NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 251

COURSE TITLE: MECHANICS 1

STUDY GUIDE

Course code MTH 251

Course Title MECHANICS 1

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Introduction

MTH 251 is one-semester course. It is a three (3) credits degree course available to all students to take towards their B.Sc. Mathematics, Physics, Computer Science, B.Sc. Physics and Mathematics Education and other related programmers in the faculty of science.

What You Learn in this Course

The course consists of units and a course guide. This course guide tells you briefly what the course is about, what course materials you will be using and how you can work with these materials. In addition, it advocates some general guidelines for the amount of time you are likely to spend on each unit of the course in order to complete it successfully.

It gives you guidance in respect of your Tutor-Marked Assignment which will be available in the assignment file. There will be regular tutorial classes that are related to the course. It is advisable for you to attend these tutorial sessions. The course will introduce you to the concept of mechanics, dynamics Vibrations etc.

Course Objectives

To achieve the aims set out, the course has a set of objectives. Each unit has specific objectives which are included at the beginning of the unit. You are should read these objectives before you study the unit. You may wish to refer to them during your study to check on your progress. You should always look at the unit objectives after completions in the unit.

Working through this Course

To complete this course you are required to read each study unit, read the textbooks and read other materials which may be provided by the National open University of Nigeria.

Each unit contains self-assessment exercise and at certain points in the course you would be required to submit assignments for assessment purposes. At the end of the course there is a final examination. The course should take you about a total of 17 weeks to complete. Below you will find listed all the components of the course, what you have to do and how you should allocate your time to each unit in order to complete the course successfully.

This course entails that you spend a lot of time to read. I would advice that you avail yourself the opportunity of attending the tutorial sessions where you have the opportunity of comparing your knowledge with that of other learners.

The Course Materials

The main components of the course are:
1. The course Guide
2. Study Units
3. Tutor Marked Assessment (TMA)
4. Presentation Schedule
5. References/Further Readings

Study Units

The study units in this course are as follows:

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Course Aims

This course aims at introducing some important facts and developments which were made to reflect Mechanics.

The first - three units of Module 1 of the course focuses on the Vectors, the Electromagnetic Field and Tensors .Module 2, focused on the Dynamics Of Systems Of Particles and categorized into three unites as follows: Discrete and Continuous Systems, Momentum of a System of Particles and Constraints, Holonomic and Non-Holonomic Constraints. However Module 3, titled The Simple Pendulum contains two units vis a vis Simple Pendulum and Hooks law. Lastly, Module 4;has three units thus: Motion along a curve, Circular Motion with Constant Speed and Force and Motion as shown above.

Each study unit consists of three hours work. Each study unit includes introduction, specific objectives, directions for study, reading materials, conclusions, summary, Tutor Marked Assignments (TMAs), references and other resources. The units direct you to work on exercise related to the required readings. In general, these exercises test you on the materials you have just covered or require you to apply it in some way and thereby assist you to evaluate your progress and to reinforce your comprehension of the material. Together with TMAs, these exercises will help you in achieving the stated learning objectives of the individual units and of the course as a whole.

Presentation Schedule

Your course materials have important dates for the early and timely completion and submission of your tutor-marked assignment and attending tutorials. You should remember that you are requested to submit all assignments by the stipulated time and date. You should guard against falling behind in your work.

Assessment

There are three aspect of the assessment of the course. First is made-up of self-assessment exercises, second consists of the tutor-marked assignments and third is the written examination/end of the course examination.

You are advised to do the exercises. In tackling the assignments, you are expected to apply information, knowledge and techniques you gathered during the course. The assignments must be submitted to your facilitator for formal assessment in accordance with the deadlines stated in the presentation schedule and the assignment file. The work you submit to your tutor for assessment will count for 30 % of your total course work. At the end of the course you will need to sit for a final or end of course examination of about two hour duration. This examination will count for 70 % of your total course mark.

Tutor-Marked Assignment

The TMA is a continuous assessment component of your course. It account for 30 % of the total score. You will be given at least four (4) TMAs to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMAs will be given by your facilitator and you are to return each assignment to your facilitator/tutor after completion. Assignment questions for the units in this course are contained in the assignment file. You will be able to complete your assignment from the information and the material contained in your reading, references and study units. However, it is desirable in all degree level of education to demonstrate that you have read and researched more into your references, which will give you a wider view point and may provide you with a deeper understanding of the subject.

Make sure that each assignment reaches your facilitator/tutor on or before deadline mentioned by the course coordinator in the presentation schedule and assignment file. If, for any reason, you cannot complete your work on time, contact your facilitator/tutor before the assignment is due to discuss the possibility of an extension. Extensions will not be granted after the due date unless there are exceptional circumstances.

Final Examination and Grading

The end of course examination for optics will be about 2 hours and it has a value of 70 % of the total course work. The examination will consist of questions, which will reflect the type of self-assessment exercise, practice exercise and tutor-marked assignment problems you have previously encountered. All areas of the course will be assessed.

You are advised to use the time between finishing the last unit and sitting the examination to revise the entire course. You might find it useful to review your self-test, tutor-marked assignments and comments on them before appear in examination.

Course Marking Scheme

Assignment	**Marks**
Assignments 1- 4	Four assignments, best three marks of the four count at 10 % each – 30 % of the course marks
End of course examination	70 % of overall course marks
Total	100 % of course materials

Facilitators/Tutors and Tutorials

There are 16 hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials as well as the name and phone number of your facilitator, as soon as you are allocated a tutorial group.

Your facilitator will mark and comment on your assignments, keep a close watch on your progress and any difficulty you might face and provide assistance to you during the course. You are expected to mail your Tutor Marked Assignment to your facilitator before the schedule date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not delay to contact your facilitator by telephone or e-mail if you need assistance.

The following might be circumstances in which you would find assistance necessary, hence you would have to contact your facilitator if:

- You do not understand any part of the study or the assigned readings
- You have difficulty with the self-tests
- You have a question or problem with an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only chance to have face to face contact with your course facilitator and to ask questions which may/may not be answered instantly. You can raise any problem encountered in the course of your study.

To gain much benefit from course tutorials prepare a question list before attending them. You will learn a lot from participating actively in discussions.

Summary

MTH 251 is a course that intends to give a comprehensive teaching on the principles and applications of Mechanics. Upon completion of this course, you will be able to explain the nature of Mechanics, gives an insight into the application and purpose of the course .

The basic thing is to understand all that you have learnt in this course and be able to apply them in solving different problems on the course.

I wish you a splendid study time as you go through the course.

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MTH 251 MECHANICS

MODULE 1 STATIC: System of live vectors.

UNIT 1 VECTORS

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1.0 INTRODUCTION

A vector could be defined as a quantity which has both magnitude and direction. The vector **a** may be represented geometrically by an arrow of length α drawn from any point in the appropriate direction. In particular, the position of a point *P* with respect to a given origin *O* may be specified by the *position* vector ***r*** drawn from *O* to *P*.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

Define a vector.
Freely discourse some elementary properties of vector.
Know about vector product.
Know about differentiation and integration of vector.
Know about Gradient, Divergence and Curl.
Know about integral theorem.
Know about Curvilinear Co-ordinates.

3.0 MAIN CONTENT

3.1 Definition and Elementary Properties

A vector **a** is a quantity specified by a magnitude, written α or $|a|$, and a direction in space. It is to be contrasted with a scalar, which is a quantity specified by a magnitude alone. The vector **a** may be represented geometrically by an arrow of length α drawn from any point in the appropriate direction. In particular, the position of a point *P* with respect to a given origin *O* may be specified by the *position* vector ***r*** drawn from *O* to *P*.

Any vector can be specified, with respect to a given set of Cartesian axes, by three components. If x, y, z are the Cartesian co-ordinates of *P*, then we write r = (x, y, z), and say that x, y, z are the components of **r**. (See Fig. A.I.). We often speak of *P* as 'the point **r**'. When *P* coincides with *O*, we have the *zero vector* **0** = (0, 0, 0) of length 0 and indeterminate director. For a general vector **a**, we write **a** = $[a_{x,} a_{y,} a_z]$.

The product of a vector **a** and a scalar *c* is c**a** = $(ca_{x,} ca_{y,} ca_z)$. If c > 0, it is a vector in the same direction as a, and of length ca; if c < 0, it is the opposite direction, and of length $|c|a$. In particular, if c = 1/a, we have the unit vector in the direction of a written as, à = a /α .

Addition of two vectors **a** and **b** may be defined geometrically by drawing one vector from the head of the other, as in Fig. A. 2. (This is the 'parallelogram law' for addition of forces). Subtraction is defined similarly by Fig. A.3. in terms of components,

a + b = $(a_x + b_{x,} a_y + b_{y,} a_z + b_z)$.

It is often useful to introduce three unit vectors i, j, k, pointing in the directions of the $x^-, y^-, z-$ axes, respectively. They form what is known as an *orthonomal triad* – a set of three mutually perpendicular vectors of unit length. It is clear from Fig. A.1 that any vector r can be written as a sum of three vectors along the three axes.

R = $xi + yj + zk$. (1)

If θ is the angle between the vectors a and b, then by elementary trigonometry the length of their sum is given by

$$[a + b]^2 = a^2 + b^2 + 2ab \cos \theta.$$

It is useful to define the scalar product **a.b** ('a dot b') as

$$a.b = ab \cos \theta. \qquad (2)$$

Note that this is equal to the length of a multiplied by the projection of **b** on **a**, or vice versa.

In particular, the square of **a** is:

$$a^2 = a.a = a^2. \qquad (3)$$

Thus we can rewrite the relation above as

$(a+b)^2 = a^2+b^2+2a.b,$

And similarly

$(a-b)^2 = a^2+b^2-2a.b.$

All the ordinary rules of algebra are valid for sums and scalar products of vectors, save one. (For example, the commutative law of addition, a + b = b + a is obvious from Fig. A. 2, and the other laws can be deduced from appropriate figures). The exception is the following: for two scalars, the equation $ab = 0$ implies that either $a = 0$ or $b = 0$ (or, of course, that both = 0), but we can find two non-zero vectors for which a.b = 0. In fact, this is the case if $\theta = \frac{1}{2}\pi$, that is if the vectors are orthogonal:

a.b = 0 if a ⊥ b.
(i.e vector a is perpendicular to vector b)

The scalar products of the unit vectors i, j, k are

$$i^2 = j^2 = k^2 = 1, \qquad (4)$$
$$i.j = j.k = k.i = 0.$$

Thus, taking the scalar product of each in turn with (1), we find

$$i.r = x, \quad j.r = y, \quad k.r = z. \qquad (5)$$

These relations express the fact that the components of r are equal to its projections on the co-ordinate axes.

More generally, if we take the scalar product of two vectors a and b, we find

$$a.b = a_x b_x + a_y b_y + a_z b_z, \qquad (6)$$

and, in particular,

$$r^2 = r^2 = x^2+y^2+z^2. \qquad (7)$$

3.2 The Vector Product

Any two nonparallel vectors **a** and **b** drawn from *0* define a unique axis through 0 perpendicular to the plane containing **a** and **b**. It is useful to define the vector product **a** ^**b** ('**a** cross **b**', sometimes written **a** x **b**) to be a vector along this axis whose magnitude is the area of the parallelogram with edges **a**, **b**,

$$|a \wedge b| = ab \sin \theta \qquad (8)$$

(See Fig. A.4.). To distinguish between the two opposite directions along the axis, we introduce a convention: the direction of $a \wedge b$ is that in which a right-hand screw would move when turned from **a** to **b**.

A vector whose sense is merely conventional, and would be reversed by changing from a right-hand to a left-hand convention is called an axial vector, as opposed to an ordinary or polar vector. For example, velocity and force are polar vectors, but angular velocity is an axial vector (see §5.1). The vector product of two polar vectors is thus an axial vector.

The vector product has one very important, but unfamiliar, property. If we interchange **a** and **b**, we reverse the sign of the vector product,

$$b \wedge a = -a \wedge b. \qquad (9)$$

It is essential to remember this fact when manipulating any expression involving vector products. In particular, the vector product of a vector with itself is the zero vector,

$$a \wedge a = 0.$$

More generally, $a \wedge b$ vanishes if θ = 0 or π ,

$$a \wedge a = 0 \quad if\ a \parallel b.$$

If we choose our co-ordinate axes to be right-handed, then the vector products of i, j, k are

$$i \wedge i = j \wedge j = k \wedge k = 0,$$
$$i \wedge k = k, \qquad j \wedge i = k, \qquad (10)$$

$$j \wedge k = i, \qquad k \wedge j = -i,$$
$$k \wedge i = j, \qquad i \wedge k = -j.$$

Thus, when we form the vector product of **a** and **b** we obtain

$$a \wedge b = i\,(a_y b_z - a_z b_y) + j(a_z b_x - a_x b_z) + k(a_x b_y - a_y b_x).$$

This relation may conveniently be expressed in the form of a determinant.

$$a \wedge b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \qquad (11)$$

From any three vectors **a, b, c**, we can form the scalar triple product $(a \wedge b).c$. Geometrically, it represents the volume V of the parallele-piped with adjacent edges a, b, c. (See Fig. A.5.) For, if φ is the angle between c and $a \wedge b$, then

$$(a \wedge b).c = |a \wedge b|c \quad \cos\,\varphi = Ah = V,$$

Where A is the area of the base, and h = c cos φ is the height. The volume is reckoned positive if a, b, c form a right-handed triad, and

Fig. A.5

Negative if they form a left-handed triad. For example, $(i \wedge j.k = 1,$ but $(i \wedge k).j = -1.$

In terms of components, we can evaluate the scalar triple product by taking the scalar product of **c** with (A.11). We find

$$(a \wedge b).c = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \qquad (12)$$

Either from this formula, or from its geometrical interpretation, we see that the scalar triple product is unchanged by any cyclic permutation of **a, b, c,** but changes signs if any pair is interchanged,

$$(a \wedge b).c = (b \wedge c).a = (c \wedge a).b$$

$$= -(b \wedge a).c = -(c \wedge b).a = -(a \wedge c).b. \quad (13)$$

Moreover, we may interchange the dot and cross,

$$(a \wedge b).c = a;(b \wedge c). \quad (14)$$

(For this reason, the more symmetrical notation [a, b, c] is sometimes used for the scalar triple product.)

Note that the scalar triple product vanishes if any two vectors are equal, or parallel. More generally, it vanishes if a, b, c are coplanar.

We can also form the vector triple product $(a \wedge b) \wedge c.$ since this vector is perpendicular to $a \wedge b,$ it must lie in the plane of **a** and **b**, and must therefore be a linear combination of these two vectors. It is not hard to show, by writing out the components, that

$$(a \wedge b) \wedge c = b(a.c) - a(b.c). \quad (15)$$

Similarly,

$$a \wedge (b \wedge c) = b(a.c) - c(a.b). \quad (16)$$

Note that these expressions are unequal, so that we cannot omit the brackets in a vector triple product. It is useful to notice that in both these formulae the term with positive sign is the middle vector b times the scalar product of the other two.

3.3 Differentiation and Integration of Vectors

We are often concerned with vectors which are functions of some scalar parameter, for example the position of a particle as a function of time, **r**(*t*). The vector distance travelled by the particle in a short time interval Δt is

$$\Delta r = r(t + \Delta t) - r(t).$$

(See Fig. A.6.). The velocity, or derivative with respect to t, is defined just as for scalars, as the limit of a ratio,

$$\acute{r} = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}. \qquad (17)$$

In the limit, the direction of this vector is that of the tangent to the path of the particle, and its magnitude is the speed in the usual sense. In terms of co-ordinates,

$$\dot{r} - (\dot{x}, \dot{y}, \dot{z}).$$

Derivatives of other vectors are defined similarly. In particular, we can differentiate again to form the acceleration vector $\ddot{r}$.

It is easy to show that all the usual rules for differentiating sums and products apply also to vectors. For example,

$$\frac{d}{dt}(a \wedge b) = \frac{da}{dt} \wedge b + a \wedge \frac{db}{dt}$$

Though in this case one must be careful to preserve the order of the two factors, because of the antisymmetry of the vector product.

Note that the derivative of the magnitude of **r**, d**r**/dt, is not the same thing as the magnitude of the derivative $\left|\frac{dr}{dt}\right|$. For example, for a particle moving in a circle, r is constant, so that $\dot{r} = 0$, but clearly $|\dot{r}|$ is not zero in general. In fact, applying the rule for differentiating a scalar product to r^2, we obtain

$$2r\dot{r} = \frac{d}{dt}(r^2) = \frac{d}{dt}(r^2) = 2r.\dot{r}$$

Which may also be written

$$\dot{r} = \hat{r}.\dot{r}. \qquad (18)$$

Thus the rate of change of the distance r from the origin is equal to the radial component of the velocity vector.

We can also define the integral of a vector. If **v** = d**r**/dt, then we also write

$$r = \int v \ dt,$$

and say that r is the integral of v. If we are given v(t) as a function of time, and the initial value of r, $r(t_0)$, then the position at any later time is given by the definite integral.

$$r(t_0) + \int_{t_0}^{t_1} v_x(t)dt. \qquad (19)$$

This is equivalent to the three scalar equations for the components, for example

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} v_x(t)dt.$$

One can show, exactly as for scalars, that the integral in (19) may be expressed as the limit of a sum.

3.4 Gradient, Divergence and Curl

There are many quantities in physics which are functions of position in space; for example, temperature, gravitational potential or electric field. Such quantities are known as fields. A scalar field is a scalar function $\emptyset$ (x, y, z) of position in space; a vector field is a vector function A (x,y,z). We can also indicate the position in space by the position vector r, and write $\emptyset$ (**r**) or **A**(**r**).

Now let us consider the three partial derivatives of a scalar field, $\partial\emptyset|\partial x, \partial\emptyset|\partial y, \partial\emptyset|\partial z$. They form the component of a vector field, known as the gradient of $\emptyset$ and written grad $\emptyset$, or $\nabla\emptyset$ $('del\ \emptyset')$. To show that they really are the components of a vector, we have to show that it can be defined in a manner which is independent of the choice of axes. We note that if **r** and **r** + d**r** are two neighboring points, then the difference between the values of $\emptyset$ at these points is

$$d\emptyset = \emptyset(r + dr) - \emptyset(r) = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz = dr'\nabla\emptyset. \qquad (20)$$

Now, if the distance $|dr|$ is fixed, then this scalar product takes on its maximum value when d**r** is in the direction of $\nabla\emptyset$. Hence we conclude that the direction of $\nabla\emptyset$ is the direction in which $\emptyset$ increases most rapidly. Moreover, its magnitude is the rate of increase of $\emptyset$ with distance in this direction. (This is the reason for the name 'gradient'.) Clearly, therefore, we could define $\nabla\emptyset$ by these properties, which are independent of any choice of axes.

We are often interested in the value of a scalar field ∅ evaluated at the position of a particle, ∅(**r**(t)). From (20) it follows that the rate of change of ∅(**r**(t)) is

$$\dot{\phi}(r(t)) = \dot{r}'\nabla\phi \quad (21)$$

The symbol ∇ may be regarded as a vector which is also a differential operator (like d/d*x*), given by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \quad (22)$$

We can also apply it to a vector field A. The divergence of A is defined to be

$$\text{Div A} = \nabla . A = \frac{\partial A_x}{\partial x} + j\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \quad (23)$$

And the curl of A to be *

$$curl\ A = \nabla \wedge A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \quad (24)$$

This latter expression is an abbreviation for the expanded form

$$A = i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

In particular, we may take A to be the gradient of a scalar field, $A = \nabla\phi$. Then its divergence is called the Laplacian of ∅,

$$\nabla . \nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (25)$$

Just as $a \wedge a = 0$, we find that the curl of a gradient vanishes,

$$\nabla \wedge \nabla\phi = 0. \quad (26)$$

For example, its *z* component is

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 0.$$

Similarly, one can show that the divergence of a curl vanishes,

$$\nabla.(\nabla \wedge A) = 0. \tag{27}$$

The rule for differentiating products can also be applied to expressions involving ∇ . For example, $\nabla.(A \wedge B)$ is a sum of two terms, in one of which ∇ acts on **A** only, and in the other on **B** only. The gradient of a product of scalar fields can be written

$$\nabla(\emptyset\psi) = \psi\nabla\emptyset + \emptyset\nabla\psi,$$

But, when vector fields are involved, we have to remember that the order of the factors as a product of vectors cannot be changed without affecting the signs. Thus we have

$$\nabla \cdot (A \wedge B) = B.(\nabla \wedge A) - A.(\nabla \wedge B),$$

And similarly

$$\nabla \wedge (\emptyset A) = \emptyset(\nabla \wedge A) - A \wedge (\nabla\emptyset).$$

An important identity, analogous to the expansion of the vector triple product (A.16) is

$$\nabla \wedge (\nabla \wedge A) = \nabla(\nabla.A) - \nabla^2 A, \tag{28}$$

Where of course

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}.$$

It may easily be proved by inserting the expressions in terms of components.

3.5 Integral Theorems

There are three important theorems for vectors which are generalizations of the fundamental theorem of the calculus,

$$\int_{x_0}^{x_1} \frac{df}{dx} dx = f(x_1) - f(x_0).$$

First, consider a curve *C* in space, running from r_0 to r_1. (see Fig. A.7.) Let the directed element of length along *C* is d**r**. If $\emptyset$ is a scalar field, then, according to (20), the change in $\emptyset$ along this element of length is

$$d\emptyset = dr.\nabla\emptyset.$$

Thus, integrating from r_0 to r_1, we obtain the first of the integral theorems,

$$\int_{r_0}^{r_1} dr.\nabla\phi = \phi(r_1) - \phi(r_0). \qquad (29)$$

The integral on the left is called the line integral of $\nabla\phi$ along C. This theorem may be used to relate the potential energy function $V(\mathbf{r})$ for a conservative force to the work done in going from some fixed point r_0, where V is chosen to vanish, to r. Thus, if $F = -\nabla V,$ then

$$V(r) = -\int_{r_0}^{r} dr.F. \qquad (30)$$

When F is conservative, this integral depends only on its end-points, and not on the path *C* chosen between them. Conversely, if this condition is satisfied, we can define V by (30), and the force must be conservative. The condition that two line integrals of the form (30) should be equal whenever their end-points coincide may be restated by saying that the line integral round any closed path should vanish. Physically, this means that no work is done in taking the particle round a loop which returns to its starting point. The integral round a closed loop *C* is usually denoted by the symbol $\oint_c$. Thus we require

$$\oint_c dr.F = 0 \qquad (31)$$

for all closed loops *C*.

This condition may be simplified by using the second of the integral theorems – Stokes' theorem. Consider a curved surface *S*, bounded by the closed curve *C*. If one side of S is chosen to be the 'positive' side, then the positive direction round *C* may be defined by the right-hand screw convention. (See Fig. A.8). Take a small element of the surface, of area d*S*, and let **n** be the unit vector normal to the element, and directed towards its positive side. Then the directed element of area is defined to be d**S** = **n**dS. Stokes' theorem states that if **A** is any vector field, then

$$\iint_s dS.(\nabla \wedge A) = \oint_c dr.A. \qquad (32)$$

The application of this theorem to (31) is immediate. If the line integral round C is required to vanish for all closed curves C, then the surface integral must vanish for all surfaces S. But this is only possible if the integrand vanishes identically. So the condition for a force F to be conservative is

$$\nabla \wedge F = 0. \qquad (33)$$

We shall not prove Stokes' theorem. However, it is easy to verify that it is true for a small rectangular surface. Suppose S is a rectangle in the *xy*-plane of area d*x*d*y*. Then d**S** = **k**d*x*dy, so the surface integral is

$$k.(\nabla \wedge A)\, dxdy = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dxdy. \qquad \text{(A.34)}$$

The line integral involves four terms, one from each edge. The two terms arising from the edges parallel to the *x*-axis involve the x component of **A** evaluated for different values of *y*. They therefore contribute

$$A_x(y)dx - A_x(y + dy)dx = -\frac{\partial A_x}{\partial y} dxdy.$$

Similarly, the other pair of edges yield the first term of (34).
We can also find a necessary and sufficient condition for a field **B**(**r**) to have the form of a curl,

$$B = \nabla \wedge A.$$

By (A. 27), such a field must satisfy

$$\nabla . B = 0. \qquad (35)$$

The proof that this is also a sufficient condition (which we shall not give in detail) follows much the same lines as before. One can show it is sufficient that the surface integral of **B** over any closed surface should vanish,

$$\iint_s dS.B = 0,$$

And then use the third of the integral theorems, Gauss' theorem. This states that if *V* is a volume in space bounded by the closed surface *S*, then for any vector field **B**,

$$\iiint_v dV\, \nabla . B = \iint_s dS.B. \qquad (36)$$

Where d*V* denotes the volume element d*V* = d*x*d*y*d*z*, and the positive side of *S* is taken to be the outside.

It is again easy to verify Gauss' theorem for a small rectangular volume d*V* = d*x*d*y*d*z*. The volume integral is

$$\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)dxdydz. \qquad (37)$$

The surface integral consists of six terms, one for each face. Consider the faces parallel to the xy-plane, with directed surface elements **k** d*x*d*y* and –**k** d*x*d*y*. Their contributions involve k.**B** = B_z evaluated for different values of z. thus they contribute

$$B_z(z+dz)dxdy - B_z(z)dxdy = \frac{\partial B_z}{\partial z}dxdydz.$$

Similarly, the other terms of (37) come from the other faces.

3.6 Curvilinear Co-ordinates

One of the uses of the integral theorem is to provide expressions for the gradient, divergence and curl in terms of curvilinear co-ordinates.

Consider a set of orthogonal curvilinear co-ordinates q_1, q_2, q_3, and denotes the elements of length along the three co-ordinate curves by h_1dq_1, h_2dq_2, h_3dq_3. For example, in cylindrical polars,

$$h_p = 1, \qquad h_\varphi = p, \qquad h_z = 1, \qquad (38)$$

and in spherical polars

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta. \qquad (39)$$

Now consider a scalar field ψ, and two neighbouring points (q_1, q_2, q_3) and (q_1, q_2, $q_3 + dq_3$). Then the difference between the values of ψ at these points is

$$\frac{\partial\psi}{\partial q_3}\, dq_3 = d\psi = dr.\nabla\psi = h_3 dq_3(\nabla[\psi)]_3.$$

Where $(\nabla[\psi)]_3$ is the component of $\nabla\psi$ in the direction of increasing q_3. Hence we find

$$(\nabla[\psi)]_3 = \frac{1}{h_3}\frac{\nabla\psi}{\partial q_3}. \qquad (40)$$

with similar expressions for the other components. Thus, in cylindrical and spherical polars, we have

$$\nabla\psi = \left(\frac{\partial\psi}{\partial p}, \frac{1}{p}\frac{\partial\psi}{\partial\varphi}, \frac{\partial\psi}{\partial z}\right). \qquad (41)$$

and

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right). \qquad (42)$$

To find an expression for the divergence, we use Gauss' theorem, applied to a small volume bounded by the co-ordinate surface. The volume integral is

$$(\nabla.A)h_1dq_1h_2dq_2h_3dq_3.$$

In the surface integral, the terms arising from the faces which are surfaces of constant q_3 are of the form $A_3h_1dq_1h_2dq_2$, evaluated for two different values of q_3. They therefore contribute

$$\frac{\partial}{\partial q_3}(h_1h_2h_3)dq_1dq_2dq_3.$$

Adding the terms from all three pairs of faces, and comparing with the volume integral, we obtain

$$\nabla.A = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}(h_2h_3A_1) + \frac{\partial}{\partial q_2}(h_3h_1h_2) + \frac{\partial}{\partial q_3}(h_1h_2h_3)\right\}. \qquad (43)$$

In particular, in cylindrical and spherical polars,

$$\nabla.A = \frac{1}{p}\frac{\partial(pA_p)}{\partial p} + \frac{1}{p}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}, \qquad (44)$$

And

$$\nabla.A = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}. \qquad (45)$$

To find the curl, we use Stokes' theorem in a similar way. If we consider a small element of a surface q_3 = constant, bounded by curves of constant q_1 and q_2, then the surface integral is

$$(\nabla\wedge A)_3h_1dq_1h_2dq_2.$$

In the line integral round the boundary, the two edges of constant q_2 involve $A_1h_1dq_1$ evaluated for different values of q_2, and contribute

$$-\frac{\partial}{\partial q_2}(h_1 A_1)dq_1 dq_2.$$

Hence, adding the contribution from the other two edges, we obtain

$$(\nabla \wedge A)_3 = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial q_1}(h_2 A_2) - \frac{\partial}{\partial q_2}(h_1 A_1)\right\}. \qquad (46)$$

With similar expressions for the other components. Thus, in particular, in cylindrical polars.

$$\nabla \wedge A = \left\{\frac{1}{p}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}, \frac{\partial A_p}{\partial z} - \frac{\partial A_z}{\partial p}, \frac{1}{p}\left(\frac{\partial (pA_\varphi)}{\partial p} - \frac{\partial A_p}{\partial \varphi}\right)\right\}.$$
(47)

And in spherical polars

$$\nabla \wedge A = \left\{\begin{matrix} \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right) \\ , \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\end{matrix}\right\}. \qquad (48)$$

Finally, combining the expressions for the divergence and gradient, we can find the Laplacian of a scalar field. It is

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right\}. \qquad (49)$$

In cylindrical polars

$$\nabla^2\psi = \frac{1}{p}\frac{\partial}{\partial p}\left(p\frac{\partial\psi}{\partial p}\right) + \frac{1}{p^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}. \qquad (50)$$

And, in spherical polars,

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

$$+\frac{1}{r^2 sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$
(51)

4.0 CONCLUSION

In conclusion, having read through this unit, the student should be able to use vector approach to solve some Engineering , mechanics and physics related problems. More so, students are advised to try all the trial exercises giving to them to enhance their comprehension of the unit. It worth to mention here that; this unit is a perquisite to other units and some other courses in mathematics and physics respectively.

5.0 SUMMARY

What you have learnt in this unit concerns:

that any vector r can be written as a sum of three vectors along the three axes thus: R = $xi + yj + zk.$

If θ is the angle between the vectors a and b, then by elementary trigonometry the length of their sum is given by

$$[a+b]^2 = a^2 + b^2 + 2ab\ \cos\ \theta.$$

Obviously, we define the scalar product **a.b** ('a dot b') as

$$a.b\ = ab\ \cos\ \theta.$$

Remark; $(a.b\ = ab\ \cos\ [\![\ \theta.]\!]$) is equal to the length of a multiplied by the projection of **b** on **a**, or vice versa.

We alsi note that the square of **a** is: $a^2 = a.a = a^2.$

Thus we can rewrite the relation above as (a+b)^2 = a^2+b^2+2a.b,

And similarly (a-b)^2 = a^2+b^2-2a.b.

All the ordinary rules of algebra are valid for sums and scalar products of vectors, save one. (For example, the commutative law of addition, a + b = b + a is obvious from Fig. A. 2, and the other laws can be deduced from appropriate figures). The exception is the following: for two scalars, the equation $ab = 0$ implies that either $a = 0\ or\ b = 0$ (or, of course, that both = 0), but we can find two non-zero vectors for which a.b = 0. In fact, this is the case if $\theta \frac{1}{2}\pi$, that is if the vectors are orthogonal:

a.b = 0 if a $\perp$ b.

(i.e vector a is perpendicular to vector b)

The scalar products of the unit vectors i, j, k are

$$i^2 = j^2 = k^2 = 1,$$
$$i.j = j.k = k.i = 0.$$

More generally, if we take the scalar product of two vectors a and b, we find

$$a.b = a_x b_x + a_y b_y + a_z b_z,$$

and, in particular,

$$r^2 = r^2 = x^2+y^2+z^2.$$

Vector product

Any two nonparallel vectors **a** and **b** drawn from *0* define a unique axis through 0 perpendicular to the plane containing **a** and **b**.

$$|a \wedge b| = ab \sin \theta$$

The vector product has one very important, but unfamiliar, property. If we interchange **a** and **b**, we reverse the sign of the vector product,

$$b \wedge a = -a \wedge b.$$

Thus, when we form the vector product of **a** and **b** we obtain

$$a \wedge b = i\,(a_y b_z - a_z b_y) + j(a_z b_x - a_x b_z) + k(a_x b_y - a_y b_x).$$

This relation may conveniently be expressed in the form of a determinant.

$$a \wedge b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

From any three vectors **a, b, c**, we can form the scalar triple product $(a \wedge b).c$. Geometrically, it represents the volume V of the parallelepiped with adjacent edges a, b, c. (See Fig. A.5.) For, if φ is the angle between c and $a \wedge b$, then

$$(a \wedge b).c = |a \wedge b|c \quad \cos \varphi = Ah = V,$$

Where A is the area of the base, and h = c cos φ is the height.

6.0 TUTOR-MARKED ASSIGNMENT

1. By drawing appropriate figures, prove the following laws of vector
   (a+b)+c = a+(b+c),
   $\lambda(a+b) = \lambda a + \lambda b,$
   $a.(b+c) = a.b + a.c.$

   Note that a, b, c need not be coplanar.)

2. Show that $(a \wedge b).(c \wedge d) = a.c\, b.d - a.d\, b.c.$
3. Evaluate $\nabla \wedge (a \wedge b).$
4. Prove that $a \wedge (b+c) = a \wedge b + a \wedge c.$ (Hint: Show first that in $a \wedge b, b$ may be replaced by its projection on the plane normal to **a**, and then prove the result for vectors in this plane).
5. Evaluate the components of $\nabla^2 A$ in cylindrical polar co-ordinates using the identity (A. 28). Show that they are not the same as the scalar Laplacians of the components of **A**.

7.0 REFERENCES/FURTHER READING

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8. Differential Games by Avner Friedman.
9. Classical Mechanics by TWB Kibble

UNIT 2 THE ELECTROMAGNETIC FIELD

CONTENTS

1.0 INTRODUCTION

Electromagnetic theory lies outside the scope of this book. However, since we have discussed various examples involving electromagnetic fields, it may be useful to summarize some relevant properties of these fields here. We shall simply quote the results without proof, and we shall not consider the case of dielectric or magnetic media. We shall use Gaussian units, but quote the forms appropriate to SI units in brackets.

2.0 OBJECTIVE

At the end of this unit, you should be able to discussed various examples involving electromagnetic fields.

2.0 MAIN CONTENT

3.1 THE ELECTROMAGNETIC FIELD

The basic equations of electromagnetic theory are Maxwell's equations. In the absence of dielectric or magnetic media, they may be expressed in terms of two fields, the electric field **E** and the magnetic field **B**. There are two equations involving these fields alone,

$$\nabla \wedge E + \frac{1}{c}\frac{\partial B}{\partial t} = 0, \qquad \left[\nabla \wedge \mathrm{E} + \frac{\partial \mathrm{B}}{\partial t} = 0\right] \tag{1}$$

$$\nabla . B = 0, \qquad [\nabla . B = 0] \tag{2}$$

And two more involving also the electric charge density p and current density **j**,

$$\nabla \wedge B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} j \qquad \left[\mu_0^{-1} \nabla \wedge B - \varepsilon_0 \frac{\partial E}{\partial t} = j\right] \tag{3}$$

$$\nabla . E = 4\pi p. \qquad [\varepsilon_0 \nabla . E = p] \tag{4}$$

The basic set of equations is completed by the Lorentz force equation, which determines the force on a particle of charge *q* moving with velocity **v**,

$$F = q\left(E + \frac{1}{c} v \wedge B\right). \qquad [F = q(E + v \wedge B)] \tag{5}$$

From (B.2), it follows that there must exist a vector potential A such that

$$B = \nabla \wedge A. \tag{6}$$

Substituting in (B.1), we then find that there must exist a scalar potential Ø such that

$$E = -\nabla \varnothing - \frac{1}{c}\frac{\partial A}{\partial t}. \qquad \left[E = -\nabla \varnothing - \frac{\partial A}{\partial t}\right] \tag{7}$$

These potentials are not unique. If $\wedge$ is any scalar field, then

$$\varnothing' = \varnothing + \frac{1}{c}\frac{\partial \wedge}{\partial t}, \qquad \left[\varnothing' = \varnothing + \frac{\partial \wedge}{\partial t}\right]$$

$$A' = A - \nabla \wedge \tag{8}$$

Define the same fields **E** and **B** as do Ø and A. The transformation (B.8) is called a *guage transformation*. In particular, we can always choose $\wedge$ so that the new potentials obey the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial \varnothing'}{\partial t} + \nabla . A' = 0. \qquad \left[\frac{1}{c^2}\frac{\partial \varnothing'}{\partial t} + \nabla . A' = 0\right]* \tag{9}$$

It is only necessary to choose $\wedge$ to be a solution of

$$\frac{1}{c^2}\frac{\partial^2 \wedge}{\partial t^2} - \nabla^2 \wedge = -\left(\frac{1}{c}\frac{\partial \varnothing}{\partial t} + \nabla . A\right). \left[= -\left(\frac{1}{c^2}\frac{\partial \varnothing}{\partial t} + \nabla . A\right)\right]$$

When the Lorentz guage condition is satisfied, we find from (3). (4) and the identity (28) that the potentials satisfy

$$\frac{1}{c^2}\frac{\partial^2 \varnothing}{\partial t^2} - \nabla^2 \varnothing = 4\pi p, \qquad [= \varepsilon_0^{-1} p] \tag{10}$$

And

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{4\pi}{c} j. \quad [= \mu_0 j] \tag{11}$$

When there is no electric charge or current density, these are three-dimensional wave equations, which describe a wave propagating with velocity *c.*

For the static case, in which all the fields are time-independent; Maxwell's equations separate into a pair of electrostatic equations,

$$\nabla \wedge E = 0, \qquad \nabla . E = 4\pi p, \qquad [\varepsilon_0^{-1} p] \tag{12}$$

Identical with (6.46) and (6.47), and a pair of magneto static equations,

$$\nabla . B = 0, \qquad \nabla \wedge B = \frac{4\pi}{c} j. \quad [= \mu_0 j] \tag{13}$$

Equation (10) reduces to Poisson's equation (6.48), and (B.11) expresses the vector potential similarly in terms of the current density. The solution of (11) for this case is similar to (6.15), namely

$$A(r) = \frac{1}{c}\iiint \frac{j(r')}{|r - r'|} d^3 r'. \tag{14}$$

[Here and below the SI form is obtained by the replacement $\frac{1}{c} \to \frac{\mu_0}{4\pi}$] Thus, given a static distribution of charges and currents, we can calculate explicitly the scalar and vector potentials, and hence find the fields **E** and **B**.

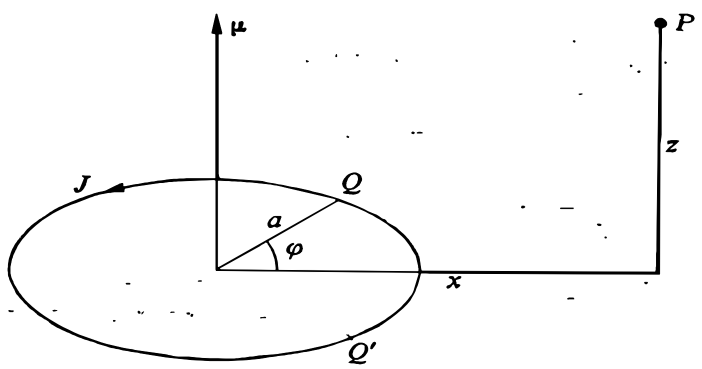

Fig. B.1

As a simple example, we consider a circular current loop of radius *a* in the *xy*-plane, carrying a current **J**. The equation (14) then reduces to a single integration round the loop,

$$A(r) = \frac{J}{c}\oint \frac{dr'}{|r - r'|}. \tag{15}$$

The evaluation of this integral is much simplified by considerations of symmetry. Since the current lies in the xy-plane, A_z is clearly zero. Now let us consider a point P with co-ordinates (x, 0, z). (See fig. B.1) For each point Q on the loop, there will be another point Q', equidistant from P. The contributions of small elements of the loop at Q and Q' to the component A_x will cancel. Thus the only non-zero component at P is A_y. Its value is

$$A_y = \frac{J}{c}\int_0^{2\pi} \frac{a\cos\varphi\, d\varphi}{(r^2 + a^2 - 2ax\cos\varphi)^{1/2}}$$

Now we shall assume that the loop is small, so that $a \ll r$. Then the denominator is approximately

$$(r^2 - 2ax\cos[\varphi)^{-\frac{1}{2}}] \approx \frac{1}{r}\left(1 + \frac{ax}{r^2}\cos\varphi\right)$$

Whence

$$A_y = \frac{J}{cr}\int_0^{2\pi}\left(a\cos\varphi + \frac{a^2x}{r^2}\cos^2\varphi\right)d\varphi = \frac{\pi J a^2 x}{cr^3}.$$

It is clear that at an arbitrary point the only non-vanishing component of A will be in the φ direction of polar co-ordinates. If we define the magnetic moment μ of the loop to be

$$\mu = \frac{\pi a^2 J}{c}, \qquad [\mu = \pi a^2 J] \tag{16}$$

Then the vector potentials is

$$A_r = 0, \qquad A_\theta = 0, \qquad A_\varphi = \frac{\mu\sin\theta}{r^2}.$$
(17)

[Here and below the SI form is obtained by $\mu \to \frac{\mu_0\mu}{4\pi}$.] The co-responding magnetic field is easily evaluated using (A.48).
It is

$$B_r = \frac{2\mu \cos\theta}{r^3}, \qquad B_\theta = \frac{\mu \sin\theta}{r^3}, \qquad B_\varphi = 0.$$
(18)

This is a magnetic dipole field. It has precisely the same form as the electric dipole field (6.11).

4.0 CONCLUSION

In conclusion the magnetic dipole field has precisely the same form as the electric dipole field.

5.0 SUMMARY

What you have learned in this unit concerns the basic equations of electromagnetic theory and these are known as Maxwell's equations. As a simple example, we consider a circular current loop of radius *a* in the *xy*-plane, carrying a current **J**.

6.0 TUTOR-MARKED ASSIGNMENT

1. Calculate the vector potential due to a short segment of wire of directed length d**s**, carrying a current *J*, placed at the origin. Evaluate the corresponding magnetic field. Find the force on another segment of length d**s**' carrying current J', at r. Show that this force does not satisfy Newton's third law. (To compute the force, treat the current element as a collection of moving charges).

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UNIT 3 TENSORS

CONTENTS

1.0 INTRODUCTION

Scalars and vectors are the first two members of a family of quantities known as tensors, and described by 1, 3, 9, 27….. Components. Scalars and vectors are called tensors of rank 0, and of rank 1, respectively. In this appendix, we shall be concerned with the next member of the family, the tensors of rank 2, often called dyadic. We shall use the word tensor in this restricted sense, to mean a tensor of rank 2.

2.0 OBJECTIVES

At the end of this unit, you should be able to recognize scalar and vector as the first two members of Tensors (i.e to mean a tensor of rank 0 and rank 1 respectively) and the recognition of dyadic to mean a tensor of rank 2

3.0 MAIN CONTENT

3.1 Elementary Properties: The DOT Product

Tensors occur most frequently when one vector **b** is defined as a linear function of another vector **a**, according to

$$\begin{aligned} b_x &= T_{xx}a_x + T_{xy}a_y + T_{xz}a_z, \\ b_y &= T_{yx}a_x + T_{yy}a_y + T_{yz}a_z, \\ b_z &= T_{zx}a_x + T_{zy}a_y + T_{zz}a_z. \end{aligned} \tag{1}$$

We have already encountered one set of equations of this type-the relations (9.17) between the angular velocity ω and angular momentum **J** of a rigid body.

It will be convenient to introduce a slight change of notation. We write a_1, a_2, a_3 in place of a_x, a_y, a_z, so that (1) may be written

$$b_i = \sum_j T_{ij} a_j. \qquad (2)$$

Where i and j run over 1, 2, 3. In this notation, the scalar product of two vectors is

$$a.b = \sum_i a_i b_i. \qquad (3)$$

Tensors are commonly denoted by sans-serif capitals, like **T**. The nine components of a tensor **T** may conveniently be exhibited in a square array, or matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \qquad (4)$$

Note that the first subscript labels the rows, and the second, the columns.

In view of the similarity between the expressions (2) and (3), it is natural to extend the dot notation, and write (2) in the form

$b = \mathsf{T}.a.$

For instance, the relation (9.17) may be written

$J = \mathsf{I}.\omega$

Where I is the inertia tensor.

We can then form the scalar product of this vector with another vector c, and obtain a scalar

$$c.T.a = \sum_i \sum_j c_i T_{ij} a_j. \qquad (5)$$

For example, it follows from (9.22) that the kinetic energy of a rigid body is

$$T = \frac{1}{2}\omega.I.\omega = \frac{1}{2}\sum_i \sum_j \omega_i I_{ij} \omega_j.$$

For any tensor T, we define the transposed tensor $\tilde{T}$ by

$\tilde{T}ij = T_{ji}$.

This corresponds to reflecting the array (4) in the leading diagonal. From (5) we see that in general c.T.a is not the same as a.T.c. In fact,

$$a.T.c = \sum_i \sum_j a_i T_{ij} c_j = \sum_j \sum_i c_j \tilde{T}_{ji} a_i .$$

So that

$$a.T.c = c.\tilde{T}.a. \qquad (6)$$

Note that he dot always corresponds to a sum over adjacent subscripts.

The tensor T is called symmetric if $\tilde{T} = T$, or, equivalently, $T_{ji} = T_{ij}$. In this case, the array (4) is unchanged by reflection in the leading diagonal. An equivalent condition is that, for all vectors a and c,

$$a.T.c = c.T.a. \qquad (7)$$

Similarly, T is called anti-symmetric (or skew-symmetric) if $\tilde{T} = -T, or\ T_{ji} = -T_{ij}$. For example, consider the relation giving the velocity of a point in a rotating body,

$$v = \omega \wedge r.$$

This is a linear relation between the components of **r** and **v**, and can therefore be written in the form

$$v = T.r,$$

Where T is some suitable tensor. It is easy to see that its components are given by

$$T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \qquad (8)$$

This tensor is clearly anti symmetric. Note that its diagonal elements T_{ij} are necessarily zero. In fact, any ant symmetric tensor may be associated with an axial vector in this way, and vice versa.

There is a special tensor **1** called the *unit tensor*, or *identity tensor*, which has the property that

$$\mathbf{1.a} = \mathbf{a} \qquad (9)$$

For all vectors **a**. Its components are

$$\mathbf{1}_{ij} = \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j, \end{cases}$$

Or, written out in detail,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (10)$$

3.2 Sums and Products; The Tensor Product

The sum of two tensors may be defined in an obvious way. The tensor $R = \alpha S + \beta T$ is the tensor with components $R_{ij} = \alpha S_{ij} + \beta T_{ij}$. Its effect on a vector **a** is given by

R.a= α (**S**.a) + β (**T**.a)

For example, it is easy to show that any tensor **T** can be written as a sum of a sysmmetric tensor **S** and an anti symmetric tensor **A**. in fact, **T** = **S** + **A**, where **S** = $\left(\frac{1}{2}\right)(\mathrm{T} + \mathrm{T}\square)$ and **A** = $\left(\frac{1}{2}\right)(\mathrm{T} + \mathrm{T}\square)$.

We can also define the dot product of two tensors, **S .T**. if c = **S.b** and **b** = **T.a**, then it is natural to write **c** = **S**.(**T.a**) = (**S .T**).a. In terms of components,

$$c_i = \sum_j S_{ij}\left(\sum_k T_{jk}\, a_k\right) = \sum_k\left(\sum_j S_{ij} T_{jk}\right) a_k.$$

Hence **S .T** = **R** is the tensor with components

$$R_{ik} = \sum_j S_{ij} T_{jk}. \qquad (11)$$

Once again, the dot signifies summation over adjacent subscripts. Note the rule for constructing the elements of the product: to form the element in the *i*th row and *k*th column of **S .T**, we take the *i*th row of **S**, and the *k*th column of **T**, multiply the corresponding elements, and sum. (This is known as the rule of matrix multiplication.) It is important to realize that, in general, **T .S** ≠ **S .T**. in fact, **T .S** = **Q** has components

$$Q_{ik} = \sum_j T_{ij} S_{jk}.$$

There is one special case in which these products are equal, namely the case **S**= **1**. It is easy to see that

1.**T** = **T**.**1** = **T**,

So that **1** plays exactly the same role as the unit in ordinary algebra. From any two vectors **a** and **b** we can form a tensor **T** whose components are $T_{ij} = a_i b_j$. This tensor is written **T** = **a b**, with no dot or cross, and is called the tensor product or dyadic product of **a** and **b**. note that

T·c = **(ab)˙c = a(b˙c),**

So that the brackets are in fact unnecessary. The use of the tensor product allows us to write some earlier results in a different way. For example, for any vector **a**,

1˙a = a = i(i.a) + j(j˙a) + k(k˙a)
= (ii + jj + kk)˙a,

So that

$$\mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{1}, \qquad (12)$$

as may easily be verified by writing out the components. Similarly, we may write (9.16) in the form

$$J = \sum m(r^2 \omega - rr\cdot\omega) = I\cdot\omega,$$

Where the inertia tensor is given explicitly by

$$\mathbf{I} = \sum m(r^2 \mathbf{1} - rr). \qquad (13)$$

It is easy to check that the nine components of this equation reproduce (9.15).

It is clear that if **T** = **ab**, then $\tilde{T} = ba$. In particular, it follows that the tensor (13) is symmetric

3.3 Eigenvalues; Diagonalization of a Symmetric Tensor

Throughout this section, we consider a given symmetric tensor **T**. A vector **a** is called an eigenvector of **T** if

$$\mathbf{T} \cdot a = \lambda a, \qquad (14)$$

Where λ is a number called *eigenvalue*. Equivalently, the equation (14) may be written

$$(\mathbf{T} - \lambda 1) \cdot a = 0,$$

Or, written out in full,

$$\begin{aligned} (T_{11} - \lambda)a_1 + T_{12}a_2 + T_{13}a_3 &= 0, \\ T_{21}a_1 + (T_{22} - \lambda)a_2 + T_{23}a_3 &= 0, \\ T_{31}a_1 + T_{32}a_2 + (T_{33} - \lambda)a_3 &= 0. \end{aligned}$$

These are the same kind of equations that we discussed in Chapter 12 in connection with normal modes. (Compare (12.15). As in that case, the equations are mutually consistent only if the determinant of the coefficients vanishes,

$$\det(T - \lambda 1) = \begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0.$$

When expanded, this determinant is a cubic equation for λ whose three roots are all real, or else one real and two complex conjugates of each other.

We shall now show that the latter possibility can be ruled out. For. Suppose λ is a complex eigenvalue, and a = (a_1, a_2, a_3) the corresponding eigenvalue, whose components may also be complex. We shall denote the complex conjugate eigenvalue by λ*. Then, taking the complex conjugate of

$$\mathbf{T} \cdot a = \lambda a,$$

We obtain

$\mathbf{T} \cdot a^* = \lambda^* a^*$,

Where $a^* = (a_x^*, a_y^*, a_z^*)$. Multiplying these two equations by **a*** and **a** respectively, we obtain

$$\mathbf{a^* \cdot T \cdot a = \lambda a^* \cdot a,}$$
$$\mathbf{a \cdot T \cdot a^* = \lambda^* a \cdot a^*.}$$

4.0 CONCLUSION

In this unit we want to conclude by considering a given symmetric tensor **T**. A vector **a which** is called an eigenvector of **T** if

$\mathbf{T} \cdot a = \lambda a,$

Where λ is a number called *eigenvalue*. Equivalently, $\mathbf{T} \cdot a = \lambda a,$ may be written $(\mathbf{T} - \lambda 1) \cdot a = 0$,

5.0 SUMMARY

The summary of what you have learnt is as in the conclusion.

SELF ASSESSMENT EXERCISE

6.0 TUTOR-MARKED ASSIGNMENT

1) define the term Tensor
2) state some properties of Tensor that you are taught
3) under what consideration can you ascertain that
$\mathbf{T} \cdot a = \lambda a,$ may be written as $(\mathbf{T} - \lambda 1) \cdot a = 0$,

7.0 REFERENCES/FURTHER READING

1.Theoretical Mechanics by Murray, R. Spiegel.

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8.Differential Games by Avner Friedman.

9.Classical Mechanics by TWB Kibble

MODULE 2 DYNAMICS OF SYSTEMS OF PARTICLES

UNIT 1 DISCRETE AND CONTINUOUS SYSTEMS

CONTENTS

1.0 INTRODUCTION

Up to now we have dealt mainly with the motion of an object which could be considered as a particle or point mass. In many practical cases the objects with which we are concerned can more realistically be considered as collections or systems of particles. Such systems are called discrete or continuous according as the particles can be considered as separated from each other or not.

For many practical purposes a discrete system having a very large but finite number of particles can be considered as a continuous system. Conversely a continuous system can be considered as a discrete system consisting of a large but finite number of particles.

2.0 OBJECTIVES

At the end of this unit, you should be able to know about the distinction between Discrete and Continuous Systems with examples.

3.0 MAIN CONTENT

3.1 Density

For continuous systems of particles occupying a region of space it is often convenient to define a mass per unit volume which is called the *volume density* or briefly *density*. Mathematically, if ΔM is the total mass of a volume Δ_T of particles, then the density can be defined as

$$\sigma = \lim_{\Delta_T \to 0} \frac{\Delta M}{\Delta_T} \qquad (1)$$

The density is a function of position and can vary from point to point. When the density is a constant, the systems is said to be of uniform density or simply uniform.

When the continuous system of particles occupy a surface, we can similarly define a surface density or mass per unit area. Similarly when the particles occupy a line [or curve] we can define a mass per unit length or linear density.

3.2 Rigid and Elastic Bodies

In practice, forces applied to systems of particles will change the distances between individual particles. Such systems are often called deformable or elastic bodies. In some cases, however, deformations may be so slight that they may for most practical purposes be considered non-existent. It is thus convenient to define a mathematical model in which the distance between any two specified particles of a system remains the same regardless of applied forces. Such a system is called a rigid body. The mechanics of rigid bodies is considered in Chapters 9 and 10.

3.3 Degrees of Freedom

The number of coordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

a) A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.

b) A system consisting of *N* particles moving freely in space requires *3N* coordinates to specify its position. Thus the number of degrees of freedom is *3N*.

c) A rigid body which can move freely in space has 6 degrees of freedom, i.e 6 coordinates are required to specify the position.

Examples on Degrees of Freedom

1. Determine the number of degrees of freedom in each of the following cases: (a) a particle moving on a given space curve; (b) five particles moving freely in a plane; (c) five particles moving freely in space; (d) two particles connected by a rigid rod moving freely in a plane.

(a) The curve can be described by the parametric equations $x = x(s), y = y(s), z = z(s)$ where s is the parameter. Then the position of a particle on the curve is determined by specifying one coordinate, and hence there is one degree of freedom.

(b) Each particle requires two coordinates to specify its position in the plane. Thus $5 \cdot 2 = 10$ coordinates are needed so specify the positions of all 5 particles, i.e. the system has 10 degrees of freedom.

(c) Since each particles requires three coordinates to specify its position, the system has $5 \cdot 3 = 15$ degrees of freedom.

(d) Method 1

The coordinates of the two particles can be expressed by (x_1, y_1) and (x_2, y_2), i.e. a total of 4 coordinates. However, since the distant between these points is a constant a [the length of the rigid rod], we have $(x_1 – x_2)^2 + (y_1 – y_2)^2 = a^2$ so that one of the coordinates can be expressed in terms of the others. Thus there are 4 – 1 = 3 degrees of freedom.

Method 2

The motion is completely specified if we give the two coordinates of the center of mass and the angle made by the rod with some specified direction. Thus there are 2 + 1 = 3 degrees of freedom

2. Prove that the center of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.

Let $\mathbf{F}_v$ be the resultant external force acting on particle v while $f_{v\lambda}$ is the internal force on particle v due to particle λ . We shall assume $\mathbf{f}_{vv} = \mathbf{0}$, i.e. particle v does not exert any force on itself.

By Newton's second law the total force on particle *v* is

$$F_v + \sum_{\lambda} f_{v\lambda} = \frac{dp_v}{dt} = \frac{d^2}{dt^2}(m_v r_v) \qquad (1)$$

Where the second term on the left represents the resultant internal force on particle *v* due to all other particles.

Summing over *v* in equation (1), we find

$$\sum_{v} F_v + \sum_{v}\sum_{\lambda} f_{v\lambda} = \frac{d^2}{dt^2}\left\{\sum_{v}(m_v r_v)\right\} \qquad (2)$$

Now according to Newton's third law of action and reaction, $f_{v\lambda} = -f_{\lambda v}$ so that the double summation on the left of (2) is zero. If we then write

$$F = \sum_{v} F_v \quad \text{and} \quad \bar{r} = \frac{1}{M}\sum_{v} m_v r_v \qquad (3)$$

(2) becomes $F = M\dfrac{d^2\bar{r}}{dt^2}$ (4)

Since **F** is the total external force on all particles applied at the center of mass $\bar{r}$, the required result is proved

3. A system of particles consists of a 3 gram mass located at 91, 0, -1), a 5 gram mass at (-2, 1, 3) and a 2 gram mass at (3, - 1, 1). Find the coordinates of the center of mass.

The positive vectors of the particles are given respectively by

$r_1 = i – k$, $r_2 = - 2i + j + 3K$, $r_3 = 3i – j + k$

then the center of mass is given by

$$\bar{r} = \frac{3(i-k)+5(-2i+j+3k)+2(3i-j+k)}{3+5+2} = -\frac{1}{10}i + \frac{3}{10}j + \frac{7}{5}k$$

Thus the coordinates of the center of mass are $\left(-\frac{1}{10}, \frac{3}{10}, \frac{7}{5}\right)$.

4. Find the centroid of a solid region $\mathcal{R}$ as in Fig. 7 – 3

Consider the volume element $\Delta\tau_v$ of the solid. The mass of this volume element is

$$\Delta M_v = \sigma_v \Delta\tau_v = \sigma_v \Delta x_v \Delta y_v \Delta z_v$$

Where σ_v is the density [mass per unit volume] and $\Delta x_v, \Delta y_v, \Delta z_v$ are the dimensions of the volume element. Then the centroid is given approximately by

$$\frac{\sum r_v \Delta M_v}{\sum \Delta M_v} = \frac{\sum r_v \sigma_v \Delta\tau_v}{\sum \sigma_v \Delta\tau_v} = \frac{\sum r_v \sigma_v \Delta x_v \Delta y_v \Delta z_v}{\sum \sigma_v \Delta x_v \Delta y_v \Delta z_v}$$

Where the summation is taken over all volume elements of the solid.

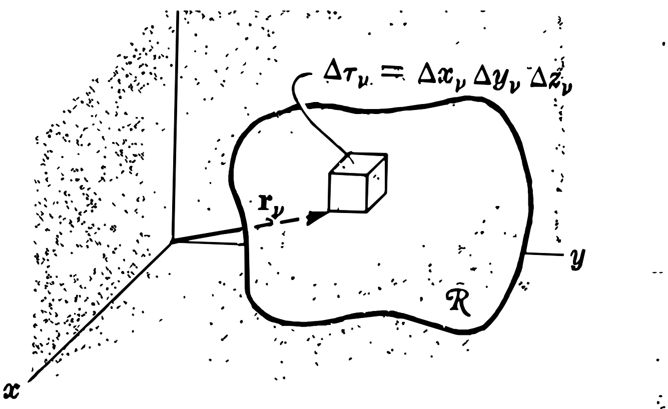

Fig. 7-3

Taking the limit as the number of volume elements becomes infinite in such a way that $\Delta\tau_v \to 0$ or $\Delta x_v \to 0, \Delta y_v \to 0, \Delta z_v \to 0,$ we obtain for the centroid of the solid:

$$\bar{r} = \frac{\int_{\mathcal{R}} r\, dM}{\int_{\mathcal{R}} dM} = \frac{\int_{\mathcal{R}} r\, \sigma d\tau}{\int_{\mathcal{R}} \sigma\, d\tau} = \frac{\iiint_{\mathcal{R}} r\, \sigma\, dx\, dy\, dz}{\iiint_{\mathcal{R}} \sigma\, dx\, dy\, dz}$$

Where the integration is to be performed over $\mathcal{R}$, is indicated.

Writing r = x**i** + y**j** + z**k**, $\bar{r} = \bar{x}i + \bar{y}j + \bar{z}k,$ this can also be written in component form as

$$\bar{x} = \frac{\int_{\mathcal{R}} r\, dM}{\int_{\mathcal{R}} dM} = \frac{\bar{y}\left(\int_{\mathcal{R}} r\, \sigma d_T\right)}{\int_{\mathcal{R}} \sigma\, d_T} = \frac{\bar{z}\left(\iiint_{\mathcal{R}} r\, \sigma\, dx\, dy\, dz\right)}{\iiint_{\mathcal{R}} \sigma\, dx\, dy\, dz}$$

5. Find the center of mass of a uniform solid hemisphere of radius *a*.

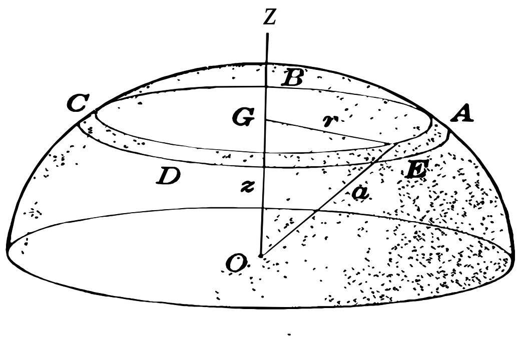

Fig. 7-7

By symmetric the center of mass lies on the *z* axis [see Fig. 7 – 7]. Subdivided the hemisphere into solid circular plates of radius r, such as *ABCDEA*. If the center *G* of such a ring is at distance z from the center *O* of the hemisphere, $r^2 + z^2 = a^2$. Then if *dz* is the thickness of the plate, the volume of each right as

$$\pi r^2 dz = \pi(a^2 - z^2)dz$$

And the mass is $\pi\sigma z(a^2 - z^2)dz$. Thus we have

$$\bar{z} = \frac{\int_{z=0}^{a} \pi\sigma z(a^2 - z^2)}{\int_{z=0}^{a} \pi\sigma(a^2 - z^2)} = \frac{3}{8}a$$

3.4 Center of Mass

Let r_1, r_2,……., r_N be the position vectors of a system of N particles of masses m_1, m_2, ……., m_N respectively [see Fig. 7 – 1].

The *center of mass* or *centroid* of the system of particles is define as that point C having position vector

$$\bar{r} = \frac{m_1 r_1 + m_2 r_2 + \cdots .. + m_N r_N}{m_1 + m_2 + \cdots .. + m_N} + \frac{1}{M}\sum_{v=1}^{N} m_v r_v \qquad (2)$$

Where $M = \sum_{v=1}^{N} m_v$ is the total mass of the system. We sometimes use $\sum_v \square$ or simply $\sum \square$ in place of $\sum_{v=1}^{N} \square$.

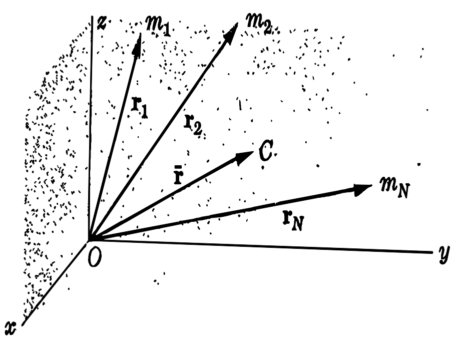

Fig. 7-1

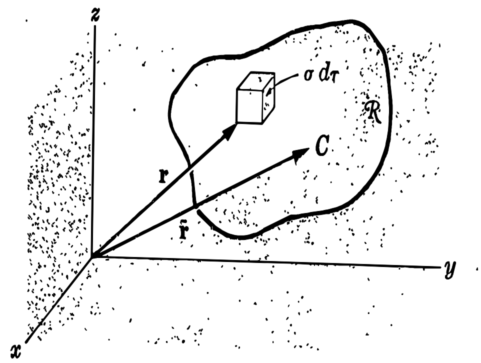

Fig. 7-2

For continuous systems of particles occupying a region $\mathcal{R}$ of space in which the volume density is σ , the center of mass can be written

$$\bar{r} = \frac{\int_{\mathcal{R}} \sigma r d_T}{\int_{\mathcal{R}} \sigma d_T} \qquad (3)$$

Where the integral is taken over the entire region $\mathcal{R}$ [see Fig. 7.2). If we write

$$\bar{r} = \bar{x}i + \bar{y}j + \bar{z}k, \qquad r_v = x_v i + y_v j + z_v k$$

Then (3) can equivalently be written as

$$\bar{x} = \frac{\sum m_v x_v}{M}, \qquad \bar{y} = \frac{\sum m_v y_v}{M}, \bar{z} = \frac{\sum m_v z_v}{M} \qquad (4)$$

And $\bar{x} = \frac{\int_{\mathcal{R}} \sigma x dr}{M}, \bar{y} = \frac{\int_{\mathcal{R}} \sigma y dr}{M}, \bar{z} = \frac{\int_{\mathcal{R}} \sigma z dr}{M}$ (5)

where the total mass is given by their

$$M = \sum m_v \qquad (6)$$

or $$M = \int_{\mathcal{R}} \sigma d_T \qquad (7)$$

The integrals in (3), (5) or (7) can be single, double or triple integrals, depending on which may be preferable.

In practice it is fairly simple to go from discrete to continuous systems by merely replacing summations by integrations. Consequently we will present all theorems for discrete systems.

3.5 Center of Gravity

If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity.

4.0 CONCLUSION

We shall conclude by saying that In practice it is fairly simple to go from discrete to continuous systems by merely replacing summations by integrations.

5.0 SUMMARY

What you have learnt in this unit concerns: centre of gravity, centre of mass, density, degree of freedom their definitions and examples of each also discussed is their real life applications. Rigid and Elastic Bodies are also taught extensively in this unit.

6.0 TUTOR-MARKED ASSIGNMENT

Show in tabular form the difference between a discrete and a continuous system. Also give examples of each.

7.0 REFERENCES/FURTHER READING

1. Theoretical Mechanics by Murray, R. Spiegel.

2. Advanced Engineering Mathematics by KREYSZIC.

3.Generalized function. Mathematical Physics by U. S. Vladinirou.

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7.Lecture Notes on Analytical Dynamics from UNILORIN(1999)

8.Differential Games by Avner Friedman.

9.Classical Mechanics by TWB Kibble

UNIT 2 MOMENTUM OF A SYSTEM OF PARTICLES

CONTENTS

1.0 INTRODUCTION

When we say system of particles, this refers to centre of mass, the motion of a rotating ax thrown between two jugglers looks rather complicated, and very different from the standard projectile motion alluded to. We deduce from experiment that one point of the ax follows a trajectory described by the standard equations of motion of a projectile. This special point is called the centre of mass of the ax.

2.0 OBJECTIVES

At the end of this unit, you should be able to know momentum of a system of particles as stated in the main contents (3.1-3.8) above.

3.0 MAIN CONTENT

3.1 Momentum of a System of Particles

If $V_v = \frac{dr_v}{dt} = \dot{r}_v$ is the velocity of m_v, the total momentum of the system is defined as

$$P = \sum_{v=1}^{N} m_v V_v = \sum_{v=1}^{N} m_v \dot{r}_v \qquad (8)$$

We can show [see Problem 7.3] that

$$P = M\bar{v} = M\frac{d\bar{r}}{dt} = M\dot{\bar{r}} \qquad (9)$$

Where $\bar{v} = \frac{d\bar{r}}{dt}$ is the velocity of the center of mass.

This is expressed in the following

Theorem 1. The total momentum of a system of particles can be found by multiplying the total mass *M* of the system by the velocity $\bar{v}$ of the center of mass.

3.2 Motion of the Center of Mass

Suppose that the internal forces between any two particles of the system obey Newton's third law. Then if F is the resultant external force acting on the system, we have

$$F = \frac{dp}{dt} = M\frac{d^2\bar{r}}{dt^2} = M\frac{d\bar{v}}{dt} \qquad (10)$$

This is expressed in

Theorem 2. The center of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.

3.3 Conservation of Momentum

Putting F = 0 in (10), we find that

$$P = \sum_{v=1}^{N} m_v V_v = \text{constant} \qquad (11)$$

Thus we have

Theorem 3. If the resultant external force acting on a system of particles is zero, then the total momentum remains constant, i.e. is conserved. In such case the center of mass is either at rest or in motion with constant velocity.

This theorem is often called the principle of conservation of momentum. It is a generalization of Theorem 2 – 8,

3.4 Angular Momentum of a System of particles

The quantity

$$\Omega = \sum_{v=1}^{N} m_v \, (r_v X V_v) \qquad (12)$$

is called the total angular momentum [of moment of momentum] of the system of particles about origin O.

3.5 The Total External Torque Acting on a System

If F_v is the external force acting on particle v, then r_v X F_v is called the moment of the force F_v or torque about O. the sum

$$\Lambda = \sum_{v=1}^{N} r_v X F_v \qquad (13)$$

is called the *total external torque* about the origin.

3.6 Relation between Angular Momentum and Total External Torque

If we assume that the internal forces between any two particles are always directed along the line joining the particles [i.e. they are central forces], then we can show as in problem 7.12 that

$$\Lambda = \frac{d\Omega}{dt} \qquad (14)$$

Thus we have

Theorem 4. The total external torque on a system of particles is equal to the time rate of change of the angular momentum of the system, provided the internal forces between particles are central forces.

6. Solved examples on Angular Momentum and Torque

Prove theorem 4: The total external torque on a system of particles is equal to the time rate of change of angular momentum of the system, provided that the internal forces between particles are central forces.

we have

$$F_v + \sum_{\lambda} f_{v\lambda} = \frac{dp_v}{dt} = \frac{d}{dt}(m_v v_v) \qquad (1)$$

Multiplying both sides of (1) by $\mathbf{r}_v$ X, we have

$$r_v X F_v + \sum_{\lambda} r_v X f_{v\lambda} = r_v X \frac{d}{dt}(m_v v_v) \qquad (2)$$

Since $$r_v X \frac{d}{dt}(m_v v_v) = \frac{d}{dt}\{m_v (r_v X v_v)\} \qquad (3)$$

(2) becomes $$r_v X F_v + \sum_{\lambda} r_v X f_{v\lambda} = \frac{d}{dt}\{m_v (r_v X v_v)\} \qquad (4)$$

Summing over *v* in (4), we find

$$\sum_{v} r_v X F_v + \sum_{v}\sum_{\lambda} r_v X f_{v\lambda} = \frac{d}{dt}\left\{\sum_{v} m_v (r_v X v_v)\right\} \qquad (5)$$

Now the double sum in (5) is composed of terms such as

$$r_v \ X \ f_{v\lambda} + r_\lambda \ X \ f_{\lambda v} \qquad (6)$$

Which becomes on writing $f_{\lambda v} = -f_{v\lambda}$ according to Newton's third law,

$$r_v \ X \ f_{v\lambda} - r_\lambda \ X \ f_{v\lambda} = (r_v - r_\lambda) X \ f_{v\lambda} \qquad (7)$$

Then since we suppose that the forces are central, i.e. $f_{v\lambda}$ has the same direction as $r_v - r_\lambda$, it follows that (7) is zero and also that the double sum in (5) is zero. Thus equation (5) becomes

$$\sum_{v} r_v \ X \ F_v = \frac{d}{dt}\left\{\sum_{v} m_v (r_v \ X \ v_v)\right\} \quad or \quad \Lambda = \frac{d\Omega}{dt}$$

Where $\Lambda = \sum_v r_v \ X \ F_v, \Omega = \sum_v m_v (r_v \ X \ v_c)$.

7. Suppose that the internal forces of a system of particles are conservative and are derived from a potential

$$V_{\lambda v}(r_{\lambda v}) = V_{v\lambda}(r_{v\lambda})$$

Where $f_{\lambda v} = -f_{v\lambda} = \sqrt{(x_\lambda - x_v)^2 + (y_\lambda - y_v)^2 + (z_\lambda - z_v)^2}$ is the distance between particles λ and v of the systems.

(a) Prove that $\sum_v \sum_\lambda f_{\lambda v} \cdot dr_v = -\frac{1}{2}\sum_v \sum_\lambda dV_{\lambda v}$ where $f_{\lambda v}$ is the internal force on particles v due to particle λ .

(b) Evaluate the double sum $\sum_v \sum_\lambda \int_1^2 f_{\lambda v} \cdot dr_v$ of problem 7.13

(a) The force acting on particle v is

$$f_{\lambda v} = -\frac{\partial V_{\lambda v}}{\partial x_v} i - \frac{\partial V_{\lambda v}}{\partial y_v} j - \frac{\partial V_{\lambda v}}{\partial z_v} k = -grad_v \ V_{\lambda v} = -\Delta_v \ V_{\lambda v} \quad (1)$$

The force acting on particle λ is

$$f_{v\lambda} = -\frac{\partial V_{\lambda v}}{\partial x_v} i - \frac{\partial V_{\lambda v}}{\partial y_v} j - \frac{\partial V_{\lambda v}}{\partial z_v} k = -grad_v \ V_{\lambda v} = -\Delta_v \ V_{\lambda v} = -f_{v\lambda}$$

The work done by these forces in producing the displacements d**r**$_v$ and dr_λ of particles v and λ respectively is

$$f_{v\lambda} \cdot dr_v + f_{\lambda v} \cdot dr_\lambda = -\left\{\frac{\partial V_{\lambda v}}{\partial x_v} dx_v + \frac{\partial V_{\lambda v}}{\partial y_v} dy_v + \frac{\partial V_{\lambda v}}{\partial z_v} dz_v + \frac{\partial V_{\lambda v}}{\partial x_\lambda} dx_\lambda + \frac{\partial V_{\lambda v}}{\partial y_\lambda} dy_\lambda + \frac{\partial V_{\lambda v}}{\partial z_\lambda}\right\}$$

$$= -dV_{\lambda v}$$

Then the total work done by the internal forces is

$$\sum_v \sum_\lambda f_{\lambda v} \cdot dr_v = -\frac{1}{2}\sum_v \sum_\lambda dV_{\lambda v} \quad (3)$$

The factor $\frac{1}{2}$ on the right being introduced because otherwise the terms in the summation would enter twice.

(b) By integrating (*3*) of part (*a*), we have

$$\sum_{v}\sum_{\lambda}\int_{1}^{2} f_{\lambda v}\cdot dr_{v} = -\frac{1}{2}\sum_{v}\sum_{\lambda}\int_{1}^{2} dV_{\lambda v} = V_{1}^{(int)} - V_{2}^{(int)} \quad (4)$$

Where $V_1^{(int)}$ and $V_2^{(int)}$ denote the total internal potentials

$$\frac{1}{2}\sum_{v}\sum_{\lambda} V_{\lambda v} \quad (5)$$

At times t_1 and t_2 respectively.

8. Prove that if both the external and internal forces for a system of particles are conservative, then the principle of conservation of energy is valid.

If the external forces are conservation, then we have

$$F_v = -\Delta V_v \quad (1)$$

From which $$\sum_{v}\int_{1}^{2} F_{v}\cdot dr_{v} = -\sum_{v}\int_{1}^{2} dV_{v} = V_{1}^{(ext)} - V_{2}^{(ext)} \quad (2)$$

Where $V_1^{(ext)}$ and $V_2^{(ext)}$ denote the total external potential

$$\sum_{v} V_v$$

At times t_1 and t_2 respectively

Using (2) and equation (4) of problem 7.14 (b) in equation (5) of problem 7.13, we find

$$T_2 - T_1 = V_1^{(ext)} - V_2^{(ext)} + V_1^{(int)} - V_2^{(int)} = V_1 - V_2 \quad (3)$$

Where
$$V_1 = V_1^{(ext)} + V_1^{(int)} \text{ and } V_2 = V_2^{(ext)} + V_2^{(int)} \quad (4)$$

Are the respective total potential energies [external and internal] at times t_1 and t_2. We thus find from (3),

$$T_1 + V_1 = T_2 + V_2 \text{ or } T + V = \text{constant} \quad (5)$$

Which is the principle of conservation of energy.

3.7 Conservation of Angular Momentum

Putting $\Lambda = 0$ in (14), we find that

$$\Omega = \sum_{v=1}^{N} m_v (r_v X V_v) = \text{constant} \quad (15)$$

Thus we have

Theorem 5. If the resultant external torque acting on a system of particles is zero, then the total angular momentum remains constant i.e. is conserved

This theorem is often called the principle of conservation of angular momentum. It is the generalization of Theorem earlier discussed.

3.8 Kinetic Energy of a System of Particles

The total kinetic energy of a system of particles is defined as

$$T = \frac{1}{2}\sum_{v=1}^{N} m_v v_v^2 = \frac{1}{2}\sum_{v=1}^{N} m_v \dot{r}_v^2 \quad (16)$$

Work

If $\mathcal{F}_v$ is the force (external and internal) acting on particle v, then the total work done in moving the system of particles from one state [symbolized by 1] to another [symbolized by 2] is

$$W_{12} = \sum_{v-1}^{N} \int_1^2 \mathcal{F}_v \, . \, dr_v \quad (17)$$

As in the case of a single particle, we can prove the following

Theorem 6. The total work done in moving a system of particles from one state where the kinetic energy T_1 to another where the kinetic energy is T_2, is

$$W_{12} = T_2 - T_1 \tag{18}$$

Potential Energy, Conservation of Energy

When all forces, external and internal, are conservative, we can define a total potential energy *V* of the system. In such case we can prove the following.

Theorem 7. If *T* and *V* are respectively the total kinetic energy and total potential energy of a system of particles, then

$$T + V = \text{constant} \tag{19}$$

This is the principle of conservation of energy for systems of particles.

Motion Relative to the Center of Mass

It is often useful to describe the motion of a system of particles about [or relative to] the center of mass. The following theorems are of fundamental importance. In all cases primes denote quantities relative to the center of mass.

Theorem 8. The total linear momentum of a system of particles about the center of mass is zero. In symbols,

$$\sum_{v=1}^{N} m_v v_v' = \sum_{v=1}^{N} m_v \dot{r}_v' = 0 \tag{20}$$

Theorem 9. The total angular momentum of a system of particles about any point O equals the angular momentum of the total mass assumed to be located at the center of mass plus the angular momentum about the center of mass. It could be expressed mathematically, thus

$$\Omega = \bar{r} X \; M\bar{v} + \sum_{v=1}^{N} m_v (r_v' X v_v') \tag{21}$$

Theorem 10. The total kinetic energy of a system of particles about any point O equals the kinetic energy of translation of the center of mass [assuming the total mass located there] plus the kinetic energy of motion about the center of mass. Thus,

$$T = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}\sum_{v=1}^{N} m_v v_v'^2 \qquad (22)$$

Theorem 11. The total external torque about the center of mass equals the time rate of change in angular momentum about the center of mass, i.e. equation (14) holds not only for inertial coordinate systems but also for coordinate systems moving with the center of mass. Consequently,

$$\Lambda' = \frac{d\Omega'}{dt} \qquad (23)$$

If motion is described relative to points other than the center of mass, the results in the above theorems become more complicated.

Impulse

If **F** is the total external force acting on a system of particles, then

$$\int_{t_1}^{t_2} \mathbf{F}\, dt \qquad (24)$$

is called the *total linear impulse* or briefly *total impulse*. As in the case of one particle, we can prove

Theorem 12. The total linear impulse is equal to the change in linear momentum. Similarly if Λ is the total external torque applied to a system of particles about origin 0, then

$$\int_{t_1}^{t_2} \Lambda\, dt \qquad (25)$$

Is called the total angular impulse. We can then prove

Theorem 13. The total angular impulse is equal to the change in angular momentum.

4.0 CONCLUSION

In conclusion, as in Theorem, 10. The total kinetic energy of a system of particles about any point O equals the kinetic energy of translation of the center of mass [assuming the total mass located there] plus the kinetic energy of motion about the center of mass. Thus,

$$T = \frac{1}{2}M\bar{v}^2 + \frac{1}{2}\sum_{v=1}^{N} m_v v_v'^2$$

5.0 SUMMARY

Some thirteen theorems are discussed in this unit thus:

1. The total momentum of a system of particles can be found by multiplying the total mass M of the system by the velocity $\bar{v}$ of the center of mass.
2. The center of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.
3. If the resultant external force acting on a system of particles is zero, then the total momentum remains constant, i.e. is conserved. In such case the center of mass is either at rest or in motion with constant velocity.
4. The total external torque on a system of particles is equal to the time rate of change of the angular momentum of the system, provided the internal forces between particles are central forces.
5. If the resultant external torque acting on a system of particles is zero, then the total angular momentum remains constant i.e. is conserved
6. The total work done in moving a system of particles from one state where the kinetic energy T_1 to another where the kinetic energy is T_2, is
7. If T and V are respectively the total kinetic energy and total potential energy of a system of particles, then T+V is a constant.
8. The total linear momentum of a system of particles about the center of mass is zero
9. The total angular momentum of a system of particles about any point O equals the angular momentum of the total mass assumed to be located at the center of mass plus the angular momentum about the center of mass.
10. The total kinetic energy of a system of particles about any point O equals the kinetic energy of translation of the center of mass [assuming the total mass located there] plus the kinetic energy of motion about the center of mass.
11. The total external torque about the center of mass equals the time rate of change in angular momentum about the center of mass.
12. The total linear impulse is equal to the change in linear momentum.
13. The total angular impulse is equal to the change in angular momentum.

Example

Prove Theorem 10, The total kinetic energy of a system of particles about any point O equals the kinetic energy of the center of mass [assuming the total mass located there] plus the kinetic energy of motion about the center of mass.

The kinetic energy relative to O is

$$T = \frac{1}{2}\sum_v m_v v_v^2 = \frac{1}{2}\sum_v m_v (\dot{r}_v \cdot \dot{r}_v) \qquad (1)$$

but

$$\dot{r}_v = \dot{\bar{r}} + \dot{r}'_v = \bar{V} + v'_v$$

Thus (1) can be written

$$T = \frac{1}{2}\sum_v m_v \{(\bar{v} + v'_v) \cdot (\bar{v} + v'_v)\}$$

$$= \frac{1}{2}\sum_v m_v \bar{v} \bullet v'_v + \frac{1}{2}\sum_v m_v v'_v \bullet v'_v$$

$$= \frac{1}{2}\left(\sum_v m_v\right)\bar{v}^2 + \bar{v} \cdot \left\{\sum_v m_v v'_v\right\} + \frac{1}{2}\sum_v m_v v'^2_v$$

$$= \frac{1}{2}M\bar{v}^2 + \frac{1}{2}m_v v'^2_v$$

Since $\sum_v m_v v'_v = 0$

6.0 TUTOR-MARKED ASSIGNMENT

1. What can be referred to as being the generalization of Theorems (2 – 8)?
2. Prof that the total angular impulse is equal to the change in angular momentum.
3. state the law of conservation of energy.

7.0 REFERENCE/FURTHER READING

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8.Differential Games by Avner Friedman.

9.Classical Mechanics by TWB Kibble

UNIT 3 Constraints, Holonomic and Non-Holonomic Constraints

CONTENTS

1.0 INTRODUCTION

Often in practice the motion of a particle or system of particles is restricted in some way. For example, in *rigid bodies* [considered in Chapters 9 and 10] the motion must be such that the distance between any two particular particles of the rigid body is always the same.

As another example, the motion of particles may be restricted to curves or surfaces.

The limitations on the motion are often called *constraints*. If the constraint condition can be expressed as an equation

$$\phi(r_1, r_2, \dots\dots, r_N, t) = 0 \qquad (26)$$

connecting the position vectors of the particles and the time, then the constrain is called *holonomic*. If it cannot be so expressed it is called *non-holonomic*.

2.0 OBJECTIVES

At the end of this unit, you should be able to discussed the following:

1.Virtual Displacements
2.Statics of a System of particles. Principle of Virtual Work
3.Equilibrium in Conservative Fields. Stability of Equilibrium
4.D'Alembert's Principle

3.0 MAIN CONTENT

3.1 Virtual Displacements

Consider two possible configurations of a system of particles at a particular instant which are consistent with the forces and constraints. To go from one configuration to the other, we need only give the $_v$th particle a displacement δr_v from the old to the new position. We call δr_v a *virtual displacement* to distinguish it from a *true displacement* [denoted by d**r**$_v$] which occurs in a time interval where forces and constraints could be changing. The symbol δ has the usual properties of the differential d; for example, $\delta(\sin\theta) = \cos\theta\,\delta\theta$.

3.2 Statics of a System of particles. Principle of Virtual Work

In order for a system of particles to be in equilibrium, the resultant force acting on each particle must be zero, i.e. $\mathbf{F}_v = 0$. It thus follows that $F_v \cdot \delta r_v = 0$ where $F_v \cdot \delta r_v$ is called the virtual work. By adding these we then have

$$\sum_{v=1}^{N} F_v \cdot \delta r_v = 0 \qquad (27)$$

If constraints are present, then we can write

$$F_v = F_v^{(a)} + F_v^{(c)} \qquad (28)$$

Where $F_v^{(a)}$ and $F_v^{(c)}$ are respectively the *actual force* and *constraint force* acting on the $_v$th particle. By assuming that the virtual work of the constraint forces is zero [which is true for rigid bodies and for motion on curves and surfaces without friction], we arrive at

Theorem 14. A system of particles is in equilibrium if and only if the total virtual work of the actual forces is zero, i.e. if

$$\sum_{v=1}^{N} F_v^{(a)} \cdot \delta r_v = 0 \qquad (29)$$

This is often called the *principle of virtual work.*

3.3 Equilibrium in Conservative Fields. Stability of Equilibrium

The results for equilibrium of a particle in a conservative force field can be generalized to systems of particles. The following theorems summarize the basic results.

Theorem 15. If V is the total potential of a system of particles depending on coordinates q_1, q_2, ……., then the system will be in equilibrium if

$$\frac{\partial V}{\partial q_1} = 0, \frac{\partial V}{\partial q_2} = 0, \dots \dots . \qquad (31)$$

Since the virtual work done on the system is

$$\delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \cdots \dots .$$

(31) is equivalent to the principle of virtual work.

Theorem 16. A system of particles will be in stable equilibrium if the potential *V* is a minimum.

In case *V* depends on only one coordinate, say q_1, sufficient are

$$\frac{\partial V}{\partial q_1} = 0, \qquad \frac{\partial^2 V}{\partial q_1^2} > 0$$

Other cases of equilibrium where the potential is not a minimum are called unstable.

3.4 D'Alembert's Principle

Although Theorem 14 as stated applies to the statics of a system of particles, it can be restated so as to give an analogous theorem for dynamics. To do this we note that according to Newton's second law of motion,

$$F_v = \dot{P}_v \quad or \quad F_v - \dot{P}_v = 0 \qquad (30)$$

Where p_v is the momentum of the $_v$th particle. The second equation amounts to saying that a moving system of particles can be considered to be in equilibrium under a force $F_v - \dot{P}_v$. i.e. the actual force together with the added force $-\dot{P}_v$ which is often called the reversed effective force on particle *v*. By using the principle of virtual work we can then arrive at

Theorem 17. A system of particles moves in such a way that the total virtual work

$$\sum_{v=1}^{N}(F_v^{(a)} - \dot{P}_v)\cdot\delta r_v = 0 \qquad (32)$$

With this theorem, which is often called D'Alembert's principle, we can consider dynamics as a special case of statics.

Example

Motion Relative to the Center of Mass

(1) Let r_v' and v_v' be respectively the position vector and velocity of particle v relative to the center of mass. Prove that (a) $\sum_v m_v r_v' = 0,$ (b) $\sum_v m_v v_v' = 0.$

(a) Let $\mathbf{r}_v$ be the position vector of particle *v* relative to *0* and $\bar{r}$ the position vector of the center of mass C relative to O. Then from the definition of the center of mass,

$$\bar{r} = \frac{1}{M}\sum_v m_v r_v \qquad (1)$$

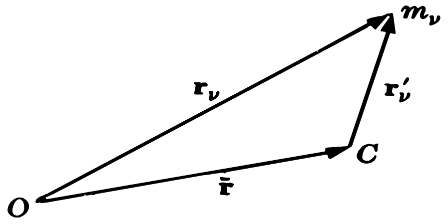

Fig. 7-8

Where $M = \sum_v m_v$. From Fig. 7 – 8 we have

$$r_v = r_v' + \bar{r} \qquad (2)$$

Then substituting (2) into (1), we find

$$\bar{r} = \frac{1}{M}\sum_{v} m_v (r_v' + \bar{r}) = \frac{1}{M}\sum_{v} m_v r_v' + \bar{r}$$

From which $\sum_{v} m_v r_v' = 0$ (3)

(b) Differentiating both sides of (3) with respect to *t*, we have $\sum_{v} m_v v_v' = 0.$

Example 2

In each of the following cases whether the constraint is holonomic or non-holonomic and give a reason for your answer: (a) a bead moving on a circular wire; (b) a particle sliding down an inclined plane under the influence of gravity; (c) a particle sliding down a sphere from a point near the top under the influence of gravity.

(a) The constraint is holonomic since the bead, which can be considered a particle, is constrained to move on the circular wire.

(b) The constraint is holonomic since the particle is constrained to move along a surface which is in this case a plane

(c) the constraint way of seeing this is to note that r is the position vector of the particle relative to the center of the sphere as origin and a is the radius of the sphere, then the particles moves so that $r^2 \geqq a^2$. This is a non-holonomic constraint since it is not of the form (26), page 170. An example of a holonomic constraint would be $r^2 = a^2$.

4.0 CONCLUSION

In conclusion, In order for a system of particles to be in equilibrium, the resultant force acting on each particle must be zero, i.e. $\mathbf{F}_v = 0$

5.0 SUMMARY

The summaries of what you have learnt are as contained in theorems 14 – 17 above thus:

Theorem 14. A system of particles is in equilibrium if and only if the total virtual work of the actual forces is zero, Called principle of virtual work.

Theorem 15. If V is the total potential of a system of particles depending on coordinates q_1, q_2, ……., then the system will be in equilibrium if

$\frac{\partial V}{\partial q_1} = 0, \frac{\partial V}{\partial q_2} = 0, \dots \dots.$ which is equally equivalent to virtual work.

Theorem 16. A system of particles will be in stable equilibrium if the potential *V* is a minimum. and

Theorem 17. A system of particles moves in such a way that the total virtual work given as

$$\sum_{v=1}^{N} \left(F_v^{(a)} - \dot{P}_v\right) \cdot \delta r_v = 0$$

which is often called D'Alembert's principle

6.0 TUTOR-MARKED ASSIGNMENT

a) Explain the term virtual displacement

b) Define D'Alembert's principle

c) define center of mass

d) what do you understand by the momentum of system of particle

e) Explain the terms holonomic and nonholonomic constraints.

7.0 REFERENCES/FURTHER READING

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17. Differential Games by Avner Friedman.

18. Classical Mechanics by TWB Kibble

MODULE 3

UNIT 1 THE SIMPLE PENDULUM

CONTENTS

1.0 INTRODUCTION

The simple pendulum is one of the most common examples of *simple harmonic motion*, at least as far as laboratory observation of oscillatory motions is concerned. A *harmonic motion* is one for which the restoring force obeys Hooke's law, provided the displacement from equilibrium position is small. In that case the displacement, velocity and acceleration towards the equilibrium position are represented by simple sinusoidal functions of time or linear combinations of them. The term *simple* comes into the definition as a result of the fact that the amplitude and therefore energy of the system is conserved (constant) when dissipative (friction type) forces are negligible. Then the curves of the dynamic variables such as displacement, velocity and acceleration will be pure sine or cosine curves.

We are interested in this type of motion because, as you will recall from your college physics, vibratory motion is one of the four fundamental motions in nature. Vibratory or periodic motion is a prototype of the motions of most physical systems. The structures of buildings, bridges and crystals such quartz used for the construction of your wrist watch are in a state of vibration at all times. The motion of electrons in an antenna that transmits or receives a radio signal is vibratory.

In this unit you will study the simple mathematical formulation of this important type of motion and discuss the properties of the solutions of its differential equation.

2.0 OBJECTIVES

At the end of this unit, you should be able to:
a. derive the equation of a simple pendulum
b. show that the equation of a simple pendulum is a particular case of the more general equation of a simple harmonic oscillator
c. demonstrate an understanding of the dependence of the period of a simple pendulum on the length and local gravitational acceleration.
d. Solve simple problems involving the simple solutions of the equation of the simple pendulum.
e. discuss elastic systems in terms of Hooke's law
f. calculate the energy stored in an elastic system
g. show that a spring force is conservative.

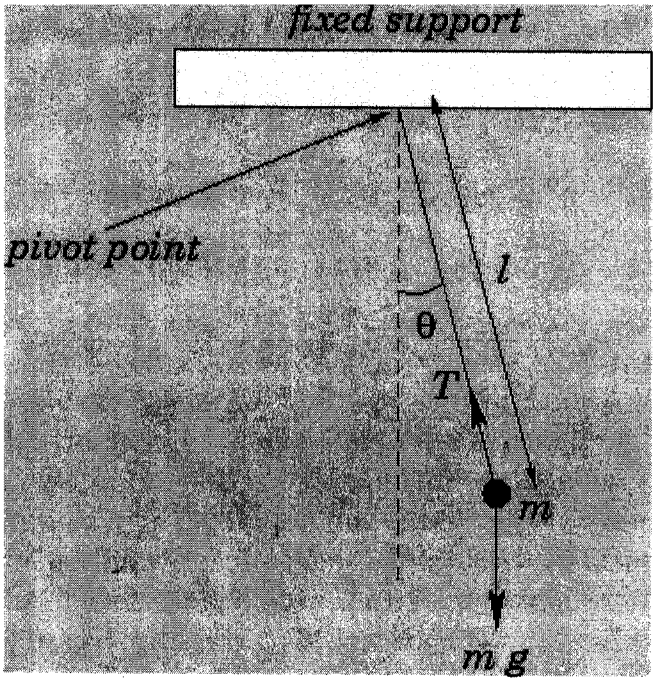
3.0 MAIN CONTENT

3.1 The Simple Pendulum

Consider a mass m suspended from a light inextensible string of length l, such that the mass is free to swing from side to side in a vertical plane, as shown in Fig. a. This setup is known as a *simple pendulum*. Let θ be the angle subtended between the string and the downward vertical. Obviously, the equilibrium state of the simple pendulum corresponds to the situation in which the mass is stationary and hanging vertically down (i.e., θ = 0). The angular equation of motion of the pendulum is simply

$$I\,\ddot{\theta} = \tau \qquad (523)$$

where I is the moment of inertia of the mass, and τ is the torque acting on the system. For the case in hand, given that the mass is essentially a point particle, and is situated a distance *l* from the axis of rotation (i.e., the pivot point), it is easily seen that $I = ml^2$.



The two forces acting on the mass are the downward gravitational force, *mg*, and the tension, *T*, in the string. Note, however, that the tension makes no contribution to the torque, since its line of action clearly passes through the pivot point. From simple trigonometry, the line of action of the gravitational force passes a distance $l \sin \theta$ from the pivot point. Hence, the magnitude of the gravitational torque is $m\, g\, l \sin \theta$. Moreover, the gravitational torque is a *restoring torque:* i.e., if the mass is displaced slightly from its equilibrium state (i.e., $\theta = 0$) then the gravitational force clearly acts to push the mass back toward that state. Thus, we can write

$$\tau = -m\, g\, l \sin \theta. \qquad 524$$

Combining the previous two equations, we obtain the following angular equation of motion of the pendulum:

$$l\, \ddot{\theta} = -g \sin. \qquad (525)$$

Unfortunately, this is *not* the simple harmonic equation. Indeed, the above equation possesses no closed solution which can be expressed in terms of simple functions.

Suppose that we restrict our attention to relatively *small* deviations from the equilibrium state. In other words, suppose that the angle θ is constrained to take fairly small values. We know, from trigonometry, that for $|\theta|$ less than about 6^0 it is a good approximation to write

$$\sin\theta \simeq \theta. \quad (526)$$

Hence, in the *small angle limit*, reduces to

$$l\,\ddot{\theta} = -g\,\theta, \quad (527)$$

which is in the familiar form of a simple harmonic equation. Comparing with our original simple harmonic equation, and its solution, we conclude that the angular frequency of small amplitude oscillations of a simple pendulum is given by

$$\omega = \sqrt{\frac{g}{l}}\,. \quad (528)$$

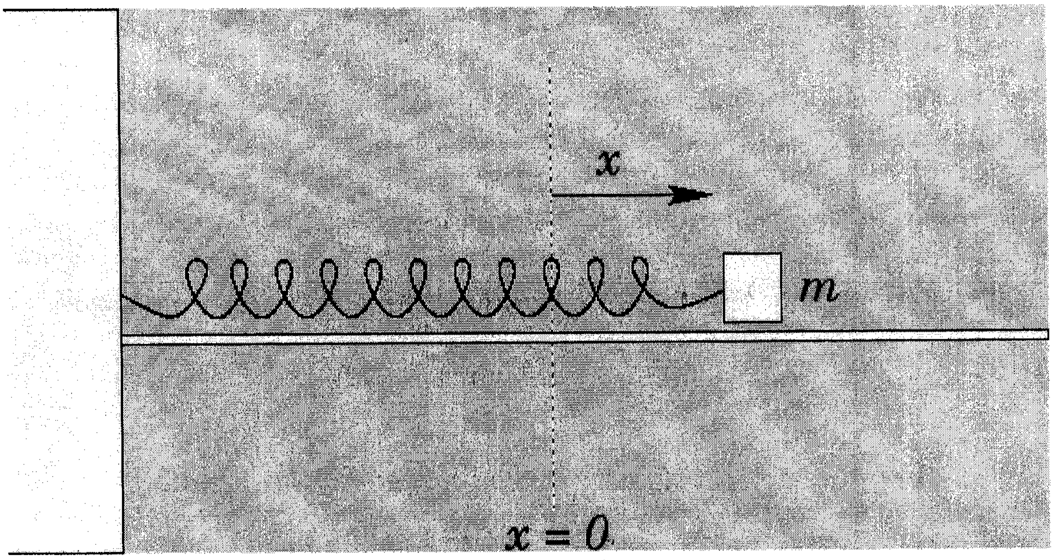
In this case, the pendulum frequency is dependent only on the length of the pendulum and the local gravitational acceleration, and is independent of the mass of the pendulum and the amplitude of the pendulum swings (provided that $\sin\theta \simeq \theta$ remains a good approximation). Historically, the simple pendulum was the basis of virtually all accurate time-keeping devices before the advent of electronic clocks. Simple pendulums can also be used to measure local variations in g.

3.2 Hooke's Law

Consider a mass *m* which slides over a horizontal frictionless surface. Suppose that the mass is attached to a light horizontal spring whose other end is anchored to an immovable object. See Fig. Let x be the extension of the spring: i.e., the difference between the spring's actual length and its unstretched length. Obviously, x can also be used as a coordinate to determine the horizontal displacement of the mass. According to Hooke's law, the force *f* that the spring exerts on the mass is directly proportional to its extension, and always acts to reduce this extension. Hence, we can write

$$f = -kx, \quad (159)$$

where the positive quantity *k* is called the *force constant* and measures the *stiffness* of the spring. Note that the minus sign in the above equation ensures that the force always acts to reduce the spring's extension: e.g., if the extension is positive then the force acts to the left, so as to shorten the spring.



Mass on a spring

According to Eq. (140), the work performed by the spring force on the mass as it moves from displacement x_A to x_B is

$$W = \int_{x_A}^{x_B} f(x)dx = -k \int_{x_A}^{x_B} x dx = -\left[\frac{1}{2}kx_B^2 - \frac{1}{2}kx_A^2\right].$$

Note that the right-hand side of the above expression consists of the difference between two factors: the first only depends on the final state of the mass, whereas the second only depends on its initial state. This is a sure sign that it is possible to associate a *potential energy* with the spring force. Equation (155), which is the basic definition of potential energy, yields

$$U(x_B) - U(x_A) = -\int_{x_A}^{x_B} f(x)dx = \frac{1}{2}kx_B^2 - \frac{1}{2}kx_A^2. \quad (161)$$

Hence, the potential energy of the mass takes the form

$$U(x) = \frac{1}{2}kx^2. \quad (162)$$

Note that the above potential energy actually represents energy stored by the spring – in the form of mechanical stresses – when it is either stretched or compressed. Incidentally, this energy must be stored *without loss,* otherwise the concept of potential energy would be meaningless. It follows that the spring force is another example of a *conservative force*.

It is reasonable to suppose that the form of the spring potential energy is somehow related to the form of the spring force. Let us now explicitly investigate this relationship. If we let $x_B \to x$ and $x_A \to 0$ then Eq. (161) gives

$$U(x) = -\int_0^x f(x')dx' \qquad (163)$$

We can differentiate this expression to obtain

$$f = -\frac{dU}{dx} \ . \qquad (164)$$

Thus, in 1-dimension, a conservative force is equal to minus the derivative (with respect to displacement) of its associated potential energy. This is a quite general result. For the case of a spring force: $U = (1/2)\,kx^2$, so $f = -dU/dx = -kx$.

As is easily demonstrated, the 3-dimensional equivalent to Eq. (164) is

$$F = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right). \qquad (165)$$

For example, we have seen that the gravitational potential energy of a mass m moving above the Earth's surface is $U = m\,g\,z$, where z measures height off the ground. It follows that the associated gravitational force is

$$f = (0, 0, -mg). \qquad (166)$$

In other words, the force is of magnitude $m\,g$, and is directed vertically downward.

The total energy of the mass shown in Fig. 42 is the sum of its kinetic and potential energies:

$$E = K + U = K + \frac{1}{2}kx^2. \qquad (167)$$

Of course, E remains constant during the mass's motion. Hence, the above expression can be rearranged to give

$$K = E - \frac{1}{2}kx^2. \qquad (168)$$

Since it is impossible for a kinetic energy to be negative, the above expression suggests that $|x|$ can never exceed the value

$$x_0 = \sqrt{\frac{2E}{k}}. \qquad (169)$$

Here, x_0 is termed the *amplitude* of the mass's motion. Note that when x attains its maximum value x_0, or its minimum value $-x_0$, the kinetic energy is momentarily zero (i.e., K = 0).

4.0 CONCLUSION

As in the summary.

5.0 SUMMARY

In this unit you have been introduced to the equation of a simple pendulum. It is a particular case of the more general equation of a simple harmonic oscillator.
The period of the motion is independent of its mass but depends on its length and the value of the local gravitational acceleration.

Simple harmonic oscillations are observed only for small displacements from their equilibrium positions. Their restoring forces will then obey Hook's law of elasticity.
For an elastic system, the work done by the elastic forces manifests as the change in its potential energy. The total energy is the sum the kinetic and potential energies interchanged as the elastic system is alternately stretched and compressed.

6.0 TMA

1) Explain the term **Hooke's Law**
2) show that a spring force is conservative.

7.0 REFERENCES / FURTHER READING

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Module 4

UNIT 1 Motion along a curve

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study the motion of an object, or particle, moving in space. The object may be a car speeding around a racetrack, an electron being propelled through a linear accelerator, or a satellite in orbit. We assume that the motion takes place in a fixed coordinate system and that the object can be located by specifying a single point, its centre of gravity

2.0 OBJECTIVES

At the end of this unit, you should be able to:

1. Understand the Motion along a curve
2. Understand and solve simple problems in Position, Velocity, and Acceleration of system of particles along curves.

3.0 MAIN CONTENT

3.1 MOTION ALONG CURVES

In this unit, we study the motion of an object, or particle, moving in space. The object may be a car speeding around a racetrack, an electron being propelled through a linear accelerator, or a satellite in orbit. We assume that the motion takes place in a fixed coordinate system and that the object can be located by specifying a single point, its centre of gravity.

3.1.1 Position, Velocity, and Acceleration

The three basic notions for analyzing motion are position, velocity, and acceleration. As a particle moves along a path, we suppose that the coordinates (x,y,z) of its position are twice differentiable functions of time

$x = x(t) \qquad y = y(t) \qquad z = z(t)$

The vector function

r(t) = x(t)**i** + y(t)**j** + z(t)**k**

from the origin to the particle is called the **position function** of the particle. Figure 14-36 shows the position of a particle at time *t* and at another time *t* + Δ*t*.

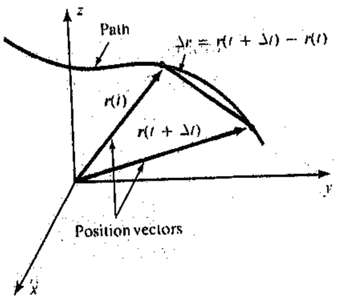


Figure 14-36: Δr is the change in position from time *t* to *t* + Δ*t*.

The displacement vector Δr = r(t + Δt) – r(t) represents the *change in position.* The scalar multiple Δr/Δt represents the *average* change in position from time *t* to t + Δt, and the average change in position is called the *average velocity* over the time period Δt. Now, just as in the case of motion along a line, we define the *(instantaneous)* **velocity** to be the limit of the average velocities as Δt approaches 0; that is

$$\text{velocity} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$$

According to (14.36) in the previous section, the limit on the right is the *vector* r’(t). If v(t) denotes the velocity at time *t*, it follows that

V(t) = r′(t)

Thus, velocity is the rate of change of position with respect to time. Furthermore, the rate of change of velocity with respect to time is called the **acceleration** and is denoted by **a;** that is,

$a(t) = v'(t) = r''(t)$

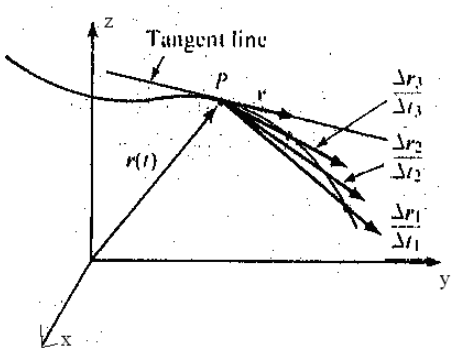


Figure 14-37: The vectors $\Delta r/\Delta t \rightarrow v$ as $\Delta t \rightarrow 0$; v is a direction vector for the line tangent to the path at *P*.

The discussion above is completely consistent with our earlier discussion of position, velocity, and acceleration. Notice, however, that all three are now considered to be vectors and can be represented by arrows. The position vector always has its tail at the origin, but the velocity and acceleration vectors are considered to have their tails at the location of the particle. Moreover, *the arrow representing the velocity is always tangent to the path.* To see why this is so, we suppose that the particle is at point *P* at time *t*. Figure 14-37 indicates that as $\Delta t \rightarrow 0$, the vectors $\Delta r/\Delta t$ approach a direction vector of the tangent line through *P*. It follows that this direction vector is the velocity vector **v.** Figure 14-38 shows some typical velocity and acceleration vectors at various points on the path. Notice that acceleration vectors usually point toward the concave side of the path.

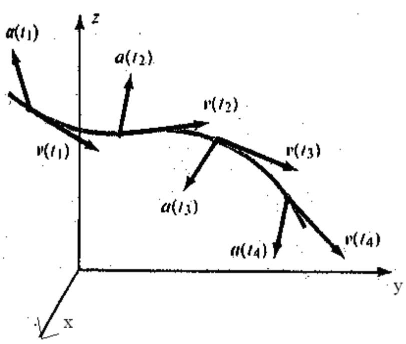


Figure 14-38: Typical velocity and acceleration vectors on the path of motion.

The **speed** v of a particle is defined to be the rate of change of distance (along the path) with respect to time. Speed has magnitude only and is, therefore, a scalar. If the particle starts at time t_0, then the distance *s* it travels along the path from t_0 to time *t* is given by the arc length formula (14.32).

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$$

It follows from the Fundamental Theorem that

$$v(t) = s'(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

But the expression on the right is the length of r′(t) = v(t) and, therefore,

$$\text{speed} = v(t) = |v(t)|$$

The entire discussion above can be summarized as follows:

For a particle travelling through space, we have

(1) r(t) is the position vector; its tail is at the origin and its tip traces out the path.
(2) v(t) = r′(t) is the velocity vector; it is tangent to the path.
(3) a(t) = v′(t) = r″(t) is the acceleration vector; it usually points toward the concave side of the path.
(4) v(t) = s′(t) = $|v(t)|$ is the speed.

Example 1

The position vector of a particle is r(t) = cos *t***i** + sin *t***j** + *t***k**.
Find its velocity, speed, and acceleration at any time *t*.

Solution:

The path of the particle is a circular helix.

v(t) = r′(t) = -sin *t***i** + cos *t***j** + **k**
speed = $|v'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$
a(t) = v′(t) = -cos *t***i** – sin *t***j**

Note: The acceleration is not **0** even though the speed is constant. The reason is that the velocity is constantly changing direction. Also notice that v(t) · a(t) = 0, which means that **v** and **a** are always orthogonal .see the diagram below.

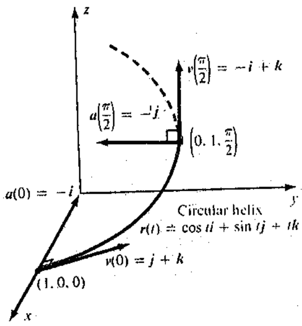


Figure . 5

4.0 CONCLUSION

In conclusion, the motion of an object, moving in space. it was assumed that the motion takes place in a fixed coordinate system and that the object can be located by specifying a single point, its centre of gravity.

5.0 SUMMARY

summarily, we have studied the motion of an object, or particle, moving in space. The object may be a car speeding around a racetrack, an electron being propelled through a linear accelerator, or a satellite in orbit.

6.0 TMA

1.Define the following **Position, Velocity, and Acceleration of a satellite in orbit.**

2.Give a brief introduction of motion in a circle.

3. The position vector of a particle is r(t) = tan *t***i** + sin *t***j** +cos*t***k**. Find its velocity, speed, and acceleration at time *t=30s*.

7.0 REFERENCES / FURTHER READING

1. Theoretical Mechanics by Murray, R. Spiegel.

2. Advanced Engineering Mathematics by KREYSZIC.

3. Generalized function. Mathematical Physics by U. S. Vladinirou.

4. Vector Analysis and Mathematical Method by S. T. Ajibola. First Published (2006).

5. Lecture Notes on Analytical Dynamics from LASU(1992).

6. Lecture Notes on Analytical Dynamics from FUTA(2008).

7. Lecture Notes on Analytical Dynamics from UNILORIN(1999)

8. Differential Games by Avner Friedman.

9. Classical Mechanics by TWB Kibble

UNIT 2 Circular Motions with Constant Speed

CONTENTS

1.0 INTRODUCTION

A Circular Motion with Constant Speed is a phenomena in which a particle moves with constant (uniform) speed (velocity) in a circular path

2.0 OBJECTIVES

At the end of this unit, you should be able to define and solve examples on Circular Motion with Constant Speed. Here the acceleration vector always points toward the center of the circle (*centripetal) acceleration and / or* the acceleration vector always points toward the circumference of the circle (centrifugal) acceleration.

3.0 MAIN CONTENT

3.1 Circular Motion with Constant Speed

If a particle moves with constant speed in a circular path, then the acceleration vector always points toward the center of the circle; this is called *centripetal acceleration*. To see why this is so, we observe that a circular path lies in a single plane, and we might as well consider the circle to be in the xy-plane with radius **r** and center at the origin (Figure 14-40). Since the speed is constant, the angle θ

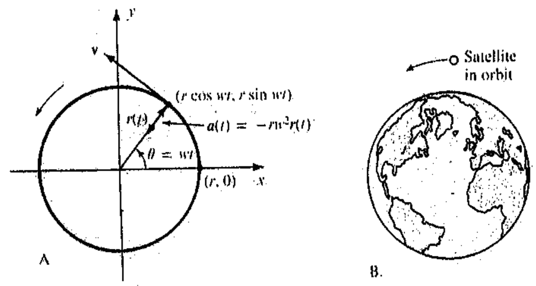


Figure 14-40. (A) Circular motion with constant speed; centripetal acceleration points toward the center. (B) Example 2, Satellite in orbit around the earth.

from the positive x – axis to the position vector r(t) is changing at a constant rate ω (the Greek letter *omega*); that is, $\theta = \omega t$. It follows that the position vector is

$$r(t) = r \cos \omega t\mathbf{i} + r \sin \omega t\mathbf{j}$$

Therefore,

$$v(t) = -r\omega \sin \omega t\mathbf{i} + r\omega \cos \omega t\mathbf{j}$$
$$a(t) = -r\omega^2 \cos \omega t\mathbf{i} + r\omega^2 \sin \omega t\mathbf{j}$$

Thus, $\mathbf{a}(t) = \omega^2\, \mathbf{r}(t)$, and this shows that **a** always points toward the center of the circle; its magnitude is $|a(t)| = r\omega^2$. Since the constant speed is $v = |v(t)| = r\omega$, we also have the important relationship $v^2 = r|a(t)|$ or

$$|a(t)| = \frac{v^2}{r}$$

This holds for all circular motion with constant speed.

Example 2

(*Satellite in orbit).* Suppose that a satellite is in circular orbit 200 miles above the earth. What is its speed and period? (Assume the radius of the earth is 4,000 mi and the acceleration due to gravity is 32 ft/sec^2.)

Solution

The radius of the circular path is r = 4,200 mi; the acceleration vector points toward the center of the earth and its magnitude is 32ft/sec^2. To find the speed, we use (14.43) making sure the units of measurement are compatible.

$v^2 = r|a(t)| = (4{,}200 \text{ mi})(32 \text{ ft/sec}^2)(5{,}280 \text{ ft/mi})$
$\approx 7.1 \times 10^8 \text{ ft}^2/\text{sec}^2$

Taking square roots, we have

$v \approx 2.7 \times 10^4$ ft/sec (about 18,409 mph)

The period is the time it takes for one revolution.

$$\text{period} = \frac{2\pi r}{speed} = \frac{2\pi(4{,}200 mi)(5{,}280 ft/mi)}{2.7 x 10^4 ft/\sec}$$
$\approx$ 5.161 seconds (about 86 minutes)

3.2 Force and Motion

Suppose a particle has constant mass *m*. Then Newton's second law of motion states that the product of *m* and the acceleration **a** of the particle equals the total external force acting on the particle.

14.44 $\mathbf{F} = m\mathbf{a}$ (force = mass x acceleration)

If the force is a given function of time, and the initial velocity and initial position are known, then it is possible to obtain the path of the particle by integration.

Example 3

The force acting on a particle at time *t* is $\mathbf{F}(t) = 6t\mathbf{i} + \mathbf{j}$. If the particle starts from the point (3,-1,2) with the velocity v(0) = 4**k**, find parametric equations of its path.

Solution

The path is obtained by finding the position vector function r(*t*). Since $\mathbf{F} = m\mathbf{a} = m\mathbf{r}''$, we start this problem with the equation

$$\mathbf{r}''(t) = \frac{1}{m}\mathbf{F} = \frac{1}{m}(6t\mathbf{i} + \mathbf{j})$$

Integration of both sides yields

(14.45) $\quad (t) = r'(t) = \frac{1}{m}(3t^2\mathbf{i} + t\mathbf{j}) + C$

We are given that v(0) = 4k; thus, C = 4k and (14.45) becomes

$$r'(t) = \frac{1}{m}(3t^2\mathbf{i} + t\mathbf{j}) + 4\mathbf{k}$$

Integrate once again;

$$\mathbf{r}(t) = \frac{1}{m}(t^3\mathbf{i} + \frac{t^2}{2}\mathbf{j} + 4t\mathbf{k} + \mathbf{C}$$

The starting point (3, -1, 2) yields r(0) = 3**i** – **j** + 2**k** = **C**. Therefore,

$$\mathbf{r}(t) = \left[\frac{1}{m}\left(t^3 i + \frac{t^2}{2} j\right) + 4tk\right] + [3\mathbf{i} - \mathbf{j} - 2\mathbf{k}]$$

$$= \left(\frac{t^3}{m} + 3\right)\mathbf{i} + \left(\frac{t^2}{2m} - 1\right)\mathbf{j} + (4t + 2)\mathbf{k}$$

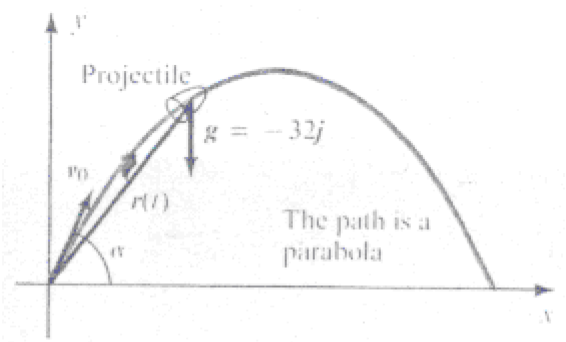
It follows that parametric equations of the path are

$$x = \frac{t^2}{m} + 3 \quad y = \frac{t^2}{2m} - 1 \quad z = 4t + 2 \qquad t \geq 0$$

The method of Example 3 can be applied to objects in motion near the surface of the earth. If air resistance is neglected, such objects are subject only to the force of gravity **F** = *m*g, which is constant. In this case, the action takes place in the plane determined by **g** and the initial velocity vector $\mathbf{v}_0$. This is the situation for the motion of a projectile; that is, an object launched into the air and allowed to move freely. The plane of motion is taken to be the xy-plane and the acceleration due to gravity is

$$g = 32\mathbf{j}$$

which points straight down.



Example 4

(*Path of a projectile*). A projectile is launched from the origin with an initial speed of v_0 ft/sec at an angle α from the horizontal (Figure 14-41); that is, the initial velocity vector is $v_0 = v_0 \cos \alpha i + v_0 \sin \alpha j$. Show that the path is part of a parabola.

Solution

The acceleration g in this case is known; it points straight down and its magnitude is always 32 ft/sec^2. Therefore,

$$\mathbf{r}''(t) = g = -32\mathbf{j}$$

Integration of both sides yields

(14.46) $\quad v(t) = \mathbf{r}'(t) = -32t\mathbf{j} + \mathbf{C}$

Since $v(0) = v_0 = C$, we have

$$C = v_0 \cos \alpha\mathbf{i} + v_0 \sin \alpha\mathbf{j}$$

and (14.46) becomes

$$\mathbf{r}'(t) = v_0 \cos \alpha\mathbf{i} + (v_0 \sin \alpha - 32t)\mathbf{j}$$

We integrate again to obtain

$$\mathbf{r}(t) = (v_0 \cos \alpha)\, t\mathbf{i} + [(v_0 \sin \alpha)t - 16t^2]\mathbf{j} + \mathbf{C}$$

Since the projectile starts from the origin, we have $r(0) = \mathbf{0} = C$; therefore,

$$\mathbf{r}(t) = (v_0 \cos \alpha)\, t\mathbf{i} + [(v_0 \sin \alpha)t - 16t^2]\mathbf{j}$$

Parametric equations for the path are

$$x = (v_0 \cos \alpha)t \text{ and } y = (v_0 \sin \alpha)t - 16t^2$$

To show that the path is a parabola, we solve the first equation for t, eliminate the parameter, and obtain

$$y = (\tan \alpha)x - \frac{16}{(v_0 \cos\alpha)^2} x^2$$

which is an equation of a parabola.

4.0 CONCLUSION

As in the summary.

5.0 SUMARRY

In summary; Circular Motion with Constant Speed, Force and Motion of particles are discussed and related examples were given and solved for the purpose of more understanding of the unit objectives.

6.0 TUTOR MARK ASSISGNMENT (TMA)

Find the velocity, speed, and acceleration of the following position vectors.

1. $\mathbf{r}(t) = \mathbf{i} - 2t\mathbf{j} + (t + 1)\mathbf{k}$
2. $\mathbf{r}(t) = -3t\mathbf{i} + t\mathbf{j} + \mathbf{k}$
3. $\mathbf{r}(t) = \cos t\mathbf{i} + \sinh t\mathbf{j} + t\mathbf{k}$

The acceleration and initial position and velocity of a particle are given. Find the position functions.

4. $\mathbf{a}(t) = t\mathbf{i} - 6t\mathbf{j} + \mathbf{k}$; $r(0) = \mathbf{0}$, $v(0) = \mathbf{i}$
5. $\mathbf{a}(t) = 2t\mathbf{i} + t\mathbf{j} - 3\mathbf{k}$; $r(0) = \mathbf{i} + \mathbf{j}$, $v(0) = \mathbf{0}$

7.0 REFERENCES / FURTHER READING

1 Theoretical Mechanics by Murray, R. Spiegel.

2 Advanced Engineering Mathematics by KREYSZIC.

3 Generalized function. Mathematical Physics by U. S. Vladinirou.

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