NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 101

COURSE TITLE: ELEMENTARY MATHEMATICS I

COURSE GUIDE

Course Developer/Writers

Ester Omonayin
Base University, Abuja
and
Babatubde Osho
National Open University of Nigeria

Content Editor

Ishola Christy,
Olorunnisola Toyin
and
Ogundipe. Olalekan
Department of Mathematics,
National Open University of Nigeria

Course Coordinators

Ishola Christy,
Olorunnisola Toyin
and
Ogundipe. Olalekan
Department of Mathematics,
National Open University of Nigeria

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos
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Course Guide

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Introduction

Introduction

MTH 101 - Elemetary Mathematics I is designed to teach you how mathematics could be used in solving problems in the contemporary Science world. Therefore, the course is structured to expose you to the skills required in other to attain a level of proficiency in Science, tecnology and Engineering Professions.

What you will learn in this Course

You will be taught the basis of mathematics required in solving scientific problems.

Course Aim

There are ten study units in the course and each unit has its objectives. You should read the objectives of each unit and bear them in mind as you go through the unit. In addition to the objectives of each unit, the overall aims of this course include:

(i) To introduce you to the words and concepts in Elementary mathematics
(ii) To familiarize you with the peculiar characteristics in Elementary mathematics.
(iii) To expose you to the need for and the demands of mathematics in the Science world.
(iv) To prepare you for the contemporary Science world.

Course Objectives

The objectives of this course are:

* To inculcate appropriate mathematical skills required in Science and Engineering.

* Educate learners on how to use mathematical Techniques in solving real life problems.

* Educate the learners on how to integrate mathematical models in Sciences and Engineering.

Working through this Course

{ You have to work through all the study units in the course. There are two modules and ten study units in all.

Course Materials

Major components of the course are:

1. Course Guide
2. Study Units
3. Textbooks
4. CDs
5. Assignments File
6. Presentation Schedule

Study Units

The breakdown of the three modules and eight study units are as follows:

MODULE ONE

Unit 1 Elementary Sets Theory

Unit 2 Basic Set Operations

Unit 3 Set of Numbers

MODULE TWO

Unit 1 : Real Sequence And Series
UNIT 2 : Qadratic equations
Unit 3 : Mathematical Induction

MODULE THREE:

UNIT !: COMPLEX NUMBERS
UNIT 2 CIRCULAR MEASURE

Recommended Texts

* {Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

* Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable, Vol. 1)

* Pure Mathematics for Advanced Level By B.D Bunday H Mulholland1970.

* Introduction to Mathematical Economics By Edward T. Dowling.

* Mathematics and Quantitative Methods for Business and Economics. By Stephen P. Shao. 1976.

* Mathematics for Commerce & Economics By Qazi Zameeruddin & V.K. Khanne 1995.
* Engineering Mathematics By K. A Stroad.
* Business Mathematics and Information Technology. ACCA STUDY MANUAL By. Foulks Lynch.
* Introduction to Mathematical Economics SCHAUM"S Out lines

Assignment File

{ In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain from these assignments will count towards the final mark you obtain for this course. Further information on assignments will be found in the Assignment File itself and later in this *Course Guide* in the section on assessment.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor-marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against falling behind in your work.

Assessment

Your assessment will be based on tutor-marked assignments (TMAs) and a final examination which you will write at the end of the course.

}

Exercises TMAS

{ Every unit contains at least one or two assignments. You are advised to work through all the assignments and submit them for assessment. Your tutor will assess the assignments and select four which will constitute the 30% of your final grade. The tutor-marked assignments may be presented to you in a separate file. Just know that for every unit there are some tutor-marked assignments for you. It is important you do them and submit for assessment. }

Final Examination and Grading

{ At the end of the course, you will write a final examination which will constitute 70% of your final grade. In the examination which shall last for two hours, you will be requested to answer three questions out of at least five questions.

Course marking Scheme

This table shows how the actual course marking as it is broken down.

Assessment	Marks
Assignments	Four assignments, Best three marks of the four count at 30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

How to Get the Most from This Course

In distance learning, the study units replace the university lecture. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suits you best. Think of it as reading the lecture instead of listening to the lecturer. In the same way a lecturer might give you some reading to do, the study units tell you when to read, and which are your text materials or set books. You are provided exercises to do at appropriate points, just as a lecturer might give you an in-class exercise. Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit, and how a particular unit is integrated with the other units and the course as a whole. Next to this is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. These learning objectives are meant to guide your study. The moment a unit is finished, you must go back and check whether you have achieved the objectives. If this is made a habit, then you will significantly improve your chances of passing the course. The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a Reading section. The following is a practical strategy for working through the course. If you run into any trouble, telephone your tutor. Remember that your tutor's job is to help you. When you need assistance, do not hesitate to call and ask your tutor to provide it.

In addition do the following:

1. Read this Course Guide thoroughly, it is your first assignment.

2. Organise a Study Schedule. Design a Course Overview " to guide you through the Course". Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the Semester is available from the study centre. You need to gather all the information into one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates and schedule of work for each unit.

3. Once you have created your own study schedule, do everything to stay faithful to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please, let your tutor know before it is too late for help.

4. Turn to Unit 1, and read the introduction and the objectives for the unit.

5. Assemble the study materials. You will need your set books and the unit you are studying at any point in time.

6. Work through the unit. As you work through the unit, you will know what sources to consult for further information.

7. Keep in touch with your study centre. Up-to-date course information will be continuously available there.

8. Well before the relevant due dates (about 4 weeks before due dates), keep in mind that you will learn a lot by doing the assignment carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the examination. Submit all assignments not later than the due date.

9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

10. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

11. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the Assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also the written comments on the ordinary assignments.

12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in the Course Guide).

Tutors and Tutorials

The dates, times and locations of these tutorials will be made available to you, together with the name, telephone number and the address of your tutor. Each assignment will be marked by your tutor. Pay close attention to the comments

your tutor might make on your assignments as these will help in your progress. Make sure that assignments reach your tutor on or before the due date.

Your tutorials are important therefore try not to skip any. It is an opportunity to meet your tutor and your fellow students. It is also an opportunity to get the help of your tutor and discuss any difficulties encountered on your reading. **Summary**

This course would train you on the concept of multimedia, production and utilization of it.

Wish you the best of luck as you read through this course

Course Code:MTH 101
Course Title: ELEMENTARY MATHEMATICS I

MODULE ONE

Unit 1 Elementary Sets Theory

Unit 2 Basic Set Operations

Unit 3 Set of Numbers

MODULE ONE

Unit 1 Elementary Sets Theory

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UNIT 1:

Elementary SetsTheory

1.0 Introduction

2.0 Objectives

3.0 Main Body

  3.1 Sets

  3.1.1 Notation

  3.1.2 Finite and Infinite sets

  3.1.3 Equality of sets

  3.1.4 Null Set

3.2 Subsets

  3.2.1 Proper subsets

  3.2.2 Comparability

  3.2.3 Set of sets

  3.2.4 Universal set

  3.2.5 Power set

  3.2.6 Disjoint sets

3.3 Venn-Euler diagrams

3.4 Axiomatic development of set theory

4.0 Conclusion

5.0 Summary

6.0 Tutor-Marked assignment

7.0 References and Further readings

1.0 INTRODUCTION

The theory of sets lies at the foundation of mathematics. It is a concept that rears its head in almost all fields of mathematics; pure and applied.

This unit aims at introducing basic concepts that would be explained further in subsequent units. There will be definition of terms and lots of examples and exercises to help you as you go along.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Identify sets from some given statements
- Rewrite sets in the different set notation
- Identify the different kinds of sets with examples

3.0 MAIN BODY

3.1 SETS

As mentioned in the introduction, a fundamental concept in all a branch of mathematics is that of set. Here is a definition **"A set is any well-defined list, collection or class of objects".**

The objects in sets, as we shall see from examples, can be anything: But for clarity, we now list ten particular examples of sets:

Example 1.1 The numbers 0, 2, 4, 6, 8

Example 1.2 The solutions of the equation $x^2 + 2x + 1 = 0$

Example 1.3 The vowels of the alphabet: a, e, i, o, u

Example 1.4 The people living on earth

Example 1.5 The students Tom, Dick and Harry **Example 1.6** The students who are absent from school **Example 1.7** The countries England, France and Denmark **Example 1.8** The capital cities of Nigeria

Example 1.9 The number 1, 3, 7, and 10

Example 1.10 The Rivers in Nigeria

Note that the sets in the odd numbered examples are defined, that is, presented, by actually listing its members; and the sets in the even numbered examples are defined by stating properties that is, rules, which decide whether or not a particular object is a member of the set.

3.1.1 Notation

Sets will usually be denoted by capital letters;

A, B, X, Y,......

Lower case letters will usually represent the elements in our sets:

Lets take as an example; if we define a particular set by actually listing its members, for example, let A consist of numbers 1,3,7, and 10, then we write

$A=\{1,3,7,10\}$

That is, the elements are separated by commas and enclosed in brackets { }.

We call this the tabular form of a set

Now, try your hand on this

Exercise 1.1

State in words and then write in tabular form

1. $A = \{x\} \mid x^2 = 4^2\}$

2. $B = \{x\} \mid x - 2 = 5\}$

3. C = {x} | x is positive, x is negative}

4. D = {x} | x is a letter in the word “correct”}

Solution:

1. It reads “A is the set of x such that x squared equals four”. The only numbers which when squared give four are 2 and -2. Hence A = {2, -2}

2. It reads “B is the set of x such that x minus 2 equals 5”. The only solution is 7; hence B = {7}

3. It read “C is the set of x such that x is positive and x is negative”. There is no number which is both positive and negative; hence C is empty, that is, C=Ø

4. It reads “D is the set of x such that x is letter in the work ‘correct’. The indicated letters a re c,o,r,e and t; thus D = {c,o,r,e,t}

But if we define a particular set by stating properties which its elements must satisfy, for example, let B be the set of all even numbers, then we use a letter, usually x, to represent an arbitrary element and we write:

B = {x | x is even}

Which reads “B is the set of numbers x such that x is even”. We call this the set builders form of a set. Notice that the vertical line “|” is read “such as”.

In order to illustrate the use of the above notations, we rewrite the sets in examples 1.1-1.10. We denote the sets by A_1, A_2, ... A_{10} respectively.

Example 2.1: A_1 = {0, 2, 4, 6, 8}

Example 2.2: A_2 = {x | $x^2+2x + 1 = 0$}

Example 2.3: A_3 = {a, e, i, o, u}

Example 2.4: A_4 = {x | x is a person living on the earth}

Example 2.5: A_5 = {Tom, Dick, Harry}

Example 2.6: A_6 = {x ∣ x is a student and x is absent from school}

Example 2.7: A_7 = {England, France, Denmark}

Example 2.8: A_8 = {x∣x is a capital city and x is in Nigeria}

Example 2.9: A_9 = {1, 3, 7, 10}

Example 2.10: A_{10} = { x∣x is a river and x is in Nigeria} It is easy as that!

Exercise 1.2

Write These Sets in a Set-Builder Form

1. Let A consist of the letters a, b, c, d and e
2. Let B = {2, 4, 6, 8........}
3. Let C consist of the countries in the United Nations
4. Let D = {3}
5. Let E be the Heads of State Obasanjo, Yaradua and Jonathan

Solution

1. A={x ∣x appears before f in the alphabet} = {xx is one of the first letters in the alphabet}
2. B = {x ∣x is even and positive}
3. C = {x ∣x is a country, x is in the United Nations}
4. D = {x ∣x – 2 = 1} = {x∣2x = 6}
5. E = {x ∣x was Head of state after Abacha}

If an object x is a member of a set A, i.e., A contains x as one of its\elements, then we write: $x \in A$

Which can be read "x belongs to A" or 'x is in A". If, on the otherhand, an object x is not a member of a set A, i.e A does not contain x as one of its elements, then we write; $x \notin A$

It is a common custom in mathematics to put a vertical line "|" or " \ " through a symbol to indicate the opposite or negative meaning of the symbol.

Example 3:1: Let A = {a, e, i o, u}. Then a∈A, b∉A, f∉A.

Example 3.2: Let B = {x< x is even}. Then 3∉B, 6∈B, 11∉B,14∈B

3.1.2 Finite & Infinite Sets

Sets can be finite or infinite. Intuitively, a set is finite if it consists of a specific number of different elements, i.e. if in counting the different members of the set the counting process can come to an end. Otherwise a set is infinite. Lets look at some examples.

Example 4:1: Let M be the set of the days of the week. The M is finite

Example 4:2: Let N = {0,2,4,6,8........}. Then N is infinite

Example 4:3: Let P = {x< x is a river on the earth}. Although it maybe difficult to count the number of rivers in the world, P is still a finite set.

Exercise 1.3: Which sets are finite?

1. The months of the year

2. {1, 2, 3, ... 99, 100}

3. The people living on the earth

4. {x | x is even}

5. {1, 2, 3,........}

Solution:

The first three sets are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two sets are infinite. If we ever try to count the even numbers, we would never come to the end.

3.1.3 Equality of Sets

Set A is equal to set B if they both have the same members, i.e if every element which belongs to A also belongs to B and if every element which belongs to B also belongs to A. We denote the equality of sets A and B by:

$$A = B$$

Example 5.1 Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B,

that is {1,2,3,4} = {3,1,4,2}, since each of the elements

1,2,3 and 4 of A belongs to B and each of the elements

3,1,4 and 2 of B belongs to A. Note therefore that a set does not change if its elements are rearranged.

Example 5.3 Let $E=\{x \mid x^2-3x = -2\}$, F={2,1} and G ={1,2,2, 1},

Then E= F= G

3.1.4 Null Set

It is convenient to introduce the concept of the empty set, that is, a set which contains no elements. This set is sometimes called the null set.

We say that such a set is void or empty, and we denote its symbol $\emptyset$

Example 6.1: Let A be the set of people in the world who are older than 200 years. According to known statistics A is the null set.

Example 6.2: Let $B = \{x \mid x^2= 4, x \text{ is odd}\}$, Then B is the empty set.

3.2 SUBSETS

If every element in a set A is also a member of a set B, then A is called subset of B.

More specifically, A is a subset of B if $x \in A$ implies $x \in B$. We denote this relationship by writing; $A \subset B$, which can also be read "A is contained in B".

Example 7.1

The set C = {1,3,5} is a subset of D = {5,4,3,2,1}, since each number 1, 3 and 5 belonging to C also belongs to D.

Example 7.2

The set E = {2,4,6} is a subset of F = {6,2,4}, since each number 2,4, and 6 belonging to E also belongs to F. Note, in particular, that E = F. In a similar manner it can be shown that every set is a subset of itself.

Example 7.3

Let G = {x | x is even}, i.e. G = {2,4,6}, and let F = {x | x is a positive power of 2}, i.e. let F = {2,4,8,16.....} Then F⊂G, i.e. F is contained in G.

With the above definition of a subset, we are able to restate the definition of the equality of two sets.

Two set A and B are equal, i.e. A = B, if an only if A⊂B and B⊂A. If A is a subset of B, then we can also write:

B⊃A

Which reads "B is a superset of A" or "B contains A". Furthermore, we write:

A⊄B

if A is not a subset of B.

Conclusively, we state:

1. The null set ∅ is considered to be a subset of every set
2. If A is not a subset of B, that is, if A⊄B, then there is at least one element in A that is not a member of B.

3.2.1 Proper Subsets

Since every set A is a subset of itself, we call B a proper subset of A if, first, is a subset of A and secondly, if B is not equal to A. More briefly, B is a proper subset of A if:

$B \subset A$ and $B \neq A$

In some books "B is a subset of A" is denoted by

$B \subseteq A$

and B" is a proper subset of A" is denoted by

$B \subset A$

We will continue to use the previous notation in which we do not distinguished between a subset and a proper subset.

3.2.2 Comparability

Two sets A and B are said to be comparable if:

$A \subset B$ or $B \subset A$;

That is, if one of the sets is a subset of the other set. Moreover, two sets A and B are said to be not comparable if:

$A \not\subset B$ and $B \not\subset A$

Note that if A is not comparable to B then there is an element in A.which is not in B and ... also, there is an element in B which is not in A.

Example 8.1: Let A = {a,b} and B = {a,b,c}. The A is comparable to B, since A is a subset of B.

Example 8.2: Let R – {a,b} and S = {b,c,d}. Then R and S are not comparable, since $a \in R$ and $a \notin S$ and $c \notin R$

In mathematics, many statements can be proven to be true by the use of previous assumptions and definitions. In fact, the essence of mathematics consists of theorems and their proofs. We now proof our first

Theorem 1.1: If A is a subset of B and B is a subset of C then A is a subset of C, that is,

$A \subset B$ and $B \subset C$ implies $A \subset B$

Proof: (Notice that we must show that any element in A is also an element in C). Let x be an element of A, that is, let $x \in A$. Since A is a subset of B, x also belongs to B, that is, $x \in B$. But by hypothesis, $B \subset C$; hence every element of B, which includes x, is a number of C. We have shown that $x \in A$ implies $x \in C$. Accordingly, by definition, $A \subset C$.

3.2.3 Sets of Sets

It sometimes will happen that the object of a set are sets themselves; for example, the set of all subsets of A. In order to avoid saying "set of sets", it is common practice to say "family of sets" or "class of sets". Under the circumstances, and in order to avoid confusion, we sometimes will let script letters A, B,..........

Denote families, or classes, of sets since capital letters already denote their elements.

Example 9.1: In geometry we usually say "a family of lines" or "a family of curves" since lines and curves are themselves sets of points.

Example 9.2: The set {{2,3}, {2}, {5,6}} is a family of sets. Its members are the sets {2,3}, {2} and {5,6}.

Theoretically, it is possible that a set has some members, which are sets themselves and some members which are not sets, although in any application of the theory of sets this case arises infrequently.

Example 9.3: Let A = {2, {1,3}, 4, {2,5}}. Then A is not a family of sets; here some elements of A are sets and some are not.

3.2.4 Universal Set

In any application of the theory of sets, all the sets under investigation will likely be subsets of a fixed set. We call this set the universal set or universe of discourse. We denote this set by U.

Example 10.1: In plane geometry, the universal set consists of all the points in the plane.

Example 10.2: In human population studies, the universal set consists of all the people in the world.

3.2.5 Power Set

The family of all the subsets of any set S is called the power set of S.

We denote the power set of S by: 2^S

Example 11.1: Let M = {a,b} Then 2^M = {{a, b}, {a}, {b}, ∅}

Example 11.2: Let T = {4,7,8} then 2^T = {T, {4,7}, {4,8}, {7,8}, {4}, {7}, {8}, ∅}

If a set S is finite, say S has n elements, then the power set of S can be shown to have 2^n elements. This is one reason why the class of subsets of S is called the power set of S and is denoted by $2^{S.}$

3.2.6 Disjoint Sets

If sets A and B have no elements in common, i.e if no element of A is in B and no element of B is in A, then we say that A and B are disjoint

Example 12.1: Let A = {1,3,7,8} and B = {2,4,7,9}, Then A and B are not disjoint since 7 is in both sets, i.e 7∈A and 7∈B

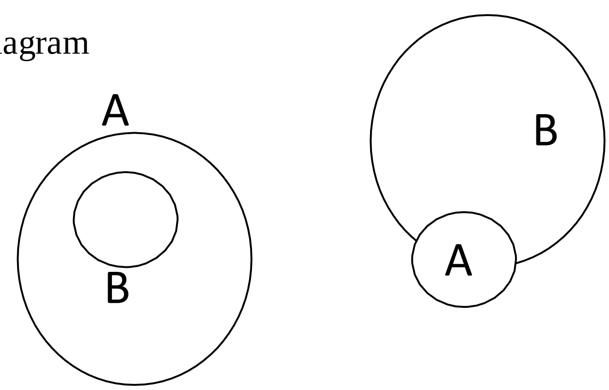
Example 12.2: Let A be the positive numbers and let B be the negative numbers. Then A and B are disjoint since no number is both positive and negative.

Example 12.3: Let E = {x, y, z} and F = {r, s, t}, Then E and F are disjoint.

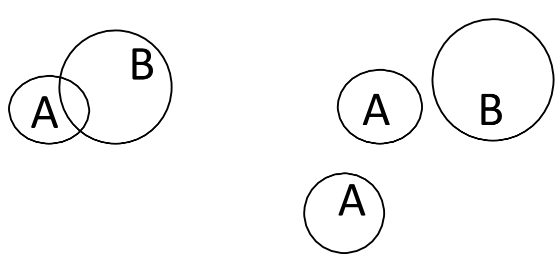
3.3 VENN-EULER DIAGRAMS

A simple and instructive way of illustrating the relationships between sets is in the use of the so-called Ven-Euler diagrams or, simply, Venn diagrams. Here we represent a set by a simple plane area, usually bounded by a circle.

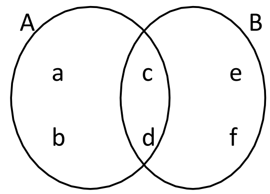
Example 13.1: Suppose $A \subset B$ and, say, $A \neq B$, then A and B can be described by either diagram



Example 13.2: Suppose A and B are not comparable. Then A and B can be represented by the diagram on the right if they are disjoint, or the diagram on the let if they are not disjoint.



Example 13.3: Let A = {a, b, c, d} and B= {c, d, e, f}. Then we illustrate these sets a Venn diagram of the form:



3.4 AXIOMATIC DEVELOPMENT OF SET THEORY

In an axiomatic development of a branch of mathematics, one begins with:

1. Undefined terms

2. Undefined relations

3. Axioms relating the undefined terms and undefined relations. Then, one develops theorems based upon the axioms and definitions **Example 14:1:**

In an axiomatic development of Plane Euclidean geometry

1. "Points" and "lines" are undefined terms

2. "Points on a line" or, equivalent, "line contain a point" is an undefined relation

3. Two of the axioms are:

Axiom 1: Two different points are on one and only one line

Axiom 2: Two different lines cannot contain more than one point in common

In an axiomatic development of set theory:

1. "Element' and "set" are undefined terms

2. "Element belongs to a set" is undefined relation

3. Two of the axioms are

Axiom of Extension: Two sets A and B are equal if and only if every element in A belongs to B and every element in B belongs to A.

Axiom of Specification: Let P(x) be any statement and let A be any set. Then there exists a set:

B= $\{a \mid a \in A, P(a) \text{ is true}\}$

Here, P(x) is a sentence in one variable for which P(a) is true or false for any $a \in A$. for example P(x) could be the sentence "$x^2 = 4$" or "x is a member of the United Nations"

4.0 CONCLUSION

You have been introduced to basic concepts of sets, set notation e.t.c that will be built upon in other units. If you have not mastered them by now you will notice you have to come back to this unit from time to time.

5.0 SUMMARY

A summary of the basic concept of set theory is as follows:

- A set is any well-defined list, collection, or class of objects.
- Given a set A with elements 1,3,5,7 the tabular form of representing this set is A = {1, 3, 5, 7}.
- The set-builder form of the same set is $A = \{x \mid x = 2n + 1, 0 \leq n \leq 3\}$
- Given the set N = {2,4,6,8,....} then N is said to be infinite, since the counting process of its elements will never come to an end, otherwise it is finite
- Two sets of A and B are said to be equal if they both have the same elements, written A = B

- The null set,∅ , contains no elements and is a subset of every set
- The set A is a subset of another set B, written $A \subset B$, if every element of A is also an element of B, i.e. for every $x \in A$ then $x \in B$
- If $B \subset A$ and $B \neq A$, then B is a proper subset of A
- Two sets A and B are comparable if $A \subset B$ and $B \subset A$
- The power set 2^2 of any set S is the family of all the subsets of S
- Two sets A and B are said to be disjoint if they do not have any element in common, i.e their intersection is a null set.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Rewrite the following statement using set notation:

1. x does not belong to A.
2. R is a superset of S
3. d is a member of E
4. F is not a subset of G
5. H does not included D

2. Which of these sets are equal: {r,t,s}, {s,t,r,s}, {t,s,t,r}, {s,r,s,t}?

3. Which sets are finite?

   1. The months of the year
   2. {1,2,3,......99, 100}
   3. The people living on the earth
   4. {x | x is even}
   5. {1,2,3........)

The first three set are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two set are infinite. If we ever try to count the even numbers we would never come to the end.

4. Which word is different from each other, and why: (1) empty, (2) void, (3) zero, (4) null?

7.0 REFERENCE AND FURTHER READING

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable, Vol. 1)

UNIT 2: BASIC SET OPERATIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall see operations performed on sets as in simple arithmetic. This operations simply give sets a language of their own.

You will notice in subsequent units that you cannot talk of sets without reference, sort of, to these operations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Compare two sets and/or assign to them another set depending on their comparability.
- Represent these relationships on the Venn diagram.

3.0 MAIN BODY

3.1 SET OPERATIONS

In arithmetic, we learn to add, subtract and multiply, that is, we assign to each pair of numbers x and y a number x + y called the sum of x and y, a number x – y called the difference of x and y, and a number xy called the product of x and y. These assignments are called the operations of addition, subtraction and multiplication of numbers. In this unit, we define the operation Union, Intersection and difference of sets, that is, we will assign new pairs of sets A and B. In a later unit, we will see that these set operations behave in a manner some what similar to the above operations on numbers.

3.1.1 Union

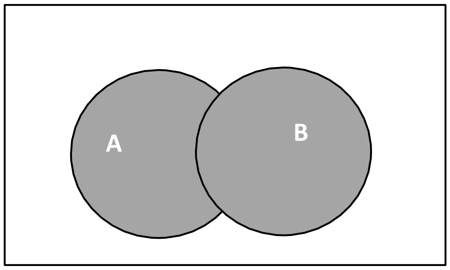
The union of sets A and B is the set of all elements which belong to A or to B or to both. We denote the union of A and B by;

$A \cup B$

Which is usually read “A union B”

Example 1.1: In the Venn diagram in fig 2-1, we have shaded $A \cup B$,

i.e. the area of A and the area of B.



A∪B is shaded

Fig 2.1

Example 1.2: Let S = {a, b. c. d} and T = {f, b, d, g}.

Then S T = {a, b, c, d, f, g}.

Example 1.3: Let P be the set of positive real numbers and let Q be the set of negative real numbers. The P∪Q, the union of P and Q, consist of all the real numbers except zero. The union of A and B may also be defined concisely by:

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Remark 2.1: It follows directly from the definition of the union of two sets that A∪B and B∪A are the same set, i.e.,

A∪B = B∪A

Remark 2.2: Both A and B are always subsets of A and B that is,

$A \subset (A \cup B)$ and $B \subset (A \cup B)$

In some books, the union of A and B is denoted by A + B and is called the set-theoretic sum of A and B or, simply, A plus B

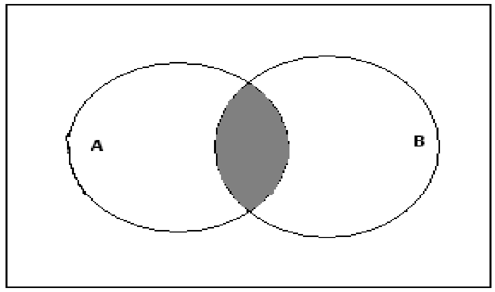
3.1.2 Intersection

The Intersection of sets A and B is the set of elements which are common to A and B, that is, those elements which belong to A and which belong to B. We denote the intersection of A and B by:

A∩B

Which is read "A intersection B".

Example 2.1: In the Venn diagram in fig 2.2, we have shaded A∩B, the area that is common to both A and B



A∩B is shaded

Fig 2.2

Example 2.2: Let S = {a, b, c, d} and T = {f, b, d, g}. Then S∩T = {b, d}

Example 2.3: Let V = 2, 3, 6, ...} i.e. the multiples of 2; and

Let W = {3, 6, 9,....} i.e. the multiples of 3. Then

V∩W = {6, 12, 18......}

The intersection of A and B may also be defined concisely by

$$A\cap B = \{x\in A, x\in B\}$$

Here, the comma has the same meaning as "and".

Remark 2.3: It follows directly from the definition of the intersection of two sets that;

$$A\cap B = B\cap A$$

Remark 2.4:

Each of the sets A and B contains A∩B as a subset, i.e.,

$$(A\cap B)\subset A \text{ and } (A\cap B)\subset B$$

Remark 2.5: If sets A and B have no elements in common, i.e. if A and B are disjoint, then the intersection of A and B is the null set, i.e. $A\cap B = \emptyset$

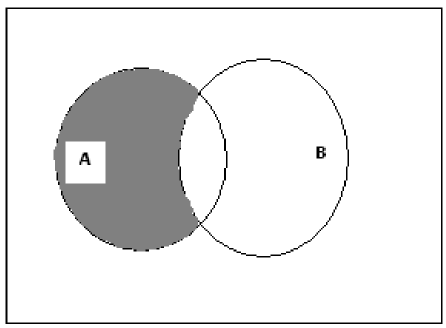
In some books, especially on probability, the intersection of A and B is denoted by AB and is called the set-theoretic product of A and B or, simply, A times B.

3.1.3 DIFFERENCE

The difference of sets A and B is the set of elements which belong to A but which do not belong to B. We denote the difference of A and B by A – B

Which is read “A difference B” or, simply, “A minus B”.

Example 3.1: In the Venn diagram in Fig 2.3, we have shaded A – B, the area



in A which is n

A – B is shaded

Fig 2.3

Example 3.2: Let R be the set of real numbers and let Q be the set of rational numbers. Then R – Q consists of the irrational numbers.

The difference of A and B may also be defined concisely by

$A - B = \{x \mid x \in A, x \notin B\}$

Remark 2.6: Set A contains A – B as a subset, i.e.,

$(A - B) \subset A$

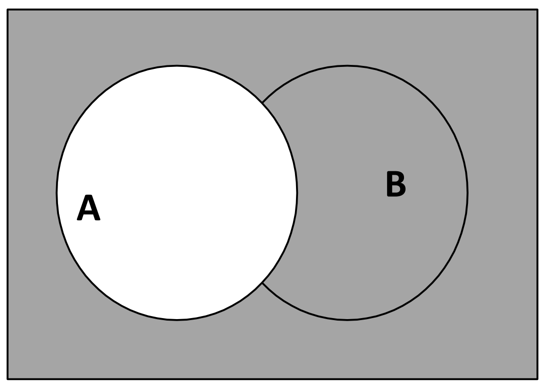
Remark 2.7: The sets (A – B), A∩B and (B – A) are mutually disjoint, that is, the intersection of any two is the null set.

The difference of A and B is sometimes denoted by A/B or A ~ B

3.1.4 Complement

The complement of a set A is the set of elements that do not belong to A, that is, the difference of the universal set U and A. We denote the complement of A by **A′**

Example 4.1: In the Venn diagram in Fig 2.4, we shaded the complement of A, i.e. the area outside A. Here we assume that the universal set U consists of the area in the rectangle.



A' is shaded

Fig. 2.4

Example 4.2: Let the Universal set U be the English alphabet and let T = {a, b, c}. Then;

T' = {d, e, f, ..., y, z}

Example 4.3:

Let E = {2, 4, 6, ...}, that is, the even numbers.

Then E′ = {1, 3, 5, ...}, the odd numbers. Here we assume that the universal set is the natural numbers, 1, 2, 3,.....

The complement of A may also be defined concisely by;

$A' = \{x \mid x \in U, x \notin A\}$ or, simply,

$A' = \{x \mid x \notin A\}$

We state some facts about sets, which follow directly from the definition of the complement of a set.

Remark 2.8: The union of any set A and its complement A′ is the universal set, i.e.,

$A \cup A' = U$

Furthermore, set A and its complement A′ are disjoint, i.e.,

$A \cap A' = \emptyset$

Remark 2.9: The complement of the universal set U is the null set ∅, and vice versa, that is,

$U' = \emptyset$ and $\emptyset' = U$

Remark 2.10: The complement of the complement of set A is the set A itself. More briefly,

$(A')' = A$

Our next remark shows how the difference of two sets can be defined in terms of the complement of a set and the intersection of two sets. More specifically, we have the following basic relationship:

Remark 2.11: The difference of A and B is equal to the intersection of A and the complement of B, that is,

$A - B = A \cap B'$

The proof of Remark 2.11 follows directly from definitions:

$A - B = \{x \mid x \in A, x \notin B\} = \{x \mid x \in A, x \notin B'\} = A \cap B'$

3.2 OPERATIONS ON COMPARABLE SETS

The operations of union, intersection, difference and complement have simple properties when the sets under investigation are comparable. The following theorems can be proved.

Theorem 2.1: Let A be a subset of B. Then the union intersection of A and B is precisely A, that is,

$A \subset B$ implies $A \cap B = A$

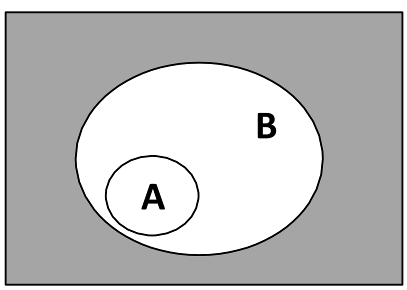
Theorem 2.2: Let A be a subset of B. Then the of A and B is precisely B, that is,

$A \subset B$ implies $A \cup B = B$

Theorem 2.3: Let A be a subset of B. Then B' is a subset of A', that is,

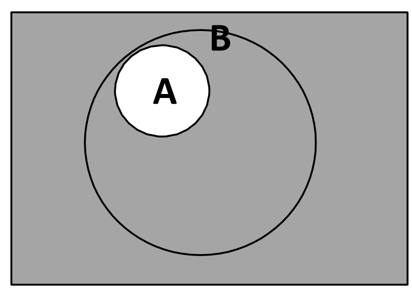
$A \subset B$ implies $B' \subset A'$

We illustrate Theorem 2.3 by the Venn diagrams in Fig 2-5 and 2-6. Notice how the area of B' is included in the area of A'.



B′ is shaded

Fig 2.5



A′ is shaded

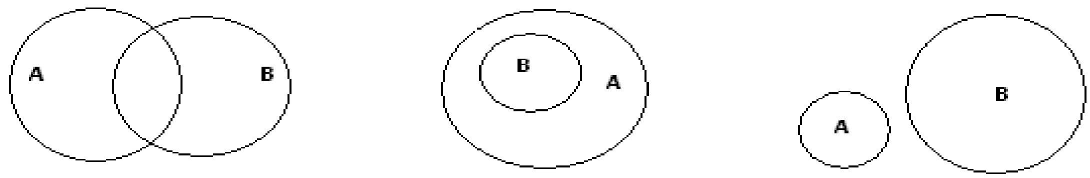
Fig 2.6

Theorem 2.4: Let A be a subset of B. Then the Union of A and (B – A) is precisely B, that is,

$A \subset B$ implies $A \cup (B - A) = B$

Exercises

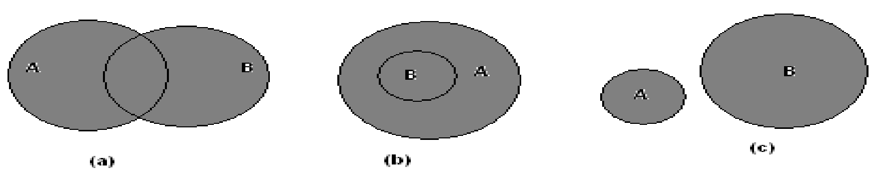
1. In the Venn diagram below, shade A Union B, that is, A∪B:



(a) (b) (c)

Solution:

The union of A and B is the set of all elements that belong to A and to B or to both. We therefore shade the area in A and B as follows:



2. Let A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find

(a) A∪B, (b) A∪C, (c) B∪C, (d) B∪B

Solution:

To form the union of A and B we put all the elements from A together with the elements of B Accordingly,

A∪B = {1,2,3,4,6,8}
A∪C = { 1,2,3,4,5,6}
B∪C = {2,4,6,8,3,5}
B∪B = {2,4,6,8}

Notice that B∪B is precisely B.

3. Let A, B and C be the sets in Problem 2. Find (1) (A∪B)∪C,(2) A∪(B∪C).

Solution:

1. We first find (A∪B) = {1,2,3,4,6,8}. Then the union of {A∪B} and C is

(A∪B)∪C = {1,2,3,4,6,8,5}

2. We first find (B∪C) = {2,4,6,8,3,5}. Then the union of A and (B∪C) is A∪(B∪C) = {1,2,3,4,6,8,5}.

Notice that (A∪B)∪C = A∪(B∪C)

4.0 CONCLUSION

You have seen how the basic operations of Union, Intersection, Difference and Complement on sets work like the operations on numbers. These are also the basic symbols associated with set theory.

5.0 SUMMARY

The basic set operations are Union, Intersection, Difference and Complement defined as:

- The Union of sets A and B, denoted by A∪B, is the set of all elements, which belong to A or to B or to both.
- The intersection of sets A and B, denoted by A∩B, is the set of elements, which are common to A and B. If A and B are disjoint then their intersection is the Null set ∅
- The difference of sets A and B, denoted by A – B, is the set of elements which belong to A but which do not belong to B.
- The complement of a set A, denoted by A’, is the set of elements, which do not belong to A, that is, the difference of the universal set U and A.

6.0 TUTOR – MARKED ASSIGNMENTS

1. Let X = {Tom, Dick, Harry}, Y = {Tom, Marc, Eric} and Z = {Marc, Eric, Edward}. Find (a) $X \cup Y$, (b) $Y \cup Z$(c) $X \cup Z$

2. Prove: $A \cap \emptyset = \emptyset$.

3. Prove Remark 2.6: $(A – B) \subset A$.

4. Let U = {1,2,3,...,8,9}, A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find (a) A', (b) B', (c) $(A \cap C)'$, (d) $(A \cup B)'$, (e) (A')', (f) (B – C)'

5. Prove: B – A is a subset of A'

7.0 REFERENCES AND FURTHER READINGS

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real – valued functions of a real variable), Vol. 1

UNIT 3: SET OF NUMBERS

CONTENTS

1.0 INTRODUCTION

Although, the theory of sets is very general, important sets, which we meet in elementary mathematics, are sets of numbers. Of particular importance, especially in analysis, is the set of ***real numbers,*** which we denote by $\mathcal{R}$

In fact, we assume in this unit, unless otherwise stated, that the set of real numbers $\mathcal{R}$ is out universal set. We first review some elementary properties of real numbers before applying our elementary principles of set theory to sets of numbers. The set of real numbers and its properties is called the ***real number system.***

2.0 OBJECTIVES

After studying this unit, you should be able to do the following:

- Represent the set of numbers on the real line
- Perform the basic set operations on intervals

3.0 MAIN BODY

3.1 REAL NUMBERS, $\mathcal{R}$

One of the most important properties of the real numbers is that points on a straight line that can represent them. As in Fig 3.1, we choose a point, called the origin, to represent 0 and another point, usually to the right, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, that is, each point will represent a unique real number and each real number will be represented by a unique point. We refer to this line as the ***real line.*** Accordingly, we can use the words point and number interchangeably.

Those numbers to the right of 0, i.e. on the same side as 1, are called the *positive numbers* and those numbers to the left of 0 are called the *negative numbers.* The number 0 itself is neither positive nor negative.

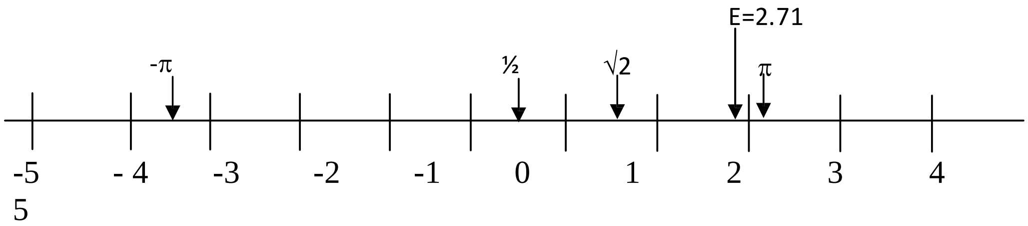


Fig 3.1

3.1.2 Integers, Z

The integers are those real numbers

…, -3, -2, -1, 0, 1, 2, 3,…

We denote the integers by Z; hence we can write

Z = { …, -2, - 1, 0, 1, 2,…}

The integers are also referred to as the "whole" numbers.

One important property of the integers is that they are "closed" under the operations of addition, multiplication and subtraction; that is, the sum, product and difference of two integers is again in integer. Notice that the quotient of two integers, e.g. 3 and 7, need not be an integer; hence the integers are not closed under the operation of division.

3.1.3 Rational Numbers, Q

The *rational numbers* are those real numbers, which can be expressed as the ratio of two integers. We denote the set of rational numbers by Q. Accordingly,

$Q = \{x \mid x = p\ q \text{ where } p \in z, q \in z\}$

Notice that each integer is also a rational number since, for example, 5 = 5/1; hence Z is a subset of Q.

The rational numbers are closed not only under the operations of addition, multiplication and subtraction but also under the operation of division (except by 0). In other words, the sum, product, difference and quotient (except by 0) of two rational numbers is again a rational number.

3.1.4 Natural Numbers, N

The *natural numbers* are the positive integers. We denote the set of natural numbers by N; hence N = {1,2,3…..}

The natural numbers were the first number system developed and were used primarily, at one time, for counting. Notice the following relationship between the above numbers systems:

$N \subset Z \subset Q \subset R$

The natural numbers are closed only under the operation of addition and multiplication. The difference and quotient of two natural numbers needed not be a natural number.

The *prime numbers* are those natural numbers p, excluding 1, which are only divisible 1 and p itself. We list the first few prime numbers: 2,3,5,7,11,13,17,19…

3.1.5 Irrational Numbers, Q'

The irrational numbers are those real numbers which are not rational, that is, the set of irrational numbers is the complement of the set of rational numbers Q in the real numbers R; hence Q' denote the irrational numbers. Examples of irrational numbers are $\sqrt{3}$, π, $\sqrt{2}$, etc.

3.1.6 Line Diagram of the Number Systems

Fig 3.2 below is a line diagram of the various sets of number, which we have investigated. (For completeness, the diagram include the sets of complex numbers, number of the form a + bi where a and b are real. Notice that the set of complex numbers is superset of the set of real numbers.)

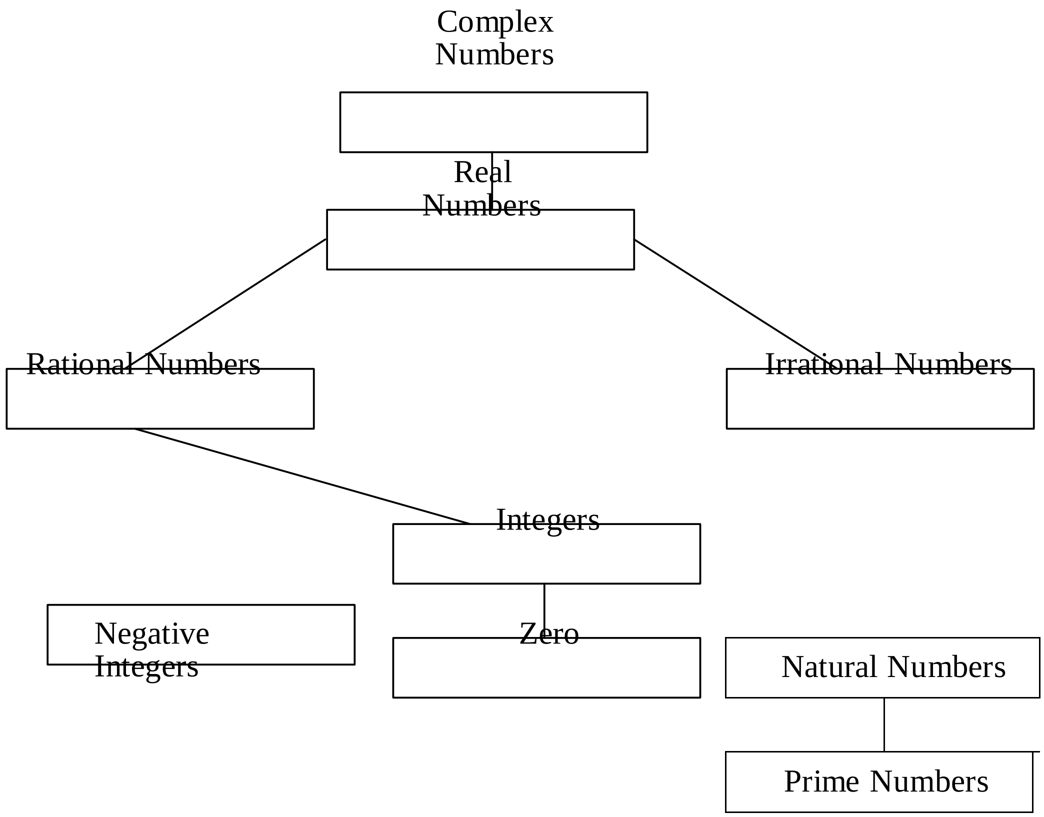


Fig 3.2

3.2 DECIMALS AND REAL NUMBERS

Every real number can be represented by a "non-terminating decimal". The decimal representation of a rational number p/q can be found by "dividing the denominator q into the numerator p". If the indicated division terminates, as for

3/8 = .375

We write 3/8 = .375000

Or 3/8 = .374999…

If we indicated division of q into p does not terminate, then it is known that a block of digits will continually be repeated; for example, 2/11 = .181818…

We now state the basic fact connecting decimals and real numbers. The rational numbers correspond precisely to those decimals in which a block of digits is continually repeated, and the irrational numbers correspond to the other non-terminating decimals.

3.3 INEQUALITIES

The concept of "order" is introduced in the real number system by the

Definition: The real number a is less than the real number b,

written $a < b$

If $b - a$ is a positive number.

The following properties of the relation $a < b$ can be proven. Let a, b and c be real numbers; then:

P_1: Either $a < b$, $a = b$ or $b < a$.

P_2: If $a < b$ and $b < c$, then $a < c$.

P_3: If $a < b$, then $a + c < b + c$

P_4: If $a < b$ and c is positive, then $ac < bc$

P_5: If $a < b$ and c is negative, then $bc < ac$.

Geometrically, if $a < b$ then the point a on the real line lies to the left of the point b.

We also denote $a < b$ by $b > a$

Which reads "b is *greater then* a". Furthermore, we write

$a < b$ or $b > a$

if $a < b$ or $a = b$, that is, if a is not greater than b.

Example 1.1: $2 < 5$; $-6 < -3$ and $4 < 4$; $5 > -8$

Example 1.2: The notation $x < 5$ means that x is a real number which is less than 5; hence x lies to the left of 5 on the real line.

The notation $2 < x < 7$; means $2 < x$ and also $x < 7$; hence x will lie between 2 and 7 on the real line.

Remark 3.1: Notice that the concept of order, i.e. the relation $a < b$, is defined in terms of the concept of positive numbers. The fundamental property of the positive numbers which is used to prove properties of the relation $a < b$ is that the positive numbers are closed under the operations of addition and multiplication. Moreover, this fact is intimately connected with the fact that the natural numbers are also closed under the operations of addition and multiplication.

Remark 3.2: The following statements are true when a, b, c are any real numbers:

1. $a < a$

2. if $a < b$ and $b < a$ then $a = b$.

3. if $a < b$ and $b < c$ then $a < c$.

3.4 ABSOLUTE VALUE

The absolute value of a real number x, denoted by ÷ x÷ is defined by the formula

$$|x| = \begin{cases} x \text{ if } x > 0 \\ -x \text{ if } x < 0 \end{cases}$$

that is, if x is positive or zero then $|x|$ equals x, and if x is negative then $|x|$ equals – x. Consequently, the absolute value of any number is always non-negative, i.e. $|x| > 0$ for every $x \in R$.

Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e. the point 0. Moreover, the distance between any two points, i.e. real numbers, a and b is $|a - b| = |b - a|$.

Example 2.1: $|-2| = 2$, $|7| = 7$. $|-p| = p$

Example 2.2: The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5, i.e. x must lies between -5 and 5 on the real line. In other words,

$|x| < 5$ and $-5 < x < 5$

have identical meaning. Similarly,

$|x| < 5$ and $-5 < x < 5$

have identical meaning.

3.5 INTERVALS

Consider the following set of numbers;

$A_1 = \{x | 2 < x < 5\}$

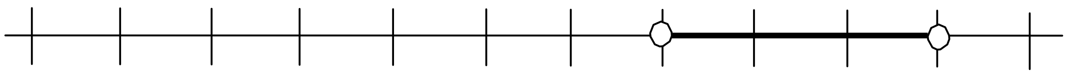
$A_2 = \{x | 2 \leq x \leq 5\}$

$A_3 = \{x | 2 < x \leq 5\}$

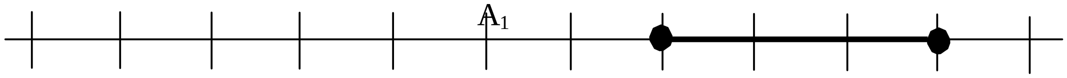
$A_4 = \{x | 2 \leq x < 5\}$

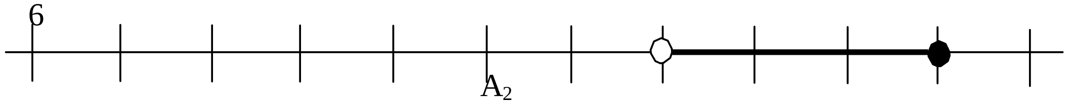
Notice, that the four sets contain only the points that lie between 2 and 5 with the possible exceptions of 2 and/or 5. We call these sets intervals, the numbers 2 and 5 being the endpoints of each interval. Moreover, A_1 is an ***open interval*** as it does not contain either end point: A_2 is a ***closed interval*** as it contains bother endpoints; A_3 and A_4 are ***open-closed*** and ***closed-open*** respectively.

We display, i.e. graph, these sets on the real line as follows.

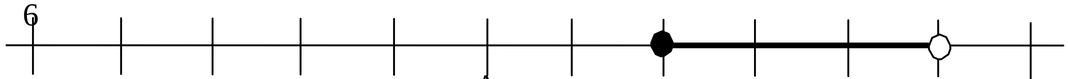


A_1





A_2



A_3

A_4

Notice that in each diagram we circle the endpoints 2 and 5 and thicken (or shade) the line segment between the points. If an interval includes an endpoint, then this is denoted by shading the circle about the endpoint.

Since intervals appear very often in mathematics, a shorter notation is frequently used to designated intervals, Specifically, the above intervals are sometimes denoted by;

A_1 = (2, 5)

A_2 = [2, 5]

A_3 = (2, 5)

A_4 = [2, 5)

Notice that a parenthesis is used to designate an open endpoint, i.e. an endpoint that is not in the interval, and a bracket is used to designate a closed endpoint.

3.5.1 Properties of Intervals

Let $\Im$ be the family of all intervals on the real line. We include in $\Im$ the null set $\emptyset$ and single points a = [a, a]. Then the intervals have the following properties:

1. The intersection of two intervals is an interval, that is, $A \in \Im$, $B \in \Im$ implies $A \cap B \in \Im$

2.The union of two non-disjoint intervals is an interval, that is, $A \in \Im$, $B \in \Im$, $A \cap B \neq \emptyset$ ipmlies $A \cup B \in \Im$

3. The difference of two non-comparable intervals is an interval, that is, $A \in \Im$, $B \in \Im$, $A \not\subset B$, $B \not\subset A$ implies $A - B \in \Im$

Example 3.1: Let A = {2, 4), B = (3, 8). Then

A Ç B = (3, 4), A È B = [2, 8)

A – B = [2, 3], B – A = [4, 8)

3.5.2 Infinite Intervals

Sets of the form

$A = \{x \mid x > 1\}$

$B = \{x \mid x \geq 2$

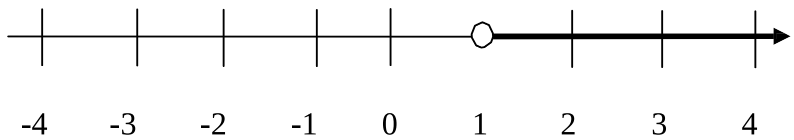
$C = \{x \mid x < 3\}$

$D = \{x \mid x \leq 4\}$

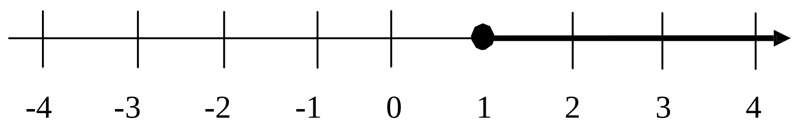
$E = \{x \mid x \in \Re\}$

Are called infinite intervals and are also denoted by

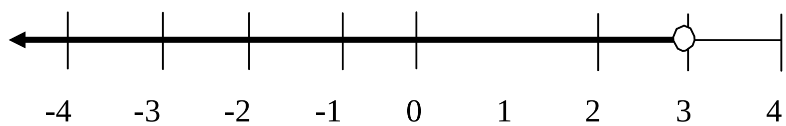
A = (1, ∞), B = [2, ∞), C = (- ∞, 3), D = (-∞, 4], E = (-∞,∞)
We plot these intervals on the real line as follows:



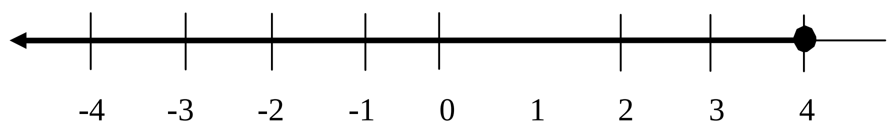
A is Shaded



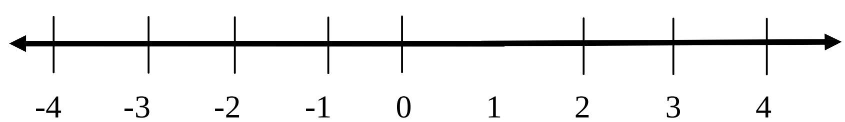
B is Shaded



C is Shaded



D is Shaded



E is Shaded

3.6 BOUNDED AND UNBOUNDED SETS

Let A be a set of numbers, then A is called ***bounded*** set if A is the subset of a finite interval. An equivalent definition of boundedness is;

Definition 3.1: Set A is ***bounded*** if there exists a positive number M such that $|x| \leq$ M.

for all x $\in$ A. A set is called ***unbounded*** if it is not bounded

Notice then, that A is a subset of the finite interval [-M, M].

***Example 4.1:** Let A = {1, ½, 1/3…}. Then A is bounded since A is certainly a subset of the closed interval [0, 1].*

Example 4.2: Let A = {2, 4, 6,…..}. Then A is an unbounded set.

Example 4.3: Let A = {7, 350, -473, 2322, 42}. Then A is bounded

Remark 3.3: If a set A is finite then, it is necessarily bounded.

If a set is infinite then it can be either bounded as in example 4.1 or unbounded as in example 4.2

4.0 CONCLUSION

The set of real numbers is of utmost importance in analysis. All (except the set of complex numbers) other sets of numbers are subsets of the set of real numbers as can be seen from the line diagram of the number system.

5.0 SUMMARY

In this unit, you have been introduced to the sets of numbers. The set of real numbers, $\Re$, contains the set of integers, Z, Rational numbers, Q, Natural numbers, N, and Irrational numbers, Q'.

Intervals on the real line are open, closed, open-closed or closed-open depending on the nature of the endpoints.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Prove: If a < b and B < c, then a < c

2. Under what conditions will the union of two disjoint interval be an interval?

3. If two sets R and S are bounded, what can be said about the union and Intersection of these sets?

7.0 REFERENCES AND FURTHER READINGS

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

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MODULE 2

UNIT 1 REAL SEQUENCE AND SERIES

CONTENTS

1.0 INTRODUCTION

A series is a succession of numbers, of which each number is formed according to a definite law which is the same throughout the series.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- find the sum of an arithmetic progressions,
- find the sum of a geometric progressions,
- apply it to simple and compound interest problems,
- commerce related problems.

3.0 MAIN CONTENT

3.1 Definition

A sequence is a succession of terms spanned by a rule or formula.

Examples

1. 1, 2, 3, 4, 5…
2. 2, 4, 6, 8, 10…
3. $a, a^2, a^4, a^8, a^{16} \ldots$

A sequence may be finite or infinite.

A finite sequence is one whose first and last element are known, while an infinite sequence is one whose terms are uncountable.

The general term of a sequence (formula) can be written as: $U_n = \frac{n+1}{2n - 1}$

We can now generate the sequence by substituting n = 1, 2, 3 …

$\Rightarrow$ When n = 1, ® $U_1 = \frac{1+1}{2(1) - 1} = \frac{2}{1} = 2$

$\Rightarrow$ When n = 2, ® $U_2 = \frac{2+1}{2(2) - 1} = \frac{3}{3} = 1$

$\Rightarrow$ When n = 3, ® $U_3 = \frac{3+1}{2(3) - 1} = \frac{4}{5} = 0.8$

$\Rightarrow$ When n = 4, ® $U_2 = \frac{4+1}{2(4) - 1} = \frac{5}{7} = 0.7143$

Hence, the sequence is: 2, 1, $\frac{4}{5}, \frac{5}{7}$........

3.2 Arithmetic Sequence (A. P)

This is a sequence in which each term differs by a common difference.

Let „a‟ be the first term,

Let „s‟ be the common difference.

The sequence is of the form:

$\Rightarrow$ a, a + d, a + d +d, a + d + d+ d, ………

$\Rightarrow$ a, a+ d, a + 2d, a + 3d…..

If u_1, u_2, u_3 ……..u_n are the n^{th} terms of an AP.

Then:

$U_1 = a$

$U_2 = a + 1d$

$U_3 = a + 2d$

$U_4 = a + 3d$

$U_5 = a + 4d$

: :

$U_n = a + (n - 1)d$. Following the same pattern

The last term or general term for an Arithmetic Sequence is

$$U_n = a + (n - 1)d.$$

Example

Find the common difference in the following sequence

i. 3, 5, 7, 9, 11…
ii. 102, 99, 96, 93 …

Solution

i. $d = u_2 – u_1 = u_3 – u_2$
$d = 5 – 3 = 7 – 5 = 2$

4 The common difference is d = 2

ii. $d = u_2 – u_1 = u_3 – u_2$
$d = 99 – 102 = 96 – 99 = -3$

4 The common difference is d = -3

Example

Find the 7th term of an A.P whose first term is 102 and common difference is -3

Solution

Let „a‟ be the first term

Let„d‟ be the common difference

The *nth* term is $U_n = a + (n – 1)\,d$

7th term, n = 7; d =-3; a = 102
® $U_7 = a + (7 - 1)d = a + 6d = 102 + 6(-3) = 102 - 18 = 84$

Example

The 7th term of an A.P is 15 and the fourth term is 9. Find the sequence, first term and the common difference.

Solution

The *nth* term of an A.P is $U_n = a + (n – 1) d$

$U_7 = a + (7 \Box 1)d = a + 6d = 15$.......... ....(1)
$U_4 = a + (4 \Box 1)d = a + 3d = 9$.......... ....(2)
Equation (1) and (2) can be solved simultaneously.

To find „a" and„d", equation (2) minus equation (1),

$(a + 6d) - (a + 3d) = 15 - 9$
$3d = 6, \ d = 2$

Substitute $d = 2$ into equation (2).

$a + 3(2) = 9, \ a + 6 = 9, \ a = 9 - 6 = 3$
$a = 3, d = 2$
The sequence is: 3, 5, 7, 9, 11, 13, 15, …

Sum of the First n-terms of an A.P

The sum of the sequence $a, a + d, a + 2d....a + (n \Box 1)d$ is:

$S_n = a + (a + d) + (a + 2d) + ... + a + (n \Box 1)d$.......(1)
$S_n = a + (n \Box 1)d + ... + (a + 2d) + a$.......... ....(2)

Summing equation (1) and (2), we have:

$2a + (n \Box 1)d + 2a + (n \Box 1)d + ... + 2a + (n \Box 1)d$
$2S_n = n[2a + (n \Box 1)d]$
$$2S_n = \frac{n}{2}[2a + (n \Box 1)d] = \frac{n}{2}[a + u_n]$$

Where *S*, is the sum of the first n –terms of the A.P sequence.

3.3 Geometric Sequence (G. P)

A geometric sequence is a sequence in which each successive terms of the sequence are in equal ratio.

Let „a" be the first term
Let „r" be the common ratio.
The sequence becomes: a, ar, ar^2, ar^3, …..

If $U_1 = a$

$U_2 = ar^{2-1}$

$U_3 = ar^{3-1}$

$U_4 = ar^{4-1}$

: :

: :

$U_n = ar^{n-1}$

The *nth* term is $U_n = ar^{n-1}$

Example

Find the common ration in each the following:

i. 2, 6, 18, 54, 162,…

ii. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$

Solution

Let r be the common ratio

i. r = $\frac{6}{2} = \frac{18}{6} = 3$

ii. r = $\frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} x \frac{2}{1} = \frac{1}{2}$

Example

Find the value P in the sequence of G.P:
$\left(\sqrt{2} - 1\right), \left(3 + 2\sqrt{2}\right), \left(5\sqrt{2} - 7\right), P.$

Solution

Let r be the common ratio.

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$= \frac{3+2\sqrt{2}}{\sqrt{2}-1} = \frac{5\sqrt{2}-7}{P} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Now, solve (2) for P:

$$P(3+2\sqrt{2} = (\sqrt{2}-1)(5\sqrt{2}-7)$$

$$P = \frac{(\sqrt{2}-1)(5\sqrt{2}-7)}{3+2\sqrt{2}} = \frac{\sqrt{2}(5\sqrt{2}-7) \square 1(5\sqrt{2}-7)}{3+2\sqrt{2}}$$

$$= \frac{10-7\sqrt{2} - 5\sqrt{2}+7)}{3+2\sqrt{2}}$$

$$= \frac{17-12\sqrt{2}}{3+2\sqrt{2}},$$

Multiplying the numerator and the denominator by the conjugate of the denominator, we have.

$$= \frac{\left(17-12\sqrt{2}\right)}{\left(3+2\sqrt{2}\right)} \frac{\left(3-2\sqrt{2}\right)}{\left(3-2\sqrt{2}\right)} = \frac{\left(17(3)-34\sqrt{2}-36\sqrt{2}+48\right)}{9-8}$$

$$= \frac{51-70\sqrt{2}+48}{1} = 99-70\sqrt{2}$$

Hence, the sequence is: $\left(\sqrt{2}-1\right), \left(3+2\sqrt{2}\right), \left(5\sqrt{2}-7\right), \left(99-70\sqrt{2}\right)$

Series

A series is the term wise summation of a sequence.

Let $u_1, u_2, u_3 \ldots . u_n$ be a sequence

$$u_1 + u_2 + u_3 + \ldots + u_n = \sum_{i=1}^{n} u_i$$

4.0 CONCLUSION

As in the summary

5.0 SUMMARY

This section introduces student to arithmetic and geometric progressions which is a good foundation for various methods of investment appraisal, with particular emphasis on discount, simple and compound interests ,commission and depreciation,

6.0 TUTOR-MARKED ASSIGNMENT

1) At what rate of interest will a single investment triple its value in 5years?

2) Calculate the present value of #5,000 at 10% p.a. for 4years.

3) In how many years will #2,000 amount to #10,200 at 5% p.a.compound intrest?

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1) Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2)Introduction to Mathematical Economics By Edward T. Dowling.

3)Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

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7) Introduction to Mathematical Economics SCHAUM"S Out lines

MODULE 2

UNIT 2 : Quadratic Equations

1. INTRODUCTION

A quadratic equation is an equation of the form $ax^2 + bx + c = y$ where a, b and c are constants. x and y are independent and dependent variables respectively. in order to determine the roots of the quadratic equation, the intersept on x- axis is sought for by seting the variable y =0 and solve for the equation.

2.0:OBJECTIVES

*is to identify quadratic equation

* to know types of methods of solving quadratic equation

*to be able to match approprate solution methods to particular quadratic equation.

*proof formular methods from comoliting the sqares methods.

3.0 MAIN CONTENT

Main topic in this section is solving a quadratic equation $ax^2+bx+c=0$. There are three ways to solve a quadratic equation. The first one is

1. By **Factoring**: This is a typical method to solve a quadratic equation whenever the polynomial ax^2+bx+c can be easily factored. Here is an example.

Example. Solve the quadratic equation $x^2-3x-4=0$ by factoring.

Solution. The polynomial x^2-3x-4 is factored as $(x-4)(x+1)$. So the equation is $(x-4)(x+1)=0$. This means that $x-4=0$ or $x+1=0$, i.e. we obtain two real solutions

$x=-1$ or $x=4$.

Example. Solve the quadratic equation $x^2-3=0$.

Solution 1. Recall the factorization formula $(a^2-b^2)=(a+b)(a-b)$. Now

$$X^2-3=[x^2-(3\surd)]^2=(x+3\surd)(x-3\surd).$$

Thus our equation becomes $(x+3\surd)(x-3\surd)=0$ whose solutions are $x = \pm 3\surd$.

Solution 2. The quadratic equation can be written as $x^2=3$. Solving this equation for x, we obtain $x = \pm 3\surd$.

Next method is

2. By **Completing the Square**:

This is a method that can be used to solve any quadratic equation. First note that

$$X^2+bx+(b/2)^2=(x+b^2)^2. \quad (1)$$

Example. Solve the equation $x^2-6x-10=0$ by completing the square.

Solution. By adding 10 to each side of the equation, we obtain

$$X^2-6x=10. \quad (2)$$

Note that half of the coefficient of x is $-6/2=-3$. Add $(-3)^2$ to each side of (2):

$$X^2-6x+(-3)^2=10+(-3)^2. \qquad (3)$$

Now notice that the LHS of (3) is exactly the same form as the LHS of the forumla (1). Hence, the quation (3) becomes

$$(x-3)^2=19.$$

Solving this for $x-3$, we obtain $x-3=\pm\sqrt{19}$.

That is, $x=3\pm\sqrt{19}$.

While completing the square can be a useful tool for some other things, I do not strongly recommend this method because there is more convenient method of solving quadratic equations.

3. By **Quadratic Formula**:

consider the quadratic equaion $ax^2+bx+c=0$ ....(1)

divide equation 1 by the coefficient of x i.e 'a' to have

$x^2+\frac{b}{a}x+\frac{c}{a}=0$ ....(2)

add (half the coefficient if x and squred the result) to both sides of (2) to obtain

Half of coefficient of x = $\frac{1}{2}\frac{b}{a}$
....(3)

Add the suare of (3) to both sides of (2) to have

$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$
....(4)

equation 4 simplified gives

$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$
....(5)

taking sqare root of both sides of 5 results to

$\left(x+\frac{b}{2a}\right)=\pm\sqrt{\left(\frac{-4ac+b^2}{4a^2}\right)}$

$x=-\frac{b}{2a}\pm\sqrt{\left(\frac{-4ac+b^2}{4a^2}\right)}$
....(6)

and futher simolification of 6 , we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
....(7)

However, applying the method by completing the square to solve the quadratic equation $ax^2+bx+c=0$, we obtain the quadratic formula as in equation (7)

Example. Solve the quadratic equation $3x^2+2x-7=0$.

Solution. $a=3$, $b=2$, and $c=-7$. Thus

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-4(3)(-7)}}{6} = \frac{-2 \pm 88}{6}$$

$$x = \frac{86}{6}. \quad or \quad x = \frac{90}{6} = 15$$

The expression inside radical b^2-4ac is called the *discriminant*. Using the discriminant, we can tell the following without solving the equation itself.

Theorem. For $ax^2+bx+c=0$ with $a\neq 0$,

- If $b^2-4ac>0$, then the equation has two distinct real solutions.
- If $b^2-4ac=0$, then the equation has only one real solution i.e equall and repeated roots.
- If $b^2-4ac<0$, then the equation has two complex solutions that are conjugate of each other.

Eq. 7 is called Quadratic formular, it is very powerful apart from solution of quadratic equations, it has a lot of applications.

two roos of equation (1) are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \textbf{or} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$$

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

using difference of two squares we obtain

$$=\left(\frac{-b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2$$

$$=\frac{b^2}{4a^2}-\frac{b^2 4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

hence, if α,β are the roots of Quadratic Equation $ax^2+bx+c=0$,

the sum of roots $\alpha+\beta=-\frac{b}{a}$ and product of roots $\alpha\beta=\frac{c}{a}$

example

Solve for x $x^2+y^2=34$ 1

$xy=15$ 2

Add Eq.1 to twice Eq.2 to have

$x^2+y^2+2xy=34+30=64$ 3

$(x+y)^2=\sqrt{64}=\pm 8$ 4

if x,y are the roots of a given quadratic equation $z^2+bz+c=0$, x+y =-b and xy=c
Thus the roots of Euqs. 1 and 2 must be the roots of Euqations

$z^2-8z+15=0$ 5

$z^2+8z+15=0$ 6

$z=\frac{8\pm\sqrt{64-60}}{2}=\frac{8\pm 2}{2}$ 7

z= 5 or z = 3...8

and

$z=\frac{-8\pm\sqrt{64-60}}{2}=\frac{-8\pm 2}{2}$ 9

z= -5 or z = -3...10

4.0 CONCLUSION

in conclusion, the sum of roots $\alpha+\beta=-\frac{b}{a}$ and product of roots $\alpha\beta=\frac{c}{a}$ can be widely used to solve quadratic equations and its applications.

5.0 SUMMARY

The quadratic equaion $ax^2+bx+c=0$ was defined and solve with examples.
Types of methods for solvinf QE was taught like:
Factorisation methods i.e (ax^2 +bx+c=0) = (x+ a_1)(x+a_2)

Also methods of completing the suare was also implored and used to moddel the QE. formular

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ which is widely used.

The several conditions for the operations of QE was also disscaused thus.

The expression inside radical b^2-4ac is called the *discriminant*. Using the discriminant, we can tell the following without solving the equation itself.

Theorem. For $ax^2+bx+c=0$ with $a\neq 0$,

- If $b^2-4ac>0$, then the equation has two distinct real solutions.
- If $b^2-4ac=0$, then the equation has only one real solution i.e equall and repeated roots.
- If $b^2-4ac<0$, then the equation has two complex solutions that are conjugate of each other.

  Lastly, sum and product of QE are also dealth with thus.

Two roos of the QEequation (1) are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \textbf{or} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$$ sum of roots

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$ = c / a product of roots

6.0 TMA

Find the nature of roots of the following:

* $x^2 - 4x + 5 = 0$
* $x^2 - 4x + 4 = 0$
* $x^2 - 4x + 3 = 0$
* $x^2 - 4x + 2 = 0$

solusion

$x^2 - 4x + 5 = 0$

$a = 1, b = -4, c = 5$

$\therefore b^2 = 16, 4ac = 20$

$\Rightarrow b^2 < 4ac$

the roots are imaginary.

2. $x^2 - 4x + 4 = 0$

solusion

$x^2 - 4x + 4 = 0$

$a = 1, b = -4, c = 4$

$\therefore b^2 = 16, 4ac = 16$

$\Rightarrow b^2 = 4ac$

The roots are equal.

3. $x^2 - 4x + 3 = 0$

solution

$x^2 - 4x + 3 = 0$

$a = 1, b = -4, c = 3$

$\therefore b^2 = 16, 4ac = 12$

$\Rightarrow b^2 - 4ac = 16 - 12 = 4$

The roots are equall

4. $x^2 - 4x + 2 = 0$

solusion

$x^2 - 4x + 2 = 0$

$a = 1, b = -4, c = 2$

$\therefore b^2 = 16, 4ac = 8$

$\Rightarrow b^2 - 4ac = 16 - 8 = 8$

The roots are real.

* Find the value of a if $4x^2 - (a - 12)x + (a - 7) = 0$ has equall roots.
* if the α, β are the roots of the QE $x^2 - 4x + 3 = 0$. find the QE whose roots are α^2, β^2

7.0 REFERENCES/FURTHER READINGS

1) Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2)Introduction to Mathematical Economics By Edward T. Dowling.

3)Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

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MODULE 2

UNIT 3:

MATHEMATICAL INDUCTION

1.0: INTRODUCTION

Mathematical Induction can be defined thus: If a result involving natural numbers is true for 1 and its validity for a natural number k implies its validity for k + 1, then the formular is true for all numbers.

2.0: OBJECTIVES

At the end of this unit, you shoul be able to:
Utilise Mathematical Induction as a tool to solve or validade any Natural number.
Utilise Mathematical Induction as a tool to validate the series of a sequence.

3.0: MAIN CONTENT

MATHEMATICAL INDUCTION

An important tool in mathematics is based on property 3 in the definition of natural numbers. This tool is called **MATHEMATICAL INDUCTION**. It is defined thus:

"If a result involving natural numbers is true for 1 and its validity for a natural number k implies its validity for k + 1, then the formular is true for all numbers."

i.e. For each n - N let $p_{(n)}$ be a statement based on the property of N.

If (1) $p_{(1)}$ is true and

(2) $p_{(k)}$ is true $p_{(k+1)}$ also is true

Then $p_{(n)}$ is true n - N

As the result holds for n = 1, it must hold for 1 + 1 i.e. n = 2. Now it has been proved for n = 2 so it must be true for 2+1 i.e. n = 3.

Following this line of reasoning, one gets that the result must hold for all natural numbers.

EXAMPLES:

1. Prove that $1^2 + 2^2 + 3^3 .... + n^2 = \frac{n(n+1)(2n+1)}{6}$

For all natural numbers.

SOLUTION

Clearly for n = 1, LHS = 1^2 = 1

RHS = $\frac{1.2.3}{6}$ = 1 .... (1)

So the formular is definitely valid for n = 1 suppose that the result is true for n = k.

i.e. $1^2 + 2^2 + 3^3 .... + k^2 = \frac{k(k+1)(2k+1)}{6}$

Then $1^2 + 2^2 + 3^3 .... + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)}{6}\left(2k^2 + k + 6k + 6\right)$$
$$= \frac{(k+1)}{6}\left(2k^2 + 7k + 6\right)$$
$$= \frac{(k+1)}{6}(k+2)(2k+3)$$
$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Hence the result is true for n = k + 1. Consequently the result is true for all natural numbers n.

2. By the method of induction, show that $10^n + 3.4^{n+2} + 5$ is divisible by 9

Solution

For n = 1, $10^n + 3.4^{n+2} + 5 \quad = \quad 10 + 3.4^3 + 5$
$= \quad 10 + 192 + 5$
$= \quad 207 \quad = \quad 9 \text{ x } 23$

Divisible for n = 1

Suppose for n = k, $10^k + 3.4^{k+2} + 5$ is divisible by 9

Consider $10^{k+1} + 3.4^{k+2} + 5 \quad = \quad 10^k.10 + 3.4^{k+2}.4 + 5$
$= \quad 10^k(9+1) + 12.4^{k+2}.4 + 5$
$= \quad 10^k(9+1) + (9+3)4^{k+2}.4 + 5$
$= \quad (10^k + 3.4^{k+2} + 5) \; 9(10^k + 4^{k+2})$

$10^{k+1} + 3.4^{k+2} + 5$ is divisible by 9 as first term on R.H.S. is divisible by 9 by our assumption and second term has 9 ad one of its factors.

Consequently, $10^n + 3.4^{n+2} + 5$ is divisible by 9 for all natural numbers n.

3. Show that for a, b, - R

$(ab)^n = a^n b^n$ n - N

Solution

$(ab)^n = a^n b^n$, $p_{(n)}$ is the statement.

Now for $p_{(1)}$, $(ab)^1$ = $a^1 b^1$

Now for or ab = ab

$p_{(1)}$ is true

Now assume that $p_{(k)}$ is true

i.e. $(ab)^k$ = $a^k b^k$

To show that $p_{(k+1)}$ is then true, we need to show that

$(ab)^{k+1}$ = $(ab)^k \, (ab)^1$

= $a^k b^k \left(a^1 b^1\right)$ according to $p_{(k)}$

= $a^k a^1 \; b^k b^1$

= $a^{k+1} b^{k+1}$ as required

If $p_{(k)}$ is true, so is $p_{(k+1)}$

And since $p_{(1)}$ is true, $p_{(n)}$ is true n - N

4.0: CONCLUSION

5.0: SUMMARY

6-0: TMA

7.0: REFFERENCES/ FURTHER READINGS

Pure Mathematics for Advanced Level By B.D Bunday H Mulholland 1970.

2)Introduction to Mathematical Economics By Edward T. Dowling.

3)Mathematics and Quantitative Methods for Business and Economics.By Stephen P. Shao. 1976.

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MODULE 3: COMPLEX NUMBERS

UNIT 1

I

1.0: INTRODUCTION

A complex variable z is of the form a + bi, where a and b are real numbers and i is called the imaginary number.

Mathematically, z = a + bi

2.0: OBJECTIVES

At the end of this unit, you must be able to:

*identify a complex nubmer
^fine the algebra of complex number
*use the conjugate to evaluate some complex problems
* plot graphs of Complex numbers

3.0: MAIN CONTENT

Complex Numbers

DEFINITION

A complex variable z is of the form a + bi, where a and b are real numbers and i is called the imaginary number.

Mathematically, z = a + bi

Two complex numbers z_1 and z_2 are equal if their real parts are equal.

If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, then $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$

Suppose a complex number is z = a + bi, then z = a – bi is called the conjugate of the complex number,

$(a+bi)\ (a-bi) = a^2 - b^2i^2 = a^2 + b^2$ (Real number), where $i^2 = -1$.

3.2.1 Properties of Imaginary Numbers

Consider the solution of the polynomial:

$x^2+1=0$, $x^2=-1$, taking square root of both sides $\sqrt{x^2}=\sqrt{-1}=i$ (imaginary)

$i=\sqrt{-1}$

$i^2 = ixi = (\sqrt{-1})(\sqrt{-1}) = -1$

$i^3 = ixi^2 = \sqrt{-1}(-1) = -i$

$i^4 = i^2xi^2 = -1x-1 = 1$

$i^5 = i^4xi = 1x\sqrt{-1} = \sqrt{-1}$

Thus

$$i = i^5 = \sqrt{-1}$$

$$i^2 = i^6 = -1$$

$$i^3 = i^7 = -i$$

$$i^4 = i^8 = 1$$

$$i^5 = i^9 = \sqrt{-1}$$

etc

$$i^2 = (\sqrt{-1})^2 = 1$$

$$i^3 = i^2 x i = 1 x \sqrt{-1} = \sqrt{-1}$$

3.2.2 Algebra of Complex Numbers

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

4 $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 - b_2)i$

4 $z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$

Example 1

If 4 z_1 + 3 + 2i and z_2 = 4 + 3i. Find (i) $z_1 + z_2$ (ii) $z_1 – z_2$

Solution

i. $z_1 + z_2 = (3 + 2i) + (4 + 3i) = (3 + 4) + (2 + 3)i = 7 + 5i$

ii. $z_1 - z_2 = (3 + 2i) - (4 + 3i) = (3 - 4) + (2 - 3)i = -1-i$

3.2.3 Scalar Multiplication in Complex Numbers

Let k be a constant or a real number and $z_1 = a + bi$,

4 $kz_1 = k (a + bi) = ka + kbi$

Example 2

Find 5 z_1 if $z_1 = 3 + 2i$

Solution

$5 z_1 = 5(3 + 2i) = 15 + 10i$

3.2.4 Multiplication of Complex Numbers

$z_1 z_2 = (a_1 + b_1 i) (a_2 + b_2 i)$

$= a_1 (a_2 + b_2 i) + b_1 i (a_2 + b_2 i)$

$= a_1a_2 + a_1b_2i + a_2b_1i + b_1b_2(i)^2$

$= a_1a_2 + (a_1b_2i + a_2b_1)i - b_1b_2$, where $i^2 = -1$

$= (a_1a_2 + b_1b_2) + (a_1b_2 + a_2b_1)i$,

When $(a_1a_2 + b_1b_2)$ and $(a_1b_2 + a_2b_1)i$ are real numbers.

Example 3

If $z_1 = 3 + 2i$ and $z_2 = 4 + 3i$. Find $z_1 z_2$

Solution

$z_1 z_2 = (3 + 2i) (4 + 3i) = 3(3 + 4) + 2i(4 + 3i) = 12 + 9i + 8i + 6(i)^2$

$= 12 + (9 + 8)i 6(-1) = (12 – 6) + 17i = 6 + 17i.$

3.2.5 Division of Complex Numbers

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

Multiply and divide through by the conjugate of the denominator, $a_2 \square b_2 i$

$$4\ \frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \text{ x } \frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)}$$

$$= \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a^2 \square b^2 2i^2)}$$

$$= \frac{(a_1 + b_1 i)(a_2 \square b_2 i)}{a_2^2 + b_2^2}$$

Example 4

If $z_1 = 3 + 2i$ and $z_2 = 4 + 3i$. Evaluate $\frac{z_1}{z_2}$

Solution

$$\frac{z_1}{z_2} = \frac{3+2i}{4+3i} \text{ x } \frac{4-3i}{4-3i} = \frac{(3+2i)(4-3i)}{4^2 - 3^2 (i)^2}$$

$$= \frac{(3+2i)(4-3i)}{16+9}$$

$$= \frac{1}{25}(3+2i)(4-3i) .$$

3.2.6 Absolute Value or Modulus of a Complex Number

The modulus of $z_1 = |z_1|$.

If $z_1 = a + bi$,

$$|z_1| = \sqrt{a^2 + b^2}$$

Example 5

If $z_1 = 3 + 2i$. Evaluate $|z_1|$

Solution

$|z_1| = \sqrt{a^2 + b^2} = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$

Also, if z_1, z_2 and z_3 are complex numbers; then for absoluteness, the following axiom holds:

- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| < |z_1| + |z_2|$
- $|z_1 + z_2| > |z_1| \square |z_2|$

3.2.7 Working Distances in Complex Numbers

The distance between two points $z_1 = (a_1 + b_1 i)$ and $z_2 = (a_2 + b_2 i)$ in complex plane is given by:

$$|z_1 - z_2| = |(a_1 + b_1 i) - (a_2 + b_2 i)| = |(a_1 - a_2) + (b_1 - b_2)i|$$

$$= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

Example 6

Find the distance between the points z_1 and z_2, given that $z_1 = 3 + 2i$ and $z_2 = 4 + 3i$.

Solution

$$|z_1 - z_2| = |(3+2i) - (4+3i)| = |(3-4) + (2-3)i| = |-1-1i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}.$$

Example 7

Evaluate $\left|\frac{z_1}{z_2}\right| = \left|\frac{(3+2i)}{(4+3i)}\right| = \left|\frac{(3+2i)}{(4+3i)} x \frac{(4-3i)}{(4-3i)}\right| = \left|\frac{(3+2i)(4-3i)}{4^2+3^2}\right|$

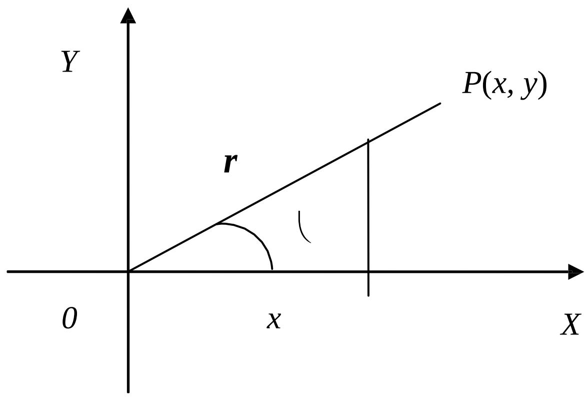
$$= \left|\frac{1}{25}(3+2i)(4-3i)\right| = \left|\frac{3(4-3i)+2i(4-3i)}{25}\right| = \left|\frac{12-9i+8i+6}{25}\right| = \left|\frac{18-i}{25}\right|$$

$$= \sqrt{(\frac{18}{25})^2 - (\frac{1}{25})^2} = \sqrt{\frac{(18^2 - 1^2)}{25^2}}$$

$$= \sqrt{\frac{18^2 - 1^2}{25}} = \sqrt{\frac{323}{25}} = 3.59$$

3.2.8 Polar Form of Complex Numbers

If p is a point in the complex plane corresponding to the complex number (x, y) or (x + iy), then:



x = r cosθ and y = r sinθ ……………………………………………………… (1)

Using Pythagoras theorem:

$x^2 + y^2 = r^2$ ……………………………………………………………… (2)

$\sqrt{x^2 + y^2}$ = r ………………………………………………………………. (3)

In complex numbers, we have distance r = $|x + iy|$, where $|x + iy|$ is called the modulus or absolute value of the complex number.

Similarly, $x^2 + y^2 = r^2$ ……………. (1).
i.e. substituting equations (1) in (2)
α $r^2 \cos^2θ + r^2 \sin^2θ$ α $r^2 (\cos^2θ + \sin^2θ)$ α $r^2 (1) = r^2$

If z = x + iy, ® z = r cosθ + r i sinθ = (cosθ + $i\sin^2θ$) is called the polar form of z.

De Moivre‟s Theorem

Given the following:

x = r cosθ and y = r sinθ ……………………………………………… (1)

$x^2 + y^2 = r^2$ ………………………………………………………………. (2)

$\sqrt{x^2 + y^2} = r$ …………………..…………………………..……..…. (3)

If $z_1 = (x_1 + iy_1) = r_1 (\cos\theta_1 + i\sin\theta_1)$ ……………………… ……..(4)
and

$z_2 = (x_2 + iy_2) = r_2 (\cos\theta_2 + i\sin\theta_2)$ …………………………………. (5)

$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ …………………..……. (6)

Also, $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ ……………………..……. (7)

$z_1 z_2 = \ldots z_n = r_1 r_2 \ldots r_n [\cos(\theta_1 + \theta_2 + \ldots\theta_n) +$
$i\sin(\theta_1 + \theta_2 + \ldots\theta_n)]$ ………………………………………..……. (8)

If $= z_1 = z_2 \ldots z_n = z$, then, $r_1 = r_2 \ldots r_n = r$

Also, if $z_1 z_2 \ldots z_n = z^n$, then, $r_1 r_2 \ldots r_n = r^n$ ……………………..……. (9)

Equations (8) and (9) becomes:

$z^n = r^n (\cos\theta + i\sin\theta) = [r(\cos\theta + i\sin\theta)]^n$

Equation (10) is called **"De Moivre"s Theorem"**

Example 8

Express the complex number in polar form:

$3 = 2+\sqrt{3i}$

Solution

The absolute value, $r = |3 + 2\sqrt{3i}| = \sqrt{3^2 + (2\sqrt{3^2})}$

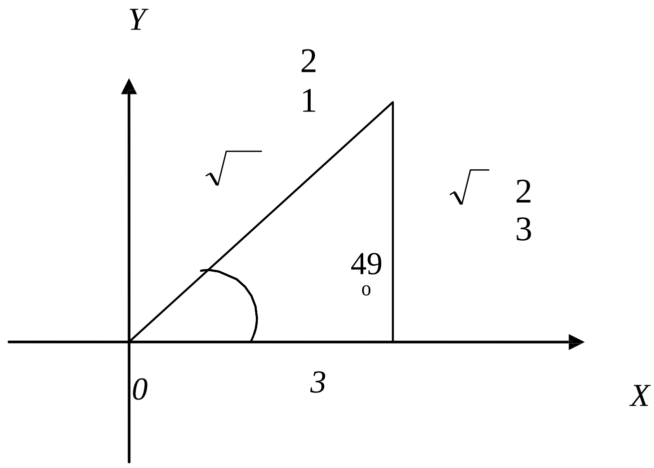
$= \sqrt{9 + 12} = \sqrt{21}$

Argument $\theta = \sin^{-1}\left(\frac{2\sqrt{3}}{\sqrt{21}}\right) = \sin^{-1}(0.7559) = 49^\circ$

Hence, $3 + 2\sqrt{3i} = r(\cos\theta + i\sin\theta) = \sqrt{9 + 12}$

$= \sqrt{21}\ (\cos 49^\circ + i\sin 49^\circ)$.

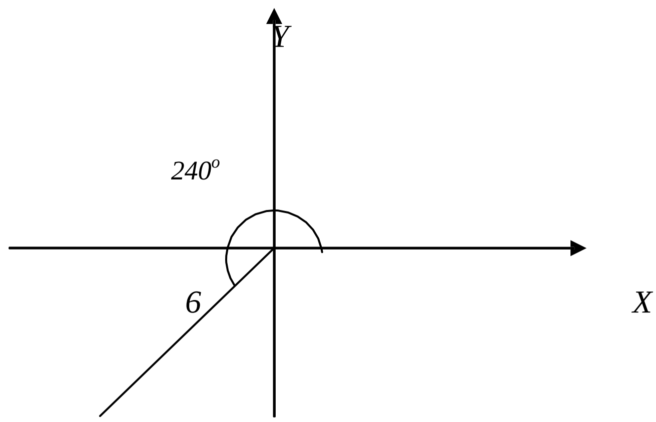
Graphically:



Example 9

Graphically, represent 6 (cos240° + isin240°)

Solution



4.0 CONCLUSION

We shall conclude as in the summary below.

5,0 SUMMARY

the following are some fundamental properties of the prolems.
giving two complex numbers $z_1, z_2 \in C$

* $|z_1 z_2| = |z_1||z_2|$
* $|z_1 / z_2| = |z_1|/|z_2|$ assuming that $|z_2| \neq 0$
* $|\bar{z}_1| = |z_1|$
* $|\text{Re}(z_1)| \leq= |z_1|$ and $|\text{Im}(z_1)| \geq= |z_1|$

* Triangle inequality] $|z_1 + z_2| \leq |z_1| + |z_2|$
* another triangle inequality is $|z_1 - z_2| \geq |z_1| - |z_2|$

6.0 TUTOR-MARKED ASSIGNMENT

1. If $\vec{a}$ = (3i + 4j) and $\vec{b}$ = (-i + j), evaluate the following:

   (i) $|a + 4b|$ (ii) $|2a \square 3b|$

2. The position vectors of points A, B and C are:
   (i) a = 3i + 2j (ii) b = -4i + 3j (iii) c = 71 + 5j respectively:
   Find k if $\overrightarrow{BC} = k\overrightarrow{AB}$, where k is a scalar.

3. Prove De Moivre‟s Theorem (cosn + isinn), where n is any positive integer.

4. Prove $\overline{z_1 + z_2}$ = z_1 + z_2, where z_1 and z_2 are conjugates.

5. If z_1 = (2 + 1), z_2 = (3 + i) and z_3 = (-1-1),
   Find:
   i. $z_1z_2z_3$
   ii. $|z_3 \square z_2|$
   iii. $|z_2 \square z_1z_3|$
   iv. $\frac{z_1}{z_2}$
   v. $|2z_1z_3|$

6. Construct graphically:

i. $(3z_1 - z_2)$
ii. $7 (\cos 240^\circ + i\sin 240^\circ)$

9. Express the complex numbers in polar form:
i. (-7 + 7i)
ii. (2 – 3i)

7.0 REFERENCES/FURTHER READINGS

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2) Introduction to Mathematical Economics By Edward T. Dowling.
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UNIT 2 CIRCULAR MEASURE

CONTENTS

1.0 INTRODUCTION

This unit will introduce you to the properties of equation of a circle, detailing how these properties could be used to solve day-day problems.

2.0 OBJECTIVES

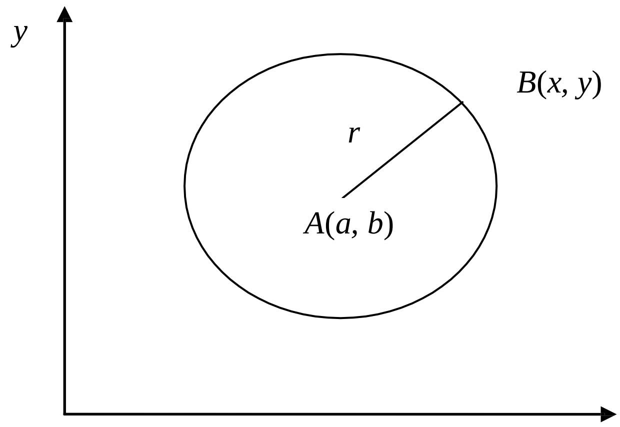
At the end of this unit, you should be able to solve simple problems involving equation of a circle.

3.0 MAIN CONTENT

3.1 Definition

A circle is the locus of a curve (equidistant from a point).

A circle could be described by its centre (fixed) and its radius. The radius is the distance between the centre of the circle and the circumference.



X

To form the equation of the circle, whose centre is the point *A* (*a*, *b*) and radius, „r‟ joined to point *B*(*x*, *y*) on the circumference.

Distance AB = r = $\sqrt{(y - a)^2 + (x - b)^2}$ …………………………… (1)

Squaring both sides, we have:

$r^2 = (y – a)^2 + (x – b)^2$……………………………………………… (2)

This is the required equation.

Suppose the centre of the circle is located at the origin meaning that a = 0, b = 0; then the equation becomes:

$r^2 = (y – 0)^2 + (x – 0)^2 = y^2 + x^2$ …………………………..………… (3)

This is equation of the circle with centre at the origin

In gradient the equation (2) can be expanded thus:

$r^2 = (y – a)^2 + (x – b)^2 = y^2 – 2ay + a^2 + x^2 – 2bx + b^2$

$y^2 + x^2 – 2ay – 2bx + a^2 + b^2 – r^2 = 0$ …………………..………… (4)

$y^2 + x^2 – 2gx – 2fy + (a^2 + b^2 – r^2) = 0$ …………………………… (5)

Where –a = f, -b = g and let $a^2 + b^2 – r^2 = c$

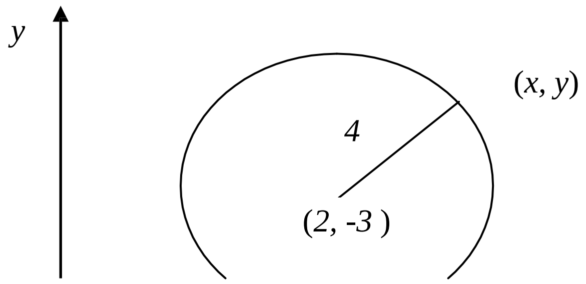
$y^2 + x^2 – 2gx – 2fy + c = 0$ …………………..…………………… (6)

Example

Find the equation of the circle centre (2, -3) and radius 4.

Solution

The equation of the circle centre (2, -3) and radius 4 is:



$y^2 + 6y + 9 + x^2 – 4x + 4 = 16$
$= [y + (3)]^2 + [x – 2]^2 = 4^2$
$y^2 + 6y + 9 + x^2 – 4x + 4 = 16$
$= y^2 + x^2 – 4x + 6y - 3 = 0.$

3.2 Properties of Equation of a Circle

1. The coefficient of the two variables x and y must be the same;
2. There must be no term in xy and;
3. The equation must be second degree or simply in second degree of x and y respectively.

Example

Find the coordinate of the centre and the radius of the circle

$y^2 + x^2 – 14x -8y + 56 = 0$

Solution

The circle $x^2 + y^2 – 14x -8y + 56 = 0$ ……………………………….. (1)

Collecting like terms:

$x^2 -14x + [\ ]\ y^2 – 8y + [\ ] = -56$ ……………………..………… (2)

Fill-up the bracket to make each equation to be perfect square.

Find the half of the coefficients of x and y and square them and add to both sides:

$\frac{1}{2}(-14)]^2 = [-7]^2 = 49$ ……………………………..…………… (3)

$$\frac{1}{2}(-8)]^2 = [-4]^2 = 16 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equation (2) becomes:

$x^2 - 14x + 49 + y^2 - 8y + 16 = -56 + 49 + 16$
$x^2 - 14x + 7^2 + y^2 - 8y + 4^2 = 9$
$(x - 7)^2 + (y - 4)^2 = 3^2$

(7, 4) are the coordinates or centre of the circle, while 3 is the radius.

Alternative Solution

Comparing $x^2 + y^2 - 14x - 8y + 56 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$, which is the general form.

$2g - 14$ implies that $g = -7$, and $2f = -8$, then $f = -4$

Where a and b are the coordinates, recall that $-a = g$ and $-b = f$

4 a = 7, b = 4 (the coordinates). Also, c

$= g^2 + f^2 - r^2$

$56 = (-7)^2 + (-4)^2 - r^2$
$56 = 49 + 16 - r^2$
$r^2 = 65 - 56$
$r^2 = 9$
$r = 3$ (the radius)

Hence, the circle has centre (7, 4), radius 3.

4.0 CONCLUSION

As in the summary

5.0 SUMMARY

In summary, this unit discourses equation of circle. Simple problems involving equation of circle and some properties of circle is also discoursed.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the equation of the circle giving the following coordinates and radii:

   i. Centre (-1, 2), radius 4 ii.
   Centre (6, 2), radius 1 iii.
   Centre (-2, -3), radius 5

2. Write down the centre and radius of the following circles:

   i. $x^2 + y^2 - 10x + 6y + 30 = 0$
   ii. $9x^2 + 9y^2 - 12x - 6y + 4 = 0$
   iii. $4x^2 + 4y^2 - 12x + 4y - 71 = 0$

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