



NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

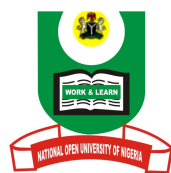
COURSE CODE: MTH303

COURSE TITLE: VECTORS AND TENSORS ANALYSIS



MTH303
VECTORS AND TENSORS ANALYSIS

Course Team Dr. Bankole Abiola (Developer/Editor) - NOUN
 Dr. J. O. Adeniran (Writer) - FUNAAB



NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
No. 5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e-mail: centralinfo@nou.edu.ng

URL: www.nou.edu.ng

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Introduction

Vector and Tensors (MTH303), requires the knowledge acquired in (MTH142-Vector and Geometry) which you studied at the 100-Level. A good mastery of the course content in Mathematical Methods 1 (MTH281) will be helpful in learning this course successfully.

This is a three-credit unit course. It is a compulsory course for all students majoring in mathematics at undergraduate level or B.Sc. (Education) Mathematics. It is also available to students offering the Bachelor of Science (B.Sc.) Computer and Information & Communication Technology. Any student with sufficient background in mathematics can also offer the course if he/she so wish though it may not count as credit towards graduation if it is not a required course in his/her field of study.

The course is divided into two modules as enumerated below:

Module 1

- Unit 1 Elementary Vector Algebra
- Unit 2 Vector Differentiation (Gradient, Divergence and Curl)
- Unit 3 The Line Integral

Module 2

- Unit 1 Green's Theorem, Divergence Theorem and Stokes's Theorem
- Unit 2 Tensor Analysis

What You Will Learn in This Course

This course guide tells you briefly what the course is about, what course materials you will be using and how you can work with these materials. In addition, it advocates some general guidelines for the amount of time you are likely to spend on each unit of the course in order to complete it successfully.

It gives you guidance in respect of the Tutor-Marked Assignment which will be made available in the assignment file. There will be regular tutorial classes that are related to the course. It is advisable that you attend these tutorial sessions. The course will prepare you for the challenges you will meet in Vectors and Tensors

Course Aim

The main aim of the course is to provide you with an understanding of Vectors and Tensors. It also aims to make clear distinctions between the ways we handle problems in higher dimensional spaces, and provide solutions to some problems that may arise in engineering, physics, and other areas of human endeavour, where the knowledge of advance vector is required.

Course Objectives

To achieve the aims set out, the course has a set of objectives. Each unit has specific objectives which are included at the beginning of the unit. You should read these objectives before you study the unit. You may wish to refer to them during your study to check on your progress. You should always look at the unit objectives at the completion of each unit. By doing so, you would have followed the instructions in the unit.

Below are comprehensive objectives of the course as a whole. By meeting these objectives, you should have achieved the aims of the course as a whole. In addition to the aims above, this course sets to achieve some objectives. Thus, after going through the course, you should be able to:

- prove continuity and establish limit functions in Vectors and Tensors
- define Divergence, Gradient of Scalar Functions, and Curl of Vectors
- establish Green's Theorem, Stokes's Theorems and Divergence Theorem
- work out answers to simple problems in Tensor Analysis.

Working through This Course

To complete this course you are required to read each study unit, read the textbooks and read other materials which may be provided by the National Open University of Nigeria.

Each unit contains self-assessment exercises and at certain points in the course you would be required to submit assignments for assessment purposes. At the end of the course there is a final examination. The course should take you about a total of 16 weeks to complete. Below you will find listed all the components of the course, what you have to do and how you should allocate your time to each unit in order to complete the course in time and successfully.

This course entails that you spend a lot of time to read and practice all related exercises. I will advise that you avail yourself of the opportunities of the tutorial classes provided by the university.

Presentation Schedule

Your course materials have important dates for the early and timely completion and submission of your TMAs and attending tutorials. You should remember that you are required to submit all your assignments by the stipulated time and date. You should guard against falling behind in your work.

Assessment

There are three aspects to the assessment of the course. The first is made up of self-assessment exercises, second consists of the Tutor-Marked Assignments and third is the written examination/end of course examination.

You are advised to do the exercises. In tackling the assignments, you are expected to apply information, knowledge and technique you gathered during the course. The assignments must be submitted to your facilitator for formal assessment in accordance with the deadlines stated in the presentation schedule and the assignment file. The work you submit to your tutor for assessment will count for 30% of your total course work. At the end of the course you will need to sit for a final or end of course examination of about three hour duration. This examination will count for 70% of your total course mark.

Tutor-Marked Assignment

The TMA is a continuous assessment component of your course. It accounts for 30% of the total score. You will be given 4 TMAs to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMAs would be given to you by your facilitator and returned after you have done the assignment. Assignment questions for the units in this course are contained in the assignment file. You will be able to complete your assignment from the information and material contained in your reading, references and study units. However, it is desirable in all Degree level of education to demonstrate that you have read and researched more into your references, which will give you a wider view point and may provide you with a deeper understanding of the subject.

Make sure that each assignment reaches your facilitator on or before the deadline given in the presentation schedule and assignment file. If for

any reason you cannot complete your work on time, contact your facilitator before the assignment is due to discuss the possibility of an extension. Extension will not be granted after the due date.

Final Examination and Grading

The end of course examination for MTH303 (Vectors and Tensors) will be for about 3 hours and it has a value of 70% of the total course work. The examination will consist of questions, which will reflect the type of self-testing, practice exercise and tutor-marked assignment problems you have previously encountered. All areas of the course will be assessed.

Use the time between finishing the last unit and sitting for the examination to revise the whole course. You might find it useful to review your self-test, TMAs and comments on them before the examination. The end of course examination covers information from all parts of the course.

Course Marking Scheme

Assignment	Marks
Assignments 1-4	Four assignments, best three marks of the four count at 10% each -30% of course marks.
End of course examination	70% of overall course marks.
Total	100% of course materials.

Facilitators/Tutors and Tutorials

There are 16 hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials as well as the name and phone number of your facilitator, as soon as you are allocated a tutorial group.

Your facilitator will mark and comment on your assignments, keep a close watch on your progress and any difficulties you might face and provide assistance to you during the course. You are expected to send your Tutor-Marked Assignment to your facilitator before the schedule date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not delay to contact your facilitator by telephone or e-mail if you need assistance.

The following might be circumstances in which you would find assistance necessary, hence you would have to contact your facilitator if:

- you do not understand any part of the study or the assigned readings
- you have difficulty with the self-tests
- you have a question or problem with an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only chance to have face to face contact with your course facilitator and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study.

To gain much benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating actively in discussions.

Summary

MTH303 (Vectors and Tensors) is a course that intends to provide solutions to problems normally encountered by engineers, physicists and mathematicians in the course of doing their normal jobs. It also serves as a tool which often enables the mathematicians to widen the frontiers of their analytical, concerns to issues that have significant mathematical implications. Nevertheless, do not forget to apply the principles you have learnt to your understanding of Vectors and Tensors.

I wish you success in the course and I hope that you will find it comprehensive and interesting.

Course Code MTH303
Course Title Vectors and Tensors Analysis

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Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
No. 5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e-mail: centralinfo@nou.edu.ng

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MODULE 1

Unit 1	Elementary Vector Algebra
Unit 2	Vector Differentiation (Gradient, Divergence and Curl)
Unit 3	The Line Integral

UNIT 1 ELEMENTARY VECTOR ALGEBRA

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2.0	Objectives
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3.1.1	Scalar Product of Two Vectors
3.1.2	Law of Scalar Product
3.1.4	Triple Products
4.0	Conclusion
5.0	Summary
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1.0 INTRODUCTION

This course is an introductory work to Vectors and Tensors. In this unit, an attempt will be made to revise some elements of vector algebra.

You should note that a vector quantity is distinguished from a scalar quantity by the fact that a scalar quantity possesses only magnitude, whereas a vector quantity possesses both magnitude and directions.

It is convenient to represent a vector geometrically as an arrow, pointing in the direction associated with the ones having length proportional to the associated magnitude. These and other properties of vector will be explained in this unit. You should read this unit carefully before proceeding to other unit.

2.0 OBJECTIVES

At the end this unit, you should be able to:

- describe vectors and scalar quantities
- list scalar product of two vectors
- state the Law of scalar product
- explain vector products
- solve problems on vector algebra.

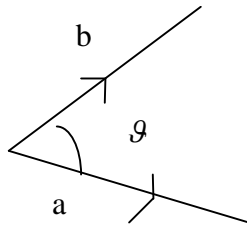
3.0 MAIN CONTENT

3.1 Revision of Elementary Vector Algebra

Definition: A scalar quantity has only magnitude while a vector quantity has both magnitude and direction.

Angle between two vectors.

Let \underline{a} and \underline{b} be two vectors such that \underline{a} and \underline{b} can be represented in the diagram below



Then the angle ϑ between the two vector in such that $0 \leq \vartheta \leq \pi$

3.1.1 Scalar Product of Two Vectors

Let \underline{a} and \underline{b} be any two vectors (of the same dimension). The scalar production of \underline{a} and \underline{b} denoted by $\underline{a} \bullet \underline{b}$ is defined by $\underline{a} \bullet \underline{b} = |\underline{a}| |\underline{b}| \cos \vartheta$ where $|\underline{a}|$ and $|\underline{b}|$ are the magnitude of vectors \underline{a} and \underline{b} respectively and ϑ is the angle between \underline{a} and \underline{b}

$\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$ then $\underline{a} \bullet \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

\therefore The angle between \underline{a} and \underline{b} is given by $\vartheta = \cos^{-1} \left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\underline{a}| |\underline{b}|} \right)$

Provided \underline{a} and \underline{b} are non-zero vectors.

If i, j, k are the usual unit vectors along the x, y, z axis respectively then

$$\underline{a} = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$$

$$\underline{b} = (b_1, b_2, b_3) = b_1 i + b_2 j + b_3 k$$

Where $i \bullet j = j \bullet i = k \bullet i = i \bullet k = j \bullet k = k \bullet j = 0$ and $i \bullet i = j \bullet j = k \bullet k = 1$

3.1.2 Laws of Scalar Products

- (i) $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$
- (ii) $m(\underline{a} \bullet \underline{b}) = (m\underline{a}) \bullet \underline{b} = (\underline{a} \bullet \underline{b})m = (\underline{a} \bullet \underline{b}m)$ ($m = \text{any scalar}$)
- (iii) if $\underline{a} \bullet \underline{b} = 0$ then \underline{a} and \underline{b} are perpendicular.

3.1.3 Vector Product

Let \underline{a} & \underline{b} be any two vectors with the same dimension and let $\hat{\underline{n}}$ be a unit vector perpendicular to both \underline{a} & \underline{b} , the vector product of \underline{a} & \underline{b} written as $\underline{a} \wedge \underline{b}$ is defined by

$$\underline{a} \wedge \underline{b} = |\underline{a}| |\underline{b}| \text{Sin } \vartheta \hat{\underline{n}}, \quad 0 \leq \vartheta \leq \pi$$

Where ϑ is the angle between \underline{a} & \underline{b} .

$$\text{If } \underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$$

$$\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$$

Then

$$\begin{aligned} \underline{a} \wedge \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \underline{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \underline{i}(a_2 b_3 - a_3 b_2) + \underline{j}(a_3 b_1 - a_1 b_3) + \underline{k}(a_1 b_2 - a_2 b_1) \end{aligned}$$

3.1.4 Triple Products

The scalar and vector products of 3 vectors $\underline{a}, \underline{b}$ and \underline{c} may have meaningful products

$$(\underline{a} \bullet \underline{b})\underline{c}, \quad \underline{a} \bullet (\underline{b} \wedge \underline{c}) \text{ and } \underline{a} \wedge (\underline{b} \bullet \underline{c})$$

Hence, the following laws are valid.

- (i) $(\underline{a} \bullet \underline{b})\underline{c} \neq \underline{a}(\underline{b} \bullet \underline{c})$
- (ii) $\underline{a} \bullet (\underline{b} \wedge \underline{c}) = \underline{b} \bullet (\underline{c} \wedge \underline{a}) = \underline{c} \bullet (\underline{a} \wedge \underline{b})$
- (iii) $\underline{a} \wedge (\underline{b} \wedge \underline{c}) + (\underline{a} \wedge \underline{b}) \wedge \underline{c}$
- (iv) $\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \bullet \underline{c})\underline{b} - (\underline{a} \bullet \underline{b}) \underline{c}$
 $(\underline{a} \wedge \underline{b}) \wedge \underline{c} = (\underline{a} \bullet \underline{c})\underline{b} - (\underline{b} \bullet \underline{c}) \underline{a}$
- (v) $\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$

Remark 1

1. The product $\underline{a} \bullet (\underline{b} \wedge \underline{c})$ is sometimes called the scalar triple product or box product and may be denoted by $[\underline{a} \ \underline{b} \ \underline{c}]$.
2. The product $\underline{a} \wedge \underline{b} \wedge \underline{c}$ is called the vector triple product.
3. Reader are advised to provide proofs to i to v above by assuming that

$$\underline{a} = a_1i + a_2j + a_3k$$

$$\underline{b} = b_1i + b_2j + b_3k$$

Remark 2

- (i) The quantity $\underline{a} \wedge \underline{b}$ is a vector quantity
- (ii) If $\underline{a} \wedge \underline{b} = 0$ then at least one of \underline{a} or \underline{b} and then by implication the angle between them is zero.
- (iii) If \underline{a} and \underline{b} are law then $\underline{a} \wedge \underline{b} = |\underline{a}| |\underline{b}|$
- (iv) $\lambda(\underline{a} \wedge \underline{b}) = \lambda \underline{a} \wedge \underline{b} = \underline{a} \wedge (\lambda \underline{b})$ where λ is any scalar.
- (v) The magnitude of $\underline{a} \wedge \underline{b}$ and $|\underline{a} \wedge \underline{b}|$ is the area of the parallelogram with sides \underline{a} and \underline{b}
- (vi) If $\underline{a} \wedge \underline{b} = 0$ and neither $\underline{a} = 0$ nor $\underline{b} = 0$ then \underline{a} and \underline{b} are parallel,

4.0 CONCLUSION

We have learnt about vector algebra in this unit. The materials in this unit are sufficient enough background to enable you understand the next unit.

5.0 SUMMARY

The following facts are to be remembered:

- that vector has magnitude and direction, unlike scalar quantity which has only magnitude
- that two vectors can be multiplied in two ways.
- scalar products which result in scalar quantity
- vector product which result in vector quantity
- if $\underline{a} \wedge \underline{b} = 0$ and $\underline{a} \neq 0$, $\underline{b} \neq 0$ then \underline{a} is parallel to \underline{b}
- that the magnitude of vectors whose product is $|\underline{a} \wedge \underline{b}|$ is the area of parallelogram with sides \underline{a} and \underline{b} .

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the length and direction cosines of the vector \underline{a} from the point $(1, -1, 3)$ to the midpoint of the line segment from the origin to the point $(6, -6, 4)$
2. Prove that $|a+b| \leq |a|+|b|$ and $|a+b| \geq |a|-|b|$
3. If θ denotes the angle between the vectors \underline{a} and \underline{b} use a theorem in elementary geometry that $|a+b|^2 = |a|^2 + |b|^2 + 2|a||b|\cos \theta$

7.0 REFERENCE/FURTHER READING

Francis, B. Hildebrand (nd). *Advanced Calculus for Application* (2nd ed.).

UNIT 2 VECTOR DIFFERENTIATION (GRADIENT, DIVERGENCE AND CURL)

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- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Differentiation of Vector
 - 3.2 Gradient, Divergence and Curl
 - 3.2.1 Directional Derivatives
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 - 3.2.3 The Curl of a Vector Function
 - 3.3 Differential Derivatives
- 4.0 Conclusion
- 5.0 Summary
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- 7.0 Reference/Further Reading

1.0 INTRODUCTION

In unit 1, we learnt about vector algebra and established some properties of vectors.

In this unit, we shall consider vector differentiation and derive some important formula and properties of vector differentiation.

2.0 OBJECTIVES

At the end this unit, you should be able to:

- differentiate vector quantities
- find the gradient of any scalar and vector field
- identify the curl and divergence of vector field.

3.0 MAIN CONTENT

Definition 1

The vector function $\underline{A}(u)$ is said to be continuous at a point u_0 if given $\varepsilon > 0$, we can find some $\delta(\varepsilon) > 0$ $= |\underline{A}(u) - \underline{A}(u_0)| < \varepsilon$ whenever $|u - u_0| \leq \delta$

This is equivalent to $\lim_{u \rightarrow u_0} \underline{A}(u) = \underline{A}(u_0)$

Definition 2

The derivative of the vector function $\underline{A}(u)$ is given as $\lim_{\Delta u \rightarrow 0} \frac{\underline{A}(u + \Delta u) - \underline{A}(u)}{\Delta u}$ provided that $\frac{d\underline{A}}{du} = \frac{dA_1}{du}i + \frac{dA_2}{du}j + \frac{dA_3}{du}k$.

Higher derivatives such as $\frac{d^2 \underline{A}}{du^2}$, $\frac{d^3 \underline{A}}{du^3}$ e.t.c. are similarly defined

Remark 1

If \underline{A} , \underline{B} and \underline{C} are differentiable vector functions of a scalar u and ϕ is a differentiable scalar functions then

$$(i) \quad \frac{d}{du}(\underline{A} + \underline{B}) = \frac{d\underline{A}}{du} + \frac{d\underline{B}}{du}$$

$$(ii) \quad \frac{d}{du}(\underline{A} \bullet \underline{B}) = \underline{A} \frac{d\underline{B}}{du} + \underline{B} \frac{d\underline{A}}{du}$$

$$(iii) \quad \frac{d}{du}(\underline{A} \wedge \underline{B}) = \underline{A} \wedge \frac{d\underline{B}}{du} + \frac{d\underline{A}}{du} \wedge \underline{B}$$

$$(iv) \quad \frac{d}{du}(\phi \underline{A}) = \phi \frac{d\underline{A}}{du} + \underline{A} \frac{d\phi}{du}$$

$$(v) \quad \begin{aligned} \frac{d}{du}(\underline{A} \bullet (\underline{B} \wedge \underline{C})) &= \underline{A} \bullet \frac{d}{du}(\underline{B} \wedge \underline{C}) + \frac{d\underline{A}}{du} \bullet (\underline{B} \wedge \underline{C}) \\ &= \underline{A} \bullet \left[\underline{B} \wedge \frac{d\underline{C}}{du} + \frac{d\underline{B}}{du} \wedge \underline{C} \right] + \frac{d\underline{A}}{du} \bullet (\underline{B} \wedge \underline{C}) \\ &= \underline{A} \bullet \left(\underline{B} \wedge \frac{d\underline{C}}{du} \right) + \underline{A} \bullet \left(\frac{d\underline{B}}{du} \wedge \underline{C} \right) + \frac{d\underline{A}}{du} \bullet (\underline{B} \wedge \underline{C}) \end{aligned}$$

$$(vi) \quad \begin{aligned} \frac{d}{du}(\underline{A} \bullet (\underline{B} \wedge \underline{C})) &= \underline{A} \wedge \frac{d}{du}(\underline{B} \wedge \underline{C}) + \frac{d\underline{A}}{du} \wedge (\underline{B} \wedge \underline{C}) \\ &= \underline{A} \wedge \left(\underline{B} \wedge \frac{d\underline{C}}{du} \right) + \underline{A} \wedge \left(\frac{d\underline{B}}{du} \wedge \underline{C} \right) + \frac{d\underline{A}}{du} \wedge (\underline{B} \wedge \underline{C}) \end{aligned}$$

3.1 Differentiation of Vector

1. $\underline{A} = A_1 i + A_2 j + A_3 k$ then
 $d\underline{A} = dA_1 i + dA_2 j + dA_3 k$
- (2) $d(\underline{A} \bullet \underline{B}) = \underline{A} \bullet d\underline{B} + d\underline{A} \bullet \underline{B}$
- (3) $d(\underline{A} \wedge \underline{B}) = \underline{A} \wedge d\underline{B} + d\underline{A} \wedge \underline{B}$
- (4) if $\underline{A} = \underline{A}(n, y, z)$ then
 $d\underline{A} = \frac{\partial \underline{A}}{\partial n} dn + \frac{\partial \underline{A}}{\partial y} dy + \frac{\partial \underline{A}}{\partial z} dz$

Examples

Given a vector $\underline{\phi} = \sin t i + \cos t j + t k$

Obtain a. $\frac{d\underline{Q}}{dt}$ b. $\frac{d^2 \underline{Q}}{dt^2}$ c. $\left| \frac{d\underline{Q}}{dt} \right|$ d. $\left| \frac{d^2 \underline{Q}}{dt^2} \right|$

$$(a) \quad \frac{d\underline{Q}}{dt} = \frac{d}{dt} (\sin t) i + \frac{d}{dt} (\cos t) j + \frac{d}{dt} (t) k$$

$$(b) \quad \frac{d^2 \underline{Q}}{dt^2} = \frac{d}{dt} \left(\frac{d\underline{Q}}{dt} \right) = \frac{d}{dt} (\cos t i - \sin t j + k)$$

$$= -\sin t i - \cos t j$$

$$(c) \quad \left| \frac{d\underline{Q}}{dt} \right| = \sqrt{\cos^2 t + \sin^2 t + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= 0. 1$$

3.2 Gradient, Divergence and Curl of Vectors

Consider the vector operators ∇ called “del” or nabla defined by

$$\nabla = i \frac{\partial}{\partial n} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

If $\phi(n, y, z)$ has continuous first partial derivatives in a particular region, we define the gradient ϕ as:

$$\text{Grad } \phi = \nabla \phi = \left(i \frac{\partial}{\partial n} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \phi$$

$$= i \frac{\partial \phi}{\partial n} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

Examples

(i) If $\phi(n, y, z) = nyz^2$ find $\nabla \phi$

$$\begin{aligned} \nabla \phi &= i \frac{\partial \phi}{\partial n} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \\ &= i \frac{\partial}{\partial n} (nyz^2) + j \frac{\partial}{\partial y} (nyz^2) + k \frac{\partial}{\partial z} (nyz^2) \\ &= nz^2 i + nz^2 j + 2nyz k \end{aligned}$$

(ii) if $\phi(n, y, z) = 3n^2 y - y^3 z^2$ find $\nabla \phi$

$$\begin{aligned} \nabla \phi &= i \frac{\partial}{\partial n} (3n^2 y - y^3 z^2) + j \frac{\partial}{\partial y} (3n^2 y - y^3 z^2) + k \frac{\partial}{\partial z} (3n^2 y - y^3 z^2) \\ &= 6nyi + (3n^2 - 3y^2 z^2)j - 2y^3 z k \end{aligned}$$

(iii) If $\phi(n, y, z)$ is a scalar function P, we know that

$$\begin{aligned} \text{Grad } \phi = \nabla \phi &= i \frac{\partial \phi}{\partial n} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \\ \text{And } d\underline{r} &= dni + dyj + dzk \\ \text{Grad } \phi \bullet d\underline{r} &= \frac{\partial \phi}{\partial n} dn + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \\ &= d\phi \end{aligned}$$

This implies

$$d\phi = \text{grad } \phi \bullet dr = \nabla \phi \bullet dr.$$

3.2.1 Directional Derivatives

We recall $d\phi = d\underline{r} \bullet \nabla \phi$, $|d\underline{r}| = ds$

$\Rightarrow \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{|d\underline{r}|}$ hence $\frac{d\underline{r}}{ds}$ is the unit vector in the direction of $d\underline{r}$

$\therefore \frac{d\phi}{ds} = \frac{d\underline{r}}{ds} \bullet \text{grad } \phi$, if $\frac{d\underline{r}}{ds} = \hat{a}$

Then $\frac{d\phi}{ds} = \hat{a} \bullet \text{grad } \phi$

NOTE:

$\frac{d\phi}{ds}$ is the propagation of grad ϕ on the unit vector \hat{a} and the directional derivative of ϕ in the direction of \hat{a}

Remark

(i) The unit Normal vector $\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|}$

(ii) The grad of sum and products of scalars.

$$\begin{aligned}\nabla(\underline{A} + \underline{B}) &= i \frac{\partial}{\partial n}(\underline{A} + \underline{B}) + j \frac{\partial}{\partial y}(\underline{A} + \underline{B}) + k \frac{\partial}{\partial z}(\underline{A} + \underline{B}) \\ &= \left(\frac{\partial \underline{A}}{\partial n} i + \frac{\partial \underline{A}}{\partial y} j + \frac{\partial \underline{A}}{\partial z} k \right) + \left(\frac{\partial \underline{B}}{\partial n} i + \frac{\partial \underline{B}}{\partial y} j + \frac{\partial \underline{B}}{\partial z} k \right) \\ &= \nabla \underline{A} + \nabla \underline{B}\end{aligned}$$

$$\begin{aligned}\nabla(\underline{A} \cdot \underline{B}) &= i \frac{\partial}{\partial n}(\underline{A} \cdot \underline{B}) + j \frac{\partial}{\partial y}(\underline{A} \cdot \underline{B}) + k \frac{\partial}{\partial z}(\underline{A} \cdot \underline{B}) \\ &= \\ i \left(\frac{\partial \underline{A}}{\partial n} \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial n} \right) &+ j \left(\frac{\partial \underline{A}}{\partial y} \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial y} \right) + k \left(\frac{\partial \underline{A}}{\partial z} \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial z} \right) \\ &= \left(i \frac{\partial \underline{A}}{\partial n} \cdot \underline{B} + j \frac{\partial \underline{A}}{\partial y} \cdot \underline{B} + k \frac{\partial \underline{A}}{\partial z} \cdot \underline{B} \right) + \left(i \underline{A} \cdot \frac{\partial \underline{B}}{\partial n} + j \underline{A} \cdot \frac{\partial \underline{B}}{\partial y} + k \underline{A} \cdot \frac{\partial \underline{B}}{\partial z} \right) \\ &= \left(i \frac{\partial \underline{A}}{\partial n} + j \frac{\partial \underline{A}}{\partial y} + k \frac{\partial \underline{A}}{\partial z} \right) \cdot \underline{B} + \underline{A} \cdot \left(i \frac{\partial \underline{B}}{\partial n} + j \frac{\partial \underline{B}}{\partial y} + k \frac{\partial \underline{B}}{\partial z} \right) \\ &= \underline{B}(\nabla \underline{A}) + \underline{A}(\nabla \underline{B})\end{aligned}$$

3.2.2 Divergence of a Vector

The divergence of a vector function $\underline{A}(n, y, z)$ is defined by $\text{div } \underline{A} = \nabla \cdot \underline{A}$

$$\begin{aligned}&= \left(i \frac{\partial}{\partial n} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (A_1 i + A_2 j + A_3 k) \\ &= \frac{\partial A_1}{\partial n} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned}$$

Examples

1. Obtain the divergence of the vector function $\underline{A} = 2nzi + yzj - ny^2k$ at the point (1, -1, 1)
2. If $\phi = n^2 yz$ a scalar function and $\underline{A} = 2nzi + yzj - ny^2k$ Find $\nabla \cdot (\phi A)$ at the point (1, -1, 1)
3. Determine the content a so that the vector $\underline{v} = (n + 3y)i + (y - 2n)j + (n + az)k$ is solenoid

Solution

$$\begin{aligned}
 1. \quad \nabla \cdot \underline{A} &= \left(\frac{\partial}{\partial n} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (2nzi + yzj - ny^2k) \\
 &= \frac{\partial}{\partial n} (2nzi) + \frac{\partial}{\partial y} (yz) - \frac{\partial}{\partial z} (ny^2) \\
 &= 2z + z = 3z
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \nabla \cdot (\phi A) &= \nabla \cdot (2n^3 yz^2 i + n^2 y^2 z j - n^3 y^3 z k) \\
 &= 6n^2 yz^2 + 2n^2 yz^2 - n^3 y^3
 \end{aligned}$$

A vector \underline{v} is solenoid if its divergence zero

$\therefore \nabla \cdot \underline{v} = 0$ implies that

$$1 + 1 + a = 0 \Rightarrow a = -2$$

3.2.3 The Curl of a Vector Function

The curl of a vector function \underline{A} define denoted by $\text{curl } \underline{A} = \nabla \wedge A$

$$\begin{aligned}
 &= \left(i \frac{\partial}{\partial n} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \wedge (A_1 i + A_2 j + A_3 k) \\
 &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial n}, \frac{\partial A_2}{\partial n} - \frac{\partial A_1}{\partial y} \right)
 \end{aligned}$$

Examples

1. Find the curl of \underline{A} if $\underline{A} = 3n^2 yi + yz^2 j - nzk$
2. If $\phi = n^2 y^2$ and $\underline{A} = 2nzi + yzj - ny^2k$ Obtain $\text{curl } (\phi A)$

Solution

$$1. \quad \text{Curl } \mathbf{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3n^2 y & yz^2 & -nz \end{vmatrix} = (0 - 2yz, +z, 0 - 3n^2)$$

$$= 2yzi + zj - 3n^2k$$

$$2. \quad \phi \mathbf{A} = 2n^3 yz^2 i + n^2 y^2 z^2 j - n^3 y^2 z k$$

$$\nabla \wedge \mathbf{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2n^3 yz^2 & n^2 y^2 z^2 & -n^3 y^2 z \end{vmatrix}$$

$$= (2ny^2 z^2 - 2n^3 z^2, 4n^3 yz + 3n^2 y^2 z, 2ny^2 z - 2n^3 z^2)$$

4.0 CONCLUSION

This section is very important for the understanding of the remaining units.

You are to master this unit very well.

5.0 SUMMARY

We have established formula to the following:

- vector differentiation
- the grad ϕ of scalar field ϕ
- the curl of a vector function
- we show that the unit normal vector $\mathbf{N} = \frac{\nabla \phi}{|\nabla \phi|}$

6.0 TUTOR-MARKED ASSIGNMENT

1. A particle moves along the curve $n = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the component of its velocity and acceleration at time $t = 1$
2. If $\underline{A} = 5t^2 i + tj - t^3 k$
And $\underline{B} = \text{Sin } t i - \text{Cos } t j$

Find (a) $\frac{d}{dt}(\underline{A} \bullet \underline{B})$

(b) $\frac{d}{dt}(\underline{A} \wedge \underline{B})$

(c) $\frac{d}{dt}(\underline{A} \bullet \underline{A})$

3. If $\phi(n, y, z) = ny^2z$ and $\underline{A} = nzi - ny^2j + yz^2k$

Find $\frac{\partial^3(\phi \underline{A})}{\partial^2 n \partial z}$ at point (2, -1, 1)

4. Let $r = (x, y, z)$ be a vector.

Prove that $\nabla \bullet (\nabla \bullet |r|^m) = 3m r^{m-2}$

5. Find the divergence of the vector

$$\underline{B} = (y^2 - 2xyz^3, 3 + 2xy - x^2z^3, 6z^3 - 3x^2yz^2)$$

6. If $\underline{A} = (2x^2 + 8xy^2z, 3x^3y - 3ny, 2x^3z)$, show that \underline{A} is not solenoidal.

7. Show that the vector $\underline{A} = \frac{r}{r^3}$ is irrotational where $r = xi + yj + zk$

8. If $\underline{A} = x^2yi + y^2zj - z^2xk$

show that $\nabla \wedge (\nabla \wedge \underline{A}) = \nabla(\nabla \bullet \underline{A}) - \nabla^2 \underline{A}$.

9. Given that \underline{E} and \underline{H} are two vectors which are assumed to have continuous partial derivatives with respect to position and time. Furthermore,

Suppose $\nabla \bullet \underline{E} = 0, \nabla \bullet \underline{H} = 0$

$$\nabla \wedge \underline{E} = \frac{-1}{c} \frac{\partial \underline{H}}{\partial t}; \nabla \wedge \underline{H} = \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

Prove that \underline{E} & \underline{H} satisfy the equation

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

Use the above relations to show that

$$\frac{\partial}{\partial t} \left[\frac{1}{2} (E^2 + H^2) \right] + c \nabla \bullet (\underline{E} \wedge \underline{H}) = 0$$

7.0 REFERENCE/FURTHER READING

Francis B. Hildebrand (nd). *Advanced Calculus for Application* (2nd ed.).

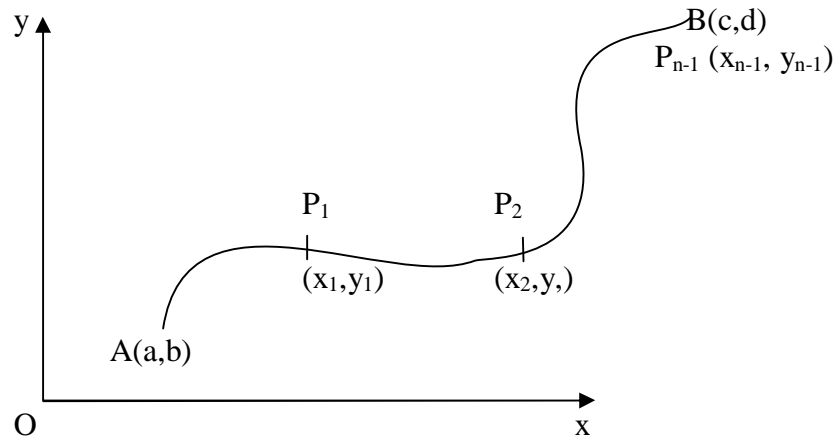
UNIT 3 THE LINE INTEGRAL

CONTENTS

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 The Line Integral
 - 3.1.1 The Surface Integral
 - 3.1.2 The Volume Integral
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

1.0 INTRODUCTION

In a line integral unlike other integrals, we have to consider two or more functions at a time, for integration purposes. Suppose these two functions are $M(x, y)$ and $N(x, y)$ such that they are single valued and continuous at every point of a curve AB. Divide the curve AB into π parts by means of $P_i(x_i, y_i) \quad i = 1, 2, 3, \dots, \pi - 1$



Let $\Delta x_i = x_i - x_{i-1}, \quad \Delta y_i = y_i - y_{i-1}$ where $x_0 = a, x_n = c, y_n = d$

Let (ξ, η) be defined by

$$x_{i-1} \leq \xi < x_i \quad y_{i-1} \leq \eta \leq y_i$$

We form the product and then add them to get

$$\sum_{i=1}^n \{M(\xi, \eta) \Delta x_i + N(\xi, \eta) \Delta y_i\}$$

Limit of this sum $\alpha\pi \rightarrow \infty$ and all $\nabla x_i \rightarrow 0, \nabla y \rightarrow 0$ simultaneously is defined as α line integral along the curve $A\beta$ of two functions M and N simultaneously.

Thus, we write

$$\begin{aligned} & \text{Limit} \\ & \left. \begin{array}{l} \pi \rightarrow \infty \\ \Delta x_i \rightarrow 0 \\ \Delta y_i \rightarrow 0 \end{array} \right\} \left[\sum_{i=1}^n \{M(\zeta, \pi)\Delta x_i + N(\zeta, \eta)\Delta y_i\} \right] = \end{aligned}$$

$$\int_{\text{Curve } AB} [M(x, y)dx + N(x, y)dy]$$

2.0 OBJECTIVES

At the end this unit, you should be able to:

- evaluate the line integral of vector functions
- identify problems relating to line integrals.

3.0 MAIN CONTENT

3.1 The Line Integral

Let $\underline{A}(x, y, z) = A_1i + A_2j + A_3k$ be a vector function of position defined and continuous along a curve C. The integral of the tangential component of \underline{A} along C is written as:

$$\int_c \underline{A} \cdot d\underline{r} = \int_c A_1dn + A_2dy + A_3dz$$

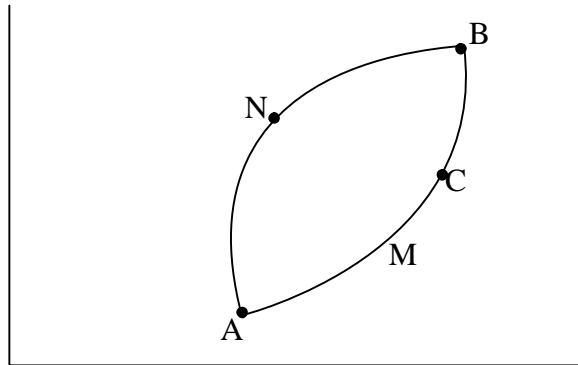
Where $d\underline{r} = dx_i + dyj + dzk$.

For instance in Aerodynamic and fluid mechanics this line integral

$\int_c \underline{A} \cdot d\underline{r}$ is called the circulation of \underline{A} about c

Where \underline{A} represents the velocity of Air or the velocity of the fluid as the case may be.

Line Integral about closes plane

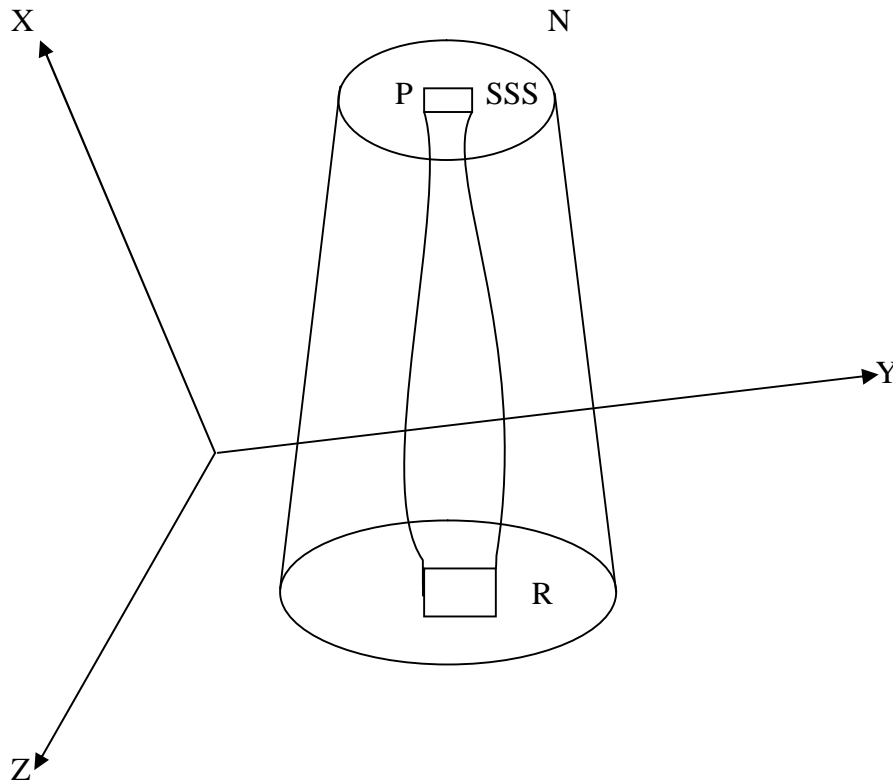


Evaluate the integral $I = \oint_C (2xydy + x^2 dx)$ where C made up of the 3 sides of triangle with vertices $O(0,0)$, $A(1,0)$, & $B(0,1)$ Ans = $\frac{1}{3}$.

SELF-ASSESSMENT EXERCISE 1

- Evaluate the line integral $I = \int_C y dx$ from $-a$ to a where C is the circle $x^2 + y^2 = a$
- Find the work done in moving a particle once around a circle C in:
 - the xy-plane if the circle C has centre at the origin and radius
 - If the force field $\underline{F} = (2x - y + z)\underline{i} + (x + y - z)\underline{j} + (3x - 2y + 4z)\underline{k}$ is as given.
- If $\underline{F} = (x - 3y)\underline{i} + (y - 2x)\underline{j}$ and C is the closed curve in the xy-plane, $x = 2 \cos t$, $y = 3 \sin t$, $t=0$ to $t=2\pi$. Evaluate $\int \underline{F} \cdot d\underline{r}$.
- Find the total work done if a particle is moved in a force field by $\underline{F} = 3xy\underline{i} - y^2 \underline{j}$ along the curve $y = 2x^2$ in the xy-plane from $(0, 0)$ to $(1, 2)$.
- If $\underline{A} = (5x^2 + 4y)\underline{i} - 14yz\underline{j} + 20xz^2\underline{k}$. Evaluate $\int \underline{A} \cdot d\underline{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the following paths.
 - $x = t$, $y = t^2$, $z = t^3$
 - The straight line from $(0, 0, 0)$ to $(1, 0, 0)$ then from $(1, 0, 0)$ to $(1, 1, 0)$ and then to $(1, 1, 1)$
- Calculate $\int_C \underline{v} \cdot d\underline{r}$ where $\underline{V} = 2\underline{j} + 3x\underline{j}$ and c is given by
- The straight line joining $(2, 0)$ to $(0, 0)$
- The arc of a circle with centre at the origin and radius 2 units

3.1.1 The Surface Integral



Let δA represent an element of R and δS the corresponding area of S at the point $P(x, y, z)$ on S. Let also $\phi(x, y, z)$ be a function of position on S and let Y denote the angle between two outward normals PN to the surface P and the positive Z axis.

$$\delta A \propto \delta S \cos Y \Rightarrow \delta S \propto \delta A \sec Y$$

And

$\sum \phi(x, y, z) \delta S$ is the total value of $\phi(x, y, z)$ taken over the surface $\zeta \rightarrow \delta_n \rightarrow 0$. This sum becomes the integral

$$I = \int_{\zeta} \phi(x, y, z) ds = \iint_R \phi(x, y, z) \sec Y dx dy$$

$$\sec Y = \sqrt{1 + z_n^2 + z_y^2}$$

This implies that

$$I_s = \int_s \phi ds = \iint_R \phi \sqrt{1 + z_n^2 + z_y^2} dx dy$$

Let S be a 2 sided surface and let one side be considered arbitrary as the positive side associated with the differential of the surface area ds . A vector $d\underline{s}$ whose magnitude is ds and direction is that of $\hat{\underline{n}}$. then $d\underline{s} = \hat{\underline{n}} ds$ where $\hat{\underline{n}}$ is the limit vector normal to any point of the positive side of S. the integral

$\int_C \underline{A} \bullet d\underline{s} = \int_S \underline{A} \bullet \hat{\underline{n}} ds$ is an example of a surface called the flux of \underline{A} over S. others are:

- i. $\iint \phi ds$
- ii. $\iint \phi \hat{\underline{n}} ds$
- iii. $\iint \underline{A} \wedge d\underline{s}$

we have $d\underline{A} = ds \text{ Sec } Y$.

where if $z = f(n, y)$ implying that $z - f(n, y) = 0$

$$\hat{\underline{n}} = \frac{\nabla[z - f(n, y)]}{|\nabla[z - f(n, y)]|} = \frac{-\frac{\partial f}{\partial n} i - \frac{\partial f}{\partial y} j + k}{\sqrt{f_n^2 + f_y^2 + 1}}$$

This implies that

$$\nabla[z - f(n, y)] = \left(\sqrt{f_n^2 + f_y^2 + 1}\right) \hat{\underline{n}} = ds$$

If the angle Y between the z-axis and $\hat{\underline{n}}$ is acute the positive sign is adopt and if the angle is obtuse we adopt the negative sign.

Hence

$$\iint \underline{A} \bullet \hat{\underline{n}} ds = \iint \underline{A} \bullet \hat{\underline{n}} \frac{d_n dy}{|\hat{\underline{n}} = k|} \text{ depending on the projection}$$

The projection

Example 1

$\iint \underline{A} \cdot \underline{\hat{n}} ds$ where $\underline{A} = 18i - 12j + 3yk$ and S is that part of the plane, $2n + 3y + 6z = 12$ which is in the ny plane (In-quadrante)

$$\iint \underline{A} \cdot \underline{\hat{n}} ds = \iint \underline{A} \cdot \underline{\hat{n}} \frac{dndy}{|\underline{\hat{n}} \cdot \underline{k}|}$$

$$\underline{\hat{n}} = \nabla(2n + 3y + 6z - 12) = 2i + 3j + 6k$$

$$\underline{\hat{n}} = \frac{2i + 3j + 6k}{7} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$$

$$\underline{\hat{n}} \cdot \underline{k} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right) \cdot (0, 0, 1) = \frac{6}{7}$$

$$\begin{aligned} \underline{A} \cdot \underline{\hat{n}} &= (18z, -12, 3y) \cdot \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right) \\ &= \frac{36z - 36 + 18y}{7} \end{aligned}$$

Now $2n + 3y + 6z = 12$ gives $z = \frac{12 - 2n - 3y}{6}$

$$\begin{aligned} \therefore \underline{A} \cdot \underline{\hat{n}} &= \frac{6(12 - 2n - 3y) - 36 + 18y}{7} \\ &= \frac{72 - 12n - 18y - 36 + 18y}{7} = \frac{36 - 12n}{7} \end{aligned}$$

This implies that

$$\begin{aligned} \iint \underline{A} \cdot \underline{\hat{n}} ds &= \iint \underline{A} \cdot \underline{\hat{n}} \frac{dndy}{|\underline{\hat{n}} \cdot \underline{k}|} = \iint \frac{36 - 12n}{7} \cdot \frac{7ndy}{6} \\ &= \iint (6 - 2n) dndy \end{aligned}$$

To obtain the limit of integration of n we set $y = z = 0$, and for y we set $z = 0$

Hence we have

$$\int_1^6 \int_{y=0}^{\frac{12-2n}{3}} (6 - 2n) dndy = 24$$

SELF-ASSESSMENT EXERCISE 2

1. Evaluate the line integral $I = \int_C y dx$ from $-a$ to a where C is the circle $x^2 + y^2 = a$
2. Find the work done in moving a particle once around a circle C in:
 - a. the xy-plane if the circle C has centre at the origin and radius
 - b. If the force field $\underline{F} = (2x - y + z)\mathbf{i} + (x + y - z)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$ is as given.
 - c. given.
3. If $\underline{F} = (x - 3y)\mathbf{i} + (y - 2x)\mathbf{j}$ and C is the closed curve in the xy-plane, $n=2 \cos t$, $y = 3 \sin t$, $t=0$ to $t=2\pi$. Evaluate $\int \underline{F} dr$.
4. Find the total work done if a particle is moved in a force field by $\underline{F} = 3xy\mathbf{i} - y^2\mathbf{j}$ along the curve $y = 2x^2$ in the xy-plane from (0, 0) to (1, 2).
5. The straight line joining (2, 0) to (0, 0)
 - ii. The arc of a circle with centre at the origin and radius 2 units

3.1.2 The Volume Integral

The volume or space integral is given by

$$\iiint \underline{A} \cdot d\mathbf{r} \quad \text{or} \quad \iiint \phi d\mathbf{r} \quad \text{for } a$$

Closed surface

Example

If $F = n^2\mathbf{i} + z\mathbf{j} + yz\mathbf{k}$. Evaluate $\iint_S F \cdot d\mathbf{s} = \iiint_S \nabla F \cdot d\mathbf{r}$ where V is the volume enclosed by the cube given by $0 \leq n \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$.

Solution

$$\iint F \cdot d\mathbf{s} = \iiint \nabla F \cdot \nabla \cdot F d\mathbf{r}$$

Where $d\mathbf{r} = dndydz$ and $\nabla \cdot F = 2n + y$

$$\iiint \nabla \cdot F d\mathbf{r} = \int_0^1 \int_0^1 \int_0^1 (2n + y) dndydz = \frac{2}{3}$$

SELF- ASSESSMENT EXERCISE 3

1. Evaluate $\iint \underline{A} \cdot \underline{\hat{n}} ds$ where $A = zi + nj - 3y^2z k$ and S is the surface of the cylinder $n^2 + y^2 = 16$ included in the 1st quadrant between $z = 0$ and $z = 5$
2. Evaluate $\iint \phi \underline{\hat{n}} ds$ in the 1st quadrant between $z = 0$ and $z = 5$.
3. If $F = yi + (n - 2nz)j - nyk$. Evaluate $\iint (\nabla \wedge F) \cdot \underline{\hat{n}} ds$ where S is the surface of the sphere $n^2 + y^2 + z^2 = a^2$ from the $ny - plane$.

4.0 CONCLUSION

The materials in this unit are very important for the understanding of subsequent units. You must understand this work thoroughly before moving to the next section.

5.0 SUMMARY

In this unit, we established the relationship between line integral of scalar field and vector functions.

Thus, we have used the line integral in finding

- Surface integral
- Volume integral or space integral which is given as:

$$\iiint \underline{A} \cdot d\underline{r} \text{ or } \iiint \phi d\underline{r}, \text{ for a closed surface.}$$

6.0 TUTOR-MARKED ASSIGNMENT

1. Let $F = 2nxi - nj + y^2k$. Evaluate $\iiint \underline{F} \cdot d\underline{v}$ where V is the region bounded by the surface $n = 0, y = 0, z = n^2, z = 4, x = 2, y = 6$
2. If $F = ny^2i + yzj + zn^2k$. Evaluate $\iint F \cdot ds$ over the sphere given by $n^2 + y^2 + z^2 = 1$
3. Evaluate $\iint (\nabla \wedge A) \cdot ds$ where $A = (n - z)i + (-nz - y)j + (y^2 + 2z)k$ and S is the surface of the sphere having centre at (3, -1, 2) and radius 3.
4. If $F = (2n^3 - 3z, -4ny, -4n)$. Evaluate $\iiint \nabla \cdot F d\underline{r}$ where \underline{r} is the closed region bounded by the plane $n = 0, y = 0, z = 0$ and $2n + 2y + z = 4$

7.0 REFERENCES/FURTHER READING

Hildebrand, F.B. (nd). *Advanced Calculus for Applications*.

Verma, P.D.S. (nd). *Engineering Mathematics*.

MODULE 2

- Unit 1 Green's Theorem, Divergence Theorem and Stokes's Theorem
Unit 2 Tensor Analysis

UNIT 1 GREEN'S THEOREM, DIVERGENCE THEOREM AND STOKE'S THEOREM**CONTENTS**

- 1.0 Introduction
- 2.0 Objectives
- 3.0 Main Content
 - 3.1 Green's Theorem
 - 3.2 Divergence Theorem
 - 3.3 Stoke's Theorem
- 4.0 Conclusion
- 5.0 Summary
- 6.0 Tutor-Marked Assignment
- 7.0 References/Further Reading

1.0 INTRODUCTION

In this unit, we will discuss some mathematical theorems such as Green's theorem, divergence theorem and stoke's theorem.

These theorems are useful in handling equations of mathematical physics, particularly in the area of velocity of a fluid in three dimensions.

We know on application of these theorems particularly the divergence theorem that the velocity of an incompressible fluid has zero divergence.

Some other applications will be considered in this section.

The references at the back of this unit throw more light on the applications of these theorems; you may wish to contact them.

2.0 OBJECTIVES

At the end this unit, you should be able to:

- apply the Green's theorem
- apply divergence theorem and stoke's theorem to solving problems arising from mathematical physics.

3.0 MAIN CONTENT

3.1 Green's Theorem

If R is a closed region on the ny -plane bounded by a simple closed curve C and if M and N are single valued functions which are continuous in n and y having continuous derivatives in R, then

$\oint Mdn + Ndy = \iint_R \left(\frac{\partial N}{\partial n} - \frac{\partial M}{\partial y} \right) dndy$ where the curve C is traversed counter clockwise

Example

Verify the Green's theorem in the plane for $\oint_c (ny + y^2)dn + n^2dy$ where C is a close curve of the region bounded by $y = n$ and $y = n^2$

Solution

The point of intersection of $y = n$ and $y = n^2$ is (0, 0) and (1, 1),
 $M = ny + y^2$ and $N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2n$$

$$\Rightarrow \oint (ny + y^2)dn + n^2dy = \iint_R (2n - x - 2y)dndy$$

L.H⁻ S is $\oint_c (ny + y^2)dn + n^2dy$ along $y = n^2, dy = 2ndx$

$$\therefore I_1 = \int_0^1 (n^3 + n^4 + 2n^3)dn = \int_0^1 (3n^3 + n^4)dn = \frac{19}{20}$$

Along $y = n, dy = dn$

$$I_2 = \int_0^1 (n^2 + n^2 + n^2)dn = -3 \int_0^1 n^2 dn = -1$$

$$I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$$

Now R.H.S.

$$\begin{aligned} \iint_R (n-2y)dn dy &= \int_0^1 \int_{n^2}^n (n-2y)dy dn \\ &= \int_0^1 (ny - y^2) \Big|_{n^2}^n dn = \int_0^1 (n^4 - n^3)dn = -\frac{1}{20} \end{aligned}$$

SELF- ASSESSMENT EXERCISE 1

Evaluate having the Green's theorem the integral $(y - \sin n)dn + \cos ndy$ where C is bounded by $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{\pi}{2}, 1)$

3.2 Divergence Theorem

If V is the volume bounded by a closed surface A and \underline{A} is a vector function of position with continuous derivatives then

$$\iiint_V \nabla A dr = \iint_S \underline{A} \cdot \hat{n} ds \text{ or } \int \text{div } F dv = \iiint_V \nabla F dv = \int_S \underline{F} \cdot d\underline{s} = \iint_R \underline{F} \cdot \hat{n} ds$$

Example

Verify the divergence theorem for

$\underline{A} = 4xi - 2y^2j + z^2k$ taken over the region bounded by $n^2 + y^2 = 4$, $z = 0$ and $z = 3$

$$\begin{aligned} \text{L.H.S. } \int_S \nabla \cdot \underline{A} dv &= \iiint_V \left(\frac{\partial}{\partial n}(4n) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) \right) dv \\ &= \iiint_V (4 - 4y + 2z) dv \\ &= \int_{-2}^2 \int_{-\sqrt{4-n^2}}^{\sqrt{4-n^2}} \int_0^3 (4 - 4y + 2z) dz dy dx = 84\pi \end{aligned}$$

R.H.S. The surface S of the cylinder consists of a base $S_1(z = 0)$ and the top $S_2(z = 3)$ and the convex position $S_3(n^2 + y^2 = 4)$. Then the surface is given by

$$\iint_S \underline{A} \cdot \hat{n} ds = \iint_{S_1} \underline{A} \cdot \hat{n} ds_1 + \iint_{S_2} \underline{A} \cdot \hat{n} ds_2 + \iint_{S_3} \underline{A} \cdot \hat{n} ds_3$$

$$\text{On } S_1(z = 0) \quad \underline{n} = -k, \quad \underline{A} = 4xi - 2y^2j \quad \underline{A} \cdot \hat{n} = 0$$

$$\text{On } S_2(z = 3) \quad \underline{n} = k, \quad \underline{A} = 4xi - 2y^2j + 9k \text{ so that}$$

$$\iint_{S_2} \underline{A} \cdot \underline{\hat{n}} ds = \iint 9 ds_2 = 9 \iint ds_3 = 9(4\Pi) = 36\Pi$$

$$\text{On } S_3 (x^2 + y^2 = 4)$$

$$\nabla(x^2 + y^2 = 4) = 2xi + 2yj \Rightarrow \underline{\hat{n}} = \frac{xi + yj}{2}$$

$$\underline{A} \cdot \underline{\hat{n}} = (4xi - 2y^2 j + z^2 k) \cdot \left(\frac{xi + yj}{2} \right)$$

$$= 2x^2 - y^3$$

$$\text{Let } x = 2 \cos \vartheta, \quad y = 2 \sin \vartheta \quad ds_3 = 2d\vartheta dz$$

This implies that

$$\begin{aligned} \iint_{S_3} \underline{A} \cdot \underline{\hat{n}} ds_3 &= \int_0^{2\Pi} \int_0^3 (2(2 \cos \vartheta) - (2 \sin \vartheta)^3) \cdot 2 dz d\vartheta \\ &= \int_0^3 \int_0^{2\Pi} [8 \cos^2 \vartheta - 8 \sin^3 \vartheta] dz d\vartheta \\ &= 16 \int_0^{2\Pi} \int_0^3 (\cos^2 \vartheta - \sin^2 \vartheta) dz d\vartheta \\ &= 48 \int_0^{2\Pi} (\cos^2 \vartheta - \sin^3 \vartheta) d\vartheta = 48\Pi \end{aligned}$$

$$\text{Hence the total surface } S = 0 + 36\Pi + 48\Pi = 84\Pi$$

SELF -ASSESSMENT EXERCISE 2

1. Evaluate $\iint_S \underline{F} \cdot \underline{n} ds$ where $\underline{F} = 4xzi - y^2 j + yzk$ and S is the surface of the cube bounded by $n = 0, x = 1, y = 1, z = 0, z = 1$
2. Evaluate $\iint_S \underline{r} \cdot \underline{n} ds$ where S is a closed surface.

3.3 Stoke's Theorem

If S is an open two-sided surface bounded by a closed non-intersecting curve C (Simple closed curve) then if A has continuous derivatives.

$$\oint_C \underline{A} \cdot d\underline{r} = \iint_S (\nabla \wedge \underline{A}) \cdot \underline{\hat{n}} ds = \iint_S (\nabla \wedge \underline{A}) \cdot d\underline{s}$$

Where C is traversed counter-clock wisely

Example:

Use stoke's theorem to determine

$$\iint_S (\nabla \wedge \underline{A}) \cdot \underline{\hat{n}} ds \text{ where } \underline{A} = (y - z + 2)i + (yz + 4)j - xzk$$

And S is the surface of the closed cube
 $n = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ the xy -plane.

Solution:

By stoke's theorem

$$\begin{aligned} I &= \iint_s (\nabla \wedge \underline{A}) \bullet \underline{\hat{n}} ds = \oint_c \underline{A} \bullet d\underline{r} \\ &= \oint_c (y - z + 2)dx + (yz + 4)dy - xzdz \end{aligned}$$

Since it is above the xy -plane then $z = 0$

$$\therefore I = \oint_c (y + 2)dx + 4dy$$

By applying the Green's theorem

$$M = y + 2, N = 4 \frac{\partial M}{\partial y} = 1 \frac{\partial N}{\partial x} = 0$$

$$\therefore I = \iint -dydx = -\int_0^2 \int_0^2 dx dy = -4.$$

SELF- ASSESSMENT EXERCISE 3

Use the stoke's theorem to determine

$\oint_s (2x^2 - y)dx + y^{2z^3} dy + y^{3z^2} dz$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ is the xy -plane.

4.0 CONCLUSION

The materials in this unit should be well understood before moving to tensor analysis which made use of all the materials we have developed in the preceding chapters.

5.0 SUMMARY

We have established the following formula:

- $\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$
- $\iiint \nabla A dr = \iint_s \underline{A} \bullet \underline{\hat{n}} ds.$
- $\oint \underline{A} \bullet d\underline{r} = \iint_c (\nabla \wedge \underline{A}) \bullet \underline{\hat{n}} ds = \iint_c (\nabla \wedge \underline{A}) \bullet d\underline{s}.$

Study the material presented in the preceding sections because the whole ideas will be used in the subsequent sections on vector and tensors.

6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate by using Green's theorem
2. $\int_C [(x^2 + y)dx + (x - y^2)dy]$ Where C is a closed curve formed by $y^3 = x^2$ and $y = x$ between (0, 0) and (1, 1)
3. Use Stokes' theorem to transform $\int ydx + xdz + zdy$ to a surface integral.
4. Prove that $\text{Grad} (\phi\psi) = \phi \text{ grad } \psi + \psi \text{ grad } \phi$
5. Prove that $\text{div} (\phi\mu) = \phi \text{ div } \mu + \mu \cdot \text{grad } \phi$

7.0 REFERENCES/FURTHER READING

Hildebrand, F.B. (nd). *Advanced Calculus for Application*.

Verma, P.D.S. (nd). *Engineering Mathematics*.

UNIT 2 TENSOR ANALYSIS

CONTENTS

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1.0 INTRODUCTION

This section introduces you to Tensor Analysis. The work will make use of the ideas developed in the previous units.

2.0 OBJECTIVES

At the end this unit, you should be able to:

- define tensors
- perform various operations on tensors
- solve correctly any exercise on tensor analysis.

3.0 MAIN CONTENT

A point in the N-dimensional space is a set of N numbers denoted by (X^1, X^2, \dots, X^N) where 1, 2, ..., N are taken not as exponents but as superscripts. The fact that we cannot visualise points in spaces of dimension higher than 3 has of course nothing to do with the existence of such space.

3.1 Transformation of Co-ordinates

Let $X^1, X^2, X^3, \dots, X^n$ and $\bar{X}^1, \bar{X}^2, \bar{X}^3, \dots, \bar{X}^N$ be co-ordinates of point in two different frames of references. Suppose there exist independent relations between the co-ordinates of the two systems having the form

$$\begin{aligned} \bar{X}^1 &= \bar{X}^1(X^1, X^2, \dots, X^N) \\ \bar{X}^2 &= \bar{X}^2(X^1, X^2, \dots, X^N) \\ &\dots\dots\dots \\ \bar{X}^N &= \bar{X}^N(X^1, X^2, \dots, X^N) \end{aligned} \quad \text{----- (1)}$$

Which can be indicated briefly as $\bar{X}^K = \bar{X}^K(X^1, X^2, \dots, X^N)$ _____ (n) where $K = 1, 2, \dots, N$. It is assumed that the functions involved are single valued, continuous and have continuous derivatives. The converse to each set of coordinates $(\bar{X}^1, \bar{X}^2, \dots, \bar{X}^N)$ there will correspond a unique set (X^1, X^2, \dots, X^N) given by $X^K = X^K(\bar{X}^1, \bar{X}^2, \bar{X}^N)$ _____ (iv)

The relation given by equation (ii) and (iii) defines a transformation of co-ordinate from one frame of reference to another frame of reference.

3.2 Cartesian Tensor

Definition: When the transformations are from one rectangular co-ordinate system to another the tensors are called Cartesian tensors.

3.3 Summation Convection

Note that $\sum_{i=1}^N a_i x^i = a_1 x^1 + a_2 x^2 + \dots + a_N x^N$ but a shorter notation is simply to write as $a_j x^j$ where the convention that whenever the index (super or subscript) is repeated in a given term we are to sum from the index from 1 to N unless otherwise specified. This is called the summation convention instead of moving the index j we could use another letter say P and the sum could be written as $a_p X^p$. Any index which is repeated in a given term so that the summation convention applied is called a dummy index.

As index occurring only once is called “free index” and can stand for any of the number 1, 2, ..., N such as k is equation (ii) and (iii)

Examples

1. If $\phi = \phi(X^1, X^2, \dots, X^N)$ then the differential of ϕ

$$\partial\phi = \frac{\partial\phi}{\partial x^1} dx^1 + \frac{\partial\phi}{\partial X^2} dx^2 + \dots + \frac{\partial\phi}{\partial X^N} dx^N$$

$$\therefore d\phi = \sum_{j=1}^N \frac{\partial\phi}{\partial X^j} dx^j = \frac{\partial\phi}{\partial X^j} dx^j$$

$$2. \quad \frac{d\bar{X}^k}{dt} = \frac{\partial\bar{X}^k}{\partial X^1} \frac{\partial X^1}{\partial t} + \frac{\partial\bar{X}^k}{\partial X^2} \frac{\partial X^2}{\partial t} + \dots + \frac{\partial\bar{X}^k}{\partial X^N} \frac{\partial X^N}{\partial t}$$

$$\frac{\partial\bar{X}^k}{\partial t} = \frac{\partial\bar{X}^k}{\partial X^M} \bullet \frac{\partial X^M}{\partial t}$$

3. If $\partial S^2 = g_{11}(\partial X^1)^2 + g_{22}(\partial X^2)^2 + g_{33}(\partial X^3)^2$

$$\partial S^2 = g_{kk}(\partial X^k)^2$$

3.4 Contravariant and Covariant Vectors

If N quantities A^1, A^2, \dots, A^N is in a co-ordinate system (X^1, X^2, \dots, X^N) are related to N other quantities $\bar{A}^1, \bar{A}^2, \dots, \bar{A}^N$. In another co-ordinate system $\bar{X}^1, \bar{X}^2, \dots, \bar{X}^N$. By the transformation equations

$$\bar{A}^P = \sum_{q=1}^N \frac{\partial \bar{X}^P}{\partial X^q} A^q$$

They are called component of a Contravariant vector or Contravariant tensor of the first rank or first order.

If N quantities A_1, A_2, \dots, A_N in a co-ordinate system X^1, X^2, \dots, X^N are related to N , other quantities $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_N$ in another co-ordinate system $\bar{X}^1, \bar{X}^2, \dots, \bar{X}^N$. By the transformation equalities

$$\bar{A}_P = \sum_{q=1}^N \frac{\partial X^q}{\partial \bar{X}^P} A_q, P = 1, 2, \dots, N$$

By convection $\bar{A}_P = \frac{\partial X^q}{\partial \bar{X}^P} A_q$ they are called component of a covariant vector or tensor of the first rank or first order.

Note: That a superscript is used to indicate contravariant component and subscript is used to indicate a covariant component.

Examples on Contravariant and Covariant Tensor

1. Write the law of transformation for:

$$\bar{A}_{jk} = \frac{\partial \bar{x}^i}{\partial x^a} \cdot \frac{\partial x^b}{\partial \bar{x}^j} \cdot \frac{\partial x^c}{\partial \bar{x}^k} = A_{bc}^k \Rightarrow$$

2. Illustrate: As an aid for remembering the transformation, note that the relative position of indices $p_1 q_2 r$ On the left side of the transformation are the same as those on the right hand side of the equation. Some these indexes are associated with \bar{x} co-ordinate and since indices i, j, k are easily written as it was done

(a) B_{ijk}^{mn} Consider $\bar{B}_{rst}^{pq} = \frac{\partial \bar{x}^p}{\partial x^m}, \frac{\partial \bar{x}^q}{\partial x^n}, \frac{\partial x^i}{\partial \bar{x}^r}, \frac{\partial x^j}{\partial \bar{x}^s}, \frac{\partial x^k}{\partial \bar{x}^t} B_{ijk}^{mn}$

(b) C^m Consider $\bar{C}^p = \frac{\partial \bar{x}^p}{\partial x^m} C^m$

3.5 Contravariant, Covariant and Mixed Tensor

If N^2 quantities A^{qs} in a co-ordinate system (X^1, X^2, \dots, X^N) are related to N^2 other quantities \bar{A}^{pr} in another co-ordinate system $(\bar{X}^1, \bar{X}^2, \dots, \bar{X}^N)$

by transformation equations $\bar{A}^{pr} = \sum_{q=1}^N \sum_{j=1}^N \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^j} A^{qs} \quad p, r = 1, 2, \dots, N$

$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$ it is called contravariant component of a tensor of the second rank. The N^2 quantities A_{qs} are called covariant component

of a tensor of the second rank is $\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \cdot \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}$ e.g. of mixed tensor

$$\delta_q^p$$

Scalar or invariant: A scalar or invariant is called a tensor of rank zero.

Symmetric and Skew Symmetric Tensor

A tensor is called symmetric w.r.t. of indices if its component remains unaltered upon. Interchange of the indices e.g. if $A_{qs}^{mpt} = A_{qs}^{pmr}$ the tensor is symmetric in p and m.

A tensor is called skew symmetric with respect to 2 contravariant or 2 covariant indices of its component change sign upon interchange of the indices e.g. $A_{qa}^{mpr} = -A_{qs}^{pmr}$ (in m & p)

Tensor of rank greater than 2

A_{kl}^{qs} are component of a mixed tensor of rank 5. (Contravariant of order 5 and covariant of order 2).

The Kronecker Delta

This written δ_k^j is defined as $\delta_k^j = \begin{cases} 0 & \text{If } j \neq k \\ 1 & \text{If } j = k \end{cases}$ is a mixed tensor of 2nd rank.

Examples

1. Evaluate (a) $\int_q^p A_s^{qr}$ (b) $\int_q^p \int_r^q$

Solution

Since $\int_q^p = 1$ if $p = q$ & 0 if $p \neq q$

$$\Rightarrow \int_q^p A_s^{qr} = A_s^{qr}$$

2. Show that $\frac{\partial x^p}{\partial x^q} = \int_q^p$

Solution

If $p = q$, then $\frac{\partial x^p}{\partial x^q} = 1$ since $x^p = x^q$

If $p \neq q$ then x^p and x^q are independent then $\frac{\partial x^p}{\partial x^q} = 0 \Rightarrow \frac{\partial x^p}{\partial x^q} = \int_q^p$

3.6 Fundamental Operation with Tensor

Addition and Subtraction: The sum of 2 or more tensor of the same rank and type i.e. same number of Contravariant indices and same number of covariant indices is also a tensor of the same rank and type thus if A_q^{mp} and B_q^{mp} are the tensor then $C_q^{mp} = A_q^{mp} + B_q^{mp}$ of vector is Enumerative and Associative. The difference of 2 tensors of the same rank and type is also a tensor of the same rank and type. i.e. $D_q^{mp} = A_q^{mp} - B_q^{mp}$

Example

If A_r^{pq} and B_r^{pq} are tensor. Prove that their sum and difference are tensor.

Solution

$$\bar{A}_i^{jk} = \frac{\partial \bar{x}^{-j}}{\partial x^p} \cdot \frac{\partial \bar{x}^{-k}}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^{-i}} A_r^{pq}$$

$$\bar{B}_i^{jk} = \frac{\partial \bar{x}^{-j}}{\partial x^p} \cdot \frac{\partial \bar{x}^{-k}}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^{-i}} B_r^{pq}$$

$$\Rightarrow \bar{A}_i^{jk} + \bar{B}_i^{jk} = \left(\frac{\partial \bar{x}^{-j}}{\partial x^p} \cdot \frac{\partial \bar{x}^{-k}}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^{-i}} \right) A_r^{pq} + B_r^{pq}$$

$$\text{And } \bar{A}_i^{jk} - \bar{B}_i^{jk} = \left(\frac{\partial \bar{x}^{-j}}{\partial x^p} \cdot \frac{\partial \bar{x}^{-k}}{\partial x^q} \cdot \frac{\partial x^r}{\partial \bar{x}^{-i}} \right) A_r^{pq} - B_r^{pq}$$

3.7 Outer Multiplication

The product of 2 tensors is a tensor whose rank is of the given tensor. This product which involves ordinary multiplication of the component of the tensor is called the outer product

e.g. $A_q^{pr} \bullet B_s^m = C_{qs}^{prm}$ is the outer product of A_q^{pr} and B_s^m

Contraction: If one contravariant and one covariant index of a tensor are set equal, the result indicates that, summation over the equal. Indices are to be taking according to the summation convention. This resulting sum is a tensor of rank 2 lesser than the original tensor. The process is called contraction.

SELF- ASSESSMENT EXERCISE

if a tensor of rank 5 A_{qs}^{mpr} i.e. set $r = s$ to obtain $A_{qs}^{mpr} = B_q^{mp}$

Tensor of Rank 3

Exercise 2 By setting the above B_q^{mp} where $p = 2$ $B_q^{mp} = D^m$

Internal Multiplication

By the process of outer multiplication of 2 tensor followed by contraction, we obtain a new tensor called an inner product of a given tensor, the process is called inner multiplication.

e.g. if A_q^{mp} and B_{st}^r the outer product $A_q^{mp} B_{st}^r$, letting $q = r$, we obtain A_r^{mp} and B_{st}^r and this is the inner product, moreover, if $p = s$ then another inner product is obtain A_q^{mp} and B_{pt}^r

Tensor form of gradient, divergent and curl

- (i) **Gradient:** If ϕ is a scalar of invariant, the gradient of ϕ is defined by $\nabla\phi = \text{grad}\phi = \delta$, $P = \frac{\partial\phi}{\partial x^p}$ where ϕ , P is the covariant derivative of ϕ w.r.t x^p
- (ii) **Divergent:** The divergent of A^p is the contraction of its covariant derivative w.r.t x_p^q be the contraction of A_q
Mathematically,

$$\text{Dir } A^p = A_q^p = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g} A^k) = \frac{\partial A^p}{\partial X^q}$$

- (iii) **Curl:** The curl of A_p is $A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial X^q} - \frac{\partial A_q}{\partial X^p}$

Tensor of rank 2, the curl is also define as $\sum^{pqr} A_{p,q}$

- (iv) **Laplacian:** The Laplacian of ϕ is the divergence of grad ϕ and $\nabla^2\phi = \text{dir } \phi$, $P = \frac{1}{\sqrt{g}} \frac{\partial}{\partial X^j} \left(\sqrt{g} g^{jk} \frac{\partial\phi}{\partial x^k} \right)$.

In case $g < 0$, \sqrt{g} must be replace by $\sqrt{-g}$ both cases $g > 0$ & $g < 0$ can be included the written $\sqrt{|g|}$ in place \sqrt{g} .

Alternating Tensor $\sum_{i,j,k}$

This is define by $\sum_{i,j,k} = \begin{cases} i & \text{if } i, j, k \text{ are in cycle} \\ -1 & \text{if } i, j, k \text{ are anti cycle} \\ 0 & \text{if } i, j, k \text{ on } 0 = k \end{cases}$

$$\begin{aligned} i - j &= \delta_{ij} & 0 - i &= \delta_{i-j} = 1 \\ i - k &= \delta_{ik} & i, j &= \delta_{i-j} = 0 \text{ for } i + j \end{aligned}$$

$$\begin{aligned}
 j - k &= \delta_{jk} \\
 i \wedge j &= |i||j| \sin \phi \text{ where } \phi \text{ is the angle between } i \text{ \& } j \\
 \delta_r \phi &= \frac{\pi}{2} \\
 |i| &= 1, |j| = 1 \\
 i \wedge j &= 1 - 1 \text{ Sum } \frac{\pi}{2} K = K
 \end{aligned}$$

Using alternating tensor

$$\begin{aligned}
 i \wedge j - \sum_{K_{21}}^3 \sum_{ijk} &= [\sum_{ij1} + \sum_{ij2} + \sum_{ij3}] k \\
 &= (0 + 0 + 1)K = K
 \end{aligned}$$

Notation

Let us denote the unit tensor by δ , δ is a matrix.

$\underline{\delta}_1 = (1, 0, 0) = \mathbf{i}$ a unit vector along x – arms

$\underline{\delta}_2 = (0, 1, 0) = \mathbf{j}$ a unit vector along y – arms

$\underline{\delta}_3 = (0, 0, 1) = \mathbf{k}$ a unit vector along z – arms

$$\underline{\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{unit } 3 \times 3 \text{ matrix}$$

Use alternating tensor to establish the result of $\underline{u} \wedge \underline{v}$ where $\underline{u} \wedge \underline{v}$ are vectors. By vector product

$$\begin{aligned}
 \underline{u} \wedge \underline{v} &= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
 &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}
 \end{aligned}$$

By alternating tensor

$$\begin{aligned}
 \underline{u} \wedge \underline{v} &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{ijk} \delta_i U_j V_k \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 [\sum_{ij1} \delta_i U_j V_1 + \sum_{ij2} \delta_i U_j V_2 + \sum_{ij3} \delta_i U_j V_3]
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^3 \sum_{i_{21}} \underline{\delta}_i U_2 V_1 + \sum_{i_{31}} \underline{\delta}_i U_3 V_1 + \sum_{i_{12}} \underline{\delta}_i U_1 V_2 + \sum_{i_{32}} \underline{\delta}_i U_3 V_2 + \sum_{i_{13}} \underline{\delta}_i U_1 V_3 + \sum_{i_{21}} \underline{\delta}_i U_2 V_3 \\
&= \left[\sum_{i=1}^3 \sum_{i_{21}} \underline{\delta}_i U_2 V_1 + \sum_{i_{31}} \underline{\delta}_i U_3 V_1 + \sum_{i_{12}} \underline{\delta}_i U_1 V_2 + \sum_{i_{32}} \underline{\delta}_i U_3 V_2 + \sum_{i_{13}} \underline{\delta}_i U_1 V_3 + \sum_{i_{23}} \underline{\delta}_i U_2 V_3 \right] \\
&\quad \sum_{321} \underline{\delta}_3 U_2 V_1 + \sum_{231} \underline{\delta}_2 U_3 V_1 + \sum_{312} \underline{\delta}_3 U_1 V_2 + \sum_{132} \underline{\delta}_1 U_3 V_2 + \sum_{213} \underline{\delta}_2 U_1 V_3 + \sum_{123} \underline{\delta}_1 U_2 V_3 \\
&= \sum_{321} \underline{\delta}_3 U_2 V_1 + \sum_{231} \underline{\delta}_2 U_3 V_1 + \sum_{312} \underline{\delta}_3 U_1 V_2 + \sum_{132} \underline{\delta}_1 U_3 V_2 + \sum_{213} \underline{\delta}_2 U_1 V_3 + \sum_{123} \underline{\delta}_1 U_2 V_3 \\
&= -U_2 V_1 K + U_3 V_1 j + U_1 V_2 K - U_3 V_2 i - U_1 V_3 j + U_2 V_3 i \\
&= (U_2 V_3 - U_3 V_2) i + (U_3 V_1 - U_1 V_3) j + (U_2 V_1 - U_1 V_2) k = \underline{U} \wedge \underline{V} \text{ proved.}
\end{aligned}$$

Defined

The vector $\underline{U} = (U_1, U_2, U_3)$, $\delta_1 = (1, 0, 0)$, $\delta_2 = (0, 1, 0)$, $\delta_3 = (0, 0, 1)$ e.t.c. is called first order tensor.

Let L^- be a second order tensor

$$L^- = \begin{pmatrix} L^-_{11} & L^-_{12} & L^-_{13} \\ L^-_{21} & L^-_{22} & L^-_{23} \\ L^-_{31} & L^-_{32} & L^-_{33} \end{pmatrix} \& \delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_{3 \times 3}$$

3.8 Multiplication of Tensors

We have various multiplications

Dynoadic product

Let $\underline{U} = (U_1, U_2, U_3)$, $\underline{V} = (V_1, V_2, V_3)$ be vectors i.e. first order tensor

Definition: we defined the dynoadic products of \underline{u} & \underline{v} written as $\underline{U} \underline{V}$

$$\underline{U} \underline{V} = \begin{pmatrix} U_1 V_1 & U_1 V_2 & U_1 V_3 \\ U_2 V_1 & U_2 V_2 & U_2 V_3 \\ U_3 V_1 & U_3 V_2 & U_3 V_3 \end{pmatrix}$$

$$\text{Hence } \delta_1 - \delta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \& \delta_1 - \delta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Let } L^{-1} = \delta_i \delta_j L^{-ij} \quad N = 3$$

Let s be scalar and L^{-} be second order tensor then

$$SL^{-} = S\delta_i \delta_j (i) \quad N = 3$$

$$SL^{-} = \sum_{i=1}^3 \sum_{j=1}^3 S \delta_i \delta_j L^{-ij}$$

Single Dot, Product of 2 Tensors

Definition:

Let $\underline{0}^{-}, L^{-}$ be tensors. We defined the single dot product of $\underline{0}^{-}$ and \underline{L}^{-}

Written $\underline{0}^{-} \bullet \underline{L}^{-}$ so that the iL component of $\underline{0}^{-} \bullet \underline{L}^{-} = \sum 0^{-ij} L^{-jl}$

$$L^{-} = \begin{pmatrix} L^{-}_{11} & L^{-}_{12} & L^{-}_{13} \\ L^{-}_{21} & L^{-}_{22} & L^{-}_{23} \\ L^{-}_{31} & L^{-}_{32} & L^{-}_{33} \end{pmatrix}, \quad \underline{0}^{-} = \begin{pmatrix} 0^{-}_{11} & 0^{-}_{12} & 0^{-}_{13} \\ 0^{-}_{21} & 0^{-}_{22} & 0^{-}_{23} \\ 0^{-}_{31} & 0^{-}_{32} & 0^{-}_{33} \end{pmatrix}$$

$$\underline{0}^{-} \bullet \underline{L}^{-} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \text{ where } S_{il} = \sum_{j=1}^3 0^{-ij} L^{-jl}$$

$$= 0^{-i1} L^{-1l} + 0^{-i2} L^{-2l} + 0^{-i3} L^{-3l}$$

Dot Product of a Tensor and a Vector

Let L^{-} be a tensor, let \underline{v} be a vector. The dot product of \underline{L}^{-} and \underline{v} written $[\underline{L}^{-}, \underline{v}]$ is defined by

$$[\underline{L}^{-} \bullet \underline{v}] = \sum_0 \delta_i (\sum L^{-ij} V_j) \text{ so that } i\text{th component of } [\underline{L}^{-} \bullet \underline{v}] = \sum L^{-ij} V_j$$

Example

$$\text{If } \underline{L}^{-} = \begin{pmatrix} L^{-}_{11} & L^{-}_{12} & L^{-}_{13} \\ L^{-}_{21} & L^{-}_{22} & L^{-}_{23} \\ L^{-}_{31} & L^{-}_{32} & L^{-}_{33} \end{pmatrix} \quad \& \quad V = (V_1, V_2, V_3)$$

The first component is $\sum_{j=1}^3 L^{-1j} V_j = L^{-11} V_1 + L^{-12} V_2 + L^{-13} V_3$

The second component is $\sum_{j=1}^3 L^{-2j} V_j = L^{-21} V_1 + L^{-22} V_2 + L^{-23} V_3$

The third component is $\sum_{j=1}^3 L^{-3j} V_j = L^{-31} V_1 + L^{-32} V_2 + L^{-33} V_3$

So $[\underline{L}^{-} \bullet \underline{V}] = (S_1, S_2, S_3)$

Let consider

$$[\underline{v} \bullet \underline{L}^{-1}] = \sum \delta_i V_i L^{-ji} \quad \text{or} \quad \sum \delta_i \sum V_j L^{-ji}$$

The i th component = $\sum V_j \underline{L}^{-ji}$

First component = $\sum_{j=1}^3 V_j \underline{L}^{-j1} = V_1 L^{-11} + V_2 L^{-21} + V_3 L^{-31}$

Second component = $\sum_{j=1}^3 V_j \underline{L}^{-j2} = V_1 L^{-12} + V_2 L^{-22} + V_3 L^{-32}$

Third component = $\sum_{j=1}^3 V_j \underline{L}^{-j3} = V_1 L^{-13} + V_2 L^{-23} + V_3 L^{-33}$

Note L^{-} is symmetric if $L^{-ij} = L^{-ji}$

Otherwise L^{-} is not symmetric hence

$$[\underline{L}^{-} \bullet \underline{V}] \neq [\underline{V} \bullet \underline{L}^{-}]$$

Example

Prove that (i) $[\underline{\delta} \bullet \underline{v}] = [\underline{v} \bullet \underline{\delta}] = \underline{v}$

(ii) $[\underline{u} \underline{v} \bullet \underline{w}] = \underline{u} [\underline{v} \bullet \underline{w}]$

3.9 Double Product of 2 Tensors

Let δ, L^{-} be tensor

Definition: we define the double product of 0^{-} and L^{-} written $(0^{-} : L^{-})$

$$\text{as } (0^{-} : L^{-}) = \sum_i \sum_j 0^{-ij} \underline{L}^{-ji}$$

In the same way

$$(\underline{L}^- \bullet \underline{uw}) = \sum_i \sum_j L^-_{ij} U_j W_i$$

Similarly

$$(i) \quad (\underline{vw} : \underline{xy}) = \sum_i \sum_j V_i W_j X_j Y_i$$

$$(ii) \quad (\underline{0}^- : \underline{L}^-) = \sum L^-_{ij} \quad , \quad \delta_{ij} = a \text{ if } i \neq j$$

Prove

$$\underline{0}^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \underline{L}^- = \begin{pmatrix} L^-_{11} & L^-_{12} & L^-_{13} \\ L^-_{21} & L^-_{22} & L^-_{23} \\ L^-_{31} & L^-_{32} & L^-_{33} \end{pmatrix}$$

$$\begin{aligned} (\underline{0}^- \bullet \underline{L}^-) &= \sum_{i=1}^3 \sum_{j=1}^3 0^-_{ij} L^-_{ji} \\ &= \sum_{i=1}^3 (0^-_{i1} L^-_{1i} + 0^-_{i2} L^-_{2i} + 0^-_{i3} L^-_{3i}) \\ &= 0^-_{11} L^-_{11} + 0^-_{22} L^-_{22} + 0^-_{33} L^-_{33} \\ &= \int_{11} \underline{L}^-_{11} + \int_{22} \underline{L}^-_{22} + \int_{33} \underline{L}^-_{33} \text{ i.e. } \int_{ij} = 0 \text{ if } 0 \neq j \text{ \& } \delta_{ij} \neq \text{uf } \delta = j \\ &\succ (\underline{0}^- \bullet \underline{L}^-) = \underline{L}^-_{11} + \underline{L}^-_{22} + \underline{L}^-_{33} \end{aligned}$$

4.0 CONCLUSION

The tensor analysis discussed in this unit is to familiarise you with high language being used in higher mathematics. You need to study this unit properly so that you can apply it to a lot of future mathematics courses.

5.0 SUMMARY

In this unit we have learnt about the following:

- summation convention in tensor analysis
- product of tensors
- covariant tensors , and contra variant tensors
- the gradient ,divergent and curl of tensors
- tensors of various ranks.

You are required to master them properly so that you will able to do various exercises and Tutor- Marked Assignments in this unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Write each of the following using the summation convention.

$$d\phi = \frac{\partial\phi}{\partial x^1} dx^1 + \frac{\partial\phi}{\partial x^2} dx^2 + \dots + \frac{\partial\phi}{\partial x^N} dx^N.$$

2. If A_k^{pr} and B_s^m are tensor. Prove that $C_{qs}^{prm} = A_q^{pr} B_s^m$ is also a tensor

$$\bar{A}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial \bar{x}^j}{\partial x^r} \cdot \frac{\partial x^q}{\partial \bar{x}^k} A_q^{pr}$$

$$\bar{B}_n^m = \frac{\partial \bar{x}^L}{\partial x^m} \cdot \frac{\partial x^s}{\partial \bar{x}^r} B_s^m$$

$$\Rightarrow \bar{A}_k^{ij} \bullet \bar{B}_n^m = \bar{C}_{kr}^{ijL} = \frac{\partial \bar{x}^i}{\partial x^p} \bullet \frac{\partial \bar{x}^j}{\partial x^r} \bullet \frac{\partial x^q}{\partial \bar{x}^k} \bullet \frac{\partial \bar{x}^L}{\partial x^m} \bullet \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}^{prm}$$

$$\frac{d\bar{x}^k}{dt} = \frac{\partial \bar{x}^k}{\partial x dt} dx^1 + \frac{\partial \bar{x}^k}{\partial x^2} d$$

7.0 REFERENCES/FURTHER READING

Hildebrand, F.B. (nd). *Advanced Calculus for Application*.

Verma, P.D.S.(nd). *Engineering Mathematics*.