NATIONAL OPEN UNIVERSITY OF NIGERIA

COURSE CODE: MTH 106

COURSE TITLE: MATHEMATICS FOR MANAGEMENT SCIENCES II

COURSE GUIDE

MTH 106

MATHEMATICS FOR MANAGEMENT SCIENCES II

Course Developer: Mr. SUFIAN JELILI BABATUNDE & BORO IRENE O.
National Open University of Nigeria
Lagos.

Course Coordinator:
National Open University of Nigeria
Lagos.

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria

Headquarters

14/16 Ahmadu Bello Way

Victoria Island, Lagos

Abuja Annex

245 Samuel Adesujo Ademulegun Street

Central Business District

Opposite Arewa Suites

Abuja.

e-mail: centralinfo@nou.edu.ng

URL: www.nou.edu.ng

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Table of Content Page

Introduction

MTH 106 - Mahematics for Management Sciences II is designed to teach you how mathematics could be used in solving problems in the contemporary business world. Therefore, the course is structured to expose you to the skills required in other to attain a level of proficiency in business management.

What you will learn in this course

You will be taught the basis of mathematics required in solving problems in business.

Course Aims

There are ten study units in the course and each unit has its objectives. You should read the objectives of each unit and bear them in mind as you go through the unit. In addition to the objectives of each unit, the overall aims of this course include:

(i) To introduce you to the words and concepts in business mathematics

(ii) To familiarize you with the peculiar characteristics in business mathematics.

(iii) To expose you to the need for and the demands of mathematics in the business world.

(iv) To prepare you for the contemporary business world.

Course Objectives

The objectives of this course are:

- To inculcate appropriate mathematical skills required in business
- Educate learners on how to use mathematical Techniques in solving problems.
- Educate the learners on how to integrate mathematical models in business.

Working through This Course

You have to work through all the study units in the course. There are two modules and ten study units in all.

Course Materials

Major components of the course are:

1. Course Guide
2. Study Units
3. Textbooks

4. CDs

5. Assignments File

6. Presentation Schedule

Study Units

The breakdown of the two modules and ten study units are as follows:

MODULE ONE

Unit 1 Sets & Subsets

Unit 2 Basic Set Operations

Unit 3 Set of Numbers

Unit 4 Functions

MODULE TWO

Unit 1 Annuity

Unit 2 Cash Flow

Unit 3 Sinking Fund

Unit 4 Mathematical Programming (Linear Programming)

MODULE THREE

Unit 1: Computation of Areas by Calculus

Unit 2: Definite Integral

Unit 3: Indefinite Integral

Unit 4: Integration of Transcendental functions

Unit 5: Integration of Powers of Trigonometric functions

MODULE FOUR

Unit 1: Mathematical Tools I: Simultaneous Equations, Linear Functions, and Linear Inequalities

Unit 2: Mathematical Tools II: Introduction to Matrix Algebra

Unit 3: Probability

References and Other Resources

Every unit contains a list of references and further reading. Try to get as many as possible of those textbooks and materials listed. The textbooks and materials are meant to deepen your knowledge of the course.

Assignment File

In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain from these assignments will count towards the final mark you obtain for this course. Further information on assignments will be found in the Assignment File itself and later in this *Course Guide* in the section on assessment.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor-marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against falling behind in your work.

Assessment

Your assessment will be based on tutor-marked assignments (TMAs) and a final examination which you will write at the end of the course.

Tutor Marked Assignments (TMA)

Every unit contains at least one or two assignments. You are advised to work through all the assignments and submit them for assessment. Your tutor will assess the assignments and select four which will constitute the 30% of your final grade. The tutor-marked assignments may be presented to you in a separate file. Just know that for every unit there are some tutor-marked assignments for you. It is important you do them and submit for assessment.

Final Examination and Grading

At the end of the course, you will write a final examination which will constitute 70% of your final grade. In the examination which shall last for two hours, you will be requested to answer three questions out of at least five questions.

Course Marking Scheme

This table shows how the actual course marking is broken down.

Assessment	Marks
Assignments	Four assignments, best three marks of the four count at 30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

How to Get the Most from This Course

In distance learning, the study units replace the university lecture. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suits you best. Think of it as reading the lecture instead of listening to the lecturer. In the same way a lecturer might give you some reading to do, the study units tell you when to read, and which are your text materials or set books. You are provided exercises to do at appropriate points, just as a lecturer might give you an in-class exercise. Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit, and how a particular unit is integrated with the other units and the course as a whole. Next to this is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. These learning objectives are meant to guide your study. The moment a unit is finished, you must go back and check whether you have achieved the objectives. If this is made a habit, then you will significantly improve your chances of passing the course. The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a Reading section. The following is a practical strategy for working through the course. If you run into any trouble, telephone your tutor. Remember that your tutor's job is to help you. When you need assistance, do not hesitate to call and ask your tutor to provide it.

In addition do the following:

1. Read this Course Guide thoroughly, it is your first assignment.

2. Organise a Study Schedule. Design a Course Overview " to guide you through the Course". Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the Semester is available from the study centre. You need to gather all the information into one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates and schedule of work for each unit.

3. Once you have created your own study schedule, do everything to stay faithful to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please, let your tutor know before it is too late for help.

4. Turn to Unit 1, and read the introduction and the objectives for the unit.

5. Assemble the study materials. You will need your set books and the unit you are studying at any point in time.

6. Work through the unit. As you work through the unit, you will know what sources to consult for further information.

7. Keep in touch with your study centre. Up-to-date course information will be continuously available there.

8. Well before the relevant due dates (about 4 weeks before due dates), keep in mind that you will learn a lot by doing the assignment carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the examination. Submit all assignments not later than the due date.

9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

10. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

11. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the Assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also the written comments on the ordinary assignments.

12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in the Course Guide).

Tutors and Tutorials

The dates, times and locations of these tutorials will be made available to you, together with the name, telephone number and the address of your tutor. Each assignment will be marked by your tutor. Pay close attention to the comments your tutor might make on your assignments as these will help in your progress. Make sure that assignments reach your tutor on or before the due date.

Your tutorials are important therefore try not to skip any. It is an opportunity to meet your tutor and your fellow students. It is also an opportunity to get the help of your tutor and discuss any difficulties encountered on your reading.

Summary

This course would train you on the concept of multimedia, production and utilization of it.

Wish you the best of luck as you read through this course

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Course Title MATHEMATICS FOR MANAGEMENT SCIENCES II

Course Developer/Writer: Mr. SUFIAN JELILI B.
National Open University of Nigeria
Lagos.

Programme Leader: Dr I.D Idrisu
National Open University of Nigeria
Lagos.

Course Coordinator: Mrs. Adegbola Eunice
National Open University of Nigeria
Lagos.

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria

Headquarters: 14/16 Ahmadu Bello Way, PMB 80067

Victoria Island, Lagos

Abuja Office: NOUN Building

No.5, Dar es Sallam Street

Off Aminu Kano Crescent

Wuse II

Abuja.

e-mail: centralinfo@nou.edu.ng

URL: www.nou.edu.ng

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MODULE ONE

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Unit 3: Probability Theory and Applications

MODULE ONE

Unit 1 Sets & Subsets

Unit 2 Basic Set Operations

Unit 3 Set of Numbers

Unit 4 Functions

UNIT 1: SETS & SUBSETS

CONTENTS

1.0 INTRODUCTION

The theory of sets lies at the foundation of mathematics. It is a concept that rears its head in almost all fields of mathematics; pure and applied.

This unit aims at introducing basic concepts that would be explained further in subsequent units. There will be definition of terms and lots of examples and exercises to help you as you go along.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Identify sets from some given statements
- Rewrite sets in the different set notation
- Identify the different kinds of sets with examples

3.0 MAIN BODY

3.1 SETS

As mentioned in the introduction, a fundamental concept in all a branch of mathematics is that of set. Here is a definition **"A set is any well-defined list, collection or class of objects".**

The objects in sets, as we shall see from examples, can be anything: But for clarity, we now list ten particular examples of sets:

Example 1.1 The numbers 0, 2, 4, 6, 8

Example 1.2 The solutions of the equation $x^2 + 2x+1 = 0$

Example 1.3 The vowels of the alphabet: a, e, i, o, u

Example 1.4 The people living on earth

Example 1.5 The students Tom, Dick and Harry **Example 1.6** The students who are absent from school **Example 1.7** The countries England, France and Denmark **Example 1.8** The capital cities of Nigeria

Example 1.9 The number 1, 3, 7, and 10

Example 1.10 The Rivers in Nigeria

Note that the sets in the odd numbered examples are defined, that is, presented, by actually listing its members; and the sets in the even numbered examples are defined by stating properties that is, rules, which decide whether or not a particular object is a member of the set.

3.1.1 Notation

Sets will usually be denoted by capital letters;

A, B, X, Y,......

Lower case letters will usually represent the elements in our sets:

Lets take as an example; if we define a particular set by actually listing its members, for example, let A consist of numbers 1,3,7, and 10, then we write

A={1,3,7,10}

That is, the elements are separated by commas and enclosed in brackets { }.

We call this the tabular form of a set

Now, try your hand on this

Exercise 1.1

State in words and then write in tabular form

1. $A = \{x\} \mid x^2 = 4^2\}$

2. $B = \{x\} \mid x - 2 = 5\}$

3. C = {x} | x is positive, x is negative}

4. D = {x} | x is a letter in the word "correct"}

Solution:

1. It reads "A is the set of x such that x squared equals four". The only numbers which when squared give four are 2 and -2. Hence A = {2, -2}

2. It reads "B is the set of x such that x minus 2 equals 5". The only solution is 7; hence B = {7}

3. It read "C is the set of x such that x is positive and x is negative". There is no number which is both positive and negative; hence C is empty, that is, C=Ø

4. It reads "D is the set of x such that x is letter in the work 'correct'. The indicated letters a re c,o,r,e and t; thus D = {c,o,r,e,t}

But if we define a particular set by stating properties which its elements must satisfy, for example, let B be the set of all even numbers, then we use a letter, usually x, to represent an arbitrary element and we write:

B = {x | x is even}

Which reads "B is the set of numbers x such that x is even". We call this the set builders form of a set. Notice that the vertical line " | " is read "such as".

In order to illustrate the use of the above notations, we rewrite the sets in examples 1.1-1.10. We denote the sets by A_1, A_2, ... A_{10} respectively.

Example 2.1: A_1 = {0, 2, 4, 6, 8}

Example 2.2: $A_2 = \{x \mid x^2+2x + 1 = 0\}$

Example 2.3: A_3 = {a, e, i, o, u}

Example 2.4: A_4 = {x | x is a person living on the earth}

Example 2.5: A_5 = {Tom, Dick, Harry}

Example 2.6: A_6 = {x │ x is a student and x is absent from school}

Example 2.7: A_7 = {England, France, Denmark}

Example 2.8: A_8 = {x│x is a capital city and x is in Nigeria}

Example 2.9: A_9 = {1, 3, 7, 10}

Example 2.10: A_{10} = { x│x is a river and x is in Nigeria} It is easy as that!

Exercise 1.2

Write These Sets in a Set-Builder Form

1. Let A consist of the letters a, b, c, d and e
2. Let B = {2, 4, 6, 8........}
3. Let C consist of the countries in the United Nations
4. Let D = {3}
5. Let E be the Heads of State Obasanjo, Yaradua and Jonathan

Solution

1. A={x │x appears before f in the alphabet} = {xx is one of the first letters in the alphabet}
2. B = {x │x is even and positive}
3. C = {x │x is a country, x is in the United Nations}
4. D = {x │x – 2 = 1} = {x│2x = 6}
5. E = {x │x was Head of state after Abacha}

If an object x is a member of a set A, i.e., A contains x as one of its\elements, then we write: $x \in A$

Which can be read "x belongs to A" or 'x is in A". If, on the otherhand, an object x is not a member of a set A, i.e A does not contain x as one of its elements, then we write; $x \notin A$

It is a common custom in mathematics to put a vertical line "|" or " \ " through a symbol to indicate the opposite or negative meaning of the symbol.

Example 3:1: Let A = {a, e, i o, u}. Then a∈A, b∉A, f∉A.

Example 3.2: Let B = {x< x is even}. Then 3∉B, 6∈B, 11∉B,14∈B

3.1.2 Finite & Infinite Sets

Sets can be finite or infinite. Intuitively, a set is finite if it consists of a specific number of different elements, i.e. if in counting the different members of the set the counting process can come to an end. Otherwise a set is infinite. Lets look at some examples.

Example 4:1: Let M be the set of the days of the week. The M is finite

Example 4:2: Let N = {0,2,4,6,8........}. Then N is infinite

Example 4:3: Let P = {x< x is a river on the earth}. Although it maybe difficult to count the number of rivers in the world, P is still a finite set.

Exercise 1.3: Which sets are finite?

1. The months of the year

2. {1, 2, 3, ... 99, 100}

3. The people living on the earth

4. {x | x is even}

5. {1, 2, 3,........}

Solution:

The first three sets are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two sets are infinite. If we ever try to count the even numbers, we would never come to the end.

3.1.3 Equality of Sets

Set A is equal to set B if they both have the same members, i.e if every element which belongs to A also belongs to B and if every element which belongs to B also belongs to A. We denote the equality of sets A and B by:

$$A = B$$

Example 5.1 Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B,

that is {1,2,3,4} = {3,1,4,2}, since each of the elements

1,2,3 and 4 of A belongs to B and each of the elements

3,1,4 and 2 of B belongs to A. Note therefore that a set does not change if its elements are rearranged.

Example 5.3 Let $E=\{x \mid x^2 - 3x = -2\}$, F={2,1} and G ={1,2,2, 1},

Then E= F= G

3.1.4 Null Set

It is convenient to introduce the concept of the empty set, that is, a set which contains no elements. This set is sometimes called the null set.

We say that such a set is void or empty, and we denote its symbol $\emptyset$

Example 6.1: Let A be the set of people in the world who are older than 200 years. According to known statistics A is the null set.

Example 6.2: Let $B = \{x \mid x^2= 4, x \text{ is odd}\}$, Then B is the empty set.

3.2 SUBSETS

If every element in a set A is also a member of a set B, then A is called subset of B.

More specifically, A is a subset of B if $x \in A$ implies $x \in B$. We denote this relationship by writing; $A \subset B$, which can also be read "A is contained in B".

Example 7.1

The set C = {1,3,5} is a subset of D = {5,4,3,2,1}, since each number 1, 3 and 5 belonging to C also belongs to D.

Example 7.2

The set E = {2,4,6} is a subset of F = {6,2,4}, since each number 2,4, and 6 belonging to E also belongs to F. Note, in particular, that E = F. In a similar manner it can be shown that every set is a subset of itself.

Example 7.3

Let G = {x | x is even}, i.e. G = {2,4,6}, and let F = {x | x is a positive power of 2}, i.e. let F = {2,4,8,16.....} Then $F \subset G$, i.e. F is contained in G.

With the above definition of a subset, we are able to restate the definition of the equality of two sets.

Two set A and B are equal, i.e. A = B, if an only if $A \subset B$ and $B \subset A$. If A is a subset of B, then we can also write:

$$B \supset A$$

Which reads "B is a superset of A" or "B contains A". Furthermore, we write:

$$A \not\subset B$$

if A is not a subset of B.

Conclusively, we state:

1. The null set ∅ is considered to be a subset of every set
2. If A is not a subset of B, that is, if $A \not\subset B$, then there is at least one element in A that is not a member of B.

3.2.1 Proper Subsets

Since every set A is a subset of itself, we call B a proper subset of A if, first, is a subset of A and secondly, if B is not equal to A. More briefly, B is a proper subset of A if:

$B \subset A$ and $B \neq A$

In some books "B is a subset of A" is denoted by

$B \subseteq A$

and B" is a proper subset of A" is denoted by

$B \subset A$

We will continue to use the previous notation in which we do not distinguished between a subset and a proper subset.

3.2.2 Comparability

Two sets A and B are said to be comparable if:

$A \subset B$ or $B \subset A$;

That is, if one of the sets is a subset of the other set. Moreover, two sets A and B are said to be not comparable if:

$A \not\subset B$ and $B \not\subset A$

Note that if A is not comparable to B then there is an element in A.which is not in B and ... also, there is an element in B which is not in A.

Example 8.1: Let A = {a,b} and B = {a,b,c}. The A is comparable to B, since A is a subset of B.

Example 8.2: Let R – {a,b} and S = {b,c,d}. Then R and S are not comparable, since $a \in R$ and $a \notin S$ and $c \notin R$

In mathematics, many statements can be proven to be true by the use of previous assumptions and definitions. In fact, the essence of mathematics consists of theorems and their proofs. We now proof our first

Theorem 1.1: If A is a subset of B and B is a subset of C then A is a subset of C, that is,

$A \subset B$ and $B \subset C$ implies $A \subset B$

Proof: (Notice that we must show that any element in A is also an element in C). Let x be an element of A, that is, let $x \in A$. Since A is a subset of B, x also belongs to B, that is, $x \in B$. But by hypothesis, $B \subset C$; hence every element of B, which includes x, is a number of C. We have shown that $x \in A$ implies $x \in C$. Accordingly, by definition, $A \subset C$.

3.2.3 Sets of Sets

It sometimes will happen that the object of a set are sets themselves; for example, the set of all subsets of A. In order to avoid saying “set of sets”, it is common practice to say “family of sets” or “class of sets”. Under the circumstances, and in order to avoid confusion, we sometimes will let script letters A, B,..........

Denote families, or classes, of sets since capital letters already denote their elements.

Example 9.1: In geometry we usually say “a family of lines” or “a family of curves” since lines and curves are themselves sets of points.

Example 9.2: The set {{2,3}, {2}, {5,6}} is a family of sets. Its members are the sets {2,3}, {2} and {5,6}.

Theoretically, it is possible that a set has some members, which are sets themselves and some members which are not sets, although in any application of the theory of sets this case arises infrequently.

Example 9.3: Let A = {2, {1,3}, 4, {2,5}}. Then A is not a family of sets; here some elements of A are sets and some are not.

3.2.4 Universal Set

In any application of the theory of sets, all the sets under investigation will likely be subsets of a fixed set. We call this set the universal set or universe of discourse. We denote this set by U.

Example 10.1: In plane geometry, the universal set consists of all the points in the plane.

Example 10.2: In human population studies, the universal set consists of all the people in the world.

3.2.5 Power Set

The family of all the subsets of any set S is called the power set of S.

We denote the power set of S by: 2^S

Example 11.1: Let M = {a,b} Then 2^M = {{a, b}, {a}, {b}, ∅}

Example 11.2: Let T = {4,7,8} then 2^T = {T, {4,7}, {4,8}, {7,8}, {4}, {7}, {8}, ∅}

If a set S is finite, say S has n elements, then the power set of S can be shown to have 2^n elements. This is one reason why the class of subsets of S is called the power set of S and is denoted by $2^{S.}$

3.2.6 Disjoint Sets

If sets A and B have no elements in common, i.e if no element of A is in B and no element of B is in A, then we say that A and B are disjoint

Example 12.1: Let A = {1,3,7,8} and B = {2,4,7,9}, Then A and B are not disjoint since 7 is in both sets, i.e 7∈A and 7∈B

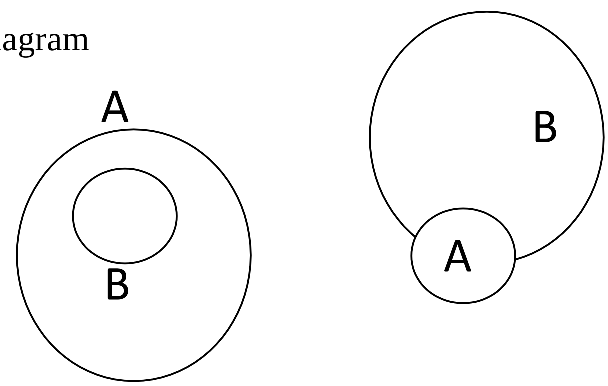
Example 12.2: Let A be the positive numbers and let B be the negative numbers. Then A and B are disjoint since no number is both positive and negative.

Example 12.3: Let E = {x, y, z} and F = {r, s, t}, Then E and F are disjoint.

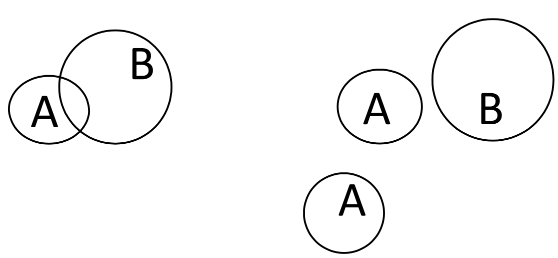
3.3 VENN-EULER DIAGRAMS

A simple and instructive way of illustrating the relationships between sets is in the use of the so-called Ven-Euler diagrams or, simply, Venn diagrams. Here we represent a set by a simple plane area, usually bounded by a circle.

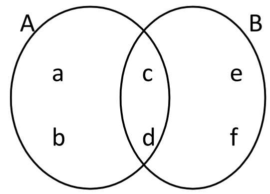
Example 13.1: Suppose A⊂B and, say, A≠B, then A and B can be described by either diagram



Example 13.2: Suppose A and B are not comparable. Then A and B can be represented by the diagram on the right if they are disjoint, or the diagram on the let if they are not disjoint.



Example 13.3: Let A = {a, b, c, d} and B= {c, d, e, f}. Then we illustrate these sets a Venn diagram of the form:



3.4 AXIOMATIC DEVELOPMENT OF SET THEORY

In an axiomatic development of a branch of mathematics, one begins with:

1. Undefined terms

2. Undefined relations

3. Axioms relating the undefined terms and undefined relations. Then, one develops theorems based upon the axioms and definitions **Example 14:1:**

In an axiomatic development of Plane Euclidean geometry

1. "Points" and "lines" are undefined terms

2. "Points on a line" or, equivalent, "line contain a point" is an undefined relation

3. Two of the axioms are:

Axiom 1: Two different points are on one and only one line

Axiom 2: Two different lines cannot contain more than one point in common

In an axiomatic development of set theory:

1. "Element' and "set" are undefined terms

2. "Element belongs to a set" is undefined relation

3. Two of the axioms are

Axiom of Extension: Two sets A and B are equal if and only if every element in A belongs to B and every element in B belongs to A.

Axiom of Specification: Let P(x) be any statement and let A be any set. Then there exists a set:

B= {a│a ∈ A, P (a) is true}

Here, P(x) is a sentence in one variable for which P(a) is true or false for any a∈A. for example P(x) could be the sentence "$x^2 = 4$" or "x is a member of the United Nations"

4.0 CONCLUSION

You have been introduced to basic concepts of sets, set notation e.t.c that will be built upon in other units. If you have not mastered them by now you will notice you have to come back to this unit from time to time.

5.0 SUMMARY

A summary of the basic concept of set theory is as follows:

- A set is any well-defined list, collection, or class of objects.
- Given a set A with elements 1,3,5,7 the tabular form of representing this set is A = {1, 3, 5, 7}.
- The set-builder form of the same set is A = {x| x = 2n + 1,0 ≤n≤3}
- Given the set N = {2,4,6,8,....} then N is said to be infinite, since the counting process of its elements will never come to an end, otherwise it is finite
- Two sets of A and B are said to be equal if they both have the same elements, written A = B

- The null set, ∅ , contains no elements and is a subset of every set
- The set A is a subset of another set B, written $A \subset B$, if every element of A is also an element of B, i.e. for every $x \in A$ then $x \in B$
- If $B \subset A$ and $B \neq A$, then B is a proper subset of A
- Two sets A and B are comparable if $A \subset B$ and $B \subset A$
- The power set 2^2 of any set S is the family of all the subsets of S
- Two sets A and B are said to be disjoint if they do not have any element in common, i.e their intersection is a null set.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Rewrite the following statement using set notation:

1. x does not belong to A.
2. R is a superset of S
3. d is a member of E
4. F is not a subset of G
5. H does not included D

2. Which of these sets are equal: {r,t,s}, {s,t,r,s}, {t,s,t,r}, {s,r,s,t}?

3. Which sets are finite?

    1. The months of the year
    2. {1,2,3,......99, 100}
    3. The people living on the earth
    4. {x | x is even}
    5. {1,2,3........)

The first three set are finite. Although physically it might be impossible to count the number of people on the earth, the set is still finite. The last two set are infinite. If we ever try to count the even numbers we would never come to the end.

4. Which word is different from each other, and why: (1) empty, (2) void, (3) zero, (4) null?

7.0 REFERENCE AND FURTHER READING

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable, Vol. 1)

UNIT 2: BASIC SET OPERATIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall see operations performed on sets as in simple arithmetic. This operations simply give sets a language of their own.

You will notice in subsequent units that you cannot talk of sets without reference, sort of, to these operations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Compare two sets and/or assign to them another set depending on their comparability.
- Represent these relationships on the Venn diagram.

3.0 MAIN BODY

3.1 SET OPERATIONS

In arithmetic, we learn to add, subtract and multiply, that is, we assign to each pair of numbers x and y a number x + y called the sum of x and y, a number x – y called the difference of x and y, and a number xy called the product of x and y. These assignments are called the operations of addition, subtraction and multiplication of numbers. In this unit, we define the operation Union, Intersection and difference of sets, that is, we will assign new pairs of sets A and B. In a later unit, we will see that these set operations behave in a manner some what similar to the above operations on numbers.

3.1.1 Union

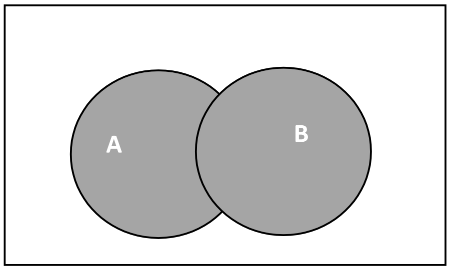
The union of sets A and B is the set of all elements which belong to A or to B or to both. We denote the union of A and B by;

$A \cup B$

Which is usually read "A union B"

Example 1.1: In the Venn diagram in fig 2-1, we have shaded $A \cup B$,

i.e. the area of A and the area of B.



A∪B is shaded

Fig 2.1

Example 1.2: Let S = {a, b. c. d} and T = {f, b, d, g}.

Then S T = {a, b, c, d, f, g}.

Example 1.3: Let P be the set of positive real numbers and let Q be the set of negative real numbers. The P∪Q, the union of P and Q, consist of all the real numbers except zero. The union of A and B may also be defined concisely by:

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Remark 2.1: It follows directly from the definition of the union of two sets that A∪B and B∪A are the same set, i.e.,

$A \cup B = B \cup A$

Remark 2.2: Both A and B are always subsets of A and B that is,

$A \subset (A \cup B)$ and $B \subset (A \cup B)$

In some books, the union of A and B is denoted by A + B and is called the set-theoretic sum of A and B or, simply, A plus B

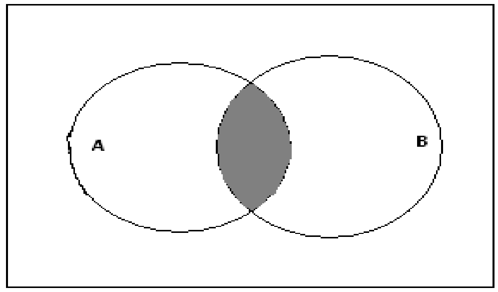
3.1.2 Intersection

The Intersection of sets A and B is the set of elements which are common to A and B, that is, those elements which belong to A and which belong to B. We denote the intersection of A and B by:

$A \cap B$

Which is read "A intersection B".

Example 2.1: In the Venn diagram in fig 2.2, we have shaded A∩B, the area that is common to both A and B



A∩B is shaded

Fig 2.2

Example 2.2: Let S = {a, b, c, d} and T = {f, b, d, g}. Then S∩T = {b, d}

Example 2.3: Let V = 2, 3, 6, ...} i.e. the multiples of 2; and

Let W = {3, 6, 9,....} i.e. the multiples of 3. Then

V∩W = {6, 12, 18......}

The intersection of A and B may also be defined concisely by

A∩B = {x∈A, x∈B}

Here, the comma has the same meaning as "and".

Remark 2.3: It follows directly from the definition of the intersection of two sets that;

A∩B =B∩A

Remark 2.4:

Each of the sets A and B contains A∩B as a subset, i.e.,

(A∩B)⊂A and (A∩B) ⊂B

Remark 2.5: If sets A and B have no elements in common, i.e. if A and B are disjoint, then the intersection of A and B is the null set, i.e. A∩B =∅

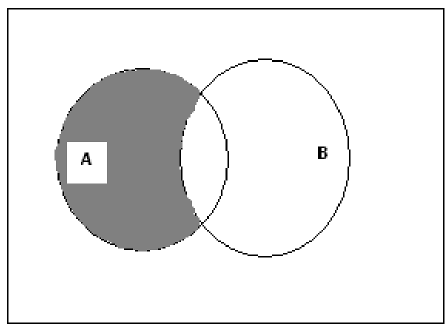
In some books, especially on probability, the intersection of A and B is denoted by AB and is called the set-theoretic product of A and B or, simply, A times B.

3.1.3 DIFFERENCE

The difference of sets A and B is the set of elements which belong to A but which do not belong to B. We denote the difference of A and B by A – B

Which is read "A difference B" or, simply, "A minus B".

Example 3.1: In the Venn diagram in Fig 2.3, we have shaded A – B, the area in A which is n



A – B is shaded

Fig 2.3

Example 3.2: Let R be the set of real numbers and let Q be the set of rational numbers. Then R – Q consists of the irrational numbers.

The difference of A and B may also be defined concisely by

$A - B = \{x \mid x \in A, x \notin B\}$

Remark 2.6: Set A contains A – B as a subset, i.e.,

$(A - B) \subset A$

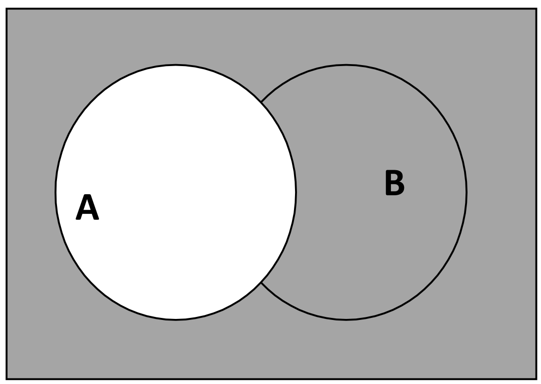
Remark 2.7: The sets (A – B), A∩B and (B – A) are mutually disjoint, that is, the intersection of any two is the null set.

The difference of A and B is sometimes denoted by A/B or A ~ B

3.1.4 Complement

The complement of a set A is the set of elements that do not belong to A, that is, the difference of the universal set U and A. We denote the complement of A by **A′**

Example 4.1: In the Venn diagram in Fig 2.4, we shaded the complement of A, i.e. the area outside A. Here we assume that the universal set U consists of the area in the rectangle.



A' is shaded

Fig. 2.4

Example 4.2: Let the Universal set U be the English alphabet and let T = {a, b, c}. Then;

T' = {d, e, f, ..., y, z}

Example 4.3:

Let E = {2, 4, 6, ...}, that is, the even numbers.

Then E′ = {1, 3, 5, ...}, the odd numbers. Here we assume that the universal set is the natural numbers, 1, 2, 3,.....

The complement of A may also be defined concisely by;

$A' = \{x \mid x \in U, x \notin A\}$ or, simply,

$A' = \{x \mid x \notin A\}$

We state some facts about sets, which follow directly from the definition of the complement of a set.

Remark 2.8: The union of any set A and its complement A′ is the universal set, i.e.,

$A \cup A' = U$

Furthermore, set A and its complement A′ are disjoint, i.e.,

$A \cap A' = \emptyset$

Remark 2.9: The complement of the universal set U is the null set ∅, and vice versa, that is,

$U' = \emptyset$ and $\emptyset' = U$

Remark 2.10: The complement of the complement of set A is the set A itself. More briefly,

$(A')' = A$

Our next remark shows how the difference of two sets can be defined in terms of the complement of a set and the intersection of two sets. More specifically, we have the following basic relationship:

Remark 2.11: The difference of A and B is equal to the intersection of A and the complement of B, that is,

$A - B = A \cap B'$

The proof of Remark 2.11 follows directly from definitions:

$A - B = \{x \mid x \in A, x \notin B\} = \{x \mid x \in A, x \notin B'\} = A \cap B'$

3.2 OPERATIONS ON COMPARABLE SETS

The operations of union, intersection, difference and complement have simple properties when the sets under investigation are comparable. The following theorems can be proved.

Theorem 2.1: Let A be a subset of B. Then the union intersection of A and B is precisely A, that is,

$A \subset B$ implies $A \cap B = A$

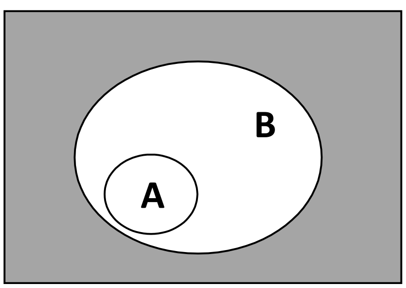
Theorem 2.2: Let A be a subset of B. Then the of A and B is precisely B, that is,

$A \subset B$ implies $A \cup B = B$

Theorem 2.3: Let A be a subset of B. Then B' is a subset of A', that is,

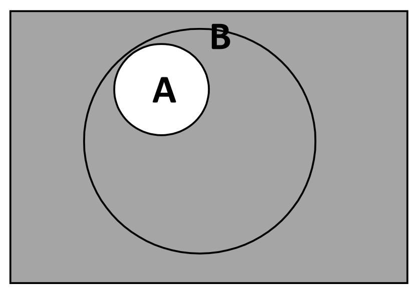
$A \subset B$ implies $B' \subset A'$

We illustrate Theorem 2.3 by the Venn diagrams in Fig 2-5 and 2-6. Notice how the area of B' is included in the area of A'.



B′ is shaded

Fig 2.5



A′ is shaded

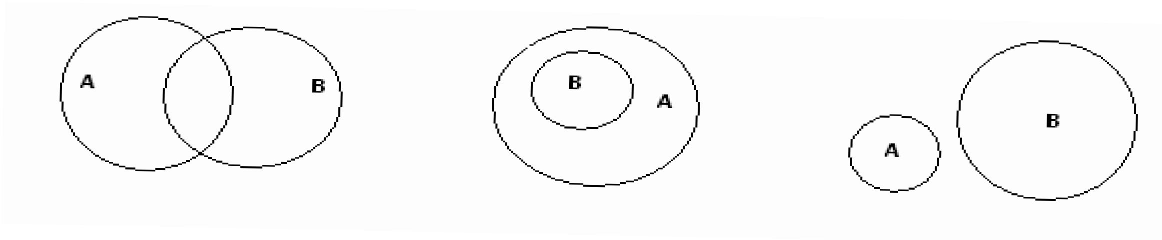
Fig 2.6

Theorem 2.4: Let A be a subset of B. Then the Union of A and (B – A) is precisely B, that is,

$A \subset B$ implies $A \cup (B - A) = B$

Exercises

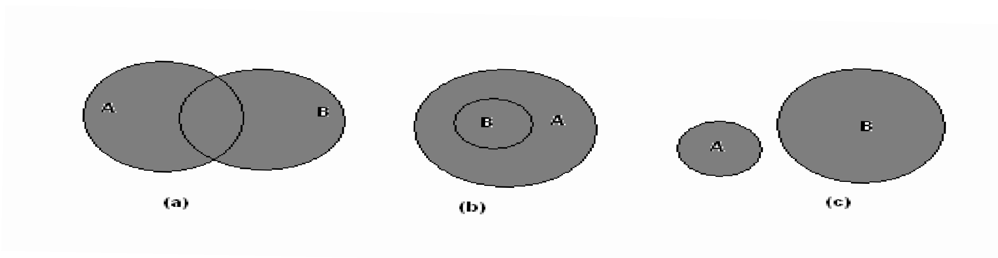
1. In the Venn diagram below, shade A Union B, that is, A∪B:



(a) (b) (c)

Solution:

The union of A and B is the set of all elements that belong to A and to B or to both. We therefore shade the area in A and B as follows:



2. Let A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find

(a) A∪B, (b) A∪C, (c) B∪C, (d) B∪B

Solution:

To form the union of A and B we put all the elements from A together with the elements of B Accordingly,

A∪B = {1,2,3,4,6,8}

A∪C = { 1,2,3,4,5,6}

B∪C = {2,4,6,8,3,5}

B∪B = {2,4,6,8}

Notice that B∪B is precisely B.

3. Let A, B and C be the sets in Problem 2. Find (1) (A∪B)∪C,(2) A∪(B∪C).

Solution:

1. We first find (A∪B) = {1,2,3,4,6,8}. Then the union of {A∪B} and C is

(A∪B)∪C = {1,2,3,4,6,8,5}

2. We first find (B∪C) = {2,4,6,8,3,5}. Then the union of A and (B∪C) is A∪(B∪C) = {1,2,3,4,6,8,5}.

Notice that (A∪B)∪C = A∪(B∪C)

4.0 CONCLUSION

You have seen how the basic operations of Union, Intersection, Difference and Complement on sets work like the operations on numbers. These are also the basic symbols associated with set theory.

5.0 SUMMARY

The basic set operations are Union, Intersection, Difference and Complement defined as:

- The Union of sets A and B, denoted by A∪B, is the set of all elements, which belong to A or to B or to both.
- The intersection of sets A and B, denoted by A∩B, is the set of elements, which are common to A and B. If A and B are disjoint then their intersection is the Null set ∅
- The difference of sets A and B, denoted by A – B, is the set of elements which belong to A but which do not belong to B.
- The complement of a set A, denoted by A', is the set of elements, which do not belong to A, that is, the difference of the universal set U and A.

6.0 TUTOR – MARKED ASSIGNMENTS

1. Let X = {Tom, Dick, Harry}, Y = {Tom, Marc, Eric} and Z = {Marc, Eric, Edward}. Find (a) $X \cup Y$, (b) $Y \cup Z$(c) $X \cup Z$

2. Prove: $A \cap \emptyset = \emptyset$.

3. Prove Remark 2.6: $(A - B) \subset A$.

4. Let U = {1,2,3,...,8,9}, A = {1,2,3,4}, B = {2,4,6,8} and C = {3,4,5,6}. Find (a) A’, (b) B’, (c) $(A \cap C)$’, (d) $(A \cup B)$’, (e) (A’)’, (f) (B – C)’

5. Prove: B – A is a subset of A’

7.0 REFERENCES AND FURTHER READINGS

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real – valued functions of a real variable), Vol. 1

UNIT 3: SET OF NUMBERS

CONTENTS

1.0 INTRODUCTION

Although, the theory of sets is very general, important sets, which we meet in elementary mathematics, are sets of numbers. Of particular importance, especially in analysis, is the set of ***real numbers,*** which we denote by $\mathcal{R}$

In fact, we assume in this unit, unless otherwise stated, that the set of real numbers $\mathcal{R}$ is out universal set. We first review some elementary properties of real numbers before applying our elementary principles of set theory to sets of numbers. The set of real numbers and its properties is called the ***real number system.***

2.0 OBJECTIVES

After studying this unit, you should be able to do the following:

- Represent the set of numbers on the real line
- Perform the basic set operations on intervals

3.0 MAIN BODY

3.1 REAL NUMBERS, $\mathcal{R}$

One of the most important properties of the real numbers is that points on a straight line that can represent them. As in Fig 3.1, we choose a point, called the origin, to represent 0 and another point, usually to the right, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, that is, each point will represent a unique real number and each real number will be represented by a unique point. We refer to this line as the ***real line.*** Accordingly, we can use the words point and number interchangeably.

Those numbers to the right of 0, i.e. on the same side as 1, are called the *positive numbers* and those numbers to the left of 0 are called the *negative numbers.* The number 0 itself is neither positive nor negative.

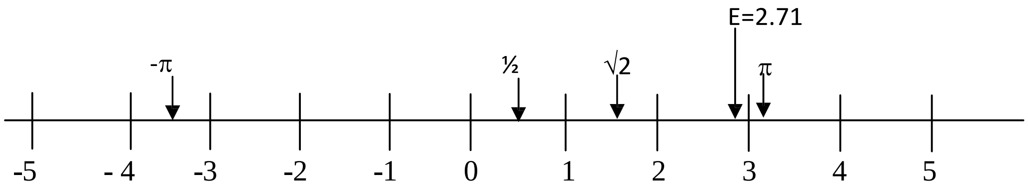


Fig 3.1

3.1.2 Integers, Z

The integers are those real numbers

…, -3, -2, -1, 0, 1, 2, 3,…

We denote the integers by Z; hence we can write

Z = { …, -2, - 1, 0, 1, 2,…}

The integers are also referred to as the "whole" numbers.

One important property of the integers is that they are "closed" under the operations of addition, multiplication and subtraction; that is, the sum, product and difference of two integers is again in integer. Notice that the quotient of two integers, e.g. 3 and 7, need not be an integer; hence the integers are not closed under the operation of division.

3.1.3 Rational Numbers, Q

The *rational numbers* are those real numbers, which can be expressed as the ratio of two integers. We denote the set of rational numbers by Q. Accordingly,

$Q = \{x \mid x = p\ q \text{ where } p \in z,\ q \in z\}$

Notice that each integer is also a rational number since, for example, 5 = 5/1; hence Z is a subset of Q.

The rational numbers are closed not only under the operations of addition, multiplication and subtraction but also under the operation of division (except by 0). In other words, the sum, product, difference and quotient (except by 0) of two rational numbers is again a rational number.

3.1.4 Natural Numbers, N

The *natural numbers* are the positive integers. We denote the set of natural numbers by N; hence N = {1,2,3…..}

The natural numbers were the first number system developed and were used primarily, at one time, for counting. Notice the following relationship between the above numbers systems:

$N \subset Z \subset Q \subset R$

The natural numbers are closed only under the operation of addition and multiplication. The difference and quotient of two natural numbers needed not be a natural number.

The *prime numbers* are those natural numbers p, excluding 1, which are only divisible 1 and p itself. We list the first few prime numbers: 2,3,5,7,11,13,17,19…

3.1.5 Irrational Numbers, Q'

The irrational numbers are those real numbers which are not rational, that is, the set of irrational numbers is the complement of the set of rational numbers Q in the real numbers R; hence Q' denote the irrational numbers. Examples of irrational numbers are $\sqrt{3}$, π, $\sqrt{2}$, etc.

3.1.6 Line Diagram of the Number Systems

Fig 3.2 below is a line diagram of the various sets of number, which we have investigated. (For completeness, the diagram include the sets of complex numbers, number of the form a + bi where a and b are real. Notice that the set of complex numbers is superset of the set of real numbers.)

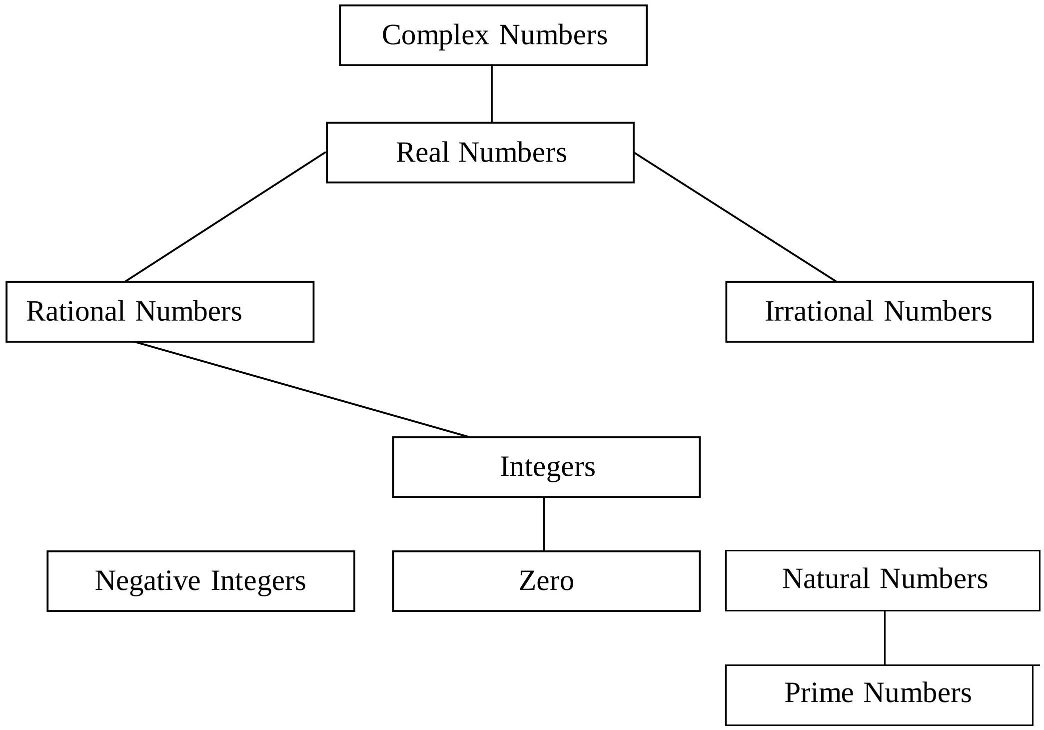


Fig 3.2

3.2 DECIMALS AND REAL NUMBERS

Every real number can be represented by a “non-terminating decimal”. The decimal representation of a rational number p/q can be found by “dividing the denominator q into the numerator p”. If the indicated division terminates, as for

3/8 = .375

We write 3/8 = .375000

Or 3/8 = .374999…

If we indicated division of q into p does not terminate, then it is known that a block of digits will continually be repeated; for example, 2/11 = .181818…

We now state the basic fact connecting decimals and real numbers. The rational numbers correspond precisely to those decimals in which a block of digits is continually repeated, and the irrational numbers correspond to the other non-terminating decimals.

3.3 INEQUALITIES

The concept of “order” is introduced in the real number system by the

Definition: The real number a is less than the real number b,

written $a < b$

If $b – a$ is a positive number.

The following properties of the relation $a < b$ can be proven. Let a, b and c be real numbers; then:

P_1: Either $a < b$, $a = b$ or $b < a$.

P_2: If $a < b$ and $b < c$, then $a < c$.

P_3: If $a < b$, then $a + c < b + c$

P_4: If $a < b$ and c is positive, then $ac < bc$

P_5: If $a < b$ and c is negative, then $bc < ac$.

Geometrically, if $a < b$ then the point a on the real line lies to the left of the point b.

We also denote $a < b$ by $b > a$

Which reads “b is *greater then* a”. Furthermore, we write

$a < b$ or $b > a$

if $a < b$ or $a = b$, that is, if a is not greater than b.

Example 1.1: $2 < 5$; $-6 < -3$ and $4 < 4$; $5 > -8$

Example 1.2: The notation $x < 5$ means that x is a real number which is less than 5; hence x lies to the left of 5 on the real line.

The notation $2 < x < 7$; means $2 < x$ and also $x < 7$; hence x will lie between 2 and 7 on the real line.

Remark 3.1: Notice that the concept of order, i.e. the relation $a < b$, is defined in terms of the concept of positive numbers. The fundamental property of the positive numbers which is used to prove properties of the relation $a < b$ is that the positive numbers are closed under the operations of addition and multiplication. Moreover, this fact is intimately connected with the fact that the natural numbers are also closed under the operations of addition and multiplication.

Remark 3.2: The following statements are true when a, b, c are any real numbers:

1. $a < a$

2. if $a < b$ and $b < a$ then $a = b$.

3. if $a < b$ and $b < c$ then $a < c$.

3.4 ABSOLUTE VALUE

The absolute value of a real number x, denoted by ÷ x÷ is defined by the formula

$$|x| = \begin{cases} x \text{ if } x > 0 \\ -x \text{ if } x < 0 \end{cases}$$

that is, if x is positive or zero then $|x|$ equals x, and if x is negative then $|x|$ equals – x. Consequently, the absolute value of any number is always non-negative, i.e. $|x| > 0$ for every $x \in R$.

Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e. the point 0. Moreover, the distance between any two points, i.e. real numbers, a and b is $|a - b| = |b - a|$.

Example 2.1: $|-2| = 2$, $|7| = 7$. $|-p| = p$

Example 2.2: The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5, i.e. x must lies between -5 and 5 on the real line. In other words,

$|x| < 5$ and $-5 < x < 5$

have identical meaning. Similarly,

$|x| < 5$ and $-5 < x < 5$

have identical meaning.

3.5 INTERVALS

Consider the following set of numbers;

$A_1 = \{x | \ 2 < x < 5\}$

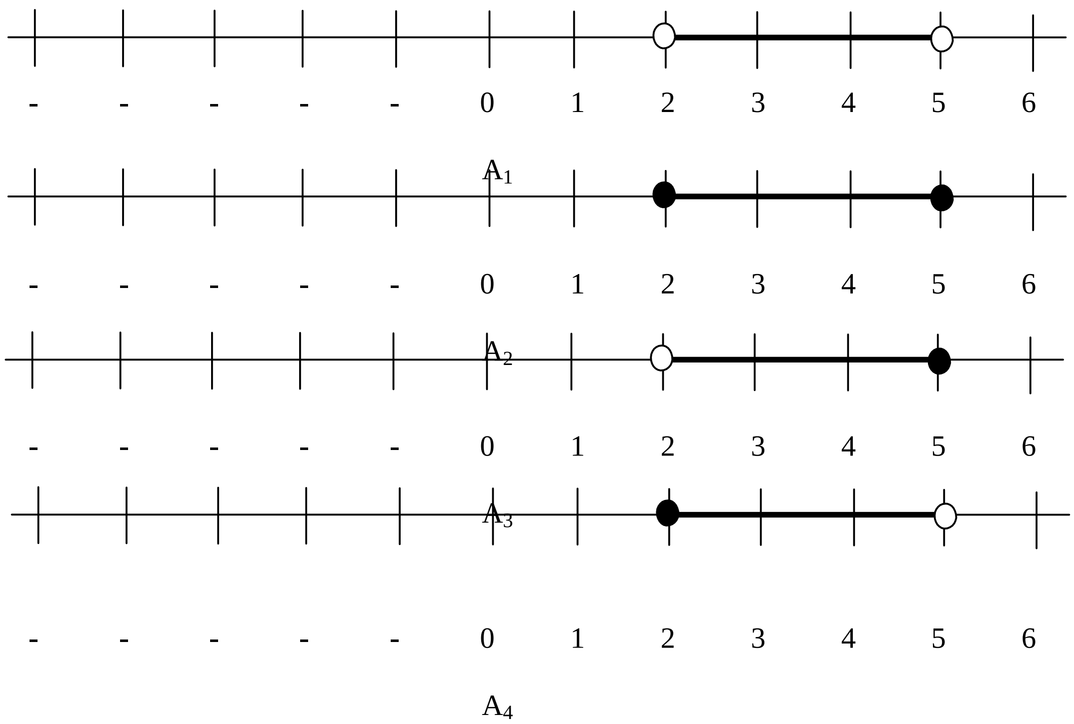
$A_2 = \{x | \ 2 \leq x \leq 5\}$

$A_3 = \{x | \ 2 < x \leq 5\}$

$A_4 = \{x | 2 \leq x < 5\}$

Notice, that the four sets contain only the points that lie between 2 and 5 with the possible exceptions of 2 and/or 5. We call these sets intervals, the numbers 2 and 5 being the endpoints of each interval. Moreover, A_1 is an ***open interval*** as it does not contain either end point: A_2 is a ***closed interval*** as it contains bother endpoints; A_3 and A_4 are ***open-closed*** and ***closed-open*** respectively.

We display, i.e. graph, these sets on the real line as follows.



Notice that in each diagram we circle the endpoints 2 and 5 and thicken (or shade) the line segment between the points. If an interval includes an endpoint, then this is denoted by shading the circle about the endpoint.

Since intervals appear very often in mathematics, a shorter notation is frequently used to designated intervals, Specifically, the above intervals are sometimes denoted by;

$A_1 = (2, 5)$

$A_2 = [2, 5]$

$A_3 = (2, 5)$

$A_4 = [2, 5)$

Notice that a parenthesis is used to designate an open endpoint, i.e. an endpoint that is not in the interval, and a bracket is used to designate a closed endpoint.

3.5.1 Properties of Intervals

Let $\Im$ be the family of all intervals on the real line. We include in $\Im$ the null set $\varnothing$ and single points a = [a, a]. Then the intervals have the following properties:

1. The intersection of two intervals is an interval, that is, $A \in \Im$, $B \in \Im$ implies $A \cap B \in \Im$

2.The union of two non-disjoint intervals is an interval, that is, $A \in \Im$, $B \in \Im$, $A \cap B \neq \varnothing$ ipmlies $A \cup B \in \Im$

3. The difference of two non-comparable intervals is an interval, that is, $A \in \Im$, $B \in \Im$, $A \not\subset B$, $B \not\subset A$ implies $A - B \in \Im$

Example 3.1: Let A = {2, 4), B = (3, 8). Then

A Ç B = (3, 4), A È B = [2, 8)

A – B = [2, 3], B – A = [4, 8)

3.5.2 Infinite Intervals

Sets of the form

$A = \{x \mid x > 1\}$

$B = \{x \mid x \geq 2$

$C = \{x \mid x < 3\}$

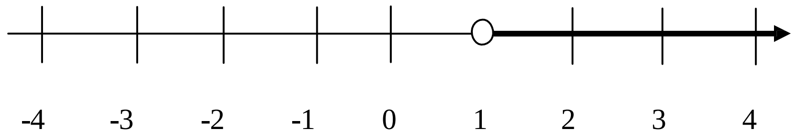
$D = \{x \mid x \leq 4\}$

$E = \{x \mid x \in \Re\}$

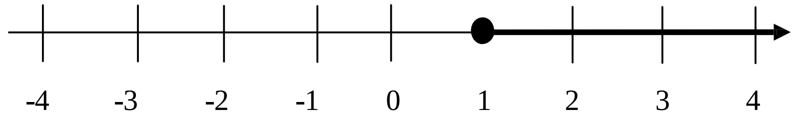
Are called infinite intervals and are also denoted by

$A = (1, \infty)$, $B = [2, \infty)$, $C = (-\infty, 3)$, $D = (-\infty, 4]$, $E = (-\infty, \infty)$

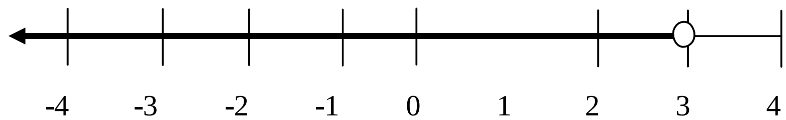
We plot these intervals on the real line as follows:



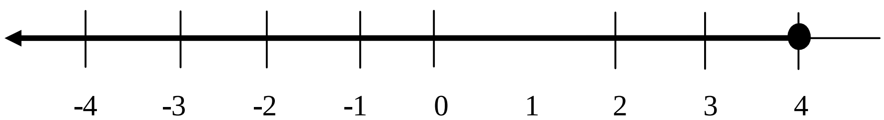
A is Shaded



B is Shaded



C is Shaded



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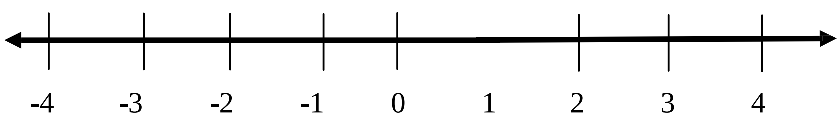

E is Shaded

3.6 BOUNDED AND UNBOUNDED SETS

Let A be a set of numbers, then A is called ***bounded*** set if A is the subset of a finite interval. An equivalent definition of boundedness is;

Definition 3.1: Set A is ***bounded*** if there exists a positive number M such that $|x| \leq M$.

for all $x \in A$. A set is called ***unbounded*** if it is not bounded

Notice then, that A is a subset of the finite interval [-M, M].

Example 4.1: Let A = {1, ½, 1/3…}. Then A is bounded since A is certainly a subset of the closed interval [0, 1].

Example 4.2: Let A = {2, 4, 6,…..}. Then A is an unbounded set.

Example 4.3: Let A = {7, 350, -473, 2322, 42}. Then A is bounded

Remark 3.3: If a set A is finite then, it is necessarily bounded.

If a set is infinite then it can be either bounded as in example 4.1 or unbounded as in example 4.2

4.0 CONCLUSION

The set of real numbers is of utmost importance in analysis. All (except the set of complex numbers) other sets of numbers are subsets of the set of real numbers as can be seen from the line diagram of the number system.

5.0 SUMMARY

In this unit, you have been introduced to the sets of numbers. The set of real numbers, $\Re$, contains the set of integers, Z, Rational numbers, Q, Natural numbers, N, and Irrational numbers, Q'.

Intervals on the real line are open, closed, open-closed or closed-open depending on the nature of the endpoints.

6.0 TUTOR-MARKED ASSIGNMENTS

1. Prove: If a < b and B < c, then a < c

2. Under what conditions will the union of two disjoint interval be an interval?

3. If two sets R and S are bounded, what can be said about the union and Intersection of these sets?

7.0 REFERENCES AND FURTHER READINGS

Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

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UNIT 4: FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, you will be introduced to the concept of functions, mappings and transformations. You will also be given instructive and typical examples of functions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Identify functions from statements or diagrams
- State whether a function is one-one or onto
- Find composition function of two or more functions

3.0 MAIN BODY

3.1 DEFINITION

Suppose that to each element in a set A there is assigned by some manner or other, a unique element of a set$\Re$. We call such assignment of ***function.*** If we let ¦ denote these assignments, we write;

$$f: A \longrightarrow B$$

which reads "f is a function of A onto B". The set A is called the ***domain*** of the function f, and B is called the ***co-domain*** of f. Further, if a $\in$ A the element in B which is assigned to a is called the ***image*** of a and is denoted by; *f*(a) which reads "f of a".

We list a number of instructive examples of functions.

Example 1.1: Let f assign to each real number its square, that is, for every real number x let $f(x) = x^2$. The domain and co-domain of f are both the real numbers, so we can write: f: $\Re$**>** $\Re$

The image of -3 is 9; hence we can also write f (-3) = 9

or f : 3→ 9

Example 1.2: Let f assign to each country in the world its capital city. Here, the domain of f is the set of countries in the world; the co-domain of f is the list of capital cities in the world. The image of France is Paris, that is, f(France) = Paris

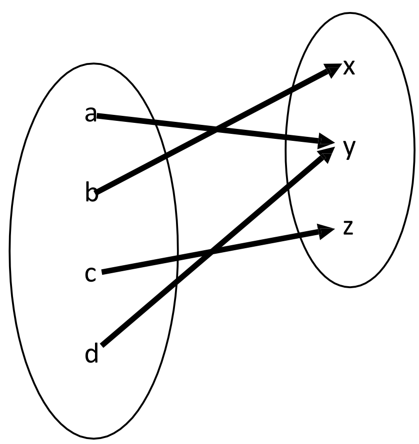
Example 1.3: Let A = {a, b, c, d} and B = {a, b, c}. Define a function f of A into B by the correspondence f(a) = b, f(b) = c, f(c) = c and f(d) = b. By this definition, the image, for example, of b is c.

Example 1.4: Let A = {-1, 1}. Let f assign to each rational number in ℜ the number 1, and to each irrational number in ℜ the number -1. Then f: ℜ→A, and f can be defined concisely by

$$\begin{cases} f(x) = 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } a \text{ is irrational} \end{cases}$$

Example 1.5: Let A = {a, b, c, d} and B = {x, y, z}. Let f:

A → B be defined by the diagram:



Notice that the functions in examples 1.1 and 1.4 are defined by specific formulas. But this need not always be the case, as is indicated by the other examples. The rules of correspondence which define functions can be diagrams as in example 1.5, can be geographical as in example 1.2, or, when the domain is finite, the correspondence can be listed for each element in the domain as in example 1.4.

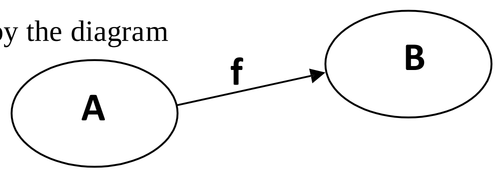
3.2 MAPPINGS, OPERATORS, TRANSFORMATIONS

If A and B are sets in general, not necessarily sets of numbers, then a function f of A into B is frequently called a mapping of A into B; and the notation

f: A ---> B is then read "f maps A into B". We can also denote a mapping, or function, f of A into B by

$$A \xrightarrow{f} B$$

Or by the diagram



If the domain and co-domain of a function are both the same set, say

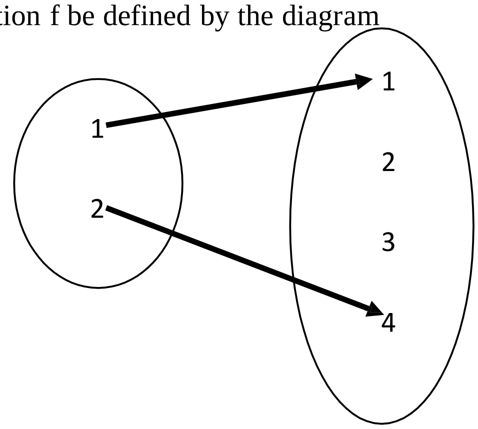
f: A ---> A then f is frequently called an ***operator*** or ***transformation*** on A. As we will see later operators are important special cases of functions.

3.3 EQUAL FUNCTIONS

If f and g are functions defined on the same domain D and if f (a) = g (a) for every a $\in$ D, then the functions f and g are equal and we write f = g

Example 2.1: Let f (x) = x^2 where x is a real number. Let g(x) = x^2 where x is a complex number. Then the function f is not equal to g since they have different domains.

Example 2.2: Let the function f be defined by the diagram



Let a function g be defined by the formula g(x) = x^2 where the domain of g is the set {1, 2}. Then f = g since they both have the same domain and since f and g assign the same image to each element in the domain.

Example 2.3: Let f: $\Re \rightarrow \Re$ and g:$\Re \rightarrow \Re$. Suppose f is defined by f(x) = x^2 and g by g(y) = y^2. Then f and g are equal functions, that is, f = g. Notice that x and y are merely dummy variable in the formulas defining the functions.

3.4 RANGE OF A FUNCTION

Let f be the mapping of A into B, that is, let f: A→ B. Each element in B need not appear as the image of an element in A. We define the range of f to consist precisely of those elements in B which appear and the image of at least one element in A. We denote the range of f: A →

By f(A)

f(A)

Notice that f(A) is a subset of B. i.e. f(A)

Example 3.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then the range of f consists of the positive real numbers and zero.

Example 3.2: Let f: A$\rightarrow$ B be the function in Example 1.3. Then f(A) = {b, c}

3.5 ONE – ONE (INJECTIVE) FUNCTIONS

Let f map A into B. Then f is called a ***one-one or Injective function*** if different elements in B are assigned to different elements in A, that is, if no two different elements in A have the same image. More briefly, f: A $\rightarrow$ B is one-one if f(a) = f(a') implies a = a` or, equivalently, a = a` implies f(a) $\neq$ f(a`)

Example 4.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^2$. Then f is not a one-one function since f(2) = f(-2) = 4, that is, since the image of two different real numbers, 2 and -2, is the same number, 4.

Example 4.2: Let the function f: $\Re \rightarrow \Re$ be defined by the formula $f(x) = x^3$. Then f is a one-one mapping since the cubes of the different real numbers are themselves different.

Example 4.3: The function f which assigns to each country in the world, its capital city is one-one since different countries have a different capital that is no city is the capital of two different countries.

3.6 ONTO (SUBJECTIVE) FUNCTION

Let f be a function of A into B. Then the range f(A) of the function f is a subset of B, that is, f(A) $\subset$ B. If f(A) = B, that is, if every member of B appears as the image of at least one element of A, then we say "f is a function of A onto B", or "f maps A onto B", of "f is an ***onto or Subjective function".***

Example 5.1: Let the function f: $\Re \rightarrow \Re$ be defined by the formula f(x) = x^2. Then f is not an onto function since the negative numbers do not appear in the range of f, that is no negative number is the square of a real number.

Example 5.2: Let f: A→ B be the function in Example 1.3. Notice that f(A) = {b, c}. Since B = {a, b, c} the range of f does not equal co-domain, i.e. is not onto.

Example 5.3: Let f: A → B be the function in example 1.5: Notice that

f(A) = {x, y, z} = B that is, the range of f is equal to the co-domain B. Thus f maps A onto B, i.e. f is an onto mapping.

3.7 IDENTITY FUNCTION

Let A be any set. Let the function f: A→ A be defined b y the formula f(x) = x, that is, let f assign to each element in A the element itself. Then f is called the identity function or the identity transformation on A. We denote this function by 1 or by ^{1}A.

3.8 CONSTANT FUNCTIONS

A function f of A onto B is called a ***constant function*** if the same element of b∈ B is assigned to every element in A. In other words, f: A → B is a constant function if the range of f consists of only one element.

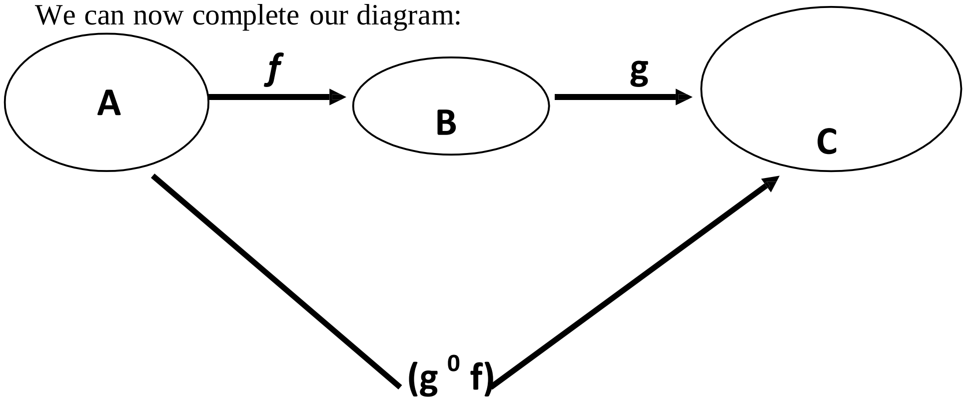
3.9 PRODUCT FUNCTION

Let F: A → B and g: B→C be two functions then the product of functions f and g is denoted

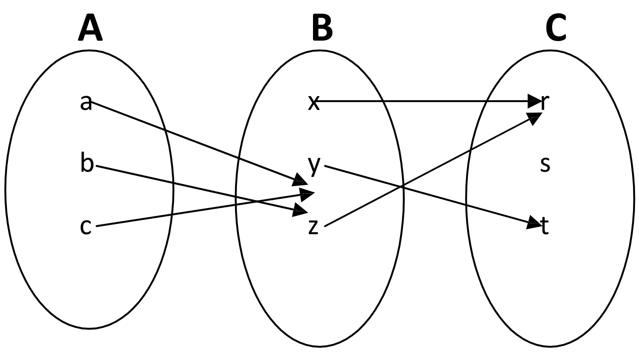
(g o f) : A → C defined by

(g o f)(a) = g (f(a))

We can now complete our diagram:



Example 7.1: Let f: A → B and g: B → C be defined by the diagrams



We compute (g o f): A → C by its definition:

(g o f)(a) = g(f(a)) = g(y) = t

(g o f)(b) = g(f(a)) = g(z) = r

(g o f)(c) = g(f(a)) = g(y) = t

Notice that the function (g o f) is equivalent to "following the arrows" from A to C in the diagrams of the functions f and g.

Example 7.2: To each real number let f assign its square, i.e. let f(x) = x^2. To each real number let g assign the number plus 3, i.e. let g(x) = x + 3. Then

(g o f)(x) = f(g(x)) = f(x+3) = $(x+3)^2 = x^2 + 6x + 9$

(g o f)(x) = g(f(x)) = g(x^2) = $x^2 + 3$

Remark 4.1: Let f: A → B. Then

$^{I}B \, ^{o} f = f$ and $f \, ^{o} \, ^{1}A = f$

that is, the product of any function and identity is the function itself.

3.9.1 Associativity of Products of Functions

Let f: A → B, g: B → C and h: c → D. Then, as illustrated in Figure 4-1, we can form the production function $g \, ^{o} f$: A → C, and then the function $h^{o} (g \, ^{o} f)$: A → D.

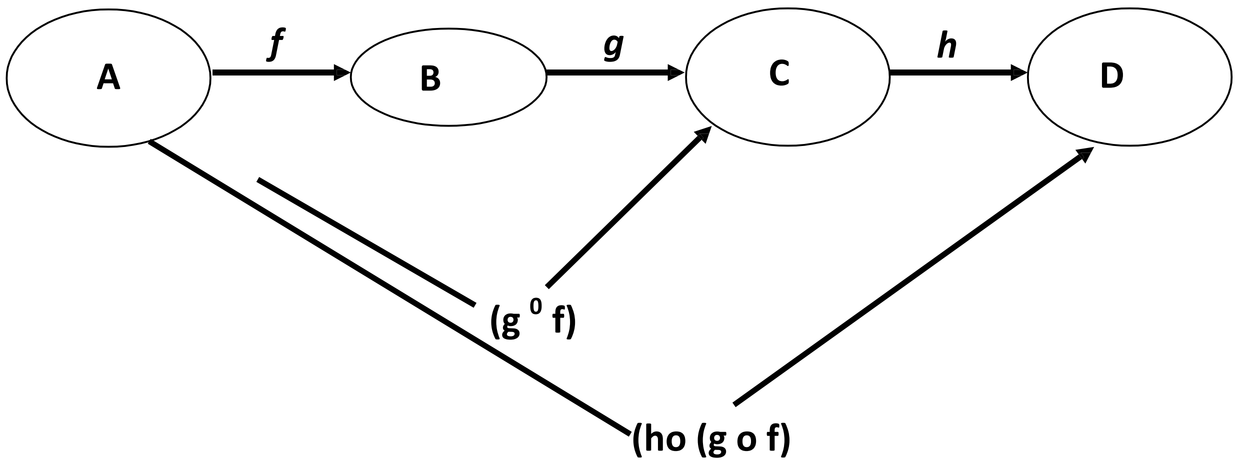


Fig. 4.1

Similarly, as illustrated in Figure 4-2, we can form the product function $h \, ^{o}$ g: B → D and then the function (h o g) o f: A → D.

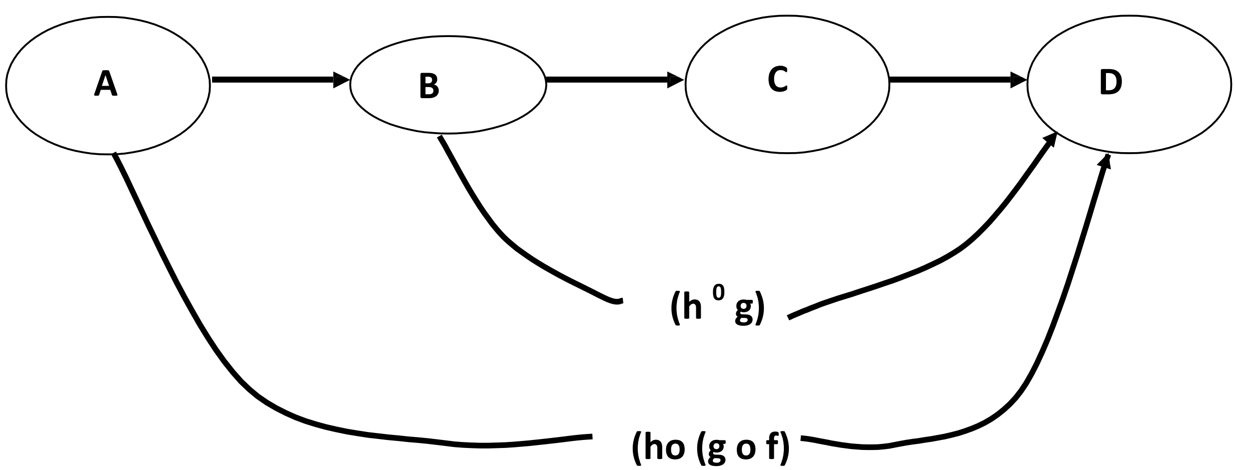


Fig 4.2

Both h ^{o} (g ^{o} f) and (h ^{o} g) o f are function of A into D. A basic theorem on functions states that these functions are equal. Specifically,

Theorem 4.1: Let f: A →B, B → C and h: C →D. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

In view of Theorem 4.1, we can write

$h^{o} g^{o} f$: A → D without any parenthesis.

3.10 INVERSE OF A FUNCTION

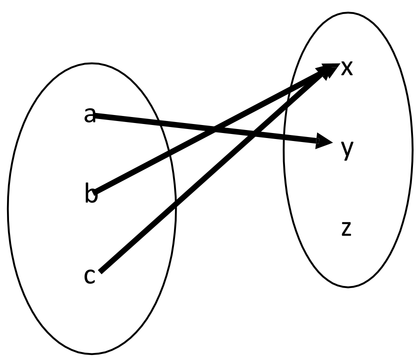
Let f be a function of A into B, and let b ∈ B. Then the ***inverse*** of b, denoted by $f^{-1}(b)$

Consist of those elements in A which are mapped onto b, that is, those element in A which have m as their image. More briefly, if f: A →B then

$f^{-1}(b) = \{x\}\ x \in A;\ f(x) = b\}$

Notice that $f^{-1}(b)$ is always a subset of A. We read f^{-1} as "f inverse".

Example 8.1: Let the function f: A → B be defined by the diagram



Then $f^{-1}(x) = \{b, c\}$, since both b and c have x as their image point.

Also, $f^{-1}(y) = \{a\}$, as only a is mapped into y. The inverse of z, $f^{-1}(z)$, is the null set ∅, since no element of A is mapped into z.

Example 8.2: Let f: $\Re \rightarrow \Re$, the real numbers, be defined by the formula f(x) = x^2. Then $f^{-1}(4) = \{2, -2\}$, since 4 is the image of both 2 and -2 and there is no other real number whose square is four. Notice that $f^{-1}(-3) = \varnothing$, since there is no element in $\Re$ whose square is -3.

Example 8.3: Let f be a function of the complex numbers into the complex numbers, where f is defined by the formula $f(x) = x^2$. Then $f^{-1}(3) = \{\sqrt{3}i, -\sqrt{3}i\}$,

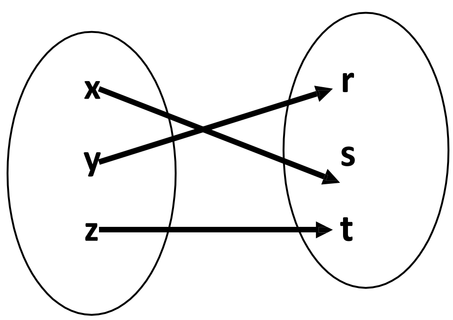
as the square of each of these numbers is -3.

Notice that the function in Example 8.2 and 8.3 are different although they are defined by the same formula.

We now extend the definition of the inverse of a function. Let f: A → B and let D be a subset of B, that is, $D \subset B$. Then the inverse of D under the mapping f, denoted by $f^{-1}(D)$, consists of those elements in A which are mapped onto some element in D. More briefly,

$f^{-1}(D) = \{x \mid x \in A, f(x) \in D\}$

Example 9.1: Let the function f: A → B be defined by the diagram



Then $f^{-1}(\{r, s\}) = \{y\}$, since only y is mapped into r or s. Also

$f^{-1}(\{r, t\}) = \{x, y, z\} = A$, since each element in A as its image r or t.

Example 9.2: Let f: $\Re \rightarrow \Re$ be defined by $f(x) = x^2$, and let

$$D = [4, 9] = \{x \mid 4 \leq x \leq 9\}$$

Then $f^{-1}(D) = \{x \mid -3 \le x \le -2 \text{ or } 2 \le x \le 3\}$

Example 9.3: Let f: A → B be any function. Then $f^{-1}(B) = A$, since every element in A has its image in B. If f(A) denote the range of the function f, then $f^{-1}(f(A)) = A$

Further, if $b \in B$, then $f^{-1}(b) = f^{-1}(\{b\})$

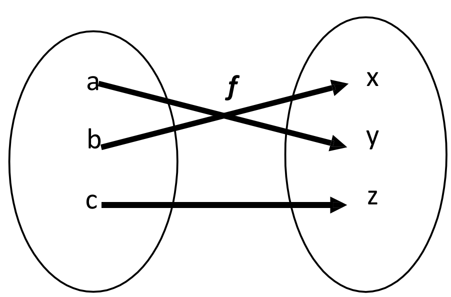
Here f^{-1} has two meanings, as the inverse of an element of B and as the inverse of a subset of B.

3.11 INVERSE FUNCTION

Let f be a function of A into B. In general, $f^{-1}(b)$ could consist of more than one element or might even be empty set ∅. Now if f: A → B is a one-one function and an onto function, then for each $b \in B$ the inverse $f^{-1}(b)$ will consist of a single element in A. We therefore have a rule that assigns to each b $\in$ B a unique element $f^{-1}(b)$ in A. Accordingly, f^{-1} is a function of B into A and we can write $f^{-1}: B \to A$

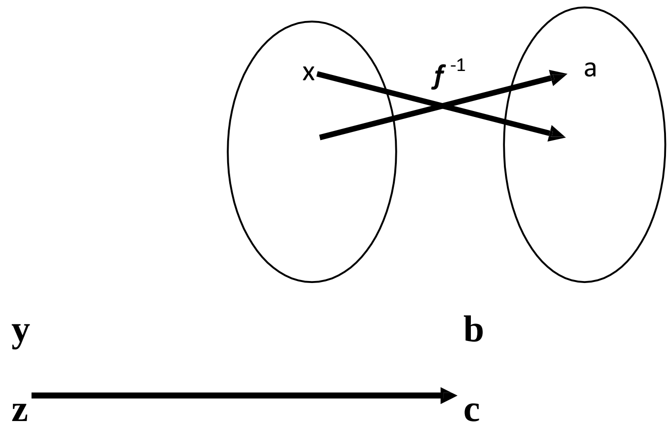
In this situation, when f: A ® B is one-one and onto, we call f^{-1} the inverse function of f.

Example 10.1: Let the function f: A → B be defined by the diagram

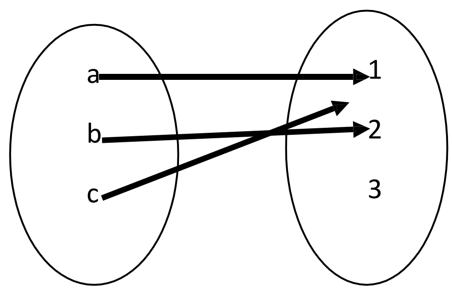


Notice that f is one-one and onto. Therefore f^{-1}, the inverse function exists.

We describe $f^{-1}: B \to A$ by the diagram



Example 6.1: Let the function f be defined by the diagram:

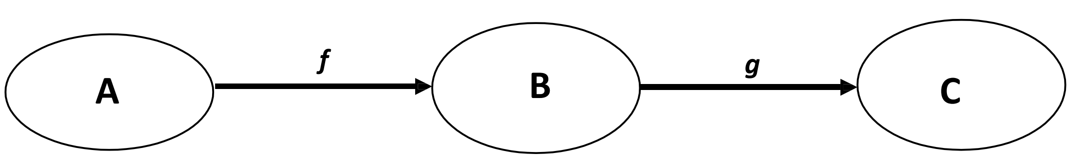


Then f is a constant function since 3 is assigned to every element in A.

Example 6.3: Let f: $\Re \rightarrow \Re$ be defined by the formula f(x) = 5. Then f is a constant Function since 5 is assigned to every element.

3.12 PRODUCT FUNCTION

Let f be a function of A and B and let g be a function of B, the co-domain of f, into C. We illustrate the function below.



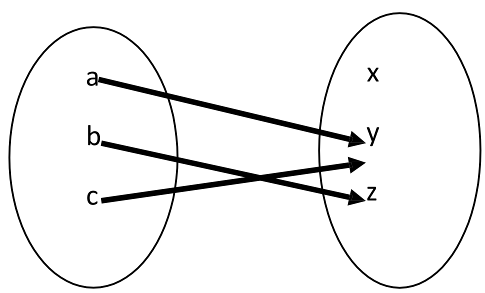
Let a ∈ A; then its image f (x) is in B which is the domain of g. Accordingly, we can find the image of f (a) under the mapping of g, that is, we can find g (f(a)). Thus, we have a rule which assigns to each element a∈ A a corresponding element (f(a)) ∈ C. In other words, we have a function of A into C. This new function is called the ***product function*** or ***composition function*** of f and g and it is denoted by (g^{o} f) or (gf)

More briefly, if f: A → B and g: B → C then we define a function

Notice further, that if we send the arrows in the opposite direction in the first diagram of f we essentially have the diagram of f^{-1}.

Example 10.2: Let the function f: A → B be defined by the diagram

Since f(a) = y and f(c) = y, the function f is not one-one. Therefore, the inverse function f^{-1} does not exist. As f^{-1} (y) = {a, c}, we cannot assign both a and c to the element y ∈ B.

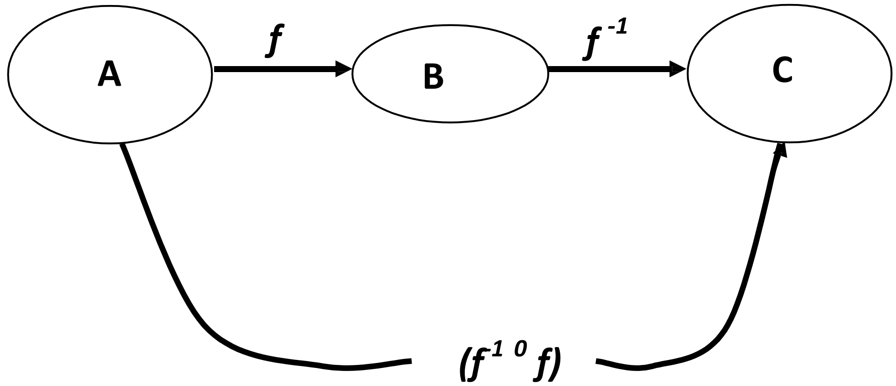


Example 10.3: Let f: ℜ →ℜ, the real numbers, be defined by f(x) = x^3. Notice that f is one-one and onto.

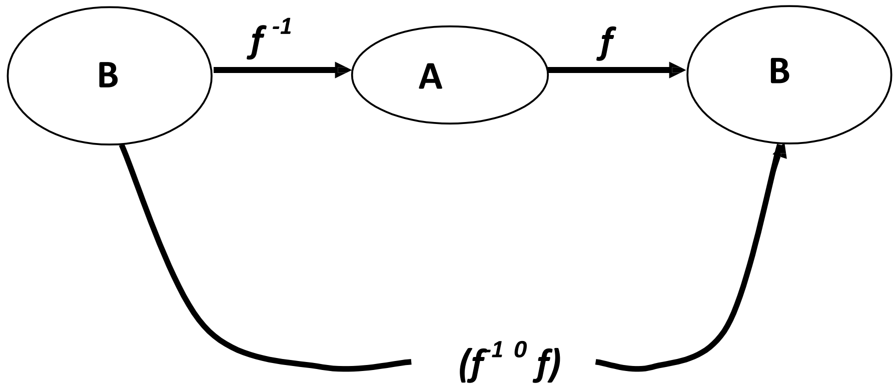
Hence f^{-1}: ℜ →ℜ exists. In fact, we have a formula which defines the inverse function, f^{-1} (x) = $\sqrt[3]{x}$.

3.12.1 Theorems on the inverse Function

Let a function f: A → B have an inverse function f^{-1}: B → A. Then we see by the diagram



That we can form the product ($f^{-1}\ ^{o}$ f) which maps A into A, and we see by the diagram



That we can form the product function (f o f^{-1}) which maps B into B. We now state the basic theorems on the inverse function:

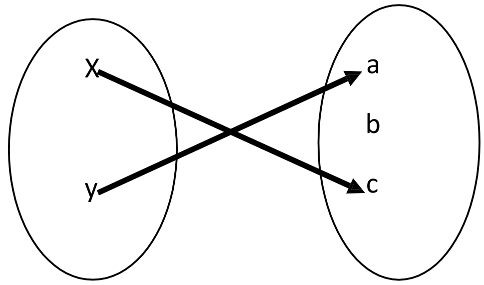
Theorem 4.2: Let the function f: A → B be one-one and onto; i.e. the inverse function $^{f\,-1}$: B → A exists. Then the product function

$$(f^{-1\,o} f): A \rightarrow A$$

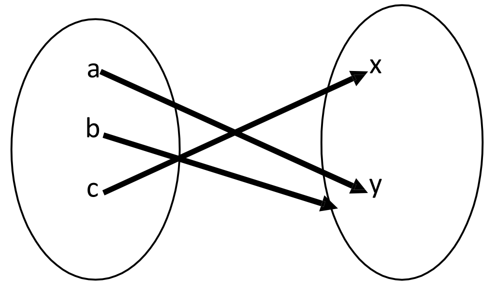
is the identity function on A, and the product function ($f\ ^{o}\ f^{-1}$): B → B is the identity function on B.

Theorem 4.3: Let f: A → B and g: B → A. Then g is the inverse function of f, i.e. g = f – 1, if the product functions ($g\ ^{o}$ f): A → A is the identity function on A and ($f\ ^{o}$ g): B → B is the identity function on B.

Both conditions are necessary in Theorem 4.3 as we shall see from the example below;



(a)



(b)

Now define a function g: B → A by the diagram (b) above.

We compute (g ^{o} f): A → A, (g ^{o} f)(x) = g (f(x)) = g(c) = x and (g ^{o} f)(y) = g (f(y)) = g(a) = y

Therefore the product function (g ^{o} f) is the identity function on A. But g is not the inverse function of f because the product function (f ^{o} G) is not the identity function on B, f not being an auto function.

4.0 CONCLUSION

I believe that by now you fully grasp the idea of functions, mappings and transformations. This knowledge will be built upon in subsequent units.

5.0 SUMMARY

Recall that in this unit we have studied concepts such as mappings and functions. We have also examined the concepts of one-to-one and onto functions. This concept has allowed us to explain equality between two sets. We also established in the unit that the inverse of f: A – B usually denoted f^{-1}, exit, if f is a one-to-one and onto function.

It is instructive to note that Inverse function is not studied in isolation but more importantly a useful and powerful tool in understanding calculus.

6.0 TUTOR – MARKED ASSIGNMENTS

1. Let the function f: R# → R# be defined by f (X) = {1 if x is rational] {-1 if x is irrational.

a. Express f in words

b. Suppose the ordered pairs (x + y, 1) and (3, x – y) are equal.

Find x and y.

2. Let M = {1, 2, 3, 4, 5} and let the function f: M → $\Re$ be defined by f(x) = x^2 + 2x -1 Find the graph of f.

3. Prove: A x (B ∩ C) = (A x B) ∩ (A x C)

4. Prove A ⊂ B and C ⊂ D implies (A x C) ⊂ (B x D).

7.0 REFERENCES AND FURTHER READINGS

Seymour L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

Sunday, O.I. (1998). Introduction to Real Analysis (Real – valued functions of a real variable), Vol. 1.

MODULE TWO

Unit 1 Annuity

Unit 2 Cash Flow

Unit 3 Sinking Fund

Unit 4 Functions

UNIT 1: ANNUITY

CONTENTS

1.0 Introduction

2.0 Objectives

3.0 Main Body

3.1 Annuity

3.1.1 Ordinary Annuity

3.1.2 Annuity Due

3.2 Valuation

3.3 Types of Annuity

4.0 Conclusion

5.0 Summary

6.0 Tutor-Marked assignment

7.0 References and Further readings

1.0 INTRODUCTION

At some point in life, one may have had to make a series of fixed payments over a period of time, such as rent or car payment or have received a series of payments over a period of time, such as bond coupons. These are called annuities. If you understand the time value of money, you are ready to learn about annuities and how their present and future values are calculated.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Define annuity and future values
- Calculate the future value of an ordinary annuity
- Calculate the present value of an ordinary annuity

3.0 MAIN BODY

3.1 MEANING OF ANNUITY

Annuities are essentially a series or fixed payments required from you or paid to you at a specified frequency over the course of a fixed time period. The most common payment frequencies are yearly, semi-annually (twice a year), quarterly and monthly. There are two types of annuities and annuities due.

A financial product sold by financial institutions that is designed to accept and grow funds from an individual and then, upon annuitization, pay out a stream of payments to the individual at a later point in time. Annuities are primarily used as a means of securing a steady cash flow for an individual during their retirement years.

Annuities can be structured according to a wide array of details and factors, such as the duration of time that payments from the annuity can be guaranteed to continue. Annuities can be created so that, upon annuitization, payments will continue so long as either the annuitant or their spouse is alive. Alternatively, annuities can be structured to pay out funds for a fixed amount of time, such as 20 years, regardless of how long the annuitant lives.

Annuities can be structured to provide fixed periodic payments to the annuitant or variable payments. The intent of variable annuities is to allow the annuitant to receive greater payments if investments of the annuity fund do well and smaller payments if its investments do poorly. This provides for a less stable cash flow than a fixed annuity, but allows the annuitant to reap the benefits of strong returns from their fund's investments.

The different ways in which annuities can be structured provide individuals seeking annuities the flexibility to construct an annuity contract that will best meet their needs.

3.1.1 Ordinary Annuity

Ordinary Annuity: Payments are required at the end of each period. For example, straight bonds usually pay coupon payments at the end of every six months until the bonds maturity date.

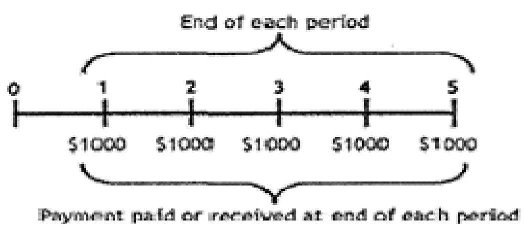
Calculating the Present and Future Value of Annuities

At some point in your life, you may have had to make a series of fixed payments over a period of time - such as rent or car payments - or have received a series of payments over a period of time, such as bond coupons. These are called annuities. If you understand the time value of money, you're ready to learn about annuities and how their present and future values are calculated.

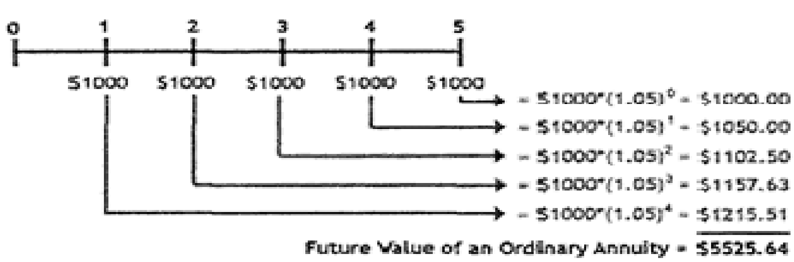
Calculating the Future Value of an Ordinary Annuity

If you know how much you can invest per period for a certain time period, the future value of an ordinary annuity formula is useful for finding out how much you would have in the future by investing at your given interest rate. If you are making payments on a loan, the future value is useful in determining the total cost of the loan.

Lets now run through Example 1. Consider the following annuity cash flow Schedule:



To calculate the future value of the annuity, we have to calculate the future value of each cash flow. Let's assume that you are receiving $1,000 every year for the next five years, and you invested each payment at 5%. The following diagram shows how much you would have at the end of the five-year period:



Since we have to add the future value of each payment, you may have noticed that if you have an ordinary annuity with many cash flows, it would take a long time to calculate all the future values and then add them together. Fortunately, mathematics provides a formula that serves as a shortcut for finding the accumulated value of all cash flows received from an ordinary annuity:

$$FV_{Ordinary\ Annuity} = C * \left[\frac{(1+i)^n - 1}{i}\right]$$

C = Cash flow per period

i = interest rate

n = number of payments

Using the above formula for Example 1 above, this is the result:

$$FV_{Ordinary\ Annuity} = \$1000 * \left[\frac{(1+0.05)^5 - 1}{0.05}\right]$$

= $1000*[5.53]

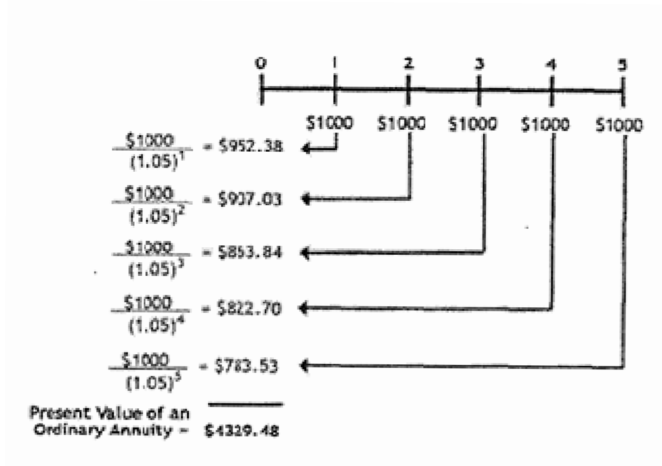
= $ 5525.63

Note that the 1 cent difference between $5,525.64 and $5,525.63 is due to a rounding error in the first calculation. Each value of the first calculation must be rounded to the nearest penny - the more you have to round numbers in a calculation, the more likely rounding errors will occur. So, the above formula not only provides a shortcut to finding FV of an ordinary annuity but also gives a more accurate result.

Calculating the Present Value of an Ordinary Annuity

If you would like to determine today's value of a future payment series, you need to use the formula that calculates the present value of an ordinary annuity. This is the formula you would use as part of a bond pricing calculation. The PV of an ordinary annuity calculates the present value of the coupon payments that you will receive in the future. More than 50% of retirement age individuals to not have enough savings

For Example 2, we'll use the same annuity cash flow schedule as we did in Example 1. To obtain the total discounted value, we need to take the present value of each future payment and, as we did in Example 1, add the cash flows.



Again, calculating and adding all these values will take a considerable amount of time, especially if we expect many future payments. As such, we can use a mathematical shortcut for PV of an ordinary annuity.

$$PV_{Ordinary\ Annuity} = C * \left[\frac{1-(1+i)^{-n}}{i}\right]$$

C = Cash flow per period

i = interest rate

n = number of payments

The formula provides us with the PV in a few easy steps. Here is the calculation of the annuity represented in the diagram for Example 2:

$$PV_{Ordinary\ Annuity} = \$1000 * \left[\frac{1-(1+0.05)^{-5}}{0.05}\right]$$

= $1000*[4.33]

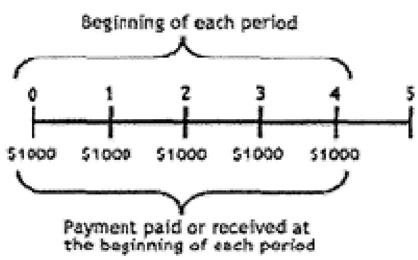
= $4329.4

3.1.2 Annuity Due:

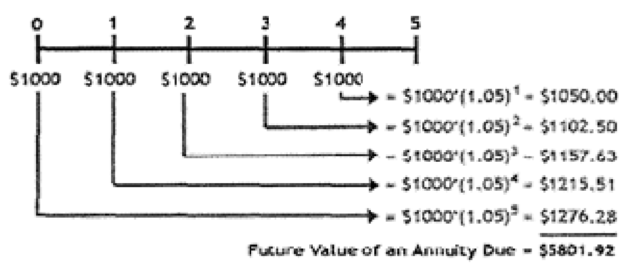
Annuity Due: Payments are required at the beginning of each period. Rent is an example of annuity due. You are usually required to pay rent when you first move in at the beginning of the month, and then on the first of each month thereafter.

Calculating the Future Value of an Annuity Due

When you are receiving or paying cash flows for an annuity due, your cash flow schedule would appear as follows:



Since each payment in the series is made one period sooner, we need to discount the formula one period back. A slight modification to the FV-of-an-ordinary-annuity formula accounts for payments occurring at the beginning of each period. In Example 3, let's illustrate why this modification is needed when each $1,000 payment is made at the beginning of the period rather than at the end (interest rate is still 5%):



Notice that when payments are made at the beginning of the period, each amount is held longer at the end of the period. For example, if the $1,000 was invested on January 1 rather than December 31 each year, the last payment before we value our investment at the end of five years (on December 31) would have been made a year prior (January 1) rather than the same day on which it is valued. The future value of annuity formula would then read:

$$FV_{\text{Annuity Due}} = C * \left[\frac{(1+i)^n - 1}{i}\right] * (1+i)$$

Therefore:

$$FV_{\text{Annuity Due}} = \$1000 * \left[\frac{(1+0.05)^5 - 1}{0.05}\right] * (1+0.05)$$

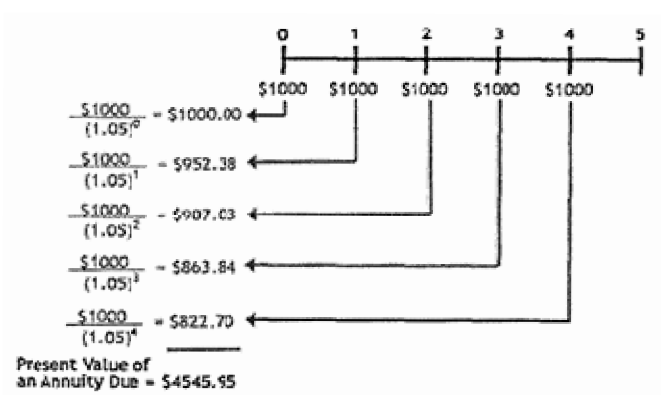
= $1000*5.53*1.05

= $5801.91

Calculating the Present Value of an Annuity Due

For the present value of an annuity due formula, we need to discount the formula one period forward as the payments are held for a lesser amount of time. When calculating the present value, we assume that the first payment was made today.

We could use this formula for calculating the present value of your future rent payments as specified in a lease you sign with your landlord. Let's say for Example 4 that you make your first rent payment at the beginning of the month and are evaluating the present value of your five-month lease on that same day. Your present value calculation would work as follows:



Of course, we can use a formula shortcut to calculate the present value of an annuity due:

$$PV_{Annuity\ Due} = C * \left[\frac{1-(1+i)^{-n}}{i}\right] * (1+i)$$

Therefore:

$$PV_{Annuity\ Due} = \$1000 * \left[\frac{1-(1+0.05)^{-5}}{0.05}\right] * (1+0.05)$$

= \$1000*4.33*1.05

= \$4545.95

Recall that the present value of an ordinary annuity returned a value of \$4,329.48. The present value of an ordinary annuity is less than that of an annuity due because the further back we discount a future payment, the lower its present value: each payment or cash flow in an ordinary annuity occurs one period further into the future.

3.2 VALUATION

The valuation of an annuity entails concepts such as time value of money, interest rate, and future value.

Annuity-immediate

If the number of payments is known in advance, the annuity is an annuity-certain. If the payments are made at the end of the time periods, so that interest is accumulated before the payment, the annuity is called an annuity-immediate, or ordinary annuity. Mortgage payments are annuity-immediate, interest is earned before being paid.

↓ ↓ ... ↓ payments

______ ______ ______ ______ __

0 1 2 ... n periods

The present value of an annuity is the value of a stream of payments, discounted by the interest rate to account for the fact that payments are being made at various moments in the future. The present value is given in actuarial notation by:

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i},$$

Where n is the number of terms and i is the per period interest rate. Present value is linear in the amount of payments, therefore the present value for payments, or rent R is:

$$PV(i,n,R) = R \times a_{\overline{n}|i}$$

In practice, often loans are stated per annum while interest is compounded and payments are made monthly. In this case, the interest I is stated as a nominal interest rate, and i=I/12.

The future value of an annuity is the accumulated amount, including payments and interest, of a stream of payments made to an interest-bearing account. For an annuity-immediate, it is the value immediately after the n-th payment. The future value is given by:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

Where n is the number of terms and i is the per period interest rate. Future value is linear in the amount of payments, therefore the future value for payments, or rent R is:

$$FV(i, n, R) = R \times s_{\overline{n}|i}$$

Example 1: The present value of a 5 year annuity with nominal annual interest rate 12% and monthly payments of $100 is:

$$PV(0.12/12, 5 \times 12, \$100) = \$100 \times a_{\overline{60}|0.01} = \$4,495.50$$

The rent is understood as either the amount paid at the end of each period in return for an amount PV borrowed at time zero, the principal of the loan, or the amount paid out by an interest-bearing account at the end of each period when the amount PV is invested at time zero, and the account becomes zero with the n-th withdrawal.

Future and present values are related as:

$$s_{\overline{n}|i} = (1+i)^n \times a_{\overline{n}|i}$$

and

$$\frac{1}{a_{\overline{n}|i}} - \frac{1}{s_{\overline{n}|i}} = i$$

Annuity-due

An annuity-due is an annuity whose payments are made at the beginning of each period. Deposits in savings, rent or lease payments, and insurance premiums are examples of annuities due.

↓ ↓ ... ↓ payments

______ ______ ______ ______ __

0 1 ... n-1 n periods

Because each annuity payment is allowed to compound for one extra period. Thus, the present and future values of an annuity-due can be calculated through the formula:

and $$\ddot{a}_{\overline{n}|i} = (1+i) \times a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{d}$$

$$\ddot{s}_{\overline{n}|i} = (1+i) \times s_{\overline{n}|i} = \frac{(1+i)^{n}-1}{d}$$

where n are the number of terms, i is the per term interest rate, and d is the effective rate of discount given by d=i/(i+1).

Future and present values for annuities due are related as:

$$\ddot{s}_{\overline{n}|i} = (1+i)^{n} \times \ddot{a}_{\overline{n}|i}$$

and

$$\frac{1}{\ddot{a}_{\overline{n}|i}} - \frac{1}{\ddot{s}_{\overline{n}|i}} = d$$

Example 2: The final value of a 7 year annuity-due with nominal annual interest rate 9% and monthly payments of $100:

$$FV_{due}(0.09/12, 7 \times 12, \$100) = \$100 \times \ddot{a}_{\overline{84}|0.0075} = \$11,730.01.$$

Note that in Excel, the PV and FV functions take on optional fifth argument which selects from annuity-immediate or annuity-due.

An annuity-due with n payments is the sum of one annuity payment now and an ordinary annuity with one payment less, and also equal, with a time shift, to an ordinary annuity. Thus we have:

$\ddot{a}_{\overline{n}|i} = a_{\overline{n}|i}(1+i) = a_{\overline{n-1}|i} + 1$ (value at the time of the first of *n* payments of 1)

$\ddot{s}_{\overline{n}|i} = s_{\overline{n}|i}(1+i) = s_{\overline{n+1}|i} - 1$ (value one period after the time of the last of n payments of 1)

Perpetuity

A perpetuity is an annuity for which the payments continue forever. Since:

$$\lim_{n\to\infty} PV(i,n,R) = \frac{R}{i}$$

even a perpetuity has a finite present value when there is a non-zero discount rate. The formula for a perpetuity are:

$$a_{\overline{\infty}|i} = 1/i; \qquad \ddot{a}_{\overline{\infty}|i} = 1/d.$$

where i is the interest rate and d=i/(1+i) is the effective discount rate.

Proof of annuity formula

To calculate present value, the k-th payment must be discounted to the present by dividing by the interest, compounded by k terms. Hence the contribution of the k-th payment R would be R/(1+i)^k. Just considering R to be one, then:

$$a_{\overline{n}|i} = \sum_{k=1}^{n} \frac{1}{(1+i)^k} = \left(\frac{1}{1+i} - \frac{1}{(1+i)^{n+1}}\right) \sum_{k=0}^{\infty} \frac{1}{(1+i)^k}$$

We notice that the second factor is an infinite geometric progression of the form, therefore, $\sum_{k=0}^{\infty} \kappa^k = \frac{1}{1-\kappa}$

$$a_{\overline{n}|i} = \left(\frac{(1+i)^n - 1}{(1+i)^{n+1}}\right)\left(\frac{1}{1-1/(1+i)}\right) = \left(\frac{1-(1+i)^{-n}}{1+i}\right)\left(\frac{1+i}{i}\right) = \frac{1-(1+i)^{-n}}{i}.$$

Similarly, we can prove the formula for the future value. The payment made at the end of the last year would accumulate no interest and the payment made at the end of the first year would accumulate interest for a total of (n−1) years. Therefore,

$$s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = (1+i)^n a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

Amortization calculations

If an annuity is for repaying a debt P with interest, the amount owed after n payments is:

$$\frac{R}{i} - (1+i)^n\left(\frac{R}{i} - P\right)$$

because the scheme is equivalent with borrowin g the amount R/i to create a perpetuity with coupon R, and putting R/i-P of that borrowed amount in the bank to grow with interest i.

Also, this can be thought of as the present value of the remaining payments:

$$R\left[\frac{1}{i} - \frac{(i+1)^{n-N}}{i}\right] = R \times a_{\overline{N-n}|i}$$

3.3 WHAT ARE THE DIFFERENT TYPES OF ANNUITIES?

The large number of annuity products on the market today can make understanding them difficult. But in fact, there are only a handful of different types of annuities, and we will help you find the best types to suit your needs. First, let's discuss the three primary considerations when thinking about annuities:

Timing of payout — immediate or deferred:

The first thing to determine is if you need an immediate or deferred annuity. Read on to learn the difference.

Immediate Annuities

In an immediate annuity, the investor begins to receive payments immediately upon investing. This is for investors that need immediate income from their annuity. When you purchase an immediate annuity you can choose between payments for a certain period of time (typically five to twenty years - "period certain"), payments for the rest of your life and/or your spouse's life, or any combination of the two. You can even choose between a fixed payment that doesn't vary or a variable payment that is based on market performance.

Deferred Annuities

In a deferred annuity, you typically receive payments starting at some future date, usually at retirement. However, most deferred annuities allow for systematic withdrawal payments beginning thirty days after the purchase of your annuity, up to 10% per year, in most cases. With a deferred annuity you can invest either a lump sum all at once, or make periodic payments, either fixed or variable. Those funds grow tax-deferred until you're ready to begin receiving payments. Deferred annuities make up a large majority of all annuity sales in the United States, and are the type of annuity that Annuity FYI generally recommends if you do not need immediate income from your annuity.

Investment type — fixed or variable:

The next decision to make is the investment type best suited to your needs: fixed or variable.

Fixed Annuities

Fixed annuities are invested primarily in government securities and high-grade corporate bonds. They offer a guaranteed rate of return, typically over a period of one to fifteen years. There are two basic types of fixed annuities: the Guaranteed Return Annuities (GRA) is a fixed annuity that offers a guarantee that you can never receive less than 100% of your investment — no penalties or fluctuations in the interest rate market can impact your principal should you surrender. The Market Value Adjustment annuity (MVA) works much like the GRA, but there is no guarantee of your principal if rates rise and you surrender your contract. MVAs work like a bond and often pay more than a GRA due to the increased short-term risk of rising rates. It is important to note that, unlike a variable annuity, where your funds are held separately from the insurance company, with a fixed annuity your assets are part of the general accounts of the insurer, and are subject to the claims-paying ability of the issuing company. For this reason it is important to understand the financial strength of the issuing insurance company before you buy a fixed annuity. See our section on financial strength ratings for more information.

Variable Annuities

A variable annuity is a contract between you and an issuer whereby you agree to give the issuer principal and in return the issuer guarantees you variable payments over time. While annuities are not insurance policies, they are issued by insurance companies. Variable annuities are different than their fixed annuity cousins, which are invested primarily in government securities and high-grade corporate bonds, and offer exclusively a guaranteed rate, typically over a period of one to ten years.

There are several distinctive features of variable annuities:

Invest Unlimited Funds, Tax-Deferred

A variable annuity is similar to a retirement plan in that you can fund it in a lump sum or a little at a time, and all capital in an annuity grows and compounds tax-deferred until you begin making withdrawals. Unlike retirement plans, however, there is no limit as to how much you can invest in annuities! With a variable annuity you can invest unlimited funds, tax deferred (unlike IRA, 401ks, and other retirement vehicles, which have annual maximum contributions). Variable annuities can grow tax deferred until withdrawn.

Wide Variety of Investment Options to Permit Stock Market Gains

Variable annuities enable you to invest in a selection of portfolios, called sub-accounts. These sub-accounts are tied to market performance, and often have a corresponding managed investment after which they are modeled. Available choices range from the most conservative, such as money market, guaranteed fixed accounts, and government bond funds, to more aggressive such as growth, small cap, mid cap, large cap, capital appreciation, aggressive growth, and emerging markets investments. You can invest in fixed accounts, money market, domestic, international, specialty, sector funds and small, medium and large cap funds. Some have as many as forty or more investment choices with ten or more managers, and allow you to switch between them at no cost and without taxes (although excessive changes to your contract could result in the imposition of a small fee, so be sure to consult your financial planner or prospectus if you are making regular changes). Variable annuities typically allow a wide variety of investment options including . Many variable annuities allow for transferring among sub-accounts free of charge.

Living Benefits

Unlike mutual funds, variable annuity products come with optional living benefits and death benefits. One special type of variable annuity benefit is the GMIB (Guaranteed Minimum Income Benefit). The most competitive GMIBs guarantee at least a 5% return over seven years, or the highest attained value on each anniversary during the surrender period, whichever is greater. In exchange for this living guarantee, the living benefit annuity has a surrender charge / penalty for early withdrawal (typically 7 years), no up-front bonus, and a slightly higher annual fee (.25% to .50% per year). Lifetime Income Benefit annuities are also very popular. The concept behind a Lifetime Income Benefit is simple. If you purchase a Lifetime Income Benefit Rider with your variable annuity, the insurance company guarantees a regular monthly, quarterly, or annual payment for your lifetime, even if your account balance goes to zero -- income you can never outlive.

Death Benefits

Another important feature of some annuities is the death benefit provision. The annuity issuer guarantees at a minimum that upon your death your total premiums invested are paid to your beneficiaries. Many annuities "step-up" on the anniversary of the date the annuity was purchased, to the highest value at any preceding anniversary; or guarantee a minimum 5% to 7% interest compounded annually, whichever is greater. Some variable annuities now even offer a combination of the aforementioned benefits, i.e. the greater of 5% or 7% compounded annually, the highest contract anniversary or the actual account value on death to the heirs (see death benefits under compare annuities for more detailed information). For example, assume you invest $10,000 in a variable annuity with an annual step-up, and over the next several years your contract grows to $40,000 on its anniversary. Now assume that the market goes down and your value drops to $25,000, and just as you thought things couldn't get any worse, you die. In this hypothetical scenario your heirs would receive the highest contract anniversary of $40,000. The enhanced death benefit options offered by insurance companies come at an additional expense, typically ranging from 0.05% to 0.50% on top of the regular annual expenses. Furthermore unlike a death benefit from a life insurance policy, the death benefit associated with an annuity does not transfer to the beneficiaries income tax free. That said, you don't have to qualify for the annuity death benefit either.

Liquidity options:

Finally, you will need to determine which liquidity option best suits your needs: those with or without withdrawal penalties.

Annuities with Withdrawal Penalties

"No-surrender" annuities allow you to withdraw either your interest earnings or up to 15% per year without a penalty (although any withdrawal from an annuity may be subject to taxes and a 10% federal penalty if taken prior to 59? years of age). Beyond that, most annuities have a surrender charge — a penalty for making an early withdrawal above the free withdrawal amount. Typically this surrender charge decreases over a seven-year period.

Why would you choose an annuity with a withdrawal penalty? Well, some annuities with surrender charges reward the investor by offering a "bonus": the insurance company adds on average 3% to 5% to the amount of your principal. For example, if you invest $10,000 in a bonus annuity the insurance company will add $300 to $500 to your annuity immediately. The trade-off is that with a bonus annuity the surrender period is usually longer (eight to nine years in most cases versus the typical seven-year surrender). Be aware, some insurance companies charge higher fees on their bonus annuities, as compared with their standard products. Be certain to compare the annual fees of a bonus annuity to the standard or traditional (no-bonus with 7 years of surrender) annuity.

Sometimes the life insurance company will raise their fees to pay for the bonus.

Annuities without Withdrawal Penalties

For investors who may need spur-the-moment access to their money, there are annuities without surrender charges (no-surrender or level load annuities) — these annuities have no penalty or charge for early withdrawal. (That said, even with a no-surrender annuity investors under the age of 59 ? are subject to a 10% federal excise tax as well as ordinary income taxes on any gains. You can avoid any taxes or penalties, however, by making a 1035 Tax-Free Exchange to another annuity, regardless of age.) No-surrender annuities do not come with bonuses, and some insurance companies charge higher fees for their no-surrender charge products, so be sure to compare all fees before you invest.

Most companies now offer no-surrender annuities. However, if you are asking a local broker or agent for their recommendation they may not share with you the no-surrender annuity, as these annuities pay the broker a much lower fee. Some agents will even try to steer the investor to annuities with surrender charges but without bonuses.

4.0 CONCLUSION

Now you can see how annuity affects how you calculate the present and future value of any amount of money. Remember that the payment frequencies, or number of payments, and the time at which these payments are made (whether at the beginning or end of each payment period) are all variables you need to account for in your calculations.

5.0 SUMMARY

Annuities offer tax-sheltered growth, which can result in significant long-term returns for you if you contribute to the annuity for a long period and wait to withdraw funds until retirement. You get peace of mind from an annuity's guaranteed income stream, and the tax benefits of deferred annuities can amount to substantial savings.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the periodic payment of an annuity due of $70000, payable annually for 3 years at 15% compounded annually.

2. Find the periodic payment of an annuity due of $250700, payable quarterly for 8 years at 5% compounded quarterly.

3. Find the periodic payment of an accumulated value of $55000, payable monthly for 3 years at 15% compounded monthly.

4. Find the periodic payment of an accumulated value of $1600000, payable annually for 3 years at 9% compounded annually.

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UNIT 2 CASH FLOW

CONTENTS

1.0 INTRODUCTION

Cash flow is the movement of money into or out of a business, project, or financial product. It is usually measured during a specified, limited period of time. Measurement of cash flow can be used for calculating other parameters that give information on a company's value and situation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

• Define cash flows

• Discuss classification of cash flow

• Discuss how cash flow could be manage

3.0 MAIN BODY

3.1 DEFINITION OF CASH FLOW

Cash flow can be used, for example, for calculating parameters: it discloses cash movements over the period.

• To determine a project's rate of return or value. The time of cash flows into and out of projects are used as inputs in financial models such as internal rate of return and net present value.

• To determine problems with a business's liquidity. Being profitable does not necessarily mean being liquid. A company can fail because of a shortage of cash even while profitable.

• As an alternative measure of a business's profits when it is believed that accrual accounting concepts do not represent economic realities. For instance, a company may be notionally profitable but generating little operational cash (as may be the case for a company that barters its products rather than selling for cash). In such a case, the company may be deriving additional operating cash by issuing shares or raising additional debt finance.

• Cash flow can be used to evaluate the 'quality' of income generated by accrual accounting. When net income is composed of large non-cash items it is considered low quality.

• To evaluate the risks within a financial product, e.g., matching cash requirements, evaluating default risk, re-investment requirements, etc.

Cash flow notion is based loosely on cash flow statement accounting standards. It's flexible as it can refer to time intervals spanning over past-future. It can refer to the total of all flows involved or a subset of those flows. Subset terms include net cash flow, operating cash flow and free cash flow.

1. A revenue or expense stream that changes a cash account over a given period. Cash inflows usually arise from one of three activities - financing, operations or investing - although this also occurs as a result of donations or gifts in the case of personal finance. Cash outflows result from expenses or investments. This holds true for both business and personal finance.

2. An accounting statement called the "statement of cash flows", which shows the amount of cash generated and used by a company in a given period. It is calculated by adding noncash charges (such as depreciation) to net income after taxes. Cash flow can be attributed to a specific project, or to a business as a whole. Cash flow can be used as an indication of a company's financial strength.

3.1.1 Business financials

The (total) net cash flow of a company over a period (typically a quarter, half year, or a full year) is equal to the change in cash balance over this period: positive if the cash balance increases (more cash becomes available), negative if the cash balance decreases. The total net cash flow is the sum of cash flows that are classified in three areas:

1. Operational cash flows

In accounting, a measure of the amount of cash generated by a company's normal business operations. Operating cash flow is important because it indicates whether a company is able to generate sufficient positive cash flow to maintain and grow its operations, or whether it may require external financing. OCF is calculated by adjusting net income for items such as depreciation, changes to accounts receivable and changes in inventory.

Financial analysts sometimes prefer to look at cash flow metrics because it strips away certain accounting effects and is thought to provide a clearer picture of the current reality of the business operations. For example, booking a large sale provides a big boost to revenue, but if the company is having a hard time collecting the cash, then it is not a true economic benefit to the company. On the other hand, a company may be highly profitable on a cash flow basis, but may not have a low net income if it has a lot of fixed assets and uses accelerated depreciation calculations.

Example: Hammett, Inc. has sales of #19,570, cost of #9460, depreciation expense of #2,130, and interest expense of #1620. If the tax rate is 35 percent, what is the operating cash flow:

How to Calculate Operating Cash Flow.

The formula is:

EBIT (earnings before interest and taxes) + Depreciation - Taxes

Therefore:

19,570 Sales

(9,460) COS

(1,620) Interest

(2,130) Depreciation

6,360 Net Income B4 Tax

(2,226) Tax @ 35%

4,134 Net Income

2,130 Depreciation (Add Back)

1,620 Interest (Add Back)

7,884 Operating Cash Flow

Cash Flow Indicator Ratios: Free Cash Flow/Operating Cash Flow Ratio

The free cash flow/operating cash flow ratio measures the relationship between free cash flow and operating cash flow.

Free cash flow is most often defined as operating cash flow minus capital expenditures, which, in analytical terms, are considered to be an essential outflow of funds to maintain a company's competitiveness and efficiency.

The cash flow remaining after this deduction is considered "free" cash flow, which becomes available to a company to use for expansion, acquisitions, and/or financial stability to weather difficult market conditions. The higher the percentage of free cash flow embedded in a company's operating cash flow, the greater the financial strength of the company.

Formula:

$$\text{FCF/OCF Ratio} = \frac{\text{Free Cash Flow (Operating Cash Flow - Capital Expenditure)}}{\text{Operating Cash Flow}}$$

$$\text{Free Cash Flow/Operating Cash Flow Ratio} = \frac{\$622.9}{\$878.2} = 70.9\%$$

As of December 31, 2010, with amounts expressed in millions, Zimmer Holdings had free cash flow of $622.9. We calculated this figure by classifying "additions to instruments" and "additions to property, plant and equipment (PP&E)" as capital expenditures (numerator). Operating cash flow, or "net cash provided by operating activities" (denominator), is recorded at $878.2. All the numbers used in the formula are in the cash flow statement. By dividing, the equation gives us a free cash flow/operating cash flow ratio of 70.9%, which is a very high, beneficial relationship for the company.

2. Investment cash flows

An item on the cash flow statement that reports the aggregate change in a company's cash position resulting from any gains (or losses) from investments in the financial markets and operating subsidiaries, and changes resulting from amounts spent on investments in capital assets such as plant and equipment.

When analyzing a company's cash flow statement, it is important to consider each of the various sections which contribute to the overall change in cash position. In many cases, a firm may have negative overall cash flow for a given quarter, but if the company can generate positive cash flow from its business operations, the negative overall cash flow may be a result of heavy investment expenditures, which is not necessarily a bad thing.

Cash Flow Statement: Analyzing Cash Flow from Investing Activities

The cash flow statement is one of the most revealing documents of a firm’s financial statements, but it is often overlooked. It shows the sources and uses of a firm’s cash as it moves both in and out. When analyzing a company's cash flow statement, it is important to consider each of the various sections that contribute to the overall change in cash position. In many cases, a firm may have negative overall cash flow for a given quarter, but if the company can generate positive cash flow from its business operations, the negative overall cash flow is not necessarily a bad thing. Below we will cover cash flow from investing activities, one of three primary categories in the statement of cash flows.

Introduction to Cash Flow from Investing Activities

An item on the cash flow statement belongs in the investing activities section if it results from any gains (or losses) from investments in financial markets and operating subsidiaries. An investing activity refers to cash spent on investments in capital assets such as plant and equipment, which is collectively referred to as capital expenditure, or capex.

Here is a simple cash flow (of investing activities) for restaurant chain Texas Roadhouse

Texas Roadhouse			
	2012	**2011**	**2010**
	$ in thousands		
Cash flows from investing activities:			
Capital expenditures—property and equipment	(84,879)	(81,758)	(45,051)
Acquisition of franchise restaurants, net of cash acquired	(4,297)	—	—
Proceeds from sale of property and equipment, including insurance proceeds	1,128	188	235
Net cash used in investing activities	(88,048)	(81,570)	(44,816)

Immediately, you can observe that the main investing activity for Texas Roadhouse is capex. Texas Roadhouse is growing briskly and spends plenty on capex to open new restaurant locations across the United States. In its 10-K filing with the SEC, it details that it spends money to remodel existing stores and build new ones, as well as acquire the land they are built on. Overall, capex is an extremely important cash flow item that investors are not going to find in reported company profits.

Texas Roadhouse also strategically buys out franchises and spent $4.3 million during 2012 to do so. Sometimes it may sell restaurant equipment that is outdated or unused, which then brings in cash instead of being an outflow like other capex. This activity amounted to just over $1 million in 2012.

Below is a more comprehensive list of cash flows that can stem from a firm's investing activities:

· Examples of Inflows:

How to find the best credit card for your lifestyle.

- Proceeds from disposal of property, plant and equipment
- Cash receipts from disposal of debt instruments of other entities

- Receipts from sale of equity instruments of other entities

Examples of Outflows:

- Payments for acquisition of property, plant and equipment
- Payments for purchase of debt instruments of other entities
- Payments for purchase of equity instruments of other entities
- Sales/maturities of investments
- Includes purchasing and selling long- term assets and other investments.

Firms with excess capital or financial institutions such as banks and insurance companies will have buying and selling activity from their investment portfolios that flow through the investing activity portion of the cash flow statement.

How do some balance sheet items relate to this cash flow section?

Analyzing the cash flow statement is extremely valuable because it provides a reconciliation of the beginning and ending cash on the balance sheet. This analysis is difficult for most publicly-traded companies because of the thousands of line items that can go into financial statements. For Texas Roadhouse, its net property and equipment increased by $34,437,000 between 2011 and 2012. Of this amount, the capital expenditure was capitalized (not expensed) on the balance sheet, net of depreciation. The other costs were expensed and reflected on the income statement. In regard to the nearly $4.3 million spent to buy out the franchised restaurant above, here is where it was allocated across the balance sheet:

	2012 $ in thousands
Current assets	64
Property and equipment, net	127
Goodwill	2,741
Intangible asset	1,510
Current liabilities	-142
Total assets	4,300

For a public company, it's going to be nearly impossible to use the original balance sheet and cash flow statements to determine each item down to the specific dollar amount. With the help of the notes in the financial statements (the above is from Texas Roadhouse's notes on acquisitions), an interested party can get a pretty clear understanding of the major items on the investing portion of the cash flow statement and what it means for a firm's financial health.

Investors Reviewing Cash Flow Statements

A firm can get itself into trouble by spending foolishly on acquisitions or capex to either maintain or grow its operations. A great guide for capex is how it relates to depreciation and amortization, which can be found in cash flow from operations on the cash flow statement. This represents an annual charge on past spending that was capitalized on the balance sheet to grow and maintain the business. For Texas Roadhouse, this amounted to $46.7 million in 2012. The fact that capex was nearly double this amount demonstrates it is a growth firm. Yet there is little worry about its financial health because it has minimal long-term debt (other than capital leases) and generated an impressive $146 million in operating cash flow for the year to easily cover capex and $29.4 million in stock buybacks for the year (a cash flow from financing activity).

3. Financing cash flows

In the previous section, we showed that cash flows through a business in four generic stages. First, cash is raised from investors and/or borrowed from lenders. Second, cash is used to buy assets and build inventory. Third, the assets and inventory enable company operations to generate cash, which pays for expenses and taxes before eventually arriving at the fourth stage. At this final stage, cash is returned to the lenders and investors. Accounting rules require companies to classify their natural cash flows into one of three buckets (as required by SFAS 95); together these buckets constitute the statement of cash flows. The diagram below shows how the natural cash flows fit into the classifications of the statement of cash flows. Inflows are displayed in green and outflows displayed in red:

	"Natural" Cash Flows		Statement of Cash Flows
			Cash Flow from Financing (CFF)
	+ Sell equity	→	+Sell equity
	+ Issue debt	→	+ Issue debt
			- Pay dividend
			Cash Flow from Investing (CFI)
	< Buy assets (PP&E) >	→	- Buy assets (PP&E)
	< Buy inventory >		
			Cash Flow from Operations (CFO)
	+ Make sales	→	+ Make sales (collect cash)
			- Buy inventory
	< Pay operating costs >	→	- Pay costs
Interest &	< Pay interest on debt >	→	- Pay interest on debt
Dividends	< Pay taxes >	→	- Pay taxes
"Repaid" to	< Pay dividend >		
Debt & Equity			
Holders			Net Cash Flow = CFF + CFI + CFO

The sum of CFF, CFI and CFO is net cash flow. Although net cash flow is almost impervious to manipulation by management, it is an inferior performance measure because it includes financing cash flows (CFF), which, depending on a company's financing activities, can affect net cash flow in a way that is contradictory to actual operating performance. For example, a profitable company may decide to use its extra cash to retire long-term debt. In this case, a negative CFF for the cash outlay to retire debt could plunge net cash flow to zero even though operating performance is strong. Conversely, a money-losing company can artificially boost net cash flow by issuing a corporate bond or by selling stock. In this case, a positive CFF could offset a negative operating cash flow (CFO), even though the company's operations are not performing well.

Now that we have a firm grasp of the structure of natural cash flows and how they are represented/classified, this section will examine which cash flow measures are best used for a particular analysis. We will also focus on how you can make adjustments to figures so that your analysis isn't distorted by reporting manipulations.

Which Cash Flow Measure Is Best?

You have at least three valid cash flow measures to choose from. Which one is suitable for you depends on your purpose and whether you are trying to value the stock or the whole company.

The easiest choice is to pull cash flow from operations (CFO) directly from the statement of cash flows. This is a popular measure, but it has weaknesses when used in isolation: it excludes capital expenditures, which are typically required to maintain the firm's productive capability. It can also be manipulated, as we show below.

If we are trying to do a valuation or replace an accrual-based earnings measure, the basic question is "which group/entity does cash flow to?" If we want cash flow to shareholders, then we should use free cash flow to equity (FCFE), which is analogous to net earnings and would be best for a price-to-cash flow ratio (P/CF).

If we want cash flows to all capital investors, we should use free cash flow to the firm (FCFF). FCFF is similar to the cash generating base used in economic value added (EVA). In EVA, it's called net operating profit after taxes

(NOPAT) or sometimes net operating profit less adjusted taxes (NOPLAT), but both are essentially FCFF where adjustments are made to the CFO component.

3.2 HOW TO MANAGE CASH FLOW

Cash is king when it comes to the financial management of a growing company. The lag between the time you have to pay your suppliers and employees and the time you collect from your customers is the problem, and the solution is cash flow management. At its simplest, cash flow management means delaying outlays of cash as long as possible while encouraging anyone who owes you money to pay it as rapidly as possible.

Measuring Cash Flow

Prepare cash flow projections for next year, next quarter and, if you're on shaky ground, next week. An accurate cash flow projection can alert you to trouble well before it strikes.

Understand that cash flow plans are not glimpses into the future. They're educated guesses that balance a number of factors, including your customers' payment histories, your own thoroughness at identifying upcoming expenditures, and your vendors' patience. Watch out for assuming without justification that receivables will continue coming in at the same rate they have recently, that payables can be extended as far as they have in the past, that you have included expenses such as capital improvements, loan interest and principal payments, and that you have accounted for seasonal sales fluctuations.

Start your cash flow projection by adding cash on hand at the beginning of the period with other cash to be received from various sources. In the process, you will wind up gathering information from salespeople, service representatives, collections, credit workers and your finance department. In all cases, you'll be asking the same question: How much cash in the form of customer payments, interest earnings, service fees, partial collections of bad debts, and other sources are we going to get in, and when?

The second part of making accurate cash flow projections is detailed knowledge of amounts and dates of upcoming cash outlays. That means not only knowing when each penny will be spent, but on what. Have a line item on your projection for every significant outlay, including rent, inventory (when purchased for cash), salaries and wages, sales and other taxes withheld or payable, benefits paid, equipment purchased for cash, professional fees, utilities, office supplies, debt payments, advertising, vehicle and equipment maintenance and fuel, and cash dividends.

"As difficult as it is for a business owner to prepare projections, it's one of the most important things one can do," says accountant Steve Mayer. "Projections rank next to business plans and mission statements among things a business must do to plan for the future."

Improving Receivables

If you got paid for sales the instant you made them, you would never have a cash flow problem. Unfortunately, that doesn't happen, but you can still improve your cash flow by managing your receivables. The basic idea is to improve the speed with which you turn materials and supplies into products, inventory into receivables, and receivables into cash. Here are specific techniques for doing this:

- Offer discounts to customers who pay their bills rapidly.

- Ask customers to make deposit payments at the time orders are taken.

- Require credit checks on all new noncash customers.

- Get rid of old, outdated inventory for whatever you can get

• Issues invoices promptly and follow up immediately if payments are slow in coming.

• Track accounts receivable to identify and avoid slow-paying customers. Instituting a policy of cash on delivery (c.o.d.) is an alternative to refusing to do business with slow-paying customers.

Managing Payables

Top-line sales growth can conceal a lot of problems-sometimes too well. When you are managing a growing company, you have to watch expenses carefully. Don't be lulled into complacency by simply expanding sales. Any time and any place you see expenses growing faster than sales, examine costs carefully to find places to cut or control them. Here are some more tips for using cash wisely:

• Take full advantage of creditor payment terms. If a payment is due in 30 days, don't pay it in 15 days.

• Use electronic funds transfer to make payments on the last day they are due. You will remain current with suppliers while retaining use of your funds as long as possible.

• Communicate with your suppliers so they know your financial situation. If you ever need to delay a payment, you'll need their trust and understanding.

• Carefully consider vendors' offers of discounts for earlier payments. These can amount to expensive loans to your suppliers, or they may provide you with a change to reduce overall costs. The devil is in the details.

• Don't always focus on the lowest price when choosing suppliers. Sometimes more flexible payment terms can improve your cash flow more than a bargain-basement price.

Surviving Shortfalls

Sooner or later, you will foresee or find yourself in a situation where you lack the cash to pay your bills. This doesn't mean you're a failure as a businessperson-you're a normal entrepreneur who can't perfectly predict the future. And there are normal, everyday business practices that can help you manage the shortfall.

The key to managing cash shortfalls is to become aware of the problem as early and as accurately as possible. Banks are wary of borrowers who have to have money today. They'd much prefer lending to you before you need it, preferably months before. When the reason you are caught short is that you failed to plan, a banker is not going to be very interested in helping you out.

If you assume from the beginning that you will someday be short on cash, you can arrange for a line of credit at your bank. This allows you to borrow money up to a preset limit any time you need it. Since it's far easier to borrow when you don't need it, arranging a credit line before you are short is vital.

If bankers won't help, turn next to your suppliers. These people are more interested in keeping you going than a banker, and they probably know more about your business. You can often get extended terms from suppliers that amount to a hefty, low-cost loan just by asking. That's especially true if you've been a good customer in the past and kept them informed about your financial situation.

Consider using factors. These are financial service businesses that can pay you today for receivables you may not otherwise be able to collect on for weeks or months. You'll receive as much as 15 percent less than you would otherwise, since factors demand a discount, but you'll eliminate the hassle of collecting and be able to fund current operations without borrowing.

Ask your best customers to accelerate payments. Explain the situation and, if necessary, offer a discount of a percentage point or two off the bill. You should also go after your worst customers-those whose invoices are more than 90 days past due. Offer them a steeper discount if they pay today.

You may be able to raise cash by selling and leasing back assets such as machinery, equipment, computers, phone systems and even office furniture. Leasing companies may be willing to perform the transactions. It's not cheap, however, and you could lose your assets if you miss lease payments.

Choose the bills you'll pay carefully. Don't just pay the smallest ones and let the rest slide. Make payroll first-unpaid employees will soon be ex-employees. Pay crucial suppliers next. Ask the rest if you can skip a payment or make a partial payment.

Financial Statements - Cash Flow Computations - Indirect Method

Under U.S. and ISA GAAP, the statement of cash flow can be presented by means of two ways:

1. The indirect method

2. The direct method

The Indirect Method

The indirect method is preferred by most firms because is shows a reconciliation from reported net income to cash provided by operations.

Calculating Cash flow from Operations

Here are the steps for calculating the cash flow from operations using the indirect method:

1. Start with net income.

2. Add back non-cash expenses.

(Such as depreciation and amortization)

3. Adjust for gains and losses on sales on assets.

Add back losses

Subtract out gains

4. Account for changes in all non-cash current assets.

5. Account for changes in all current assets and liabilities except notes payable and dividends payable.

In general, candidates should utilize the following rules:

Increase in assets = use of cash (-)

Decrease in assets = source of cash (+)

Increase in liability or capital = source of cash (+)

Decrease in liability or capital = use of cash (-)

The following example illustrates a typical net cash flow from operating activities:

XYZ Company

Cash Flow from Operating Activities

Indirect Method

	#
Net income	66,800
Adjustments:	
Depreciation and amortization	2,000
Deferred taxes	50
Decrease in accounts receivable	200
Increase in inventories	(4,000)
Increase in accounts payable	1,150
Increase in accrued interest receivable	(350)
Increase in accrued interest payable	100

Gain on sale of property	(600)
Net cash flow operating activities	65,350

Cash Flow from Investment Activities

Cash Flow from investing activities includes purchasing and selling long-term assets and marketable securities (other than cash equivalents), as well as making and collecting on loans.

Here's the calculation of the cash flows from investing using the indirect method:

XYZ Company

Cash Flow from Investment Activities

Indirect Method

Cash from sale of land	(5,000)
Purchase of plant & equipment	50,000
Cash flow from Investment Activities	**45,000**

Cash Flow from Financing Activities

Cash Flow from financing activities includes issuing and buying back capital stock, as well as borrowing and repaying loans on a short- or long-term basis (issuing bonds and notes). Dividends paid are also included in this category, but the repayment of accounts payable or accrued liabilities is not.

Here's the calculation of the cash flows from financing using the indirect method:

XYZ Company

Cash Flow from financing Activities

Indirect Method

Sale of Bonds	5,000
Stock Repurchase	(6,000)
Cash Dividend	(3,500)
Issue of Preferred shares	80,000
Cash Flow from Financing Activities	**75,000**

Calculating Total Cash flows.

Greene Co. shows the following information on its 2012 income statement:

Sales = #138,000

Costs = #71,500

Other expenses = #4,100

Depreciation expense = #10,100

Interest expense #7,900

Taxes = #17,760

Dividends = #5,400.

In addition, you're told that the firm issued #2,500 in new equity during 2012, and redeemed #3,800 in outstanding long-term debt

a. What is the 2012 operating cash flow?

b. What is the 2012 cash flow to creditors?

c. What is the 2012 cash flow to stockholders?

d. If net fixed assets increased by #17,400 during the year, what was the addition to NWC?

SOLUTION:

a. To calculate the OCF, we first need to construct an income statement. The income statement starts with revenues and subtracts costs to arrive at EBIT. We then subtract out interest to get taxable income, and then subtract taxes to arrive at net income. Doing so, we get:

Income Statement

Sales	#138,000
Costs	71,500
Other Expenses	4,100
Depreciation	10,100
EB	#52,300
Interest	7,900
Taxable income	#44,400
Taxes	17,760
Net income	#16,640
Dividends	#5,400
Addition to retained earnings	21,240

Dividends paid plus addition to retained earnings must equal net income, so:

Net income = Dividends + Addition to retained earnings

Addition to retained earnings = #26,640 — 5,400

Addition to retained earnings = #21,240

So, the operating cash flow is:

OCF = EBIT ± Depreciation — Taxes

OCF = #52,300 + 10,100 — 17,760

OCF = #44,640

b. The cash flow to creditors is the interest paid, plus any new borrowing. Since the company redeemed long debt, the new borrowing is negative. So, the cash flow to creditors is:

Cash flow to creditors = Interest paid — Net new borrowing

Cash flow to creditors = #7,900 — (—#3,800)

Cash flow to creditors = #11,700

c. The cash flow to stockholders is the dividends paid minus any new equity. So, the cash flow to stockholders is:

Cash flow to stockholders = Dividends paid — Net new equity

Cash flow to stockholders = #5,400 —2,500

Cash flow to stockholders = #2,900

d. In this case, to find the addition to NWC, we need to find the cash flow from assets. We can then use the cash flow from assets equation to find the change in NV/C. We know that cash flow from assets is equal to cash flow to creditors plus cash flow to stockholders. So, cash flow from assets is:

Cash flow from assets = Cash flow to creditors + Cash flow to stockholders

Cash flow from assets = #11,700 + 2,900

Cash flow from assets = #14,600

Net capital spending is equal to depreciation plus the increase in fixed assets, so:

Net capital spending = Depreciation + Increase in fixed assets

Net capital spending = #10,100 + 17,400

Net capital spending = #27,500

Now we can use the cash flow from assets equation to find the change in NWC. Doing so, we

Find:

Cash flow from assets = OCF — Change in NV/C — Net capital spending

#14,600 = #44,640 — Change in NWC — #27,500

Change in NV/C = #2,540

4.0 CONCLUSION

The theoretical case for a CFT is clear. It is simpler than an income tax, with lower economic costs. It can be made progressive, thus satisfying equity criteria for good tax design.

Even if the considerable difficulties with a transition could be overcome, integrating a CFT into a world where most of the economy is subject to an income tax would also pose difficulties. There is a risk that the rules needed to maintain CFT treatment, while at the same time protecting the income tax base, might negate significant portions of the simplification gains from a CFT.

5.0 SUMMARY

Cash flow from operations (CFO) should be examined for distortions in the following ways:

- Remove gains from tax benefits due to stock option exercises.
- Check for temporary CFO blips due to working capital actions. For example, withholding payables, or "stuffing the channel", to temporarily reduce inventory.
- Check for cash outflows classified under CFI that should be reclassified to CFO.
- Check for other one-time CFO blips due to nonrecurring dividends or trading gains.

Aside from being vulnerable to distortions, the major weakness of CFO is that it excludes capital investment dollars. We can generally overcome this problem by using free cash flow to equity (FCFE), which includes (or, more precisely, is reduced by) capital expenditures (CFI). Finally, the weakness of FCFE is that it will change if the capital structure changes. That is, FCFE will go up if the company replaces debt with equity (an action that reduces interest paid and therefore increases CFO) and vice versa. This problem can be overcome by using free cash flow to firm (FCFF), which is not distorted by the ratio of debt to equity.

6.0 TUTOR-MARKED ASSIGNMENT

How cash flow calculated with operating, investing and financial flow.

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UNIT 3 SINKING FUND

CONTENTS

1.0 INTRODUCTION

The sinking fund was first used in Great Britain in the 18th century to reduce national debt. While used by Robert Walpole in 1716 and effectively in the 1720s and early 1730s, it originated in the commercial tax syndicates of the Italian peninsula of the 14th century, where its function was to retire redeemable public debt of those cities.

The fund received whatever surplus occurred in the national Budget each year. However, the problem was that the fund was rarely given any priority in Government strategy. The result of this was that the funds were often raided by the Treasury when they needed funds quickly.

In 1772, the nonconformist minister Richard Price published a pamphlet on methods of reducing the national debt. The pamphlet caught the interest of William Pitt the Younger, who drafted a proposal to reform the Sinking Fund in 1786. Lord North recommended "the Creation of a Fund, to be appropriated, and invariably applied, under proper Direction, in the gradual Diminution of the Debt." Pitt's way of securing "proper Direction" was to introduce legislation that prevented ministers from raiding the fund in crises. He also increased taxes to ensure that a £1 million surplus could be used to reduce the national debt.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Define Sinking Fund
- Discuss Simple and Compound Interest
- Discuss features and types of debt instrument

3.0 MAIN BODY

3.1 DEFINITION OF SINKING FUND

A fund into which a company sets aside money over time, in order to retire its preferred stock, bonds or debentures. A fund into which a company sets aside money over time, in order to retire its preferred stock, bonds or debentures. In the case of bonds, incremental payments into the sinking fund can soften the financial impact at maturity. Investors prefer bonds and debentures backed by sinking funds because there is less risk of a default.

A sinking fund is a fund established by an economic entity by setting aside revenue over a period of time to fund a future capital expense, or repayment of a long-term debt.

In North America and elsewhere where it is common for public and private corporations to raise funds through the issue of bonds, the term is only normally used in this context. However in the United Kingdom and elsewhere where the issue of bonds (other than government bonds) is unusual, and where long-term leasehold tenancies are common, the term is only normally used in the context of replacement or renewal of capital assets, particularly the common parts of buildings.

3.1.1 Types of Sinking Fund

A sinking fund may operate in one or more of the following ways:

- The firm may repurchase a fraction of the outstanding bonds in the open market each year.
- The firm may repurchase a fraction of outstanding bonds at a special call price associated with the sinking fund provision (they are callable bonds).
- The firm has the option to repurchase the bonds at either the market price or the sinking fund price, whichever is lower. To allocate the burden of the sinking fund call fairly among bondholders, the bonds chosen for the call are selected at random based on serial number. The firm can only repurchase a limited fraction of the bond issue at the sinking fund price. At best some indentures allow firms to use a doubling option, which allows repurchase of double the required number of bonds at the sinking fund price.
- A less common provision is to call for periodic payments to a trustee, with the payments invested so that the accumulated sum can be used for

retirement of the entire issue at maturity: instead of the debt amortizing over the life, the debt remains outstanding and a matching asset accrues. Thus the balance sheet consists of Asset = Sinking fund, Liability = Bonds.

Benefits and drawbacks

For the organization retiring debt, it has the benefit that the principal of the debt or at least part of it, will be available when due. For the creditors, the fund reduces the risk the organization will default when the principal is due: it reduces credit risk.

However, if the bonds are callable, this comes at a cost to creditors, because the organization has an option on the bonds:

- The firm will choose to buy back discount bonds (selling below par) at their market price,
- while exercising its option to buy back premium bonds (selling above par) at par.

Therefore, if interest rates fall and bond prices rise, a firm will benefit from the sinking fund provision that enables it to repurchase its bonds at below-market prices. In this case, the firm's gain is the bondholder's loss – thus callable bonds will typically be issued at a higher coupon rate, reflecting the value of the option.

We can obtain the sinking fund formula by writing R in terms of A in future value of the annuity formula

$A = R [(1+i)n-1]/i$ $R = Ai/(1+i)n -1$

GENERAL FORMULA FOR SINKING FUND PAYMENT

$R = Ai / (1+i)n -1$

R: periodic payment.

A: periodic payment R is required to accumulate a sum of A dollars.

n: number of periods.

i: interest rate.

EXAMPLE 7: A company wishes to spend #40000 for new equipment and decides to set up a sinking fund to accumulate this money over a 3 – year period. If payments are to be made to the fund quarterly, with interest compounded quarterly at an annual rate of 5%, how large should the payments be?

SOLUTION: A = 40000, i=0.05/4 = 0.0125 and n = 4*3=12 (payments). We use the formula

$40000 = R*[(1.0125)12 - 1]/0.0125 = 12.86R$

Solving for R we get R = #3110.42 (each payment should be #3110.42).

3.2 SIMPLE INTEREST AND COMPOUND INTEREST

Interest is the fee charged for the privilege of borrowing money. If you get a loan from a bank or other financial institution, you are charged interest for the privilege of using the institution's money. If you invest money in a savings account, the bank pays you interest for the privilege of using your money. If you buy an item like a car or stereo ”on time,” you must pay interest for the privilege of postponing full payment until after you have taken possession of the item.

SIMPLE INTEREST

Interest that is computed on the principal* alone is called simple interest.

- the total amount of money borrowed (or invested).

Suppose you have to pay simple interest at a rate of %5 per year on a 2-year loan of #300. The interest charged will be 300(0.05)(2)=#30 and the total amount owed at the end of the 2-year period will be;

300+30 = #330

GENERAL FORMULA FOR SIMPLE INTEREST

$A=P\ (1+rn)$

P: principal for which the principal is borrowed

r: yearly interest rate

n: number of years

A: total amount at the end of n years.

COMPOUND INTEREST

Interest that is computed on the previously accumulated interest as well as on the principal is called compound interest.

If the original principal is p dollars and if the interest is compounded annually at the rate of r per year, then at the end of the first year the new principal will be $A_1 = P+Pr = P(1+r)$

the interest earned at the end of the second year is computed on A1, so at the end of the second year, the accumulated amount will be

$$A_2 = A_1+A_1r = A_1(1+r) = P(1+r)^2$$

Continuing, we see that at the end of n years, the accumulated amount will be

$An= P(1+r)n$

The interest we computed was compounded annually. In practice, however, interest is usually compounded more than once a year. In either case, the quoted rate of interest per year is called the stated, or nominal, rate and the interval of time between successive interest calculations is called the conversion period.

If interest at the nominal rate of r per year is compounded m times a year on a principal of P dollars, then the simple interest rate per conversion period is

i= r/m. (annual interest rate/periods per year)

GENERAL FORMULA FOR COMPOUND INTEREST

An= P(1+i)n where i= r/m, n = m*l

P: amount of principal

r: nominal interest rate per year

m: number of conversion periods per year

l: number of years

An: accumulated amount at the end of n conversion periods.

EXAMPLE 1: Suppose you invest #2000 at an annual interest rate of % 6. Find your balance at the end of 1 year if interest is compounded;

a) yearly b) semiannually c) quarterly d) monthly

SOLUTION: A = P (1+i)n with P=2000

a) i = 0.06/1 n = 1*1 A = 2000(1.06)= #2120

b) i = 0.06/2 n = 1*2 A = 2000(1.03)2= #2121.80

c) i = 0.06/4 n = 1*4 A = 2000(1.015)4= #2122.73

d) i = 0.06/12 n =1*12 A = 2000(1.005)12= #2123.36

the more often interest is compounded, the faster your balance will grow. Banks sometimes compete on the basis of how frequently they compound interest.

The time taken for an investment to double in values is called the doubling time for that investment. Similarly, the time taken for an investment to triple in values is called the tripling time.

3.3 DEBT INSTRUMENT

A paper or electronic obligation that enables the issuing party to raise funds by promising to repay a lender in accordance with terms of a contract. Types of debt instruments include notes, bonds, certificates, mortgages, leases or other agreements between a lender and a borrower.

Investopedia explains 'Debt Instrument'

Debt instruments are a way for markets and participants to easily transfer the ownership of debt obligations from one party to another. Debt obligation transferability increases liquidity and gives creditors a means of trading debt obligations on the market. Without debt instruments acting as a means to facilitate trading, debt is an obligation from one party to another. When a debt instrument is used as a medium to facilitate debt trading, debt obligations can be moved from one party to another quickly and efficiently.

Characteristics and properties of bonds

Bonds are often referred to as debt instruments, debt obligations, or fixed income securities (despite the fact that some bonds pay variable rates of interest, or none at all).

Coupon Rate

The rate of interest that a bond pays to the bondholder is called the coupon rate. This term dates back to when bonds were issued as paper certificates (known as bearer bonds) with coupons attached. Bearer bonds became the property of whoever had physical possession of the bond. The bondholder would clip a coupon from the certificate and redeem it with the bond's paying agent (usually a commercial bank) to collect their interest payment. Now virtually all bonds are held in computerized book-entry form (all treasuries), or a physical certificate without coupons.

The coupon rate is the stated annualized interest rate that the bond issuer is contractually obligated to pay to the bondholder. The coupon payment is usually paid on a quarterly, semi-annual, or annual basis and is based on the maturity date. For example, if the maturity date is September 15, 2020, a bond that pays quarterly interest will make a payment on March 15, June 15, September 15, and December 15 each year until 2020.

The principal payment received at maturity and the basis for determining the amount of the coupon payment is the stated par amount over the life of the bond, regardless of the general level of interest rates. of the bond. A bond that is paying a 7% coupon rate will pay #1,000 X 0.07, or #70 annually. If the bond pays on a semi-annual basis, the bondholder will receive a #35 payment twice a year on the coupon payment date. The coupon payment will not change (assuming the bond is not a floating-rate bond)

Zero Coupon Bonds

Zero coupon bonds do not make coupon payments during the life of the bond. Zero coupon bonds are issued at a discount to par value, and the return to the investor comes in the difference that the bondholder pays for the bond (discount to par value), and the amount of principal payment made at maturity (par value).

Floating Rate Bonds

A floating rate bond has a coupon rate that is pegged to a benchmark, such as libor, and is adjusted periodically. Floating rate bonds have the advantage of being less volatile in price, but the disadvantage of providing an unpredictable stream of income. Floating-rate bonds are attractive to investors that are more interested in preservation of principal than in having an assured amount of income.

Par Amount

The par amount of a bond is the amount of principal that the bondholder will receive at maturity. Par amount is also referred to as face amount.

Bonds are typically issued and traded in units of #1,000, though some issues have minimum amounts, such as #5,000. The par amount of a bond being traded will determine how close to fair market price a particular bond will trade for. The par amount that trades at the best market price is known as a round lot, and smaller sizes are known as odd lots.

Maturity

The maturity dates is the specific day, month, and year that the bond issuer is obligated to pay back the principal to the bondholder. There are rare examples of extendable bonds that give the issuer or bondholder the option to extend the maturity date of the bond.

In the bond market, the maturity of a bond is expressed as the number of years remaining until the bonds maturity date. For example, a 30-year bond that was issued ten years ago would be referred to as a 20-year bond.

When bonds are issued, there is a direct correlation between the maturity date and the coupon rate of the bond. The longer the length of time to maturity, the higher the coupon rate will be. There are a number of reasons for this correlation; one of them is the time value of money. The time value of money stipulates that an individual would rather receive money now than at some time in the future due to the potential earnings capacity of the money. Another reason for this is that the longer the time period, the more uncertainty is involved with holding the bond. There is always a chance that the price of a bond can decline, or an adverse event can befall the issuer of the bond that will jeopardize their ability to pay the interest and/or principal. Normally, this correlation also holds true for bonds trading in the secondary market, but there are rare occasion when it does not.

Bond Pricing and Yield

As marketable debt instruments, bonds can be traded and their price can fluctuate over time. Bond prices are quoted as a percentage of par amount. For example, a bond price of 99 indicates a price of 99% of par, which would be #990 for a par amount of #1,000. A bond trading at 101 3/8 is 101.375% of par or #1013.75 (every 1/8th of a point is worth #1.25 for every #1,000 in face amount). Bonds priced above par (such as 102 ½) are referred to as trading at a premium, while bonds below par are at a discount. Bonds trading at 100 are always quoted as trading at par.

An entity that is offering bonds in the primary or new issue market, it sets the coupon rate to reflect the coupon rate of issues that are similar in maturity, credit quality, etc. During the initial underwriting or offering period, the bonds should trade close to par, but when the offering period ends, the price can freely fluctuate on the secondary market. We will examine what factors influence bond prices in detail in future lessons but they include:

- The general level of interest rates;
- The expected rate of inflation;
- The economic outlook;
- The issuer's credit rating;
- The supply of new debt issues; and
- The demand for bonds versus alternative investments.

While we have stated that the coupon rate remains constant for fixed rate bonds, the market yield will fluctuate with the price of the bond. If a bond has a coupon rate of 7 ¾% and the market yield for equivalent bond is 7 ¾%, then the price of the bond would be par. But, if some event caused the market yield on equivalent bonds to rise to 8%, the coupon yield of the bond at 7 ¾% is not as attractive to potential buyers who can get a bond with the higher yield. The price of the less desirable, lower yielding bond would drop in price until its market yield was also 8%. This is a very important concept: When market yields increase, bond prices decrease; when market yields decrease, bond prices increase. That is to say that there is an inverse relationship between changes market yield and changes in bond prices.

When a bond is purchased at a price other than par, the interest payment is no longer the single factor that determines the rate of return on the investment. If a bond is purchased at 90, the investor pays #900 for the bond, but will receive #1,000 at maturity. In addition to receiving the interest payments over the life of the bond, the investor will also earn a #100 profit at maturity. Conversely, bonds that are purchased at a premium will cause the investor to experience a loss at maturity. The measure of a bond's return that takes this principal gain or loss at maturity is the bond's yield to maturity. The market yield that we have been referring to is the bond's yield to maturity. Yield to maturity is the measurement of the present value of a bond's future cash flows (coupon payments and any principal gain or loss at maturity) based on the current market price.

We could go through the mathematics of calculating yield to maturity and how to determine what the price change of a bond would be for a given change in interest rates, but it is much easier to just use bond calculators, like those found here. For those that are curious enough to want to explore the mathematics, you can find it here.

Another yield measurement that bond investors use is a bond's current yield. The current yield is determined by simply dividing the annual coupon payment by the market value of the bond. A #1,000 par bond trading at 96 would have a market value of #960. If the bond had a stated coupon rate of 7 ½% it would pay #75 annually: 75/960= .078125, so the current yield would be 7.8125%. The current yield of a bond trading at a discount will always be higher than the coupon rate, while the current yield of a premium bond will always be less than the coupon rate. This also holds true for yield to maturity. Because yield to maturity takes into account the time value of money, and current yield does not, it is a much better measure of a bond's true return.

Other Bond Properties

It is essential for anyone investing in bonds to carefully review all aspects of a bond before purchasing. Here are some important properties of bands that should be carefully examined.

4.0 CONCLUSION

The theoretical case for a sinking fund is clear. It is simpler than an income tax, with lower economic costs. It can be made progressive, thus satisfying equity criteria for good tax design.

5.0 SUMMARY

A naira expected sometimes in the future is not equivalent to a naira today because of time value of money. Interest is the payment made foe the use of money. The concepts of time value of money forms the basis of discussion on financial mathematics principles such as simple interest, compound interest, and present value.

6.0 TUTOR-MARKED ASSIGNMENT

- If a bond with an annual coupon of 10% and a face value of #1,000 has an infinite life, calculate the value if the bond yield is 12%.
- Suppose you are considering two investments: one that pays 10% simple interest per annum and one that pays 10% compound interest per annum, which would you choose? Why?

7.0 REFERENCES AND FURTHER READINGS

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UNIT 4: MATHEMATICAL PROGRAMMING (LINEAR PROGRAMMING)

1.0 INTRODUCTION

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equations and/or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resources such as market demand, production process and equipment, storage capacity, raw material availability, etc. By linearity is meant a mathematical expression in which the expressions among the variables are linear e.g., the expression a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n is linear. Higher powers of the variables or their products do not appear in the expressions for the objective function as well as the constraints (they do not have expressions like x_1^3, $x_2^{3/2}$, x_1x_2, $a_1x_1 + a_2 \log x_2$, etc.). The variables obey the properties of proportionality (e.g., if a product requires 3 hours of machining time, 5 units of it will require 15 hours) and additivity (e.g., amount of a resource required for a certain number of products is equal to the sum of the resource required for each).

It was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while they were working on a project for U.S. Air Force. This technique consisted of representing the various activities of an organization as a linear programming (L.P.) model and arriving at the optimal programme by minimizing a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as "simplex method" to solve linear programming problems.

2.0 OBJECTIVES

At the end of this study unit, you should be able to

- Explain the requirements for Linear Programming
- Highlight the assumptions of Linear Programming
- Identify the Areas of application of Linear Programming
- Formulate a Linear Programming problem
- Solve various problems using Linear Programming

3.0 MAIN CONTENT

3.1 REQUIREMENTS FOR A LINEAR PROGRAMMING PROBLEM

All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. If the supply of these resources were unlimited, the need for management tools like linear programming would not arise at all. Supply of resources being limited, the management must find the best allocation of its resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent. However this involves a number of problems which can be overcome by quantitative methods, particularly the linear programming.

Generally speaking, linear programming can be used for optimization problems if the following conditions are satisfied:

1. There must be a well-defined objective function (profit, cost or quantities produced) which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.

2. There must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities in terms of variables.

3. There must be alternative courses of action. For example, a given product may be processed by two different machines and problem may be as to how much of the product to allocate to which machine.

4. Another necessary requirement is that decision variables should be interrelated and nonnegative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are generally illogical.

5. As stated earlier, the resources must be in limited supply. For example, if a firm starts producing greater number of a particular product, it must make smaller number of other products as the total production capacity is limited.

3.2 ASSUMPTIONS IN LINEAR PROGRAMMING MODELS

A linear programming model is based on the following assumptions:

1. Proportionality: A basic assumption of linear programming is that proportionality exists in the objective function and the constraints. This assumption implies that if a product yields a profit of #10, the profit earned from the sale of 12 such products will be # (10 x 12) = #120. This may not always be true because of quantity discounts. Further, even if the sale price is constant, the manufacturing cost may vary with the number of units produced and so may vary the profit per unit. Likewise, it is assumed that if one product requires processing time of 5 hours, then ten such products will require processing time of 5 x 10 = 50 hours. This may also not be true as the processing time per unit often decreases with increase in number of units produced. The real world situations may not be strictly linear. However, assumed linearity represents their close approximations and provides very useful answers.

2. Additivity: It means that if we use t_1 hours on machine A to make product 1 and t_2 hours to make product 2, the total time required to make products 1 and 2 on machine A is $t_1 + t_2$ hours. This, however, is true only if the change-over time from product 1 to product 2 is negligible. Some processes may not behave in this way. For example, when several liquids of different chemical compositions are mixed, the resulting volume may not be equal to the sum of the volumes of the individual liquids.

3. Continuity: Another assumption underlying the linear programming model is that the decision variables are continuous i.e., they are permitted to take any

non-negative values that satisfy the constraints. However, there are problems wherein variables are restricted to have integral values only. Though such problems, strictly speaking, are not linear programming problems, they are frequently solved by linear programming techniques and the values are then rounded off to nearest integers to satisfy the constraints. This approximation, however, is valid only if the variables have large optimal values. Further, it must be ascertained whether the solution represented by the rounded values is a feasible solution and also whether the solution is the best integer solution.

4. Certainty: Another assumption underlying a linear programming model is that the various parameters, namely, the objective function coefficients, R.H.S. coefficients of the constraints and resource values in the constraints are certainly and precisely known and that their values do not change with time. Thus the profit or cost per unit of the product, labour and materials required per unit, availability of labour and materials, market demand of the product produced, etc. are assumed to be known with- certainty. The linear programming problem is, therefore, assumed to be deterministic in nature.

5. Finite Choices: A linear programming model also assumes that a finite (limited) number of choices (alternatives) are available to the decision-maker and that the decision variables are interrelated and non-negative. The non-negativity condition shows that linear programming deals with real-life situations as it is not possible to produce/use negative quantities.

Mathematically these non-negativity conditions do not differ from other constraints. However, since while solving the problems they are handled differently from the other constraints, they are termed as non-negativity restrictions and the term constraints is used to represent constraints other than non-negativity restrictions and this terminology has been followed throughout the book.

3.3 APPLICATIONS OF LINEAR PROGRAMMING METHOD

Though, in the world we live, most of the events are non-linear, yet there are many instances of linear events that occur in day-to-day life. Therefore, an understanding of linear programming and its application in solving problems is utmost essential for today’s managers.

Linear programming techniques are widely used to solve a number of business, industrial, military, economic, marketing, distribution and advertising problems. Three primary reasons for its wide use are:

1. A large number of problems from different fields can be represented or at least approximated to linear programming problems.

2. Powerful and efficient techniques for solving L.P. problems are available.

3. L.P. models can handle data variation (sensitivity analysis) easily.

However, solution procedures are generally iterative and even medium size problems require manipulation of large amount of data. But with the development of digital computers, this disadvantage has been completely overcome as these computers can handle even large L.P. problems in comparatively very little time at a low cost.

3.4 AREAS OF APPLICATION OF LINEAR PROGRAMMING

Linear programming is one of the most widely applied techniques of operations research in business, industry and numerous other fields. A few areas of its application are given below.

1. INDUSTRIAL APPLICATIONS

(a) Product mix problems: An industrial concern has available a certain production capacity (men, machines, money, materials, market, etc.) on various manufacturing processes to manufacture various products. Typically, differents products will have different selling prices, will require different amounts of production capacity at the several processes and will, therefore, have different unit profits; there may also be stipulations (conditions) on maximum and/or minimum product levels. The problem is to determine the product mix that will maximize the total profit.

(b) Blending problems: These problems are likely to arise when a product can be made from a variety of available raw materials of various compositions and prices. The manufacturing process involves blending (mixing) some of these materials in varying quantities to make a product of the desired specifications.

For instance, different grades of gasoline are required for aviation purposes. Prices and specifications such as octane ratings, tetra ethyl lead concentrations, maximum vapour pressure etc. of input ingredients are given and the problem is to decide the proportions of these ingredients to make the desired grades of gasoline so that (i) maximum output is obtained and (ii) storage capacity restrictions are satisfied. Many similar situations such as preparation of different kinds of whisky, chemicals, fertilizers and alloys, etc. have been handled by this technique of linear programming.

(c) Production scheduling problems: They involve the determination of optimum production schedule to meet fluctuating demand. The objective is to meet demand, keep inventory and employment at reasonable minimum levels, while minimizing the total cost Production and inventory.

(d) Trim loss problems: They are applicable to paper, sheet metal and glass manufacturing industries where items of standard sizes have to be cut to smaller sizes as per customer requirements with the objective of minimizing the waste produced.

(e) Assembly-line balancing: It relates to a category of problems wherein the final product has a number of different components assembled together. These components are to be assembled in a specific sequence or set of sequences. Each assembly operator is to be assigned the task / combination of tasks so that his task time is less than or equal to the cycle time.

(f) Make-or-buy (sub-contracting) problems: They arise in an organization in the face of production capacity limitations and sudden spurt in demand of its products. The manufacturer, not being sure of the demand pattern, is usually reluctant to add additional capacity and has to make a decision regarding the products to be manufactured with his own resources and the products to be sub-contracted so that the total cost is minimized.

2. MANAGEMENT APPLICATIONS

(a) Media selection problems: They involve the selection of advertising mix among different advertising media such as T.V., radio, magazines and newspapers that will maximize public exposure to company's product. The constraints may be on the total advertising budget, maximum expenditure in each media, maximum number of insertions in each media and the like.

(b) Portfolio selection problems: They are frequently encountered by banks, financial companies, insurance companies, investment services, etc. A given amount is to be allocated among several investment alternatives such as bonds, saving certificates, common stock, mutual fund, real estate, etc. to maximize the expected return or minimize the expected risk.

(c) Profit planning problems: They involve planning profits on fiscal year basis to maximize profit margin from investment in plant facilities, machinery, inventory and cash on hand.

(d) Transportation problems: They involve transportation of products from, say, n sources situated at different locations to, say, m different destinations. Supply position at the sources, demand at destinations, freight charges and storage costs, etc. are known and the problem is to design the optimum transportation plan that minimizes the total transportation cost (or distance or time).

(e) Assignment problems: They are concerned with allocation of facilities (men or machines) to jobs. Time required by each facility to perform each job is given and the problem is to find the optimum allocation (one job to one facility) so that the total time to perform the jobs is minimized.

(f) Man-power scheduling problems: They are faced by big hospitals, restaurants and companies operating in a number of shifts. The problem is to allocate optimum man-power in each shift so that the overtime cost is minimized.

3. MISCELLANEOUS APPLICATIONS

(a) Diet problems: They form another important category to which linear programming has been applied. Nutrient contents such as vitamins, proteins, fats, carbohydrates, starch, etc. in each of a number of food stuffs is known. Also the minimum daily requirement of each nutrient in the diet as well as the cost of each type of food stuff is given and the problem is to determine the minimum cost diet that satisfies the minimum daily requirement of nutrients.

(b) Agriculture problems: These problems are concerned with the allocation of input resources such as acreage of land, water, labour, fertilisers and capital to various crops so as to maximize net revenue.

(c) Flight scheduling problems: They are devoted to the determination of the most economical patterns and timings of flights that result in the most efficient use of aircrafts and crew.

(d) Environment protection: They involve analysis of different alternatives for efficient waste disposal, paper recycling and energy policies.

(e) Facilities location: These problems are concerned with the determination of best location of public parks, libraries and recreation areas, hospital ambulance depots, telephone exchanges, nuclear power plants, etc.

Oil refineries have used linear programming with considerable success. Similar trends are developing in chemical industries, iron and steel industries, aluminium industry, food processing industry, wood products manufacture and many others. Other areas where linear programming has been applied include quality control inspection, determination of optimal bombing patterns, searching of submarines, design of war weapons, vendor quotation analysis, structural design, scheduling military tanker fleet, fabrication scheduling, steel production scheduling, balancing of assembly lines and computations of maximum flows in networks.

In fact linear programming may be used for any general situation where a linear objective function has to be optimised subject to constraints expressed as linear equations/inequalities.

3.5 FORMULATION OF LINEAR PROGRAMMING PROBLEMS

First, the given problem must be presented in linear programming form. This requires defining the variables of the problem, establishing inter-relationships between them and formulating the objective function and constraints. A model, which approximates as closely as possible to the given problem, is then to be developed. If some constraints happen to be nonlinear, they are approximated to appropriate linear functions to fit the linear programming format. In case it is not possible, other techniques may be used to formulate and then solve the model.

EXAMPLE 9.1 (Production Allocation Problem)

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below.

TABLE 9.1

Machine	Time per unit (minutes)			Machine capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is #4, #3 and #6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P) model that will maximize the daily profit.

Formulation of Linear Programming Model

Step 1:

From the study of the situation find the key-decision to be made. It this connection, looking for variables helps considerably. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2:

Assume symbols for variable quantities noticed in step 1. Let the extents. (mounts) of products, 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 units respectively.

Step 3:

Express the feasible alternatives mathematically in terms of variables. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1, x_2 and x_3, where x_1, x_2, x= ≥ 0, since negative production has no meaning and is not feasible.

Step 4:

Mention the objective quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., maximize $Z = 4x_1 + 3x_2 + 6x_3$.

Step 5:

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$2x_1 + 3x_2 + 2x_3 \leq 440$,

$4x_1 + 0x_2 + 3x_3 \leq 470$,

$2x_1 + 5x_2 + 0x_3 \leq 430$.

EXAMPLE 9.2 (Diet Problem)

A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in table 2.2.

TABLE 9.2

Food type	Yield per unit			Cost per unit (#)
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate linear programming model for the problem.

Formulation of L.P Model

Let x_1, x_2, x_3 and x4 denote the number of units of food of type 1, 2, 3 and 4 respectively.

Objective is to minimize the cost i.e.,

Minimize Z = #($45x_1 + 40x_2 + 85x_3 + 65x_4$).

Constraints are on the fulfilment of the daily requirements of the various constituents.

i.e., for protein, $3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$,

for fats, $2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$,

and for carbohydrates, $6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$,

EXAMPLE 9.3 (Blending Problem)

A firm produces an alloy having the following specifications:

(i) specific gravity ≤ 0.98,

(ii) chromium ≥ 8%,

(iii) melting point ≥ 450°C.

Raw materials A, B and C having the properties shown in the table can be used to make the alloy.

Table 9.3

Property	Properties of raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	440^{O}C	490^{O}C	480^{O}C

Costs of the various raw materials per ton are: #90 for A, #280 for B and #40 for C. Formulate the L.P model to find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Formulation of Linear Programming Model

Let the percentage contents of raw materials A, B and C to be used for making the alloy be x_1, x_2 and x_3 respectively.

Objective is to minimize the cost

i.e., minimize $Z = 90x_1 + 280x_2 + 40x_3$.

Constraints are imposed by the specifications required for the alloy.

They are;

$0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$,

$7x_1 + 13x_2 + 16x_3 \geq 8$,

$440x_1 + 490x_2 + 480x_3 \geq 450$,

and $x_1 + x_2 + x_3 = 100$,

as x_1, x_2 and x_3 are the percentage contents of materials A, B and C in making the alloy.

Also x_1, x_2, x_3, each ≥ 0.

EXAMPLE 9.4 (Advertising Media Selection Problem)

An advertising company wishes to plan its advertising strategy in three different media television, radio and magazines. The purpose of advertising is to reach as large a number of potential customers as possible. Following data have been obtained from market survey:

TABLE 9.4

	Television	Radio	Magazine I	Magazine II
Cost of an advertising unit	# 30,000	# 20,000	# 15,000	# 10,000
No. of potential customers reached per unit	200,000	600,000	150,000	100,000
No. of female customers reached per unit	150,000	400,000	70,000	50,000

The company wants to spend not more than #450,000 on advertising. Following are the further requirements that must be met:

at least I million exposures take place among female customers, advertising on magazines be limited to #150,000,

at least 3 advertising units be bought on magazine I and 2 units on magazine II, the number of advertising units on television and radio should each be between 5 and 10.

Formulation of Linear Programming Model

Let x_1, x_2, x_3 and x4 denote the number of advertising units to be bought on television, radio, magazine I and magazine II respectively.

The objective is to maximize the total number of potential customers reached.

i.e., maximize $Z = 10\ (2x_1 + 6x_2 + 1.5x_3 + x_4)$.

Constraints are;

on the advertising budget: $30{,}000x_1+20{,}000x_2+15{,}000x_3+10{,}000x_4 \leq 450{,}000$

or $30x_1+20x_2+15x_3+10x_4 \leq 450$,

on number of female customers reached by

the advertising campaign: $150{,}000x_1+400{,}000x_2+70{,}000x_3+50{,}000x_4= \ \geq$ 100,000

or $15x_1+40x_2+7x_3+5x_4 \geq 100$

on expenses on magazine

advertising: $15{,}000x_3+10{,}000x_4 \leq 150{,}000$ or $15x_3+10x_4 \leq 150$

on no. of units on magazines: $x_3 \geq 3$,

$x_4 \geq 2$,

on no. of units on television: : $5 \leq x_1 \leq 10$ or $x_1 \geq 5$, $x_1 \leq 10$

on no. of units on radio: $5 \leq x_2 \leq 10$ or $x_2 \geq 5$, $x_2 \leq 10$

where x_1, x_2, x_3, x_4, each ≥ 0.

EXAMPLE 9.5 (Inspection Problem)

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1,500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are #5 per hour while those of grade II inspector are #4 per hour. Any error made by an inspector costs #3 to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company find the optimal assignment of inspectors that minimizes the daily inspection cost.

Formulation of L.P Model

Let x_1 and x_2 denote the number of grade I and grade II inspectors that may be assigned the job of quality control inspection.

The objective is to minimize the daily cost of inspection. Now the company has to incur two types of costs: wages paid to the inspectors and the cost of their inspection errors. The cost of grade I inspector/hour is

(5 + 3 x 0.04 x 20) = #7.40.

Similarly, cost of grade II inspector/hour is

(4 + 3 x 0.08 x 14) = #7.36.

: The objective function is

minimize $Z = 8(7.40x_1 + 7.36x_2) = 59.20x_1 + 58.88x_2$.

Constraints are on the number of grade I inspectors : $x_1 \leq 10$,

on the number of grade II inspectors : $x_2 \leq 15$,

on the number of pieces to be inspected daily:

$20 \text{ X } 8x_1 + 14 \text{ X } 8x_2 \geq 1{,}500$

or $160x_1 + 112x_2 \geq 1{,}500$,

where $x_1, x_2 \geq 0$.

EXAMPLE 9.6 (Product Mix Problem)

A chemical company produces two products, X and Y. Each unit of product X requires 3 hours on operation I and 4 hours on operation IL while each unit of product Y requires 4 hours on operation I and 5 hours on operation II. Total available time for operations I and 11 is 20 hours and 26 hours respectively. The production of each unit of product Y also results in two units of a by-product Z at no extra cost.

Product X sells at profit of #I0/unit, while Y sells at profit of #20/unit. By-product Z brings a unit profit of #6 if sold; in case it cannot be sold, the destruction cost is # 4/unit. Forecasts indicate that not more than 5 units of Z can be sold. Formulate the L.P. model to determine the quantities of X and Y to be produced, keeping Z in mind, so that the profit earned is maximum.

Formulation of L.P Model

Let the number of units of products X, Y and Z produced be x_1, x_2, x_Z, where

x_Z = number of units of Z produced

= number of units of Z sold + number of units of Z destroyed

= $x_3 + x_4$ (say).

Objective is to maximize the profit. Objective function (profit function) for products X and Y is linear because their profits (#10/unit and #20/unit) are constants irrespective of the number of units produced. A graph between the total profit and quantity produced will be a straight line. However, a similar graph for product Z is non-linear since it has slope +6 for first part, while a slope of -4 for the second. However, it is piece-wise linear, since it is linear in the regions (0 to 5) and (5 to 2Y). Thus splitting x into two parts, viz, the number of units of Z sold (x_3) and number of units of Z destroyed (x_4) makes the objective function for product Z also linear.

Thus the objective function is

maximize Z= $l0x_1 + 20x_2 + 6x_3 - 4x_4$.

Constraints are

on the time available on operation I: $3x_1 + 4x_2 \leq 20$,

on the time available on operation II: $4x_1 + 5x_2 \leq 26$,

on the number of units of product Z sold: $x_3 \leq 5$,

on the number of units of product Z produced: 2Y = Z

or $2x_2 = x_3 + x_4$ or $-2x_2 + x_3+x_4 = 0$,

where x_1, x_2, x_3, x_4, each ≥ 0.

EXAMPLE 9.7 (Product Mix Problem)

A firm manufactures three products A, B and C. Time to manufacture product A is twice that for B and thrice that for C and if the entire labour is engaged in making product A, 1,600 units of this product can be produced. These products are to be produced in the ratio 3: 4: 5. There is demand for at least 300, 250 and 200 units of products A, B and C and the profit earned per unit is #90, #40 and #30 respectively.

Formulate the problem as a linear programming problem.

TABLE 9.5

Raw material	Requirement per unit of product (kg)			Total availability kg
	A	B	C	
P	6	5	2	5,000
Q	4	7	3	6,000

Formulation of L.P. Model

Let x_1, x_2 and x_3 denote the number of units of products A, B and C to be manufactured.

Objective is to maximize the profit. i.e.,

maximize $Z = 90x_1 + 40x_2 + 30x_3$.

Constraints can be formulated as follows:

For raw material P, $6x_1 + 5x_2 + 2x_3 \leq 5{,}000$,

and for raw material Q, $4x + 7x_2 + 3x_3 \leq 6{,}000$.

Product B requires 1/2 and product C requires 1/3rd the time required for product A.

Let t hours be the time to produce A. Then $t/_2$ and $t/_3$ are the times in hours to produce B and C and since 1,600 units of A will need time 1,600t hours, we get the constraint,

$t\,x_1 + t/2\,x_2 + t/3x_3 \leq 1{,}600t$ or $x_1 + x_2/2 + x_3/3 \leq 1{,}600$.

Market demand requires.

$x_1 \geq 300$,

$x_2 \geq 250$,

and $x_3 \geq 200$.

Finally, since products A, B and C are to be produced in the ratio 3: 4: 5, x_1: x_2: x_3:: 3: 4: 5

or $x_1/3 = x_2/4$,

and $x_2/4 = x_3/5$.

Thus there are two additional constraints

$4x_1 - 3x_2=0$,

$5x_2- 4x_3 = 0$,

where $x_1, x_2, x_3 \geq 0$.

EXAMPLE 9.8 (Trim Loss Problem)

A paper mill produces rolls of paper used in making cash registers. Each roll of paper is 100m in length and can be used in widths of 3, 4, 6 and 1 (km. The company production process results in rolls that are 24 cm in width. Thus the company must cut its 24cm roll to the desired widths. It has six basic cutting alternatives as follows:

Cutting alternatives	Width of rolls (cm)				Waste (cm)
	3	4	6	10	
1	4	3	-	-	-
2	-	3	2	-	-
3	1	1	1	1	1
4	-	-	2	1	2
5	-	4	1	-	2
6	3	2	1	-	1

The minimum demand for the four rolls is as follows:

Roll width (cm)	Demand
2	2,000
4	3,600
6	1,600
10	500

The paper mill wishes to minimize the waste resulting from trimming to size. Formulate the L.P model.

Formulation of L.P. Model

Key decision is to determine how the paper rolls be cut to the required widths so that trim losses (wastage) are minimum.

Let x, (j = 1, 2, ..., 6) represent the number of times each cutting alternative is to be used.

These alternatives result/do not result in certain trim loss.

Objective is to minimize the trim losses.

i.e., minimize $Z = x_3 + 2x_4 + 2x_5 + x_6$.

Constraints are on the market demand for each type of roll width:

For roll width of 3cm, $4x_1 + x_3 + 3x_6 \geq 2{,}000$,

for roll width of 4 cm, $3x_1 + 3x_2 + x_3 + 4x_5 + 2x_6 \geq 3{,}600$,

for roll width of 6cm, $2x_2 + x_3 + 2x_4 + x_5 + x_6 \geq 1{,}600$,

and for roll width of 10cm, $x_3 + x_4 \geq 500$.

Since the variables represent the number of times each alternative is to be used, they cannot have negative values.

∴ $x_1, x_2, x_3, x_4, x_5, x_6$, each ≥ 0.

EXAMPLE 9.9 (Production Planning Problem)

A factory manufactures a product each unit of which consists of 5 units of part A and 4 units of part B. The two parts A and B require different raw materials of which 120 units and 240 units respectively are available. These parts can be manufactured by three different methods. Raw material requirements per production run and the number of units for each part produced are given below.

TABLE 9.6

Method	Input per run (units)		Output per run (units)	
	Raw material 1	Raw material 2	Part A	Part B
1	7	5	6	4
2	4	7	5	8
3	2	9	7	3

Formulate the L.P model to determine the number of production runs for each method so as to maximize the total number of complete units of the final product.

Formulation of Linear Programming Model

Let x_1, x_2, x_3 represent the number of production runs for method 1, 2 and 3 respectively.

The objective is to maximize the total number of units of the final product. Now, the total number of units of part A produced by different methods is $6x_1 + 5x_2 + 7x_3$ and for part B is $4x_1 + 8x_2 + 3x_1$. Since each unit of the final product requires 5 units of part A and 4 units of part B, it is evident that the maximum number of units of the final product cannot exceed the smaller value of $\frac{6x_1 + 5x_2 + 7x_3}{5}$ and $\frac{4x_1 + 8x_2 + 3x_3}{4}$

Thus the objective is to maximize

$$Z = \text{Minimum of } \left\{ \frac{6x_1 + 5x_2 + 7x_3}{5}, \frac{4x_1 + 8x_2 + 3x_3}{4} \right\}$$

Constraints are on the availability of raw materials. They are, for

raw material 1, $7x_1 + 4x_2 + 2x_3 \leq 120$,

and raw material 2, $5x_1 + 7x_2 + 9x_3 \leq 240$.

The above formulation violates the linear programming properties since the objective function is non-linear. (Linear relationship between two or more variables is the one in which the variables are directly and precisely proportional). However, the above model can be easily reduced to the generally acceptable linear programming format.

Let $y = \left(\frac{6x_1 + 5x_2 + 7x_3}{5}, \frac{4x_1 + 8x_2 + 3x_3}{4} \right)$

It follows that $\frac{6x_1 + 5x_2 + 7x_3}{5} \geq y$ and $\frac{4x_1 + 8x_2 + 3x_3}{4} \geq y$

i.e., $6x_1 + 5x_2 + 7x_3 - 5y \geq 0$, and $4x_1 + 8x_2 + 3x_3 - 4y \geq 0$.

Thus the mathematical model for the problem is

Maximize $Z = y$,

subject to constraints $7x_1 + 4x_2 + 2x_3 \leq 120$,

$5x_1 + 7x_2 + 9x_3 \leq 240$,

$6x_1 + 5x_2 + 7x_3 - 5y \geq 0$,

$4x_1 + 8x_2 + 3x_3 - 4y \geq 0$,

where $x_1, x_2, x_3, y \geq 0$.

EXAMPLE 9.10 (Fluid Blending Problem)

An oil company produces two grades of gasoline P and Q which it sells at #30 and #40 per litre. The company can buy four different crude oils with the following constituents and Costs:

TABLE 2.7

Crude oil	Constituents			Price/litre (#)
	A	B	C	
1	0.75	0.15	0.10	20.00
2	0.20	0.30	0.50	22.50
3	0.70	0.10	0.20	25.00
4	0.40	0.10	0.50	27.50

Gasoline P must have at least 55 per cent of constituent A and not more than 40 per cent of C. Gasoline Q must not have more than 25 per cent of C. Determine how the crudes should be used to maximize the profit.

Formulation of Mathematical Model

Key decision to be made is how much of each crude oil be used in making each of the two grades of gasoline. Let these quantities in litres be represented by x_{IJ}, where i = crude oil 1, 2, 3, 4 and j = gasoline of grades P and Q respectively. Thus

x_{1p} = amount in litres of crude oil 1 used in gasoline of grade P

x_{2p} = amount in litres of crude oil 2 used in gasoline of grade P

… …

… …

x_{1q} = amount in litres of crude oil 1 used in gasoline of grade Q

X_{2q} = amount in litres of crude oil 2 used in gasoline of grade Q

… …

… …

Objective is to maximize the net profit.

i.e., maximize $Z = \# [30(x_{ip} + x_{2p} + x_{3p}, + x_{4p}) + 40(_{xiq} + x_{2q} + x_{3q} + x_{4q})$

$-20(x_{1p} + X_{ia}) - 22.50 (x_{2p} + x_{2q}) - 25 (x_{3p} x_{3q}) - 27.50(x_{4p} - x_{4q})$

or maximize $Z = \#[10x_{1p} + 7.50x_{2p} + 5x_{3p}, + 2.50x_{4p} + 20_{1q} + l7.50x_{2q} + l5x_{3q})$

Constraints are on the quantities of constituents A and C to be allowed in the two grades of gasoline.

i.e., $0.75x_{1p} + 0.20x_{2p} + 0.70x_{3p} + 0.40x_{4p} \geq 0.55 (x_{1p}, + x_{2p} + x_{3p} + x_{4p})$,

$0.10x_{1p} + 0.50x_{2p} + 0.20x_{3p} + 0.50x_{4p} \leq 0.40 (_{x1} + x_{2p} + x_{3p} + x_{4p})$,

and $0.10x_{1q} + 0.50x_{2q} + 0.20x_{3q} + 0.50x_{4q} \leq 0.25 (x_{1q} + x_{2q} + x_{3q} + x_{4q})$,

where $x_{1p}, x_{2p}, x_{3p}, x_{4p}, x_{1q}, x_{2q}, x_{3q}, x_{4q}$, each ≥ 0.

EXAMPLE 9.11 (Production Planning Problem)

A company manufacturing air coolers has, at present, firm orders for the next 6 months. The company can schedule its production over the next 6 months to meet orders on either regular or overtime basis. The order size and production costs over the next six months are as follows:

Month:	1	2	3	4	5	6
Orders:	640	660	700	750	550	650
Cost/unit (#) for						
regular production:	40	42	41	45	39	40
Cost/unit (#) for						

overtime production: 52 50 53 50 45 43

With 100 air coolers in stock at present, the company wishes to have at least 150 air coolers in stock at the end of 6 months. The regular and overtime production in each month is not to exceed 600 and 400 units respectively. The inventory carrying cost for air coolers is #12 per unit per month. Formulate the L.R model to minimize the total cost.

Formulation of L.P. Model

Key decision is to determine the number of units of air coolers to he produced on regular as well as overtime basis together with the number of units of ending inventory in each of the six months.

Let x_{ij} be the number of units produced in month j (j = 1, 2, ..., 6), on a regular or overtime basis (i = 1, 2). Further let y_j represent the number of units of ending inventory in month j (j= 1, 2, ..., 6).

Objective is to minimize the total cost (of production and inventory carrying).

i.e., minimize $Z = (40_{x11} + 42_{x12} + 41_{x13} + 45_{x14} + 39_{x15} + 40_{x16})$

$+ (52_{x21} + 50_{x22} + 53_{x23} + 50_{x24} + 45_{x25} + 43_{x26})$

$$+12(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

Constraints are

for the first month, $100 + x_{11} + x_{21} - 640 = y_1$,

for the second month, $y_1 + x_{12} + x_{22} - 660 = y_2$,

for the third month, $y_2 + x_{13} + x_{23} – 700 = y_3$

for the fourth month, $y_3 + x_{14} + x_{24} – 750 = y_4$

for the fifth month, $y_4 + x_{15} + x_{25} – 550 = y_5$

and for the sixth month, $y_5 + x_{16} + x_{26} – 650 = y_6$

Also, the ending inventory constraint is

$$Y_6 \geq 150$$

Further, since regular and overtime production each month is not to exceed 600 and 400 units respectively,

$x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$, each ≤ 600,

and $x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}$, each ≤ 400.

Also $x_{ij} \geq 0$ (i=1, 2; j=1, 2,..., 6), $y_j \geq 0$.

EXAMPLE 9.12 (Transportation Problem)

A dairy firm has two milk plants with daily milk production of 6 million litres and 9 million litres respectively. Each day the firm must fulfil the needs of its three distribution centres which have milk requirement of 7, 5 and 3 million litres respectively. Cost of shipping one million litres of milk from each plant to each distribution centre is given, in hundreds of naira below. Formulate the L.P model to minimize the transportation cost.

Distribution Centres

	1	2	3	Supply
	2	3	11	6
	1	9	6	9
Plants	7	5	3	

Demand

Formulation of L.P Model

Key decision is to determine the quantity of milk to be transported from either plant to each distribution centre.

Let x_1, x_2 be the quantity of milk (in million litres) transported from plant I to distribution centre no. 1 and 2 respectively. The resulting table representing transportation of milk is shown below.

Distribution Centres

1	2	3
x_1	x_2	$6-x_1-x_2$
$7-x_1$	$5-x_2$	$9-(7-x_1)$ $(5-x_2)$
7	5	3

Objective is to minimize the transportation cost.

i.e., minimize $Z = 2x_1 + 3x_2 + 11(6 – x_1 – x_2) + (7 - x_1) + 9(5 - x_2)$

$+ 6[9 - (7 - x_1) - (5 - x_2)] = 100 - 4x_1 - llx_2$.

Constraints are

$6 – x_1 - x_2 \geq 0$ or $x_1 + x_2 \leq 6$,

$7 - x_1 \geq 0$ or $x_1 \leq 7$,

$5 - x_2 \geq 0$ or $x_2 \leq 5$,

and $9 - (7 - x_1) - (5 - x_2) \geq 0$ or $x_1+x_2 \geq 3$, where $x_1, x_2 \geq 0$.

EXAMPLE 9.13 (Product Mix Problem)

A plant manufactures washing machines and dryers. The major manufacturing departments are the stamping department, motor and transmission deptt. and assembly deptt. The first two departments produce parts for both the products while the assembly lines are different for the two products. The monthly deptt.capacities are;

Stamping deptt. : 1,000 washers or 1,000 dryers

Motor and transmission deptt. : 1,600 washers or 7,000 dryers

Washer assembly line : 9,000 washers only

Dryer assembly line : 5,000 dryers only.

Profits per piece of washers and dryers are #270 and #300 respectively. Formulate the

L.P model.

Formulation of Linear Programming Model

Let x_1 and x_2 represent the number of washing machines and dryers to be manufactured each month.

The objective is to maximize the total profit each month.

i.e. maximize $Z = 270x_1 + 300x_2$.

Constraints are on the monthly capacities of the various departments.

For the stamping deptt., $\frac{x_1}{1,000} + \frac{x_2}{1,000} \leq 1$,

For the motor and transmission deptt.,

$$\frac{x_1}{1,600} + \frac{x_2}{7,000} \le 1$$

for the washer assembly deptt., $x_1 \le 9,000$

and for the dryer assembly deptt., $x_2 \le 5,000$

where $x_1 \ge 0$, $x_2 \ge 0$.

EXAMPLE 9.14 (Product Mix Problem)

A certain farming organization operates three farms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. Following are the data for the upcoming season:

Farm	Usable acreage	Water available in acre feet
1	400	1,500
2	600	2,000
3	300	900

The organization is considering three crops for planting which differ primarily in their expected profit per acre and in their consumption of water. Furthermore, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available.

Crop	Minimum acreage	Water consumption in	Expected profit
acre feet per acre		per acre	
A	400	5	# 400
B	300	4	# 300

C	300	3	# 100

However, any combination of the crops may be grown at any of the farms. The organization wishes to know how much of each crop should be planted at the respective farms in order to maximize expected profit. Formulate this as a linear programming problem.

Formulation of Linear Programming Model

The key decision is to determine the number of acres of each farm to be allotted to each crop.

Let x, (i = farm 1, 2, 3; j = crop A, B, C) represent the number of acres of the ith farm to be allotted to the jth crop.

The objective is to maximize the total profit.

i.e., maximize $Z = \# \left[400 \sum x_{1A} + 300 \sum x_{1B} + 100 \sum x_{1C}. \right]$

Constraints are formulated as follows:

For availability of water in acre feet,

$5x_{1A} + 4x_{1B} + 3x_{1c} \leq 1{,}500,$

$5x_{2A} + 4x2_{B} + 3x_{2C} \leq 2{,}000,$

$5x_{3A} + 4x_{3B} + 3x_{3C} \leq 900.$

For availability of usable acreage in each farm,

$x_{1A} + x_{1B} + x_{1C} \leq 400,$

$x_{2A} + x_{2B} + x_{3C} \leq 600,$

$x_{3A} + x_{3B} + x_{3C} \leq 300.$

For availability of acreage for each crop,

$x_{1A} + x_{2A} + x_{3A} \geq 400,$

$x_{1B} + x_{2B} + x_{3B} \geq 300,$

$x_{1C} + x_{2C} + x_{3C} \geq 300.$

To ensure that the percentage of usable acreage is same in each farm,

$$\frac{x_{1A} + x_{1B} + x_{1C}}{400} \times 100 = \frac{x_{2A} + x_{2B} + x_{2C}}{600} \times 100 = \frac{x_{3A} + x_{3B} + x_{3C}}{300} \times 100$$

or $3(x_{1A} + x_{1B} + x_{1C}) = 2(x_{2A} + x_{2B} + x_{2C})$, and

$(x_{2A} + x_{2B} + x_{2C}) = 2(x_{3A} + x_{3B} + x_{3C})$.

where $x_{1A}, x_{1B}, x_{IC}, x_{2A}, x_{2B}, x_{2C}, x_{3A}, x_{3B}, x_{3C}$, each ≥ 0.

The above relations, therefore, constitute the L.P. model.

3.6 ADVANTAGES OF LINEAR PROGRAMMING METHODS

Following are the main advantages of linear programming methods:

1. It helps in attaining the optimum use of productive factors. Linear programming indicates how a manager can utilize his productive factors most effectively by a better selection and distribution of these elements. For example, more efficient use of manpower and machines can be obtained by the use of linear programming.

2. It improves the quality of decisions. The individual who makes use of linear programming methods becomes more objective than subjective. The individual having a clear picture of the relationships within the basic equations, inequalities or constraints can have a better idea about the problem and its solution.

3. It also helps in providing better tools for adjustments to meet changing conditions. It can go a long way in improving the knowledge and skill of future executives.

4. Most business problems involve constraints like raw materials availability, market demand, etc. which must be taken into consideration. Just because we can produce so many units of products does not mean that they can be sold. Linear programming can handle such situations also since it allows modification of its mathematical solutions.

5. It highlights the bottlenecks in the production processes. When bottlenecks occur, some machines cannot meet demand while others. remain idle, at least part of the time. Highlighting of bottlenecks is one of the most significant advantages of linear programming.

3.7 LIMITATIONS OF'LINEAR PROGRAMMING MODEL

This model, though having a wide field, has the following limitations:

1. For large problems having many limitations and constraints, the computational difficulties are enormous, even when assistance of large digital computers is available. The approximations required to reduce such problems to meaningful sizes may yield the final results far different from the exact ones.

2. Another limitation of linear programming is that it may yield fractional valued answers for the decision variables, whereas it may happen that only integer values of the variables are logical.

For instance, in finding how many lathes and milling machines to be produced, only integer values of the decision variables, say x_1 and x_2 are meaningful. Except when the variables have large values, rounding the solution values to the nearest integers will not yield an optimal solution. Such situations justify the use of special techniques like integer programming.

3. It is applicable to only static situations since it does not take into account the effect of time. The O.R. team must define the objective function and constraints which can change due to internal as well as external factors.

4. It assumes that the values of the coefficients of decision variables in the objective function as well as in all the constraints are known with certainty. Since in most of the business situations, the decision variable coefficients are known only probabilistically, it cannot be applied to such situations.

5. In some situations it is not possible to express both the objective function and constraints in linear form. For example, in production planning we often have non-linear constraints on production capacities like setup and takedown times which are often independent of the quantities produced. The misapplication of linear programming under non-linear conditions usually results in an incorrect solution.

6. Linear programming deals with problems that have a single objective. Real life problems may involve multiple and even conflicting objectives. One has to apply goal programming under such situations.

When comparison is made between the advantages and disadvantages/limitations of linear programming, its advantages clearly outweigh its limitations. It must be clearly understood that linear programming techniques, like other mathematical tools only help the manager to take better decisions; they are in no way a substitute for the manager.

3.8 GRAPHICAL METHOD OF SOLUTION

Once a problem is formulated as mathematical model, the next step is to solve the problem to get the optimal solution. A linear programming problem with only two variables presents a simple case, for which the solution can be derived using a graphical or geometrical method. Though, in actual practice such small problems are rarely encountered, the graphical method provides a pictorial representation of the solution process and a great deal of insight into the basic concepts used in solving large L.P. problems. This method consists of the following steps:

1. Represent the given problem in mathematical form i.e., formulate the mathematical model for the given problem.

2. Draw the x_1 and x_2-axes. The non-negativity restrictions $x_1 \geq 0$ and $x_2 \geq 0$ imply that the values of the variables x_1 and x_2 can lie only in the first quadrant. This eliminates a number of infeasible alternatives that lie in 2nd, 3rd and 4th quadrants.

3. Plot each of the constraint on the graph. The constraints, whether equations or inequalities are plotted as equations. For each constraint, assign any arbitrary value to one variable and get the value of the other variable. Similarly, assign another arbitrary value to the other variable and find the value of the first variable. Plot these two points and connect them by a straight line. Thus each constraint is plotted as line in the first quadrant.

4. 1dentify the feasible region (or solution space) that satisfies all the constraints simultaneously. For type constraint, the area on or above the constraint line i.e., away from the origin and for type constraint, the area on or below the constraint line i.e., towards origin will be considered. The area common to all the constraints is called feasible region and is shown shaded. Any point on or within the shaded region represents a feasible solution to the given problem. Though a number of infeasible points are eliminated, the feasible region still contains a large number of feasible points

5. Use iso-profit (cost) function line approach. For this plot the objective function by assuming Z = 0. This will be a line passing through the origin. As the value of Z is increased from zero, the line starts moving to the right, parallel to itself. Draw lines parallel to this line till the line is farthest distant from the origin (for a maximization problem). For a minimization problem, the line is nearest to the origin. The point of the feasible region through which this line passes will be optimal point; It is possible that this line may coincide with one of the edges of the feasible region. In that case, every point on that edge will give the same maximum/minimum value of the objective function and will be the optimal point.

Alternatively use extreme point enumeration approach. For this, find the co-ordinates each extreme point (or corner point or vertex) of the feasible region. Find the value of the objective function at each extreme point. The point at which objective function is maximum/minimum optimal point and its co-ordinates give the optimal solution.

4.0 CONCLUSION

Linear programming involves with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equations and/or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resources such as market demand, production process and equipment, storage capacity, raw material availability and so on.

5.0 SUMMARY

All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. If the supply of these resources were unlimited, the need for management tools like linear programming would not arise at all. Supply of resources being limited, the management must find the best allocation of its resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent. However this involves a number of problems which can be overcome by quantitative methods, particularly the linear programming. Generally speaking, linear programming can be used for optimization problems if the following conditions are satisfied- there must be a well-defined objective function; there must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities in terms of variables; there must be alternative courses of action; decision variables should be interrelated and nonnegative; and the resources must be in limited supply. Linear Programming has the following assumptions- Proportionality, Additivity, Continuity, Certainty, and Finite Choices. LP solution methods can be applied in solving industrial problems, management related problems, and a host of other problem areas.

6.0 TUTOR MARKED ASSIGNMENT

1. Briefly discuss what linear programming involves.
2. Identify and discuss five assumptions of linear programming.
3. List and explain three areas where linear programming can be applied.
4. Highlight four limitations of linear programming.
5. Give five advantages of the linear programming method.
6. A manufacturer has two milk plants with daily milk production of 9 million litres and 11 million litres respectively. Each day the firm must fulfil the needs of its three distribution centres which have milk requirement of 9, 6 and 4 million litres respectively. Cost of shipping one million litres of milk from each plant to each distribution centre is

given, in hundreds of naira below. Formulate the L.P model to minimize the transportation cost.

Distribution Centres

	1	2	3	Supply
	3	4	12	9
	2	10	7	11
Plants	9	6	4	

Demand

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MODULE THREE

Unit 1: Computation of Areas by Calculus

Unit 2: Definite Integral

Unit 3: Indefinite Integral

Unit 4: Integration of Transcendental functions

Unit 5: Integration of Powers of Trigonometric functions

UNIT 1: COMPUTATION OF AREAS BY CALCULUS

1.0 Introduction

2.0 Objectives

3.1 Area under a Curve

3.2 Partition of a Closed Internal

3.3 Computation of Area as Limits

4.0 Conclusion

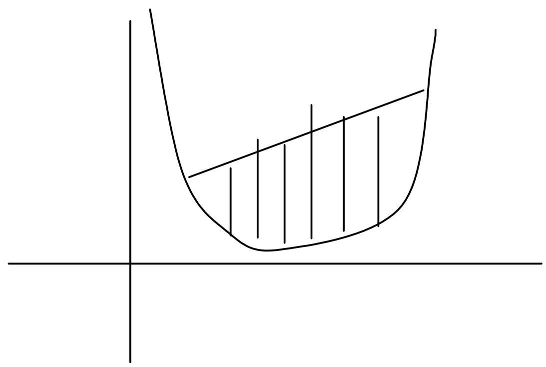
5.0 Summary

6.0 Tutor Marked Assignment

7.0 References/Further Reading

1.0 INTRODUCTION

One of the early mathematicians that attempted to find the area under a curve was a Greek named Archimedes. He used ingenious methods to compute the area bounded by a parabola and a chord. See Fig (1.1).



In this unit, you will study how to develop necessary tools of calculus to compute areas under curve as a mere routine exercise. The area under a curve gave birth to the second branch of calculus known as integration. The tools that will be developed here will naturally lead to the definition of integration in the next unit – unit 2. Recall that the word to integrate connotes "whole of" which could be interpreted to mean "find the whole area of". This concept is what will be introduced in this unit and this will be fully developed in the next unit.

2.0 OBJECTIVES

After studying this unit, you should be able to correctly

i. Approximate area under a curve by the sum of areas of rectangles inscribed in the curve.

ii. Approximate the area under a curve by the sum of the areas of rectangles circumscribed over the curve.

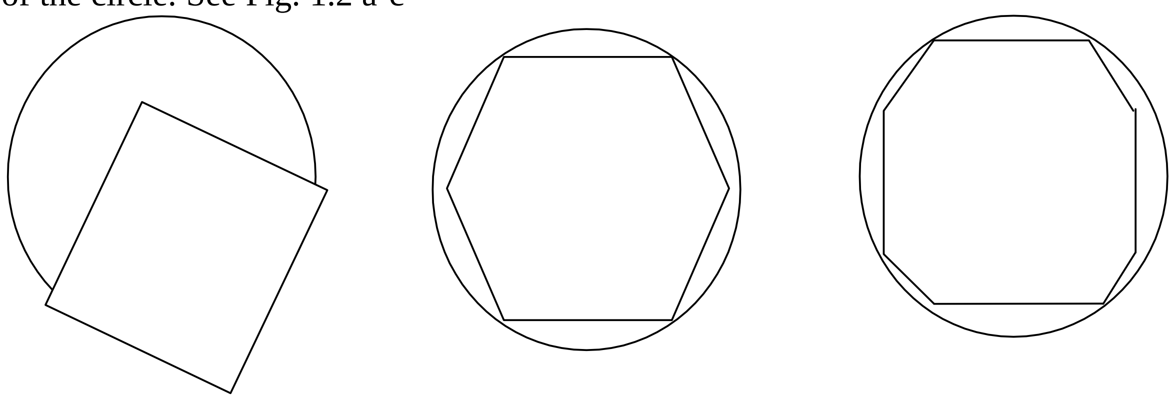
iii. Define a partition of a closed interval (a, b)

iv. Compute the exact value of the area under a curve by the limiting process.

3.1 AREA UNDER A CURVE

You are quite familiar with the computation of the areas of plane figures such as triangles, parallelogram trapezium, regular polygons etc. Interestingly, you studied in elementary geometry that the area of a regular polygon can be computed by cutting it into triangles and sum up the areas of the triangles.

You are also aware t hat the area of a circle is π^{r2}. This formula was derived by the method of limit. You could recall that the limit of the areas of inscribed regular polygons as the number of sides approaches infinity is equal to the area of the circle. See Fig. 1.2 a-c



Inscribed polygons polygons | **Inscribed polygons** | **Inscribed**

of 4 sides | of 6 sides | of 8 sides

Fig. 1.2a | **Fig. 1.2b** | **Fig. 1.2c**

Let y = f(x) be a continuous function (see the first course on calculus i.e. calculus I unit 4 for definition of continuous function) of x on a closed interval [a, b]. In this case for better understanding, you assume that the f(x) is positive in the closed interval i.e. f(x) $\geq$ 0. For all XE [a,b]. Then the problem to be considered is to calculate the area bounded by the graph y = f(x) and the vertical lines y = f(a) and y = f(b) and below by the x – axis as shown in Fig. 1.3.

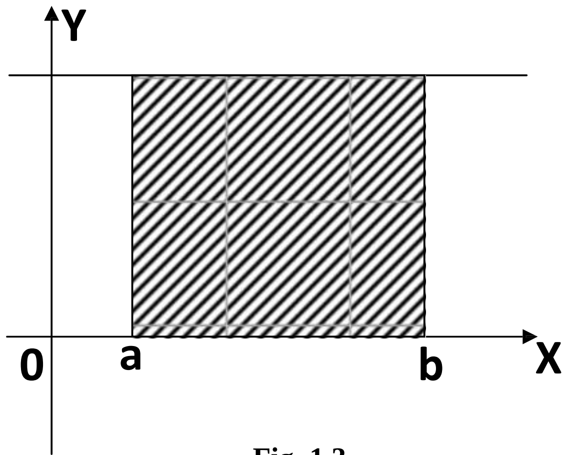


Fig. 1.3

You can start by dividing the area into n thin strips of uniform width

$\Delta x = \frac{(b-a)}{n}$ by lines perpendicular to the x – axis at the end points x = a and

x = b and many intermediate points which can be numbered as X_1, X_2, Xn_{-1}

(see fig 1.4)

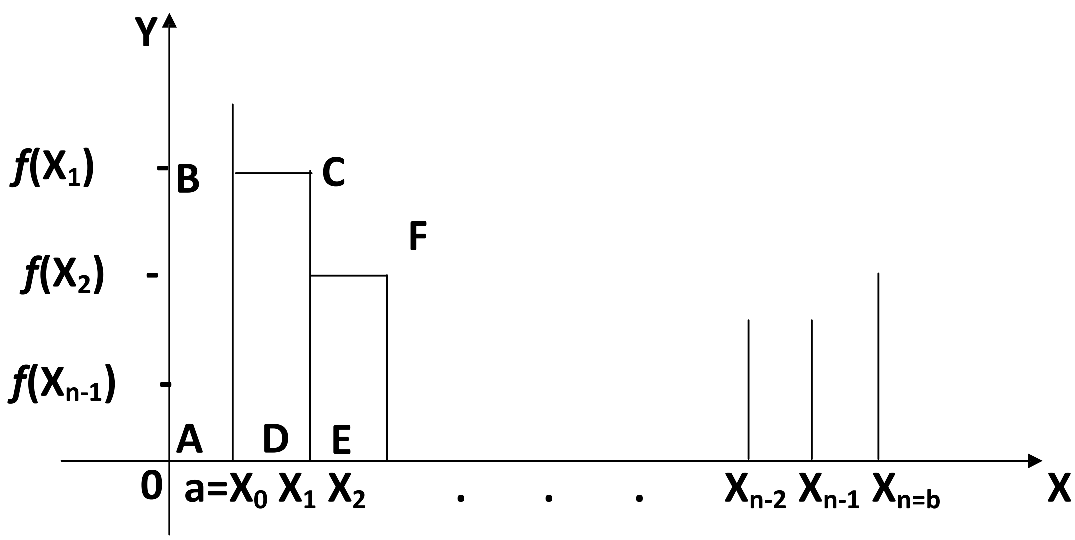


Fig. 1.4

The sum of the areas of these n rectangular strips gives an approximate value for the area under the curve. To put the above more mathematically, you can define the area of each strip in terms of f(x) and x. Given that $\Delta x = x_1 - a = x_2 - x_1 = .. = b - x_{n-1}$. For example the area of the rectangular strip ABCD in Fig. 1.4 above is given as:

Area of ABCD = $f(x_2) \cdot (x_1 - x_0) = f(x_2)\ \Delta x$

Example:

Suppose $f(x) = x^2 - 3$ in Fig 1.5 with n = 6 were a = 2, b = 8, dx = 8-2 = 6

Therefore: dx = 1 i.e. you have 6 rectangular strips.

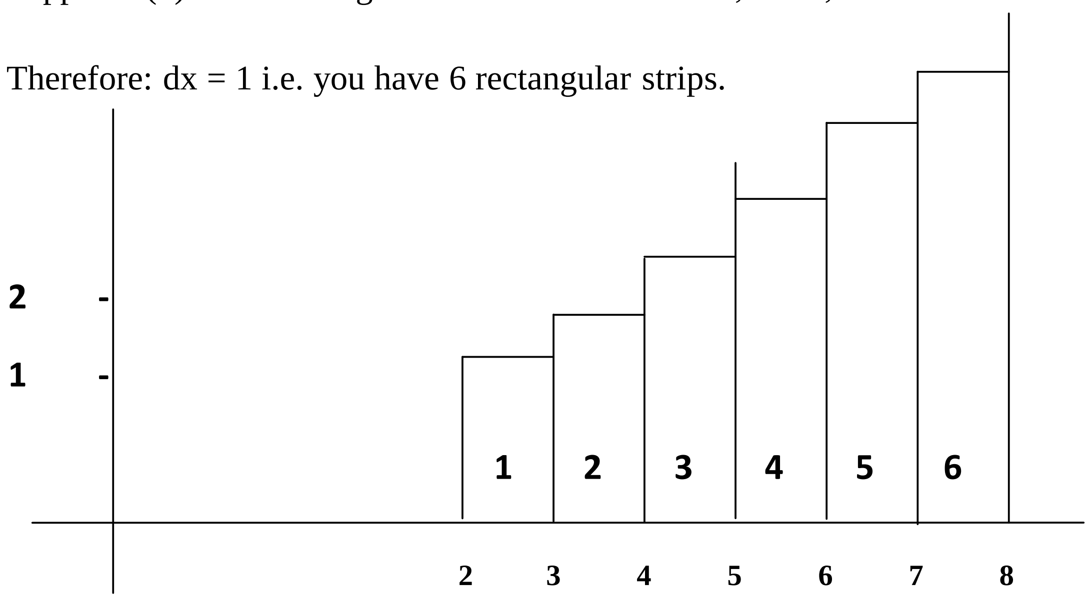


Fig. 1.5

Area is given as sum of

f(2) Δx = 1.1 = 1

f(3) Δx = 6.1 = 6

f(4) Δx = 13.1 = 13

f(5) Δx = 22.1 = 22

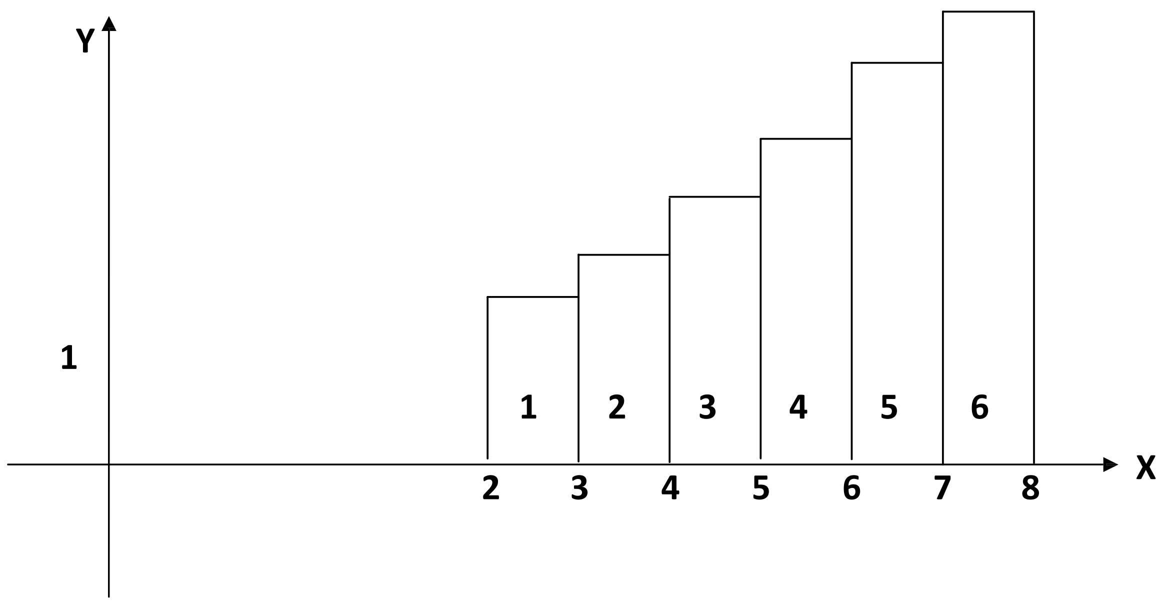
f(6) Δx = 33.1 = 33

$f(7)\ \Delta x = 46.1 = 46$

In fig. 1.4 above the area under the curve is larger than the sum of the areas of the inscribed rectangular strips numbered 1 to 6 i.e. sum of areas of strips = 1+6+13+22+33+46 = 121 which is less than area under curve.

Example:

Using the same example $Y = x^2-3$ use circumscribed rectangular strips instead of inscribed ones to compute the area under the curve. See Fig. 1.6



Area is given as the sum of

$f(x)\ .\ \Delta x = 6.1 = 6$

$f(x)\ .\ \Delta x = 13.1 = 13$

$f(x)\ .\ \Delta x = 22.1 = 22$

$f(x)\ .\ \Delta x = 33.1 = 33$

$f(x)\ .\ \Delta x = 46.1 = 46$

$f(x) . \Delta x \quad = \quad 61.1 \quad = \quad 61$

Area = 6+13+22+33+46+61 = 181

As should be expected this area is greater than the area under the curve $f(x) = x^2-3$.

In the computation with the circumscribed rectangular strips the sides of the rectangles are assumed in this case to be the points of the function in their respective subintervals. In the case of the inscribed rectangles, the sides of the rectangles are the minimum values of the function in their respective subintervals.

Therefore the area under the curve lies between the sum of the areas of the inscribed rectangles and the sum of the areas of the circumscribe rectangles. This takes to the issue of limit. Therefore it will be right to say as $n \rightarrow \alpha$ $\Delta x \rightarrow 0$ this implies that the Lim (Max Area – Min Area) = 0 as Δx 0.

From the foregoing, you can now define the area under curve as the limit of the sums of the areas of inscribed (circumscribed) rectangles as their common base of length dx approaches zero and the number of rectangles increases without bound. In symbols you can write the above limit as:

$A = \lim [f(x_0) \Delta x + f(x_7) \Delta x +..+ f(x_{n-1})] \Delta x \; n \rightarrow \infty$

$= \lim [f(x_1) \Delta x + f(x_2) f +..+ f(x_{n-1})] \Delta x \; n \rightarrow \infty$

OR

$$A = \lim_{n \rightarrow X_1} \sum_{K=10}^{n^{-1}} f(x_k)\, dx = \lim_{n \rightarrow X_1} \sum_{K=1}^{n} (fx_k)\, dx$$

Exercise

Repeat the above example using n = 10. Find the difference between the sum of areas of the inscribed rectangle (i.e. the minimum area) and the sum of areas of the circumscribed rectangles (i.e. the maximum area).

3.2 PARTITION OF A CLOSED INTERVAL

Let [a, b] be a bounded closed interval of real numbers. A partition of a closed interval [a, b] is a finite set of points

$P = \{a = x_0, x_1, x_2, .., x_{n-1}, x_{n-b}\}$ where

$a = x_0 < x_1 < x_2 < .. < x_{n-1}, x_n = b$

Example:

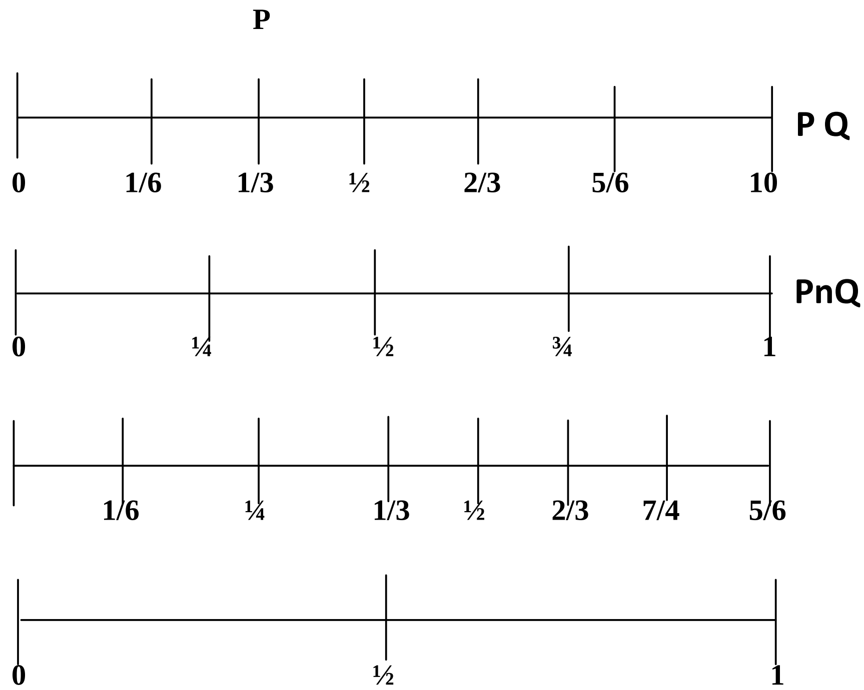
P = {0, 1/6, 1/3, ., 2/3, 5/6, 1} and

Q = {0, ¼ , ½ , ¾ , 13 are both partition of {0, 13}

PUQ = {0, 1/6, ¼ . 1/3 , ½ , 2/3 , ¾ , 5/6 1} is a partition of [0, 1]

PnQ = {0, ½ , 1} is a partition of [0,1]

See fig. 1.6(a) to (c)



A partition of P = {x_0, x_1, .., x_n} of [a, b] divides [a, b] into n closed sub interval [x_0, x_1], [x_1, x_2], …, [x_{n-1}, X_n]

The closed interval

[x_{r-1}, x_r] is called the r^{th} subinterval of the partition.

Given a partition of P[a = x_0, x_2, …, x_n = b] the length of the subinterval s are the same and it is denoted by $\Delta x_r = x_r - x_{r-1}$

This equal to the length of the interval [a, b] divided by the number of subintervals n

i.e. $\Delta x_r = \frac{b-a}{n}$

Example: Δx for p is $\frac{1-0}{6} = 1/6$

Δx for Q is $\frac{1-0}{4} = ¼$

Not in all case you will get subintervals of the same length. Example is PUQ

The length of $x_1 - X_0 = 1/6 - 0 = 1/6$

The length of $x_2 - X_1 = - 1/6 = 1/12$

Such partitions in which the subintervals are not of the same length are called irregular partition.

Exercise: Write down a regular partition for

(1) [2, 8], n = 12

(2) [1, 8], n = 7

Ans:

(i) [2, 5/2, 6/2, 7/2, 8/2, 9/2, 10/2, 11/2, 12/2, 13/2, 14/2, 15/2, 16/2]

(ii) [1, 2, 3, 4, 5, 6, 7, 8]

3.3 COMPUTATION OF AREAS AS LIMITS

In this section you will combine the results of section 3.1 and 3.2 to compute the areas under curves using the limiting process.

Example

A good starting point is to consider the area under the curve Y = X (see Fig. 1.7)

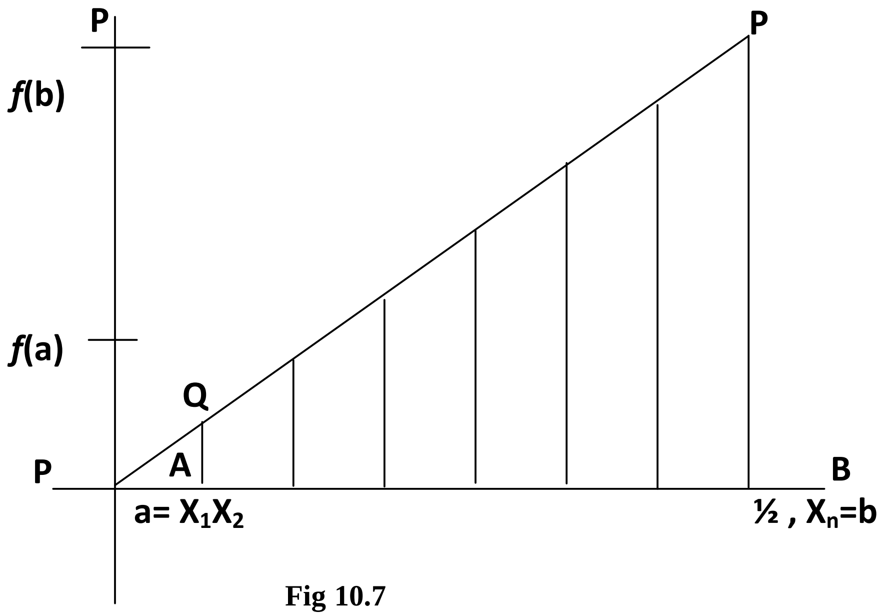


Fig 10.7

which the interval Xϵ [a, b] let there be n-regular partition of [a, b] i.e.

$$\Delta x = \frac{b - a}{n}$$

P [a, x_1, x_2, .., x_{n-1}, x_n = b]

x_1 = a + xx

$x_2 = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$x_{n-1} = a + (n-1)\ \Delta x$

Areas of inscribed rectangles are

$f(a)\ .\ \Delta x = a.\Delta x$

$f(x_1)\ .\ \Delta x = (a+\Delta x).\Delta x$

$$f(x_2) \cdot \Delta x = (a+2\Delta x) \cdot \Delta x$$

$$f(x_{n-1}) \Delta x = (a+(n-1)\Delta x) \cdot \Delta x$$

Sum of the areas of the rectangles is given as

$$\sqrt{} = (a \cdot \Delta x + (a + \Delta x) + \ldots + (a+ (n-1) \Delta x).\Delta x$$

$$= [a+((1+2+3+\ldots(n-1)\ dx$$

$$= na + (\sum_{K=1}^{n-1}, K) \Delta x) \Delta x$$

($\sum^{n-1} K = \frac{(n-1)n}{n}$ (The sum of an arithmetic 1 progression with different d=1)

$$S= [na + \frac{(n-1)n}{2} \underline{dx}]\ dx$$

but $dx = \frac{b - a}{n}$ therefore

$$S = [na + \frac{(n-1)n}{2} \frac{b - c)}{n}] \frac{b - c}{n}$$

$$= [a + \frac{b - a}{2} \frac{n - 1}{n}] (b - a)$$

Taking limit as $n \to \infty$ you get

$$\lim_{n \to \infty} S = \lim_{n \to \infty} (a + \underline{b - a} \cdot \frac{n - 1}{(2 n)} (b - a)$$

$$= (a + \frac{b - a}{2}) (b - a) \text{ lims}_{n\ \infty \to} \frac{n - 1}{n}$$

$$= \quad \left(a + \frac{(b-a)}{2}\right)(b-a) \,.\, 1$$

$$= \quad \frac{a+b}{2} \,.\, (b-a)$$

In fig. 10.8, the area of trapezium AQPB is the same as the area under the curve and as you know the area of trapezium is given as:

½ base x sum of two parallel sides

= ½ (b – a) x f(a) + f(b)

= ½ (b – c) (b + a)

Example

Find the area under the graph Y = x + 1 $0 \leq x \leq 6$

Solution:

Let n be a positive integer that there be a partition of [a, b] into n regular partition.

Therefore; $\Delta x = \frac{b}{n}$

$x_1 = \Delta x$

$x_2 = 2\Delta x$

$x_3 = 3\Delta x$

$x_{n-1} = (n-1)\,\Delta x$

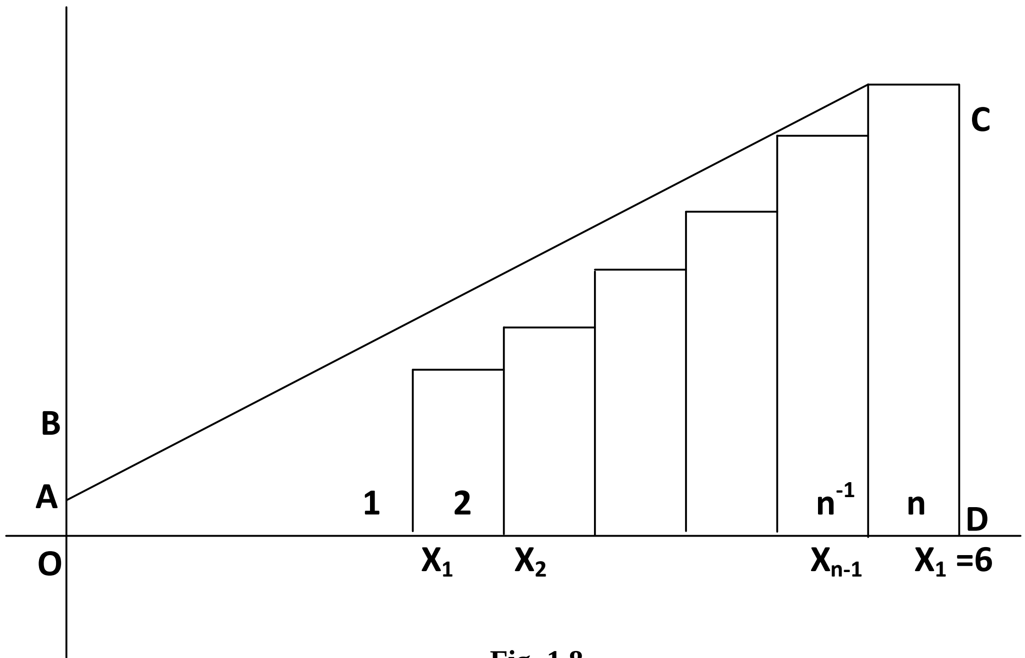


Fig. 1.8

Area of (n-1) rectangles is given as

$f(0).\Delta x = 1.\Delta x$

$f(x_1).\Delta x = (\Delta x + .1)\ \Delta x$

sum of areas of rectangles is

$S = \Delta x + (\Delta x+1)\Delta x+ (2\Delta x+1)\ \Delta x+(3\Delta r+1)\Delta x$

$+ \ldots + (n\text{-}1)\ \Delta x +1)\ \Delta x$

$= (\Delta x+ (\Delta x+1)+(2\Delta x+1) + : (n\text{-}1)(\Delta x+)\ \Delta x$

$= [\Delta x + n +) \sum_{K=1}^{n-1} k\Delta x]\ \Delta x$

$S = [\Delta x + (n+ \frac{(n-1)}{2} n\ \Delta x]\ \Delta x$ let $\Delta x = \frac{l}{n}$

then

S = [b] + [(n + (n-1) n b)] b
n] [(2) n)] n

= b + (1 + b . n -1)
n 2) . b

Taking limits as n → ∞

Lim S = lim b + b lim (1 + b (n-1)
n 2 n

n → ∞ n – 1 ∞ n → ∞

= 0 + b (1 + b/2)

lim S = b + b 2

n → ∞ 2

4.0 CONCLUSION

In this unit, you have studied how to find an approximate value of the area under a curve by computing the sums of areas of rectangles inscribed under the cure and circumscribed over if you have defined a partition of a closed interval. You have studied that as the number of partitions of a closed interval [a, b] is increased without bound the value of the sum of the areas of the rectangles (inscribed or circumscribed) approaches the exact value of the area under the curve in the given interval [a, b] that is the limit of the sum of areas of the rectangles is equal to the exact area under the given curve as the number n of partition tends to infinity or the length dx of the subinterval of the partition tends to zero.

5.0 SUMMARY

In this unit you studied how to

- Compute the minimum value of area under a curve i.e. sum of area of rectangles inscribed under a curve within an interval

- Compute the maximum value of the area under a curve i.e. the sum of areas of rectangles circumscribed over the curve.

- Define a partition of a closed interval [a, b] i.e. $a = x_1 < x_2 < \ldots < x_2 = b = P[a, b]$

- Compute the exact area under a curve in a given interval [a, b] by taking the limit of the sum of the areas of the rectangles (inscribed or circumscribed) as the number n of partition of [a, b] is increased without bound i.e.

$$A = \lim_{n \to \infty} \int_n \text{ where } dx = \frac{b-a}{n}$$

6.0 TUTOR MARKED ASSIGNMENT

1. Show that the sets

   {0, 1} {0, ., ½ ,1}, {0, ¼, ½, 1} and

   {0, 1/4 , 1/3, ½, 5/8, 1} are partition of {0, 1}

2. Which of the partition of [0, 1] in exercise (1) above are regular?

3. Find the minimum and maximum values of the area under the curve $f(x) = 2x$ for $x \in [0, 1]$ and P(0, ¼, ½, 1)

2. Find the minimum value of the area under the curve f(x)= 1- x xe [0,2] P($0_1$1/3, ¾ 1, 2).

3. Find the area under the curve Y = x^2 X€[0, b] by taking appropriate limits.

4. Find the area under the curve Y = mx a ≤ x ≤ b by taking appropriate limits.

5. Sketch the graph of Y = X + 1. Divide the internal into n = 6 subintervals with dx = (b-a)/6. Sketch the inscribed rectangles.

6. Repeat Σ x 7 but this time sketches the circumscribed rectangle.

7. Compute the sums of areas in € x 7 and € x 8 above.

8. Find the area under the curve Y = x + 1 a ≤ b by taking appropriate limits of results of exercise 9 above.

7.0 REFERENCES

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UNIT 2: DEFINITE INTEGRAL

1.0 INTRODUCTION

In unit 1, you studied how to compute the area under a curve and showed how you could estimate it by computing sums of area of rectangles. Using the above estimate you applied the concept of limit to get the exact value of the area under a curve. These methods were applied to functions or graphs that could easily be sketched i.e. not too complicated functions. In this unit, you will be introduced to the famous path taken by Leibniz and Newton in showing how exact areas can be computed easily by using integral calculus. It is necessary you refer once more to unit 1 of this course before embarking on this one. It will help you have a proper grasp of this unit if you do so.

2.0 OBJECTIVES

After studying this unit you should be able to:

- Define the definite integral of a function within an interval [a, b].
- Evaluate definite integrals of function.
- State the fundamental theorem of integral calculus.

3.0 DEFINITION OF DEFINITE INTEGRAL

In unit 1, you studied that the sum of the areas of inscribed rectangles gives a lower (minimum) approximation of the area under the curve of the function f(x). If you list all the values of the function f(x) in a given interval [a, b] and take the least among these value you will have what is known as the infimum of f(x) for all xЄ [a, b]

i.e. Inf f(x) XЄ[a, b].

let Inf f(x) = M_r and XЄ P[a, b]

when $dx_r = x_r - x_{r-1}$. then the area is $M_r . dx_r$. The sum of such area is

$A_L = \Sigma M_r (x_r - x_{r-1})$ is called the Lower Sum of the function f(x).

If you take the maximum value of f(x) within [a, b] and find their areas i.e. M_r = ∫up f(x) XЄ [x_{r-1} x_r] then the Upper sum for the areas is given as

$$A_u = \sum_{r=1} M_r(x_r - x_{r-1})$$

No known concept has been introduced. You are rewriting sum of areas of a rectangles inscribed under the curve f(x) as $A_L = \Sigma M_r (x_r - x_{r-1})$ and the sum of areas of rectangles circumscribed over f(x) as $A_u = \Sigma M_r\ x_r - x_{r-1})$

Once you keep the fact you will not run into any difficulty understanding what follow next. Definition: The unique number I which satisfies the inequality $A_L(P) \leq I \leq A_u(P)$ for all partitions P of [a, b] is called the definite integral (or more simply the integral of f on [a, b] and is written as: $I = \int_a^b f(x)dx$

This symbol S dates back to Leibniz and it is called integral sign. It is an elongated S, which represents sum. The numbers in this case are called the limits of integration. The expression $\int_a^b f(x)\, dx$ (read integrating from b to a with respect to x)

In the above definition, it has been assumed that f(x) is continuous in the closed [a, b]. This condition guarantee the existence of a number I such that $A_L(p) \leq I \leq A_L(p)$.

The prove of the above theorem could be found in the text suggested for further reading given at the end of this course.

If $f(x) \geq 0 \forall X \in [a, b]$ then

$I = \int_a^b f(x)dx$ = Area under the curve f(x)

Example

Given that $f(x) = K \ \forall \ X \in [a, b]$ show that $\int_a^b f(x)dx = K(b - a)$

Solution: Let $P = \{u, x_o, x_1, x_2, .., x_n = b\}$ be any partition of [a, b]

Since $f(x) = K \forall X \in [a, b]$ the $f(x_0) = f(x_1) = \ldots = f(x_n)$

Let $A_L (P) = \Sigma\, m\Delta X_r = K\Delta X_1 + K + \Delta X_2 + ..\, K\Delta X_n$

$= K(\Delta X_1 + ... + \Delta X_n) = K(b-a)$

Also

$A_u(P) = \Sigma M_r \Delta X_r = K(b-a)$

But

$A_L(P) \leq \int_a^b f(x)dx \leq A_u(P)$

then $K(b-a) \leq \int_a^b f(x)dx \leq K(b-a)$

$\rightarrow \int_a^b f(x)dx = K(b-a)$

Example

Given that $f(x) = x$ show that $\int_a^b xdx = ½ (b^2-a^2)$

Solution: Let $P\{ax_o, x_1, .. , x_n, = b\}$ be an arbitrary partition of [a, b].

$f(x) = x$ for $X Є [X_r, X_{r2},]$ for all such subintervals.

So $M_r \leq f(x)\ m_r\ X Є [x_r, x_{rt1}]$ such m_r and m_r exist for each subinterval.

Let $M_r = x_r$ and $m_r = x_{r-1}$

then

$$A_u(P) = \sum_{r=1}^{n} mr\Delta x_r = \sum_{r=1}^{n} x_r\Delta x$$

$= x_1(x_1-x_o)+x_2(x_2-x_1)+\ldots+ x_n(x_n-x_{r-1})$

and

$$AL(P) = \sum_{r-1}^{n} m_r\Delta x_r = \sum_{r-1} x_{r-1}\Delta x_r$$

$= x_o(x_1-x_o) + x1(x_2-x_1) + .. x_{2-1}(x-x_{n-1})$

For each index,

$x_{r-1} \leq ½ (x_r + x_{r-1}) \leq x_r$

Therefore

$A_L(P) \leq ½ (x_1+x_o) (x_1-x_o) + ½ (x_2+x_1) (x_2-x_1) + \ldots+ ½ (x_n+x_{n-1})$

(x_n-x_{r-1})

$\leq A_u(P)$

But $½ (x_1+x_o) (x_1-x_o) + ½ (x_2+x_1) (x_2-x_1) + .. + ½ (x_n+x_{n-1})$

(X_n+X_{n-1})

$= ½ (x_1^2 – x_0^2 + x_2^2 – x_2^2 + … + x^2 – x2^{n-1}) = . (x^{2n} – x^2o)$

➔ $A_L (P) \leq ½ (x^2_n – x^2_o) \leq A_u(P)$

➔ $A_L (P) \leq ½ (b^2-a^2)$

➔ $\int_a^b x\, dx = ½ (b^2 - a^2)$

The following properties of definite integral are hereby stated without their proofs are beyond the scope this course:

1. If $a < c < b$ then $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

2. If $a < b$ then $–\int_a^b f(x)dx = \int_b^a f(x)dx$

3. $\int_a^a f(x)\, dx = 0$

Example: Given that

$\int_0^1 f(x)dx = 6, \int_1^3 f(x)dx = 5$

$\int_3^7 f(x)dx = 2$

Find (i) $\int_1^7 f(x)dx$ (ii) $\int_1^3 f(x)dx$ (iii) $\int_1^1 f(x)dx$ (iv) $\int_7^1 f(x)dx$

Solution:

(i) $\int_0^7 f(x)dx$

$\int_0^7 f(x)dx = \int_0^t f(x)dx + \int_x^7 f(x)dx$

let t = 3 i.e. $\int_0^7 f(x)dx = \int_0^3 f(x)dx + \int_3^7 f(x)dx = 5 + 2 = 7$

(ii) $\int_1^3 f(x)dx = \int_1^t f(x)dx + \int_t^3 (x)dx$

let t = 0 $\int_1^3 f(x)dx \int_1^0 f(x)dx + \int_0^3 f(x)dx = -6 + 5 = 1$

(iii) $\int_1^1 f(x)dx = 0$

(iv) $\int_7^1 f(x)dx = -\int_1^7 f(x)dx = [\int_1^3 f(x)dx + \int_1^7 f(x)dx = -[-1+2]$

Exercise: Given that $\int_2^2 f(x)dx = 2$, $\int_0^3 f(x)dx = 4$ and $\int_2^4 f(x)dx = 7$

Find (i) $\int_0^4 f(x)dx$ (ii) $\int_4^3 f(x)dx$ (iii) $\int_3^2 f(x)dx$

(iv) $\int_2^3 f(x)dx$

Answer: (i) 9 (ii) -5 (iii) -2 (iv) 2

3.2 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

To find the value of the function $F(x) = \int_a^b f(x)dt$ for some simple function it could easily be evaluated. Either by the limiting process discussed in unit or by direct evaluation as was done in the previous section. Such process might prove very laborious for certain classes of functions. In this section you will examine the direct connection between differential calculus and integral calculus. This connection was made possible by looking at the summation process of finding areas and volumes and the differentiation process of finding the scope of a target to a curve. It is quite interesting that the process of carrying out inverse differentiation yields an easy tool of solving the summation problem.

So you will now discuss the proof of the fundamental theorem concept behind the theorem is that before you can evaluate a definite integral $S_a^b f(x)dx$ you will first of all find a function Fl(x) whose derivative is f(x). i.e. F1(x) = f(x) ∀Є (A, B). You will now as first step study the proof of the following theorem:

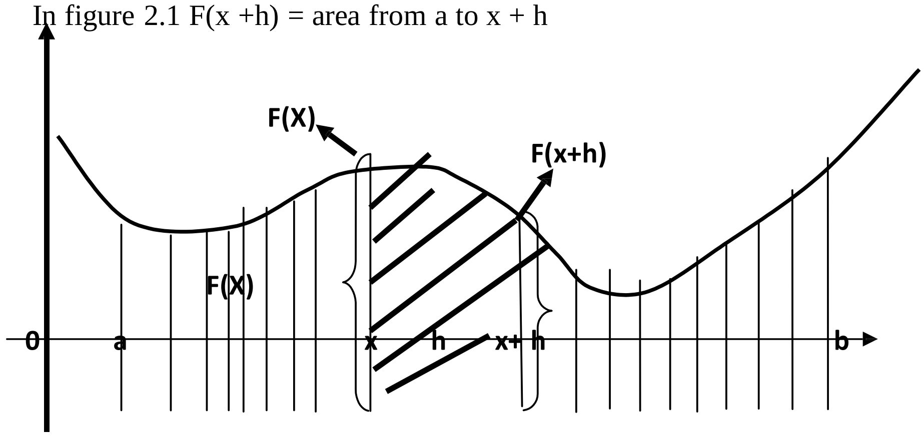
Theorem 1: If f(x) is a continuous function on [a, b], the function F(x) defined on [a, b] by setting $F(x) = \int_a^x f(t)\, dt$ is a(i) continuous function on [a, b] and (ii) satisfies $F^1(x) = f(x)$ for all x in (a, b).

Proof: You will begin with xЄ [a, b] and show that

Lim $\frac{F(x+h) - F(x)}{h} = f(x)$

$h \to O^-$

In figure 2.1 F(x +h) = area from a to x + h



f(x) = area from a to x

f(x+h) –F (x) = area from x to x + h

(Area = base x height)

$\frac{F(x + h) - F(x)}{h} = \frac{\text{area from x to x + h}}{h} \cong f(x)$ if $h \to 0$

(Note f(x) = height of the area under curve in Fig. 2.1

If $x < x + h \leq b$ then

$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$

(since from statement of theorem $F(x) = \int_a^x f(t)dt$)

It follows therefore that

$F(x+h) - F(x) = \int_a^{x+h}(f(t)dt + \int_x^a f(t)dt$

$= \int_x^{x+h}(f(t)dt$

Let M_h = maximum value of f(x) on [x, x+h]

and m_h = minimum value of f(x) on [x, x+h]

since $M_h (x+h - x) = M_h.h$

and $m_h (x+h - x) = m_h.h$

therefore, M_h = upper sum (see UNIT 1) and m_h = lower sum (see UNIT 1)

therefore

$m_h.h \leq \int_x^{x+h} f(t)dt \leq M_h.h$

$= m_h.h \leq \frac{F(x+h - F(x)}{h} \leq M_h.h$

since f(x) is a continuous function on [x, x + h] therefore

$\lim_{H \to 0^-} m_h.h = f(x) = \lim_{h \to 0^-} M_h.h$

thus $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$.- I

In a similar manner you can show that if X Є (a, b), then

$\lim_{h\, 0^+} \frac{F(x+h) - F(x)}{h} = f(x)$ -I

Now if xЄ (a, b) then equation (I) and (II) hold

Thus $\lim_{H \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$

and $\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F^1 (x)$

therefore $F^1(x) = f(x)$

since $F^1(x)$ exists then F(x) must be continuous on (a, b). Before you prove the fundamental theorem of calculus. Look at this definition.

Definition:

A function F(x) is called an anti-derivative for f(x) on (a. b) if and only if

(i) F(x) is continuous on (a, b) and

(ii) F1(x) = f(x) for all XЄ (a, b)

Using the above definition you can rewrite theorem 1 as

If f is continuous on (a, b) then

$F(x) = \int^x f(t)dt$

The above now says to you that you can construct or find an anti-derivative for f(x) by integration f(x). The next theorem you are going to study will tell you that you can evaluate the definite integral $\int_a^h f(x)dx$ by finding an anti-derivative for f(x).

The Fundamental Theorem of Integral Calculus:

Let f(x) be continuous for all xЄ (a, b) If P(x) is an anti-derivative of f(x) for all xЄ (a, b) then $\int_a^a f(x)dx = P(b) – P(a)$

Proof: In theorem 1, the function $F(x) = \int_a^x f(t)dt$ is an anti-derivative for f(x) for all XЄ (a, b). If P(x) is another anti-derivative for f(x) for all xЄ (a, b), then it implies that both P(x) and F(x) are continuous for all XЄ (a, b) and also will satisfy that P1(x) = $F^1(x)$ for all XЄ (a, b). There exist a constant C such that

F(x) – P(x) = C

Since $F^1(x) = P^1(x)$ and derivative of a constant is zero

i.e. F(x) – P(x) = C →$F^1(x) – P^1(x) = 0$

Since F(a) = 0 then P(a) + C = 0 and C = -P(a)

This implies that

F(x) = P(x) – P(a) for all XЄ (a, b)

Thus F(h) = P(b – P(a) (x = b)

Since $F(b) = \int_a^b f(t)dt = P(b)-P(a)$

which is the required result.

3.3 EVALUATION OF DEFINITE INTEGRAL

You are now set to seek or construct anti-derivates F(x) which will evaluate the definite integral given as $\int_a^b f(x)dx$

Example: Find $\int_a bxdx$

Solution

Let $F(x) = ½x^2$ as an anti-derivative

Then $\int_a^b xdx = ½ (b^2 – a^2)$

Find the $\int_a^b x^n dx$ when n is a positive integral the anti-derivative to use is

$$F(x)= \frac{1}{n+1} x^{n+1}$$

$\rightarrow F^1(x) = x^n \rightarrow \int_a^b x^n = F(b) – F(a)$

$$= \frac{1}{n+1} (b^{n+1} – a^{a+1})$$

Notation:

$\int_a^b f(x)dx = [F(x)]^b = F(b) – F(a)$

thus $\int_a^b x^4 dx \left[\frac{1}{4+1} x^{4+1}\right]^b = 1/5 (b^5 – a^5)$

Example:

$\int^2(6x^5-2x^3-x)dx$

$$\text{Let } F(x) = x^6 – \frac{2x^4}{4} – \frac{x^2}{2}$$

then $\int_1^2(6x^5 – 2x^3 – x)dx = [x^6 – \underline{1x^4} – \underline{x^2}]_1^4$

$$\quad\quad 2 \quad\quad 2$$

= 3960.

Example:

Evaluate the following integrals by applying the fundamental theorem.

(i) $\int_3^0(x-1)\ (x-2)dx$

(ii) $\int_3^7 \underline{dx}$

$(x-2)^2$

(iii) $\int_0^1(x^{3/4} + ½\ x\ ½)\ dx$

(iv) $\int_a^9(a^2\ x – x^4\ )dx$

(v) $\int_1^3 \underline{2 - x}\ dx$

x^3

(vi) $\int_1^8(\sqrt{t} – \underline{1})dt$

t^2

(vii) $\int_1^3\underline{6 – t}dt$

(viii) $\int_1^2x^2(x-1)dx$

(ix) $\int_1^4\sqrt{x+1}dx$

(x) $\int_0^1(x-1)^{17}dx$

Solution: To evaluate $\int_{-3}^0(x-1)\ (x-1)dx$

you expand the function (x - 1) (x -1)

$= x^2 – 2x + 1$

(i) $\int_0^3(x-1)\ (x-1)dx = \int^{03}(x2-2x+1)\ dx$

let F(x) = 1/3 x3 - x2 + x serve as anti-derivative

therefore $\int_0^3\ (x^2-2x+1) = [1/3x^3 – x^2 + x]_0^3 = 3$

(ii) $\int_3^7 \frac{dx}{(x-2)}$

construct a function with derivative as $\frac{1}{(x-2)^2}$ it is not difficult to see that

$$\frac{d}{dx}\left(\frac{-1}{x-2)}\right) = (x-2)^2$$

therefore: $\int_3^7 \frac{dx}{(x-2)} = \left[\frac{-1}{x-2}\right]_3^7 = \frac{8-}{10}$

(iii) $\int_0^1 (x^{3/4} + ½ x ½)dx$

let $F(x) = 4/7x^{3/4} + ½ x ½) = 4/7 x 7/4 + 1/3 x 3/2$

therefore $\int_0^1 (x^{3/4} + ½ x ½) = [4/7 x^{7/4} + 1/3 x^{3/4})^1 = 19/0$

(iv) $\int_0^9 (a^2x^2 – x^4)dx$

let $F(x) = (a^2 x^3 - x^5)$

$\int a(a^2x^2-x^4)dx = \left[\frac{a^2 x^3}{3} – \frac{x^5}{5}\right]_0^a$

$= \frac{a^3 x^3}{3} - \frac{a^5}{5}$

(v) $\int_1^3 \frac{2 – x}{x3} dx = \int^3 (\frac{1}{} – \frac{1}{})dx$

Let $F(x) = \frac{1}{x} - \frac{1}{x_2}$

then $\int_1^3 \frac{2 – x}{x^3} = \left[\frac{1}{x} – \frac{1}{x^2}\right]_1^3 = \frac{9}{2}$

(vi) $\int^8 (\sqrt{E} – 1/t^2)dt$

Let $F(t) = \frac{2t^{3/2}}{3} + \frac{1}{t}$

4.0 CONCLUSION

In this unit, you have studied how to define a definite integral. You have seen the connection between the summation process of finding the area under a curve and the differentiation of the function representing the area under the curve. You have studied that the fundamental theorem of integral calculus is the bridge between the summation process and the differentiation process i.e. you can find the area under a curve by finding an anti-derivative for the curve. You have applied the theorem in evaluation of definite integrals.

5.0 SUMMARY

You have studied the following in this unit:

1. How to define a definite integral

2. How to evaluate definite, integral using the following properties:

   (i) $\int_a^a f = 0$, (ii) $\int_a^b f + \int_b^c = \int_a^c$ and (iii) $\int_a^b f = -\int_b^a f$

3. How to apply the fundamental theorem of integral calculus in evaluating the definite integral of rational functions.

6.0 TUTOR MARKED ASSIGNMENTS

Evaluate the following integrals by applying the fundamental theorem of integral calculus.

(1) $\int_0^1 (4x - 3)dx$

(2) $\int_1^0 5x - 3dx$

(3) $\int_0^1 (3x + 2)dx$

(4) $f^5\sqrt{x}$

(5) $\int_{-c}^a (x - a)^2 dx$

(6) $\int_1^2 (\underline{5} + t^x)dx$

7.0 REFERENCES/FURTHER READING

Gupta, P.K., and Hira, D.S. (2012). Operations Research, New – Delhi: S. Chand & Company.

Lucey, T. (1988). Quantitative Techniques: An Instructional Manual, London: DP Publications.

UNIT 3: INDEFINITE INTEGRAL

1.0 INTRODUCTION

You have studied rules for differentiation of various functions such as polynomids functions, rational functions, trigonometric function of sines, cosines, tangent etc. hyperbolic functions and then inverses, exponent and logarithm functions. All these you studied in the first course in calculus. However, the reverse process i.e. anti-differentiation is somehow not as straight forward process as the differentiation. The reasons being that there are no systematic rules or procedures for anti-differentiation. Rather success on techniques of anti-differentiation depends much more on your familiarity with differentiation itself. So before embarking on the study of this unit, it might be worth the time to practice some of the differentiation in calculus I. Do not be discouraged when you come across functions whose derivatives are not very common. In this unit and subsequent ones you will study some basic methods of anti-differentiation.

2.0 OBJECTIVES

After studying this unit, you should be able to:

(i) Evaluate indefinite integral as anti-differentiation.

(ii) Recall notations for integration and

(iii) Recall properties of indefinite integration

(iv) Evaluate indefinite integrals using the properties of indefinite integration.

(v) Integrate differential equations that are separable.

3.0 INDEFINITE INTEGRATION

In this section an informal definition of what is anti-differentiation will be given. Suppose that the derivative of the function is given as:

$$\frac{dy}{dx} = f(x)$$

and you were asked to find the function y = F(x). For example you are given the differential equation.

$$\frac{dy}{dx} = 2x.$$

From your experience with differentiation you can easily know that $y = x^2$ since $\frac{dy}{dx} = 2x$

Interestingly, it is not only $y = x^2$ that can be differentiated to give $\frac{dy}{dx} = 2x$.

Other function like $y = x^2 - 1$, $y = x^2 + 2$, $y = x^2 + a$, $y = x^2 + 4$ can be differentiated to yield

$$\frac{dy}{dx} = 2x$$

In general any function of this form $y = x^2 + c$, where C is any constant will yield a differential equation of this type

$$\frac{dy}{dx} = 2x$$

You are now ready to take this definition.

Definition 1: An equation such as $\frac{dy}{dx} = f(x)$ which specifies the derivative as a function of x (or as a function of x and y) is called a differential equation.

For example $\frac{dy}{dx} = \text{sine}$ is differential equation

Definition 2: A function $y = f(x)$ is called a solution of the differential equation

$\frac{dy}{dx} = f(x)$ if over domain a<xcb f(x) is differentiable and

$$\frac{d}{dx} f(x) = f^1(x) = f(x)$$

in this case f(x) is called an integral of f(x) with respect to x.

Definition 3: If F(x) is an integral of the function f(x) with respect to x so is the function F(x) + C an integral of f(x) with respect to x, where c is an arbitrary constant.

If $\frac{d}{dx}$ f(x) = f(x) so also is f(x) + C i.e. $\frac{d}{dx}$ [f(x) + C] =

$\frac{d}{dx}$ f(x) + C $\frac{dc}{dx}$ = $\frac{df(x)}{dx}$ + 0 = $f^1(x)$ = f(x)

From the above if y = f(x) is any solution of dy = f(x) then all other solutions are contained in the formula y= f(x)+C dx where C is an arbitrary constant this gives rise to the symbol. ∫f(x) dx = f(x) + C – (1) where the symbol *S* is called an integral sign (see unit 2). Equation 1 is read the integral of f(x)dx is equal to f(x) plus C since $\frac{dy}{dx}$ = 2x and a typical

solution is f(x) = x^2 + C. then $\frac{d\,f(x)}{dx}$ = 2x = $\frac{d(x^2+C)}{dx}$

= y = x^2 + C

and $\frac{dy}{dx}$ = $\frac{d}{dx}$ (x2 + C) = 2x

Example: If $y = x$ $\frac{dy}{dx} = 1$

$= \int \frac{dy}{dx} dx = \int 1 dx = x + C$

In other words, when you integrate the differential of a function you get that function plus an arbitrary constant.

Example: Solve the differential equation $\frac{dy}{dx} = 4x^3$

Solution: let $\frac{dy}{dx} = 4x^3$

then $dy = 4x^3dx$ integrate both side you get $\int dy = \int 4x^3dx$ but $\frac{d}{dx}(x^2) = 4x^3dx$.

therefore $y = \int 4x^3dx = \int d(x^4) = x^4C$.

Example: Solve the differential equation $\frac{dy}{dx} = 2x + 1$

$= dy = (2x + 1)\, dx$

but $\frac{d}{dx}(x^2 + x) = 2x + 1$

therefore $\int dy = \int (2x + 1)dx$ becomes $y = \int dx^2 + x) = x^2 + x + C$.

compare $\int d(F(x)) = F(x)$ with the result of UNIT 2.

Example: Solve the following differential equation:

(1) $\frac{dy}{dx} = x^2 - 1$ (2) $\frac{dy}{dx} = \frac{1}{x^2} + x$

(3) $\frac{dy}{dx} = \frac{x}{y}$

(4) $\frac{dy}{dx} = 2x + 3$

(5) $\frac{dy}{dx} = (x^2 + \sqrt{x})dx$

(6) $\frac{dy}{dx} = 3x^2-2x+3$

(7) $\frac{ds}{dt} = 3t^2-2t-6$

(8) $\frac{dv}{du} = 5u4-3xu2-1$

(9) $\frac{dx}{dt} = 8\sqrt{x}$

(10) $\frac{dy}{dx} = (2x2- \frac{1}{x^2})$

Solution: $\frac{dy}{dx} = x^2- 1$

$=$ $dy\ (x^2 – 1)dx$

$\int dy = S(x^2-1)dx$

but $d(\frac{x^3}{3} – x) = (x^2-1)dx$

therefore: $y = \int d(\frac{x^3}{3} -x) = \frac{x^3}{3} – x + C$

(2) $\frac{dy}{dx} = \frac{1}{x^2} + x$

$\int dy = S(\frac{1}{x^2} + x)dx$

$d(\frac{-1}{x} + \frac{x^2}{2}) = (\frac{1}{x^2} + x)\ dx$

$y = \int d(\frac{x^2}{2} – \frac{1}{x}) = \frac{x^2}{2} - \frac{1}{x} + c$

(3) $\frac{dy}{dx} = \frac{x}{y}$

$\int ydy = \int x\, dx$

$d\left(\frac{y^2}{2}\right) = ydy$ and $d\left(\frac{x^2}{2}\right) = xdx$

therefore: $\int ydy = \int d\left(\frac{y2}{2}\right)$

$\int d\left(\frac{y^2}{2}\right) = \int d\left(\frac{x^2}{2}\right)$

$= \frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 = x^2 + 2C_1$

$y^2 = x^2 + C$

(4) $\frac{dy}{dx} = 2x + 3$

$\int dy = \int (2 + 3)\, dx$

$y = \int d(x^2 + 3x) = x^2 + 3x + C$

(5) $\frac{dy}{dx} = (x^2 + \sqrt{x})$

$dy = (x^2 + \sqrt{x})\, dx$

$\int dy = \int (x^2 + \sqrt{x})dx$

$y = \int d\left(\frac{x^3}{3} + \frac{2x^{3/2}}{3}\right) = \frac{x^3}{3} + \frac{2x^{3/2}}{3} + C$

(6) $\frac{dy}{dx} = 3x^2 – 2x – 5$

$dy = 3x^2 – 2x – 5$

$\int dy = \int (3x^3 - x^2 - 5x) = x^3 - x^2 - 5x + C$

(7) $\underline{ds} = 3t^2 - 2t - 6$

dt

$\int ds = \int (3t^2 - 2t - 6)dt$

$\int = \int d(t^3 - t^{2-}6t) = t^3 - t^2 - 6t + C$

(8) $\underline{dv} = 5u^4 - 3u^2 - 1$

du

$\int dv = \int (5u^4 - 3u^2 - 1)du$

$V = \int d(u^5 - U^3 - U) = U^5 - U^3 - U + C$

(9) $\underline{dx} = 8\sqrt{x}\ dt$

dt

$dx = 8\sqrt{x}\ dt$

$= \int \underline{dx} = \int 8\ dt$

$\sqrt{x}$

$\int d(2\sqrt{x}) = \int d(8t)$

$2\sqrt{x} + C_x = 8t + Ct$

$2\sqrt{x} = 8t + C_x + C_t$

$2\sqrt{x} = 8t + C$, where $C = C_x + C_t$

(10) $\underline{dy} = (4x^2 - \underline{1})$

dx x2

$\int dy = \int (4x^2 - \underline{1})dx$

x

$y = \int d(\underline{4x^3} - \underline{1}) = \underline{4x^3} - \underline{1} + C$

3 x 3 x

3.1 PROPERTIES OF INDEFINITE INTEGRAL

So far, you would have been doing much of guess work to find an appropriate anti-derivative that will fit the answers above you will now be given some properties of indefinite integral. It would help reduce the amount of guesswork when evaluating integrals.

(1) The integral of the differential of a function U is U plus an arbitrary constant.

$\int du = u + c$

(2) A constant may be moved across the integral sign $S adu_1 = a S du$

(3) The integral of the sum of two differentials is the sum of their integrals

$\int (du + dv) = \int du + \int dv$

(4) The integral of difference of two differential is the difference of their integrals

$\int (du - dv) = \int du - \int dv$

(5) As a consequent of 2, 3 and 4 above, you have that $\int a(du \pm dv) = a\int du \pm a \int dv$

$\int du_1 \pm du_2 \pm du3 \ldots du_h = \int du_1 \pm \int du_2 \pm \ldots + \int du_h$

(7) If n is not equal to minus 1, the integral of $U^n du$ is obtained by adding one to the exponent dwindling by the new exponent and adding an arbitrary constant

$$\int u^n du = \frac{U^n+1}{n + 1} = C$$

Find the following

Example (1) $\int (5x^{10} - x^8 + 2x)dx = \int 5x^{10}dx - \int x^8 dxt \int 2xdx$

$$= \frac{5x^{10+1}}{10+1} - \frac{x^{8+1}}{8+1} + \frac{2x^{1+1}}{1+1}$$

$$= \frac{5x^{11}}{11} - \frac{x^9}{9} + x^2 + C$$

(2) $$\int x^{3/2}dx = \frac{x^{3/2+1}}{3/2^{+1}} = \frac{x^{5/2}}{5/2}$$

$$= 2/5\ x^{5/2}$$

(3) $\int 3x + 1dx$

Let $u = 3x+1$

then $\frac{du}{dx} = 3 = du = 3dx$

Therefore $\int\sqrt{3x+1}dx = \int u^{\frac{1}{2}}\ \frac{du}{3}$

here $dx = \frac{du}{3}$

therefore $1/3 S u^{\frac{1}{2}}\ du = 1/3\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$= \frac{1}{3}\ \frac{2u^{3/2}}{3} = \frac{2U^{3/2}}{9}$

$= \frac{2(3x+1)}{9}$

(4) $\int\sqrt{-}\ 1\ dx$

Let $U = 4x - 1$ $\frac{du}{dx} = 4 = dx = \frac{du}{4}$

then $\int\sqrt{4x-1}dx = \int u^{\frac{1}{2}}\ \frac{du}{4}$

$\frac{1}{4}\int u^{\frac{1}{2}} du = \frac{U^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2(4x-1)^{3/2}}{12} + C$

Examples: Evaluate the following integrals

(i) $\int\sqrt{1=4x}$

(ii) $\int 3\sqrt{1 + x\ dx}$

(iii) $\int\sqrt{2x+1}$

(iv) $\int\sqrt{4x - 2}$

Solution:

(i) $\int \sqrt{1-4x}dx$ let $U = 1- 4x$

then $\frac{du}{dx} = 4$, $dx = -\frac{du}{4}$

therefore $\int\sqrt{1-4x}\, dx = \int u^{½} (-du) = - ¼ \int u^{½}du$

$= - \frac{1}{4} [2u^{3/2}] = - \frac{1}{6}(1-4x)^{3/2} + C$

4 [3]

(ii) ∫ 3 1 + x dx [redacted]

then $\frac{du}{dx} = 1$ $du = dx$

therefore ∫ [redacted] + x dx $= \int u^{1/3}du = \frac{3u^{4/3}}{4} = \frac{3}{4}(1 + x)^{4/3} + C$

(iii) ∫[redacted]2x 1 dx let $U = 2x + 1$

then $du/dx = 2$ → $= dx = du/2$

therefore ∫ [redacted]x +1 dx $= \int U^{1/5} \frac{du}{2} = ½ [\frac{5U^{6/5}}{6}] = 5(2x+1)^{6/5}C$

$= \frac{5}{12}(2x + 1)^{6/5} + C$

(iv) ∫ [redacted]4x-2 dx let $U = 4x - 2$

then $du = 4dx$ →$dx = du/4$

∫ 4 4x -2dx = . ∫U1/4 du = . [4/5(4x-2)5/4]

$= 1/5\ (4x-2)^{5/4}$

Exercise: Evaluate the integrals

(1) $\int (8x^7 - 6x^5 - x^4 + 3x^3 + 2)\, dx$

(2) $\int (6x + 1)^{1/6}\, dx$

(3) $\int (1 - 4x)^{1/4}\, dx$

(4) $\int (4 - 10x)^{1/10}\, dx$

(5) $\int (x - 1)^{1/3} dx$

3.2 APPLICATION OF INDEFINITE INTEGRATION

Most elementary differential equation could be solved by integrating them.

Example: Solve the differential equation given as $\frac{dy}{dx} = f(x)$

→ $dy = f(x)\, dx$

→ $\int dy = \int f(x)\, dx$

→ $y = \int f(x)\, dx$

Such class of differential equation is used to solve various types of problems arising from Biology, all branches of engineering, physics, chemistry and economics.

In application of indefinite integral the value of the arbitrary constant must be found by applying the initial conditions of the problem that is being solved. Therefore before continuing it is important that you know more about this arbitrary constant C.

Example: Let $\frac{dy}{dx} = 2x$

then $y = x^2 + C$

The graph of $y = x^2$ for C = 0 is given in Fig. 3.1

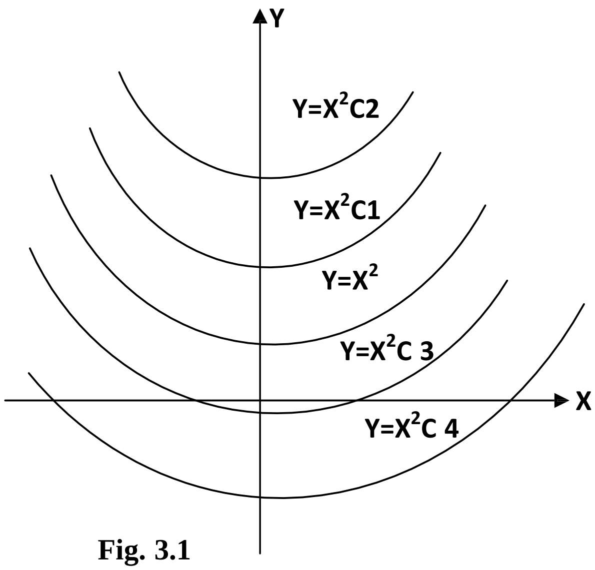


Fig. 3.1

Any other integral curve y2 + C can be obtained by shifting this curve $y = x^2$ through a vertical displacement C. In Fig.3.1 such vertical displacements give rise to a family of parallel curves. They are parallel since the slope of each curve is equal to 2x. This family of curves has the property that for any given part (x_o, y_o) where $x_o \in {}^1D$ (i.e. D is the domain of definition) there is only one and only one curve from the family of curves that passes through the part (x_o, y_o). Hence the part (x_o, y_o) must satisfy the equation

$Y_o = x_o^2 + C$

i.e. $C = y_o - x_o^2$ so for any particular point $(x_o\, y_o)$ C can be uniquely be determined.

This condition that $y = y_o$ and $x = x_o$ imposed on the differential equation du/dx = 2x is referred to as initial condition. You will use this method to solve problems on application of integration.

Example: Total profit P(x) from selling X units of a product can be determined by integrating the differential equation of the marginal profit dp/dt and using some initial conditions based on the market forces to obtain the constant of integration. Given that

$dp/dt = 2 + 3/(2x-1)^3$

Find P(x) for $0 \leq x \leq C$ if P(1) = 1.

Solution:

$$dp = \int \left(2 + \frac{2}{(2x-1)^3}\right)dx = \int \left(2 + \frac{2}{(2x-1)^3}\right) dx$$

$$\int dp = \int \left(2 + \frac{2}{(2x-1)^3}\right) dx$$

$$P = 2x - -\frac{1}{2(2x-1)^2} + C$$

Since $P_o = 1$ $X_o = 1$

$$1 = 2.1 - \frac{1}{2(2-1)^2} + C$$

$\rightarrow$ C = -1/2

Therefore $P(x) = 2x - \frac{1}{2(2x-1)} - ½$

Example: Given that $dy/dx = 8x^7$

Find y when y = -1 and x = 1

Solution:

$\int dy = \int 8x^7\, dx$

$y = x^8 + C$

$-1 = 1 + C$

$C = x^8 - 2$

Exercises:

Solve the following equations subject to the prescribed initial conditions:

(1) $dy/dx = 4x^2 – 2x – 5$ $\quad x = -1, y = 0$

(2) $dy/dx = 4(x-5)^3$ $\quad x = 0, y = 2$

(3) $dy/dx = \frac{x^2 +1}{x^4} dx$ $\quad x = 1, y = 1$

(4) $dy/dx = x\sqrt{1+x2}$ $\quad x = 0, y = 0$

(5) $dy/dx = x^{½} + x^{1/5}$ $\quad x = 0, y = 2$

You will study more on application of indefinite integration in the last unit in the course.

Ans: (i) $y = 4/^3x3-x^2+5x+22/3$ (ii) $y = (x-5)^4-623$

(iii) $y = -1/x – 1/3x^3+7/3$ (iv) $y = 1/3(x^2+1)^{3/2}-1/3$

(v) $y = 2/3x^{3/2}$

4.0 CONCLUSION

In this unit emphasis has been on techniques of finding anti-derivative. Therefore, you have studied numerous solved examples on method of finding anti-derivatives of functions. You have known the notation for indefinite integration as $\int f(x)dx = f(x) + C$. You have studied properties of indefinite integration and how to use them to evaluate integrals. You have studied how to integrate simple differential equations.

5.0 SUMMARY

You have studied:

(1) The definition of indefinite integral

(2) Properties and notation of indefinite integration

(3) To evaluate integrals using both the notation and properties of indefinite integration.

(4) To integrate differential equation that are separable.

6.0 TUTOR MARKED ASSIGNMENTS

Evaluate the following integrals:

(1) $\int \sqrt{} dx$

(2) $\int \overline{4x\ 1}$

(3) $\int (7x^6 - 4x^3 + 4x^6 - 2x)$

(4) $\int dx/x^7\ dx$

(5) $\int \frac{x^4 - 1}{X^6\ 1 + x} dx$

(6) $\int (x + 1)\ dx$

Solved the differential equation at the specified points:

(13) $dy/dx = \frac{x^2 - 1}{x^4}$ y = 0 x = 1

(14) $dy/dx = \frac{1}{\sqrt{1+7x}}$ y = 2, x = 1

(15) $dy/dx = (1 - 4x)^{1/4}$ y = 1, x = -3

7.0 REFERENCES

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Lucey, T. (1988). Quantitative Techniques: An Instructional Manual, London: DP Publications.

UNIT 4

INTEGRATION OF TRANSCEDENTAL FUNCTIONS

1.0 INTRODUCTION

In the previous unit, you studied the integration of polynomial function and simple rational function. However, there are some functions whose derivatives are not very common. Integration of such functions uncommon derivatives can only be possible by using derivatives of known functions to do the evaluation. In this unit integration of transcendental and rational function are discussed. These integration will form part of the basic tools that will be needed in applying techniques of integration that will be studied in the next unit.

2.0 OBJECTIVES

After studying this unit you should be able to correctly

(i) Derive the formula for integrating rational functions, exponential function and trigonometric functions

(ii) Evaluate definite and indefinite integrals of sinx, cos x, e^x and any combination of them

(iii) To evaluate integrals by using the derivatives of inverse trigonometric functions of sinx and tan x.

3.0 INTEGRATION OF RATIONAL AND EXPONENTAL FUNCTION

3.0.1 The integral ∫du/u =, u ≠0 Recall that d/dx 1nu = du/u **-** I

then the integral counterpart of equation I above is that ∫du/u = ln1u1+C

In the above u is a differentiable function of x and u>0 for all values of x in the specified domain.

Example: Find ∫<u>8x</u> dx

2^{x-1}

Solution: let u = x^{2-1}, du = 2x dx

then

<u>du</u> = x dx

2

therefore

∫ 8x dx = 8 ∫ <u>du</u>

x^{2-1} 2

= 4∫<u>du</u> | = 4/u/u∫+C

x

Example: Find ∫ <u>x^2</u> - dx

$1+3x^3$ |

let u = $1+3x^3$, du = $9x^2$dx

➔ x^2dx = du/9

$\int \frac{x^2 \, dx}{1_{3x}^{3}} = \int \frac{\frac{du}{9}}{u} = \frac{1}{9} \int \frac{du}{u}$

$= \frac{1}{9} \ln/u/ + C = \frac{1}{9} \ln/1+3x^3/+C$

Example: Find

$\int \frac{8x^3 - 2}{x^4-x+1} \, dx$

let $u = x^4 - x + 1$, $du = (4x^3 - 1)\, dv$

but $(8x^3 - 2)dx = 2(4x^3-1)\, dx$

therefore: $\int \frac{(8x^3 - 2)dx}{x^4-x+1} = \int \frac{2(4x^3 -1)dx}{x^4-x+1} = \int \frac{2du}{u}$

$= 2\ln/u/+C$

$= 2\ln/x^4 - x + 1/ + C$

Example: Find $\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$

let $u = x + 1$ and $v = x + 2$

$du = dx \qquad du = dx$

$\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx \quad = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \int \frac{du}{u} - \int \frac{dv}{v}$

$= \ln/u/ - \ln/v/ + C$

$= \ln/x+1/ - \ln/x+2/ + C$

Example: Find $\int \frac{\log(x+1)}{x + 1} dx$

let $u = \log(x+1)$ $du = \frac{1}{x+1} dx$

therefore: (x+1) du = dx → $\int \frac{\log(x+1)}{x+1} dx = \int \frac{u}{x+1} .(x+1)du$

$= \int u du = ½ u^2 + C$

$= ½ \log (x+1)^2 + C$

Exercise: Evaluate the following integrals

(1) $\int \frac{dx}{3-4x}$

(2) $\int \frac{3}{x-5} dx$

(3) $\int \frac{x}{x^2-2} dx$

(4) $\int \frac{\log x}{x} du$

(5) $\int \frac{4x-2}{x^2-x+1} dx$

Ans: (1) $\frac{-1}{4}$ ln/(3-4x) /+C

(2) 3ln/(x-5) + C

(3) ½ ln/x^2-2/ + C

(4) ½ $\log x^2$ + C

(5) 2ln/x^2 – x + 1/

The method adopted above is to differentiate the denominator and check if it is a factor of the numerator; if so with appropriate algebraic manipulation, the derivative of the denominator will be made to look like the numerator. This method was used in UNIT 3.

i.e. $\int \frac{g(x)}{P(x)} dx$ let u = P(x)

and du = $P^1(x)$dx = g(x)dx

then $\int \frac{g(x)}{P(v)} = \int \frac{du}{u}$ = ln/u/ + C

→ Ln/P(x) + C

3.0.2 THE INTEGRAL $\int e^x du$

Recall that $\frac{de^u}{dx} = \frac{de^u}{du} \cdot \frac{du}{dx} = e^u \frac{du}{dx}$

then $\frac{de^u}{dx} = e^u \frac{du}{dx}$

→ $de^u = e^u du$ then $\int de^u$

$= \int e^u du$ therefore $\int e^u du$

$= e^u + C$

Example: Find $Se^{-x}dx$.

Let u = -x, du = 1dx

➔ dx -du

therefore $\int e^{-x} dx = \int e(-du) = -\int e^u du$

$= -e^u + C = e^{-x} C.$

Example: Find $\int e^{2x}dx$. Let u = 2x → du = 2dx

$dx = \frac{du}{2}$

therefore $\int e^{2x}dx = \int e^u\frac{(du)}{2} = ½ \int e^u du$

$= ½\, e^{2x} + C$

Example: Find $\int e^{x/3dx}$ let $u = \frac{x}{3}$, $du = \frac{dx}{3}$

$dx = 3du$, $Se^{x/3}$ $\int e^{x/3}du = \int e^u.(3du)$

$\int e^{x/3} dx = 3Se^u du = 3e^u C$

$= 3e^{x/3} + C$

Example: $\int 4e^{2x}dx$ Let $U = e^{2x}$ $du = 2e^{2x}dx$.

$\int 4e^{2x}du = 2\int 2e^{2x}dx = 2\int du = 2u + C$

$= 2e^{2x} + C$

Example: $\int (e^x x)^2 (e^x 1)dx$

Let $u = e^x + x = du = (e^x 1)dx$

$\int (e^x + x)^2 (e^x + 1)\,dx = \int u^2 du$

$= \frac{U^3}{3} + C = \frac{(e^x + x)^3}{3} + C$

Example: $\int xe^{x2}dx$

Let $u = |x^2$ $du = 2x\,dx$

➔ $\frac{du}{2} = x\,dx$

then $\int xe^{x2}\,dx = ½\,Se^u du$

$= ½\,e^u + C = ½\,e^{x2} + C$

Exercise: Evaluate the following integrals

(1) $\int e3xdx$ (2) $\int e^{\frac{sx}{2}}$

(3) $\int 8e^{4x}\,dx$ (4) $\int (e^x – x)^2 (e^x – 1)\,dx$

(5) $\int 3x^2\,e^{x3}$

Ans: (1) $1/3e^{3x} + C$ (2) $\frac{2e^{5x}}{5^2} + C$ (3) $2e^{4x} + C$

(5) $e^{x3} + C$

3.1 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Recall from UNIT 8 of the first course on calculus that for any differentiable function U of X that

$\frac{d}{dx}$ (sinu) = cosu$\frac{du}{dx}$

$\frac{d}{dx}$ (cosu) = sin^2U $\frac{du}{dx}$

$\frac{d}{dx}$ (tanu) = -$cosec^2u\frac{du}{dx}$

$\frac{d}{dx}$(cotu) = -$cosec^2u\frac{du}{dx}$

$\frac{d}{dx}$(secu) = secu tan u $\frac{du}{dx}$

$\frac{d}{dx}$ (asecu) = -wsecucotu$\frac{du}{dx}$

Using the above you will integrate the following trigonometric function as

(1) ∫sinudu = -∫-sinudu = -∫$\frac{d}{dx}$(cosu)

= - cosu C

therefore

∫sinu du = -Cosu +C

(ii) ∫Cosu du = ∫dsinu) = sinu C

therefore

∫cosudu =sinu = Sinu+C

Given that ∫$\frac{1}{f(x)}$. $\frac{d}{dx}$ [f(x)] dx = log/f(x) + C

then

(iii) ∫tanudu = ∫$\frac{sinu}{cosu}$ du = -∫$\frac{1}{cosu}$.d cosu)

$= -\int \frac{dv}{v} = \ln v + C$, where v = cosu

$= -\ln|\cos u| + C = \ln\left(\frac{1}{\cos u}\right) = \ln|\sec U| + C$

therefore $\int \tan u \, du = \ln|\sec u| + C$

(iv) $\int \sec u du = \int \sec u \frac{(\sec u + \tan u)}{(\sec u + \tan u)} du$

$= \int \frac{\sec^2 + \sec u \tan u \, du}{\sec u + \tan u}$

Let V = tan u + sec u, $dv = \sec^{2u} + \tan u \sec u \, du$

therefore: $\int \frac{\sec^{2u} + \tan u \sec u \, du}{\tan u + \tan u} = \int \frac{dv}{v}$

(v) $S\text{cotudu} = \int \frac{\cos u}{\sin u} du = \int \frac{1}{\sin u} d(\sin u)$

$= \ln|\sin u| + C$

(vi) $\int \operatorname{cosec} u \, du = \int \operatorname{cosec} u \frac{\operatorname{cosec} u - \cot u}{\operatorname{cosec} U - \cot U}$

$= \int \frac{\operatorname{cosec}^2 u - \cot u \operatorname{cosec} u}{\operatorname{cosec} u - \cot u} du$

$= \int \frac{dv}{v}$, v = cosec u – cot u

$dv = \operatorname{cosec}^2 u - \cot \operatorname{cosec} u \, du$

➔ $\ln|v| + C = \ln|\operatorname{cosec} u - \cot u| + C.$

Example: Find $\int \sec^2 u \, du = \int d(\tan u)$ C

= tan u + C

Example: Find $\int \operatorname{cosec}^2 u \, du = -\int -\operatorname{cosec}^2 u \, du - \int (d(\cot u) = -\cot u + C$

Example: Find $\int \sec u \tan u \, du \int \sec u \tan u = \int d(\sec u) = \sec u + C$

Example: Find $\int \cos x \sin x\, dx$

Let $u = \sin x$ $du = \cos x\, dx$

therefore $\int \sin x \cos x\, dx = \int u\, du$

$= \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$

Example: Find $\int \sec^3 x \tan x\, dx$

Let $u = \sec x$ $du = \sec x \tan x\, dx$

therefore $\int \sec^3 x \tan x\, dx = \int \sec^2 x \sec x \tan x\, dx$

$= \int u^2 du = \frac{u^3}{3} + C$

$= \frac{\sec^3 x}{3} + C$

Example: Find $\int \mathrm{cosec}^3 x \cot x\, dx$. Let $u = \mathrm{cosec}\, x$ $du = -\mathrm{cosec}\, x \cot x\, dx$

Therefore $\int \mathrm{cosec}^3 x \cot x\, dx = \int \mathrm{cosec}^2 x\ \mathrm{cosec}\, x \cot x\, dx$

$= -\int u^2 du = -\frac{u^3}{3} + C$

$= -\frac{\mathrm{cosec}^3 x}{3} + C$

Example: Find $\int x \cos ax^2\, dx$

Let $U = ax^2$ $du = 2ax dx$

$\int x \cos ax^2 dx = \int \frac{1}{2} a (\cos ax^2) (2ax)\, dx$

$= \frac{1}{2} \int \cos U . du = \frac{1}{2} a (\sin U + C)$

$= \frac{1}{2} a \sin ax^2 + C$

Example: Find $\int \frac{\sec^2 x}{1+\tan x} dx$

let $U = 1 + \tan x$ $du = \sec^2 x\, dx$

therefore $\int \frac{\sec^2 xdx}{1+\tan x} = \int \frac{du}{u} = \ln/u/+C$

$= \ln/1+ \tan x1 + C$

Exercises: Find the following integrals

(i) $\int \sin (2x-1)\, dx$ (ii) $\int \sin ½ ax\, dx$

(ii) $\int 2\cos^2 x \sin x\, dx$

Ans: (i) $- ½ \cos (2x – 1) + C$

(ii) $\frac{-2}{a} \cos ½ ax + C$

(iii) $\frac{-2}{3}\cos^3 x + C$

3.2 INTEGRATION OF INVERSE TRIGONOMETRIC FUNCTION

Recall that $\frac{d}{dx}(\text{arc} \sin u) = \frac{1}{\sqrt{1-x^2}} \frac{du}{dx}$

to evaluate ∫arc sin u du you have to know how to integrate by part which is one of the techniques of integration that you will study next unit. For now

$\int \text{arc} \sin u\, du = u\, \text{arc} \sin u + \sqrt{1 – u^2} + C$ and

$\int \text{arc} \tan u\, du = u\, \text{ar} \tan u = - ½ \ln/1+u^2/+C$

You can proceed to make use of the derivative of arc tan x to evaluate special integrals.

Recall $\frac{d}{Du}(\text{arc} \tan u) = \frac{1}{1+u^2}$

$u^2 = a^2v^2$

therefore: $\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{adv}{\sqrt{a^2-a^2v^2}} = \int \frac{adv}{a\sqrt{1-v^2}}$

$= \int \frac{dv}{} = \text{arc} \sin v + C$

$\sqrt{1-v^2}$

$= \arcsin \frac{u}{a} + C$

Example: Find $\int \frac{du}{\sqrt{4-u^2}}$

Solution: $\int \frac{du}{\sqrt{4-u^2}} = \int \frac{du}{\sqrt{(2)^2 - u^2}} = \arcsin u + C$ $= \arcsin \frac{u}{2} + C$

Example: Find (1) $\int \frac{dx}{a^2+(x+2)^2}$

Solution let $u = (x + 2)$, $du = dx$

therefore $\frac{dx}{a^2 + (x+2)^2} = \int \frac{du}{a^2 + u^2}$

$= \frac{1}{a} \arctan \frac{u}{a} + C$

$= \frac{1}{a} \arctan \frac{(x+2)}{a} + C$

Example: Find $\int \frac{dx}{\sqrt{a^2 + (x-1)^2}}$

Let $u = x-1$ $du = dx$

therefore $\int \frac{dx}{\sqrt{a^2+u^2}} = \arcsin \frac{u}{a} + C$

$= \arcsin \frac{x-1}{a} + C$

Exercises: Find the following integrals:

(i) $\int \frac{dx}{16 + 4x^2}$ (ii) $\int \frac{dx}{\sqrt{9 - 64x^2}}$

(iii) $\int \frac{dx}{49 + (x+2)^2}$ (iv) $\int \frac{dx}{\sqrt{25 - 9x^2}}$

(v) $\int_0^5 \frac{dx}{25+x^2}$

Ans: (i) ¼ arc tan $\frac{x}{8}$ (ii) arc sin $\frac{8x}{3}$

(iii) 1/7 art tan $\frac{x+2}{7}$ (iv) arc sin $\frac{3x}{5}$

(v) Π/20

4.0 CONCLUSION

In this unit you have derived the formula for common rational functions and how to find their integrals. You studied how to derive the integration formula of trigonometric functions. Evaluation and trigonometric functions were treated. You also find the integrals of special functions using the inverse functions of sin x and tan x. The formulas derived in this unit will be used to study methods and techniques of integration which will be studied in the next unit of this course.

5.0 SUMMARY

In this unit you have studied how to;

1. Derive formula such as:

   (i) $\int \frac{1}{u}\,du$ = ln/u/+C

   (ii) ∫sin u du = -cos u + C

   (iii) ∫cos du = sin u + C

   (iv) ∫tan u du = ln/sec u/+ C

(v) $\int \cot u \, du = \ln/\sin u/ + C$

(vi) $\int \sec u \, du = \ln/\tan u + \sec u/$

(vii) $\int \operatorname{cosec} u \, du = \ln/\operatorname{cosec} u - \cot u/ + C$

(viii) $\int e^u \, du = e^u + C$

2. Evaluate integral of this form $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$

and $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

3. How to use the formula in (i) above to evaluate integrals.

6.0 TUTOR MARKED ASSIGNMENT

Find the following integrals

(1) $\int \frac{dx}{5-7x}$ (2) $\int \frac{1}{x-6} dx$ (3) $\int \frac{x \, dx}{x^2-4}$

(4) $\int \frac{10x+5}{5x2+5x+1} dx$ (5) $\int e^4 \, dx$ (6) $\int \sin(4x-1) \, dx$

(7) $\int \sin^c x \operatorname{cov}x \, du$ (8) $\int \frac{du}{\sin^2 x}$ (9) $\int \sin^4 ax \cos ax \, dx$

(10) $\int x \cot(x)^2 dx$

UNIT 5

INTEGRATION OF POWERS OF TRIGONOMETRIC FUNCTIONS

1.0 Introduction

2.0 Objectives

3.0 Basic Formulas

3.1 Powers of Trigonometric Function

3.2 Even Powers of Sines and Cosines

3.3 Powers and Products of Other Trigonometric Functions

4.0 Conclusion

5.0 Summary

6.0 Tutor Marked Assignment

7.0 Further Reading

1.0 INTRODUCTION

So far what you have studied in the last two units is to find the function whose derivative gives you the integral of another function. This process is summed up in the fundamental theorem of integral calculus. For a review, consider evaluating the integral $\int f(x)dx$ what you have studied in unit 2 and 3 is to find a function f(x) such that d/dx f(x) = f(x) – 1 then $F(x)+C = \int f(x)dx$. The process of finding f(x) that satisfies equation 1 above is the difficult aspect and that is why differentiation is taught before integration. So far, all you have been doing is making a good guess for the function F(x) which is dependent on how familiar you are with differentials of functions. In this unit you will study how to make the guesswork a lot easier. This will be done by introducing firstly the use of differentiation formulas alongside their integration formulas, second, by applying some techniques that will be developed here based on the knowledge of function as well as their respective derivative. Since it is the anti-derivative that gives the solution to the integral it is necessary once again you review basic rules and formulas for derivatives of function in the course calculus I.

The emphasis in this unit would be on developing skills rather than finding specific answer to any given problem. Therefore as was done in the previous units a particular example might be solved several times with different methods. Therefore the examples in this unit have been kept fairly simple so that you would be able to develop the necessary skills expected of you.

2.0 OBJECTIVES

After studying this unit you should be able to correctly

1. Recall differential formulas and their corresponding integrals

2. Evaluate integrals involving powers of trigonometric functions

3. Evaluate integrals involving products of even powers of sines and cosines

4. To develop techniques and methods for evaluating integrals of any function formed by functions of the trigonometric functions.

3.0 BASIC FORMULA

The first requirement for skill in integration is a thorough mastery of the formulas for differentiation. Therefore, a good starting point for you to develop the skill required of you in this course is for you to build your own table of integral. You may make your own note in which the various sections are headed by standard form like $SU^{n}du$ and then under each heading include several examples to illustrate the range of application of the particular formula.

Therefore, what will be done in this unit is to list formulas for differentiation together with their integration counterparts.

Summary of Differential Formulas and Corresponding Integrals

1.	du = du dx	1.	$\int du = u\ C$
2.	d(au) = a du	2.	$\int a\, du = a \int du$
3.	d(u + v) = du + dv	3.	$\int (du + dv) = \int du + \int dv$
4.	$d(u^n) = nu^{n-1}du$	4.	$\int u^n\, du = \frac{u^{n+1}}{n + 1} + C,\ n \neq - 1$
5.	d(ln u) = du	5.	$\int \frac{du}{} = \ln /u/ + C$
6.	a) $d(e^u) = e^u\, du$	6.	a) $\int e^u\, du = e^u + C$
b)	$d(a^u) = a^u \ln a\, du$		b) $\int a^u\, du = \frac{a^u}{\ln a} + C$
7.	d(sin u) = -sin u du	7.	$\int \cos u\, du = \sin u + C$
8.	d(cos u) = -sin u du	8.	$\int \sin u\, du = \tan u + C$
9.	d(tan u) = sec u tan u du	9.	$\int \sec^3 u\, du = \tan u + C$
10.	$d(\cot u) = -\csc^2 u\, du$	10.	$\int \csc^2 u\, du = -\cot u + C$
11.	d(sec u) = sec u tan u du	11.	$\int \sec u \tan u\, du = \sec u + C$
12.	d(csc u) = -csc u cot u du	12.	$\int \csc u \cot u\, du = -\csc u + C$
13.	$d(\sin^{-1} u) = \frac{du}{\sqrt{1 - u^2}}$	13.	$\int \frac{du}{\sqrt{1 - u^2}} = \{\sin^{-1} u + C$ and $\{-\cos{-1}\ u + C$
14.	$d(\cos^{-1} u) = \frac{du}{\sqrt{1 - u^2}}$	14.	

3.1 INTEGRATION INVOLVING POWERS OF TRIGONOMETRIC FUNCTIONS

From the above basic formula you have that:

(1) $\int u^n \, du = \frac{u^{n+1} + C}{n + 1}$ for $n \neq -1$

and

(2) $\int u^n \, du = \ln/u/+C \; n = 1$

This could be used to evaluate integrals involving powers of trigonometric functions.

Example: Find $\int \sin^n ax \cos ax \, dx$

Let $u = \sin ax \; du = a\cos ax \, dx$

then $\frac{du}{a} = \cos ax \, dx$, $u^n = \sin^n ax$

therefore: $\int \sin^n ax \cos ax \, dx = \int u^n \frac{du}{a}$

$= \frac{n^{n+1}}{a(n + 1)} + C$

using equation (1) above you get

(3) $\int \sin^n ax \cos ax \, dx = \frac{\sin^{n+1} ax}{a(n + 1)} + C$

with equation (2) you get n + 1

(4) $\int \frac{\cos ax \, dx}{\sin ax} = \frac{1}{a} \ln /\sin ax/+C$

Interestingly this is the same result arrive at when you derive the formula for

$\int \cot u \, du = \int \frac{\cos ax}{\sin ax} dx = \ln/\sin u/ + C$

In a similar manner you can find ∫cosn ax sin ax dx

Let u = cos ax du = -a sin ax

$U^n = \cos^n$ ax then

$\int \cos^n ax \sin ax\, dx = \int u^u(-du) = - \frac{U^{n+1} C}{n+1}$

for n ≠ 1

therefore $\int \cos^n ax \sin ax\, dx = -\frac{\cos^{n+1} ax\ C}{(n+1)a}$

for n = 1 $\int \frac{\sin ax}{\cos ax} dx - \frac{1}{a}\ln/\cos ax/ + C$

this is the same as ∫ tan ax dx

i.e. $\int \tan ax\, dx = -\frac{1}{a}.\ \ln/\cos ax/ + C$

$= \frac{1}{a} \ln/\sec ax/ + C$

Example: Try finding $\int \sin^3 x\, dx$ you find out that the above method does not work because there is cos x side of it to give d (sin x)/ therefore, another method has to be tried.

Recall that $\sin^3 x = \sin^2 x \sin x$

$= (1 - \cos^2 x) \sin x$

$\sin^3 x = \sin x - \cos^2 x \sin x$

then let u = cos x du = -sin x

$\int \sin^3 dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx$

$= -\cos x + \frac{\cos^3 x}{3} + C$

The above give rise to a formula or technique for integrating odd powers of sin x or cos x

i.e. $\cos 2^{n+1} x = \cos^{2n} x \cos x$

but $\cos^{2n} x = (\cos^2 x)^n = (1 – \sin^2 x)$

therefore $\cos^{2n+1} x = (1-\sin^2 x)^{n.} \cos x$

let u = sin x du = cos x dx

therefore $\int \cos^{2n+1} x\, dx = \int (1 – \sin^2 x)^n \cos x\, dx$

$= \int (1 – u^2)\, du.$

What follows next is to expand the expression $(1-u^2)$ du where u = cos x smf

$\int \cos^{2n+1} x\, dx = -\int(1-u^2)^n\, du$ where u = sin x

Example: Find (1) $\int\cos^3 x\, dx$ ii $\int \sin^5 x\, dx$

Solution: $\int \cos^{2n+1} x\, dx = \int(1 – u^2)^n \mid du$ 2n + 1 = 3 → n = 1, u = sin x

therefore: $\int \cos^3 x\, dx = -\int (1 – u^2)\, du = u – \frac{u^3}{3}$

$= \sin x – \frac{\sin^3 x}{3} + C$

(optimal) $= \sin x – \frac{\sin x}{3} + \frac{\sin 2 x \cos x}{3}$

$= \frac{\sin^2 x \cos x}{3} - \frac{2}{3} \sin x$

(ii) $\int \sin^5 x\, dx$

2n + 1 = 5→ n 2, u = cos x

therefore $\int \sin^5 x\, dx = \int (1 – u^2)^2\, du$

$= \int (1 – 2u^2 + u^4)\, du$

$= u – \frac{2u^3}{3} + \frac{u^5}{5} + C$

therefore $\int \sin^5 x\,dx = \cos x - \frac{2}{3}\cos^3 + \frac{1}{5}\cos^5 x + C$

(optimal) $= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C$

Example: Find $\int \sec x \tan x\,dx$

Solution: $\int \sec x \tan x\,dx = \int \frac{1}{\cos x}\frac{\sin x}{\cos x} = \int \frac{\sin x}{\cos^2 x}$

then $\int \sec x \tan x\,dx = \int \cos^{-2} x \sin x\,dx$

therefore $\int \cos^{-2} x \sin x = -\frac{\cos^{-2+1}}{-2+1} + C$

$= \frac{\cos^{-1}}{-1} x + C$

$= \frac{-1}{\cos^x} + C$

$= \sec x + C$

Example: Find $\int \tan^4 x\,dx$

recall that $\sin^2 + \cos^2 x = 1$

therefore $\tan^2 x = \sec^2 x - 1$

then

$\int \tan^4 x\,dx = \int \tan^2 x.\tan^2 x\,dx$

$= \int \tan^2 x(\sec^2 x - 1)\,dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x)\,dx$

$= \int \tan^2 x \sec^2 x - \int (\sec^2 x - 1)\,dx$

$= \int (\tan^2 x \sec^2 x)\,dx - \int \sec^2 x\,dx + \int dx$

let $u = \tan x$ $du = \sec^2 x\,dx$

therefore $\int \tan^4 x\,dx = \int u^2\,du - \int du - \int dx$

$$= \frac{u^3}{3} - u - x$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Therefore, for n = even you can derive the formula using the technique above.

$$\int \tan n\ x\ dx = \int \tan^{n-1} x\ (\sec^2 x - 1)\ dx$$

$$= \int \tan^{n-2} x \sec^2 x)\ dx - \int \tan^{n-2} x\ dx$$

$$= \int (\tan^{n-2} x \sec^2 x)\ dx - \int (\sec^{n-2} x - 1)\ dx$$

$$= \int (\tan^{n-2} x \sec^2 x\ )\ dx - \int \sec^{n-2} + dx$$

$$= \int \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x\ dx$$

Exercises:

(i) $\int \sin^3 x\ dx$ (ii) $\int \tan^2 4x\ dx$ (iii) $\int \cos^5 x\ dx$

(iv) $\int \cot^3 x\ dx$ (v) $\int \cos^3 x \sin^2 x\ dx$ (vi) $\int \sec^u x \tan u\ du$

Ans:

(i) $1/3 \cos^3 x - \cos x + C$ (ii) $\tan^4 x - 4x + C$

(iii) $\sin x - \frac{2}{3} \sin^3 x + 1/5 \sin^5 x + C$ (iv) $\frac{-\cot^2 x}{2} - \ln/\sin x/ + C$

(v) $\frac{\sin^3}{3} x - \frac{\sin^5 x}{5} + C$ (vi) $\ln(\operatorname{cosec} x - \cot x) + C$

3.2 INTEGRATION OF EVEN POWERS OF SINES AND COSINES

In the previous section you have studied how to integrate odd powers of sin x and cos x. You will attempt to evaluate integrals of even powers of sines and cosines by applying the same technique used above for odd powers i.e.

$\int \sin^n x \cos^m x\ dx$ where m or n is an even numbers.

that $\int \cos^{½} x \sin^3\ dx$ evaluate the integral.

Recall that 3 is odd as such $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$.

therefore: $\int \cos^{½} x \sin^3 x\, dx = \int \cos^{½} x\ 91 - \cos^2 x) \sin x\, dx$

for u = cos x du = sin x dx.

therefore: $\int \cos ½ x (1 - \cos^{12} x) \sin dx = \int u^{½} (1 - u^2)\, du$

$= \int (u^{½} - u)\, du = 2 \frac{u^{3/2}}{3} - \frac{u^2}{2}$

$= \frac{2\cos^{3/2} x}{3} = \frac{\cos^2 x}{2} + C$

If in the above you have $\sin^4 x$ instead of $\sin^3 x$ then you have to evaluate

$\int \cos^{½} x \sin^4 x\, dx$

Then using the above method will fail because $\sin^4 x = (1 - \cos^2 x)^2$ which give

$\int \cos^{½} x \sin^4 x\, dx = \int \cos^{½} x (1-\cos^2 x)^2\, dx$

missing above is –sin x dx = du that goes with the cos x. Therefore, there is a need to use another trigonometric identity. The one that will be used is given as

$\sin^2 x = .\ (1 - \cos^2 x)$ and $\cos^2 x = .\ (1 + \cos 2x)$.

Note: The above identities are derived by adding or subtracting the equations

$\cos^2 x + \sin^2 x = 1$ and $\cos^2 x - \sin^2 x = \cos 2x$

Recall

$$\begin{aligned}
\int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\
&= \int [\ ½ (1 - \cos 2x]^2\, dx \\
&= \int ¼ (1 - 2 \cos^2 x + \cos^2 2x)\, dx \\
&= ¼ \int (1 - 2 \cos 2x + ½ (1 - \cos 4x)\, dx \\
&= ¼ [x - \sin 2x + \frac{x}{2} - \frac{1}{8} \sin 4x] \\
&= \frac{3}{8} x - ¼ \sin 2x = \frac{1}{32} \sin 4x + C
\end{aligned}$$

Example: Find $\int \sin^2 x \cos^2 x \, dx$

Here both powers are even.

Let $\sin^2 x = (1 - \cos^2 x)$

Therefore $\sin^2 x \cos^2 x = (1 - \cos^2 x) \cos^2 x$

➔ $\int \sin^2 x \cos^2 x \, dx = \int (\cos^1 x - \cos^4 x) \, dx$

$\int \cos^2 x \, dx - \int \cos^4 x \, dx$

$\int \cos^2 x \, dx = \int ½ (1 + \cos {}^2x) \, dx = \frac{x}{2} + \frac{\sin 2x}{4}$

$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx = \int [\ .\ (1 + \cos 2x)]^2 \, dx$

$= \int ¼ \ [1 + 2 \cos 2x + \cos^2 2x] \, dx$

$= ¼ \ [1 + 2\cos 2x + .\ (1 + \cos 4x)] \, dx$

$= \frac{3x}{8} + .\ \sin 2x + \frac{1}{32} \sin 4x$

$\int \sin^2 x \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + \frac{3x}{8} + ¼ \sin 2x + \frac{1}{32} \sin 4x$

$= \frac{7}{8} x \ ½ \sin 2x + \frac{1}{32} \sin 4x + C$

Example: Find $\int \cos^6 x \, dx$

➔ $\int \cos^6 x \, dx = \int (\cos^2 x)3 \, dx = \int \frac{1}{8} (1 + \cos 2x)^3 \, dx$

$= \frac{1}{8} \int (1 + 3 \cos 2x + 3\cos^2 2x + \cos^3 2x) \, dx$

$= \frac{5}{16} x + ¼ \sin 2x + \frac{3}{64} \sin 4x \ \frac{-1}{48} \sin^3 2x + C.$

3.3 POWERS AND PRODUCTS OF OTHER TRIGONOMETRIC FUNCTIONS

In this section, you shall evaluate two types of integrals

(1) $\int \tan^m x \sec^n x \, dx$ and

(2) $\int \cot^m x \operatorname{cosex}^n dx$

Example: When n is even you write $\tan^m x \sec^n x = \tan^m x \sec^{n-2} x \sec^2 x$ and then express $\sec^{n-2}$ in terms of $\tan^2 x$ using $\sec^2 x + 1 = \tan^2 x$.

Example: $\int \tan^3 x \sec^2 x \, dx$

let $u = \tan x$ $du = \sec^2 x \, dx$.

then $\int \tan^3 x \sec^2 x \, dx = \int u^3 \, du$

$= \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$

When n and m are both odd you write

$\tan x \sec^n x = \tan^{m-1} x \sec^{n-1} \sec x$ 1an u

and express $\tan^{m-1} x$ in terms of $\sec^2 x$ using $\tan^2 x = \sec^2 x - 1$

Example: $\int \tan^3 x \sec^3 x \, dx$

$\tan^3 x \sec^3 x = \tan^2 x \sec^2 x \tan x \sec x$

and $\tan^2 x = (\sec^2 x - 1)$

therefore: $\int \tan^3 x \sec^3 x \, dx = \int (\sec 3x - 1) \sec^2 x \sec x \tan x \, dx$

$= \int (\sec^4 x - \sec^2 x) \sec x \tan x \, dx$

(but u = sec x, du = se x 1 an x du)

therefore $\int \tan^3 x \sec^3 x \, dx = \int (u^4 - u2) \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^3}{5} x - \frac{\sec^3}{3} x + C$

you can do the same for $\cot^m x \operatorname{cosec}^n x$ in a similar manner. That is for $\int \cot^m x \operatorname{cosec}^n x\, dx$ when n is even you write out $\cot^m x \operatorname{cosec}^n x = \cot^m x \operatorname{cosec}^{n-2} \operatorname{cosec} x$ and express $\operatorname{cosec}^{n-2}$ in terms of $\cot^2 x$ using $\operatorname{cosec}^2 x = \cot^2 x + 1$

Example: $\int \cot^5 x \operatorname{cosec}^4 x\, dx$

$= \int \cot^5 x \operatorname{cosec}^2 x \operatorname{cosec}^2 x\, dx$

$= \int \cot^5 x (\cot^2 x + 1) \operatorname{cosec}^2 x\, dx$

$= \int \cot^7 \operatorname{cosec}^2 x\, dx + \int \cot^5 x \operatorname{cosec}^2 x\, dx$

$(u = \cot x \quad du = -\operatorname{cosec}^2 x\, dx).$

$= \frac{-\cot^8 x}{8} - \frac{-\cot^6 x}{6} + C$

In similar manner when m and n are both odd you have $\cot^m x \operatorname{cosex}^n x = \cot^{m-1} x \operatorname{cosec}^{n-1} x \operatorname{cosec} x \cot x$ and then express $\cot^{m-1} x$ in terms of $\operatorname{cosec}^2 x$ using $\cot^2 x = \operatorname{cosec}^2 x - 1$

Example: $\int \cot^5 x \operatorname{cosec}^3 x\, dx$

$= \int \cot4 x \operatorname{cosec}2 x \operatorname{cosec} x \cot x\, dx$

$= \int (\operatorname{cosec}^2 x - 1)^2 \operatorname{cosec}^2 x \operatorname{cosec} x \cot x\, dx$

$= \int (\operatorname{cosec}^6 x - 2 \operatorname{cosec}^4 x + \operatorname{cosec}^2 x) \operatorname{cosec} x \cot x\, dx$

$u = \operatorname{cosec} x \quad du = -\operatorname{cosec} x \cot x\, dx$

$= \int (u^6 - 2u^4 + u^2)(-du)$

$= \frac{-u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C$

$= \frac{-\operatorname{cosec}^7 x}{7} + \frac{2\operatorname{cosec}^5 x}{5} - \frac{\operatorname{cosec}^3 x}{3} + C$

Exercises Find

1. $\int \cot^3 x \operatorname{cosec}^3 x\, dx$
2. $\int \cot^3 x \operatorname{cosec}^2 x\, dx$
3. $\int \tan^5 x \sec^2 x\, dx$

Ans:

1. $\frac{-\text{cosec}^5 x}{5} + \frac{\text{cosec}^3 x}{3} + C$

2. $\frac{-\cot^4 x}{4} + C$

3. $\frac{\tan^6 x}{6} + C$

4.0 CONCLUSION

In this unit, you have reviewed differential formulas and their corresponding integrals. These basic formulas will be used throughout the remaining part of the course. You have developed techniques of finding integrals of powers of trigonometric functions by using the trigonometric identities;

(i) $\cos^2 x + \sin^2 x = 1$ and

(ii) $1 + \tan^2 x = \sec^2 x$ etc.

You have also studied how to evaluate the products of even powers of sines and cosines functions. These integrals will be used when developing other techniques of integration in the next unit of this course.

5.0 SUMMARY

You have studied in the unit how to

(i) Recall basic differential formulas and corresponding integrals

(ii) Use these basic formulas to develop techniques of integration of powers of trigonometric function

(iii) Evaluate the integrals of odd powers of trigonometric function such as $\int \sin^n x \, dx$, $\int \cos^n x \, dx$

(iv) Evaluate the integrals of trigonometric function such as ∫ $\tan^n x\,dx$, ∫ $\cot^n x\,dx$ where n is odd or even

(v) Evaluate the integrals of even powers of sec x and cosec x

(vi) Evaluate the integrals of products of even powers of sin x and cos x such as ∫ cosn x dx, ∫ $\sin^n x\,dx$, ∫ $\cos^n x\,dx\,\sin^m x\,dx$ where n or m is even or both are even.

6.0 TUTOR MARKED ASSIGNMENT

(1) Find ∫ $\sin^2 x \cos^2 x\,dx$

(2) Show that ∫ tan ax dx = $\frac{1}{a}$ ln/cosax/+C

(3) Find ∫ $\sin^3 4x\,dx$

(4) Find ∫ $\tan^5 x \sec^3 x\,dx$

(5) Show that ∫ $\sec^2 n\,x\,dx = \int (1+u^2)^{n-1}\,du$ where u = tan x

7.0 REFERENCES

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MODULE FOUR

Unit 1: Mathematical Tools I: Simultaneous Equations, Linear Functions, and Linear Inequalities

Unit 2: Mathematical Tools II: Introduction to Matrix Algebra

Unit 3: Probability

UNIT 1: MATHEMATICAL TOOLS I: SIMULTANEOUS EQUATIONS, LINEAR FUNCTIONS, AND LINEAR INEQUALITIES

1.0 Introduction
2.0 Objectives
3.0 Main Content
   3.1 Simultaneous Equations
   3.2 Linear Functions
   3.3 Linear Inequalities
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Reading

1.0 INTRODUCTION

This unit is a continuation of our discussions on mathematical tools. This focuses on the other aspects of equations and inequalities that are useful in business and economic decisions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain what simultaneous equation systems are all about
- describe how simultaneous equation systems are formulated and solved
- analyse business conditions using linear functions and linear inequalities.

3.0 MAIN CONTENT

3.1 Simultaneous Equations

A simultaneous equation system is a set of equations with two or more unknown. The mostly used method of solving a simultaneous equation system is the so-called "Addition-Subtraction" or "Elimination" method.

Consider the following example:

Solve for X and Y in the system of equations:

$3X - 4Y = 13$ (1)

$3Y + 2X = 3$ (2)

Solution:

Rearranging equation (1) and (2), we get:

$3X - 4Y = 13$ (1)

$2X + 3Y = 3$ (2)

Multiplying equation (1) by 2 (the coefficient of X in equation (2)) and equation

(2) by 3 (the coefficient of X in equation (1), we get:

$$6X - 8Y = 26 \quad (3)$$

$$6X + 9Y = 9 \quad (4)$$

Subtract equation (4) from equation (3) to get:

$6X - 8Y = 26$

$\underline{-(6X + 9Y = 9)}$

$0 - 17Y = 17$

$-17Y = 17$

$Y = -1$

Substituting the value of Y = -1 to equation (1), we get:

$3X - 4(-1) = 13$

$3X + 4 = 13$

$3X = 13 - 4$

$3X = 9$

$X = 3$

The solution is therefore X = 3; Y = -1.

Note that this substitution will yield the same result if equation (2) is used in the substitution.

Application:

1. Suppose a factory manager is setting up a production schedule for two models, A and B, of a new product. Model A requires 4 units of labour input and 9 units of capital input. Model B requires 5 units of labour input and 14 units of capital input. The total available labour input for the production of the product is 335 man-hours per day, and that of capital is 850 units per day. How many of each model should the manager plan to make each day so that all the available labour hours and capital inputs are used.

Solution:

Tabulating the schedule, we get:

Input	Model A	Model B	Total Available
Labour	4	5	335
Capital	9	14	850

Let X = number produced of model A per day

Y = number produced of model B per day

These require a total of:

$4X + 5Y = 335$ (labour)

and $9X + 14Y = 850$ (capital)

We now have the simultaneous equations:

$4X + 5Y = 335$ (1)

$9X + 14Y = 850$ (2)

Solving for X and Y simultaneously, we get: Multiplying eq. (1) by 9 and eq. (2) by4:

$36X + 45Y = 3015$ (3)

$36X + 56Y = 3400$ (4)

eq. (1) – eq. (2):

$36X + 45Y = 3015$

$-\underline{(36X + 56Y) = 3400)}$

$0 – 11Y = -385$

$-11Y = -385$

$Y = \underline{-385} = 35$

-11

Substituting for y in eq. (1):

4X + 5(35) = 335

4X + 175 = 335

4X = 335 – 175

4X = 160

X = 40

The solution values, Y = 35 and X = 40 indicate that, according to the specified constraints, the manager should plan to make 40 units of model A and 35 units of model B.

3.2 LINEAR FUNCTIONS

A function, f, is said to be a linear function if f(x) can be written in a form:

$$f(x) = ax + b; \; a \neq 0$$

The function: Y = f(x) ax+b is an equation of a straight line, with slope = a and Y – intercept = b.

Examples:

(1) Suppose f is a linear function with slope of 2 and f(4) = 8, find f(x).

Solution:

The linear function is of the form:

f(x) = ax + b Here, a = 2 and

f(4) = 8 = 2(4) + b

8 = 8 + b

B = 8 – 8 = 0

Thus, f(x) = 2x + b

= 2x + 0

=2x

f(x) = 2x

(2) In testing an experimental diet for hens, it was determined that the average live weight, w, (in kgs) of a hen was statistically a linear function of the number of days, d, after the diet was begun,

where 0 δ d δ 50. Suppose the average weight of a hen beginning the diet was 4kg. And 25 days later, it was 7kg.

a) Determine w as a linear function of d.

b) What is the average weight of a hen for a 10 days period?

Solutions

a) The required function is of the form:

$W = f(d) = md + b,$

where m is the slope and b is the constant intercept.

By definition, $m = \frac{-W}{-d} = \frac{W_2 - W_1}{d_2 - d_1}$

From the given information:

when d = 0; w = 4 and when d = 25; w = 7.

It follows that, $W_1 = 4$; $w_2 = 7$, $d_1 = 10$, $d_2 = 25$

$$m = \frac{7 - 4}{25 - 0} = \frac{3}{25}$$

Using the so-called point-slope form of a linear function:

$w - w_1 = m(d - d_1)$, we get:

$$w - 4 = \frac{3}{25}(d - 0)$$

w – 4 = 0.12d

w = 0.12d + 4

Thus, the required linear function is

W = 0.12d + 4

(b) When d = 10, w = 0.12 (10) + 4

= 5.2

Thus, the average weight of a hen for a 10 days period is 5.2 kg.

3.3 LINEAR INEQUALITIES

An Inequality is simply a statement that two numbers are not equal. A **linear inequality** in the variable, X, is an inequality that can be written in the form:

aX + b<0 **or** aX + b δ 0; (a 0).

Examples

Solve the inequalities:

(1) 2(X - 3) < 4

(2) 3 – 2X δ 6

Solutions

(1) 2(X - 3) < 4

2X - 6 < 4

2X < 4 + 6

2X < 10

$\underline{2X} < \underline{10}$

2 2

X < 5

(2) $3 - 2X \leq 6$

$-2X + 3 \leq 6$

$-2X \leq 6-3$

$-2X \leq 3$

$\frac{-2X}{-2} \leq \frac{3}{-2}$

X ε -3/2

The reverse in inequality sign is due to the negative effect.

Application:

1. The current ratio of any business organisation is the ratio of its current assets to its current liabilities. The Managing Director of ACE Equipment Co. has decided to obtain a short-term loan to build up inventory. The company has current assets of N350, 000 and current liabilities of N80, 000. How much can the Managing Director borrow if the company's current ratio must be not less than 2.5?

Note that funds received are considered current assets and loans are considered current liability.

Solution

Let X = the amount to be borrowed Then,

current assets = 350,000 +X Current liability =

80,000 + X

By definition,

$$\text{Current ratio} = \frac{\text{Current Assets}}{\text{Current Liabilities}}$$

$$= \frac{350{,}000 + X}{80{,}000 + X}$$

Thus, according to the specification:

$$\frac{350,000+X}{80,000+X} \; \varepsilon \; 2.5$$

Solving, we get:

350,000 + X ε 2.5 (80,000 + X)

350,000 + X ε 200,000 + 2.5X

X – 2.5 X ε 200,000 – 350,000

-1.5 X ε - 150,000

$$\frac{-1.5\,X}{-1.5} \; \varepsilon \; \frac{-150,000}{-1.5}$$

X ≤ 100,000

Thus, the Managing Director can borrow not more than or as much as N100, 000 and still maintain a current ratio of not less than 2.5.

2. A publishing company finds that the cost of publishing each copy of a magazine is N0.38. The revenue from dealers of the magazine is N0.35 per copy. The advertising revenue is 10% of the revenue received from dealers for all copies sold beyond10,000 units. What is the least number of copies which must be sold so as to have a positive profit?

Solution

Let x = number of copies to be sold.

By definition,

Profit (∪) = R-C

Where R = Revenue from dealers + Revenue from adverts.

Revenue from dealers = N0.35x

Revenue from adverts = 0.10 [0.35(x-10,000)]

Total Cost (C) = 0.38x

Thus,

Profit (Ս) = 0.35x+0.10[0.35(x-10,000)] – 0.38 x > 0

Solving, we get:

0.35x + 0.10 [0.35x – 3500] – 0.38 x > 0

0.35x + 0.035x – 350 – 0.38x > 0

0.35x + 0.035x – 0.38x > 350

0.005 x > 350

$$\frac{0.005}{0.005} x > \frac{350}{0.005}$$

x > 70,000

Thus, the total number of copies to be sold must be greater than 70,000. That is, at least 70,001 copies.

SELF ASSESSMENT EXERCISE

1. Solve the inequality: $\frac{9Y + 1}{4} \leq 2Y - 1$

2. Using inequality symbols symbolise the statement: The number of man-hours, X, to produce a commodity is not less than 2.5 nor more than 4.

4.0 CONCLUSION

This unit has focused on simultaneous equations, linear functions and linear inequalities as extension to the most basic mathematical tools of quantitative analysis. The unit looked at these subjects with some emphasis on the practical business applications.

5.0 SUMMARY

1. A simultaneous equation system is a set of equations with two or more unknown. The mostly used method of solving a simultaneous equation system is the so-called

“Addition- Subtraction” or “Elimination” method.

2. A function, f, is said to be a linear function if f(x) can be written in a form:

$$f(x) = ax + b; \; a \; \bar{} 0$$

The function: Y = f(x) ax+b is an equation of a straight line, with slope = a and Y – intercept = b.

3. An inequality is simply a statement that two numbers are not equal. A **linear inequality** in the variable, X, is an inequality that can be written in the form:

$$aX + b<0 \; \textbf{or} \; aX + b \; \delta \; 0; \quad (a \; \bar{} 0).$$

6.0 TUTOR-MARKED ASSIGNMENT

Let P = 0.09q + 50 be the supply equation for a manufacturer. The demand equation for his product is:

P = 0.07q + 65

a) If a tax of N1.50/unit is to be imposed on the manufacturer, how will the original equilibrium price be effected if the demand remains the same?

b) Determine the total revenue obtained by the manufacturer at the equilibrium point, both before and after the tax.

7.0 REFERENCES/FURTHER READING

Haessuler, E.F. &Paul, R.S. (1976). *Introductory Mathematical Analysis for Students of Business and Economics,* (2nd edition.) Reston Virginia: Reston Publishing Company.

UNIT 2: MAHEMATICAL TOOLS II: INTRODUCTION TO MATRIX ALGEBRA

CONTENTS

1.0 INTRODUCTION

An understanding of matrices and their operations is essential for input output analysis in business and economic decisions. It is also essential for solving complicated problems in simultaneous equation systems. In this unit, you will just be introduced to the basic rudimentary of matrices, with some simple applications.

A matrix is a rectangular array of numbers, called entries. Examples are:

Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Matrix $B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

Matrix C = $\begin{bmatrix} 1 & 5 & 8 \end{bmatrix}$ = row matrix

Matrix D = $\begin{bmatrix} 7 \\ 0 \\ 9 \end{bmatrix}$ = column matrix

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- State the basic principles of matrix algebra
- analyse the important mathematical operations of matrices
- apply matrices in business decisions.

3.0 MAIN CONTENT

3.1 Equality of Matrices

Two matrices, A and B, are said to be equal, A = B, if they have the same dimension and their corresponding entries are equal.

Example:

$$\text{Given } A = \begin{bmatrix} X & Y+1 \\ 2Z & 5W \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 7 \\ 4 & 2 \end{bmatrix}$$

For A = B, X = 2; Y + 1 = 7; 2Z = 4; 5w = 2

3.2 Matrix Addition

If A and B are two matrices with the same dimension, then the sum, A + B is the matrix obtained by adding the corresponding entries in A and B.

3.3 Scalar Multiplication

If A is an (m x n) matrix and K is a real number (or scalar), then KA is a scalar multiplication.

Example

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} \text{ and } K = 10$$

$$\text{then } KA = 10 \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 40 & -20 \end{bmatrix}$$

3.4 Matrix Multiplication

If A is an (m x n) matrix and B is an (n x p) matrix, the product AB = C is of dimension (m x p). For this product to exist, the number of columns in A (that is, n) must equal the number of rows in B (that is, n). It follows that if:

$$A = (2 \times 3) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad B = (3 \times 3) \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\text{then, } AB = C = (2 \times 3) \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{bmatrix}$$

Where: $C_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$
$C_{12} = a_{11}b_{12} + a_{12}b_{12} + a_{13}b_{32}$
$C_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$
$C_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$
$C_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$
$C_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$

Examples

1.Suppose the prices in N per unit for products A,B, and C are represented by the price matrix:

Price of

$$\underset{(1x3)}{P} = \begin{matrix} A & B & C \\ [2 & 3 & 4] \end{matrix}$$

The quantities purchased are given by the quantity matrix:

$$\underset{(3 X 1)}{Q} = \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix} \begin{matrix} \text{units of A} \\ \text{units of B} \\ \text{units of C} \end{matrix}$$

Compute the total expenditure on the products.

Solution

Required to compute PQ:

$$PQ = [2 \quad 3 \quad 4] \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix}$$

$= 2(7) + 3(5) + 4(11)$
$= \underset{(1x1)}{[\ 73\ ]}$

Thus, the total expenditure on products A,B, and C is N73.

2. A manufacturer of calculators has an East Coast and a West Coast plant, each of which produces Business Calculators and Standard Calculators. The manufacturing time requirements (in hours per calculator) and the assembly and packaging costs (in naira per hour) are given by the following matrices:

Hours per Unit
Assembly

T = ⌊0.2 ⌈0.2

Packaging

$$\begin{bmatrix} 0.1 \end{bmatrix} \text{Business Calculators} \qquad \begin{bmatrix} 0.1 \end{bmatrix} \text{Standard Calculators}$$

Naira per Hour

$$C = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} \begin{matrix} \text{Assembly} \\ \text{Packaging} \end{matrix}$$

(columns: East Coast, West Coast)

a) What is the total cost of manufacturing a business calculator on the East Coast?

b) Calculate the manufacturer's total cost of producing the calculators in the two plant locations.

Solutions

Question (a) and (b) requires the product matrix:

$$TC = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} = D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

Where $D_{11} = 0.2(5) + 0.1(4) = 1.4$

$D_{12} = 0.2(6) + 0.1(5) = 1.7$

$D_{21} = 0.3(5) + 0.1(4) = 1.9$

$D_{22} = 0.3(6) + 0.1(5) = 2.3$

It follows that the cost matrix:

$$\begin{bmatrix} 1.4 & 1.7 \\ 1.9 & 2.3 \end{bmatrix} \begin{matrix} \text{Business Calculator} \\ \text{Standard Calculator} \end{matrix}$$

(columns: East Coast, West Coast)

a) The total cost of manufacturing a Business Calculator in the East Coast is the entry in D11 = N1.40.

b) The cost matrix, D, indicates that the total cost of producing 1 unit of Business Calculator is N1.40 in the East Coast, and N1.70 in the West Coast. The cost of producing 1 unit of Standard Calculator is N1.90 in the East Coast, and N2.30 in the West Coast.

SELF ASSESSMENT EXERCISE 1

A square matrix, M, of dimension, (3x3), has elements Mij = 3j
Where i represents row and j represents column. Write out the matrix.

4.0 CONCLUSION

This unit had exposed you to the basic principles of matrices. Of most importance are the operations in matrices, similar to mathematical operations, including:

(i) equality of matrices; (ii) addition of matrices;
(iii) scalar multiplication; and,
(iv) matrix multiplication.

The unit also made some simple applications of such matrix operations.

5.0 SUMMARY

You had learned the followings from the unit’s discussions:

1.Two matrices, A and B, are said to be equal, A = B, if they have the same dimension and their corresponding entries are equal.

2. If A and B are two matrices with the same dimension, then the sum, A + B is the matrix obtained by adding the corresponding entries in A and B.

3. If A is an (m x n) matrix and K is a real number (or scalar), then KA is a scalar multiplication.

4. If A is an (m x n) matrix and B is an (n x p) matrix, the product AB = C is of dimension (m x p). For this product to exist, the number of columns in A (that is, n) must equal the number of rows in B (that is, n).

6.0 TUTOR-MARKED ASSIGNMENT

Suppose a building contractor has accepted orders for five ranch style houses, seven cape cod-style houses and twelve colonial-style houses. These orders can be represented by the row matrix

$$Q = [5 \quad 7 \quad 12]$$

Furthermore, suppose the raw materials that go into each type of house are steel, wood, glass, paint, and labour. The entries in the matrix R below give the number of units of each raw material going into each type of house:

$$R = \begin{array}{r} \text{Ranch} \\ \text{Cape Cod} \\ \text{Colonial} \end{array} \overset{\begin{array}{ccccc} \text{Steel} & \text{Wood} & \text{Glass} & \text{Paint} & \text{Labour} \end{array}}{\begin{bmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 31 \end{bmatrix}}$$

a) Compute the product, QR, the amount of each raw material needed for the contract.

b) Suppose that steel costs N15/unit, wood costs N8/unit, glass, paint and labour cost N5, N1, and N10 per unit respectively. These costs are represented in the column matrix:

$$C = \begin{bmatrix} 15 \\ 8 \\ 5 \\ 1 \\ 10 \end{bmatrix}$$

Compute the product RC, that is, the cost of each type of house.

7.0 REFERENCES/FURTHER READING

Haessuler, E.F. and Paul, R.S. (1976), *Introductory Mathematical Analysis for Students of Business and Economics*, *2nd edition*. Reston Virginia: Reston Publishing Company.

UNIT 3 PROBABILITY

CONTENTS

1.0 INTRODUCTION

So far, you have learnt about measures of location and variability. This prior knowledge will enhance your understanding of probability. Simply put, all humans are faced with uncertainty about life; thus, probability theory is a basic component of decision-making, under uncertainty. From the sciences, arts to the humanities we are confronted with decision making under uncertainty. The objective of this unit then, is to present to you the basic concept of probability needed for an understanding of statistical inference. We shall introduce the subject by considering probabilities that are based on observed data.

2.0 OBJECTIVES

At the end of the unit, you should be able to:

• highlight the role of probability in drawing inferences, in the face of uncertainty

• assign probabilities using classical, relative, Bayesian and axiomatic models

• describe permutation and combination.

3.0 MAIN CONTENT

3.1 DEFINITION OF TERMS

Here, let us look at a number of terms

a. Random equipment

This is an experiment which the outcome is unpredictable- for example, tossing a coin, rolling a die, lottery card game etc.

b. Sample space

This is the set of all possible outcome of a random experiment. This sample space is denoted by, and a generic element of the sample space is denoted by **Ω**. For example, driving to work, a commuter passes through a sequence of three intersections with traffic lights. At each light, he either stops, (s) or continues (c). The sample space is the set of all possible outcomes.

Ω= {CCC, CCS, CSS, CSC, SSS, SSC, SCC, SCS} Where, *CSC*- for example, denotes the outcome of the movement of the commuter- through the first light, stopping at the second light and continuing through the third light.

c. Event Here, we are interested in particular subset o*f* **Ω**, which in probability language are called events. We denote events by capital alphabets A, B, C,… X, Y, Z. In the example above, the event that the commuter stops at the first light is the subset of Ω denoted as follows

A = {SSS, SSC, SCC, SCS}.

d. Intersection of two events A set *M* is said to be intersection of two sets *A* and *B* if the elements of *M* are the elements common to both *A* and *B*. *M* is denoted as *M* = A n B. For example, if *A* is the event that the commuter stops at the first light (listed above) and if *B* is the event that he stops at the third light, therefore-

B = {CCS, CSS, SSS, SCS) M = A n B = {SSS, SCS)

e. The union of two sets *A* and *B*, is the event *M* that either *A* occurs or *B* occurs or both occur- *M = A U B*. For example, if *A* is the event that the commuter stops at the first light (see above) and *B* is the event that he stops at the third light (see above), therefore: M = {SSS, SSC, SCC, SCS, CCS, CSS}.

f. Impossible events An impossible event is an event that will never happen. It is represented by φ. For example, *T = {x: 0 < 1 x < , x Σ N}* is an impossible event, where *N* is the set of natural numbers.

g. Sure events or certain event An event that must happen is called sure or certain event. For example, in tossing a die- *T* = *{x : 1 < x < 6}* is a sure event.

h. Mutually exclusive events Two events- *P* and *Q* are called mutually exclusive event or disjointed events if the occurrence of one of them precludes (prevents) the occurrence of the other. That is, *P n Q* = φ. For example, in rolling a die, where *P* and *Q* are numbers on the upturned face of the die. Let P = {x : P is even} Q = {x : Q is odd) Then *P n Q* = φ, which- when interpreted, *P* and *Q* are mutually exclusive.

i. Complementary events

The complement of an event- *Ac*, is the event that *A* does not occur, and thus consist of all those elements in the sample space that are not in *A*. That is, if *A U Ac = or Ac* = Ω – *A*. For example, the complement of the event that the commuter stops at the first light is the event that he continues at the first light. :. Ac = {CCC, CCS, CSC}

j. Event space (A̶)

The set of all possible subset of a sample space is called the event space, it is most often called the power set in elementary mathematics. For example, 4 Ω= { a, b) 78 = {a}, {b}, {a,b}, {φ}} = {(a}, {b} (Ω){φ}}

k. Independent event Two events *P* and *Q* are said to be independent if the occurrence (or non-occurrence) or *P* has no effect on the occurrence (or non-occurrence) of *Q*. For example, toss a coin twice, and each time, observe the face that shows up. The events- P = {heads in first toss} Q = {heads in second toss}

3.2 DEFINITION OF PROBABILITY

The earliest definition of probability is the classical or objective definition, which is based on repeated trial of tests and the assumption that various outcome of a trial are equally, likely and mutually exhaustive events. For example, if there are *n* exhaustive, equally, likely and mutually exclusive cases- *"y"* of which are favourable to the occurrence of event T, then *Pr (T) = y/n* For example, if *S* = {1, 2, 3, 4, 5, 6}. Find the probability of getting a number less than 6. These are six possible cases, out of these 6 cases, only 5 numbers are less than 5 i.e: T = { 1, 2, 3, 4, 5}. Therefore, Pr (T) = y/n = 5/6

3.3 FREQUENTIST OR RELATIVE PROBABILITY

When the number of trails is carried out in an infinite number of repeated tests, we can define the relative frequency *f* the event (P) in terms of Rf(P) = Number of trials in which P occurs Total number of trails This ratio *Rf(P)* is called the empirical probability. This definition assumes that probability of an event has a limiting value on the long run. Pr (P) = lim = y . n -7 n

We, therefore, conclude that the definitions by the classical and relative frequency schools of thought are based on objectivity.

3.4 BAYESIAN OR SUBJECTIVE PROBABILITY

According to the Bayesian model, probability is not based on repeated trials of tests. For Bayesian then, probability is a model for quantifying the strength or personal options. Baye's rule describes how personal opinion evolves with experience. Suppose that the prior probability of *A* is *P (A)*. On observation of an event *P*, the opinion about *P* changes to *Pr (A/P)* according to Baye's rule-

$$\Pr(A/P) = \frac{\Pr\{P/A\} \Pr\{A\}}{\Pr(P)}$$

You will learn more about this, as we proceed in this course.

3.5 AXIOMATIC PROBABILITY

An axiom is a rule or principle that is, generally, believed to be true. It does not need to be proved or debunked. In the axiomatic approach, we develop an idealised model from which we can predict the probability of the occurrence of various events. This model establishes abstract relationship between the events of a random experiment, and as such, could be used in calculating their probabilities. For example- Pr (A or B) = Pr (A) + Pr (B) - P (A n B) – non mutually exclusive event is an axiomatic approach.

Example 1

(1) A die is rolled once; what are the probabilities of getting:

i. an even number

ii. an odd number

iii. a prime number

iv. an event prime number

v. an odd prime number.

Solution

Let S = {1, 2, 3, 4, 5, 6}

Let P = event that even number occurs

Q = event that odd number occurs

M = event that prime number occurs

L = event that even prime number occurs

F = event that odd prime number occurs

(i) n (S) = 6

P {2, 4, 6}

n (P) = 3

$Pr(P) = \frac{n(P)}{n(S)} = \frac{3}{6} = 1/2$

(ii) Q {1,3,5}

n (Q) = 3

$Pr(Q) = \frac{n(Q)}{n(S)} = \frac{3}{6} = 1/2$

(iii) M{2,3,5}

$Pr(M) = \frac{n(M)}{n(S)} = \frac{3}{6} = 1/2$

Example 2 If four fair coins are tossed, find the probability of getting:

i. at least, 3 heads

ii. at most, one head

iii. exactly, four heads

Solution Let the sample space be *S* and at least 3 heads, at most one head and exactly four heads be *M*, *N* and *O* respectively.

Then- S = {(H H H H), (HHHT), (HHTH), (HHTT), (HTHT)} {(HTTH), (HT,TT), (THHH),(THHT)(THTH) {(THTT), (TTHH), (TTHT),(TTTH),(TTTT)}

n (S) = 16 M = {(HHHH), (HHHT), (HTHH),(HHTH),(THHH)}

n (M) = 5 N = {(HTT),(THTT),(TTHT),(TTTH)}

N (N) = 4

O = {HHH}

n (O) = 1

1. Pr (M) = n(M)/n(S) = 5/6

2. Pr (N) = n(N)/n(S) = 4/16 = ¼

3. Pr (O) =n(O)/n(S) = 1/16

3.6 BASIC LAWS OF PROBABILITY

You are to note the following

a. $0 \leq P(E) \leq 1$

b. P (Ω) = 1, Pr(φ) = 0

c. For any E Ω and its complement

Ec, we have: Pr (E) + Pr (Ec) = 1

d. For any events E1 and E2 Pr (E1, U, E2) = P (E1) + Pr (E2) – (E2) – Pr (E1 n E2)

3.7 CONDITIONAL PROBABILITY

The conditional probability of a given *B* is denoted by Pr (A/B). Similarly, the conditional probability of *B* given *A* is represented by *Pr(B/A)*.

.

That is- Pr (A/B) = $\frac{\text{Pr (A n B)}}{\text{Pr (B)}}$ provided Pr (B) ≠ 0

Example 3

If *A* and *B* are independent events with Pr(A) = 0.05 and Pr(B) = 0.65 Find- Pr (A/B)

Solution- Pr (A/B) = Pr (A n B)/ Pr (B)

0.05 Pr(B)=0.65 Pr (A) x Pr (B)/ Pr (B) = Pr (A) = Pr(A/B) = Pr(A) x Pr(B)

= 0.05 x 0.65 = 0.325

4.0 CONCLUSION

By now, your understanding of the concept of probability should have been enhanced. Decision- making is paramount in life, in business and research work. By applying the principles of probability, quality decisions can be easily arrived at. Your lives are full of choices and better alternative choices indicate a better chance for improvement and optimum performance. Applying the mathematics of probabilities you learned in this unit will make a difference. You are going to now study probability distributions in the next unit.

5.0 SUMMARY

You have learnt the following key facts in this unit. • Addition rule for mutually exclusive events- if *A* and *B* are two mutually exclusive events, then the probability of obtaining either *A* and *B* is equal to the probability of obtaining *A* plus the probability of obtaining *B*. Pr (A or B) = Pr (A) + Pr (B) • Addition rule for non-mutually exclusive events- if *A* and *B* are non- mutually exclusive events, then we must subtract the probability of the joint occurrence of *A* and *B* from the sum of their probabilities. Pr (A or B) = Pr (A) + Pr (B) - Pr (A ∩ B) • Bayes's rule for conditional probabilities. A general method for revising prior probabilities in the light of new information, to provide posterior probabilities. Two events: Pr (B /A) = Pr (A and B) Pr (A)

General form: Pr (B /A) = Pr (A/Bj) Pr (Bj) n Σ Pr (A/Bj) Pr (Bj) j=1

6.0 TUTOR-MARKED ASSIGNMENT

i. A residential sub-division developer has a house styles to build on 11 adjacent lots. How many distinguishable arrangements are possible if the developer decides to build 2 houses of each style?

ii. An investment firm is interested in the following- A = {Common stock in *NYZ* corporation gains 10% next year} B = {Gross national product gains 10% next year}. The firm has assigned the following probability distribution on the bases of available information- Pr(A / B) = 0.8 (P (B) 0.3

Compute Pr (A n B)

iii. In a large sub-urban community, 30% of the large household use Brand A toothpaste, 27% use Brand B, 25% use Brand C, and 18% use brand D. In the four groups of households, the proportion of resident who learned about the brand they use through television advertising are as follows= Brand A, 0.10, Brand B, 0.05, Brand C, 0.02 and Brand D, 0.05; In a household selected at random from community, it is found that i. what is the probability that the brand of toothpaste used in the household is (a) A (b) B (c) C (d) D? ii. Use diagram to show the prior probability, conditional probabilities, and joint probability.

7.0 REFERENCES/FURTHER READING

Brian, P.M., & Philip, M.N. (2006). *Applied Statistics for Public Policy*. New Delhi: Prentice-Hall.

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