NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 302-

COURSE TITLE: ELEMENTARY DIFFERENTIAL EQUATIONS II

MTH 302- ELEMENTARY DIFFERENTIAL EQUATIONS II

Course Team: Dr. O.J. Adeniran (Writer) – UNI. of Agric. Abeokuta
Dr. Bankole Abiola (Editor) – NOUN
Dr. Bankole Abiola (Programme Leader) – NOUN
Dr S.O. Ajibola (Course Coordinator) – NOUN

NATIONAL OPEN UNIVERSITY OF NIGERIA

CONTENT

Module 1: Series Solution of Ordinary Differential Equation

ODULE 1: SERIES SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

UNIT 1: SERIES SOLUTION OF DIFFERENTIAL EQUATIONS

1.0. Introduction: A large class of ordinary differential equations possesses solution expressible, over a certain interval, in terms of power series. In this unit we are going to investigate methods of obtaining such solutions.

2.0. Objectives: At the end of this unit you should be able to
- determine radius of convergence of series
- apply series solution method to solving differential equation
- determine ordinary point, and singular points of the differential equation

3.0 MAIN CONTENT

3.1. **Series Solution of Ordinary Differential Equation**

An expression of the form

$$A_0 + A_1(x - x_0) + \ldots A_n(x - x_0)^n + \ldots = \sum_{n=0}^{\infty} A_n(x - x_y)^n + \ldots \qquad (1)$$

is called the power series.

To determine for what values of x the series (1) converges we use ratio test

$$\rho = \lim_{n\to\infty}\left|\frac{A_{n+1}(x - x_0)^{n+1}}{A_n(x - x_0)^n}\right| = \lim_{n\to\infty}\left|\frac{T_{n+1}}{T_n}\right| = L|x - x_0|$$

Where $L = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ (2)

The series is convergent when $\rho < 1$, divergent when $\rho > 1$. The test fails if $\rho = 1$. $\rho = \frac{1}{L}$ is called the radius of convergence

The series converges when

$|x - x_0| < \frac{1}{L} = R$ (radius of convergence)

diverges when

$|x - x_0| > \frac{1}{L} = R$

(i) If L is zero, the series converges for all Values of x

(ii) If L is infinite, the series converges only at the point $x = x_0$

(iii) If L is finite, then the series converges, when

$|x - x_0| < \frac{1}{L} = R$ (radius of convergence) and diverges if

$|x - x_0| > \frac{1}{L}$

If $\sum_{n=0}^{\infty} a_n(x - x_y)^n$ and $\sum_{n=0}^{\infty} b_n(x - x_y)^n$

Converge to $f(x)$ and $g(x)$ respectively, for $|x - x_0| < \rho_1$ (radius of convergence) $\rho_1 > 0$, then the following are true for $|x - x_0| < \rho_1$.

(i)Two series can be added and subtracted term wise, and

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_y)^n$$

(ii) The series can be multiplied and

$$f(x)g(x) = \sum_{n=0}^{\infty} a_n (x - x_y)^n \sum_{n=0}^{\infty} b_n (x - x_0)^n = \sum_{n=0}^{\infty} C_n (x - x_0)^n$$

Where $C_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_n b_0$

If $g(x_0) \neq 0$, the series

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n (x - x_y)^n$$

although formula for d_n is complicated if $f(x) = \sum_{n=0}^{\infty} a_n (x - x_y)^n$,then $f(x)$ is continuous

has derivates of all orders for $|x - x_0| < \rho_1$. and f', f'', f''' … can be computed by differentiating the series. Thus

$$a_n = \frac{f^n(x_0)}{n!} \quad \text{or} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_y)^n \qquad \text{……. (3)}$$

(3) is called the Taylor series for function f at $x = x_0$

A function f that has Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{Ln}(x - x_y)^n$$

With a radius of convergence $\rho > 0$ is said to be analytic at $x = x_0$.

The polynomial is analytic at every point, thus sums, differences, products, quotients (excepts at the zeroes of the denominator) of polynomials are analytic at every point.

(i) Determine the radius of convergence of the power series

(i) $\sum_{n=0}^{\infty} 2^n x^n$ (ii) $f(x) = \sum_{n=0}^{\infty} \frac{(2x+1)^n}{n^2}$ (iii) $\rho = \lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \right| = \frac{1}{2}$

(iv) $\rho = \lim_{n\to\infty} \left| \frac{A_n}{A_{n+1}} \right| = \frac{1}{2}$ (v) $\rho = \lim_{n\to\infty} \left| \frac{2^n}{n^2} \frac{(n+1)^2}{2^{n+1}} \right| = \lim_{n\to 0} \left| \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \right| = \frac{1}{2}$

3.2. **Determining the Radius of Convergence**

If we obtain the Taylor series of a function $f(x)$ about a point x_0, then the radius of convergence of the series is equal to the distance of the point x_0 from the nearest singularity.

Remark about a change in the index of summation

(a) $\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2} = \sum_{k=0}^{\infty} a_{k+2} x^{k+2}$

(b) $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

(c) $\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$

(d) $\sum_{n=k}^{\infty} a_{n+m} x^{n+p} = \sum_{n=0}^{\infty} a_{n+k+m} x^{n+p+k}$

3.3 **Ordinary Points and Singular Points of the Differential Equations**

We consider the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \qquad (4)$$

(we assume that $P(x)$, $Q(x)$ and $R(x)$ are polynomials)

(a) if $P(x_0) \neq 0$, then x_0 is an ordinary point of the equation (1), or

$$P = \frac{Q(x)}{R(x)} \quad , \quad Q = \frac{R(x)}{P(x)}$$

y, P, Q are analytic at the point $x = x_0$, then x_0 is the ordinary point of the equation.

(b) If the functions $P(x)$, $Q(x)$ and $R(x)$ are polynomials having no common factors, the singular points of equation (1) are the points for which

$$P(x) = 0 \qquad (5)$$

(c) If $\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}$ is finite

and $\lim_{x \to x_0} (x - x_0)^2 \frac{Q(x)}{P(x)}$ is finite

Then the point $x = x_0$ is called the REGULAR SINGULAR POINT of equation (3)

(d) Any singular point of equation (3) that is not regular singular point is called an <u>irregular singular point</u>.

3.3.1 Solution Near An Ordinary Point

Let us consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (6)$$

Where $P(x)$, $Q(x)$ and $R(x)$ are polynomials. x_0 is the ordinary point of the equation (6).

Assuming that $y = \phi(x)$ is a solution of (6) and $\phi(x)$ has a Taylor Series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_y)^n \qquad (7)$$

Now we know that

$$a_m = \frac{\phi^m(x)}{m!} \qquad (8)$$

We can write (1)

$$y'' + P(x)y' + q(x)y = 0$$

where $P = \dfrac{Q(x)}{R(x)}$, $q = \dfrac{R(x)}{P(x)}$

$$\therefore \quad y'' = -Py' - qy \qquad (9)$$

or

$$y''' = -p'y'' - P'y' - q'y - qy' \qquad (10)$$

(It is natural to assume that $y = y(x)$, $y' = y'(x)$ at $x = x_0$ and $y(0) = a_1$, $y'(0) = a_2$, we can easily calculate the coefficient a_n a, provided that we could compute infinitely many derivatives of p and q existing at x_0 . Thus p and q must have some condition for line calculation of a_n. It has been proved that.

$P = \dfrac{Q(x)}{R(x)}$, $q = \dfrac{R(x)}{P(x)}$ are analytic at x_0., then the general solution of (6) is $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x)$

Where a_n and a_1 are arbitrary y_1 and y_2 are linearly independent series solutions which are analytic at x_0.

We shall illustrate the method by examples.

Example 1. Solve the equation

$$y'' + 4y = 0$$

near the ordinary point $x = 0$

Solution: we assume the solution as

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad (1)$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \qquad (2)$$

Substituting these values in the equation yields

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^n = 0 \qquad (3)$$

or $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0 \qquad (4)$

or $\sum_{n=0}^{\infty} [\; n(n-1)a_n + 4\sum_{n=2}^{\infty} a_{n-2} \;] x^{n-2} = 0 \qquad (5)$

Because the first two terms of the first sum in (4) are zero.

We now use the fact that for a power series to vanish identically over any interval, each coefficient in the series must be zero

Recurrence relation :

$n(n-1)a_n + 4a_{n-2} = 0$ or $a_n = \dfrac{-4a_{n-2}}{n(n-1)}$, $n \geq 2$

Now we calculate in coefficients

$$a_2 = \frac{-4a_0}{2.1} \;,\quad a_3 = \frac{-4a_1}{3.2} \;,\quad a_4 = \frac{-4a_2}{4.3} \;,\quad a_5 = \frac{-4a_3}{5.4}$$

$$a_{2k} = \frac{-4a_{2k-2}}{2k(2k-1)} \;,\quad a_{2k+1} = \frac{-4a_{2k-1}}{(2k+1)(2k)}$$

From above we have

$$a_2.a_{n...}a_{2k} = \frac{(-1)^k 4^k}{2k!} a_0.a_2...a_{2k-2}$$

$$a_{2k} = \frac{(-1)^k 4^k}{2k!} a_0 \;,\quad a_{2k+1} = \frac{(-1)^k 4^k}{(2k+1)!} a_1$$

Hence we can write in solution

$$y=\sum_{n=0}^{\infty} a_n x^n = a_0+\sum_{k=1}^{\infty} a_{2k}x^{2k}+a_1x+\sum_{k=1}^{\infty} a_{2k+1}x^{2k+1}$$

$$=a_0+\sum_{k=1}^{\infty} \frac{(-1)^k 4^k}{2k!}a_0x^{2k}+a_1x+\sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k+1)!}a_1x^{2k+1}$$

$$=a_0[\ 1+\sum_{k=1}^{\infty} \frac{(-1)^k}{2k!}(2x)^{2k}\ ]+\frac{1}{2}a_1[\ 2k+\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(2x)^{2k+1}\ ]$$

$$=a_0 Cos2x+\frac{1}{2}a_1 Sin2x$$

Example 2: Solve the equation

$$(1-x^2)y''-6xy'-4y=0$$

near the ordinary point $x=0$

Solution: we assume the solution

$$y=\sum_{n=0}^{\infty} a_n x^n \qquad (1)$$

The only singular points of the equation in the finite plane are $x=1$ and $x=-1$. Hence we show that the solution is valid in $|x|<1$ with a_0 and a_1 arbitrary coefficients

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} n(n-1)a_n x^n-\sum_{n=0}^{\infty} 6na_n x^n-\sum_{n=0}^{\infty} 4a_n x^n=0$$

or $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} (n^2+5n+4)a_n x^n$

or $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} (n+1)(n+4))a_n x^n$

Let us shift the index the second series.

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=2}^{\infty} (n-1)(n+2))a_{n-2}x^{n-2}=0 \qquad (2)$$

In equation (2), the coefficient of each power of x must the zero.

$$a_n=\frac{n+2}{n}a_{n-2}\ , \qquad \text{for} \quad n\geq 2 \qquad (3)$$

(3) <u>is called recurrence relation</u>. A recurrence relation is a special kind of difference equation.

$n=2,4,6,...$ and $n=3,5,7,...$

$a_2=\frac{4}{2}a_0$, $\qquad a_3=\frac{5}{3}a_1$

$a_4=\frac{6}{4}a_2$ $\qquad a_5=\frac{7}{5}a_3$

. .

$$a_{2k} = \frac{2k+1}{2k} a_{2k-2} \qquad a_{2k+1} = \frac{2k+3}{2k+1} a_{2k-1}$$

$k \geq 1$

$a_{2k} = (k+1)a_0$

Similarly, $k \geq 1$

$$a_{2k+1} = \frac{2k+3}{3} a_1$$

Hence the solution $y = \sum_{n=0}^{\infty} a_n x^n$

$$y = a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k} + a_1 x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 [\ 1 + \sum_{k=1}^{\infty} (k+1) x^{2k}\ ] + a_1 [\ x + \sum_{k=1}^{\infty} \frac{2k+3}{3} x^{2k+1}\ ]$$

$$= \frac{a_0}{(1-x^2)} + \frac{a_1 (3x - x^3)}{3(1-x^2)^2}$$

Example 3. Solve the equation

$$y'' + (x-1)^2 y' - 4(x-1) y = 0$$

about the ordinary point $x = 1$

Solution: we assume the solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad (1)$$

We first translate the axes, putting

$$x - 1 = u \ , \quad \frac{dy}{du} . \frac{du}{dx} = \frac{dy}{dx}$$

$\frac{dy}{du} .1 = \frac{dy}{dx}$ The equation becomes

$$\frac{d^2 y}{du^2} + u^2 \frac{dy}{du} - 4uy = 0$$

Then we assume the solution

$$y = \sum_{n=0}^{\infty} a_n u^n$$

$$\sum_{n=0}^{\infty} n(n-1) a_n u^{n-2} + \sum_{n=0}^{\infty} n a_n u^{n+1} - \sum_{n=0}^{\infty} 4 a_n u^{n+1} = 0$$

Collecting the terms

$$\sum_{n=0}^{\infty} n(n-1) a_n u^{n-2} + \sum_{n=0}^{\infty} (n-4) a_n u^{n+1} = 0$$

Shifting the index from n to $n-3$ in the second series

$$\sum_{n=0}^{\infty} n(n-1)a_n u^{n-2} + \sum_{n=3}^{\infty} (n-7)a_{n-3}u^{n-21} = 0$$

Therefore a_0 and a_1 are arbitrary and for remainder, we have

$2a_2 = 0$

$n \geq 3$

$$a_n = \frac{n-7}{n(n-1)}a_{n-3}$$

a_{01} arbitrary $\quad a_{11}$ arbitrary $\quad a_{01} = 0$

$a_3 = \frac{-4}{3.2}a_0 \quad a_4 = \frac{-3}{4.3}a_1 \quad a_5 = \frac{-2}{5.4}a_2 = 0$

$a_6 = \frac{-1}{6.5}a_3 \quad a_7 = \frac{0}{7(6)}a_4 = 0 \quad a_8 = ()a_5 = 0$

$a_9 = \frac{-2}{9.8}a_6 \quad a_{10} = \frac{-3}{10.9}a_7 = 0 \quad a_1 = 0$

- - -

- - -

$a_{3k} = \frac{3k-7}{3k(3k-1)}a_{3k-3} \quad a_{3k+1} = 0, k \geq 2 \quad a_{2k+2} = 0, k \geq 1$

$$k \geq 1 : \quad a_{3k} = \frac{(-1)^k[\ (-4)(-1)...2...(3k-7)\ ]}{[\ 3.6.9....(3k)\ ][\ 2.5..8...(3k-1)\ ]}a_6$$

$$y = a_0[\ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k[\ (-4)(-1)...2...(3k-7)\ ]}{[\ 3.6.9....(3k)\ ][\ 2.5..8...(3k-1)\ ]}u^{3k} + a_1(u + \frac{1}{4}u^4)$$

Now substitute $u = x - 1$

$$y = a_0[\ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k[\ (-4)(-1)...2...(3k-7)\ ]}{[\ 3.6.9....(3k)\ ][\ 2.5..8...(3k-1)\ ]}u^{3k} + a_1(u + \frac{1}{4}u^4)$$

4.0 **Conclusion**: In this unit we have attempted the series solution method to Ordinary Differential Equations. In the subsequent unit we are going to discuss more about this method in greater details. You are supposed to master this unit properly to be well equipped for the next unit.

5.0 **Summary**: Recall that in this unit we discuss power series and radius of convergence for the series. We also applied the series to solve differential equations. We derived the singular and ordinary points for each of the series solutions. Study this unit properly before going to the next unit.

6.0. **Tutor Marked Assignments**

1. Determine a lower bound for the radius of convergence of series solution about each given point x_0 for each of the following differential equations.

(i) $(x^2 - 2x - 3)y'' + xy' + 4y = 0$, $x_0 = 4$, $x_0 = -4$ and $x_0 = 0$

(ii) $(1 + x^3)y'' + 4xy' + y = 0$, $x_0 = 0$, and $x_0 = 2$

2. Determine whether each of the points $-1, 0$ and 1 is an ordinary point, or regular singular point or irregular singular point for the following differential equation,

(i) $2x^4(1 - x^2)y'' + 2xy' + 3x^2 y = 0$

(ii) $(x + 3)y'' - 2xy' + (1 - x^2)y = 0$

7.0 REFERENCES/FURTHER READINGS

EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London.

UNIT 2: EULER EQUATION.

1.0 Introduction: In this unit we deal with a class of differential equation normally refer to as Euler Equation. This type of equation usually possesses solutions that are classified as regular singular points of the differential equations. Series solution of this class of equation must be attempted with different approach. We shall see this in our treatment of this system of equation in this unit.

2.0 Objective: At the end of this unit you should be able to
- differentiate Euler equations from others.
- use series solution approach to solve these categories of equations
- solve problems relating to Euler equation

3.0. MAIN CONTENT

3.1. Euler Equation

$$L(y) = x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0 \qquad (1)$$

is known as Euler equation.

It is easy to see that $x = 0$ is a regular singular point of (1)

In any interval not including the origin, (1) has a general solution of the form.

$$y = c_1 y_1(x) + c_2 y_2(x) \quad ,$$

y_1 and y_2 are linear, independent solution.

Here we assume that (1) has a solution of the form

$$y = x^2$$

$L(x^2) = x^2(x^2)'' + \alpha x(x^2)' + \beta x^2 = x^2 F(r)$

Where

$F(r) = r(r-1) + \alpha r + \beta$ (2)

If r r is a root of the equation

$$r_1 = \frac{-(\alpha-1) + \sqrt{(\alpha-1)^2 - u\beta}}{2} \quad (3)$$

$$r_2 = \frac{-(\alpha-1) - \sqrt{(\alpha-1)^2 - u\beta}}{2} \quad (4)$$

$\therefore F(r) = (r - r_1)(r - r_2)$

Case I $(\alpha-1)^2 - 4\beta > 0$, then the roots are real and unequal and $W(x^{r_1}, x^{r_2})$ is non-vanishing for $r_1 \neq r_2)$ and $x > 0$. Thus the general solution is

$y = c_1 x^{r_1} + c_2 x^{r_2}$ $\quad x > 0$

case ll $(\alpha-1)^2 - 4\beta = 0$, then $r_1 = r_2 = -\frac{(\alpha-1)}{2}$

and we have only one solution

$y_1(x) = x^{r_1}$

of the differential equation. We can obtain the second solution by the method of reduction. We consider a different approach to obtain the solution.

$L(x^r) = x^r F(r)$

If $r = r_1$, then

$L(x^{r_1}) = x^{r_1} F(r_1) = 0$

Now $F(r) = (r - r_1)^2$, if we differentiate

$F(r)$ i.e $\therefore F'(r) = 2(r - r_1)$ and then set $r = r_1$, if given $F'(r) = 0$,it suggest that

$$\frac{\partial}{\partial r} L(x^r) = \frac{\partial}{\partial r}[x^r F(r)]$$

$L(x^r \log x) = x^r \log x F(r) + r(r - r_1)x^r$

We set $r = r_1$, thus

$L(x^{r_1} \log x) = 0$

$\therefore y_2 = x^{r_1} \log x$ $\quad x > 0$

is the second solution of (1)

Thus the general solution is

$y_2 = (c_1 + c_2 \log x)x^{r_1}$, $\quad x > 0$

Case III $(\alpha-1)^2 - 4\beta < 0$), in this case, the root are complex, say

$$r_1 = \lambda + i\mu, \quad r = \lambda + i\mu$$

Thus the general solution is

$$y_2 = (c_1 x^{\lambda+i\mu} + c_2 x^{\lambda-})x^{r_1 i\mu}$$
$$= x^{\lambda}[c_1 x^{i\mu} + c_2 x^{-i\mu}]$$
$$= x^{\lambda}[c_1 e^{i\mu\log x} + c_2 e^{-i\mu\log x}]$$
$$= x^{\lambda}[c_1 \cos(\mu\log x) + c_2 \sin(\mu\log x)]$$

It is always possible to obtain a real valued solution of Euler equation (1) in the interval, by making the following changes

$$\frac{d}{dx} = (-1)\frac{d}{\xi}, \frac{d^2}{dx^2} = \frac{d^2}{d\xi^2}$$

in the equation, we have

$$\xi^2 \frac{d^2u}{d\xi^2} = \xi^2 \frac{du}{d\xi} + \beta u = 0,\ \xi^2 \qquad \xi^2 > 0$$

It is obtained as above. Since

$$|\ x\ | x \qquad \text{for } x > 0$$
$$-x = \xi \qquad \text{for } x > 0$$

It follows that we read one, to replace $\text{for}\, x$ by $|\ x\ |$ in the above solution to obtain real valued solution valid in any interval not containing the origin

To solve the Euler equation (1)

$$x^2 y'' + x\alpha y' + \beta y = 0$$

in any interval not containing the origin substitute $y = x^r$ and compute the root r^1 and r^2 of the equation

$$F(r) = r^2 + (\alpha - 1)r + \beta = 0$$

If the roots are real and unequal

$$y = c_1 |\ x\ |^{r_1} + c_2 |\ x\ |^{r_2}$$

If the roots are real and equal

$$y = (c_1 + c_2 \log|\ x\ | x)|\ x\ |^{r_1}$$

If the roots are complex

$$y = |\ x\ |^{\lambda}(c_1 \cos(\mu\log|\ x\ | + c_2 \sin(\mu\log|\ x\ |$$

For an Euler equation of the form

$(x-x_0)^2 y'' + \alpha(x-x_0)y' + \beta y = 0$

Change the independent variable by

$t = x - xo)^r$

or suppose the solution

$y = (x-xo)^r$

Note: The situation for a general second order differential equation with a regular singular point is similar to that for an Euler equation.

2. Another method of obtaining the solution of Euler Equation

$$x^2 y'' + \alpha x y' + \beta y = 0$$

Solution: We make the change of variable $x = e^z$ or $z = \log x$ and $x > 0$.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz}\frac{dz}{dx} - \frac{1}{x^2} = \frac{dy}{dz}$$

Substituting there value in the equation

$$\frac{d^2y}{dx^2} + (\alpha-1)\frac{dz}{dx} + \beta y = 0$$

This is an equation with constants coefficients

The auxiliary equation is

$$r^2 + (\alpha-1)r + \beta y = 0$$

(i) If r_1 and r_2 are real and unequal. $y = c_1 e^{zr1} + c_2 e^{zr_2} = c_1 x^{r_1} + c_2 x^{r_2}$

(ii) If the roots are equal i.e $\therefore y = (c_1 + c_2 z)e^{zr}$.

$$= (c_1 + c_2 \log x)\alpha^{r_1}$$

(iii) If the roots are complex

$$y = e^{\lambda z}(c_1 \cos \mu z + c_2 \sinh z)$$

$$= x^{\lambda}(c_1 \cos(\mu + \log x) + c_2 \sin(\mu \log x)$$

3.2 Series solution near a regular singular point

Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (1)$$

Assume that $x=0$ is a regular singular point of (1) means that $xP(x)=\frac{xQ(x)}{P(x)}=$ and $x^2q(x)=\frac{x^2R(x)}{P(x)}$ have finite limits as $x\to 0$ and are analytics at $x=x_0$ for some interval about the origin

(i) can be written

$$x^2y''+x[xp(x)]y'+[x^2q(x)]y=0$$

But $xp(x)=\sum_{n=0}^{\infty} P_nx^n$

$$x^2q(x)=\sum_{n=0}^{\infty} q_nx^n$$

$$x^2y''+x[p_o+p_1x+.....p_nx+......]y'+[q_0+q_1x+........+q_xx^n+..........]y=0 \quad (2)$$

If all the coefficient use zeros, except $P0$ and q_n, then (2) reduces to Euler equation, which was discussed previously.

If some of the P_n and q_n. $n\geq$ will not be zero. However the essential character of the solution remains the same. It is natural to seek the solution of the form of "Euler Solution" the power series.

$$y=x^r\sum_{n=0}^{\infty} a_nx^n \qquad (3)$$

As part of our problem we have to determine

(1) The values of r for which equation (1) has a solution of the form (3)

(2) The recurrence relation for the a_n

(3) The radius of convergence of the series $\sum_{n=0}^{\infty} a_nx^n$

We shall illustrate the method by example

Example I: Find the series solution of the equation

$$2xy''+(1+x)y'-2y=0 \qquad (1)$$

Solution: $x=0$ is the regular singular point of the equation.

We assume line solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad (2)$$

Direct substitution of y in (2) given

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+z} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Now we shift the index of the second series in (3). We get

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r-l} + \sum_{n=0}^{\infty} (n+r-2)a_n x^{n+r} = 0 \qquad (3)$$

Once more we reason that the total coefficient of each power of x in the left member of (4) must vanish. The second summation does start the contribution, until $n=1$. Hence the equation determinants c and an are given by

$$n=0 \quad r(2r-1)a_0 = 0 \text{ , but } a_0 \neq 0 \qquad (5)$$

$$\therefore r(2r-1) = 0 \qquad (6)$$

(6) is called the **<u>indicial equation.</u>**

$\therefore r_1 = \frac{1}{2}, \; r_2 = 0 \qquad n \geq 1$

$n+r)(2n+2r-1)a_n + (v+r-3)a_{n-1} = 0$

Recurrence relation

$$\therefore a_n = -\frac{n+r-3}{(n+r)(2n+2r-1)} a_{n-1} \qquad (7)$$

(i) Take $r = \frac{1}{2}$

$$a_1 = \frac{(-3)a_0}{2.3}$$

$$a_2 = \frac{(-1)a_1}{4.5}$$

$$a_3 = \frac{(-1)a_2}{6.7}$$

$$\ldots\ldots\ldots\ldots$$

$$a_n = -\frac{(2n-5)a_{n-1}}{2.n(2n+1}$$

$$a_n = \frac{(-1)^n[(-3])(-1)(1)\ldots(2n-5)]a_0}{[2.4.6........(2n)][3.5.7.....(2n+1)]}$$

Omitting the constant a_0, we may write the particular solution as

$$y_1 x^{yz} + \sum_{n=1}^{\infty} \frac{(-1)^n 3x^{n+\frac{1}{2}}}{2^n n1(2n-3)(2n-1)(2n+1)} \qquad (8)$$

Next task is to find the solution corresponding to the root $c_2 = 0$.

The recurrence relation becomes $n \geq 1$.

$$n(2n-1)b_n + (n-3)b_{n-1} = 0$$

0r $\quad b_n = \frac{(n-3)}{n(2n-1)} b_{n-1} = 0$

$$b_1 = \frac{(-2)b_0}{1.1}$$

$$b_2 = \frac{(-1)b_1}{2.3}$$

$$b_2 = \frac{(-1)b_2}{3.5}$$

$$b_n = \frac{(0)b_{n-1}}{n(2n-1)}$$

$$\therefore b_n = 0 \text{ if } b_n = -\frac{(n-3)b_{n-1}}{n(2n-1)}$$

$$\therefore b_n = 0 \text{ if } n \geq 3$$

$$\therefore b_1 = b_0, \ \therefore b_2 = \frac{1}{6} b_1 = \frac{1}{6} b_0$$

The solution is

$$y_2 = b_0 [1 + 2x + \frac{1}{3} x^2]$$

The general solution is

$$y = Ay_1 + By_2$$

Note: The roots of indicial equation are unequal and do not differ by <u>integer</u>

3.3**<u>Indicial equation with equal roots</u>**

Example 2. Solve the equation

$$x^2 y'' + 3xy' + (1-2x)y = 0 \qquad (1)$$

Solution: $x = 0$ is regular singular point of (i)

We assume the solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad (2)$$

Substituting this value in (2), we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + 3\sum_{n=0}^{\infty} (n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - 2\sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Shifting the index

$$\sum_{n=0}^{\infty} [(n+r)^2 + 2(n+r)+1]a_n x^{n+r} - 2\sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0 \qquad (3)$$

The indicial equation

$$r^2 + 2r + 1 = 0$$

$$\therefore r = -1$$

The recurrence relation is

$$a_n = \frac{2a_{n-1}}{(n+r+1)^2} \qquad (4)$$

$n \geq 1,$

In which

$$n \geq 1,\ a_n = \frac{2^n a_0}{[(r+2)(2+3)........(2+n+1)]^2} \qquad (5)$$

$$\therefore y(x,r) = x^2 + \sum_{n=1}^{\infty} a_n(r)x^{n+r}, \qquad (6)$$

in which

$$n \geq 1,\ a_n(r) = \frac{2^n}{[(r+2)(2+3)........(r+n+1)]^2} \qquad (7)$$

Let as write

$$L(y) = x^2 y'' + 3xy' + (1-2x)y$$

The y of equation (6) has been so determined that for that y the Eight member of (8) reduced to a single term the $n = 0$.

Thus

$$L(y) = x^2 y'' + 3xy' + (1-2x)y \qquad (8)$$

A solution of the original differential equation is a function y for which $L(y) = 0$. Now taking $r = -l$ makes $L[y(x,-1)] = 0$

Now differentiate each member of (9) with respect to

$$\frac{\partial}{dr}[y(x,r)] = \frac{\partial}{\partial r}[(r+1)^2 x^2]$$

$$l[\frac{\partial}{dr} y(x,r)] = 2(r+1)x^2 + (r+1)^2 x^2 \log x \qquad (10)$$

From (9) and 10, it can be seen early that the two solution of the equation $L(y)=0$ are

$$y_1=[y(x,r)] \qquad y=[y(x,-1) \tag{11}$$

and $y_2=[\frac{\partial}{\partial r}y(x,r)]_{r=-1}$ (12)

$$y(x,r)=x^r+\sum_{n=1}^{\infty} a_n(r)x^{n+r}$$

$$\frac{\partial}{dr}y(x,r)]=x^2\log x+\sum_{n=1}^{\infty} a_n'(r)x^{n+r}+\sum_{n=1}^{\infty} a_n(2)x^{n+r}\log x$$

$$=y\ (x,r)\log x+\sum_{n=1}^{\infty} a_n'(r)x^{n+r}$$

$$\therefore = y_1\log x+\sum_{n=1}^{\infty} a_n(-1)x^{n-r}$$

$$\therefore = y_1 x^{-1}+\sum_{n=1}^{\infty} a_n'(-1)x^{n-r}$$

$$a_n(r)\frac{2^n}{[(r+2)(r+3)...(2+n+1)]^2)}$$

$$\log a_n(r)=\log 2^n+2[\log(r+2)+......\log(2+n+r)]$$

$$a_n''(r)=2a_n(r)[\frac{1}{r+2}+\frac{1}{r+3}+........\frac{1}{r+n+1}]$$

$r=-1$, we obtain

$$a_n(-1)=\frac{2^n}{(n1)^2}$$

$$\therefore a_n'(-1)=-2\frac{2^n}{(n1)^2}[1+\frac{1}{2}+\frac{1}{3}+.......\frac{1}{n}]$$

We write

$$H_n=\frac{1}{2}1+.......\frac{1}{n}$$

The solutions are

$$=y_1x^{-1}+\sum_{n=1}^{\infty}\frac{2^n x^{n-1}}{(n!)^2}$$

$$=y_2x_1\log x-\sum_{n=1}^{\infty}\frac{2^{n+1}H_n x^{n-1}}{(n!)^2}$$

The general solution, valid for all finite $x\neq 0$ is $y=Ay_1+By_2$.

3.3: Indicial Equation with difference of Roots a Positive Integer, non Logarithmic case

Solve line equation

$$xy'' - (4+x)y' + 2y = 0 \qquad (1)$$

Solution: We assume the solution

$$y = \sum_{n=0}^{\infty} a^n x^{n+r} \qquad (2)$$

$$\therefore L(y) = \sum_{n=0}^{\infty} (n+r)(n+r-6)a_n x^{n+r-1} \sum_{n=1}^{\infty} (n+r-3)a_{n-1} x^{n+r-1} \qquad (3)$$

The Indicial equation is

$c(c-5) = 0$

$\therefore c_2 = 0$, $\quad c_1 = 5$

$\therefore s = 5 - 0 = 5$

We reason that we hope for two power series solutions, one starting with an x^0 and term, one with an x^5 term.

If we use the longer root $c = 5$, then the x^0 term would never enter. Thus we use the smaller roots $c = 0$, then the trial solution of the form.

$$y = \sum_{n=0}^{\infty} a^n x^n \qquad (4)$$

has a chance of picking up both solutions because the $n = 5$ $n = s$ term does contain x^5

$$\therefore L(y) = \sum_{n=0}^{\infty} n(n-5)a_n x^{n-1} - \sum_{n=1}^{\infty} (n-3)a_{n-1} x^{n-1} L(y) = 0$$

$L(y) = 0$

$$n(n-5)a_n = (n-3)a_{n-1} \qquad (5)$$

$n = 1$	$-4a_1 + 2a_0 = 0$	$\therefore a_1 = \frac{1}{2}a_o$
$n = 2$	$-6a_2 + a_1 = 0$	$a_2 = \frac{1}{12}a_o$
$n = 3$	$-6a_3 + 0a_1 = 0$	$a_3 = 0$
$n = 4$	$-4a_4 + a_3 = 0$	$a_4 = 0$
$n = 5$	$0.a_5 + 2a_4 = 0$	$a_4 = 0$
$n \geq 6$		

$a_6 = \frac{3a_6}{6.1}$

$a_7 = \frac{4a_6}{7.2}$

$a_n = \frac{(n-3)(a_{n-1})}{n(n-5)}$

$a_n = \frac{3.4.5....(n-3)a_5}{[6.7.8......n](n-5)1}$

$a_n = \frac{3.4.5a_5}{(n-2)(n-1)n(n-5)1}$

Therefore, with a_0 and a^5 arbitrary, the general solution may be written

$y = a_0(1+\frac{1}{2}x+\frac{1}{12}x^2)$

$a_5[x^5 + \sum_{n=6}^{\infty} \frac{60x^n}{(n-5)1n(n-1\_(n-2)}$

$a_0 = (1+\frac{1}{2}x+x^2)$

$a_5 = [x^5 + \sum_{n=0}^{\infty} \frac{60x^{n+5}}{n1(n+5)(n+4)(n+3)}]$

Example 2 Solve the equation

$$xy'' + (4+3x)y' + 3y = 0 \qquad (1)$$

Solution $x = 0$ is the regular singular point of (1).

We assume the solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad (2)$$

$\therefore L(y) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1}$

$4\sum_{n=0}^{\infty} a_n x^{n+r-1} + 3\sum_{n=0}^{\infty} (n+r)(a_n + r-1)a_n x^{n+r}$

$+3\sum_{n=0}^{\infty} a_n x^{n+r}$

$L(y)\sum_{n=0}^{\infty} a_n(n+r)(a_n + r+3)x^{n+r-1}$

$$+\sum_{n=0}^{\infty} 3a_n(n+r+1)x^{n+r} \qquad (3)$$

$n = 0$, we get the indicial equation

$r(r+3) = 0$

$r = 0, -3$

$$L(y)\sum_{n=0}^{\infty} a_n(n+r)(a+r+3)x^{n+r-1}$$

$$+\sum_{n=1}^{\infty} 3a_{n-1}(n+r)x^{n+r-1}$$

Using the smaller root $r_1 = -3$

$$L(y)\sum_{n=0}^{\infty} a_n(n-3)n^{n-4}$$

Recurrence relation is

$a_n(n-3)n = -3(n-3)a_{n-1}$

$n = 1$ $\qquad a_1(-2)(1) = (-3)(-2)a_o$

$n = 2$ $\qquad a_2(-1)(2) = (-3)(-1)a_1$

$n = 3$ $\qquad a_3(0)(3) = (-3)(0)a_2$

$n > 4$ $\qquad a_n = \frac{-3}{n}a_{n-1}$

$$a_4 = \frac{-3}{4}a_3$$

$$a_5 = \frac{-3}{5}a_4$$

....

$$a_n = \frac{-3}{n}a_{n-1}$$

$$\therefore a_n = \frac{(-3)}{nl}6a_3$$

$y = (a_0 xt^{-3} a_1 x^{-2} + a_2 x^{-1}$

$$a_0(x^{-3} - 3x^2 + a_2 + \frac{9}{2}x^{-1}) + 6a_3\sum_{n=3}^{\infty} \frac{(-1)^{n-3}x^{n-3}}{n1}$$

This is the required solution

4.0. Conclusion

We have looked at various problems involving Euler equations in this unit their various form of indicial equations. In the next unit we shall consider indicial equation of positive integer and logarithmic case.

5.0 Summary

You will recall in this unit that a general form of Euler equation was given. We also consider various form of Euler equations. You are required to master this unit very well before proceeding to other units.

6.0. Tutor Marked Assignment:

(1). Solve the equation

$$x^2 y'' + 3xy' + (1-2x)y = 0$$

(2).Solve the equation

$$xy'' - (4+x)y' + 2y = 0$$

7.0 REFERENCES/FURTHER READINGS

1. EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

2. FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

3. EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London.

UNIT3: INDICIAL EQUATION WITH DIFFERENCE OF ROOTS A POSITIVE INTEGER, LOGARITHMIC CASE

1.0. Introduction

In unit 2 we have considered indicial equations where logarithm case is not considered. We shall undertake to consider the positive and logarithm cases in this unit.

2.0 Objectives

At the end of this unit you should be able to
- to solve differential equation whose indicial equation has a positive integer
- to solve differential equation whose indicial equation has roots with logarithmic case.

3.0. MAIN CONTENT

3.1. Indicial Equation with Difference of roots a positive integer, logarithmic case.

We illustrate this method by an example
Solve the equation
$x^2 y'' + x(1-x)y' - (1+3x)y = 0$
Solution: We assume the solution

$$\therefore L(y)\sum_{n=0}^{\infty} 9n+r+1)(n+r-1)a_n x^{n+r} - \sum_{n=1}^{\infty} (n+r+2)a_{n-1}x^{n+r}$$

The indicial equation is

$(r+1)(r-1)=0$ (Putting $n=0$) (1)

$r=1,-1$

$n \geq 1$, the recurrence relation

$$\therefore a_n = \frac{(r+n+2)a_0}{(r+2)[r(r+1).....(r+n-1)]} \quad (2)$$

$$\therefore y = a_0 x^r \sum_{n=1}^{\infty} \frac{(r+n+2)a_0 x^{n+r}}{(r+2)[r(r+1).....(2+n-1)]}$$

It follows that

$\therefore L(y)(r+1)(r-1)a_o x^r$

For $r=1$, only one solution can be obtained.

Note: for $r=-1$, since there is no power series with skilling x^{-1}, we suspect the presence $r=-1$.

Choose $a_0 = r+1$, we have

$$y(x,r) = (r+1)x^r + \sum_{n=1}^{\infty} \frac{(r+1(r+n+2)x^{n+r}}{(r+2)[2(2+1).....(2+n-1]}$$

We can obtain two solutions with respect to

For which

$\therefore L[y(x,r) = (r+1)^2(r-1)x^c$

We use the same argument as that of equal roots

Putting

$y_1 = y(x,-1)$

$$y_2 = [\frac{\partial y}{\partial r}(x,r)]_{r=-1}$$

$$y = y(x,r) = (r+1) + \frac{(r+1)(r+3)x^{r+1}}{r(r+2)} + \frac{(r+4)}{(r+2)_r} x^{r+2}$$

$$+ \sum_{n=3}^{\infty} \frac{(r+n+2)x^{n+r}}{(r+2)[(r+2).....(2+n-1)]}$$

Differentiate with respect to r.

$$\frac{\partial}{\partial r} = y(r,x)\log x + x^r + \frac{(r+1)(r+3)x^{r+1}}{r(r+2)}$$

$$\{ \frac{1}{r+1} + \frac{1}{r+3} - \frac{1}{r+2} - \frac{1}{2} \} + \frac{(r+4)x^{2+2}}{r(r+2)} \{ \frac{1}{r+u} - \frac{1}{r+2}$$

$$+\sum_{n=3}^{\infty} \frac{(2+n+2)x^{n+r}\dfrac{1}{2+n+2}-\dfrac{1}{r+2}-\dfrac{1}{r}-(\dfrac{1}{r+2}+\dfrac{1}{r+3}+.....\dfrac{1}{r+n-1}}{(r+2)[(r+2)(r+3)........(2+n-1)}$$

Putting $e=-1$, get

$$y_1 = 0.x + 0.x^o - 3x + \sum_{n=-3}^{\infty} \frac{(n+1)x^{n-1}}{(-1)[1.2...(n-2)]}$$

$$y_2 = y_1 \log x + x^1 - 2x^0 - 3x\{\frac{1}{3} - 1 + 1\} + \sum_{n=3}^{\infty} \frac{(n+1)x^{n-1}\{\dfrac{1}{n+1} - 1 + 1 - (1 + \dfrac{1}{2} + ... + \dfrac{1}{n-2}\}}{(-1)[1.2...(n-2)]}$$

$$y_1 = -3x + \sum_{n=3}^{\infty} \frac{(n+1)x^{n-1}}{(n-2)!}$$

$$y_2 = y_1 \log x + x^{-1} - 2 - x - \sum_{n=3}^{\infty} \frac{[1-(n+1)H_{n-2}]x^{n-1}}{(n-2)]}$$

Problem: Find line general series solution of the D.E

$$4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$$

and show that it can be expressed in line form

$$y_1 = 0.x + 0.x^o - 3x + \sum_{n=-3}^{\infty} \frac{(n+1)x^{n-1}}{(-1)[1.2...(n-2)]}$$

Solution: $x = 0$ is a regular singular point of D.E. Assuming the solution

Substituting in the D.E

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Changing the index

$$\sum_{n=0}^{\infty} 2(n+r)(2n+2r+1)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_n x^{n+r-1} = 0.$$

The indicial equation is

$$2r(2r+1) = 0, \quad r = 0, \frac{1}{2}$$

The recurrence relation is

$$a_n = \frac{(1)}{2(n+2)(2n+2r+1} a_{n-1}, \qquad n \geq 1$$

(i) $r = 0$, then

$$a_n = \frac{(-1)}{(2n(2n+1)} a_{n-1}$$

$$n = 1, \qquad a_1 = \frac{(-1)}{(2)(3)} a_0$$

$$a_2 = \frac{(-1)}{(4)(5)} a_1$$

Thus

$$a_1 = \frac{(-1)}{(2n+1)!)(3)} a_0$$

Hence the solution is

$$y_1 = a_0 \sum_{n=0}^{\infty} \frac{(-1)x^h}{(2n+1)!}$$

$$= \frac{a_0}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n (x\frac{1}{2})^{2n+1}}{(2n+1)!} = \frac{a_0}{\sqrt{x}} Sin\sqrt{x}$$

(ii) $r = -\frac{1}{2}$, then

$$a_n = \frac{(-1)}{(2n-1)(2n)} a_{n-1}$$

$$n = 1, \qquad a_1 = \frac{(-1)a_0}{(1)(2)}$$

$$a_2 = \frac{(-1)a_1}{(3)(4)}$$

$$a_n = \frac{(-1)^n a_0}{2n!}$$

$$y_2 = \frac{a_0}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)x^n}{(n-2)]} = \frac{a_0}{\sqrt{x}} \cos\sqrt{x}$$

Hence the general solution is

$$y = \frac{1}{\sqrt{x}} (ACos\sqrt{x} + BSin\sqrt{x})$$

3.2. Fourier series

1. Orthogonality: A set of function is $\{f_0(x), f(x), ... f_n(x), ...\}$ said to be an orthogonal set with respect to the weight function $w(x)$ over the interval $a \le x \le b$ if

$$\int_a^b w(x) f_n(x) f_m(x) dx = 0 \qquad \text{for } m \ne n$$

$$\ne 0 \qquad \text{for } m = n$$

Orthogonality is a property widely encountered in certain branches of mathematics. Much use is made of the representation of functions in series of the form

$$\sum_{n=0}^{\infty} c_n f_n(x)$$

In which the c_n are numerical coefficients and $\{f_n(x)\}$ is an orthogonal set is

3.3 Orthogonality of a set of series and cosines:

We shall consider the set of function

$$Sin(\frac{n\pi x}{c}), \qquad n = 1,2,3..........$$

$$Cos(\frac{n\pi x}{c}), \qquad n = 0,1,2,3..........$$

or

$$Sin(\frac{\pi x}{c}), \quad Sin(\frac{2\pi x}{c}), \quad Sin(\frac{3\pi x}{c}), \ldots\ldots\ldots\ldots\ldots Sin(\frac{n\pi x}{c}) \ldots\ldots\ldots\ldots\ldots\ldots$$

$$1,\ Cos(\frac{n\pi x}{c}), \quad Cos(\frac{2\pi x}{c}), \quad Cos(\frac{3\pi x}{c}), \ldots\ldots\ldots\ldots\ldots Cos(\frac{n\pi x}{c}) \ldots\ldots\ldots\ldots\ldots\ldots..$$

is orthogonal with respect to the weight function $w(x) = 1$ over the interval $-c \le x \le c$

i.e.

$$\int_{-c}^{c} Sin\frac{n\pi x}{c} Cos\frac{k\pi x}{c} dx = 0, \qquad \text{where } k \ne n.$$

: Before we prove the result, we give some definition to shorten the proof.

(a) **Even function**: A function $y = g(x)$ is said to be even y
$g(-x) = g(x)$
For all x.

(b) **Odd function**: A function $y = h(x)$ is odd if $y = h(x) = -h(-x)$
For all x.

Example: $Sinx$ is an odd function

$Cosx$ is an even function

(c) Most function are neither even or odd example x. (is in one function $f(x)=0$

(d) If $g(x)$ is an <u>even</u> function then as well as even)

$$\int_{-c}^{c} g(x)dx = 2\int_{0}^{c} g(x)dx$$

Consider the integral.

$$I_1 = \int_{-c}^{c} Sin\frac{n\pi x}{c}Cos\frac{k\pi x}{c}dx = 0$$ for all k and n.

Follows at once from the facts that the integrand is an odd function of x. It does not dependent upon one fact and k and n are integers

$$I_2 = \int Sin\frac{n\pi x}{c}Sin\frac{k\pi x}{c}dx, \qquad k \neq n$$

Take

$$\beta = \frac{\pi x}{c}, dx = \frac{c}{\pi}d\beta$$

$$\frac{c}{2\pi}\int_{-\lambda\pi}^{\pi} Cos(n-k)\beta - Cos(n+k)\beta]d\beta \ \frac{c}{\pi}\int_{-\lambda\pi}^{\pi} Sinn\beta Sink\beta d\beta$$

$=$

$$= \frac{c}{2\pi}[\ \frac{Sin(n-k)}{n-k}\beta - \frac{Sin(n+k)\beta}{n+k}\ ]_{-\pi}^{\pi}$$

Since $n-k$ and $n+k$ are $+ve$ integers.

$=0$

Finally we consider the integral

Where $n=0$, 1,........... $n \neq 0$,

$$= \frac{c}{2\pi}[\ \frac{Sin(n-k)\beta}{n-k} - \frac{Sin(n+k)\beta}{n+k}\ ]_{-\pi}^{\pi}$$

$=0$

$$I_4 = \int_{-c}^{c} Sin^2\frac{n\pi x}{c}dx = \int_{-c}^{c} Sin^2\frac{nx}{c}dx$$

let f(x) the continuous and differentiable at every point in an interval $-c \leq x \leq c$ except for a most finite number of points and at more points, let $f(x)$ and $f'(x)$ have right and left-hand limits

Note:

The notation $f(c+0)$ is used to denote line right-hand limit of $x \to c$ as $f(x)$ from alone, i.e.

$f(c+0)$ and $= \lim_{n \to c^+} f(x)$

Similarly

$f(c-0) = = \lim_{n \to c^-} f(x)$

Denotes, the limit of f(x) as approaches c.

Since Fourier series for $f(x)$ may not converge to the value $f(x)$ every where. It is customary to replace the equals sign in equation (8) by the symbol ~which may be read "has for its Fourier Series" we write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c},$$

Where a_n and b_n are given by (a) and (10)

Example: Construct the Fourier series, over the interval $-2 \le x \le 0,$ for the function defined by

$f(x) = 2\,, \qquad -2 \le x \le 0,$

$x = 2 \quad , \qquad 0 < x < 2$

Solution: Now $f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2})$

In which

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx; \quad n = 0,1,......2$$

and

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx;\ n =,1,......2$$

$$\therefore a_n = \frac{1}{2}\int_{-2}^{0} \cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx \qquad \text{(a)}$$

If $n \ne 0$, then

$$\therefore a_n = \frac{2}{\pi}[\sin\frac{n\pi x}{2}\ ]_{-2}^{0} + \frac{1}{2}[\frac{2}{\pi}x\sin\frac{n\pi x}{2} + (\frac{2}{\pi})^2\cos\frac{n\pi x}{2}]^2{}_0$$

$$= \frac{-2(1-\cos n\pi)}{n^2\pi^2}$$

For $n = 0$, from (a), we get

a_0 =3

$\therefore b_n = \frac{1}{2}$

Thus we write

$$f(x) \sim \frac{3}{2} - 2\sum_{n=1}^{\infty} [\frac{1-(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2} + \frac{1}{n\pi}\sin\frac{n\pi x}{2}].$$

Example 2 Obtain the Fourier series over the interval $-\pi$ to π for the function x^2

Solution: We know

$$x^2 \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos^{nx} + b_n \sin nx] \qquad \text{for } -\pi < x < \pi \text{, where}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nxdx; \qquad n = ,1,2,.......$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nxdx; \qquad n = ,1,2,.......$$

x^2 is an even function, $\sin nx$ is an odd function, thus $x^2 \sin nx$ is an odd function Hence.

$b_n = 0$. for every n.

$$a_n = \frac{2}{\pi}[\int_{-\pi}^{\pi} \frac{x^2 \sin nx}{n} + \frac{2x\cos x}{n^2} - \frac{2\sin nx}{n^3}]^{\pi}{}_0 \text{ for } n \neq 0$$

From which

$$a_n = \frac{2}{\pi}[\frac{2\pi\cos n\pi}{n^2}] = \frac{4(-1)}{n^2}, \qquad n = 1,2,........$$

$$a_n = \frac{2}{\pi}[\frac{2\cos\pi}{n}] = \frac{4(-1)^{\pi}}{n^2} \qquad n = 1,2,........$$

for $n = 0$

$$a_0 = \frac{2}{\pi}[\int_0^{\pi} x^2 dx = \frac{2}{\pi}\frac{\pi^3}{3} = 2\frac{\pi^2}{3}$$

Therefore, in the interval; $-\pi < x < \pi$

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} = \frac{(-1)^n \cos nx}{n^2}$$

Indeed, because of condition of line function involved, we write

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} = \frac{(-1)^n \cos nx}{n^2}, \text{for } -\pi \leq x \leq \pi.$$

2. Fourier Sine Series: Sometimes it is desirable to expand the function $f(x)$ in a series involving Sine function only.

In order to get a Sine series for $f(x)$ we introduced a function $g(x)$ defined as follows

$$g(x) = f(x) \qquad 0 < x < c$$
$$= -f(-x) \qquad -c < x < 0$$

Thus $g(x)$ is an odd function over the interval $-c < x < 0$.

Hence

$$g(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{c} dx = 0, \qquad n = 0,1,......$$

It follows that

$$a_n \frac{1}{c_0} \int_{-c}^{c} g(x)\cos\frac{n\pi x}{c} dx = 0 \qquad n = 0,1,......$$

(Note integrand is an odd function) and that

$$b_n = \frac{1}{c}\int_{-c}^{c} g(x)\sin\frac{n\pi x}{c} dx$$
$$= \frac{2}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}$$

Thus

$$f(x) \sim +\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c} \qquad o < x < c$$

$f(x)$ Where $b_n = \frac{2}{c_0}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c} dx \qquad n = 1,2,.................$

Example: Expand $f(x) == x^2 =$ in a Fourier Sine Series over the interval $0 < x < 1$

Solution: At once we write, for $0 < x < 10$

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin n\pi x$$

In which

$$b_n = \int_{-c}^{c} x^2 \sin n\pi x dx$$
$$= 2\int_0^1 x^2 \frac{\cos n\pi x}{n\pi} + \frac{2x\sin n\pi x}{(n\pi)^2} + \frac{2\cos n\pi x}{(n\pi)^3}$$

$$= 2[-\frac{\cos n\pi}{n\pi} + \frac{2\cos \pi x}{n^3\pi^3} + \frac{2\cos n\pi x}{n^3\pi^3}$$

Hence the Fourier Sine Series, over $0 < x < 1$ for is x^2

$$= 2[-\frac{\cos n\pi}{n\pi} + \frac{2(-1)^n}{n^3\pi^3}\frac{2}{n^3\pi^3}]$$

3. Fourier Cosine Series: In order to expand the function $f(x)$ in a series involving cosine function only, such series is called Fourier Cosine Series. We define

$$h(x) = f(x) \qquad 0 < x < c$$
$$= f(-x) \qquad -c < x < 0$$

It follows that $h(x)$ is an even function of x.

$$h(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}$$

$$a_n \frac{1}{c}\int_{-c}^{c} h(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

$0 < x < c$

But $b_n = \frac{i}{c}\int_{-c}^{c} h(x)\sin\frac{n\pi x}{c}dx = 0$

Thus we have

$$f(x) \sim \frac{1}{2}a_0 + \int_{0}^{c} (f(x)\cos\frac{n\pi x}{c}dx$$

in which

$$a_n = \frac{2}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

Example:

Solution: At once we have

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$ in which

in which

$$a_n \frac{2}{c}\int_{0}^{c} \cos\frac{n\pi x}{c}dx$$

$$= \frac{2}{c}[\frac{c.}{n\pi}x\sin\frac{n\pi x}{c} + (\frac{c}{n\pi})^2 \cos\frac{n\pi x}{c}]_0^c$$

$$= \frac{2}{c}[(\frac{c.}{n\pi})^2 \cos n\pi - \frac{n\pi x}{c}]_0^{\ c}$$

$$= \frac{-2}{n^2\pi^2}(1 - \cos n\pi),\ n \neq 0$$

The coefficient a_0 is readily obtained

$$a_n \frac{2}{c}\int_{-c}^{c} xdx = \frac{2}{c}\frac{c^2}{2} = c$$

Thus the Fourier Cosine Series over the interval $0 < x < c$ the function $f(x) = x$ is

$$f(x) \sim \frac{1}{2}c - \frac{uc}{\pi^2}\sum_{k=1}^{\infty} \frac{1-(-1)}{n^2}\cos\frac{n\pi x}{c}$$

4.0. Conclusion: You have learnt about indicial equations where the roots are positive and logarithmic. You have also learnt about Fourier series and odd functions.

5.0 Summary: You are required to study materials in this unit very well before proceeding to the next units.

6.0 Tutor Marked Assignments:

(1). Find the general series solution of the D.E

$$4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$$

(2).Construct the Fourier series, over the interval $-2 \leq x \leq 0,$ for the function defined by

$f(x) = 2,\qquad -2 \leq x \leq 0,$

$x = 2\ \ ,\qquad 0 < x < 2$

7.0 REFERENCES/FURTHER READINGS

1. EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India

2. FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey

3. EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London.

UNIT 4: BOUNDARY VALUE PROBLEMS

1.0 Introduction

In this unit, we will discuss some of the properties of boundary valve problems for linear second order equation. This class of differential equations is very useful for practical applications. We shall devote some time in studying them in this unit.

2.0 Objectives: At the end of this unit you should be able to

- classify second order differential equations into homogeneous and non-homogeneous.
- differentiate between eigen values and eigen functions
- solve related eigen value problems

3.0 MAIN CONTENT

3.1 Boundary Value Problems

The linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = g(x) \qquad (1)$$

was classified homogeneous if, $g(x) = 0$, and non-homogeneous otherwise.

Similarly, a linear boundary condition

$$a_1 y(0) + a_2 y''(0) = c \qquad (2)$$

A boundary value problem is homogeneous if both its differential equation and in boundary conditions are homogeneous. If not then it is non-homogeneous.

A typical linear homogeneous second order boundary value problem is of the form.

$P(x)y'' + Q(x)y' + R(x)y = 0$ (3)

$0 < x < 1$,

$a_1 y(0) + a_2 y''(0) = 0$ (4)

$b_1 y(I) + b_2 y'(0) = 0$ (5)

Most of the problems, we will discuss are of the form given by (3) to (5).

3.2. Eigen Values and Eigen Functions

Consider the differential equation

$y'' + p(x,\lambda)y' + q(x,\lambda)y = 0 \qquad 0 < x < 1$ (1)

The boundary conditions

$a_1 y(0) + a_2 y'(0) = 0$ (2)

$b_1 y(I) + b_2 y'(0) = 0$ (3)

Where λ is arbitrary parameter.

Clearly the solution of (1) depends on x and λ and can be written as

$y = c_1 y_1(x,\lambda) + c_2(x,\lambda)$, (4)

Where y_1 and y_1 are a fundamental solution of (1). Substituting for y_1 in the boundary condition (2) and (3), yield.

$c_1[a_1 y_1(o,\lambda) + a_2 y'(o,\lambda)] + c_2[a_1 y_2'(x,\lambda)] = 0$ (5)

$c_1[b_1 y_1(i,\lambda) + b_2 y'(i,\lambda)] + c_2[b_2 y_2'(i,\lambda)] = 0$ (6)

A set of two linear homogeneous algebraic equations for the constant. Such a set has solutions (other than $c_1 = c_2 = 0$ if and only if the determinant of coefficients $D(\lambda)$ vanishes i.e.

$$D(\lambda) = \begin{vmatrix} c_1[a_1 y_1(o,\lambda) + a_2 y'(o,\lambda)] & a_1 y_2(o,\lambda) + a_2 y_2'(o,\lambda) \\ b_1 y_1(i,\lambda) + b_2 y'(i,\lambda)] & b_1 y_2(i,\lambda) + b_2 y_2'(i,\lambda)] \end{vmatrix} = 0$$

Values satisfying this determinant equation are the eigenvalues of the boundary- value problems (1), (2) and (3)

Corresponding to each Eigen value is at least one non- trivial solution. An E**igen function**

Note: We will consider problems namely only real eigen value
Example I consider the equation

$$y'' + \lambda y = 0 \qquad (1)$$

$$y(0), \qquad y(1) = 0 \qquad (2)$$

Solution: $y'' + \lambda y = 0$, the solution is

$$y = c_1 \cos\sqrt{x} + c_1 \sin\sqrt{x}\,, \qquad (3)$$

By the boundary conditions

$c_1 = 0$ s

$c_2 \sin\sqrt{\lambda} = 0$

$c_2 \neq 0$, otherwise $y = 0$ is the solution

$\sin\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = n\pi\,,\ n = 1,2,................$

or $\qquad \lambda = n\pi^2, 4n\pi^2 \qquad (4)$

(4) gives the eigen values of (1). If we consider $\lambda = 1$

$\therefore y = c_1 \cos x + c_2 \sin x$

$0 = c_1$ by (2)

$\therefore y = c_2 \sin I = 0 \Rightarrow c_2 = 0$

$0 = c_1$ by (2)

$\therefore y = 0$ Hence $\lambda = 1$ is the eigen -function

The eigen function are

$\therefore y_n = c_2 \sin n\pi x$ .... (5)

$n = 1,2,..........c_2$ .... c_1 is an arbitrary constant

Example 2: Find the real eigen values eigen -function of the boundary value problem

$y'' + \lambda y = 0$

$y(0) = 0 \qquad y'(l) = 0$

.The solution is

$y'' + \lambda y = 0$

$\therefore y = c_1 \cos\sqrt{\lambda} x \qquad (1)$

$y(0) = 0$, given

$c_1 = 0$. Also

$y' = c_2 \cos\sqrt{\lambda} \cos\sqrt{\lambda x}$

But $y' = (l) = 0$, yields

$c_2 = \sqrt{\lambda} \cos\sqrt{\lambda l = 0} \Rightarrow$

$c_2 = \sqrt{\lambda} \cos l = 0$

$\sqrt{\lambda} l \frac{(2n-1)\pi}{2l}, \qquad n = 1,2,\ldots\ldots\ldots\ldots$

(2) gives eigen value

$y_n = \sin[\frac{2n-1)\pi x}{2l}], \qquad n = 1,2,\ldots\ldots\ldots\ldots \qquad (2)$

(3) gives eigen functions

Examples $y'' + \lambda y = 0$

$y' = (0) = 0 \qquad y' = (1) = 0$

Solution: The solution is

$$y' = c_1 \cos\sqrt{\lambda} x + c_2 c \sin\sqrt{\lambda x} \qquad (1)$$

$$y' = -c_1 \sin\sqrt{\lambda} x + c_2 \sqrt{x} \cos\sqrt{\lambda x} \qquad (2)$$

$$c_1 + c_2 \mu = 0$$

$$\begin{vmatrix} 1 & \mu \\ \cos\mu & \cos\mu \end{vmatrix} = 0$$

Thus the eigen value are given by the equation

$$\mu = \tan\mu . \qquad (3)$$

$y = -c_2 \mu + \mu_n c_2 \sin \mu x$

If μ_n is the root of (3), then eigen function is

$y_n = \sin\mu_n - \mu_n \cos\mu_n x \qquad (4)$

If $\lambda = 0$, then the solution is

$y = -c_2 + c_{2}$ y

$y' = c_1$

$\therefore c_1 + c_2 = 0$

Hence the solution is

$y = c_1(x-1)$

thus $\lambda = 0$ is also an eigen value

$\sqrt{\mu} = \tan\sqrt{\mu}$

$\sqrt{\lambda_1} \sim 4.49, \quad \sqrt{\lambda_N} \sim \frac{(2n+1)\pi}{2}$

Example 5.

$$y'' + \lambda y = 0$$

$$y = 0, \qquad y(1) + y'(0) = 0$$

Solution

$$y = c_1 \cos\sqrt{\lambda} \sin\sqrt{\lambda}x + c_2 + \sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$c_2 = 0,$$

$$\therefore \sqrt{\lambda} = \cot\sqrt{\lambda}$$

The eigen values are given by equation (3). The eigen function are

$y_n = \sqrt{\lambda}_n x$ Where the root of is λ_n is the root of the equation

$\sqrt{\lambda} = \cot\sqrt{\lambda}$

$\sqrt{\lambda} = x, \qquad y = \cot x$

$\sqrt{\lambda}_n = (n-1)\pi \qquad n \geq 2,3,$...........

$\lambda_n = (n-1)^2\pi^2$ for large n

Example 6.

Consider the problem

$y'' + \lambda y = 0$

$y = (0), \qquad y'(0) = 0$

Show that if ϕ_m ,and ϕ_n are eigen function corresponding to the eigen value λ_m and λ_n Respectively, then

$$\int_0^l \phi_m(x)\phi(x)dx = 0$$

Provided that $\lambda_m \neq \lambda_n$.

Solution:

$$\phi_m'' + \lambda_m\phi_m = 0$$
$$\phi_n'' + \lambda_m\phi_m = 0$$
$$\phi_n'' + \lambda_m\phi_n = 0$$

$$\phi_m''\phi_n + \lambda_m\phi_m\phi_n = 0 \qquad (1)$$
$$\phi_n''\phi_m + \lambda_n\phi_n\phi_n = 0 \qquad (2)$$

$$\int_0^l \phi_m''(x)\phi_n''(x)dx + \lambda_m\int_0^l \phi_m\phi_n dx = 0 \qquad (3)$$
$$\int_0^l \phi_n''\phi_m'' dx + \lambda_n\int_0^l \phi_n\phi_m dx = 0 \qquad (4)$$
$$\int_0^l \phi_m'(x)\phi_n''(x) - \int_0^l \phi_m'(x)\phi_n(x)dx + \lambda_m\int_0^l \phi_m\phi_n dx = 0$$
$$\phi_m'(l)\phi_n''(l) - \phi_m'(o)\phi_n'(0) - \phi' - [\phi_m'(x)\phi_n'(x) - \int_0^l \phi_m'(x)\phi_n''(x)dx] + \lambda_m\int_o^l \phi_m'(x)\phi_n''(x)dx$$

or

By boundary value conditions

$$\int_0^l \phi_m(x)\phi_n''(x)dx + \lambda_m\int_0^l \phi_m(x)\phi_n(x)dx = 0 \qquad (5)$$

Subtract (4) from (5), we have

$$(\lambda_n - \lambda_m\int_0^l \phi_m(x)\phi_n(x)dx = 0$$

If $\lambda_n \neq \lambda_m$, then

$$\int_0^l \phi_m(x)\phi_n''(x)dx = 0$$

Example 14.

Hyperbolic function

$\cos x\dfrac{e^x e^{-x}}{2}$, $\qquad$ $\sinh\dfrac{e^x - e^{-x}}{2}$

$$\frac{d}{dx}(\cosh) = \sinh(x)$$
$$\frac{d}{dx}(\sin x) = \cosh h(x)$$

(a) Solution of the problem is

$r^4 - \lambda = 0$, Take $\lambda = \mu^4$

$r^4 - \mu^4 = 0$

The solution is

$$y = c_1 \cos \mu x + c_2 \sin \mu x + c_3 \mu^2 \cos \mu x + c_2 \sinh \mu x \qquad (1)$$

The boundary condition

$c_1 + c_3 = 0$

$c_1 - c_3 = 0$

$\Rightarrow c_1 = 0$ and $c_3 = 0$

$\therefore y = c_2 \sin \mu + c_4 \sin \mu l = 0 - c_2 \sin \mu l + c_4 \sinh \mu l = 0$

$\therefore y = c_2 \sin \mu + c_4 \sin \mu l = 0 - c_2 \sin \mu l + c_4 \sinh \mu l = 0$

$\therefore y = c_2 \sin \mu + c_4 \sin \mu l = 0 - c_2 \sin \mu l + c_4 \sinh \mu l = 0$

$\therefore y = c_2 \sin \mu + c_4 \sin \mu l = 0 - c_2 \sin \mu l + c_4 \sinh \mu l = 0$

$\therefore \sin \mu l = 0 \sinh \mu l = 0 \qquad n = 1, 2, \ldots\ldots\ldots\ldots$

$\therefore \sin \mu l = n\pi$

$\therefore y_n = \sin \frac{n\pi}{l} x \qquad n = 1, 2, \ldots\ldots\ldots\ldots$

Is the eigen-function

4.0 Conclusion

We have been able to study some eigen-value problems in this unit. This unit must be mastered properly before moving to the next unit.

5.0 Summary

Recall that the linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = g(x) \qquad (1)$$

was classified homogeneous if, $g(x) = 0$, and non-homogeneous otherwise.

Similarly, a linear boundary condition

$$a_1 y(0) + a_2 y''(0) = c \qquad (2)$$

A boundary value problem is homogeneous if both its differential equation and in boundary conditions are homogeneous. If not then it is non-homogeneous. We also classified some equations into eigen value problem depending upon whether the determinant of the eigen value of the problem is zero or not. Read carefully and re work all exercises and problems in this unit for better understanding.

6.0 Tutor Marked Assignment:

1. Consider the problem

$y'' + \lambda y = 0$

$y = (0)$, $\qquad y'(0) = 0$

Show that if ϕ_m ,and ϕ_n are eigen function corresponding to the eigen value λ_m and λ_n Respectively, then

$\int_0^l \phi_m(x)\phi(x)dx = 0$

Provided that $\lambda_m \neq \lambda_n$.

2. Find the real eigen- values and eigen -function of the boundary value problem

$y'' + \lambda y = 0$

$y(0) = 0 \qquad y'(l) = 0$

7.0 REFERENCES/FURTHER READINGS

1. EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India
2. FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey
3. EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London.

Module 2: Sturm Liouville Boundary Value Problems and Special Functions

UNIT1: Sturm and Liouville Problem

1.0. Introduction

We solved some partial differential equations by the method of separation of variables. In the last step we expanded a certain function in a Fourier series, i.e. as the sum of an infinite series of sine and cosine functions. It is of fundamental importance that the eigen functions of a more general class of boundary values problems can be used as a basis for series expansions, which have properties similar to Fourier Series.

Such eigen- functions series are useful in extending the method of separation of values to a larger class of problems in partial differential equation.
The class of boundary value problem we will discuss is associated with the names of Sturm and Liouville.

2.0 Objectives:

After studying this unit you should be able
- to solve partial differential equation using Sturm and Liouville methods
- solve correctly the associated Tutor Marked Assignments

3.0 MAIN CONTENT

3.1. Sturm and Liouville Problem

We introduce the operator

$$L[y] = -[p(x)y'] + q(x)y \qquad (1)$$

$$L[y] = \lambda r(x)y \qquad (2)$$

$[P(x)y']' - q(x)y + \lambda r(x)y = 0$ (3)

on the interval $0 < x < 1$, together with the boundary condition

$a_1 y(0) + a_2 y'(0) = 0$ (4)

$b_1 y(I) + b_2 y'(1) = 0$ (5)

We shall assume that p, q and r are continuous functions in the interval $[0,1]$. $P(x) > 0, r(x) > 0$ for all x in $0 \le x \le 1$.

(i) Lagrange's identity: let u and v be functions having continuous second derivatives on the interval $0 \le x \le 1$. Then

$$\int_0^1 (UL[U] - UL[u])dx$$
$$= -p(x)[u'(x)u(x) - u(x)u'(x)] \quad (6)$$

Solution 1:

$$\int_0^1 UL[U]dx \int_0^1 \{-u[p(x)u'q(x)u\}dx$$

$$= -u(py') + u'p(x)u(x) + \int_0^1 \{-u(pu') + uqu\}dx$$

$$\therefore \int_0^1 (uL[u] - UL[U])dx = -(px)[u'(x)u(x) - u(x)u'(x)$$

This is known as Lagrange's identity if u and u satisfy (5) and (4)

R.H.S = $- p(1)[u'(1)u(1) - u(1)u'(1)]$

$+ p(0)[u'(0)u(0) - u(0)u'(0)]$

$- p(1)[-\frac{b_1}{b_2}u(1)u(1) + \frac{b_1}{b_2}u(1)u(1)]$

$+ p(0)[-\frac{a_1}{a_2}u(0)u(0) + \frac{a_1}{a_2}u(0)u(0)]$

$= 0$

Thus we have

$$\int_0^1 \{u[u] - uL[u]dx = 0$$

(ii) Show that all the eigen value of the Sturm-Liouville problem

$L(y) = \lambda r(x) y$ A

With boundary conditions

$$\left.\begin{aligned} a_1 y(0) + a_2 y'(0) = 0 \\ b_1 y(I) + b_2 y'(1) = 0 \end{aligned}\right\} \quad B$$

are real.

Proof: let us suppose there exists a complex eigen value $\lambda = \mu + iv$ will $v \neq 0$ and corresponding to this value is the eigen function $Q(x) = U(x) + IV(x)$ Where at least one of them is not identically zero.

Now Q satisfies the differential equation

$$L[Q) = \lambda r Q$$
$$L[Q] = \pi R Q$$

or

$u = Q$ and $u = Q$

$\int_0^1 \{QL(Q) - QL(Q)]dx = \int_0^1 (\lambda - \lambda')$

$r(x)Q(x)Q(x)dx = 0$

or $2r\int_0^1 r(x)[V^2(x) + V^2(x)dx = 0$ ………. (1)

Since $r(x) > 0$ for all x in $0 \leq x \leq 1 (1) \Rightarrow v = 0$

This contradicts the original hypothesis. Hence the eigen value of Sturm-Liouville problem are real.

(iii) If Q_1 and Q_2 are eigenvalues of the Sturm-Liouville problem (A) and (B), corresponding to eigen valves λ_1 and λ_2, respectively , and $\lambda_1 + \lambda_2$, then

$$\int_0^1 r(x)Q_1(x)Q_2(x)dx = 0$$

[$r(x)$ is called the weight function and it is an orthogonal property of eigenfunction]

Proof: - $L[Q_1]\lambda_1 r Q_1$

$$L[Q_2]\lambda_2 rQ_2$$

If we let = $U = Q_1$ and $U = Q_2$ then

$$\int_0^1 \{uL[U] - UL[U]\}dx$$

$$\lambda_1 - \lambda_2 \int_0^1 r(x)Q_1(x)Q_2(x)dx = 0$$

Hence the result

(iv) Let us now consider a more general boundary value problem for the differential equation

$$L[y] = \lambda M[y], \qquad 0 < x < 1$$

Where L and M are linear homogeneous differential operations of orders n and n respectively.

$$L[y] = p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + ... + p_{n-1}(x)y' + p_n(x)y$$

$$M[y] = r_0(x)y^{(m)} + r_1(x)y^{(m-1)} + ... + 2_{m-1}(x)y' + r_m(x)y$$

Where $n > m$.

In addition to the differential equation a set of n linear homogeneous boundary conditions at $x = 0$, $x = 1$is also prescribed. If the relations

$$\int_0^1 (uL[u] - uL[u]dx = 0$$

$$\int_0^1 (uM[u] - uM[u]dx = 0$$

are line for every pair of functions u and u, which are $n-$lines continuously differentiable on $0,l$ and which satisfy un given boundary conditions, then the given boundary valve problem is said to be <u>self adjoint.</u>

Problem I. Show that the Sturm-Liouville problems

$$L(y) - [P(x)y'] + q(x)y$$

$$M(y) = \lambda r(x)y$$

(i) $\int_0^1 [UM[u] - uM[u]dx$

$\int_0^1 [U\lambda r(x)u - u\lambda r(x)u]dx$

$= 0$

For every pair of u, u

$$\int_0^1 [UL[u]-uL[u]dx = 0$$

as shown previously. Hence it is self-adjoint

Problem

(a) $y'' + y' + 2y = 0$ $y = 0$, $y = (I) = 0$

Solution $L(y) = y'' + y' + 2y$

(i) $\int_0^1 [U(u'' + u' + 2u) - u(u'' + u' + 2u)dx$

$= -2\int_0^1 u'udx,$ are true for every pair of function u and u, which are n-times continuously differentiable on $[o,l]$ which satisfy un given boundary value problem is said to be self-adjoint.

Solution: Sturm-Liouville problems

$L(y) = -[p(x)y'] + q(x)y$

$M(y) = \lambda r(x)y$

(i) $\int_0^1 [UM(u) - uM(u)]dx$

$\int_0^1 [U\lambda r(x) - u\lambda r(x)]dx$

$= 0$

as shown previously. Hence it is not self-adjoint.

Problem

$y'' + y' + 2y = 0$ $y = 0$, $y = (I) = 0\backslash$

Solution $L(y) = y'' + y' + 2y$ $y = 0$, $y = (I) = 0$

$\int_0^1 [U(u'' + u' + 2u) - u(u'' + u' + 2u)]dx$

$\int_0^1 [UL(u) - uL(u)]dx = 0$

as shown previous. Hence it self-adjoint

Problem (a)

Problem

$y'' + y' + 2y = 0$ $y = 0$, $y = (I) = 0\backslash$

Solution $L(y) = y'' + y' + 2y$ $y = 0$, $y = (I) = 0$

$\int_0^1 [U(u'' + u' + 2u) - u(u'' + u' + 2u)]dx$

$= -2\int_0^1 u'u dx = 0$, u' and u' are continuous in the interval $0 \le x \le 1$. Hence it is not zero.

Thus it is not self-adjoint.

Problem(b)

$(x + x^2)y'' + 2xy' + y = 0$ $\quad$ $y' = 0$, $y = (1) + 2y'(1) = 0$

$L(y) = (1 + x^2)U'' + 2xy'' + y = 0$, $y' = (0) = 0$ $y = (1) + 2y'(1) = 0$,

$L(y) = (1 + x^2)y'' + 2xy' + y$

$M(y) = 0$

$\int_0^1 [U[(1 + x^2)u'(u'' + u' + 2u)]dx = 0$

It is Sturm-Liouville problem.

(c) $y'' + y = \lambda y,$ $\quad$ $(y) = 0 - y'(0) - y'(1) = 0$

$y' = 0 - y'(0) - y'(1) = 0$

Solution

$L(y) = (1 + x^2)y'' + 2xy' + y$

$M(y) = y$

(i)

$\int_0^1 (uM(\mu) - UM(\mu)dx$

$\int_0^1 (uu - uu)dx = 0$

(ii) $\int_0^1 (u''u) - U(\mu'')dx]$

$= \int_0^1 (uu'' - u''u)dx$

$= uu' - uu'$

$= [u(1)u'(1) - u(1)u'(1)$

$-[u(1)u(0) - u(1)u'(0)]$

$= [u(1)u(0) - u(I)u(0)$

The right side is not zero. Hence it is not self-adjoint.

Problem 3 Consider the differential equation

$y'' + \lambda y + 2y = 0$ \

With boundary conditions

$y = (0) - y(I) = 0\,,\quad y' = (0) - y'(1) = 0$

(a) Show that the problem is self-adjoint even though it is not a Sturm-Liouville problem.

(b) Find all eigenvalues and eigenfunctions of the given problem

Solution: $L(y) = y''$

$M(y) = -y$

(i) $\int_0^1 [UM(u) - u(-u)dx] = 0$

$\int_0^1 [U(-u) - u(-u)]dx = 0$

(ii) $\int_0^1 [u(uu'' - uu'']dx$

$= (uu') - \int_0^1 (u'u'dx - u'u + \int_0^1 u'u'dx$

$= [(1)u'(I) - u'(I)u(I)]$

$-[(0)u'(0) - u'(0)u(0)]$

$-[(0)u'(0) - u'(0)u(0)]$

$-[(0)u'(0) - u'(0)u(0)] = 0$

Hence it is self-adjoint

The solution of the equation is

$y = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{\lambda}x$

Applying the boundary conditions, we have

$c_1\sqrt{\lambda}\sin\sqrt{\lambda} + c_2 \sin\sqrt{\lambda}(1 - \cos\sqrt{\lambda} = 0$

$c_1(\cos\sqrt{\lambda} - 1 + c_2 \sin\sqrt{\lambda} = 0$

Thus

$$\begin{vmatrix} \sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}(1-\cos\sqrt{\lambda} \\ \cos\sqrt{\lambda} - 1 & \sin\sqrt{\lambda} \end{vmatrix} = 0$$

Or

$\sqrt{\lambda}(1 - \cos\sqrt{\lambda} = 0 \Rightarrow$

$\lambda = 0$ or $\lambda = (2n - \pi)^2$, $n = 1,2,\ldots\ldots\ldots$

$\lambda_0 = 0$ $\varphi_0(x) = 1$

$\lambda_n = (2n - \pi)^2$

$\therefore Q_n(x) = \cos 2n\pi x, \quad \therefore Q_n(x) = \sin 2n\pi x,$

$y_1 = \cos 2n\pi x, \qquad y_2 = \sin 2n\pi x,$

$y_1 = \cos 2n\pi x, \qquad y_2 = 2n\pi x \cos 2n\pi x,$

$$W(y_1 y_{2)} = \begin{vmatrix} \cos 2n\pi x & \sin 2n\pi x \\ -2n\pi \sin n\pi x & 2n\pi \cos 2n\pi x \end{vmatrix}$$

$2n\pi x \cos^2 2n\pi x + 2n\pi \sin^2 n\pi x$

$2n\pi \neq 0 \qquad 0 \leftarrow x$

Between $0 \leq x \leq 1$

Thus the eigen functions are lining independent.

Problem 5:

Consider the Sturm-Liouville problems

$-[p(x)y'] + q(x)y = \lambda r(x)y$ a1

$a_1 y(0) + a_2 y'(0) = 0, b, y(1) + b_2 y'(1) = 0$

Where p, q and z continuous function in the interval $0 \leq x \leq 1$.

(a) show that if λ is an eigen-value and φ a corresponding eigen function, then

$$\lambda \int_0^1 eQ^2 dx = \int_0^1 (p\phi'^2 + q\phi^2)dx + \frac{b_1}{b_2} p(1)\phi^2(1) - \frac{a_1}{a_2} p(0)\phi^2(0)$$

Provided that $a_2 \neq 0$ and $b_2 \neq 0$ How this result be modified if $a_2 = 0$ or $b_2 = 0$

(b) Show that if $q(x) \geq 0$ and if $\frac{b_1}{b_2}$. and $\frac{-a_2}{a_1}$ are non-negative, then the eigen-value λ Is non negative

(c) Show that the eigen-value λ is strictly $0 \leq x \leq 1$ under q(x) = 0 for each x in 0,x,1

Solution

$\lambda r(x) Q^2 = -[p(x)Q']'Q + q(r)Q^2$

Thus

$\lambda\int_0^1 r(x)Q^2dx = \int_0^1 (qQ^2 - p(x)Q')Q)dx$

Integrating by parts, we have

$\int_0^1 qQ^2dx - Q[p(Q'] + \int_0^1 pQ'^2)dx$

From boundary condition, we have obtain the result

$Q'(1) = \frac{b_1}{b_2}Q'(1)$

$Q'(0) = -\frac{a_1}{a_2}Q'(0)$

Putting these values on the right side and we obtain the result

if or $a_2 = 0$ or $b_2 = 0$, then the first boundary condition reduces to

$y = 0$

$\Rightarrow Q(1) = 0$ or $\Rightarrow$ $Q(o) = 0$

The result reduces to

$\lambda\int_0^1 rQ^2dx = \int_0^1 (qQ^2 + pQ'dx\frac{b_1}{b_2}p(1)Q^2(I)$

or

$\lambda\int_0^1 rQ^2dx = \int_0^1 (qQ^2 + pQ'^2dx\frac{a_1}{a_2}Q^2(0)Q^2(I)$

(b) In a Sturm Lioville problem, we always assume that $p(x) > 0,\ r(x) > 0,$

By given condition $r(x) > 0$ for all x in $0 \le x \le 1$ $Q^2 > 0$ for all $0 \le x \le 1$.

Now we impose condition, so that right side of the equation in (a) is $+ve$ The second and third term are $+ve$ y $\frac{b_1}{b_2}$ and $\frac{-a_1}{a_2}$ are non-negatives

Now

$\int_0^1 qQ^2dx$ is $+ve$ in order that

(c) If $q(x) = 0$ for all x $0 \le x \le 1$ then λ is strictly.

4.0 Conclusion

We have studied the Sturm Lioville problem in this unit . You are to master this unit properly so that you will be able to solve the problems that follow.

5.0 Summary

Recall that Sturm Lioville problems are usually problems associated with eigen values problems of partial differential equations which we have dealt with in this unit. In our subsequent course in mathematics in this Programme, we will have cause to deal with it again particularly when will shall study Partial Differential Equation.

6.0 Tutor Marked Assignment

Consider the problem

$y'' - 2y' + (1+\lambda)u = 0$

$Y(0) = 0, y(1) = 0$

1) Show that this problem is not self-adjoint
2) Show that all eigenvalues are real
3) Show that the eigenfunctions are not orthogonal. (with respect to the weight function arising from the coefficients of …….. in the differential equation.

7.0 REFERENCES/FURTHER READINGS

1. EARL. A. CODDINGTON: An Introduction to Ordinary Differential Equations. Prentice-Hall of India
2. FRANCIS B. HILDEBRAND: Advanced Calculus for Applications, Prentice-Hall, New Jersey
3. EINAR HILLE: Lectures on Ordinary Differential Equations, Addison – Wesley Publishing Company, London.