreased from 1 to 9/10

3. Use the differential to estimate √1004

4. A ball thrown straight up from the top of a building at height of $180 + 64t - 16t^2$ after t sec. Compute;
   (a) It velocity after 1 sec.
   (b) Its maximum height
   (c) Its velocity as it hits the ground.

5. A ball is $180 + 64t - 16t^2$ meters above the ground at time t sec. Find its acceleration at time t.

6. A ball thrown above the ground attains a height of $f(x) = 20 = 4t - t^2$ after t seconds. Find;
   (a) The maximum height
   (b) The velocity after 3 sees.
   (c) The velocity when the ball hits the ground.
   (d) The acceleration of the ball

7.0 REFERENCES/FURTHER READINGS

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UNIT 4 NORMAL AND TANGENTS

CONTENTS

1.0 INTRODUCTION

In this unit you will apply the differentiation of a function y =f(x) to find the slope of a tangent to a curve at a point. You could recall that this idea was extensively discussed in unit 6.You then use the slope of the tangent to compute the slope of the normal to the curve at the same point. So it is necessary you review the slope f a curve studied in unit 6.

2.0 OBJECTIVES

After studying this unit you should be able to correctly:

(i) Find the slope of a tangent to a curve by method of differentiation.
(ii) Find the slope of a normal to a curve by the method of differentiation
(iii) Derive the point-slope equation of a tangent at a given point of a curve.
(iv) Derive the point - slope equation of a normal at a given point of curve.

3.0 MAIN CONTENT

3.1 The Point-Slope Equation of a Line

Let a line L pass through the point P (x_l, y_l) and let Q (x, y) any other pint on the curve.

The slope of the line L is given as:

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1) \quad \text{------ I}$$

The above equation I is known as the point-slope form of the equation of the line L. Since it givens the equation of the line is terms of a single point P(x_1 y_1) on the line and the slope m of the line. That is why it is called the point -slope equation of a line.

Therefore once you know the coordinate of just one point on the curve and you can determine the slope of the tangent-line at that point by method of differentiation. Then you can easily form the equation of the tangent -line by using equation I above.

Example: Given that the slope of a line is 2 and the line passes through the point P (2, - 2). Write the equation of the line

Solution

Using point-slope formula

$y - y_1 = m(x-x_1)$
and given that $m = 2_1$ $x_1= 2_1$ and $y_1 = -2$

then you have that:

$y- (-20) = 2(x-2)$
$y = 2 = 2x-4y = 2x -6$

which is the required equation of a line.

3.2 Equation of a Tangent to a Curve

In unit 5 you studied that the slope of the curve y = f (x) at any given point is given as:

$$m= \frac{(dy)}{dx} = f^1 (x)$$

Which is also the slope of the tangent to curve y = f (x) at the given point (x_1 y_1).

Therefore the equation of a tangent - line to a given curve y = f(x) at a given point (x_1, y_1). On the curve can be written as;

$y - y_1 = m(x - x_1)$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

(x_1, y_1)

$= f^1(x_1)(x - x_1).$

Therefore tangent is given as;

$y = f^1(x_1)(x - x_1) + y_1$

3.3 Equation of Normal to Curve

The normal is the line that is peculiar to the tangent. As such if m is the slope of the tangent passing through the point (xl, yl) then the slope of the normal passing through the [point (xl yl) is given as;

$$M_N = \frac{-1}{M_T}$$

Since $M_T = f^1(X^1)$

Then $M_N = \dfrac{-1}{f(x_1)}$

Therefore the equation of the normal line at point (x_1, y_1) is given as;

$$y = -1 \frac{(x - x_1)}{f(x_1)} + y_1$$

Where $y = f(x)$.

Examples

Write the equation of the tangent to the following curves at the given points.

(i) $y = x^2$ (-2, 4)

(ii) $y = x^3$ (-1,-1)

(iii) $y = {}^1/_x$ (1,1)

(iv) $y^2 = x^2$ (2,4)

Solution

(i) $\frac{dy}{dx} = 2x$ at x_1 _ - 2, $2x_1 = 2\ (-2)$

$m = -4$

$y = -4(x-(-2) = 4$

$y = -4x - 8 + 4 = -4x - 4$

$y = -4x - 4.$

(ii) $\frac{dy}{dx} = 3x^2$, $m = 3(x_1)^2 = 3\ (-1)^2 = 3.$

$y = 3(x-(-2) = (-1) = 3x = 3 - 1$

$y = 3x = 2$

(iii) $\frac{dy}{dx} = -\frac{1}{x^2}$ $m = \frac{-1}{(x_1)^2} = -1$

$y = -(9x - 2) + 1 = -x + 2 + 1$

$y = 3 - x$

(iv) $2y = \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow m = \frac{2}{2} = 1$

$y = (x-2) + 2 = x$

$y = x$

(v) $xy = 2x + xy = 1$ (12)

(vi) $x\frac{dy}{dx} + y + 2 + 4\frac{dy}{dx}$ 0 (differentiating)

$\Rightarrow (x + 4)\frac{dy}{dx} + y + 2 = 0$ (collecting like terms)

$\frac{dy}{dx} = -\frac{(y + 2)}{x + 4} = -\frac{(2 = 2)}{1 = 4} = \frac{-4}{5}$

Equation of tangent is

$Y = \frac{-4}{5}(x-1) = 2$

$\Rightarrow$ 5y = -4x = 1 + 10 = -4x + 11.
5y = -4x + 11

Equation of normal is

$Y = \frac{5}{4}(x - 1) + 2$

4y = sx – 5 =
4y = 5x -3.

(iii) $y^2 - 2x - 4y + x^2 = 4$ (-1, -2)

Differentiating

$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} + 2x = 0$

Collecting like terms

$(2y - 4)\frac{dy}{dx} = 2 - 2x$

$\frac{dy}{dx} - \frac{2-2x}{2y-4} = \frac{1-x}{y-2} = \frac{1-(-1)}{-2-2} = \frac{2}{-4}$

m = - ½

Equation of tangent

Y= - ½ (x-(-1))+(-2)
2y=-x-1-4=-x-5
2y+x+5=0.

Equation of normal
Y=2(x+1)-2=2x+2-2
Y=2x

SELF ASSESSMENT EXERCISE 1

1. Find the equation of the tangent and normal to the curves at the given points. (for exercises a - d)

(a) $y = 2x^2 - 1$ x=1
(b) $y = x^2 - 2x$ x=2
(c) $y = \frac{x - 1}{x + 1}$ x = 1
(d) $2x^2 - xy = 16 = y^2$ at (2,4)

2. The slope of the tangent at a point $P(8_1, y_1)$ on the curve $y = 2x^2 - 6x + 1$ is 10. Find x_1 and y_1.

3 The curve where equation $y = ax^2 - bx - 6$ passes through the point (-1, - 4) and the slope of the curve at the is 2 find the value a and b.

Solution

(1). (a) y=4x-3, 4y=3-x
(b) y=2x-4, 2y=xn-2
(c) 2y = x-1, y=2-2x
(d) 5y = 2x + 16 2y = 5x -2.

(2). (4,9)
(3). a = 2, b = -6

4.0 CONCLUSION

In this unit you have reviewed the point - slope equation formula for the point slope equation of the tangent line by using differentiation to calculate the slope of the tangent at a given point. You have used the slope of a tangent to determine the slope of a normal and consequently derived the formula for the point slope equation of a the normal line to a curve at a given point. You have solved example on the above.

5.0. SUMMARY

You have studied in this unit how to:

(1) Determine point slope equation of a line $(y - y_1) = m(x - x_1)$
(2) Derive the point - slope equation of the tangent to a curve at a given point.
(3) Derive the point - slope equation of the normal to a curve at a given point.
(4) Determine the slope of the a tangent and normal to a curve by differentiation.

6.0 TUTOR MARKED ASSIGNMENT

Find the tangent and normal to the curves at the specified points.

1. $y^2+x^2-4x+3y=1$ x=1,y-1

2. $y^2=x+\frac{4}{x}$ $x=1, y=\sqrt{5}$

3. $y = \frac{2-+}{3-x}$ (2,5)

4. $x^2 + y^2 = 25$ (3,4)

5. $y=x^3-x$ $x_0=-2$

6. Derive the point - slope from of equation of a tangent and normal to a curve y = f(x) at the point (x_1, y1).

7.0 REFERENCES/FURTHER READINGS

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