NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE : MTH 305

COURSE TITLE: Complex Analysis II

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MODULE 1

UNIT 1 FUNCTION OF COMPLEX VARIABLES

CONTENT

1.0 INTRODUCTION

The set of real number is not adequate to handle some of the numbers we come across in mathematics. We need another set – the complex numbers.

In this course we will do analysis on complex variables and establish those results which are analogue to the real number systems.

2.0 OBJECTIVES

At the end of this unit, you will learn about:

- variables and functions of complex number
- functions and transformation or complex variables.

3.0 MAIN CONTENT

A symbol, such as z, which can stand for any complex number is called a complex variable. If to each value a complex variable z can assume there correspondence one or more values of a complex variable w, we say that W is a function of z and write $w = f(z)$ or $w = g(z)$ etc. The variable z is sometimes called an independent variable while w is called a dependent variable. The value of a function at $z = a$ is often written as $f(a)$.

e.g. $f(z) = z^z$, for $z = 3i$, $f(z) = f(3i) = -9$. If one value of w corresponds to each value of z, we sat that w is a single-valued function of z or that $f(z)$ each value of z, we say w is a multiple-valued or many-valued function of z.

Example: if $w = z^3$, then to each value of z there is only one value of w. $w = f(z) = z^3$ is a single-valued function of z.

Example

If $w = z^{1/2}$

3.2Transformations

If $w = u + iv$ (where u and v are real) is a single-valued function of $z = u = iy$ (where x and y are real), we can write $u + iv = f(x + iy)$. By equating real and imaginary parts this is equivalent to

$u = u(x, y),\ v = v(x, y)$. ………………………………………..… ……….(1)

Hence, given a point (x, y) in the z –plane, there corresponds a point (u, v) in the w plane. The set of equations (1) [or the equivalent, $w = f(z)$] is called a transformation.

Example 1

If $w = z^2$**, then**

$f(z^2) = (x + iy)^2 = x^2 - y^2 + 2xy$

Hence, $u(x, y) = x^2 - y^2$

and $v(x, y) = 2xy$

Example 2

Let $w = f(z) = \dfrac{1}{z}$ for

z (.)

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^3}$$

Hence,

$u(x,y)=\dfrac{x}{x^2+y^2}$ and $v(x,y)=\dfrac{-y}{x^2+y^2}$

3.3The Elementary Functions

1. **Polynomial Functions**: Polynomial functions $P(z)$ are defined as $P(z)=a_0z^{\eta}+a_1z^{\eta-1}+................+a_{\eta-1}z+a_{\eta}$ where $a_0\neq 0,\quad a_1,.........,a_n$ are complex constants and η is a positive integer called the degree of the polynomial $P(z)$.

2. **Rational Algebraic Function** are defined by $F(z)=\dfrac{P(z)}{\vartheta(z)}$ where $P(z)$ and $\vartheta(z)$ are polynomials.

3. Exponential Functions are defined by

$w=f(z)=\ell^{x+iy}=\ell^{x}$ (Cos y - i Sin y)

where e is the natural base of logarithms. (e=2.71828). complex exponential functions have properties similar to those of real exponential functions.

For example $\ell^{z1}\bullet\ell^{z2}=\ell^{zi+z2},\ell^{z1}/\ell^{z2}=\ell^{z1-z2}$ $\ell^{z_1}\bullet\ell^{z_2}=\ell^{x_1}$(Cos y_1+ i $\sin y_1$)$\bullet\ell^{x_2}$ (Cos y_2+i Sin y_2)

$$=\ell^{x_1}+\ell^{x_2}(\text{ Cos } y_1+i\,Sin\,y_1)\ (Cis\,y_2+i\,Siny_2)$$
$$=\ e^{x_1}-e^{x_2}(Cos\ yi+1\ Sin\,yi)(Cos\ y_2+1\ Sin\ y_2)$$
$$=e^{x_1}+n_2[Cos\ y_1\ Cos\ y_2+i\ Cos\ y_i\ Sin\ y_2+1Siny_1\ Cos\,y_2-Sin\ y_1\ Sin\,y_2]$$
$$=\ e^{x_1+x_2}\ [(Cos\ y_1\ Cos\ y_2-Sin\,y_1\ Sin\,y_2)+i\,(Cos\,y_1\ Sin\,y_2+Sin\,y_1\ Cos\,y_2)]$$
$$=\ e^{x_1+x_2}Cos(y_1+y_2)+i\ Sin(y_1+y_2)$$
$$=\ e^{z_1+z_2}$$

Note that when $w=e^z$*, the number w can be written as*

$w=\rho e^{i\phi}$ where $\rho=e^x$ and $\phi=y$

If we think of $w=e^z$ as a transformation from z to the w plane, we thus find that any non zero point $w=\rho e^{i\phi}$ is the z- Log $\rho+i\phi$

Therefore the range of the exponential function $w=e^z$ id the entire nonzero point $w=pe^{i\phi}$ is actually the image of an infinite number of points in the z plane under the transformation $w=e^z$. For in general, ϕ may have any one of the values

$\phi = \Phi + 2n\dot{-}(n = 0, \pm 1, \pm 2...)$ where Φ denotes the principal value of arg w. It then follows that w is the image of all the points.

$z = \log\ \rho + i\Phi + i\pi \quad (1\ n = 0, \mp 1, \pm 2,....)$

***Example**: find all values of z such that* $e^z = -1$

Solution

$e^z = e^x e^{iy}$,and $-1 = 1e^{i\pi}$ so that

$e^x e^{iy} = 1e^{i\pi}$

By equality of two complex numbers in exponential form, this means that

$e^x - 1$ *and* $y = \pi + 2n\pi$ where n is an integral.

$n = \log 1 = 0,$ then

$z = (2n+1)\pi \qquad (n = 0, \pm 1, \pm 2.....)$

Example: Find the values of z for which $e^{4z} = 1$

Solution:

$e^{4z} = i$
$e^{4x}.e^{4yi} = e\ 1.i$

So that, by equality, w have

$e^{yx} = e^0 \Rightarrow 4x = 0 \Rightarrow n = 0$ and

$4y = 2n\bar{\pi} + \frac{\pi}{2}$

$y = \frac{1^{n\pi}0}{2} + \frac{1\pi 0}{8}$ for $(\eta = 0, \pm 1, \pm 2,...)$

The solution is then $\frac{1}{2}n\pi\, i\, c \pm \frac{1}{8}\pi c$

SELF ASSESSMENT EXERCISE

1 . Show that (i) $\left|e^z\right| = e^x$ (ii) $e^{z+2k\pi i}$

2. Find the value of z for which $e^{3z} = 1$

4. **Trigonometric Function**: are defined in terms of exponential functions as follows:

$$Sin\, z = \frac{e^{iz} - e^{-1z}}{2i}, \qquad Cos\, z = \frac{e^{iz} + e^{-iz}}{2}$$

$$Sec\, z = \frac{1}{Cos\, z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad CSC\, Z = \frac{1}{Sin\, z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{Sin\, z}{Cos\, z} = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}, \qquad Cot\, z = \frac{Cos\, z}{Sin\, z}$$

$$= \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}}$$

Many properties satisfied by real trigonometric functions are also satisfied by complex trigonometric function.

e.g.

$$Sin^2 z + Cos^2 z = 1, \quad 1 + 1\tan^2 z = Sec^2 z, \quad 1 + wt^2 z = \csc^2 z.$$

$$Sin(-z) = -Sin\, z \quad Cos(-z) = Cos\, z \quad \tan(-z) = -\tan z$$

$$Sin(z_1 \pm z_2) = Sin\, z_1\ Cos z_2 \pm Cos\, z_1\, Sin\, z_2$$

$$Cos\left(z_1 \pm z_2\right) = Cos\ z_1\ \ Cos\, z_2 \pm Sin\, z_1\ \ Sin\, z_2$$

$$\tan\left(z_1 \pm z_2\right) = \frac{\tan z_1 \pm \tan z_2}{1 - \tan z_1\ \tan z_2}.$$

Exercise

Prove that $Sin^2 z_0 + Cos^2 z_0 = 1$

Proof

By definition, $Sin\, z \frac{e^{iz} - e^{-1z}}{2i}$, $Cos\ z = \frac{e^{iz} + e^{-iz}}{2}$

Then $Sin^2 z + Cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)$

$$= \left(\frac{e^{2iz} - 2 + e^{-2iz}}{4}\right) + \left(\frac{e^{2iz} + 2 + e^{-2iz}}{4}\right)$$

= 1.

6. Hyperbolic Function: Are defined as follows:

$$Sin\,hz = \frac{e^z - e^{-z}}{2}, \quad Coshz = \frac{e^z + e^{-z}}{2}, \quad \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$Sechz = \frac{1}{Coshz}, \quad Co\sec hz = \frac{1}{Sinhz}, \quad Coth = \frac{Coshz}{Sinhz}$$

The following properties hold:

$$Cosh^2 z - Sinh^2 z = 1, \quad 1 - \tanh^2 z = \sec h^2 z, \ Coth^2 z - 1 = \csc h^2 z$$
$$Sin\,h(-z) = -Sinhz, \quad Cosh(-z) = Cosh\,z, \ \tanh(-z) = -\tanh(z)$$
$$Sinh(z_1 \pm z_2) = Sinhz_1 \ Coshz_2 \ \pm Cos\,hz_1 \ \ Sin\,hz_2$$
$$Cosh(z_1 \pm z_2) = Cosh\,z_1 \ Coshz_2 \pm Sinh\,z_1 \ Sinh\,z_2$$
$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}.$$

These properties can easily be proved from the definitions. For example, to show that:

$Cos\,h^2 z - Sin\,h^2 z = 1$, we observed that,

$$Cosh^2 z - Sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$
$$= \tfrac{1}{4}\left(e^{2z} + 2e^2 e^{-z} + e^{-2z}\right) - \tfrac{1}{4}\left(e^{2z} - 2e^z e^{-z} + e^2\right)^2$$
$$= \tfrac{1}{4}\left(e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}\right)$$
$$= \tfrac{4}{4} = 1.$$

Exercise:

The proofs of others are left as exercise

Trigonometric and hyperbolic functions are related

For instance:

$$Sin\,iz = i\,Sin\,hz, \quad Cosi\,z = Cosh\,z, \ \tan i\,z = i \tanh z.$$
$$Sinhiz = i\,Sin\,z, \quad Cos\,hiz = Cos\,z, \ \tan h\,i\,z = i \tan z$$

SELF ASSESSMENT EXERCISE

1. If $Cos\, z = 2$, Find
   (a) $Cos\, 2\, z$
   (b) $Cos\, 3\, z$

2. Find $U(x, y)$ and $V(x, y)$ such that
   (a) $Sinh\, 2z = \mu + iv$
   (b) $z\, Cosh\, z = \mu + iv$

3. Evaluate the following
   (a) $Sinh(\frac{\pi}{8})i$
   (b) $\cosh \frac{2n+1)}{2} \pi$

(3) $Tan \cosh \frac{\pi i}{2}$

4. Show that $\left| Tanh \frac{\pi(1+i)}{4} \right| = 1$
5. If $\tan z = u + iv$ show that

$$\mu = \frac{Sin\, 2u}{Cos\, 2u + Cos\, h\, 2\, y}, \quad v = \frac{Sin\, h\, 2\, y}{Cos\, 2u + Cos\, h\, 2\, y}$$

6. Logarithmic Functions

The natural logarithm function is the reverse of the exponential function and can be defined as:

$$w = 1n\, z = 1n\, r + 1(\phi + 2k\pi),\ \ k = 0,\ \pm 1,\ \pm 2.$$

$Where\ \ z = re^{iQ} = re^{i(Q+2k\bar{\pi})}$

$1 \neq z$ is a multiple valued function with the principled value. In $i + i\phi$ where $0 \leq \phi \leq 2\pi$ or its equivalent.

For $z = a^w$ where a is real, $w = \log,\ z\ \ where\ \ a > 0,$ and $a \neq 0,1$, in this case, $z = e^{w\, In\, a}$ and so $w = \frac{1nz}{In\, a}$.

Exercises

Evaluate

(1)In (-40

(2)In $\left(\sqrt{3-i}\right)$

Solution

(i) In (-4)

$z = -4 + 0i, \quad r = |z| = \sqrt{-4^7 + 0^2} = \pm 4.$

$\arg z = \tan^{-1}\frac{0}{4} = \tan^{-1} 0 = 0 = \pi = \pi + 2k$

In (-4) = In $\left[4e^{i(\pi+2k\pi)}\right] =$ In 4+ $\left(\overline{\pi} + 2k\overline{\pi}\right)i$ for $k = 0, \pm 1, \pm 2.......$

(ii) In $\left(\sqrt{3} - i\right)$

$z = \sqrt{3} + -u, \quad r = |z| = \sqrt{\sqrt{3^2} + (-1)^2} = 2.$

$\arg z = \tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{26}{180}\pi = \frac{334\pi}{180} + 2\pi k \neq \frac{11\pi}{6} + 2k\pi$

In $\left(\sqrt{3} - i\right) =$ In $\left(2e\frac{11}{6} + 2k\pi\right) =$ In 2 + $\left(\frac{11\pi}{6} + 2k\pi\right)i$

SELF ASSESSMENT EXERCISE

1. Evaluate

(a) In $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

(b) In $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

(c) In $\left(\sqrt{3} - 2i\right)$

7. Inverse Trigonometric Functions

To define the inverse sin function $Sin^{-1}z$, we write $w = Sin^{-1}z$ when $z = Sin\,w$. That is

$w = Sin^{-1}z$, when $z = \frac{e^{iw} - e^{-iw}}{21}$

Which is equivalent to:

$\left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0$.

This is quadratic in w^{iw}. *Solving for* e^{iw} *one have*

$$e^{iw} = 1z + \left(1 - z^2\right)^{1/2}$$

Taking logarithms of both sides and recalling that $w = Sin^{-1}/_z$, we have

$$Sin^{-1}z = -i\ In\left[iz - \sqrt{1-z^2}\ \right]$$

Which is a multiple-valued function with infinitely many values at each z

.

Similarly,

$$Cos^{-1}z = -i\ \ In\left[z + i\sqrt{1-z^2}\ \right]$$

$$\tan^{-1} z = \ \frac{i}{2}\ In\left\{\frac{1+z}{1-z}\right\}$$

Which are also multiple valued functions?

Exercise

Find the values of $Sin^{-1}2$

Solution

$$Sin^{-1}2 = -i\ \ In\left[2i + \sqrt{1-2^2}\right)$$

$$= \ -i\ \ In\left(2i + \sqrt{3}i\right) = -2\ \ In\ \left(2 + \sqrt{3}\right)i$$

$$= \ -i\ In(2 + j3)e^{\left(\pi/2 + 2k\pi\right)i}$$

$$= \ \ -i\ In(2 + j3) + \left(\frac{\pi}{2} + 2k\pi\right)\ i$$

$$-i\ In\ (2 + j3) + \frac{\pi}{2} + 2k\pi$$

SELF ASSESSMENT EXERCISE

1. Evaluate

(a) $Cos^{-1}2$

(b) $Cos^{-1}2$

8. Inverse Hyperbolic Functions

If $z = Sinhw$ then $w = Sinh^{-1}z$ is called the inverse hyperbolic sine of Z. Other inverse hyperbolic functions are similarly defined.

$$Sin\,h^{-1}z = In\ \left\{z + \sqrt{z^2}\ + 1\right\}$$

$Cos\,h^{-1}z = In\{z + \sqrt{z^2} - 1\}$

$\tan h^{-1}z = \frac{1}{2}\,In\left(\frac{1+z}{1-z}\right)$

In each case, the constant $2k\bar{\pi}i$, k = 0, $\pm 1, \pm 2...$ has been omitted. They are all multiple valued functions.

$Cos\,h^{-1}i = In\ \{i + \sqrt{-1-1}\ =\ In\{i = \sqrt{2i}\}$

$= \text{In}\ (1 + j2)i\ =\ In(1+\sqrt{2})\exp\left\{\frac{\pi}{2} + 2n\pi\right\}i$

$= \text{In}\ (1 = \sqrt{2}) + i\frac{\pi}{2} + 2n\pi$

SELF ASSESSMENT EXERCISE

1. Find all the values of

(a) $Sinh^{-1}\ i$

(b) $Sin\,h^{-1}\ [In\ (-1)]$

4.0 CONCLUSION

In this unit we considered in general, functions of complex variables and considered various functions in these categories. Practice all exercises in this unit to gain mastery of the topic.

5.0 SUMMARY

What we have learnt in this unit can be summarized as follows:

(a) Definition of Complex Variables
(b) Some Elementary functions of Complex Variables
© Transformation of Complex variables.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that $Cos^{-1}z = -i\ In\ \left[z + i\ \sqrt{1-z^2}\right]$

2. Show that: In $(2-1) = \frac{1}{2}$ In $\{(u-1)^2 + y^2\} + i\tan^{-1}\frac{y}{x-1}$

3. Evaluate the following

(a) $Sinh(\frac{\pi}{8})i$

(b) $\cosh \frac{2n+1)}{2}\pi$

(3) $Tan \cosh \frac{\pi i}{2}$

4. Show that $\left| Tanh \frac{\pi(1+i)}{4} \right| = 1$

5. If $\tan z = u + iv$ show that

$$\mu = \frac{Sin\,2u}{Cos\,2u + Cos\,h\,2\,y}, \quad v = \frac{Sin\,h\,2\,y}{Cos\,2u + Cos\,h\,2\,y}$$

7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand (1976) Advanced *Calculus For Application* 2nd Edition

UNIT 2 LIMITS AND CONTINUITY OF FUNCTION OF COMPLEX VARIABLES

CONTENTS

1.0 INTRODUCTION

In this unit, we will learnt about limits, and continuity in complex variables,

We shall establish some relevant theorems on limits and continuity.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- limit and continuity of functions of complex variables
- theorems related to limits and continuity of complex variables
- Answer all related questions on limits and continuity.

3.0 MAIN CONTENT

3.1 Limits

Definition: Let a function f be defined at all point Z in some neighborhood Z_0, except possibly for the point Z_0 itself. A complex number L is said to be the limit of $f(z)$ as Z approaches Z_0 if for each positive number ε there is a positive number δ such that
$|f(z) - L| < \varepsilon$ whenever $0 < |Z - Z_0| < \delta$

We write

$$\lim_{z \to z} f(z) = L$$

Example: Show that

$$\lim_{z \to 2i} \left(2x + iy^2\right) = Atl \qquad z = x + iy$$

Solution

For each positive number ε. We must find a positive number ε such that $\left|2x + iy^2 - 4i\right| < \varepsilon$ whenever $0 < |z - 2i| < \delta$

To do this, we must write

$$\left|2x + iy^2 - 4i\right| \le 2|x| + \left|y^2 - 4\right| = 2|x| + |y-2|\ |y+2|$$

and thus note that the first of inequalities will be satisfied if

$2|x| < \dfrac{\varepsilon}{2}$ and $|y-2|\ |y+2| < \varepsilon/2$

The first of these inequalities is, of course, satisfied if $|x| < \varepsilon/2$. To establish conditions on y such that the second holds, we restrict y so that $|y-2| < \varepsilon$ and then observe that

$$|y+2| = |(y-2)+4| \le |y-2| + 4 < 5$$

Hence if $|y-2| < \min\{\varepsilon/10, 1\}$, *if follows that* $|y-2|\ |y+2| < \left(\dfrac{\varepsilon}{10}\right)5 = \varepsilon/2$

An appropriate value of δ

is now easily seen from the conditions that $|x|$ be less than $\varepsilon/4$ and that $|y-2|$

be less then min $\{\varepsilon/10, 1\}$

$\delta =$ min $\{\varepsilon/10, 1\}$

Note that the limit of a function $f(z)$

at a point z_0

if it exists is unique. For suppose that

$$\lim_{z \to z_0} f(z) = L_0 \text{ and } \lim_{z \to z_0} f(z) = L_1$$

Then for an arbitrary positive number δ_0

and δ_1

© such that

$|f(z) - L_0| < \varepsilon$ whenever $0 < |z - L_0| \quad \delta_0$

and $|f(z) - L_1| < \varepsilon$ whenever $0 < |z - L_1| < \delta_1$

So if $0 < |z - z_0| < \delta$ where δ denotes the smaller of the two numbers δ_0 and δ_1, then

$$|(f(z) - Lo) - (f(z) - L_1)| \leq |f(z) - Lo| + |f(z) - L_1| < 2\varepsilon$$

That is

$$|L_1 - Lo| < 2\ \varepsilon$$

But

$L_1 - L_0$ is a constant, and ε can be chosen arbitrarily small. Hence,

$L_1 - L_0 = 0$ *or* $L_i = L_0$

***Definition**: The statement*

$$\lim_{z \to \infty} f(z) = L_0$$

Means that for each positive number ε *there is a positive number* δ *such that*

$|f(z) - L_0| < \varepsilon$ whenever $|z| < \dfrac{1}{\delta}$

That is, the point $L = f(z)$ lies in the ε *nbd* $|l - L_0| < \varepsilon$ of L_0 whenever the point z lies in the *nbd* $|z| > 1/\delta$ of the point at infinity.

***Example**: Observe that*

$$\lim_{z \to \infty} \frac{1}{z^2} = 0$$

Since

$$\left| \frac{1}{z^2} - 0 \right| < \varepsilon \text{ whenever } |z| < \frac{1}{\sqrt{\varepsilon}}$$

Hence $\delta = \sqrt{\varepsilon}$

When L_0 *is the point at infinity and* z_0 *lies in the finite plane, we write*

$$\lim_{z \to z_0} f(z) = \infty$$

If for each ε there is a corresponding δ such that $|f(z)| > \frac{1}{\varepsilon}$ whenever $0 < |z - z_0| < \delta$

***Example**: As expected*

$$\lim_{z \to 0} \frac{1}{z^2} = \infty$$

for $\left| \frac{1}{z^2} \right| > \frac{1}{\varepsilon}$ whenever $0 < |z - \varepsilon| < \sqrt{\varepsilon}$

3.2 Theorems on Limits

Theorem 1: Suppose that

$f(z) = U|x, y| + V(x, y)$, $z_0 = x_0 + iy_0$ and $L_0 = u_0 + iv_0$ Then

$$\lim_{z \to z_0} f(z) = Lo \text{ ....(1)}$$

If and only if

$$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and}$$

$$\lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0 \text{ ....(2)}$$

Proof

Assume (1) is true, by the definition of limit, there is for each positive number ε, a positive number δ such that $|(u-u_0)+i(v-v_0)| < \varepsilon$ whenever $0 < |(n-n_0)+i(y-y_0)| < \delta$

Since $|u-u_0| \leq |(u-u_0)+i(v-v_0)|$ and

$$(u-u_0) + i\ (v-v_0)$$

$$|v-v_0| \leq |(v-v_0) + i\ (v-v_0)|$$

,

It follows that

$|u-u_0| < \varepsilon$ and $|v-v_0| < \delta$

whenever

$0 < |(n+iy)-(n=iy_0)| < \delta$

which is statement (1), hence the proof

Theorem 2: Suppose that

$\lim_{z \to z_0} f(z) = L_0$ and $\lim_{z \to z_0} f(z) = L_0$

then

$$\lim_{z \to z_0} [f(z) + f(z)] = l_0 + L_0$$

$$\lim_{z \to z_0} [f(z) f(z)] = l_0 L_0$$

if $L_0 \pm 0$,

$$\lim_{z \to z_0} \frac{f(z)}{f(z)} = \frac{l_0}{L_0}$$

Proof: (Left as exercise)

SELF ASSESSMENT EXERCISE

1. Evaluate the following using theorems on limits

(a) $\lim_{z \to z+i} (z^2 + 10z - 15)$

(b) $\lim_{z \to z2i} \frac{(4z+3)(z-1)}{z^2 - 2z + 4}$

(c) $\lim_{z \to 2e^{\pi i/3}} \frac{z^3+8}{z^4+4z^2+16}$

(d) $\lim_{z \to zi} \left(iz^4 + z^2 - 10i\right)$

(e) $\lim_{z \to e^{\pi i/4}} \frac{z^2}{z^4+z+1}$

Show that

$$\lim_{z \to 2e^{\pi i/3}} \frac{z^3+\delta}{z^4+4z^2+16} = \frac{3}{8} - \frac{\sqrt{3}}{8} i$$

3.3 Continuity

Definition: A function f is continuous at a point z_0 if all the following conditions are satisfied.

(1) $\lim_{z \to z_0} f(z)$ exists

(2) $f(z_0)$ exists

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

Note that statement (3) contains (1) and (2) and it say that for each positive number ε there exist a positive number δ such that

$\left| f(z) - f(z_0) \right| < \varepsilon$ whenever $\left| z - z_0 \right| < \delta$

a function of complex variable is said to be contours in a region R if it is continuous at each point

Example: The function

$f(z) = xy^2 + i(2x - y)$

is its everywhere in the complex plane because the component functions are polynomials on x and y and are therefore continuous at each point (n, y)

Example: If

$$f(z) = \begin{cases} z^2 & z \neq l \\ u & z - 1 \end{cases}$$

$\lim_{z \to i} f(z) = -1$ But $f(i) = 0$. Hence, $\lim_{z \to i} f(z) \neq f(i)$

Therefore the function is not continuous at $z = 1$

Example: The function
$f(z) = e^{xy} + i\ Sin\left(n^2 - 2ny^3\right)$
is continuous for all z because of
the continuity of the polynomials on n

and y
as well as the continuity of the exponential and sine functions.

Theorem on Continuity

1. if $f(z)$ **and** $g(z)$ are its at $z - z_0$. So also are the functions $f(z) * g(z)$, $f(z) - g(z)$, $f(z)\ g(z)$, $f(z)/g(z)$, the last only $g(z_0) \pm 0$.
2. A function of a continuous function is its $w = g[f(z)]$ is its $f(z)$ is its
3. If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ and also it's in the region.
4. If a function $f(z)$ is its in a closed region, it is bounded in the region, i.e. there exists a constant M such that $|f(z)| < M$ for all points z in the region.

SELF ASSESSMENT EXERCISE

1. Let $f(z) = \dfrac{z^2 + 4}{z - 2i}$ if $z \pm 2i$ while $f(2i) = 3 + 4i$

(a) Prove that $\lim_{z \to i} f(z)$ exists and determine its value

(b) Is $f(z)$ its at $z = 2i$? Explain?

(c) Is $f(z)$ its at point $z \neq 2i$?. Explain

2. Find all possible points of discontinuity of the following function

   (a) $f(z) = \dfrac{2z-3}{z^2+2z+2}$

   (b) $f(z) = \dfrac{3z^2+4}{z^2-16}$

(c) $f(z) = Cot\ z$

Answer

(a) $-1 \pm i$

(b) $\pm 2, \pm 2i$

(c) $k\pi,\quad k \neq v, \pm, \pm 2.$

3. For what values of z are each of the following function continuous

   (a) $f(z) = \dfrac{z}{z^2+1}$

   (b) $f(z) = \dfrac{1}{Sin\, z}$

4.0 CONCLUSION

In this unit we have studied limits of functions, continuity of functions of complex variables in a manner similar to that of real variables. You are required to master them properly so that you can be able to apply them when necessary.

5.0 SUMMARY

Recall the following points;

- Continuity in Complex variables can be treated analogously as in the real variables
- If $f(z)$ is a continuous complex variable so also its real and imaginary parts.
- A complex function $f(z)\bullet$ is bounded if there exist a constant M>0 such that $|f(z)| < M$

6.0 TUTOR-MARKED ASSIGNMENT

1. Prove that

$$\lim_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - 1} = 4 + 4i$$

Is the function its at $z - i$?

2. Factorized
   (i) $z^3 + 8$
   (ii) $z^4 + 4z^2 + 16$

(b) (i) Show that

$$\lim_{z \to 2e^{\pi/3}} \frac{z^3 + 8}{z^4 + 4z^2 + 16} = \frac{3}{8} - \frac{\sqrt{3}}{8} i$$

(ii) Discuss the continuity of

$$f(z) = \frac{z^3 + 8}{z^4 + 4z^2 + 16} \text{ at } z = 2e^{\pi i/3}$$

7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand (1976) Advanced *Calculus For Application* 2nd Edition

UNIT 3 CONVERGENCE OF SEQUENCE AND SERIES OF COMPLEX VARIABLES

CONTENTS

1.0 INTRODUCTION

In this unit, you will learn about sequences and series of complex variables. You will also learn about the convergence of these series and sequences.

All related theorems in real variables will be established for complex variables. We shall consider Taylor and Laurent series.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- to define convergence of sequences and series on complex variables
- solve related problems on series and sequence.

3.0 MAIN CONTENT

3.1 Definition

An infinite sequence of complex numbers, $z_1, z_2, ...z_n, ...$ has a limit z if for each positive number ε there exists a positive integral number such that

$|z_n - z| < \varepsilon$ whenever $n > n_0$.

If the limit exists, it is unique.

When the limit z exists, the sequence is said to converge to z; and we write

$$\lim_{n \to \infty} z_n = z$$

If the sequence has no limit, it diverges.

Theorem: Suppose that $Z_n = x_n + iy_n \ (n = 1, 2, ...)$ and $z = x + iy$. Then

$$\lim_{n \to \infty} z_n = z \quad \text{...(i)}$$

If and only if

$$\lim_{n \to \infty} x_n = x \text{ and}$$

$$\lim_{n \to \infty} y_n = y \quad \text{.......................................(ii)}$$

Proof:

Assume (i) is true, for each positive number ε there exists a positive integer number such that

$|(x_n - x) + i(y_n - y)| < \varepsilon$ Whenever $n > n_0$

But

$|x_n - x| \le |(x_n - x) + i(y_n - y)|$

And

$|y_n - y| \le |(x_n - x) + i(y_n - y)|$

Consequently,

$|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ whenever $n > n_{0i}$ and (3) are satisfied.

Conversely, form (3), for each positive number ε ©, there is positive numbers n_1 and n_2 such that

$|x_n - x| < \dfrac{\varepsilon}{2}$ whenever $n > n_1$

And

$|y_n - y| < \dfrac{\varepsilon}{2}$ whenever $n > n_2$

Hence if number is the larger of the two integers n_1 and n_2,

Then

$|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$ whenever $n > n_0$

But

$|(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|,$

And so

$|z_n - z| < \varepsilon$ whenever $n > n_0$

Which is condition (2)

Definition: An infinite series of complex numbers

$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + ... + z_n + ...$ Converges to a sum S, called the sum of the series, if the sequence

$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + ... + z_n \quad (N = 1, 2, ...)$ of partial sums

converges to S, we then write $\sum_{n=1}^{\infty} z_n = S$

Note that since a sequence can have at most one limit, a series can have at must one sum, when a series does not converge, we say that it diverge,

Theorem: Suppose that $z_n = x_n + iy_n \quad (n = 1, 2, ...)$ and $S = X + iY$. then

$$\sum_{n=1}^{\infty} z_n = S$$

If and only if

$$\sum_{n=1}^{\infty} Xn = X \quad and \quad \sum_{n=1}^{\infty} Yn = Y$$

Definition: An infinite sequence of single valued functions of complex variable

$U_1(z), U_2(z), U_3(z), ..., U_n(z), ...$

Denoted by $\{U_n(z)\}$, has a limit $U(z)$ as $n \to \infty$, if given any positive number ε we can find a number N (depending in general on both ε and Z) such that $|U_n(z) - U(z)| < \varepsilon$ for all $n > N$.

We write $\lim_{n \to \infty} U_n(z) = U(z)$. In such case, we say that the sequence converges or is convergent to $U(z)$.

If a sequence converges for all values of Z (points) in a region R, we call R the region of convergence of the sequence. A sequence which is not convergent at some value (point) Z is called divergent at Z.

Definition: The sum of $\{U_n(z)\}$, denoted by $\{S_n(z)\}$ is symbolized by

$U_1(z)+U_2(z)+ \quad \neq \sum_{n=1}^{\infty} U_n(z)$ is called an infinite series

If $\lim_{n \to \infty} Sn(z) = S(z)$, the series is said to be convergent and S(z) is its sum, otherwise the series is said to be divergent. If a series converges for all values of Z (points) in a region R, we call R the region of convergence of the series.

Definition (absolute convergence): A series $\sum_{n=1}^{\infty} U_n(z)$ is called absolutely convergent of the series of absolute values.

i.e. $\sum_{n=1}^{\infty} |U_n(z)|$, converges

If $\sum_{n=1}^{\infty} U_n(z)$ converges but $\sum_{n=1}^{\infty} |U_n(z)|$ does not converge, we call $\sum_{n=1}^{\infty} U_n(z)$ conditionally convergent.

Definition: In the definition, if a number N depends only in ε and not in Z, the sequence $U_N(z)$ is said to be uniformly convergent.

3.2 Taylor Series

Theorem (Taylor's Theorem): Let f be analytic everywhere inside a circle C with center at Z_0 and radius R. Then at each point Z inside C.

$$f(z) = f(z_0) + \frac{f^1(z_0)}{1}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + ... + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + ...$$

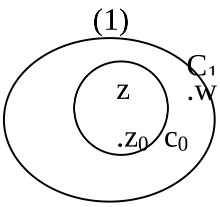
That is, the power series have converge to $f(z)$ *when* $|z - z_0| < R$.

Proof

Let Z_0 be any point inside C. Construct a circle C, with centre at z_0 and enclosing Z. Then by Cauchy's integral formula

$$f(z) = \frac{1}{\ell ni}\oint_a \frac{f(w)}{w-z}dw$$

For any point w on C_1



We have

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{(w-z_0)}\left\{\frac{1}{1-\frac{(z-z_0)}{w-z_0}}\right\}$$

$$= \frac{1}{w-z_0}\left\{1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\ldots\ldots\ldots+\left(\frac{(z-z_0)}{w-z_0}\right)^{n-1}+\left(\frac{z-z_0}{w-z_0}\right)^n\right\}$$

Or $$\frac{1}{w-z} = \frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)}+\frac{(z-z)^2}{(w-z)^3}+\ldots\ldots+\frac{(z-z_0)^{n-1}}{(w-z_0)^2}+\left(\frac{z-z_0}{w-z_0}\right)\frac{1}{w=z}$$

Proof

We first prove the theorem when $z_0 = 0$ *and then extends to any* z_0.

Let z be any fixed point inside the circle C, centred now at the origin. Then let $|z| = r$ and note that $r < R$ where R is the radius of C. Let S denote points lying on a positively oriented circle C_1about the origin with radius R_1 where $r < R_1 < R;$ then $|S| = R_1$. Since Z is interior to C_1, and f is analytic within and on the circle, the Cauchy integral formula gives

$$f(z) = \frac{1}{\ell \bar{\pi} i}\oint_{c_1}\frac{f(s)ds}{s-z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, we can write

$\frac{1}{s-z} = \frac{1}{s}\left[\frac{1}{1-\left(z/s\right)}\right]$ and using the first that

$\frac{1}{1-c} = 1 + c + c^2 + ... + c^{n-1} + \frac{c^n}{1-c}(n = 1,\ 2, ...)$ where C is any complex number other than unity. Hence

$$\frac{1}{s-z} = \frac{1}{s}\left[1 + z/s + (z/s)^2 + ... + (z/s)^{n-1} + \frac{(z/s)^n}{1-(z/s)}\right]$$

and consequently

$$\frac{1}{s-z} = \frac{1}{S} + \frac{1Z}{S^2} + \frac{1}{S^3}Z^2 +, ... + \frac{1\ Z^{N-1}}{S^N} + \frac{Z^N}{(s-z)S^N}$$

………………………(2)

Multiply this equation through by $f(s)/2\wedge-i$ and integrate wrt S, we have

$$\frac{1}{2\overline{\pi} i}\int_c \frac{f(s)ds}{s-z} = \frac{1}{\ell\overline{\pi} i}\int_{c_1}\frac{f(s)}{s}ds + \frac{z}{\ell\overline{\pi} i}\int_{c_1}\frac{f(s)ds}{s^2} + \frac{z^2}{\ell\pi i}\int_{c_1}\frac{f(s)}{s^3}ds + \frac{z^{n-1}}{\ell\overline{\pi} i}\int_{c_1}\frac{f(s)}{s^n} +$$

$$\frac{z^n}{\ell\overline{\pi} i}\int_{c_1}\frac{f(s)ds}{(s-z)S^N}$$

In view of expression (2) and applying the this that

$$\frac{1}{\ell\pi i}\int_{C_1}\frac{f(s)ds}{S^{n+1}} = \frac{1}{\ell\pi i}\int_{C_1}\frac{f(s)ds}{(S-i)^{N+1}} = \frac{f^{(n)}(0)}{n1}$$

We can write the result as

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f'(0)}{\partial!}z^2 + ... + \frac{f^{(n-1)}(0)}{(n-1)!}Z^{N-1} + f(z)$$

Where

$$f(z) = \frac{z^n}{\ell\overline{\pi} i}\int_{c_1}\frac{f(s)ds}{(s-z)s^N} \qquad ………………………………………………..(4)$$

- Recalling that $|z| = r$ and $|\delta| = R_1$
  , where $r < R_1$, we note that $|s - z| \geq |s| - |z| = R_1 - r$
- It follows from (4) that when M_1 denotes the maximum of $|f(s)|$ on C_1,
- $|\rho_n(z)| \leq \frac{r^n}{\ell\overline{\pi}}\left(\left(\frac{M_1}{(R_1 - r)R_1^N}\ell\overline{\pi}R_1\frac{M_1R_1}{R_1 - r}\left(\frac{r}{R_1}\right)^N\right)\right)$............................(5)

But $\left(\frac{r}{R_1}\right) < 1$, and therefore

$$\lim_{n \to \infty} \rho_N(z) = 0$$

So that

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f'(0)}{\ell!}z^2 + ... + \frac{f^{(n)}(0)}{n!}z^n + \qquad \text{.....................(6)}$$

In the open disk $|z| < R$

.

This is a special case, of (1) and it is called the maclurin series.

Suppose now that f is as in the statement of the theorem, since $f(z)$ is analytic when $|z - z_0| < R$, the composite function $f(z + z_0)$ is analytic when $|(z + z_0) - z_0| < R$. But the last inequality is simply $|z| < R$; and if we write $g(z) = f(z + z_0)$, the analyticity of g inside the circle $|z| = R$ implies the existence of a Mcdanrim series representation.

$$g(z) = \sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}z^n$$

$$(|z| < R)$$

That is

$$f(z + z_0) = \sum_{n=0}^{\infty}\frac{f^{(n)}z_0}{n!}z^n$$

Using z by $z - z_0$

in this equation, we arrive at the desired Taylor series representation for $f(z)$

about the point z_0.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$(|z - z_0| < R)$$

.

Example: If $f(z) = Sin\, z$, then $f^{(2n)}(0) = 0 \;\; (n = 0, 1, 2.........)$

and $f^{(2n+1)}(0) = (-1)^n \quad (n = 0, 1, 2,..........)$

hence

$$Sin\; z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)}, \quad (|z| < \infty)$$

The condition $|z| < \infty$ follows from the fact that the function is entire.

Differentiating each side of the above equation with respect to and interchanging the symbols for differentiation and summation on the right-hand side, we have the expression

$$Cos\; z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2\pi)}$$

!

Because $Sin\, h\, z = -i \sin(iz)$

, replacing z

by iz

in each side of (∞) and multiply through the result by –c, we have

$$Sin\, h\, z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Differentiating each side of this equation gives

$$Cos\, h\, z = \sum_{n=0}^{\infty} \frac{Z^{2n}}{(2n)!}$$

3.3 Laurent Series

Theorem: Let C_0 and C_1 denote two positively oriented circles centred at a point Z_0, where C_0 is smaller than C_1. if a function f is analytic on C_0 and C, and throughout the annular domain between them, then at each point Z is the domain $f(z)$

is represented by the equation.

$$f(z) = \sum_{n=0}^{\infty} C_{in}(z - z_0)^n + \sum_{n=1}^{\infty}$$

Where

$$a_n = \frac{1}{\ell\overline{\pi}i}\int_{C_1} \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, ...)$$

And

$$b_n \frac{1}{\ell\overline{\pi}i}\int_{C_0} \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, ...)$$

The series here is called a Laurent series

We let R_0 and R_1 denote the radius of C_0 and C_1 respectively. Thus R_0 and R_1 and if f is analytic at every point inside and on C_1 except at the point Z_0 itself, the radius R_0 may be taken arbitrarily small, expansion (1) then valid when $0 < |z - z_0| < R_1$ If f

Is analytic at all points inside and on C_1, we need only write the integral in expansion (3) as $f(z)(z - z_0)^{n-1}$ to see that it is analytic inside and on C_0. For $n - 1 \geq 0$ when n is a positive integer. So all the coefficient bn are zero, and because

$$\frac{1}{\ell\overline{\pi}i}\int_{C_1} \frac{f(z)dz}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z)}{n!} \quad (n = 0, 1, 2, ...)$$

Expansion (1) includes to a Taylor series about Z_0

4.0 CONCLUSION

In this unit we have established condition for convergence of series in complex variables. You are required to study this unit properly to be able to understand subsequent units.

5.0 SUMMARY

The following definition is hereby recalled, to stress the importance of convergence of series in complex variables

(1)An infinite sequence of complex numbers, $z_1, z_2, ...z_n, ...$ has a limit z if for each positive number ε there exists a positive integral number such that $|z_n - z| < \varepsilon$ whenever $n > n_0$.

If the limit exists, it is unique.

When the limit z exists, the sequence is said to converge to z; and we write

$$\lim_{n \to \infty} z_n = z$$

If the sequence has no limit, it diverges.

2. We have also stated theorems that can help us in proofing convergence of series.
3. The Taylor and Laurent series have been applied in treating convergence of series.

6.0 TUTOR-MARKED ASSIGNMENT

1. Expand the following complex variable using Taylor series about z=$\frac{\pi}{2}$

   (a) $Tanz$ (b) $Cosz$

2 State the Laurent series for the above.

7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand (1976) Advanced *Calculus For Application* 2nd Edition

UNIT 4 SOME IMPORTANT THEOREMS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider some related theorems on complex variables.

We shall consider theorems on test of convergence of complex variables

We shall also learn about singularities and classifications or singularities.

OBJECTIVES

At the end of this unit, you should be able to:

- state the important theorems on convergences of sequences and series of complex variables
- be able to classify singularities on complex variables
- work problems on complex variables.

3.0 MAIN CONTENT

Theorem 1: The limit of a sequence, if it exists, is unique.

Theorem 2: Let $\{a_n\}$ be a real sequence with the property that

(i) $a_{nti} \geq a_n$ or $a_{nt1} \leq a_n$
(ii) $|a_n| < M(a_{nt1})$

Then $\{a_n\}$*converges.*

That is, every bounded monotonic (increasing or decreasing) sequence has a limit.

Theorem 3: A necessary and sufficient conditions that $\{U_n\}$converges is that given $\varepsilon > 0,$ we can find a number N such that $|U_n - U_q| < \varepsilon$ for all $\rho > N,\ q > N.$ This is called Cauchy's convergence criterion.

3.1 Special Tests for Convergence

Theorem 1: (comparison tests)

(a) If $\sum |V_n|$ converges and $|U_n| \leq |V_n|$, then $\sum U_n$ converges absolutely
(b) If $\sum |V_n|$ diverges and $|U_n| \geq |V_n|$, then $\sum |U_n|$ diverges but $\sum U_n$ may or may not converge.

Theorem 2: (Ratio Test)

(a) If $\lim_{n \to \infty} \left| \frac{U_{nt1}}{U_n} \right| = L,$ then $\sum U_n$ converges (absolutely)
(b) If $L < 1$ and diverges if $L > 1$. If L – 1, the test fails.

Theorem 3: (nth Root Test)

(a) If $\lim_{n \to \infty} \sqrt[n]{|U_n|} = L$, then $\sum U_n$ converges (absolutely)
(b) if L < 1 and diverges if L > 1. If L = 1, the test fails

Theorem 4: (Integral Test)

(a) If $f(x) \geq 0$ for $x \geq a,$ then $\sum f(x)$converges or diverges if $\lim_{m \to \infty}$ $\int_a^M f(x)dx$ converge diverges.

Theorem 5: (Raabe's Test)

(a) If $\lim_{n \to \infty} n\left(1 - \left|\frac{U_{nt1}}{U_n}\right|\right) = L,$ then $\sum U_n$ converges (absolutely)
(b) If L > 1 and diverges or converges conditionally if L < 1.
(c) If L = 1, the test fails.

Theorem 6: (Gauss' Test)

(a) If $\left|\frac{U_{nt1}}{U_n}\right| = 1 - \frac{L}{n} + \frac{C_n}{n^2}$ where $|C_n| < M$ for all $n > N$, then $\sum U_n$ converges (absolutely) if L > 1 and diverges or converges conditionally if L ≤ 1.

3.2 Theorems on Power Series

Note that a series of the form

$$a_0 + a_1(z - z_0) + a_2(z - z_0) + ... = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$ is called a power series in $z - z_0$

Theorem 1: A power series converges uniformly and absolutely in any region which lies entirely inside its circle of convergence.

Theorem 2: (Abel's Theorem)

Let $\sum a_n z^n$ have radius of convergence $\Re$ and suppose that z_0 is a point on the circle of convergence such that $\sum a_n z_0^n$ converges.

Then $\lim_{z - z_0} \sum a_n z^n = \sum a_n z_0^n$ where $z \to z_0$ from within the circle of convergence.

Theorem 3: If $\sum a_n z^n$ converges to zero for all Z such that $|z| < R$ where $R > 0$, then $a_n = 0$. Equivalently. If $\sum a_n z^n = \sum b_n z^n$ for all Z such that $|z| < R$, then $a_n = b_n$.

3.3 Laurent Series

If a function f fails to be analytic at a point z_0, we cannot apply Taylor's theorem at that point. If is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$.

Theorem (Laurent Theorem): Let C_0 and C_1 denote two positively oriented circles centred at a point z_0, where C_0 is smaller than C_1. If a function f is analytic at C_0 and C_1, and throughout the annular domain between them, then at each point z in that domain $f(z)$ is represented by the expansion.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)dz}{(z - z_0)^{nt1}} \quad (n - 0, 1, 2, \ldots\ldots\ldots\ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

And

$$b_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)dz}{(z - z_0)^{-nt1}} \quad (n = 1, 2, \ldots\ldots\ldots\ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

The series here is called a Laurent series.

Since the

two integrands $\frac{f(z)}{(z - z_0)^{nt1}}$ and $\frac{f(z)}{(z - z_0)^{-nt1}}$ in expressions (2) and (3) are analytic throughout the annular domain $R_0 |z - z_0| < R_1$, and in its boundary, any simple closed contour C around the

domain in the

positive direction can be used as a path of integration instead of the circular paths C_0 and C_1. Thus the Laurent series (1) can be written as

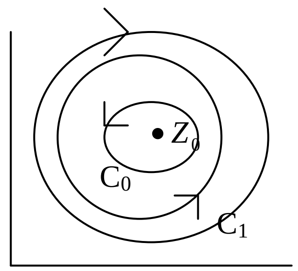
$$f(z) = \sum_{n=-0}^{\infty} C_n (z - z_0)^n \quad (R_0 < |z - z_0| < R_1)$$

Where

$$C_n = \frac{1}{2\overline{\wedge} i} \int_C \frac{f(z)dz}{(z - z_0)^{nt1}}$$

$$(n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots\ldots)$$

Particular cases, of course, some of the coefficient may be zero.



Example: The expansion

$$\frac{e^z}{z^z} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots\ldots\ldots\ldots \qquad 0 < |z| < \infty$$

Follows from the Maclurin series representation

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots\ldots\ldots$$

$(|z| < \infty)$

3.4 Classification of Singularities

1. Poles: If $f(z)$ has the form

$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots\ldots + \frac{a-1}{z - z_0} + \frac{a-2}{(z - z_0)^{2z}}$ In which the principal part has only a finite number of terms given by

$\frac{a_{+1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \ldots\ldots\ldots + \frac{a_{-n}}{(z - z_0)^n}$ Where $a_{-n} \neq 0$, then $z = z_0$ is called a pole of order n.

If n = 1, it is called a simple pole.

If $f(z)$ has a pole at $z = z_0$, then $\lim_{z \to z_0} f(z) = \infty$.

2. **Removable Singularities**: If a single valued function $f(z)$ is not defined at $z = z_0$ but $\lim_{z \to z_0} f(z)$ exist, then $z = z_0$ is a removable singularities. In such case, we define $f(z)$ at $z = z_0$ as equal to $\lim_{z \to z_0} f(z)$.

Example: If $f(z) = \frac{Sin\, z}{z}$, then $z = 0$ is a removable singularities since $f(0)$ is not defined but $\lim_{z \to 0} \frac{Sin\, z}{z} = 1$

Note that $\frac{Sin\, z}{z} = \frac{1}{z}\left\{z - \frac{z^2}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!}\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} +$

3. **Essential Singularities**: If $f(z)$ is single valued, then any singularity which is not a pole or removable singularity is called an essential singularity. If $z = a$ u an essential singularity of $f(z)$, the principal part of the Laurent expansion has infinitely many terms

Example: Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!}z^3$

$z = 0$ is an essential singularity.

4. **Branch Points**: A point $z = z_0$ is called a branch point of the multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when Z describes a closed path about z_0. Since each of the branches of a multiple-valued function is analytic, all the theorems for analytic functions, in particular Taylor's theorem apply.

Example: The branch of $f(z) = z^{1/2}$ which has the value 1 for $z = 1$, has a Taylor series of the form

$a_0 + a_1(2-1) + a_2(z-1)^2 + ...$ With radius of convergence $R = 1$ [the distance from Z=1 to the nearest singularity, namely the branch point z=0].

5. Singularities at Infinity: By letting $z = 1/w$ in $f(z)$ we obtain the function $f(1/w) = f(w)$. Then the nature of the singularity at $z = \infty$ [the point at infinity] is defined to be the same as that of $f(w)$ at $w = 0$.

Example: If $f(z) = z^3$ has a pole of order 3 at $z = \infty$, since $f(w) = f(1/w) = 1/w^3$ has a pole of order 3 at $w = 0$.

Similarly, $f(z) = e^z$ has an essential singularity at $z = \infty$, since $f(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$.

4.0 CONCLUSION

This unit is a very important unit which must be studied properly and understood before proceeding to other units.

5.0 SUMMARY

Recall that in this unit we discussed very important theorems in the solution of complex variables. We also discussed singularities, Laurent series and application, we discussed branch. These are to aid in tackling any exercises on complex variables.

6.0 TUTOR-MARKED ASSIGNMENT

1. State all the Convergent Tests listed in this unit
2. If $f(z) = \frac{Sinz}{z}$ determine the removable singularity and carry out the expansion.
3. Define the essential singularity and determine the essential singularity for $f(z) = e^{\frac{1}{z}}$

7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand (1976) Advanced *Calculus For Application* 2nd Edition

MODULE 2

UNIT 1 SOME EXAMPLES ON TAYLOR AND LAURENT SERIES

CONTENTS

1.0 INTRODUCTION

This unit considers examples on Taylor and Laurent series of complex variables.

The aim is to expose the students to more workable examples on complex variables.

2.0 OBJECTIVES

The students will be able to:

- Solve problems successfully on complex variables using Taylor's Series and Laurent Series.

3.0 MAIN CONTENT

3.1 Examples on Taylor and Laurent Series

Example: Expand $f(z) = Cos\ z$ in Taylor series about $z = \frac{\pi}{4}$ and determine its region of convergence

Solution:

By Taylor series.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z - z_0)^2}{2!} + \ldots\ldots\ldots$$

$$f(z) = Cos\ z, \quad f'(z) = -Sin\, z, \quad f''(z) = -Cos\, z, \quad f'''(a) = Sin\, z$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}/2, \; f''\left(\frac{\pi}{4}\right) = -\sqrt{2}/2, \; f''\left(\frac{\pi}{4}\right) = -\sqrt{2}/2, \; f'''\left(\frac{\pi}{4}\right) = \sqrt{2}/2, \ldots\ldots.$$

$$f(z) = \sqrt{2}/2 - \sqrt{2}/2\left(z - \frac{\pi}{4}\right) - \frac{\sqrt{2}/2\left(z - \frac{\pi}{4}\right)^2}{2} + \sqrt{2}/2\frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \sqrt{2}/2\frac{\left(z - \frac{\pi}{4}\right)^4}{4!} - \ldots\ldots$$

$$f(z) = \sqrt{2}/2\left(1 - \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^2}{2} + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^4}{4!} - \right)$$

$$f(z) = \sqrt{2}/2\left[\left(1 - \frac{\left(2 - \frac{\pi}{4}\right)^2}{2} + \frac{\left(z - \frac{\pi}{4}\right)}{4!} - \ldots\ldots\right) - \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^5}{5!}\right]$$

$$\ldots\ldots\ldots\ldots(-1)^{n-1}\frac{\left(z - \frac{\pi}{4}\right)^{2n-2}}{(2_n - z)!}\ldots\ldots.\frac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-1}}{(2n-1)!}$$

$$f(z) = \frac{\sqrt{2}}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-2}}{(2n-2)!} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(z - \frac{\pi}{4}\right)^{2n-1}}{(2n-1)!}\right]$$

For the region of convergence, using ratio test

Let $U_n = \frac{(-1)^{n-1}\left(z-\frac{\pi}{4}\right)^{2n-1}}{(2n-2)!}$, $U_n + 1 = \frac{(-1)^2\left(z-\frac{\pi}{4}\right)^{2n}}{2n!}$

Also

Let $\frac{(-1)^{n-1}\left(z-\frac{\pi}{4}\right)^{2n-1}}{(2n-1)!} = V_n$, $V_{n+1} = \frac{(-1)^n\left(z-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\left(z-\frac{\pi}{4}\right)}{2n} + \frac{(2n-2)!}{(-1)^{n-1}\left(z-\frac{\pi}{4}\right)^{2n-2}}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{(z-\frac{\pi}{4})^{2n}}{2n(2n-1)\left(z-\frac{\pi}{4}\right)^{2n-2}}\right|$$

$$= \lim_{n\to\infty} -\frac{1}{2n(2n-1)}\left|z-\frac{\pi}{4}\right|^2$$

$$= \lim_{n\to\infty}\frac{\left|z-\frac{\pi}{4}\right|^2}{-(2n(2n-1))} = 0$$

Similarly $\lim_{n\to\infty}\left|\frac{V_{n+1}}{V_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\left(z-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} + \frac{(2n-1)!}{(-1)^{n-1}\left(z-\frac{\pi}{4}\right)^{2n-1}}\right|$

$$= \lim_{n\to\infty} -\frac{\left|z-\frac{\pi}{4}\right|^2}{2n(2n+1)} = 0$$

This shows that the singularity of $Cos\, z$ nearest to $\frac{\pi}{4}$ is at infinity. Hence the series converges for all values of z i.e. $|z| < \infty$

Example: Expand $f(z) = \frac{1}{z-3}$ is a Laurent series valid for

(a) $|a| < 3$

(b) $|z| > 3$

Solution:

For $|z| < 3$

$$\frac{1}{(z-3)} = \frac{1}{-3+z} = \frac{1}{-3(1-z/3)} = \frac{1}{-3}(1-z/3)^{-1}$$

$$= -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}.......\right] = -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}$$

For $|z| > 3$

$$\frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} = \frac{1}{z}\left[1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+.......\right]$$

$$= \frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\frac{27}{z^4}+........$$

Example: Expand $f(z) = \dfrac{z}{(z-1)(z-2)}$ in Laurent series valid for $|z| <|$

Solution

$$\frac{z}{(z-1)(z-2)} = \frac{1}{z-1}+\frac{2}{2-z}$$

For $|z| <|$, $\dfrac{1}{z-1} = \dfrac{1}{1(1-z)} = -[1+z+z^2+z^3+z^4+....]$

$$= -1-z-z^2-z^3-z^4-$$

and

$$\frac{2}{2-z} = \frac{2}{2(1-z/2)} = 1(1-z/2)^{-1}$$

$$= 1+z/2+z^2/4+z\,3/8+z^4/16+.........$$

Adding, we have

$$\frac{z}{(z-1)(2-z)} = -\frac{1}{2}z - \frac{3}{4}z^2 - \tfrac{7}{8}z^3 - \tfrac{15}{16}z^4 - \ldots\ldots\ldots.$$

Example: Find the Laurent series for the function $f(z) = (z-3)\,Sin\frac{1}{z+2}$ about $z=-2$. Also state that type of singularity and the region of convergence for the series.

Solution:

$(z-3)\,Sin\frac{1}{z+2}$; $z=-2$. Let $z+2=u$ or $z=u-2$.

Then

$$(z-3)\,Sin\frac{1}{z+2} = (u-5)\,Sin\frac{1}{u} = (u-5)\left\{\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} + \ldots\ldots\ldots\right\}$$

$$= 1 - \frac{5}{u} - \frac{1}{3!u^2} + \frac{5}{3!u^3} + \frac{1}{5!u^{4-}}$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+12)^2} + \frac{5}{6(z+12)^3} + \frac{1}{120(z+2)^4}\ldots\ldots\ldots$$

$z=-2$ is an essential singularity. The series converges for all values of $z+-2$.

4.0 CONCLUSION

In this unit we discussed Laurent series and Taylor series. We applied them to solve some problems .You are to learn this unit very well . You may wish to attempt the Tutor- Marked Assignment.

5.0 SUMMARY

Recall in this unit that while Taylor series can be useful to analyze functions , Laurent Series gives clearer and simple ways of handling functions of complex variables. These were clearly demonstrated in the examples we considered in this unit. Answer the Tutor Marked Assignment at the end of this unit, for more understanding of the concept.

6.0 TUTOR-MARKED ASSIGNMENT

1. Expand the function in each of the following series:

(a) a Taylor series of powers of z for $|z| < 1$

(b) a Laurent series of powers of z for $|z| > 1$

(c) a Taylor series of power of $z+1$ for $|z| < 1$

7.0 REFERENCES/FURTHER READINGS

Francis B Hildebrand (1976): Advanced Calculus for Application **2nd Edition.**

UNIT 2 ANALYTIC FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit we shall study analytic functions of complex variables. We shall establish the condition for functions to be analytic.

All related theorems on analytic function will be considered.

2.0 OBJECTIVES

At the end of this unit, you should be to have learnt about:

- derivatives of complex variables
- Cauchy – Riemann equations
- polar form of complex variables
- harmonic functions.

3.0 MAIN CONTENT

3.1 Derivatives

Definition: Let F be a …..whose domain of definition contains a nbd of a point Z_0. The derivative of f at Z_0, written as $f^1(Z_0)$, is defined as

$$f^1(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad ..(3.1.1)$$

Provided this limit exists. The function f is said to be differentiable at z_0 when its derivative at z_0 exists.

Note that (3.1.1) is equivalent to

$$f^1(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \ldots (3.1.2)$$

Where $\Delta z = z - z_0$

Which is also the same as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Where $f^1(z) = \frac{dw}{dz}$, $\Delta w = f(z_0 + \Delta z) - f(z_0)$ write $z - z_0$

Example: Suppose that

$$f(z) = z^2$$

At any point z

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Hence, $\frac{dw}{dz} = 2z$ or $f^1(z) = 2z$

Example: For the function $f(z) = |z|^2$

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \Delta \bar{z}) - z\bar{z}}{\Delta z}$$

$$= \bar{} + \Delta \bar{z} + z \frac{\Delta \bar{z}}{\Delta z}$$

When $z = 0$, this reduces to $\frac{\Delta w}{\Delta z} = \Delta \bar{z}$. Hence $\frac{dw}{dz} = \lim_{\Delta z \to 0} = 0$. at the origin $dw/dz = 0$

If the limit of $\Delta w/\Delta z$ exists when $z \neq 0$, this limit may be found by letting the variable $\Delta z = \Delta x + 1\Delta y$ approach 0 in any manner. In particular, when Δz approaches 0through the real values $\Delta z = \Delta z + i0$, we may write $\Delta \overline{Z} = \Delta Z$. Hence if the limit of $\Delta w/\Delta z$ exists, its value must be $\bar{z} + z$.
However, when ΔZ approaches 0 through the pure imaginary
Value, so that $\Delta \overline{Z} = -\Delta Z$, the limit if found to be $\overline{Z} - Z$. Since a limit is unique,
it follows that $\overline{Z} + Z = \overline{Z} - Z$, or $Z = 0$, if dw/dz exists. But $Z \neq 0$, and we may conclude from this contradiction that dw/dz exists only at the origin.

From example above, it follows that:

(1) A function can be differentiable at a certain point but nowhere else in any nbd of that point.

(2) Since the real and imaging parts of $f(z) = |z|^2$ are $\mu(n, y) = n^2 + y^2$ and $v(n, y) = 0$.
Respectively, it also shows that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point and yet the function many not even be differentiable there.

(3) The function $f(z) = |z|^2$ is its at each point in the plane since its components functions are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there.

It is, however, true that the existence of the derivative of a function at a point implies the continuity of the function at that point.

3.2 Differentiation Formulae

Definition: Let F be a …..whose domain of definition contains a nbd of a point Z_0. The derivative of f at Z_0, written as $f^1(Z_0)$, is defined as

$$f^1(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \text{……………………………………..(3.1.1)}$$

Provided this limit exists. The function f is said to be differentiable at z_0 when its derivative at z_0 exists.

Note that (3.1.1) is equivalent to

$$f^1(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \text{(3.1.2)}$$

Where $\Delta z = z - z_0$

Which is also the same as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Where $f^1(z) = \frac{dw}{dz}$, $\Delta w = f(z_0 + \Delta z) - f(z_0)$ write $z - z_0$

Example: Suppose that

$$f(z) = z^2$$

At any point z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Hence, $\frac{dw}{dz} = 2z$ or $f^1(z) = 2z$

Example: For the function $f(z) = |z|^2$

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta 2)(\overline{z} + \Delta \overline{z}) - Z\overline{z}}{\Delta Z}$$

$$= \overline{Z} + \Delta \overline{Z} + Z \frac{\Delta \overline{Z}}{\Delta Z}$$

When z=0, this reduces to $\frac{\Delta w}{\Delta z} = \Delta \overline{Z}$. Hence $\frac{dw}{dz} = \lim_{\Delta z \to 0}$ = 0. at the origin $dw/dz = 0$

If the limit of $\Delta w/\Delta z$ exists when $z \neq 0$, this limit may be found by letting the variable $\Delta z = \Delta x + 1\Delta y$ approach 0 in any manner. In particular, when ΔZ approaches 0 through the real values $\Delta Z = \Delta n + i0$, we may write $\Delta \overline{Z} = \Delta Z$. Hence if the limit of $\Delta w/\Delta z$ exists, its value must be $\overline{Z} + Z$.

However, when ΔZ approaches 0 through the pure imaginary value $\Delta Z = 0 + i\Delta y$, so that $\Delta \overline{Z} = -\Delta Z$, the limit if found to be $\overline{Z} - Z$. Since a limit is unique, it follows that $\overline{Z} + Z = \overline{Z} - Z$, or $Z = 0$, if $^{dw}/_{dz}$ exists. But $Z \neq 0$, and we may conclude from this contradiction that dw/dz exists only at the origin.

From example above, it follows that:

(1) A function can be differentiable at a certain point but nowhere else in any nbd of that point.

(2) Since the real and imaging parts of $f(z) = |z|^2$ are $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$.
Respectively, it also shows that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point and yet the function many not even be differentiable there.

(3) The function $f(z) = |z|^2$ is its at each point in the plane since its components functions are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there.

It is, however, true that the existence of the derivative of a function at a point implies the continuity of the function at that point.

3.3 Cauchy-Riemann Equations

Suppose that
$f(z) = u(x.y) + iv(x, y)$ and that $f^1(z_0)$ exists at a point $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v wrt n and y must exist at (u_0, y_0), and they must satisfy.

$U_x(x_0 y_0) = Vy(x_0, y_0)$ and $Uy(x_0, y_0) = -V_x(x_0, y_0)$ at that point. … (1)
Also $f^i(z_0)$ is given in terms of the partial derivatives by either
$f^1(z_0) = U_x(x_0, y_0) + iv(x_0 y_0)$
or $f^1(z_0) = Vy(x_0, y_0) - Uy(x_0, y_0)$

Equation (1)… is referred to as Cauchy Riemann equation.

Example: the derivative of the function $f(z) = z^2$ exists everywhere.

To verify that the Cauchy-Riemann equations are satisfied everywhere, we note that

$f(z) = z^2 = x^2 - y^2 + i2xy$ so that

$U(x, y) = x^2 - y^2$ and $V_x(xiy) = 2x$

$U_x(x, y) = 2x$, $V_x(xiy) = 2y$

$Uy(x, y) = 2y$ $Vy(x, y) = 2x$

So that

$U_n(n, y) = Vy(n, y) = 2x$

$Uy(x, y) = -V_n(x, y) = -2y$

Also

$f^1(z) = U(x_0, y_0) + iV(x, y) = 2x + i2y = 2z$

3.4 Sufficient Conditions

Satisfaction of the Cauchy – Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. The following theorem gives sufficient conditions.

Theorem: (Sufficiency Theorem) :
Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε - nbd of a point $z_0 = x_0 - iy_0$ suppose that the first-order partial derivatives of the functions U and V with respect to n and y exist everywhere in that nbd they are continuous at (x_0, y_0). Then if these partial derivatives satisfy the Cauchy-Riemann equations.

$U_x = V_y, and, U_y = -Vx$

At (x_0, y_0), the derivative $f^1(z_0)$ exists.

Proof: We shall leave the proof as exercise.

Example: suppose that
$f(z) = e^x$ 9Cos y + i Sin y)

Where y is to be taken in radius when Cos y and Sin y are evaluated then

$U(x, y) = e^x Cosy$ and $V(x, y) = e^x Siny$

Since $U_x = V_y$ and $Uy = -Vx$ everywhere and since those derivatives are everywhere continuous, the conditions in the theorem are satisfied at all points in the complex plane. Thus, $f^1(z)$ exists everywhere and

$f'(z) = U_x(x, y) = iV_x(x, y) = e^x(\cos x + i\sin y)$

Note that $f^1(z) = f(z)$

Example: for the function
$f(z) = |z|^2 = U(x, y) = x^2 + y^2$ and $V(x, y) = 0$ So that $U_x(x, y) = 2x$ and $V_y(x, y) = 0$ while $U_y(x, y) = xy$ and $V_x(x, y) = 0$. Since $U_x(x, y) \neq V_y(x, y)$ unless $x = y = 0$ Cauchy-Riemann equations are not satisfied unless $x = y = 0$ the derivative $f'(z)$ cannot exist if $z \neq 0$ and besides, the existence of $f'(0)$ is not guaranteed unless conditions of theorem (3-4-1) are satisfied.

If follows from the theorem (3.4.1) that the further $f(z) = |z|^2 = (x^2 + y^2) + 10$ has derivative at $z = 0$; in fact, $f'(0) = 0 + 0 = 0$.

3.5 Polar Form

Cauchy-Riemann equations can be written in polar form. For $z = n + iy \ or \ z = r\,(\mathrm{Cos}\,\theta + i\,Sin\theta)$, we have $n = r\cos\theta, \quad y = r\sin\theta, \ r = \sqrt{n^2} + y^2. \ \theta = \tan^{-1} y/n$
Then,

$$Ur = Ur\frac{\partial r}{\partial r} + U\theta\frac{\partial\theta}{\partial x} = Ur\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + U\theta\left(\frac{-y}{x^2 + y^2}\right)$$

So that

$$Uy = Ur \ Sin\theta + \frac{1}{r} \ U\theta \ Cos \ \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$$Vx = Vr\frac{\partial r}{\partial x} + V\theta\frac{r\theta}{rx} = Vr \ Cos\,\theta - \frac{1}{r}V\theta \ Sin\,\theta$$

So that

$$Vn = Vr \ Cos\theta = \frac{1}{r}V\theta \ Sin\,\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$y = Vr\frac{\partial r}{\partial y} + V\theta\frac{\partial\theta}{\partial y} = Vr \ Sin\,\theta\frac{1}{r} \ V\theta \ \ Cos \ \theta$$

So that

$Vy = Vr\ \ Sin\theta + \frac{1}{r}\, V\theta\ Cos\theta$ ……………………………………………..(4)

From the Cauchy-Riemann equation, $Un = Vy$, equating (1) and (4), we have

$\left(Ur - \frac{1}{r}V_\theta\right)\ Cos\theta - \left(Vr + \frac{1}{r}U_\theta\right)\ Sin\theta = 0$ ………………………………..(5)

From the Cauchy-Riemann equation, $Uy = -Vn$, equating (2) and (3), we have

$\left(Ur - \frac{1}{r}V_\theta\right)\ Sin\theta + \left(Vr + \frac{1}{r}U_\theta\right)\ Cos\ = 0$ …………………………………(6)

Multiplying (5) by cos θ, (6) by Sin θ and adding given

$(Ur = -\frac{1}{r}U_\theta)$ ………………………………………………………..(7)

Also, multiplying (5) by $-\sin\theta$, (6) by $Cos\theta$ and adding given

$Vr = -\frac{1}{r}U\theta$ ………………………………………………………..(8)

Equations (7) and (8) are the Cauchy-Riemann equations in polar form.

Theorem: Let the function
$f(z) = U(r,\theta) + i\ v(r,\theta)$

Be defined throughout some ε neighborhood of a no zero point
$f(z) = r_0(Cos\ \theta_0 + i\,Sin\,\theta_0)$.

Suppose that the first order partial derivatives of the functions U and V wrt r and θ exist everywhere on that neighborhood and that they are continuous at (r_0, θ_0). Then if those partial derivatives satisfy polar forms (7) and (8) of the Cauchy-Riemann equations at (r_0, θ_0), the derivatives $f^1(z_0)$ exists.

The derivative $f^1(z_0)$ is given as

$$f^1(z_0) = e^{-1\theta}\left[Ur(r_0, \theta_0) + i\,Vr(z_0, \theta_0)\right]$$

Example: Consider the function
$f(z) = \frac{1}{r} = \frac{1}{re^\theta}$,

$U(r,\theta)=\frac{Cos\theta}{r}$ and $V(r,\theta)=\frac{-Sin\theta}{r}$ and the condition of the theorem are satisfied at any nonzero point $z=re^{i\theta}$ in the plane. Hence the derivative of f exists there: and according to (9)

$$f^1(z)=e^{-1\theta}\left(-\frac{Cos\theta}{r^2}+i\frac{Sin\theta}{r^2}\right)=-\frac{1}{\left(re^{i\theta}\right)^2}=-\frac{1}{z^2}$$

3.6 Analytic Functions

Definition: A function f of the complex variables z is analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighborhood of z_0. A function f is said to be analytic in a region R if it is analytic at each point in $\Re$. The term halomorphic is also used in literature to denote analyticity.

If $f(z)=z^2$, then f is analytic everywhere. But the function $f(z)=|z|$ is not analytic at any point since its derivative exists why at $z=0$ and not throughout any nbd.

An entire function is a function that is analytic at each point in the entire plane. E.g. polynomial functions.

If a function f fails to be analytic at a point z_0, buy is analytic at some point in every nbd of z_0, then z_0 is called a singular point or singularity of f. For example, the function $f(z)=1/z$, where derivative is $f(z)=-1/z^2$ is analytic at every point except $z=0$ hence it is not even defined. Therefore the point $z=0$ is a singular point.

If two functions are analytic in domain D, their sum and their product are both analytic in D. similarly, their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.

3.7 Harmonic Functions

A real-valued function h of two real variables x and y is said to be harmonic in a given domain in the xy plane if throughout that domain it has continuous partial derivatives of first and second order and satisfies the partial differential equation.

$h_{xx}(x, y) + h_{yy}(x, y) = 0$ ……………………………………………(3.7.1)

Known as Laplace's equation
If a function
$f(z) = u(x, y) + i\,v(x, y)$ …………………………………………...(3.7.2)

Is analytic in a domain D, then its component functions U and V are harmonic in D. to show this,

Since f is analytic in D, the first order partial derivatives of its component functions satisfy the Cauchy-Riemann equations throughout D.
$Ux = Vy,\ Uy = -Vx$ ……………………………………….……..(3.7.3)

Differentiating both sides of these equating with respect to x, we have
$Uxy = Vyy \qquad Uyy = -Vxy$ …………………………………………… (3.7.4)

The continuity of the partial derivatives ensures that $Uyx = Uxy$ and $Vyx = Vxy$. It then follows from (3.7.4) and (3.7.5) that $Uxx(n, y) + Uyy(x, y) = 0$ and $Vxx(x, y) + Vyy(x, y) = 0$.

Thus, if a function $f(z) = U(x, y) + iV(x, y)$ is analytic in a domain D, its component functions U and V are harmonic in D.

3.8 Solved Problems

1. Verify that the real and imaginary parts of the function $f(z) = z^2 + 5i\,z + 3 = i$ satisfy Cauchy-Riemann equation and deduce the analyticity of the function.

Solution

$f(z) = z^2 + 5i\,z + 3 - 1$

$= (x + iy)^2 + 5i(x + iy) + 3 = 1$

$= x^2 - y^2 - 5y + 3 + i(2xy + 5x - 1)$

So that
$U(x, y) = x^2 - y^2 - 5y + 3, \qquad V(x, y) = 2xy + 5x - 1$
$Ux(x, y) = 2x,\ \ Uy(x, y) = -2y - 5 = -(2y + 5)$
$Vx(x, y) = 2y + 5 \qquad Vy(x, y) = 2x$
And since $Ux(x, y) = Vy(x, y) = 2x$
And $Uy(x, y) = -Vx = -(2y + 5)$

The function satisfies Cauchy Riemann equation. Also, since the partial derivatives are polynomial functions which are continuous, then the function is analytic.

2. (a) Prove that the function $U = 2x(1-y)$ is harmonic
   (b) Find a function V such that $f(z) = u + iv$ and express $f(z)$ in terms of z .

Solutions

(a) $U = 2x(1-y)$.
The function is harmonic if $Uxx + Uyy = 0$
$Ux = 2(1-y), \quad Uxx = 0$
$Uy = -2x \qquad Uyy = 0$
$Uxx + Uyy = 0 + 0 = 0$. Hence the function is harmonic

(b) By Cauchy-Riemann equation

Example 2: show that the function $U(x, y) = y^3 - 3x^2 y$ is harmonic and find its harmonic conjugate.

Solution

$U(x, y) = y^3 - 3x^2 y$
$Ux = -6xy, \quad Uxx = -6y$
$Uy = 3y^2 - 3x^2 \qquad Uyy = 6y$
And since
$Uxx + Uyy = -6y + 6y = 0$

The function
$U(x, y) = y^3 3x^2 y$ is harmonic

To find the harmonic conjugate,
From
$Ux(x, y) = -6xy,$ since $Ux = Vy$,
$Vy(x, y) = -6xy$

Find x, and integrate both sides with respect to y,
$V(x, y) = -3xy^2 + \phi(x)$
And since $Uy = -Vx$ must hold, it follows from (x) and (xr) that

$3y^2 - 3x^2 = 3y^2 + \phi^1(x)$

So that

$\phi^1(x) = 3x^2$ and $\phi(x) = 6x + C$

$V(x, y) = -3xy^2 + 6x + C$.

Is the harmonic conjugate of $x / x, y)$

The corresponding analytic function u

$f(z) = (y^3 - 3x^2 y) + i(x^3 - 3xy^2 + C)$

Which is equivalent to

$f(z) = i(z^3 + 1)$

SELF ASSESSMENT EXERCISES

1. Verify that the real and imaginary parts of the following functions satisfy the Cauchy-Riemann equations and thus deduce the analyticity of each function
   (a) $f(z) = z^2 + 5iz + 3 = 1$
   (b) $f(z) = ze^{-z}$
   (c) $f(z) = Sin 2z$

2. (a) Prove that the function $U = 2x(1 - y)$ is harmonic
   (b) Find a function $v \quad s.\ t \ f(z) = u + iv$ is analytic
   (c) Express $f(z)$ in terms of z

3. Verify that C – R equation are satisfied for the functions
   (a) e^{z^2}
   (b) $Cos 2z$
   (c) $Sinh 4z$

4. Determine which of the following functions are harmonic and find their conjugates.
   (a) $3x^2 y + 2x^2 - y^3 - 2y^2$
   (b) $2xy + 3xy^2 - 2y^3$
   (c) $xe^x \cos y - ye^x Sin y$
   (d) $e^{-2xy} Sin(x^2 - y^2)$

5. (a) Prove that $\psi = In[(x - 1j^2) + (y - 2j^2)]$ is harmonic in every region which does not include the point (1, 2)
   (b) Find a function $\phi \quad s.\ t \quad \psi + i\psi.1$ analytic

(c) Express $\psi\, x\, i\, \psi$ as a function of Z

6. If U and V are harmonic in a region R, prove that
$(Uy - Vx) + i\ (Ux + Vy)$ is analytic in R.

4.0 CONCLUSION

This unit had been devoted to treatment of special class of function usually dealt with both in real and complex functions. You are required to master these functions so that you can be able to solve problems associated with them.

5.0 SUMMARY

Recall that in this unit we considered derivatives in complex variables, we derived the Cauchy Riemann equations for determining analytic functions in complex variables, we also studied harmonic functions etc. Examples were given to illustrate each of these functions.

6.0 TUTOR-MARKED ASSIGNMENT

1. (a) Prove that the function $U = 2x(1-y)$ is harmonic
   (b) Find a function $v \;\; s.t \;\; f(z) = u + iv$ is analytic
   (c) Express $f(z)$ in terms of z

2. Verify that C – R equation are satisfied for the functions
   (a) e^{z^2}
   (b) $Cos\, 2\, z$
   (c) $Sinh\, 4\, z$

3. Determine which of the following functions are harmonic and find their conjugates.

(a) $3x^2 y + 2x^2 - y^3 - 2y^2$

(b) $2xy + 3xy^2 - 2y^3$

(c) $xe^x \cos y - ye^x Sin\, y$

(d) $e^{-2xy} Sin\,(x^2 - y^2)$

4. (a) Prove that $\psi = In[(x-1j^2) + (y-2j^2)]$ is harmonic in every region which does not include the point (1, 2)

(b) Find a function $\phi \;\; s.\, t \;\; \psi + i\psi.1$ analytic

(c) Express $\psi\, x\, i\psi$ as a function of Z

5. If U and V are harmonic in a region R, prove that $(Uy - Vx) + i\;(Ux + Vy)$ is analytic in R

7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand (1976), Advanced Calculus for Application 2nd Edition.

UNIT 3 PRINCIPLES OF ANALYTIC CONTINUATION

CONTENTS

1.0 INTRODUCTION

We shall examine in this unit principle of analytic continuation and establish conditions under which functions of complex variables will be analytic in some regions.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define residues and residues theorem
- do calculations of residues
- answer questions on residues.

3.0 MAIN CONTENT

Suppose that inside some circle of convergence C_1 with centre at a $,f(z)$ is represented by a Taylor series expansion defined by:

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ...(1)$

If the value of $f(z)$ is not known, choosing a point b inside C_1, we can find the value of $f(z)$ and its derivatives at b. from (1) and thus arrive at a new series

$b_0 + b_1(z-b) + b_2(z-b)^2 + ... + ...(2)$

Having circle of convergence C_2. If C_2 extends beyond C_1, then the values of and its durations can be obtained in this extended portion. In this case, we say that $f(z)$ has been extended analytically beyond C_1 and the process is called analytic continuation or analytic extension. This process can be repeated indefinitely.

Definition: Let $F_1(z)$ be a function of z which is analytic in a region R_1. Suppose that we can find a function $F_2(z)$ which is analytic in a region R_2 and which is such that $F_1(z) = F_2(z)$ in the region common to R_1 and R_2. Then we say that $F_2(z)$ is an analytic continuation of $F_1(z)$.

3.1 Residues and Residues Theorems

Recall that a point z_0 is called a singular point of the function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. A singular point z_0 is said to be isolated if in addition, there is some nbd of z_0 throughout which f is analytic except at the point itself.

When z_0 is an isolated singular point of a function f, there is a positive number R, such that f is analytic at each point z for which $0 < |z - z_0| < R_1$ consequently the function is represented by a series.

$$f(z) = \sum_{n=0} an(z-z_0)^\eta + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + ... + \frac{b_\eta}{(z-z)^\eta} + ...$$

$$0 < |z - z_0| < R_1$$

Where the coefficients $a\eta$ and $b\eta$ have certain integral representations. In particular

$$b_\eta = \frac{1}{\eta\bar{\eta}i}\int_c \frac{f(z)dz}{(z-z_0)^{\eta+1}} \qquad (\eta = 1, \quad 2..........) \qquad ....(2)$$

When C is any positively oriented simple closed contour around Z_0 and lying in the domain $0 < |z - z_\eta| < R$

When $\eta = 1,$ this expression for b_η can be written $\int_c f(z)dz = 2\eta i\, b_1$ ...(3)

The complex number b_1 which is the coefficient of $\frac{1}{(z-z_0)}$ in expansion (1) called the residue of f at the isolated singular point z_0

Equation (3) provides a powerful method for conducting certain integral around simple closed ……

Example

Consider the integral

$$\int_c \frac{e^{-z}}{(z-1)^2}dz$$

Is analytic within and on C except at the isolated singular point z =1. Thus, according to equation (3), the value of integral (4) is ……times the …..of f at z = 1. To determine this residue, we recall the maclaurin series expansion.

$$e^z = \sum_{n=o}^{\infty} \frac{z^\eta}{n1} \qquad (|z| < \infty)$$

From which it follows that

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}e^{-(z-1)}}{(z-1)^2} = \sum_{n-0}^{\infty} \frac{(-1)^\eta (z-1)^{n-2}}{\eta\ i\ e} \quad (0 < |z-1| < \infty)$$

In this Laurent series expansion, which can be written in the form (1), the coefficient of $\frac{1}{z-1}$ is $-1/e$. that is, the residue of f at z = 1 is $-1/e$. Hence $\int_c \frac{e^{i-z}}{(z-1)^2}dz = \frac{)n(}{e}$

3.2 Calculation of Residues

If $z = z_0$ is a pole of order K, there is a formula for b_n given as

$$b_n = \frac{\lim}{z \to z_0} \frac{1}{(n-1)!} \frac{d^{ii-i}}{dz^{n-1}} \left\{(z-z_0)^n f(z)\right\} \ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

If $\eta = 1$ (simple pole), the result is given as

$$b_1 = \frac{\lim}{z \to z_0}(z-z_0)f(z)$$

$z \to z_0$

Which is a special case $- f$ (5) with $\eta = 1$ if one defines $0! = 1$.

Example

For each of the following functions, determine the poles and the residues at the poles.

(a) $\frac{2z+1}{z^2-z-2}$ (b) $\left(\frac{z+1}{z-1}\right)^2$

Solution

(a) $\frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z+1)(z-2)}$ ……the function has two poles at $z = -1$ and $z = 2$ both of under 1.

Residue at $z = -1$,

$$\frac{\lim}{z \to -1}(z+1)f(z) = \frac{\lim}{z \to -1}\frac{(z+1)(2z+1)}{(z+1)(z-2)}$$

$$= \frac{\lim}{z \to -1}\frac{2z+1}{z-2} = \frac{1}{3}$$

Residue at z = 2,

$$\frac{\lim}{z=2}\frac{(z-2)(2z+1)}{(z+1)(z-2)}=\frac{\lim}{z\to 2}\frac{2z+1}{z+1}=\frac{5}{3}$$

(b) z = 1 is a pole of order 2.
Residue at z = 1 is

$$\frac{\lim}{z\to 1}\frac{y}{dz}\left\{(z-1)^2(z+1)^2/(z-1)^2\right\}$$

$$\frac{\lim}{z\to 1}\frac{d}{dz}(z+1)^2=\frac{\lim}{z\to 1}2(z+1)=4.$$

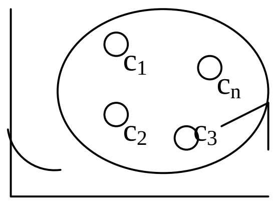
3.3 Residue Theorem

Theorem: Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ interior to C. If $B_1, B_2, \ldots, B_n$ denote the residues of f at these points respectively, then

$$\int_c f(z)dz = 2\pi i\,(B_1+B_2+\ldots\ldots\ldots+B_n)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Proof

Let the singular points $zj\ (j=1,2,\ldots.n)$ be centers of positively oriented circles Cj which are interior to C and are so small that no two of the circles have points in common.



The circles Cj together the simple closed contour C form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain. Hence, according to the extension of the Cauchy-Goursat theorem to such regions.

$$\int_c f(z)dz = \int_{c_1} f(z)dz - \int_{c_2} f(z)dz - \ldots\ldots\ldots - \int_{c_n} f(z)dz = 0$$

This reduces to equation (1) because

$\int_{c_1} f(z)dz = 2\overline{\pi} iBj \qquad (1, 2, ...n)$

And the proof is complete.

Example: Let us use the theorem to evaluate

$$\int_c \frac{5z-2}{z(z-1)}\ dz$$

Where C is the circle $|z| = 2$, described counterclockwise. The integrated has the two singularities $z = 0$ and $z = 1$, both of which are interior to C. We can find the residues B_1 at $z = 0$ and B_2 at $z = 1$ with the aid of the maclurin series.

$$\frac{1}{1-z} = 1 + z + z\tfrac{2}{1}............ \qquad (|z| < 1)$$

We first write the Laurent expansion

$$\frac{5z-2}{z(z-1)} = \left(\frac{5z-1}{z}\right)\left(\frac{-1}{1-z}\right) = \left(5 - \frac{2}{z}\right)\left(-1 - z - z^2.........\right)$$

$$= \frac{2}{z} - 3 - 3z.............................. \qquad (0 < |z| < 1)$$

Of the integrand and conclude that $B_1 = 2$. Next, we observe that

$$\frac{5z-2}{z(z-1)} = \left[\frac{5(z-1)+3}{z}\right]\left[\frac{1}{1+(z-1)}\right]$$

$$= \left(5 + \frac{3}{z-1}\right)\ (1-(z-1)) + (z-1)^2........$$

When $0 < |z-1| < 1$. The coefficient of $1/(z-1)$ in the Laurent expansion which is valid for $0 < |z-1| < 1$ is therefore 3

Thus $B_2 = 3$, and

$$\int_c \frac{5z-2}{z(z-1)} dz = 2\overline{\pi} i(B_1 + B_2) = 1\,0\ \overline{\pi i}\ .$$

An alternative and simple way of solving the problem is to write the integrand as the sum of its partial fractions. Then

$$\int_c \frac{5z-2}{z(z-1)} dz = \int_c \frac{2}{z} dz + \int_c \frac{3}{z-1} dz = 4\pi i + 6\pi i = 10\pi\ i$$

4.0 CONCLUSION

The residue method learnt in this units allows us to handle integration with ease. You are required to master this method very well.

5.0 SUMMARY

Recall that we started this unit by defining the residue theorem which is now recalled for your understanding:

Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points z_1, z_2,…, z_n interior to C. If B_1, B_2,…,B_n denote the residues of f at these points respectively, then

$$\int_c f(z)dz = 2\pi i\,(B_1 + B_2 + ...+ B_n).$$

This theorem form the basis for solves complex integration. You may wish to answer the following tutor-marked assignment question.

6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate the integral

$$\int_c \frac{7z-5}{z-1}dz$$

2. Evaluate

$$\int \frac{z+2}{z^2-5z+6}dz$$

3. $\displaystyle\int_c \frac{5z-2}{z^2(z-1)}dz$

7.0 REFERENCES/FURTHER READING

Francis B. Hildebrand (1976), Advanced Calculus for Application 2nd Edition.

UNIT 4 COMPLEX INTEGRATION

CONTENTS

2.0 INTRODUCTION

This unit will examine complex integration. The theorem on line integral, such as greens theorem will also be examined.

3.0 OBJECTIVES

At the end of this unit, you should be able to:

- define integration on complex variables
- define the complex form of green's Theorem
- learn about Cauchy-Goursat theorem
- learn about Cauchy integral
- solve related problems on complex integrations.

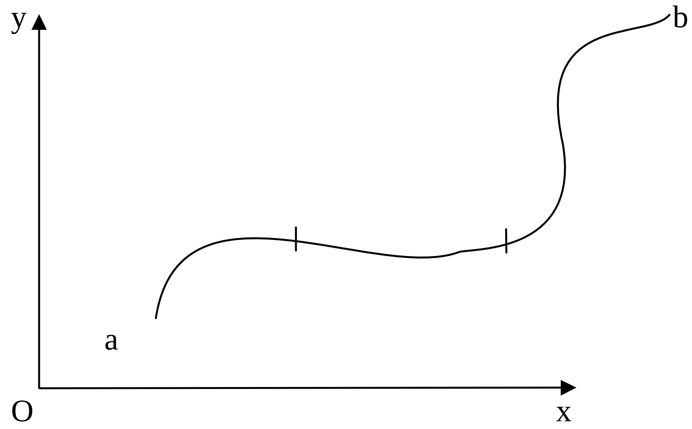
3.0 MAIN CONTENT

3.1 Curves

If $\phi(t)$ and $\psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \le t \le t_2$, the parametric equations

$$Z = x + iy = \phi(t) + i\psi(t) = Z(t) \qquad t_1 \le t \le t_2$$

Define a continuous curve or arc in the Z-plane joining points $a = Z(t_1)$ and $b = Z(t_2)$ as shown below



If $t_1 \mp t_2$ while $Z(t_1) = Z(t_2)$, i.e. $a = b$, the end point is coincide and the curve is said to be closed. A close curve which does not intersect itself anywhere is called a simple closed curve.

If $\phi(t)$ and $\psi(t)$ have its derivations in $t_1 \le t \le t_2$, the curve is often called a smooth curve or arc. A curve which is composed of a finite number of smooth arcs is called a piecewise or sectionally smooth curve or sometimes a contour. For example, the boundary of a square of a piecewise smooth curve or contour.

3.2 Simply and Multiply Connected Regions

A region R is called simply connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply connected is called multiply-connected e.g. $|z| < 2$. $|<||z| < 2$.

3.3 Complex Line Integrals

Suppose that the equation

$z = z(t) \qquad (a \le i \le b)$…………………………………………….. (4.1.1)

Represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. Let the function $f(z) = \mu(x, y) + iv(x, y)$ be piecewise continuous on C. If $z(t) = \mu(t) + iy(t)$ the function

$f[z(t)] = \mu[x(t), y(t)] + i\,v[x(t), y(t)]$ is piecewise continuous on the interval $a \le t \le b$. We define the line integral or contour integral, of f along C as follows:

$$\int_c f(z)dz = \int_a^b [z(t)] \ z^1(t)dt \ldots\ldots (4.1.2)$$

Note that since C is a contour, $z(t)$ is also piecewise continuous on the integral $a \le t \le b$, and so the existence of integral (4.1.2) is ensured.

The integral on the right-hand side in equation (4.1.2) is the product of the complex-valued functions.

$\mu[x(t), y(t)] + i \ v\,[n(t), y(t)], \ \ n^1(t) + y^1(t).$
Of the real variable t. Thus

$$\int_c f(z)dz = \int_a^b \left(\mu x^1 - vy^1\right)dt + i\int_a^b \left(vn^1 + uy^1\right)dt \ldots\ldots (4.1.3)$$

In terms of line integrals of real-valued functions of two real variables, then

$$\int_c f(z)dz = \int_c \mu dx - vdy + i \int vdu + udy \ldots\ldots (4.1.4)$$

Example: Find the value of the integral

$$I_1 = \int_{c_1} z^2 dz$$

Where C_1 is the line segment from $z = 0$ to $z = 2 + i$

Proof

Points of C_1 lie on the line $y = x/2$ or $x = 2y$. If the coordinate y is used as the parameter, a parametric equation for C_1 μ

$$z = 2y + i\ y \qquad (0 \le y \le 1)$$

Also, in C_1 the integral z^2 becomes

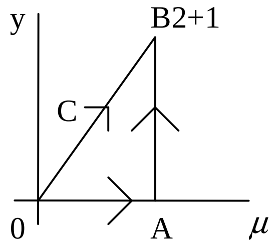
$$z^2 = (2y + iy)^2 = 3y^2 + i4y^2$$

Therefore,

$$I_1 = \int_0^1 \left(3y^2 + i4y^2\right)\ (2+i)\ dy$$

$$= (3+4i)(2+i)\int_0^1 y^2 dy = \tfrac{2}{3} + \tfrac{11}{3}\ i$$

Example: Let C_2 denote the contour $0AB$ shown below



Evaluate

$$I_2 = \int_{c_2} z^2 dz$$

Solution

$$I_2 = \int_{c_2} z^2 dz = \int_{0A} z^2 dz + \int_{AB} z^2 dz$$

The parametric equation for path 0A is $z = n + i0(1 \le x \le 2)$ and for the path AB one can write $Z = 2 + iy(0 \le y \le 1)$.

Hence

$$I_2 = \int_0^2 x^2 dx + \int_0^1 (2+iy)^2 idy.$$

$$= \int_0^2 x^2 dx + 2\left[\int_0^1 (4 - y^2)dy + 4i\int_o^1 ydy\right]$$

$$= \frac{2}{3} + \frac{11}{3} i$$

Green's Theorem in the plane

Let $P(x, y)$ and $Q(x, y)$ be its and have its partial derivatives in a region R and on its bounding C. Green's theorem states that

$$\oint_c Pdx + Qdy = \int_R \int (Q_x - P_y)dndy$$

The theorem is valid for both simple and multiple connected regions.

3.4 Complex Form of Green's Theorem

Let $F\left(z,\bar{z}\right)$ be its and have its derivations in a region R and on its bounding C, where $z = x + iy$, $\bar{z} = x - iy$ are complex conjugate coordinates. The Green's theorem can be written in the complex form as

$\oint_c F\left(z,\bar{z}\right)dz = 2i\iint \frac{\partial F}{\partial \bar{z}}\, dA$ where dA represents the element of area $dndy$

Proof

Let $F(z,z) = P(x) + iQ(x,y)$. Then using Green's theorem, we have

$$\oint_c F\left(z_1\bar{z}\right)dz = \oint_c (P + iQ)(x,y)dz \qquad = \oint_c Pdn -, Qdy + i\oint_c Qdn + P\,dy$$

$$= -\oiint_R \left(\frac{\partial Q}{\partial^R} + \frac{\partial P}{\partial y}\right) dndy + i\oiint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dxdy$$

$$= i\iint_R \left[\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy$$

$$= 2i\iint_R \frac{\partial F}{\partial \bar{Z}}\, dndy$$

Example: Evaluate the integral

$$I = \int \bar{z}dz$$

Where

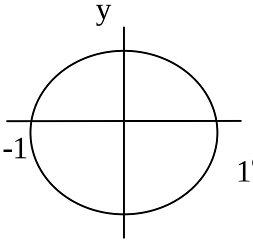
(i) The path of integration C is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$.

(ii) Same points but along the lower semi circle C.

Solution

(i) The parametric representation $z = e^{i\not\in}$ $(0 \le \not\in \le i)$ and since $d\left(e^{iQ}\right)/\,d\not\in = ie^{i\not\in}$

$$I = \int_c \bar{z}dz = -\int_0^{\hat{\bar{}}} e^{-i\not\in}\, i\, e^{i\not\in}\, d0 = -\pi i$$



(ii) $I = \int \bar{z}dz = \int_{\hat{\bar{}}}^{2\hat{\bar{}}} e^{-1Q} i\; e^{i\not\in}\; d\phi = \pi i$

Example: Evaluate $\int \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by (a) $z = t^2 + it$ (b) the line from $z = 0$ and $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$

Solution

(a) The given integral equal,

$$\int_c (n - iy)(dn + idy) = \int_c ndk + ydy + i \int_c ndy - ydn$$

The parametric equations of C are $n = t^2, y = t$ from $t = 0$ to $t = 2$
Then the line integral equal

$$\int_{t=0}^{2} \left(t^2\right)(2tdt) + (t)(dt) + i\int_{t=0}^{2} \left(t^2\right)(dt) - (t)(dt - dt)$$

$$= \int_0^2 \left(2i^3 + t\right)dt + i\int_0^2 \left(-t^2\right)dt = 10 - \frac{8i}{3}$$

(b) $\int_c (x - iy)(dx + idy) = \int_c xdx + ydy + i\int_c ndy - ydx$

The line from $Z = 0$ *to* $Z = 2i$ is the same as $(0{,}0)$ to $(0{,}2)$ for which $x = 0,\ dn = 0$ and the line integral equals.

$$\int_{y=0}^{4} (Q)(0) + ydy + i\int_{y=0}^{2} (0)dy - y(0) = \int_{y=0}^{2} ydy = 2\,.$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y = 2, dy = 0$ and the line integral equals $\int_0^4 xdx + 2 \bullet 0 + i\int_{x=0}^4 n \bullet 0 - 2dn = \int_0^4 xdx + i\int_0^4 \frac{-2xdx}{8i} = 8 - 8i$
Then the requires value = $2 + (8 - 8i) = 10$.

3.4 Cauchy-Goursat Theorem

Suppose that two real-valued function $P(n, y)$ and $Q(n, y)$ together with their partial derivatives of the first order, are continuous throughout a closed region R consisting of points interior to and on a simple closed contour C in the ny plane. By Green's theorem, for line integrals,

$$\int_c Pdn + Qdy = \iint_R (\phi_x - P_y) dndy.$$

Consider a function

$f(z) = u(x, y) + i\ v\ (x, y)$

Which is analytic throughout such a region R in the ny, or Z, plane, the line integral of f along C can be written

$$\int_c f(z) dz = \int_c ndn - vdy + i \int_c vdx + udy \text{ ....}(1)$$

Since f is its in R, the functions u and v are also its theorem i and if the derivative f^1 of f is its in R, so are the first order partial derivatives of u and v. By Green's theorem, (1) could be written as

$$\int_c f(z) dz = \iint_R (-v_x - u_y) dndy + i \iint_R (u_x - v_y) dndy \text{ .... } (2)$$

But in view of the Cauchy-Goursat equations

$U_x = V_y, U_y = -V_x$

The integrals of these two double integral are zero throughout R. So

Theorem: If f is analytic in R and f^1 is continuous then, $\int_c f(z) dz = 0$.

This is known as Cauchy theorem.

Goursat proved that the condition of continuity of f^1 in the above Cauchy theorem can be omitted.

Theorem: (Cauchy-Goursat theorem)

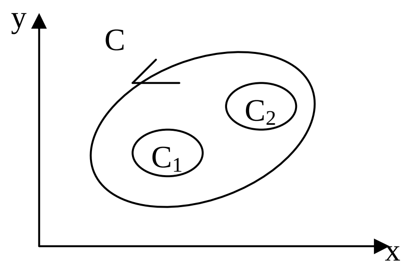
If a function f is analytic at all points interior to and in a simple closed contour C, then

$\int_c f(z)dz = 0$.

Cauchy-Goursat theorem can also be modified for the …… B of a multiply connected domain.

Theorem: Let C be a simple closed contour and let $Cj \quad (j = 1,2,....n)$ be a finite number of simple closed contours inside C such that the regions interior to each Cj have no points in common. Let R be the closed region consisting of all points within and on C except for points interior to each Cj. Let B and all the contours oriented boundary of R consisting of C and all the contours Cj, described in a direction such that the interior points of R lie to the left of B. Then, if f is analytic throughout R_1.

$\int_B f(z)dz = 0$

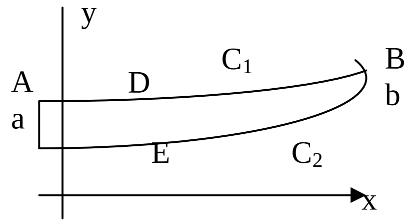


As a consequence of Cauchy's theorem, we have the following

Theorem: If $f(z)$ is analytic in a simply-connected region R, then $\int_a^b f(z)dz$ is independent of the path in R joining any two points a and b in R.

Proof

Consider the figure below



By Cauchy's theorem

$$\int_{ADBCA} f(z)dz = 0$$

Or $\int_{ADB} f(z)dz + \int_{BEA} f(z)dz \quad = \; 0$

Hence

$$\int_{ADB} f(z)dz = - \int_{BEA} f(z)dz = \int_{AEB} f(z)dz$$

Thus

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_a^b f(z)dz$$

This yields the required result.

Example: If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1, 1) and (2, 3), show that

$\int_c (12z^2 - 4iz)dz$ is independent of the path joining (1, 1) and (2, 3)

Solution:

A (1, 1) $\rightarrow$ B (2, 1) $\rightarrow$ C (2, 3)

Along A (1, 1) to B (2, 1), y = 1, dy = 0. So that $z = x4i$ and $dz = dx$. Then

$$\int_{x-1}^{2} \{12(x4i)^2 - 4i(x+i)\}dx = 20 + 30i$$

Along B (2, 1) to C (2, 3), $x = 2,\ dx = 0$ so that $z = 2 + iy$ and $dz = idy$. Then

$$\int_{y=1}^{3} \left\{12(2+iy)^2 - 4i(2+iy)\right\} idy = -176 + 8i$$

$y = 1$

So that

$$\int_c (12z^2 - 41z)dz = 20 + 30i - 176 + 8i = -156 + 38i$$

The given integral equals

$$\int_{1+i}^{2+3i} \left(12z^2 - 4iz\right)dz = \left(4z^3 - 2iz^2\right)\Big|_{1+i}^{2+3i} = -156 + 38i$$

Morera's Theorem

Let $f(z)$ be continuous in a simply connected region R and suppose that

$$\oint_c f^1(z)dz = 0$$

Around every simple closed curve C in R. Then $f(z)$ is analytic in R.

This theorem is called the converse of Cauchy's theorem and it can be extended to multiply-connected regions.

Indefinite Integrals (Anti-derivatives)

Let $f(z)$ be a function which is continuous throughout a domain D, and suppose that there is an analytic function F such that $F^1(z) = f(z)$ at each point in D. The function F is said to be an anti derivative of f in the domain D.

Cauchy Integral Formula

Theorem: Let f be analytic everywhere within and in a simple closed contour C taken in the positive sense. If Z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{Z - z_0}$$

This formula is called the Cauchy integral formula. It says that that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f in C.

When the Cauchy integral formula is written as

$$\int_c \frac{f(a)dz}{Z - z_0} = 2\pi \bar{i}\ f(z_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

It can be used to evaluate certain integrals along.

Simple closed contours

Example: Let C be the positively oriented circle $|z| =$ since the function $f(z) = {}^2 / (9 - z^2)$ is analytic within and in C and the point $Z_0 = -i$ is interior to C, then by Cauchy Integral formula

$$\int_c \frac{zdz}{(9 - z^2)(z + i)} = \int_c \frac{z/(9 - z^2)}{z - (-i)} = 2\pi c\left(\frac{-i}{10}\right) = \frac{\pi}{\delta}$$

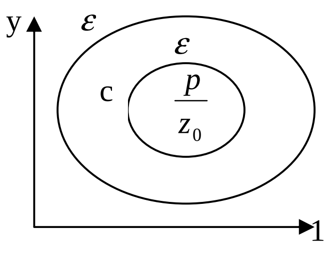
Proof

Since f is its at Z_0, there corresponds to any positive number ε, however small, a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| = \rho \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Observe that the function $f(z) / (z - z_0)$ is analytic at all points within and in C except at the point z_0. Hence, by Cauchy-Goursat theorem for multiply connected domain, it's integral around the oriented boundary of the region between C and C_0 has value zero.

$$\int_c \frac{f(z)dz}{Z - z_0} - \int_{C_0} \frac{f(z)dz}{Z - z_0} = 0$$



That is

$$\int_c \frac{f(z)dz}{Z - Z_0} = \int_{C_0} \frac{f(z)dz}{Z - Z_0}$$

This allows us to write

$$\int_C \frac{f(z)dz}{Z-Z_0} - f(z_0)\int_{C_0} \frac{dz}{Z-Z_0} = \int_{C_0} \frac{f(z)-f(z_0)}{Z-Z_0}dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\int_{C_0} \frac{dz}{Z-Z_0} = 2\pi i$$

And so equation (Z) becomes

$$\int_C \frac{f(z)dz}{Z-Z_0} - 2\pi i\ f(z_0) = \int_{C_0} \frac{f(z)-f(z_0)}{Z-Z_0}dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

By (1) and noting that the length of C_0 is $2\bar{\wedge}\rho$, by properties of integrals

$$\left|\int_{C_0} \frac{f(z)-f(z_0)}{Z-Z_0}dz\right| < \frac{\varepsilon}{\rho}2\bar{\pi}\rho = 2\pi\varepsilon$$

In view of (3) then

$$\left|\int_C \frac{f(z)dz}{Z-Z_0} - 2\pi i\ f(z_0)\right| < 2\bar{\pi}\varepsilon .$$

Since the left hand side of this inequality is a non negative constant which is less than an arbitrary small positive number, it must be equal to zero. Hence, equation for it valid and the theorem is proved.

Cauchy's integral formula can also be extended to a multiply connected region. With the understanding that $f\genfrac{}{}{0pt}{}{(v)}{z}$ denotes $f(z)$ and that 0! = 1, we can use mathematical induction to verify that

$$f^{(n)}(z_0) = \frac{n!}{2\bar{\pi}i}\int_C \frac{f(z)dz}{(Z-Z_0)^{n+1}}(n = v, 1, 2.)$$

When n = 0, this is just the Cauchy integral formula stated earlier.

Example: Find the value of $\oint_C \frac{Sin^6 Z}{(Z-\bar{\wedge}\!/_6)^3}dz$

Where C is a circle $|z| = 1$

Solution:

$$\int_C \frac{Sin^6 z}{\left(Z - \pi/6\right)^3} dz = \frac{2\pi i\ f^{2}\left(Sin^6 \pi/6\right)}{2^{1}}$$

$$= \frac{6x2\pi i}{2}\left[5\,Sin^4 \pi/6\ \ Cos^2 \pi/6 - Sin^6 \pi/6\right]$$

$$= 2\,1\pi i/16$$

<u>Other Important Theorems</u>

1. **Cauchy's inequality**:
If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z \neq a$, then $\left|f^{(n)}(a)\right| \leq \frac{M \bullet n!}{r^n}$ $n = 0, 1, 2, \ldots\ldots\ldots$
Where M is a constant such that $\left|f(z)\right| < M$ on C, i.e. M is an upper bound of $\left|f(z)\right|$ on C.

2. **Lowville's Theorem:**
Suppose that for all Z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded, i.e. $\left|f(z)\right| < M$ for some constant M, then $f(z)$ must be a constant

3. Fun**damental Theorem of Algebra:**
Every polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \quad a_n Z^n = 0$ with degree $\pi \geq 1$, and $a_n \neq 0$ has at least one root.

4. **Maximum Modulus Theorem:**
If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum values of $\left|f(z)\right|$ occurs on C.

SELF ASSESSMENT EXERCISES

1. Evaluate $\int_{(0,1)}^{(2,5)} (3x + y)dx + (xy - x)dy$ along
   (a) the curve $y = x^2 + 1$
   (b) the straight line joining (0, 1) and (2, 5)
   (c) the straight line from (0, 1) to (0, 5) and then (0, 5) to (2, 5)

2. Evaluate $\int_C \left(x^2 - iy^2\right)dz$
   (a) along the parabola…… $y = 4n^2$ from (1,4) to (2, 16)
   (b) straight line from (1, 1) to (1, 8) and then from (1, 8) to (2, 8).

3. Evaluate $\int_{-2+i}^{2-i} \left(3xy + iy^2\right)dz$
   (a) along the curve $x = 2t - 2\ i - y = 1 + t - t^2$
   (b) along the straight line joining $x = -2 + i$ and $z = 2 - i$

4. Evaluate
   (a) $\oint_C \frac{Sin\,\pi Z^2 + Cos\,\pi Z^2}{(Z-1)(Z-2)}dz$, where C is the circle $|Z| = 3$.
   (b) $\oint_C \frac{e^{2z}}{(Z+1)^4}dz$ where C is the circle $|Z|$=3

5. Evaluate $\oint_C \frac{Sin\,3z}{Z + \pi/2}dz$ if C is the circle $|Z| = 5$

4.0 CONCLUSION

The materials in this unit must be learnt properly because they will keep on re occurring as progress in the study of mathematics at higher level.

5.0 SUMMARY

We recap what we have learnt in this unit as follows:

You learnt about Cauchy-Goursat equations, Moreras Theorem and applied it to indefinite integrals. We also consider Cauchy integral Formula

We considered some solved examples to illustrate the theory we have learnt in this unit. You may which to answer the following tutor-marked assignment

6.0 TUTOR-MARKED ASSIGNMENT

1. Evaluate $\int_{-2+i}^{2-i}\left(3xy+iy^{2}\right)dz$
   (a) along the curve
   (b) along the straight line joining $x=-2+i$ and $z=2-i$

2, Evaluate
   (a) $\oint_{C}\frac{Sin\,\pi z^{2}+Cos\,\pi z^{2}}{(z-1)(z-2)}dz$, where C is the circle $|z|=3$.
   (b) $\oint_{C}\frac{e^{2z}}{(z+1)^{4}}dz$ where C is the circle $|z|=3$

3.. Evaluate $\oint_{C}\frac{Sin\,3z}{z+\pi/2}dz$ if C is the circle $|z|=5$

7.0 REFERENCES/FURTHER READINGS

Francis B. Hildebrand (1976), Advanced Calculus for Application 2nd Edition.