NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 301

COURSE TITLE: Metric Space Topology

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CONTENTS
PAGES

MODULE 1

UNIT 1 TOPOLOGICAL SPACES

CONTENTS

1.0 INTRODUCTION

The aim of this course is two-fold

(i) To give the students a good expositism of metric space topology
(ii) To develop fundamental notions of topological spaces

In Topological spaces motion of open sets is fundamental. Although this course is metric space topology, however, it is better to start this course on a more general note, so that we could particurlarise these notions in a metric spaces.

We shall topological spaces in a general setting. We shall also give examples of topological spaces.

We shall consider the real Value Concepts, norms and state useful theorems on those concepts.

2.0 OBJECTIVES

At the end of this study you should be able to:

- define topological spaces
- give examples of topological
- solve some questions on topological spaces.

3.0 MAIN CONTENT

3.1 Topological Spaces

3.1.1 Definition

A topological space (X, τ) is a non-empty set X of points together with a family τ of subsets. (which we shall call open) possessing the following properties:

(i) $X \varepsilon \tau$, $\phi \varepsilon \tau$

(ii) $A_1 \varepsilon \tau$, $A_2 \varepsilon \tau$ imply $A_1 \cap A_2 \varepsilon \tau$

(iii) Given $A\alpha \varepsilon \tau$, then $\frac{\cup}{\alpha} A\alpha \varepsilon \tau$.

The family τ is called a topology for the bet X.

Remark: The properties in this definition as we shall see in unit 2 are all satisfied by open sets in a metric space (X, d) where d is a metric of X. In fact we can associate a topological spaces (X, τ) where τ is the family of open sets in (X, d). A topological space which is associated in this manner to some metric space is called metricable and the metric d is to be a metric for the topological space.

Examples

(i) Let X be any set. Let $\tau = \{X, \phi\}$ then τ is a topology on X. τ is called indiscrete topology.

(ii) Let X be any set Define $\tau = \{P(X) = 2^X\}$ = collection of all subset of X. τ is a topology on X called the discrete topology, and (X, τ) is the discrete topological space.

(iii) Let $\Re$ - set of real numbers. A set G C $\Re$ is open if the each XεG τ r >o such that B(x,r) = $\{y/1y-\Re 1<r\}$ CG

Let τ be set of all such open sets in $\Re$. τ is a topology on $\Re$ called the usual topology on $\Re$.

Exercise

Show that the examples i, ii, and iii are topological spaces by verifying properties. (i), (ii) and (iii) in definition (3.1.1).

3.1.2 Real Number System

A system ($(\Re,+,\bullet)$ is an ordered field if the following axioms are satisfied.

I. Axiom of Addition.
Given , x, y and $\neq \varepsilon 1\Re$

(i) x + y = y + x (commutative law)
(ii) R + (y + z) = (x + y + z) (associative law of addition)
(iii) τ 0$\varepsilon\Re$ such that
x + 0 = x
(iv) $\forall x\varepsilon\Re$ $\exists -$ x such that
x + (- x) = 0

II. Axiom of Multiplication

(i) x. y = y. x
(ii) x. (y. z) = (x.y).z
(iii) $\Im$ 1 $\varepsilon\Re$ such that x.1 = 1.x = x
(iv) ∀ x ≠0 $\Im$ x^{-1} such that
$x.x^{-1} = 1$
(v) x.(y+z) = x.y + x.z

III Order Axiom

(i) If x$\leq y$ and $y \leq z \Rightarrow x \leq z$
(ii) If $x \leq y$ and $y \leq x \Rightarrow$ $x = y$
(iii) If x, y$\varepsilon\Re$, them either $x \leq y$ or $y \leq x$
$\Rightarrow$ $x\varepsilon\Re$ only one of the following holds
x>0, x = 0, $\Re$<0.
(iv) if x ≤y then
X + z ≤ y + z.

(v) x ≥ 0, y ≥ 0 then
x y≥ 0.

3.1.3 Absolute Value Concept

Let x$\varepsilon\Re$ then the absolute value of x is demote by $|x|$ and defined as:

$$|x| = \int_{-x \;\; x\leq 0}^{x \;\; x\geq 0}$$

This can be written as
x if x<0
0 if x=0
-x if x<0.

If x, yε $\Re$, the distance between them is demoted by
$|x-y|$

The following property is satisfied.
If x, yε $\Re$ then
$|x\#y| \leq |x| + |y|$
We call this property the triangular inequality

3.1.4 $\Re^n$ Dimensional Euclidean Space

Definition: The Euclidean space $\Re^n$ consist of n-topples of real numbers i.e

$\Re^n$ = {(x_1, x_2,…………x_n)}
Such that
xi$\varepsilon\Re$, $1 \leq i \leq n$

$\Re^n$ is the cartesian product of $\Re$.
Such that
$\Re^n = \Re \, x \, \Re \, x$…………x $\Re$ (n times)

Therefore, xε $\Re^n$ can be represented as
x= (x_1, x_2, x_3, ……………….x_n).

We shall can $\underline{x}$ a point or a vector in $\Re^n$.

Example: Let x$\varepsilon\Re^n$ then
x = (x_1, x_2)

For any x, y$\varepsilon\Re^n$ where
x = (x_1, x_2,……………x_2), y = (y_1, y…………y_n)

We define addition as

x+y = (x_1 +y_1, x_2 + y_2, x_3 + y_3 +………………..x_n + y_n)

We can also define sealar multiplication as follows:

Let $\alpha \varepsilon$ K where K is sealar field.

Then for $\underline{x} \varepsilon \Re^n, \alpha \underline{x} = \alpha \left(x_1, x_2 .... x_n \right)$
= $\left(\alpha x_1, \alpha x_2 .... \alpha x_n \right)$

The following statements are obvious.

(i) $((\Re^n, +,)$ is a vector space of dimension n.

(ii) We can enumerate the basis vector as
ℓ_1 = (1, 0 0 ………)
$(\ell_2 = (0\ 1\ 0..............)$
en = (0 0 0 ……1)

Therefore for $x \varepsilon \Re^n, x = (x_1, x_2 .... x_n)$

And x = $\sum_{i=1}^{n} x_i \varepsilon_i$

Definition: The length or norm of a vector $x \varepsilon \Re^n$ is defined as the number: $\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$

For example let n = 2

Then
$\|x\| = (x_i^2 + x_2^i)^{1/2}$

Other ways of defining norm are
$\|x\|_1 = \max \left\{ |x_1|_3 |x_2|, ... |x_n| \right\}$.

With the above definition of norms, we define the distance as follows:

Let x_1 $y \varepsilon \Re^n$, then the distance between x and y is defined as $\|x - y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$

This is the metric define by
d(x, y) = $\left(\sum (x_i - y_i)^2\right)^{1/2}$
Hence
$d(x, y) = \|x - y\|$

We can view $\|.\|$ as a function such that
$\|.\| : \Re^n \to [0\ v \infty]$..

We can also define the inner product on $\Re^n x \Re^n$ and denote v as $\langle \circ \imath \circ \rangle$ by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$
Now
$\langle x, x \rangle = \sum_{i=1}^{n} x_i^{\ 2} = \|x\|^2$

3.1.4 Some Theorems

Theorems 1: For a vector x $\varepsilon \Re^n$
(i) $\|x\| \geq 0$
(ii) $\|x\| = 0$ iff x = 0
(iii) $\|\alpha x\| = |\alpha| \|x\|, \ \alpha \varepsilon \Re$.
(iv) $\|x + y\| \leq \|x\| + \|y\|$ (D – inequality)

Theorems 2: For a vector x $\varepsilon \Re^n$

(i) d(x, y) ≥ 0
(ii) d(x, y) = > iff x = y.
(iii) d(x, y) = d(y, x) (symmetric property)
(iv) d(x, y) $\leq$ d(x, z) + d(z, y) $\forall$ x, y, z $\varepsilon \Re^n$ (D – inequality)

Theorems 3: For x $\varepsilon \Re^n$.

(i) $\langle x, x \rangle \geq 0$
(ii) $\langle x, x \rangle = 0$ iff x = 0
(iii) $\langle x, y \rangle = \langle y, x \rangle$
(iv) $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$
(v) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$.
(vi) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Cauchy and wartz inequality)

4.0 CONCLUSION

The materials developed so far as well as the three theorems stated are sufficient background to allow us go into the details of our course. We shall be making use of them as we go on in this course.

5.0 SUMMARY

In this unit we have been introduced to concepts of Topological spaces, the real numbers system the absolute value concept the $\Re^n$ dimensional Euclidean space and some theorems on $\Re^n$. The real space $\Re$ and the product space $\Re^n$ are good examples of Metric spaces and these spaces carry the usual topological structure.

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READINGS

UNIT 2 METRIC SPACES

CONTENTS

1.0 INTRODUCTION

The real number system has two types of properties, namely the algebraic property which deals with addition, multiplication, etc.

The property which deals with the notion of distance between two numbers and with the concepts of a limited.

The second property is called topological property ins paces in which the notion of distance is defined. You will recall that in unit 1 we defined concept of topology in general and distance function on $\Re^n$.

Theorem 2 in unit 1 is particularly very instructive in this regard.

2.0 OBJECTIVES

As the end of this unit, you will be able to:

- define and give examples of metric space
- distinguish between a metric and pseudometric
- answer questions at the end of the unit.

3.0 MAIN CONTENT

3.1 Metric Spaces

You will recall in unit 1 that given x, y $\varepsilon \Re^n$ we can define the distance between the two vectors as follows:

$$\|x-y\| = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If Let we $\|x-y\| = d(x, y)$ then

d(x, y) = $\left(\sum(x_i - y_i)^2\right)$ ……………………………………….(2)

From theorem (2) it is very important to note that

(i) d (x, y) $\geq$ 0
(ii) d (x, y) = 0 iff x = y
(iii) d (x, y) = d (y, x)
(iv) d(x, y) $\leq$ d(x, z) + (z, y)

The above properties can be verified to be true of equation (2) on the bans of the above. We have the following definitions.

3.1.1 Definition

A metric space $\langle x, d\rangle$ is a non-empty sit x of elements together with a real-valued function d defined on X x X such that for all x, y –z ε X.

(i) d(x, y) $\geq$ 0
(ii) d(x, y) = 0 if and only if x = y
(iii) d(x, y) = d(y, x) (symmetric property)
(iv) d(x, y) $\leq$ d(x, z) + d(z, y) D- inequality

The function d is called a metric.

3.1.2 Examples

(1) The first examples that is very obvious is the metric space of $\Re$ - set of real number with
D(x, y) = $|x-y|$

(2) Let $\Re^n$ be n-dimensional Euclidean space whose points are n tuples.

$x = (x_1, x_2 \ldots\ldots\ldots\ldots x_n)$ of real numbers and
$d(x, y) = [(x_1-y_1)^2 + \ldots\ldots\ldots\ldots + (x_n-y_n)^2]^{½}$

Remark: We need to emphasis that a metric space is not the set X of its points. It is the pair (X, d), consists of the set of its point together with metric d.

For example we can define another metric $d^* (x, y) = |x^1-y_1| + \ldots\ldots\ldots + |x_n - y_n|$ which is another metric on $\Re^n$.

If we have two metric spaces $\langle x, d\rangle$ and $\langle y, d^*\rangle$. We can form a new metric space called the Cartesian product X x Y whose set points is the set X x Y = $\{\langle x, y\rangle : x\varepsilon X, y\varepsilon Y\}$ and whose metric τ is given by
$\tau (x_1,y_1), (x_2, y_2) = [d(x_1, x_2)^2 + d(y_1, y_2)^2]^{½}$

SELF ASSESSMENT EXERCISE 1

Verify that τ $\langle (x_1, y_1), (x_2, y_2)\rangle = \left[d(x_1, x_2)^2 + d^{\bullet}(y_1, y_2\right]^{½}$ is a metric defined on $\Re^1 x \Re^2$.

3.1.3 Pseudometics

A pair (X, d) is called a pseudometric space if d satisfies all the conditions of a metric except that d (x, y) = 0 need not imply x = y.

SELF ASSESSMENT EXERCISE 2

Show that d (x, y) = 0 is an equivalence relation, and if under this a relation, then d (x, y) depends only on the equivalence classes of x and y and defines a metric on X^*.

4.0 CONCLUSION

In this unit we have studied metric spaces and consider some examples of metric spaces. The structure of metric spaces make it easier to construct space set on the space of $\Re^N$, this we shall see in the subsequent unit.

5.0 SUMMARY

Recalled that a metric space is a set X together with a distant function d defined as X such that (X, d) form a pair and satisfies the following properties.

(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0$ iff $x = y$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ – D – inequality

A metric becomes a pseudometrics if in property (ii) $d(x, y) = 0$ need not imply $x = y$.

The exercises in this units are designed to reveal more properties of the metric spaces.

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READINGS

UNIT 3 OPEN SET AND CLOSED SET, INTERIOR, EXTERIOR, FRONTIER, LIMIT POINT AND CLOSURE OF A SET.

CONTENTS

1.0 INTRODUCTION

The simplest types of set encountered on the real lines are intervals. It is sometimes important to distinguish between intervals which include their endpoints and intervals which do not.

Suppose $\alpha < b$, the open interval (α, b) is defined to be the set.

$(\alpha, b) = \{x/ \alpha < x < b\}$.

The closed interval $[\alpha, b]$ is the set $[\alpha, b] = \{x/ \alpha \leq b\}$. Half open interval (α, b) and (α, b) are similarly defined, using the inequalities,

(i) $\alpha < x \leq b$ and
(ii) $\alpha \leq x < b$, respectively.

Infinite intervals are defined as follows:

(iii) $(\alpha, + \infty) = \{x|\alpha<x\}$
(iv) $(\alpha, + \infty) = \{x| \alpha\leq x\}$
(v) $(-\infty,\alpha) = \{x| x < \alpha\}$
(vi) $(-\infty,\alpha) = \{x| x \leq\alpha\}$

The real line is sometimes refer to as open interval $(-\infty,\alpha)$.
A single point is also considered α “degemerate” closed interval.

In this unit we shall find that α number of the properties of the set of real numbers apply immediately to set in a metric spaces. Throughout the present unit are sets mentioned are subsets of a given metric space (X, d).

2.0 OBJECTIVES

At the end of the unit, you should be able to:

- open set, closed interior and exterior point, a limit points and closure of a set
- characterised them by their properties
- answer questions on the above concepts.

3.0 MAIN CONTENT

3.1 Open Set and Closed Set

Definition (1): Let x$\varepsilon\Re$ be a fixed point, and $\varepsilon > 0$, then B (x, ε) = $\{y\varepsilon\Re^n : d(x, y) < \varepsilon\}$ is called an open ball or ε- disk or ε- neighbourhood.

For example in $\Re$, for x $\varepsilon\Re$,
x-ε<y<x+ε is defined as {y : |x-y|<ε }.

In open set we can write B (x, ε) = $\{y\varepsilon\square = \|x - y\| < \varepsilon\}$.
We now have the following definition

Definition (2): (Open set): A set AC$\square^n$ is said to be open if about each point xεA, $\Im$ ε0, such that B (x, 0) CA.

Examples

1. Consider the interval (0,1),
   (0,1) is open in $\square$. To set this let
   xε(0,1), choose
   (x-$\varepsilon, x+\varepsilon$)C (0,1).
   (0,1) C $\square^2$ but not open in $\square^2$ since there does not exist set (0,1) such that B (x, ε) C (0,1) there (0,1) is not an open set in $\square^2$.

2. The interval [0,1] is not open in $\square$ since B(0, $1/n$) and B (1, $1/n$) are balls not entirely in [0, 1].

Theorems (3,1): Let $x \varepsilon \mathbb{R}^n$, then the set B(x, ε), is open.

Proof: Let yε B(x, ε), we need to find ε_1 such that B(y, ε_1) C B(x, ε).
Since yε B(x,ε) then
d(x, y) $<\varepsilon$.
$\Rightarrow \varepsilon - d(x,y) > 0$. Take ε_1 such that $\varepsilon_1 = \varepsilon - d(x,y)$,
Let z $\leftarrow B(y,\varepsilon_1) \Rightarrow$
d(z, y) < ε_1

d(x, z) $\leq$ d(x, y) + d(y, z) <
d(x, y) + ε_1
< d(x, y) + ε - d(x, y) = ε.
Therefore
Z $\leftarrow B(x,\varepsilon)$.

Remarks (1): In $\mathbb{R}^n$, the empty set and $\mathbb{R}^n$ are open set. Prove!

Theorem (3.1):

(i) In $\mathbb{R}^n$, the union of arbitrary collection of open set is open.
(ii) The finite intersection of collection of open set is open.

Proof: Let $\{G_i\} i\varepsilon$ I be arbitrary collection of open set.

Let G = $\bigcup_{i \varepsilon I} Gi$

Let x<G $\Rightarrow$ xε Gi$_0$ for some
(i). since Gi is open for every
$\Im$ $\varepsilon > 0$, $\ni B(x,\varepsilon) C\, Gi$
C $\bigcup i\, Gi = G$

(ii) Let $G^1 = \bigcap_{i=1}^{k} G_i$

If x G^1 then xεGi $\forall i \varepsilon$ I.
$\therefore$ For each i $\Im$ $\varepsilon i > 0$ $\Im$
B (x, ε) C Gi
Define ε = num {ε_i)
$1^\circ \leq 1^\circ \leq$K
Then B (x, ε) C B (x, εi)
$\forall c' \varepsilon$ (1, 2,…………… κ).

And so
B (x, ε) εGi .

Remark 2: Arbitrary intersection of open set is not open. Prove!

Definitions (3) Closed Set

A set is closed if its compliment is open.
For example
$B^c = \{\mathbb{R}^n - B\}$,
$\{x\} = \mathbb{R}^n$ is closed in $\mathbb{R}^n$ since it contains compliment which is open in $\mathbb{R}^1$

The set defined as
(1) $\{(x, y) \mid x^2 + y^2 = 1\}$ is a closed set in $\mathbb{R}^2$. It is illustrated as

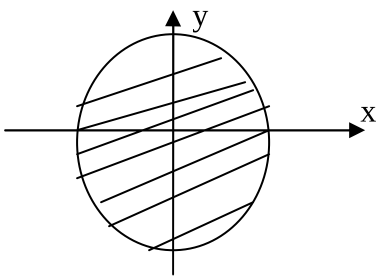

Fig. (1)

(2) The set $[0, 1] \leq \mathbb{R}^1$ is a closed set.

(3) The set $\Phi \mathbb{R}^n$ are both open and closed

Theorem (3.3): Let $F = \bigcap_{i \varepsilon 1} F_i$ be arbitrary intersection of closed set, then

(i) F is closed

(ii) $\bigcup_{i \varepsilon 1}^{k} F_i$

SELF ASSESSMENT EXERCISE

(1) Prove Remark 1 and 2
(2) Prove Theorem (3.3).

3.2 Interior, Frontier, Exterior Closure and Limit Point

Definition ($3_0 1_0 1$): For any set $AC\mathbb{R}^n$ xA, is an interior point of A if $\Im$ an open set $\cup$ such that $x\varepsilon \cup CA$

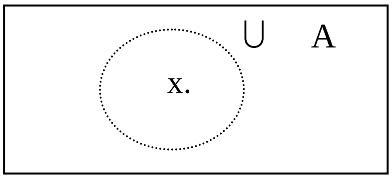

Fig. 2

The implication of the above is that given a set A, and a point x, x is an interior point of A if we can find ε>0, such that
B(x, ε) C A.

We shall denote interior point of A as Int A.

Examples

(i) Let A = {x} then
Interior A = ϕ.

(ii) Let A = $x^2 + y^2 < 1$ be a disk in $\mathbb{R}^2$

Then the Int A is defined as

Int A = {(x, y) : $x^1 + y^2 < 1$}.

Remark: We note that

Int A = {collection of all interior point of A}.

On the basis of the above remark, we have the following definition:

Definition (3.1.2): The interior of set A in $\mathbb{R}^n$ is defined as the union of all open subset of A.

Remark:
(i) From the above definition Ant A is open
(ii) Int A is the largest open subset of A
(iii) A set which does not contain an open set has an empty interior.

Theorem: A is op miff Ant A = A
Prove: Trivial

FRONTIER OR BOUNDARY POINT

Definition (3.1.3) Let (X, d) be a metric space, let S be a subset of X, a point of X is called a boundary point $\Im$. If every open set containing this point also contains a point of S and a point not in S (see fig 3 below).

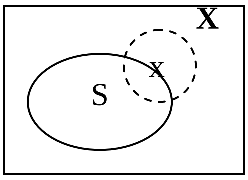

Fig. 3

Definition (3.1.4): by a closure of a subset S of X, we mean the union of S and all its boundary point.

The closure of S is denoted by $\overline{S}$.

Definition (3.1.5): $\overline{S}$ is closed and is equal to the intersection of all closed set containing S.

In particular

$\overline{\overline{S}} = \overline{S}$.

SELF ASSESSMENT EXERCISE

Show that for the subsets
S, T of X then

(i) $\overline{S} \cup \overline{T}$ = $\overline{S \cup T}$

(ii) $\overline{S \cap T}$ C $\overline{S} \cap \overline{T}$

Definition (3.1.6): A point xε□^{n} is an accumulative point or a limit point of SC □^{n} if every open set $\cup$ containing x interest with A at point other than x.

This implies

$\{(\cup - \{x\}) \cap A \pm \phi$.

Examples: Let (0, 1) ε□ let $\varepsilon \rangle 0$, then (0 - ε, 0 + ε) $\pm \phi$ 0 is a point of accumulation, but 0 $\in (0,1)$.

Definition (3.1.7): A set $SC\mathbb{R}^n$ close iff S contains all its accumulation points.

We may characterized the closure of S as follows.

Definition (3.1.7): Let $SC\mathbb{R}^n$, and Q_i are closed set, such that $SCQ_i \ \forall i$. Then $\underset{i\varepsilon I}{\cap} Q_i = \overline{S}$ (closure of S).

Theorem: A set S C $\mathbb{R}^n$ is closed iff $\overline{S} = S$.

Theorem: Let S C$\mathbb{R}^n$, then $\overline{S}$ consist of the union of S and its accumulation points.

Remark: We normally denote the frontier of S by ∂S.

Theorem: ∂S is a closed set.

4.0 CONCLUSION

In this section we have defined open set, closed set closure, limit point and closure of a set.

We have also characterized them, with the properties of each. You are required to master those properties very well. Work through all the grades exercises and prove all the theorems left unproved.

5.0 SUMMARY

Recall that:

- Open ball is B (x, ε) = {y $\varepsilon\mathbb{R}^n$: d(x, y) < ε}
- Set A C X is open if B(x, ε) < A.
- Finite intersection of open set is open
- Arbitrary union of open set is open
- A set S, is closed if its compliment is open
- Arbitrary intersection of closed set is closed
- Finite union of closed set is closed
- Interior of A is the union of all open subsets of A
- A set $SC\mathbb{R}^n$ is closed if S contain all its accumulation points
- A set S is closed if $\overline{S} = S$.

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READINGS

UNIT 4 DENSE SUBSET AND SEPARABLE SPACES, BAIRE CATEGORY

CONTENTS

1.0 INTRODUCTION

This unit relied heavily on the concept of closed sets and properties of closure studied in unit 3.

You are to master unit 3 properly before venturing into this unit.

2.0 OBJECTIVES

At the end of this study, you should be able to:

- define correctly what is meant by dense set
- explain what is separability of sets
- proof theorem relating to separability
- understand the Baire – category theorem
- solve – problems on this unit correctly.

3.0 MAIN CONTENT

3.1 Separable Set

Before explaining the concept of separability we need to define some concepts.

Definition (3.1.1): Countable Set: A set A is said to be countable if it is equivalent to the set of all positive integers or to some (finite or infinite) subset of the positive integers.

Example

(i) The set of rational number denoted by $\mathbb{Q}$ is countable.
(ii) The set of real number $\mathbb{R}$ is not countable

Definition (3.1.2): Dense Subset: Let X be a non-empty set, suppose DCX, we say D is dense in X if $\overline{D}$ = X. i.e. the closure of D is the closure of X.

Example:

The set of rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$.

Definition 3.1.3: (Separability): a metric space (X, d) is said to be separable, if it has a countable dense subset.

Example:

Let $\mathbb{R}$ be real let $\mathbb{Q} C \mathbb{R}$ such that $\mathbb{Q}$ = {x $\varepsilon \mathbb{R}$: x is a rational number}
Now $\overline{Q} = \mathbb{R}$, and since $\mathbb{Q}$ is countable the $\mathbb{R}$ is separable.

Theorem: $\mathbb{R}^n$ is separable.

Proof: Take $\mathbb{Q}^n$ = {x $\varepsilon \mathbb{R}^n$, x = (x_1….x)} with x_i, is national.
We know that $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$
Therefore $\mathbb{R}^n$ is separable.

Theorem: In $\mathbb{R}^n$, every family of disjoint non-empty open set is countable.

Proof: Let {x_n} be countable dense subset. Let {B_i} be a family of non-empty disjoint open sets. Biε {Bi}, ∃ ∀ n, $x_n \varepsilon Bi$.

$\phi:\{Bi\} \rightarrow Ni$ such that $\phi(Bi)$ = smallest n, to which $x_n \varepsilon Bi$
Then ϕ mass {Bi} into a subset of N hence it is countable.

3.2 Baire Category

In this section, we shall go deeply, by examining certain aspect of metric spaces.

We first consider the following definition.

Definition (3.2.1): A set E is said to be <u>nowhere dense</u> if $\overline{(\in)^c}$ is dense.

The definition above is equivalent to saying that $\bar{\in}$ contain no spheroid.

Example: Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{Z}$ be the set of integers then $\overline{(\mathbb{Z})^c}$ is dense in $\mathbb{R} \Rightarrow \mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

Definition (3.2.2): First category: A set ε is said to be of first category (or meager), if it is the union of countable collection of nowhere dense sets.

Definition (3.2.3): A set which is not of first category is said to be of second category.

Definition (3.2.4): (complete metric space) Let (X, d) be a metric space (X, d), we shall say that (X, d) is a complete metric space if all the Cauchy sequence $\{x_n\}$ converges to points in the metric space.
(Refer to unit 6).

Our intention is to show that a complete metric space is of second category.

We begin with the following theorem

Theorem: Let X be a complete metric space and $\{O_n\}$ is countable collection of dense open subset of X, then $\bigcup 0_n$ is not empty.

Proof: Let x_1 be a point of 0_1 and S_1 a spheroid of radius $r_{1,}$ which is countered at x_1 and contained in 0_1. Since 0_1 is dense, there must be a point x_1 in $0_2 \cap S_1$. Since 0_2 is open, there is a spheroid S_2 countered at x_1, and contained in 0_2, and we may take the radius r_2 of S_2 to be smaller than ½ r_1 and smaller than $r_1 - \partial(x_1, x_2)$. Then $\overline{S_2} C S_{1.}$

Proceeding inductively, we obtain a sequence $\langle S_n \rangle$ of spheroid such that $\overline{S_n}$ C S_{n-1} and $SnC0_{n1}$ and whose radii $\langle r_n \rangle$ tend to zero.

Let $\langle x_n \rangle$ be the sequence of centres of these sphere. Then for n, $m \geq N$. we have $x_n \varepsilon$ S_N and dense S_N. Hence ∂(xn, xm) $\leq$ ∂r_n and $\{x_N\}$ is a Cauchy sequence, since $r_N \rightarrow 0$. By the completeness of X there is a point such that $x_n \rightarrow x,$ since x_n εS_{N+1} for n>N, we have $x \varepsilon \overline{S}_{N+1}$ C S_N C 0_N. Hence $x \varepsilon \bigcap 0_{n.}$

Corollary (Biare Category Theorem). A complete metric space is not the union of a countable collection of nowhere dense set.

An application of Baire category.

Theorem, we established the following theorem, which is known as uniform boundedness principle.

We will state that theorem without prove!

Theorem: Let τ be family of real-valued continuous functions on a complete metric space X and suppose that for each $x \varepsilon$ X there is a number M_k such that if cxV<M_k $\forall$ $f \varepsilon$ τ, there is a non-empty open set OCX and a constant M such that $|f(x)| \leq$ M for all $f \varepsilon \tau$ and all $x \varepsilon$ O.

4.0 CONCLUSION

Baire category had been used extensively in establishing some powerful proves of some mathematical theorem in Degrel Theory. Although this is advance mathematics, but nonetheless it is a very useful tools in analysiss.

5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READINGS

MODULE 2

Unit 1 Continuous Functions and Homeo Morphisms
Unit 2

UNIT 1 CONTINOUS FUNCTIONS AND HOMEO MORPHISMS

CONTENTS

3.0 INTRODUCTION

In our previous lessons on real analysis, we made attempt to a point. In this section we shall consider functions defined on a whole set and open sets.

4.0 OBJECTIVES

At the end of this study, the students should be able to:

- define correctly functions of several variables
- explain concept of continuity in a metric spaces
- explain concept pf moreomorphsim
- solve related exercise correctly.

3.0 MAIN CONTENT

3.1 Functions from $\square^{N}$ to $\square^{M}$

Definition (1): Let X and Y be arbitrary non-empty set. A function τ from X into Y is a singled-valued relation such that
DOMfCX and range fC Y.

Definition (2): Two functions f and g are identical if
(i) Domf = Domg
(ii) f(x) = g(x) $\forall$ xε Domf.

Definition (3): Function with the Range fC$\mathbb{R}$ are called real-valued functions, while those with Rang fC $\mathbb{R}^n$ are called vector-valued function.

If X = $\mathbb{R}^N$, y = $\mathbb{R}^M$ there
fC $\mathbb{R}^n$x $\mathbb{R}^{M'}$ (ii) (f: $\mathbb{R}^N \to \mathbb{R}^M$).

Definition (4): If in definition 3 above
m- τ, then we have
f: $\mathbb{R}^N \to \mathbb{R}^1$, f is called a function of several variables.
N = 1 implies
f: $\mathbb{R} \to \mathbb{R}^M$ and f(τ) $\varepsilon \mathbb{R}^M$
f is called a vector valued functions. ∍$\forall$ τ $\varepsilon\mathbb{R}$
f(τ)$\varepsilon\mathbb{R}^M$ can be expressed as f(τ) = {f_1(τ), f_2(τ),……..f_n(τ)}

Definition (5): Let f: $\mathbb{R}^N \to \mathbb{R}^M$
Dom f $\varepsilon\mathbb{R}^N$, and Range f$\varepsilon\mathbb{R}^M$ with let x_0 be an accumulative point of Dom (f). then we say f(x)$\to$b $\varepsilon\mathbb{R}^M$ as x $\to x_0$ if $\varepsilon > 0$, τ $\varepsilon > 0$, τ $\delta(\varepsilon) > 0$, such that $\|f(x) - b\| < \varepsilon \forall x\varepsilon$ Dom f such that $\|x - x_0\| < \delta$.

We write this as
lim f(x) = b
$x \to x$

Remark (1): lim f(x) = b $\Rightarrow$
x $\to x_\varepsilon$
f(x) ε B$[b, \varepsilon] \to x\varepsilon$B$[x_{0,}\delta]$
$\|x - x\| < \delta$

The implication of this is that given a neighbourhood $\bigcup$ of B in $\mathbb{R}^m$ τ a neighbourhood V of x_0 with $\vee\cap$ Dom f $\pm\phi$ such that xε $\vee\cap$ Dom (f) $\Rightarrow f(x)\varepsilon\bigcup$.

Theorem (1): Let f be a function with domain fC$\mathbb{R}^N$, and Range fC$\mathbb{R}^M$.
If f(n) $\to b_1$ as x$\to x_0$ and f(x) $\to b_2$ as x$\to$$x_0$.
Then $b_1 = b_2$.

Theorem (2): Let f and g be real-valued functions with Domain (f) = Range (g) = DC$\mathbb{R}^N$.

Let x_0 be a point of accumulation on D, if the
lim f(x) = ℓ and limit g(x) = n
$n \to x$ $x \to x_0$

(i) for $\alpha, \beta \varepsilon \Re$, then
Lim $(\alpha f + \beta g)(x) = \alpha \ell + \beta m$.
n = x_0

(ii) limt (fg) (x) = ℓm
n $\to R_0$

(iii) if g(x) $\pm$ 0 for x εD and m$\pm$0
Lim $({}^{f}\!/\!_{g})(x) = {}^{\ell}\!/\!_{m}$ = $x \to x_0$

$$\frac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$$

Example

(i) f: $\square^2 \to \square$
f(x, y) = $x^2 + y^2 + 1$
$(x_0, y_0) = (1, 3)$

Solution

lim f(x, y) = 1 + 9 + 1 = 11
$(x_0, y,) \to (1,3)$

(ii) f: $\square^2 \to \square$ such that

$$f(x, y) = \frac{2x}{x^2 + y^2 + 1}$$

as (R, y) = (1,3)

$$\lim f(x, y) = \lim \frac{2x}{x^2 + y^2 + 1}$$

$(x_0, y_0) \to (1,3)$ $(x_1\, y_0) \to (1,3)$
= $({}^{2}\!/\!_{11})$

(iii) $$fcx, y = \left\{ \frac{xy}{x^2 + y^2} \right\} x^2 + y^2 = 0$$

$$\{0 \quad (x, y) = 0\}$$

Show that the limit fcx, y)
(x, y) $\rightarrow (0,0)$
Does not exist.

Solution: f(x,y) = $\frac{x^2}{x^2+x^2} = \frac{1}{2}$
As x = y
Let (x, y) $\rightarrow 0$ along the x-axis,
$\Rightarrow$ f(x, y) = (x, 0) = 0.
This show that the limit does not exist since 0 = ½

Self assessment exercise 1

Let f(x, y) = $\left\{ \begin{matrix} x^2 \sin \frac{1}{y} + y^2 \sin \frac{1}{x} \\ x \pm 0, \quad y \pm 0 \\ 0 \qquad x = 0, y = 0 \end{matrix} \right\}$
Find lim f(x, y)
(x, y) $\rightarrow (0,0)$.

3.2 Continuity of Functions

Definition 3.2.1: Let A, B be metric spaces, with metric d_A, d_B respectively. Let f : $A \rightarrow B$ be a map.

(a) f is continuous at $x_0 \varepsilon A$ if, given ε >0, there exist δ >0 such that d_A (x_1 x_0) $< \delta$ implies d_B)fcx), f(cx,) $< \varepsilon$

(b) f is continuous if it is continuous at x_0 for every point $x_0 \varepsilon A$

Remark(2): the definition that we have proposed for continuity of maps between the choice of the particular metric.

The closer study of continuity and its independence from specific choice of metric leads naturally to the idea of a topology.

We will re-cast the definition of continuity given above.

Definition (3.2.2): Let S be a metric space with metric d defined on S. given any $x \varepsilon S$, let $B_\alpha(x)$ denote set of all points in S, whose distance from x is less than α, i.e $B_\alpha(x)$ $\{x\varepsilon S | d(x_1 x^1) < \alpha\}$
We call $B_\alpha(x)$ open α =ball around x

A map f: $A \to B$ (metric spaces) is said to be continuous at x_0 if given $\varepsilon > 0,$ there exists $\delta > 0$.
Such that
f($(B\delta(x_0))C\, B\varepsilon CY_0$)
We can write as
$B_\delta(x_0)cf^{-1}(B_\varepsilon(y_0))$

Theorem: f: $A \to B$ between metric spaces is continuous if and only if f^{-1} (V) is an open set in A whenever V is open set in B.

Remark (3): The above theorem does not say that if U is open in A then f(U) is open in B.

For example f: R$\to$R defined by f(x) = x^2 let U = R, then f(U) = R^+_0 which is not open in R.

Proof of the theorem: Assume f is continuous. Let V be an open set in B. We need to show that f^{-1} (V) is open in A. choose any x in f^{-1}(V) then f(x) ε V and so since V is open then we can find some $\varepsilon > 0,$ with B_ε (f(x)) CV. Now the continuity of f guarantee the existence of some $B\delta(x)C\,F^{-1}(V)$. This argument applies to each at f^{-1}(V) and then shows that f^{-1}(v) is open.

Conversely, assume the property about open set and show that f is continuous. To see this, let x εA. for any $\varepsilon > 0$ then $\varepsilon - ball$ $B\varepsilon(f(x))$ is and an open set in B and by hypothesis, f^{-1} $(B\varepsilon(f(x))$ is open in A. this means that since x ε f^{-1} $(B\varepsilon(f(x))$ there is some $\delta > 0$, with $B\delta(x)cf^{-1}(B\varepsilon(f(x))$ or in other words $f(B\delta(x))$ C $B\varepsilon(f(x))$.

This applies for each $\varepsilon > 0$ and so proves the continuity of f at x. Since x was an arbitrary point of A, we have shown that f is continuous.

Remark (4): In view of the above theorem is a clear that in the study of continuity of maps between metric spaces, it is the family of open set in each space which is important, rather than the actual metric.

More precisely, if two different metrics give rise to the same family of open sets then any map which is continuous using one metric will automatically be continuous using the other.

The family of open sets of a metric space is called topology.

4.0 CONCLUSION

5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READINGS

UNIT 2 CONVERGENCE IN METRIC SPACES

CONTENTS

1.0 INTRODUCTION

In a previous course in Real analysis students have been made to be familiar with the notion of convergence of sequence of real numbers. It is defined as follows. The sequence x_1, x_2………x_n…….of real numbers is said to be converge to the real number x if given $\varepsilon > 0$ there exists a number n_0 such that for all $n \geq n_0$, $|x_n - x| < \varepsilon$.

From this, it is obvious that we can extend this definition from the set of real number with the Euclidean metric to any metric space.

2.0 OBJECTIVES

At the end of this study, you should be able to:

- defined the convergence of metric spaces
- be able to differentiate between convergence of sequence of real numbers and metric spaces
- solve some questions on the convergence of metric spaces.

3.0 MAIN CONTENT

3.1 Convergence of Metric Spaces

Definition: Let (x, d) be a metric space and x_1 +………x_n…. a sequence of points in X. then the sequence is said to converge to $x \varepsilon X$ if given an $\varepsilon > 0$ there exists an integer no such that for all n $\geq n_0$, d(x, x_n) < ε. This is denoted by $x_n \to n$.

The sequence y, y_2…….y_n of points in (X, d) is said to be convergent if there exists a point y εX such that $y_n \to y$.

Remark

Let x_1, x_2, ………..x_n be a sequence of point in a metric space (X, d). Furthermore if x and y are points in (X, d) such that $x_n \to x$ and $x_n \to y$, then n = y.

The implication of this is that the point of convergence of sequence of points in metric space in always unique.

Proposition (4.1) = Let (X, d) be a metric space. A subset A of X is closed in (X, d) if and only if every convergent sequence of points in A converges to a point in A. (In order words, A is closed in (X, d) if and only if $a_n \to x$ where $x \varepsilon X$ and a_n is a sequence of points in A $\forall$ n implies that $x \varepsilon A$).

Proof: We assume that A is closed in (X, d) and let $a_n \to n$, where a_n εA for all positive integers n. Suppose that x ε (X- A). Then, as X-A is open set containing x, there exists an open ball B_ε(x) such that x ε $B_\varepsilon(x) \leq (X - A)$. Nothing that each a_n εA implies that d(x, a_n)>ε for each n. thence the sequence a_1, a_2, ………a_n……does not converge to x. This is a contradiction hence x εA as required.

Conversely, we assume that every convergent sequence of points in A converges to a point of A. Suppose that X-A is not open. Then there exist a point y ε x – A such that for each $\varepsilon > 0$ $B_\varepsilon(y) \cap A \pm \Phi$. For each positive integer n, Let x_n be any point in $B_{1/n}(y) \cap A$. Then we claim that $x_n \to$y. To see this let ε be any positive real number, and n_0 any integer greater that $1/\varepsilon$. Then for each $n \geq n_0$.

$x_n \varepsilon B_{1/n}(y) \subseteq B_{1/no}(y) \subseteq B_\varepsilon(y)$

So $x_n \to y$ and by our assumption $y\varepsilon$ (X-A). This is also a contradiction and so (X – A) is open and thus A is closed in the space (X, d).

Proposition 4.2:

Let (X, d) and (Y, d,) be metric spaces and fa mapping of X into Y. Let τ and τ_1 be the topologies determined by d and d_1 respectively. Then f(x, τ) $\to$ (y, τ,) is continuous if and only if $x_n \to x \to f(x_n) \to f(x)$: that is if x_1, x_2,……..x_n,….. is a sequence of points in (X, d) converging to x, then the sequence of points $f(x_1)$, $f(x_2)$,…………….$f(x_n)$,……..in (Y, d) Convergence to x.

Proof: Assume that $x_n \to x \Rightarrow f(x_n) \to f(x)$. To show that f is continuous if suffices to prove that the inverse image of every closed set in (Y, τ_1) is closed in (X, τ). So let A be closed in (Y, τ,). Let x_1, x_2,……x_n…….. be a sequence of points in $f'(A)$ convergent to a point x εX. As $x_n \to x,\ \ f(x_n \to f(x)$.

But since each $f(x_n)\ \varepsilon A$ and A is closed proposition 4. L then implies that $f(x)\varepsilon A$. Thus $x\varepsilon f^{-1}(A)$. Hence we have shown that every convergent sequence of points from $f^{-1}(A)$ converges to a point of $f^{-1}(A)$. Thus A is closed and hence f is continuous.

Conversely, let f be continuous and $x_n \to x$. Let ε be any positive real number. Then the open ball $B_\varepsilon(f(x))$ as an open set in (Y, τ_1). Therefore $f^{-1}(B_\varepsilon f(x))$ in an open set in (X, τ) and it contains x. Therefore there exists a $\delta \succ 0$ such that
$x\varepsilon B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))$.

As $x_n \to x$, there exists a positive integer n_0 such that for all $n \geq n_0, x_n\ \varepsilon B_\delta(x)$. Therefore
$f(x_n)\varepsilon f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$, for all $n \geq n$,
Thus $f(x_n) \to f(x,)$

Corollary 4.1

Let (X, d) and (Y, d,) be metric spaces, f is mapping of X into Y and τ and τ_1 the topologies determined by d and d, respectively. Then f: (x, τ) $\to$ (Y, τ_1) is continuous if and only for each $x_0 \varepsilon X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that x εX and d(x, x_0) $< \delta \to 0\ d,(f(x), f(x)) < \varepsilon$

SELF ASSESSMENT EXERCISE

4.0 CONCLUSION

5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

1. Let C[0, 1] denote the set of continuous functions from [0, 1] into ℝ. Define a metric on this set by $d(f,g)=\sum_{0}^{1} 1\, f(x)-g(x)|\,dx$

   Where g and f are are in $\subset[1,0]$.

   Define a sequence of functions f_1, f_2,……….f_n……… in $(\subset[0,1],d)$ by $f_n(x)=\frac{\sin(nx)}{n}$, $n=1,2,.........x\varepsilon[0,1]$

   Prove that $f_n \to f_0$ where $f_0(x)=0\,\forall\, x\varepsilon[0,1]$

2. Let (X, d) be a metric space and x_1, x_2,…………., x_n…….a sequence such that $x_n \to n$ if and $x, \to y$. Prove that x = y .

3. (i) Let (X, d) be a metric space, and an τ the induced topology on X and x_1, x_2 , ………..x_n,….. a sequence of points in X. Prove that $x_n \to n$ if only if for every open set $\cup\tau$ x, there exists a positive integer n_0 τ $x_n\varepsilon\cup$ for all $n \ge n_0$.

   (ii) Let x be a set and d and d, be equivalent metric on X. deduce from (i) that if $x_n \to x$ in (X, d) then $x_n \to x$ in (X, d,).

4. Let (X, τ) be a topological space and let $x_1, x_2, ...x_n$.......be a sequence of points in X. we say that $x_n \to x$ if for each open set $\cup\tau x$ there exists a positive integer n_0 such that $x_n\varepsilon\cup$ for all $n \ge n_0$. Find an example of a topological space and a sequence such that $x_n \to x$ and $x_n \to y$ be $x \pm y$.

5. Let A and B be non-empty set in a metric space (X, d). Define P(A, B) = in $f\{d(a,b): a\varepsilon A\, and\, b\varepsilon B\}$.

   $(\alpha(A,B)$ is the distance between two sets A and B)

i. If S is non-empty subset of (X, d), prove that $\overline{S}$ = $\{x : x \varepsilon X \; and \; \alpha(\{x\}, S\} = 0$

ii. If S is any non-empty subset of (X, d), then the function f(X, d) $\rightarrow$ defined by $f(x) = \alpha(\{x\}, S), x \varepsilon X$. is continuous.

7.0 REFERENCES/FURTHER READINGS

UNIT 3 CONNECTEDNESS AND COMPACTNESS

CONTENTS

1.0 INTRODUCTION

Neighbourhoods.

2.0 OBJECTIVES

3.0 MAIN CONTENT

3.1

Definition (1): Let X a metric space and N a subset of X and P a point in X. Then N is said to be a neighbourhood of the point P if there exist open set $\cup$ such that $P \cup \subseteq N$

Examples 1

The closed interval in is a neighbourhood of the point ½. Since ½ ε(¼, ¾) $\subseteq$[0,1].

Examples 2

The interval [0, 1] in is a neighbourhood of the point ¼ as ¼ ε(0, ½) $\subseteq$(0,1]. But (0, 1] is not a neighbourhood of point 1.

Examples 3

If X is a metric space and $\cup$ is a subspace of X then from definition 1 above it follows that $\cup$ is a neighbourhood of every point P$\varepsilon\cup$ for example every open interval (a, b) in is a neighbourhood of every point that it contains.

Definition 1: Let X be a topological space. X is said to be connected if the only open and closed subsets of X are X and ϕ.

Example 1

The topological space is connected.

Example 2

If X is a discrete space into more than one element, then X is not connected as each singleton set is open.

Remark:

From definition 1 above it follows that a topological space X is not connected if and only if there are non-empty open sets A and B such that A$\bigcap B=\phi$ and A$\bigcup B=X$.

Compactness

Definition

Let (X, d) be a metric space and let SCX. An open cover for S is a collection U, if open subsets of X such that S C $\bigcup\{U:U\varepsilon\psi\}$.

Definition:

A subset K of a metric space (X, d) is called compact if for each open cover U of K there exist U_1, U_2, ……………U_n $\varepsilon\cup$ such that K C U, $\cup.......\cup u$.

Definition can be restated as "A set is compact if and only if each open cover has a finite subcover".

Examples:

1. Let (X, d) be a metric space and let S C X be definite that is, S = {X_1, X_2……….X_n}. let U be an open cover of X. Then for each $x_j\,\varepsilon U_j$. If follows that S C U, $\cup........\cup\bigcup_{n.}$ hence, S is compact.

2. Let (X, d) be a compact metric space and let $\phi\pm k\,c\,x$ be compact. Fix n_0 εk. Since {B_r (x_0) : r > 0 } is an open cover of K, there are $r_1,r_2,........,r_n>0$ such that $K\,C\,B_{r_1}(x_0)\cup.......\cup B_{r_n}(x_0)$.

   With R : = max {$r_1,r_2,........,r_n$}, we observed that K C B_R ($x_{0)}$ so that the diameter of K claim (k) $\leq 2R<\infty$. This means, for example that any unbounded subset ^{n} (or more generally of any normal space) cannot be compact. Infact the only normal space that is compact is {0}.

3. Let X = (0, 1) be equipped with the usual metric. For $r\varepsilon(0,1)$ let $u_r:(r,1)$. then $\{u_r : r\,\varepsilon\,(0,1)\}$ is an open cover for (0, 1) which has no finite subcover.

Proposition

Let X be a metric space and let Y be a subspace of X than

(i) If X is compact and Y is closed in X, then Y is compact
(ii) If Y is compact, then it is closed in X.

Proof: For (i): Let U be an open cover for Y. Since Y is closed in X, then family $u \cup \{$ X/Y$\}$ is an open cover for X. since X is compact, it has a finite subcover i.e. there are $u_1, u_2, ..., u_n\, \varepsilon\, u$ such that $X = u_1 \cup u_2, ...u_n \cup$ X/Y.

By intersecting this into Y it is observed that $Y\,C\,U,\bigcup U_2 \bigcup, ... \bigcup u_{n.}$

For (ii): Let $x\varepsilon X/Y$. For each y ε Y there are $\varepsilon_y, \delta_y, > 0$ such that $B_{\varepsilon y}(x) \cap B_{\delta y}(y) = \phi$. Since $\{B_{\delta y}(y) : y\varepsilon Y\}$ is an open cover for Y, there are y_1, y_n ε Y such that
Y $\subset B_{\delta_{y,}}(y,) \bigcup ...... \bigcup B_{\delta_{y_n}}(y_n)$

Letting $\varepsilon :=$ min $\{\varepsilon_{y_1}, ..., \varepsilon_{y_n}\}$ we obtain that $B_\varepsilon(x) \bigcap Y\,C\,B_\varepsilon(x) \bigcap (B_{\delta_{y_1}}(y_1) \cup .... \cup B_{\delta_{y_n}}(y) \quad = \phi$. And thus $B_\varepsilon(x) \subset X/Y$. Since $x\varepsilon\, X/Y$ is arbitrary this implies that X/Y is open hence Y is closed in X.

Proposition: Let (k, d_k) be a compact metric space. Let (Y, d_y) be any metric space and let f: K$\rightarrow$Y be continuous. Then f(k) in compact.

Proof. Let U be an open cover for f(k). Then $\{ f^{-1}(\upsilon) : \upsilon\,\varepsilon U \}$ is an open cover for k. hence there are $\upsilon_1, ...\upsilon_n$ εu with
K = f^{-1} (ư) : ưεu with K = f^{-1} (ư, $\cup$.......$\bigcup f^{-1}$(ư$_n$). and thus f(k) C ư, $\cup$ ư$_2$ $\cup ... \cup$ ư$_n$.
This proves the claim.

Lemma: Let (k, d) be a compact metric space. Then every sequence in k has a convergent subsequence.

Proof: Let $\{x_n\}_{n=1}^{\delta}$ be a sequence in k. Assume that $\{x_n\}_{n-1}^{\delta}$ has no convergent subsequence. This means that for each x εX (It cannot be the limit of any subsequences of $\{x_n\}_{n=1}^{\delta}$) there is $\varepsilon_n > 0$ such that $B_{\varepsilon_n}(x)$ contain infinitely many terms of $\{x_n\}_{n=1}^{\delta}$; that is there is n_n ε such that $x_n \not\subset B_{\varepsilon_x}(x)$ for $\cap \geq n_{x.}$ Since $\{B_{\varepsilon_n}(x) : x \varepsilon k\}$ is an open cover for k there are $x_1^1, x_2^1, \ldots.., x_m^1 \,\varepsilon\, k$ with k = $B_{\varepsilon_{n_1^1}}(x_1^1) \cup \ldots \cup B_{\varepsilon_{k^1 m}}(x_m^1)$

For n $\geq$ max $\{n_{n_1^1}, \ldots., n_{n_m^1}\}$ this means that

$x_n \not\subset B_{\varepsilon_{n_1^1}}(x_1^1) \cup \ldots. \cup B_{\varepsilon_{n^1 m}}(x_m^1) = k$. This is contradiction hence every sequence in k has a convergent sequence.

Definition: Let (x, d) be a metric space. Then:

(a) X is called totally bounded if for each $\varepsilon > 0$ there are n_1, n_2,……, x_n ε X with X = $B_\varepsilon(x_1) \cup \ldots.. \cup B_\varepsilon(x_n)$

(b). X is called sequentially compact if every sequence is X has a convergent subsequence.

Theorem: The following properties are equivalent for a metric space (X, d)

(i) X is compact
(ii) X is complete and totally bounded
(iii) X is sequentially compact.

Proof: Left as exercise

Corollary: Let (X,d) be a totally bounded metric space. Then it's completion is compact.

Corollary: (Hence – Bored theorem)

Let K $\subset \mathbb{R}^n$. Then K is compact if and only if it is bounded and closed in $\mathbb{R}^n$.

SELF ASSESSMENT EXERCISE

4.0 CONCLUSION

5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READINGS