NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH315

COURSE TITLE: ANALYTICAL DYNAMICS

COURSE GUIDE

MTH315
ANALYTICAL DYNAMICS

Course Team	Dr. S.O. Ajibola (Developer/Writer) - NOUN Dr. B. Abiola (Coordinator) - NOUN Dr. S.O. Ajibola (Co-coordinator) - NOUN

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
No. 5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja
Nigeria

e-mail: centralinfo@nou.edu.ng
URL: www.nou.edu.ng

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Introduction

MTH315 - Analytical Dynamics is designed to teach you how mathematics could be used in solving problems in the contemporary Science/Technology and Engineering world. Therefore, the course is structured to expose you to the skills required to attain a level of proficiency in Analytical Dynamics.

What you will Learn in this course

You will be taught the basics of Analytical Dynamics; an aspect of Applied Mathematics.

Course Aims

There are thirteen study units in the course and each unit has its objectives. You should read the objectives of each unit and bear them in mind as you go through the unit. In addition to the objectives of each unit, the overall aims of this course include to:

(i) introduce you to the words and concepts in Applied Mathematics
(ii) familiarise you with the peculiar characteristics in Analytical Dynamics
(iii) expose you to the need for and demands of mathematics in the Science/ Technology and Engineering world
(iv) prepare you for the contemporary Science/Technology and Engineering world.

Course Objectives

By the end of the course, you should be able to:

- define the term “constraint”
- mention the types of constraint
- differentiate between the various types of constraint
- state D’Alambert’s Principle and relevant theorems
- apply the Lagrange’s equations to find the differential equations
- state the Lagrange function of particle(s) moving in a conservation force field
- derive Lagrange’s equations for holonomic and non-holonomic constraint respectively
- define and explain simple harmonic motion
- state the forces causing simple harmonic motion
- explain the suspension by an elastic string
- define conical pendulum

- relate discreet and continuous system to degree of freedom
- define conical pendulum
- identify the forces causing simple harmonic motion
- discuss the suspension by an elastic string

Working through this Course

You have to work through all the study units in the course. There are seven modules and thirteen study units in all.

Course Materials

Major components of the course are:

1. Course Guide
2. Study Units
3. Textbooks
4. CDs
5. Assignments File
6. Presentation Schedule

Study Units

The breakdown of the seven modules and thirteen study units are as follows:

Module 1

Unit 1 Degree of Freedom
Unit 2 Constraints

Module 2

Unit 1 Lagrange's Equation

Module 3

Unit 1 Impulsive Motion

Module 4

Unit 1 Simple Harmonic Motion
Unit 2 Collation of Smooth Spheres

Module 5

Unit 1 Newton's Law of Motion
Unit 2 Work, Power and Energy
Unit 3 Rectilinear Motion

Module 6

Unit 1 Reduction of Coplanar Forces Acting on a Rigid Body to a Force and a Couple
Unit 2 Moment of a Force

Module 7

Unit 1 The Hamiltonian
Unit 2 The Calculus of Variation

Textbooks and References

Every unit contains a list of references and further reading. Try to get as many as possible, of those textbooks and materials listed. The textbooks and materials are meant to deepen your knowledge of the course.

Assignment File

In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain from these assignments will count towards the final mark you obtain for this course. Further information on assignments will be found in the Assignment File itself and in the section on assessment of this Course Guide.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor-marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against falling behind in your work.

Assessment

Your assessment will be based on Tutor-Marked Assignments (TMAs) and a final examination which you will write at the end of the course.

Tutor-Marked Assignments (TMAs)

Every unit contains at least one or two assignments. You are advised to work through all the assignments and submit them for assessment. Your tutor will assess the assignments and select four which will constitute the 30% of your final grade. The tutor-marked assignments may be presented to you in a separate file. Just know that for every unit there are some tutor-marked assignments for you. It is important you do them and submit for assessment.

Final Examination and Grading

At the end of the course, you will write a final examination which will constitute 70% of your final grade. In the examination which shall last for two hours, you will be requested to answer three questions out of at least five questions.

Course Marking Scheme

This table shows how the actual course marking is broken down.

Assessment	Marks
Assignments	Four assignments, best three marks of the four count at 30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

How to Get the Best from this Course

In distance learning, the study units replace the university lecture. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suits you best. Think of it as reading the lecture instead of listening to the lecturer. In the same way a lecturer might give you some reading to do, the study units tell you when to read, and which are your text materials or set books. You are provided exercises to do at appropriate points, just as a lecturer might give you an in-class exercise. Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit, and how a particular unit is integrated with the other units and the course as a whole. Next to this is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. These learning objectives are meant to guide your study. The moment a unit is finished, you must go back and check whether you have achieved the objectives. If this is made a habit, then you will significantly improve your chances of passing the course. The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a Reading section. The following is a practical strategy for working through the course. If you run into any trouble, telephone your tutor. Remember that your tutor's job is to help you. When you need assistance, do not hesitate to call and ask your tutor to provide it.

In addition, do the following:

1) Read this Course Guide thoroughly, it is your first assignment.

2) Organise a Study Schedule. Design a 'Course Overview' to guide you through the Course. Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the semester is available from the study centre. You need to gather all the information in one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates and schedule of work for each unit.

3) Once you have created your own study schedule, do everything to stay faithful to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please, let your tutor know before it is too late for help.

4) Turn to unit 1, and read the introduction and the objectives for the unit.

5) Assemble the study materials. You will need your set books and the unit you are studying at any point in time.

6) Work through the unit. As you work through the unit, you will know what sources to consult for further information.

7) Keep in touch with your study centre. Up-to-date course information will be continuously available there.

8) Well before the relevant due dates (about 4 weeks before due dates), keep in mind that you will learn a lot by doing the assignment carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the examination. Submit all assignments not later than the due date.

9) Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

10) When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

11) When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also the written comments on the ordinary assignments.

12) After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in the Course Guide).

Facilitators/Tutors and Tutorials

The dates, times and locations of these tutorials will be made available to you, together with the name, telephone number and the address of your tutor. Each assignment will be marked by your tutor. Pay close attention to the comments your tutor might make on your assignments as these will help in your progress. Make sure that your assignments reach your tutor on or before the due date.

Your tutorials are important. Therefore, try not to skip any. It is an opportunity to meet your tutor and your fellow students. It is also an opportunity to get the help of your tutor and discuss any difficulties encountered on your reading.

Summary

Analytical Dynamics is designed to teach you how mathematics could be used in solving problems in contemporary Science/Technology and Engineering world. Therefore, MTH315 is duly structured to expose you to the skills required to attain a level of proficiency in Analytical Dynamics.

Wishing you the best of luck as you read through this course.

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Headquarters
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MODULE 1

UNIT 1 DEGREE OF FREEDOM

CONTENTS

1.0 INTRODUCTION

Elementary classical dynamics evolved as a result of the dynamics of an object which may be seen as a point-mass or a particle. Practical situations often involve studying the dynamics of collections or systems of particles. If the particles of such a system are separated from each other, the system is said to be discrete; otherwise, called a continuous system. However, a discrete system having a large but finite number of particles can be said to be continuous as well.

Conversely, in practical cases a discrete system having a very large but finite number of particles can be called a continuous system.

2.0 OBJECTIVE

By the end of this unit, you should be able to:

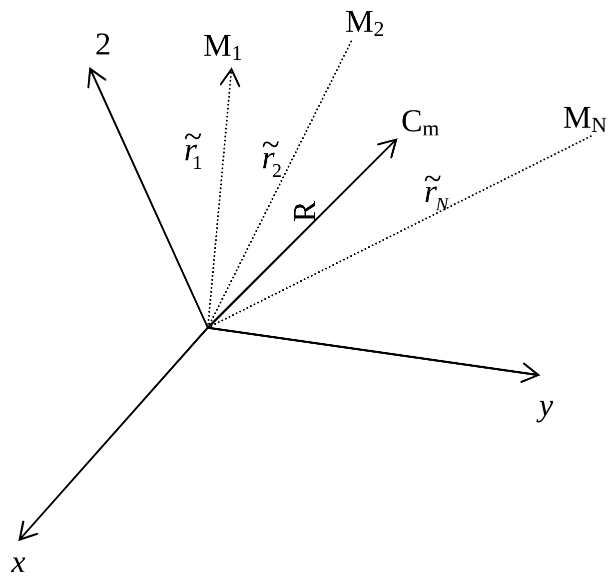
- relate discreet and continuous system to degree of freedom.

3.0 MAIN CONTENT

3.1 Degree of Freedom

The number of coordinates required to specify the position of a system of particle is called the number of degrees of freedom of the system.

For instance, a system of N particles, moving freely in space has 3N independent coordinates or degrees of freedom. Consider the system represented below:



Where C_m denotes the centre of mass

Let $\tilde{r}_1, \tilde{r}_2, ...\tilde{r}_N$ denote the positive vectors of System of N particles of constants masses m_1, m_2… m_N. If the centre of the system of the particle (i.e. centre of mass) is defined as the point C_m where its position vector $\tilde{R}$, then $\tilde{R}$ is given as:

$$\tilde{R} = \frac{\sum_{i=1}^{N} M_i r_i}{\sum_{i=1}^{N} M_i} = \frac{1}{M}\sum_{i=1}^{N} M_i r_i \qquad (1)$$

Where M = $\sum_{i=1}^{N} M_i$. This is the total mass of the system.

Then the total momentum $\underset{\sim}{P}$ of the system is given as:

$$\underset{\sim}{P} = \sum_{i=1}^{N} M_i r_i = \sum_{i=1}^{N} M_i V_i$$

Where $\underset{\sim}{Vi} = \frac{dr}{dt} = \underset{\sim}{\tilde{r}}$

But $\underset{\sim}{P} = \sum_{i=1}^{N} M_i V_i = \frac{d}{dt}\sum_{i=1}^{N} M_i r_i$

$$= \frac{d}{dt} M\tilde{R}$$

$$= M\frac{d\tilde{R}}{dt} = MV \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Where = $\frac{d\tilde{R}}{dt}$ is the velocity of the centroid.

Consequently,

$$\underset{\sim}{P} = M\underset{\sim}{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

3.2 Total Kinetic Energy

The total kinetic energy T of a system of N particles is given by:

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i |V_i|^2 = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N} M_i \left|\underset{\sim}{r}\right|^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}$$

Motion of the Centre of Mass

In considering the motion of the centre of mass of a system of N particles, we must distinguish between external forces acting on the particles due to the influence outside the system and internal forces. Then, by Newton's second law of motion, the equation of motion for the *ith* particles is obtained as:

$$\overset{o}{\underset{\sim}{P_i}} = \sum_{\substack{j=i \\ j+i}}^{N} \underline{F}\, ji + \underset{\sim}{F_i}(e) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}$$

Where $\underset{\sim}{F_i}(e)$ is the resultant external force on the *ith* particles due to the *jth* particle but $\underset{\sim}{F_{ii}} = 0$ …………………………………………….. (7)

Assuming that the system obeys Newton's third Law of Motion which states that "If particle *I* acts on particle *J* with a for F_{ij} in a direction along the line joining the *ith* and *jth* particles, while particle *j* acts on particle *i* with a force F_{ji}, conversely, then action and reaction are equal and opposite.

Mathematically, it can be shown that:

$$F_{12} = - F_{21} \ldots\ldots(8)$$

$$\text{But } \tilde{F}_{12} + \tilde{F}_{21} = 0 \Rightarrow \tilde{F}_{12} + \tilde{F}_{21} = 0 \ldots\ldots (9)$$

Consequently,

$$\sum_{\substack{i=j \\ i+j}}^{N} \underset{\sim}{F} ji = 0 \ldots\ldots (10)$$

In view of equation (10), equation (6) can.

Substituting equation (1) into equation (6), we obtained:

$$\sum_{i=1}^{N} \underset{\sim}{\dot{P}_i} = \frac{d^2}{dt^2} \sum_{i=1}^{N} M_i r_i = \sum_{i=1}^{N} F_i(e) \ldots\ldots (11)$$

Consequently, equation (1) can be written as

$$M \frac{d^2}{dt^2} R = \sum_{i=1}^{N} \underset{\sim}{F_i}(e) \equiv \underset{\sim}{F}(e) \ldots\ldots (12)$$

Remarks

Equation (12) above states that the centroid (centre of mass) moves as if the total external force $\underset{\sim}{Fe}$ were acting on the centre of mass. Thus, purely internal forces therefore have no effects on the motion of the centre of mass. It is clear that if $\underset{\sim}{F}(e) = 0$.

Then:

$$\frac{d}{dt} M\left(\frac{dR}{dt}\right) = \frac{d}{dt}\left(M \underset{\sim}{V}\right) = 0 \ldots\ldots(13)$$

Equation (12) implies that the total linear momentum is conservative. This is called the Conservative Theorem for linear momentum of a system of particle.

3.3 Total Angular Momentum

The angular momentum $\underset{\sim}{L_i}$ of the *ith* particle is given by:

$$\underset{\sim}{L_i} = \underset{\sim}{r_i} \, x \, \underset{\sim}{P_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Hence, the total angular momentum $\underset{\sim}{L_i}$ of the system is given by:

$$\underset{\sim}{L_i} = \sum_{i=1}^{N} \underset{\sim}{L_i} = \sum_{i=1}^{N} \underset{\sim}{r_i} \, x \, \underset{\sim}{P_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Thus, $\dot{\underset{\sim}{L}} = \sum_{i=1}^{N} \frac{d}{dt}(\underset{\sim}{r_i} \, x \, \underset{\sim}{P_i}) = \sum_{i=1}^{N} (r_i \, x \, \dot{P}_i) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$

Using equation (6) equation (10) becomes:

$$\dot{\underset{\sim}{L}} = \sum_{i=1}^{N} \underset{\sim}{r_i} \, x \, \underset{\sim}{F_i}(e) + \sum_{i=1}^{N} \underset{\sim}{r_i} \, x \, \underset{\sim}{F_{ji}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

But, the last term on the right hand side of (17) may be considered as a sum of pairs of the form

$$\underset{\sim}{r_i} \, x \, \underset{\sim}{F}_{ji} + \underset{\sim}{r_i} \, x \, \underset{\sim}{F}_{ij} = \left(\underset{\sim}{r_i} - \underset{\sim}{r_j}\right) x \, \underset{\sim}{F_{ji}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

Going by the equality of action and reaction, since $\underset{\sim}{r_i} - \underset{\sim}{r_j} = \underset{\sim}{r_i}$ is the vector form *j* to *i*, the law of action and reaction yields

$$\underset{\sim}{r_{ij}} \, x \, \underset{\sim}{F_{ji}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

Since $\underset{\sim}{F_{ji}}$ is along the line between the two particles,

$$\sum_{i=1}^{N} \underset{\sim}{r_i} \, x \, \underset{\sim}{F_{ji}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

Substituting (20) into (17) we can now write:

$$\dot{\underset{\sim}{L}} = \frac{dL}{dt} = \sum_{i=1}^{N} \underset{\sim}{r_i} \, x \, \underset{\sim}{F_i}(e) \equiv \underset{\sim}{N}(e) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Where $\underset{\sim}{N}(e)$ is the applied (i.e. external) torque of the system?

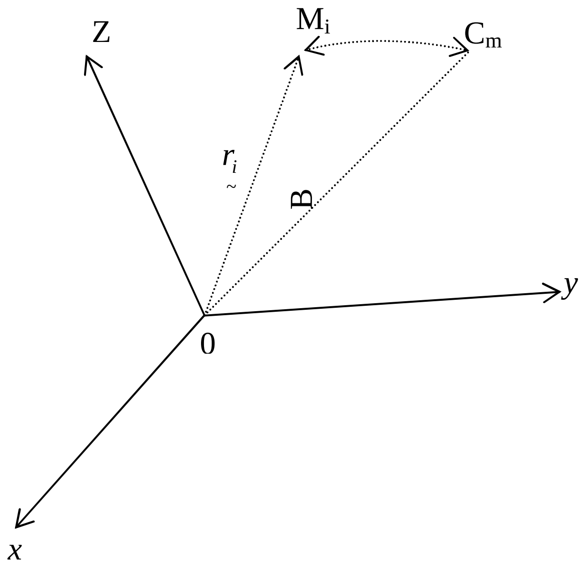
3.4 Conservation Theory of Total Angular Momentum

$\underset{\sim}{L_i}$ is said to be conserved if the applied (external) torque is zero. In other words, the total angular momentum remains constant if the resultant external torque acting on a system of particles is zero.

Remarks

Note that the conservation of angular momentum of a system in the absence of applied torques holds, only if the law of action and reaction is valid. In a system involving moving charges, where the law is isolated, it is not the total mechanical angular momentum which is conserved but the sum of the mechanical and the electromagnetic "angular momentum" of the field.

Considering the diagram below:



Let $\underset{\sim}{r_i}$ be the radius vector from the centre of mass to the *ith* particles. Then:

$$\underset{\sim}{r_i} = \underset{\sim}{r_i'} + \underset{\sim}{R} \quad \text{.......} \quad (22)$$

$$\text{and } \underset{\sim}{V_i} = \underset{\sim}{V_i'} + \underset{\sim}{V}, \quad \text{.......} \quad (23)$$

where $\underset{\sim}{V_i'} = \dfrac{d\,\underset{\sim}{r_i}}{dt}$, and $\underset{\sim}{V} = \dfrac{d\,\underset{\sim}{R}}{dt}$, ………………………….. (24)

In view of (22), we have that

$$\underset{\sim}{R} = \sum_{i=1}^{N} \frac{M_i\,\underset{\sim}{r_i}}{\sum_{i=1}^{N} M_i} = \sum_{i=1}^{N} \frac{M_i\,\underset{\sim}{r_i}}{\sum_{i=1}^{N} M_i} + R. \quad \ldots\ldots \quad (25)$$

Hence, $\sum_{i=1}^{N} M_i\,\underset{\sim}{r_i} = 0$ ……………………………………………………. (26)

This in turn implies

$$\sum_{i=1}^{N} M_i\,\underset{\sim}{V_i'} = 0 \quad \ldots\ldots \quad (27)$$

Next, with $\underset{\sim}{L} = \sum_{i=1}^{N} r_i P_i$,

from equation (22) we obtain

$$\underset{\sim}{L} = \sum_{i=1}^{N} \left(r_i' \underset{\sim}{R}\right) x M_i \left(\underset{\sim}{V_i'} + \underset{\sim}{V}\right)$$

$$= \sum_{i=1}^{N} \underset{\sim}{R}\, x M_i\, \underset{\sim}{V} + \sum_{i=1}^{N} r_i' x M_i\, \underset{\sim}{V_i'} + \left(\sum_{i=1}^{N} M_i\, \underset{\sim}{r_i}'\right) x \underset{\sim}{V} + \underset{\sim}{R}\, x \frac{d}{dt} \sum_{i=1}^{N} M_i\, \underset{\sim}{r_i}' \quad \ldots\ldots \quad (28)$$

Hence, using equation (22) and (26) we obtain

$$\underset{\sim}{L} = \underset{\sim}{R}\, x M\, \underset{\sim}{V} + \sum_{i=1}^{N} \underset{\sim}{r_i}\, x\, \underset{\sim}{P'}_i \quad \ldots\ldots \quad (29)$$

Thus, the total angular momentum about a point O is the angular momentum of the system concentrated at the centre of mass plus the angular momentum of motion about the centre of mass. Equation (29) shows that in general, $\underset{\sim}{L}$ depends on the region O through vector $\underset{\sim}{R}$ only if the centre of mass is at rest with respect to O will $\underset{\sim}{L}$ be independent of the point of reference.

In the latter case, $RxM\,\underset{\sim}{V} = 0$ and $\underset{\sim}{L}$ reduces to the angular momentum about the centre of mass.

4.0 CONCLUSION

We have been able to show that if the net external torque acting on a particle is zero, the angular momentum will remain unchanged. This is often called the principle of conservation of angular momentum.

5.0 SUMMARY

This unit has briefly discussed elementary classical dynamics of an object viewed as a particle (point mass). Discrete and continuous systems, degree of freedom were focused mathematically.

These terms are briefly defined below:

- **Discrete and Continuous System**

A discrete system having a very large but finite number of particles can be considered as a continuous system. On the other hand, a continuous system can be considered as a discrete system consisting of a large but finite number of particles.

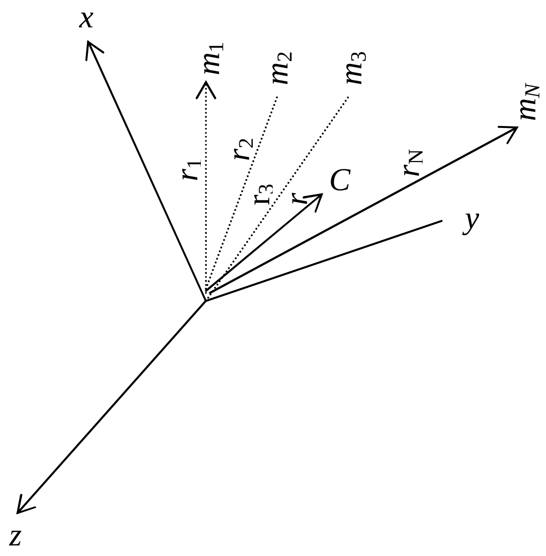
- **Degree of Freedom**

This is the number of coordinates required to specify the position of a system of one or more particles. For instance, a particle moving freely = space requires 3 coordinates e.g. (x, y, z) to specify its position. Thus, the number of degrees of freedom is 3.

We also discussed the centroid (centre of mass) of the system of particle.

- **Centre of Mass (Centroid)**

Let $r_1, r_2, ... r_N$ be the position vectors of a system of N particles of masses $m_1, m_2, ... m_N$ respectively.



Hence, the center of mass or centroid of the system of particles is defined as that point C having position vector.

$$\tilde{r} = \frac{m_i r_i + m_2 r_2 + \dots\dots\dots + m_N r_N}{m_i + m_2 + \dots\dots\dots\dots m_N}$$
$$= \frac{1}{M}\sum_{i=1}^{N} m_i r_i$$

Where M = $\sum_{i=1}^{N} m_i$ is the total mass of the system.

- The kinetic energy of a system of particles can be defined as:

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^{\,2}$$

- We further discussed extensively the total angular momentum of the system of particles about origin O as
$$\sum_{i=1}^{N} m_i (r_i x V_i)$$
- The total external torque acting on a system. The total external torque about the origin is
$\sum_{i=1}^{N} r_i x F_i$. Where F_i is the external force acting on the particle *i*.

- **Conservation of Angular Momentum**

If the resultant external torque acting on a system of particles is zero, then the total angular momentum remains constant i.e. conserved that is:

$$\sum_{i=1}^{N} m_i (r_i x V_i) = \text{constant.}$$

SELF ASSESSMENT EXERCISE 1

Describe the motion of a particle in free space.

Solution

Here, no constraints are involved and hence, we may use the Cartesian coordinates *x, y, z* as the components of the position vector of the particle, i.e. $\underset{\sim}{r} = x_i + y_i + z_k$

Let $\underset{\sim}{F}$ be the force applied on the particle. The compounds of generalised force are then:

$$\left.\begin{aligned} Q_x &= \underset{\sim}{F} . \frac{dr}{dx} = \underset{\sim}{F} . \underset{\sim}{i} = F_x \\ .Q_y &= \underset{\sim}{F} . \frac{dr}{dy} = \underset{\sim}{F} . \underset{\sim}{j} = F_y \\ Q_z &= \underset{\sim}{F} . \frac{dr}{dz} = \underset{\sim}{F} ..k = F_z \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Thus, they coincide with the components of $\underset{\sim}{F}$, the applied or motive force.

The kinetic energy T of the particle is

$$T = \frac{1}{2} m(x^2 + y^2 + z^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Using equation (1) and (2) we obtain

$$\frac{d}{dt}\left(\frac{dT}{dx}\right) - \frac{dT}{dx} = F_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

i.e. $m\ddot{x} = F_x$, $m\ddot{y} = F_z$, and $m\ddot{z} = F_z$.

These equations may be combined to obtain

$$F_x i + F_y j + kF_z = m\ddot{x} + m\ddot{y} + m\ddot{z}$$

$$= m(i\ddot{x} + j\ddot{y} + k\ddot{z})$$

$$F = ma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Where = $\underset{\sim}{a} = i\underline{\ddot{x}} + j\underline{\ddot{y}} + \underline{k}\ddot{z}$ and

$$\underset{\sim}{F} = iF_x + jF_y + kF_z.$$

Equation (4) is Newton's second law of motion.

SELF ASSESSMENT EXERCISE 2

1. Describe the motion of a particle in space using Polar Coordinates (r, θ).
2. Describe the motion of a system of two particles connected by a string over a fixed, frictionless pulley.
3. Determine the number of degree of freedom in fine particles moving freely in a plane.
4. Prove that the total momentum of a system of particles can be found by multiplying the total mass *M* of the system by the velocity $\underline{v}$ of the centroid.

6.0 TUTOR-MARKED ASSIGNMENT

1. Define the following terms with examples:

a. Centre of mass
b. Degree of freedom.

2. Determine the number of degrees of freedom in a particle moving on a given space curve.
3. A system of a particles consists of a 2 gram mass located at (0, 0, 1), a 1 gram mass at (-1, 0, 1) and 3 gram mass at (2, 1, -1). Find the coordinates of the centre of mass.
4. Prove that if the total momentum of a system is constant i.e. is conserved, then the centre of mass i.e. either at rest or in motion with constant velocity.
5. State and prove the conservation theorem for Total Angular Momentum.

7.0 REFERENCES/FURTHER READING

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UNIT 2 HOLONOMIC AND NON-HOLONOMIC CONSTRAINTS

CONTENTS

1.0 INTRODUCTION

The motion of a particle or system of articles is restricted in some ways. For example, gas molecules in a container are constrained by the wall of the vessel to move only inside the container, while a particle placed on surface of a solid sphere is restricted by the constraint so that it can move on the surface or in the region exterior to the sphere. Consequently, a constraint can be defined as the limitations on the motion of a particle.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the term constraint
- state the types of constraint
- distinguish the different types of constraint
- define D'Alambert's principle and relevant theorems.

3.0 MAIN CONTENT

3.1 Holonomic and Non-Holonomic Constraints

In practice, equation (6) in Module 1 Unit1, may not completely describe the motion of the *ith* particles of a system of N particles; for it may be necessary to take into account the constraints which limit the motion of system. For example, in rigid bodies the motion must be such that the distance between any two particular particles of the rigid body is always the same. Such limitations on the motion of a particle are referred to as constraints. The constraint condition can be expressed mathematically as

$$\varnothing(r_1, r_2, \ldots\ldots\ldots r_N, t) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

If the position vectors of the particle are considered along with the time taken, then the nature of constraint condition involved could be regarded as holonomic otherwise and non-holonomic constraints shall be discussed in detail next.

Furthermore, if the constraints are independent of time, we say they are **Scleronomous**; but if they depend explicitly on time. It can be said that the constraint condition is **rheonomous**.

Remarks

In the presence of constraints, the coordinates $r_i = |r_i|$ are no longer all independent, hence the equations given by equation (6) in Module 1 Unit 1 above are not all independent; the forces of constraints are no longer known.

They are among the unknown of problem and must be obtained from the solution we seek. In fact, the presence of constraints is an acknowledgement of the fact that there are forces acting on the system which cannot be specified directly but are known only in terms of their effect on the motion of the system.

3.2 Generalised Coordinate

Remarks

If the constraints imposed on a system of N particle are holonomic and are expressed by means of k equations of the form as in equation (30), then the equations may be used to eliminate k of the 3N coordinates which describe the system. There are then left only 3N–k independent coordinates and the system thus possesses 3N–k degrees of freedom.

The elimination of k dependent coordinates may be regarded as an exercise in transformation theory involving the introduction of 3N–k new independent variables q_1, q_2, ……….q_{3N-k} connected to the coordinates $\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots\ldots\ldots \underset{\sim}{r}_N$. Thus,

$$\left.\begin{aligned}
\underset{\sim}{r}_1 &= \underset{\sim}{r}_1(q_1, q \ldots\ldots\ldots\ldots, q_{3N-K}, t)\\
\underset{\sim}{r}_2 &= \underset{\sim}{r}_2(q_1, q \ldots\ldots\ldots\ldots, q_{3N-K}, t)\\
\underset{\sim}{r}_3 &= \underset{\sim}{r}_3(q_1, q \ldots\ldots\ldots\ldots, q_{3N-K}, t)\\
&\vdots \qquad \vdots \qquad \vdots\\
\underset{\sim}{r}_N &= \underset{\sim}{r}_N(q_1, q \ldots\ldots\ldots\ldots, q_{3N-K}, t)
\end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (31)$$

These equations contain the constraints explicitly. The new variable q_1, q_2, ………..q_{3N-k} are called generalised coordinates. We shall employ them a great deal in the sequel.

3.3 D'Alambert's Principle

Virtual Displacement (Assumed Infinitesimal)

A virtual displacement of a system of N particle with position vectors $\left(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots\ldots\ldots \underset{\sim}{r}_N\right)$, is a change in the configuration of the system arising from an arbitrary infinitesimal changes δr_i of the coordinates $\underset{\sim}{r}_i$ the changes δr_i being assumed consistent with the forces and constraints imposed on the system at the given time t.

The symbol has the usual properties of the differential δ. For instance,

$\delta(\sin\theta) = \cos\theta\ \delta\ \theta.$

Consequently, the displacement described in equation (30) above is called virtual displacement to distinguish it from an actual displacement of the system occurring in a time interval [t, t + dt] during which the forces and constraints may be changing.

Principle of Virtual Work

For a system of N particles, with position vectors $\underset{\sim}{r}_i$ to be in equilibrium, the resultant force acting on each particle must be zero. Then $F_i = 0$, for $i = 1, 2, \ldots\ldots, N$.

However, it is obvious that the virtual work $F_i.\delta r_i = 0,$ for $i = 1,2,......,N$. But if the total virtual work is considered, we obtained

$$\sum_{i=1}^{N} F_i.\delta r_i = 0 \ldots\ldots (31)$$

Remarks

If constraint are present, the motion, then we obtain the total force involved in the displacement is sum of the actual force and the constraint force acting on the *ith* particle. Then

$$F_i = F_i\text{(actual)} + F_i\text{(virtual)} \ldots\ldots (32)$$

In view of equation (30) and (31) we obtain

$$\sum_{i=1}^{N} r_i \text{ (actual). } \delta r_i + \sum_{i=1}^{N} F_i \text{ (virtual) } \delta.r_i = 0 \ldots\ldots (32)$$

Theorem

Suppose in the following that we are considering only systems of particles for which the virtual work of the forces of constraint kanish. This theorem is valid, for example for all rigid bodies. On the strength of the mentioned theorem, we obtain from equation (32) that

$$\sum_{i=1}^{N} \underset{\sim}{F_i} \text{ (actual). } \delta r_i = 0 \ldots\ldots (33)$$

This equation is generally referred to as the Principle of Virtual work. Note that since the coordinates $\underset{\sim}{r_i}$ for i = 1, 2 …….N are connected by the equations of constraints, the infinitesimal changes δr_i are not independent, and hence

$$\underset{\sim}{F_i} \text{ (actual). } \neq 0, \text{ in general} \ldots\ldots (33)$$

In order to obtain an equation of the form (34) in which the coefficient of δr_i may be set to zero, we must recast (34) by using the independent coordinates q_i. To achieve this, we may write the equations of motion

$$\underset{\sim}{F_i} - \underset{\sim}{\dot{P}_i} = 0, \text{ (for i = 1, 2 …….N)} \ldots\ldots (34)$$

Consequently, equation (32) may be re-written in the form

$$\sum_{i-1}^{N}\left(\underset{\sim}{F}-\underset{\sim}{\dot{P}_i}\right).\delta \underset{\sim}{r_i} = 0 \ldots\ldots (35)$$

This becomes:

$$\sum_{i-1}^{N}\left(\underset{\sim}{F_i}\,(actual)-\underset{\sim}{\dot{P}_i}\right).\delta \underset{\sim}{r_i}+\sum_{i-1}^{N}\left(\underset{\sim}{F_i}\,(virtual)-\underset{\sim}{\dot{P}_i}\right).\delta \underset{\sim}{r_i} = 0 \ldots\ldots (36)$$

From the remark that led to equation (32) we can deduce another theorem thus. A system of particles with position vector r_i.for i = 1, 2 …….N moves in such a way that the total virtual work

$$\sum_{i-1}^{N}\left(F_i(actual)-\underset{\sim}{\dot{P}_i}\right).\delta \underset{\sim}{r_i} = 0 \ldots\ldots (37)$$

With the theorem, equation (36) is referred to as the D'Alambert's Principle; hence we can consider dynamics as a special case of statics.

4.0 CONCLUSION

From the remark that led to equation (32), one obtains that a system of particles moves in such a way that defines the total virtual work as in equation (36).

5.0 SUMMARY

This unit has briefly discussed the motion of particles, the constraint of motion, types of constraints, which are defined as follows:

- The limitation to the motion of a particle is called a constraint. The components are usually expressed in the form $Ø(r_1, r_2, \ldots\ldots r_N, t) = 0$ where $r_1, r_2, \ldots\ldots r_N$ are position vector of the particles.
- Holonomic Constraint is a typical constraint condition that is involved if the position vector of the particle is considered along the time-taken.
- But for non-holonomic it means that the position vector of the particle is not considered along the time taken.

- Scleronomous constraint condition occurs when the constraint is independent of time.
- While rheonomous constraint is a constraint condition which is explicitly dependent on time.

We further discussed the transformation theory involving the introduction of $_{3N-k}$ new independent variables q_1, q_2 ……..q_{3N-k} that are connected to the coordinates $\underset{\sim}{r_i}, \underset{\sim}{r_2}, ...\underset{\sim}{r_N}$. where k are dependent coordinates.

Thus,

$$\left.\begin{aligned} \underset{\sim}{r_1} &= \underset{\sim}{r_1}(q_1, q..........., q_{3N-K}, t) \\ \underset{\sim}{r_2} &= \underset{\sim}{r_2}(q_1, q..........., q_{3N-K}, t) \\ \underset{\sim}{r_3} &= \underset{\sim}{r_3}(q_1, q..........., q_{3N-K}, t) \end{aligned}\right\}$$

As in equation (31) in Module 1, Unit 1

Consequently, the generalised coordinates are referred to as the new variables q_1, q_2 ……..q_{3N-k} – the above.

We discussed the equation of the Principle of Virtual work as:

$\sum_{i=1}^{N} \underset{\sim}{F_i}$ (actual). $\delta r_i = 0$ also, for total virtual work equation is

$$\sum_{i-1}^{N} \left(F_i (actual) - \underset{\sim}{\dot{P}_i} \right) . \delta \underset{\sim}{r_i} = 0$$

This is called D'Alambert's Principle.

SELF ASSESSMENT EXERCISE 1

Two particles having masses m_1 and m_2 are located on a frictionless double inclined curves and connected by an inextensible mass less string passing over a smooth pulley. Use the principle of virtual work to show that for equilibrium we must have $\frac{Sin \propto_1}{Sin \propto_2} = \frac{m_1}{m_2}$, where $\propto_1$ and $\propto_2$ are the angles of the incline.

Solution

Let $r_1$and r_2 the respective position vectors of masses m_1 and m_2 relative to O.

The actual forces (due to gravity) acting on m_1 and m_2 are respectively:

$$F_1^{(a)} = m_1 g, \qquad F_2^{(a)} = m_2 g \ldots\ldots\ldots\ldots (1)$$

According to the principle of virtual work,

$$\sum F_v^a . \delta r_v = 0.$$

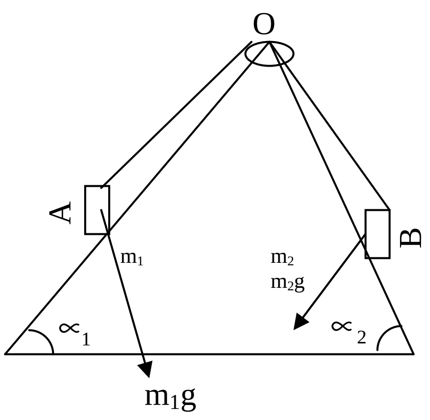
Alternative solution

$$F_1^{(a)} . \delta r_1 + \qquad F_2^{(a)} . \delta r_2 = 0 \ldots\ldots\ldots\ldots (2)$$

Where δr_1 and δr_2 are virtual displacement of m_1 and m_2 down the incline.

Using equation (1) in (2)

We obtain,



$$m_1 g . \delta r_1 + m_2 g . \delta r_2 = 0 \ldots\ldots\ldots\ldots (3)$$

Or $\qquad m_1 g . \delta r_1 \text{ Sin} \propto_1 + m_2 g . \delta r_2 \text{ Sin} \propto_2 = 0 \ldots\ldots\ldots\ldots (4)$

Then, since the string is inextensible,
i.e. $\delta r_1 + \delta r_2$

Consequently, equation (4) becomes

$$\left(m_1 g.\delta r_1 \text{ Sin} \propto_1 - m_2 g.\delta r_2 \text{ Sin} \propto_2\right) \delta r_1 = 0.$$

But since δr_1 is arbitrary we must have $m_1 g.\delta r_1 \text{ Sin} \propto_1 + m_2 g.\delta r_2 \text{ Sin} \propto_2 = 0.$

i.e. $\dfrac{Sin \propto_1}{Sin \propto_2} = \dfrac{m_1}{m_2}$ ………………………………………………….………….. (5)

SELF ASSESSMENT EXERCISE 2

Use D'Alambert's principle to describe the motion of the masses in example 1.

Solution

We introduce the reversed effective forces $m_1\ddot{r}_1$ and $m_2\ddot{r}_2$ in the equation (3) of example 1 to obtain:

$$\left(m_1 g - m_1\ddot{r}_1\right).\,\delta r_1 + \left(m_2 g - m_2\ddot{r}_2\right).\,\delta r_2 = 0. \text{ ……………………….. (1)}$$

This can be written as:

$$\left(m_1 g\, Sin \propto_1 - m_1\ddot{r}_1\right).\,\delta r_1 + \left(m_2 g\, Sin \propto_2 - m_2\ddot{r}_2\right).\,\delta r_2 = 0. \text{ …...…….(2)}$$

Now since the string is inextensible so that

$r_1 + r_2$ = constant, we have

$$\delta r_1 + \delta r_2 = 0, \qquad \ddot{r}_1 + \ddot{r}_2 = 0. \text{ …………………..……… (3)}$$

Consequently, in view of (3), equation (2) becomes

$$m_1 g\, Sin \propto_1 - m_1\ddot{r}_1 - m_2 g\, Sin \propto_2 - m_2\ddot{r}_2 = 0.$$

$$\ddot{r}_1 = \frac{m_1 g\, Sin \propto_1 - m_2 g\, Sin \propto_2}{m_1 + m_2}$$

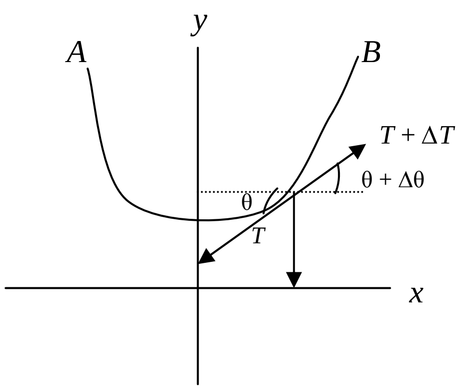
In conclusion, particle 1 goes down or up the incline with constant acceleration according as $m_1 g \text{ Sin} \propto_1 > m_2 g \text{ Sin} \propto_2$ .While particle 2 goes up or down respectively with the same constant acceleration.

6.0 TUTOR-MARKED ASSIGNMENT

1. Two particles having masses m_1 and m_2 move so that their relative velocity is V and the velocity of their centre of mass is $\bar{V}$. If M = $m_1 + m_2$ is the total mass and $M = \frac{m_1 m_2}{(m_1 + m_2)}$ is the reduced mass of the system, prove that the total kinetic energy is $\frac{1}{2}M\dot{V}^2 - \frac{1}{2}MV^2$.

2. Find the centroid of a uniform semi circular wire of radius $\propto$.

3. A uniform chain has its ends suspended from two fixed points at the same horizontal level. Find an equation for the curve in which it hangs.



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MODULE 2

UNIT 1 LAGRANGE'S EQUATION

CONTENTS

1.0 INTRODUCTION

Scientists (applied mathematicians) have dealt primarily with the formulation of problems in mechanics by Newton's Laws of Motion.

However, it is possible to give treatments of mechanics from rather general view points, in particular those due to Language and Hamilton. Although such treatments reduce to Newton's laws, they are characterised not only by the relative ease with which many problems can be formulated and solved but by their relationship in both theory and application to such advanced fields as quantum mechanics, statistical mechanics, celestial mechanics and electrodynamics.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- apply Lagrange's equations in solving dynamical problems
- trace find the Lagrange function of particle(s) moving in a conservation force field
- derive Lagrange's equations for holonomic and non-holonomic constraint respectively.

3.0 MAIN CONTENT

3.1 Lagrange's Equations

Definition

Consider a system of *N* particles, with position vectors $\underset{\sim}{r_i}$ and constant masses m_i. suppose that *k* holonomic constraints are imposed on the system such that there are only n independent coordinates. Then *n* + *k* = 3*N* and we may introduce *n* new independent coordinates *q*i for *i* = 1, 2…, *n*, by means of equation

$$\underset{\sim}{r_1} = r_1(q_1, q_2, \ldots\ldots\ldots\ldots, q_n, t)$$
$$\underset{\sim}{r_2} = r_2(q_1, q_2, \ldots\ldots\ldots\ldots, q_n, t)$$
$$\underset{\sim}{r_3} = r_3(q_1, q_2, \ldots\ldots\ldots\ldots, q_n, t)$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$\underset{\sim}{r_n} = r_n(q_1, q_2, \ldots\ldots\ldots\ldots, q_n, t)$$

We shall assume that there are sufficient differentiable transformations. From the general equation:

$$\underset{\sim}{r_i} = \underset{\sim}{r_i}\ (q_1, q_2, \ldots\ldots\ldots\ldots, q_n, t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(37)}$$

We have

$$\underset{\sim}{v_i} = \dot{r}_i = \sum_{j=1}^{n} \frac{\delta \underset{\sim}{r_i}}{\delta q_j}\dot{q}_j + \frac{\delta \underset{\sim}{r_i}}{\delta y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(38)}$$

Also, we have the virtual displacement

$$\delta \underset{\sim}{r_i} = \sum_{j=1}^{N} \frac{\delta r_i}{\delta q_i}\delta q_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(39)}$$

No time variation δt is involved in equation (39) above since only a virtual displacement is involved in the displacements of coordinates.

$$\sum_{i=1}^{N}(\underset{\sim}{F_i}(\text{actual}) - \underset{\sim}{\dot{P}_i}).\delta \underset{\sim}{r_i}$$ and set the result to zero.

Now, $\sum_{i=1}^{N}(\underset{\sim}{F_i}(\text{actual})-\underset{\sim}{\dot{P}_i}).\delta\underset{\sim}{r_i}$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\underset{\sim}{F_i}(\text{actual}).\frac{\delta dr_i}{\delta q_j}.\delta q_j.$$

$$= \sum_{j=1}^{N}\underset{\sim}{F_i}\; Q_j\;\delta q_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

Where $Q_j = \sum_{i=1}^{N}F_i(\text{actual}).\frac{\delta r_i}{\delta q_j}$ ……………………………………… (41)

The scalars Q_j are called components of the generalised force.

Similarly,

$$\sum_{i=1}^{N}\underset{\sim}{\dot{P}_i}.\delta\underset{\sim}{r_i} = \sum_{i=1}^{N}m_i\,\underset{\sim}{\ddot{r}_i}.\delta\underset{\sim}{r_i}.$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\underset{\sim}{m_i}\,\underset{\sim}{\ddot{r}_i}.\frac{\delta dr_i}{\delta q_j}.\delta q_j.$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\delta q_j\left[\frac{d}{dt}\left(m_i\,\underset{\sim}{\dot{r}_i}.\frac{\delta dr_i}{\delta q_j}\right)-m_i\,\underset{\sim}{\dot{r}_i}.\frac{d}{dt}\left(\frac{\delta r_i}{\delta q_j}\right)\right]\ldots\ldots\ldots \quad (42)$$

From equation (39), we obtain

$$\frac{d}{dt}\left(\frac{d\underset{\sim}{r_i}}{\delta q_j}\right) = \sum_{k=1}^{N}\frac{\delta^2\underset{\sim}{r_i}}{\delta q_j.\delta q_k}\dot{q}_k + \frac{\delta^2\underset{\sim}{r_i}}{\delta q_j.\delta t}\ldots\ldots\ldots\ldots \quad (43)$$

$$= \frac{\delta\underset{\sim}{V_i}}{\delta q_j}\text{ by (38)}$$

Also from equation (38), we obtain

$$\frac{\delta\underset{\sim}{V_i}}{\delta q_j} = \frac{\delta\underset{\sim}{r_i}}{\delta q_j}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (44)$$

This may be viewed as "cancellation of dots", since

$$\frac{\delta \underset{\sim}{V_i}}{\delta q_j} = \frac{\delta \dot{r}_i}{\delta \dot{q}_j} = \frac{\delta r_i}{\delta q_j}.$$

Hence, using equation (43) and (44) in (42) we have:

$$\sum_{i=1}^{N} \underset{\sim}{\dot{P}_i} . \delta \underset{\sim}{r_i} = \sum_{i=1}^{N} \sum_{j=1}^{N} \delta q_j \left[\frac{d}{dt}\left(m_i \underset{\sim}{V_i} . \frac{\delta V_i}{\delta \dot{q}_j} \right) - m_i \underset{\sim}{V_i} . \frac{\delta V_i}{dt}\left(\frac{\delta r_i}{\delta q_j} \right) \right] \delta q$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \delta q_j \left[\frac{d}{dt}\left(\frac{1}{2} m_i \underset{\sim}{V_i} . \frac{\delta V_i}{\delta \dot{q}_j} \right) - \frac{\delta}{\delta q_j}\left(\frac{1}{2} m_i \underset{\sim}{V}^2{}_i \right) \right] \delta q_j$$

$$= \sum_{j=1}^{N} \delta q_j \left[\frac{\delta}{\delta \dot{q}_j}\left(\sum_{i=1}^{N} \frac{1}{2} m_i \underset{\sim}{V}^2{}_i \right) - \frac{\delta}{\delta q_j}\left(\frac{1}{2} m_i \underset{\sim}{V}^2{}_i \right) \right] \delta q_j$$

$$= \sum_{j=1}^{N} \left[\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j} \right) - \frac{\delta T}{\delta q_j} \right] \delta q_j$$

Where we have set

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \underset{\sim}{V}^2{}_i \text{ ....(44a)}$$

which is referred to as total kinetic energy of the system.

From the preceding; D'Alambert's Principle now yields

$$\sum_{j=1}^{N} \left[Q_j - \left\{ \frac{d}{dt}\left(\frac{\delta T}{\delta q_j} \right) - \frac{\delta T}{\delta q_j} \right\} \right] \delta q_j = 0 \text{ .... (45)}$$

Since the constraints have been assumed holonomic, the coordinates qi for i = 1, 2, 3, ……N, have that:

$$\frac{d}{dt}\left(\frac{\delta d}{\delta \dot{q}_j} \right) - \frac{\delta T}{\delta q_j} = Q_j \text{ .... (46)}$$

for j = 1, 2…, n.

Equation (44a) are often referred to as the Lagrange's equation.

Remarks

Suppose that the system considered above is in fact conservative. With this assumption, there is a scalar potential function ø called the potential energy of the system such that

F_i (actual) = $-\nabla_i ø$ ……………………………………..………….. (47)

Then the components Q_j of generalised force become:

$$Q_j = \sum_{i=1}^{N} F_i (\text{actual}).\frac{\delta r_i}{\delta q_j} = -\sum_{i=1}^{N} F_i \nabla ø.\frac{\delta r_i}{\delta q_j}$$

$$= \quad \frac{\delta ø}{\delta q_j} \text{ ………………………………..……………………..…….. (48)}$$

Using equation (48) in (47), we now have:

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta (T - ø)}{\delta q_j} = 0 \text{ ……………………………………….…………. (49)}$$

If ø depends only on $q_1, q_2, q_3........q_N$ i.e. ø is independent of z, then equation (49) may be written thus:

$$\frac{d}{dt}\left(\frac{\delta (T - ø)}{\delta \dot{q}_j}\right) - \frac{\delta (T - ø)}{\delta q_j} = 0 \text{ ……………………….………………. (50)}$$

for j = 1, 2…, N.

Consequently, setting

$$T - ø = L \text{ ……………………………………………………………. (51)}$$

Equation (50) may now be written thus:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_j}\right) - \frac{\delta L}{\delta q_j} = 0, \text{ ……………………………………………….... (52)}$$

for j = 1, 2…, n.

The equation given by (52) rather than equations given by (46) are what are often referred to as LAGRANGE'S EQUATIONS. The function L = T - ø in equation (51) is referred to as the Lagrangian function or simply the Lagrangian.

Remarks

The equations given by (52) are Lagrange's equations for a conservative system.

Now, suppose that the system is non-conservative, but that the components Q_j of generalised force are given by a velocity – dependent function.

$\left(q_j \dot{q}_j\right) \to u\left(q_j \dot{q}_j\right)$ as follows $Q_j =$

$$\frac{\delta u}{\delta q_j} x \frac{d}{dt}\left(\frac{\delta u}{\delta \dot{q}_j}\right) \ldots\ldots (53)$$

Then equation (52) remains valid with

$$L = T - u. \ldots\ldots (54)$$

The function U is called a generalised potential or a velocity – dependent potential, and it is a dissipation function.

Remarks

i) The function $P_j = \frac{\delta L}{\delta \dot{q}_j}$, where L is the Lagrangian and is called the generalised momentum associated with the generalised coordinates q_j.

ii) The kinetic energy T is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \left|\underset{\sim}{V}\right|^2 \ldots\ldots (55)$$

But from equation (3)

$$\underset{\sim}{V_i} = \underset{\sim}{\dot{r}_i} = \sum_{j=1}^{N} \frac{\delta \underset{\sim}{r_i}}{\delta q_i} \dot{q} + \frac{\delta \underset{\sim}{r_i}}{\delta_i t}$$

Substituting this in the expression

For T, we have

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \left| \underset{\sim}{V} \right|^2$$

$$= \sum_{i=1}^{N} \frac{1}{2} m \left(\sum_{j=1}^{N} \frac{\delta r_i}{\delta q_j} \dot{q}_j + \frac{\delta \underset{\sim}{r_i}}{\delta t} \right)^2$$

$$= \lambda + \sum_{j=1}^{N} \lambda_j \dot{q}_j + \sum_{j,k=1}^{N} \lambda_{jk} \dot{q}_j \dot{q}_k \qquad \ldots\ldots (56)$$

Where $\lambda = \sum_{j=1}^{N} \frac{1}{2} m_i \frac{\delta r_i}{\delta t} \cdot \frac{\delta r_i}{\delta q_j}$

$$\lambda j_k = \sum_{j=1}^{N} \frac{1}{2} m_i \frac{\delta r_i}{\delta q_t} \cdot \frac{\delta r_i}{\delta q_k}$$

If the transformation equations (37) do not involve t, i.e. in the presence of Scleronomous constraints then $\lambda = 0 \quad = \lambda = j$, for $j = 1, 2, \ldots, n$ and T is reduced to the following homogenous quadratic expression in the generalised velocities.

$$T = \sum_{j,k=1}^{N} \lambda jk \, \delta \dot{q}_j \delta \dot{q}_k \qquad \ldots\ldots (57)$$

Examples on Lagrange's Equation

SELF ASSESSMENT EXERCISE 1

Find the motion of a system of two particles connected by a string over a fixed, frictionless pulley.

Solution

Let L be the length of the string connecting the masses and let D denote the diameter of the pulley.

Setting $L - D = \ell$.

Then, it is clear that there is only one independent variable, *x*, say since from the diagram we must always have distances indicated.

Now the potential energy *V* of the system is:

$$V = -m_1gx - m_2g(L - x)$$

And the kinetic energy *T* of the system is

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Hence, the Lagrangian L of the system is

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + mg(L - x)$$

By equation (52), we now obtain

$$\frac{d}{dt}.\frac{(\delta L)}{\delta x} - \frac{\delta L}{\delta x} = (m_1 + m_2)\ddot{x}^2 - (m_1 + m_2)g = 0$$

Thus: $(m_1 + m_2)\ddot{x} - (m_1 + m_2)g$

This is the equation of motion of the system.

Remarks

i) The above problem is an example of a conservative system with holonomic, Scleronomous constraints. The holonomic constraint is given by:

$$x + y = \ell$$

Where *x*, *y* are the distances of the masses m_1 and m_2 respectively from the horizontal plane through the centre of the pulley.

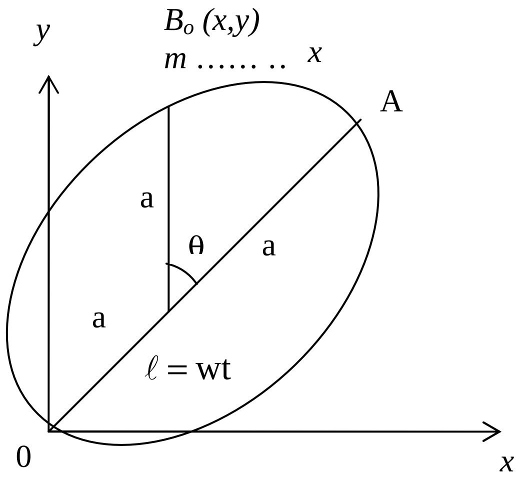
This problem also shows that the forces of constraint i.e. the tension in the string do not appear in the Lagrangian formulation.

Hence, the tension in the string cannot be found directly by the Lagrangian method.

SELF ASSESSMENT EXERCISE 2

Investigate the motion of a bead of mass *m* sliding freely on a smooth circular wire of radius ‘a’ which rotates in a horizontal plane about one of its point ‘O’ with constant angular velocity, ‘w’.

Solution



The plane of rotation is that of the paper and rotation is anti-clockwise, with ‘O’ as the centre of revolution. At time *t* the wire rotates through an angle $u = wt$ as shown. The angle made by the bead with OAC (where C is the centre of wire) is also shown. The coordinates (x, y) of the bead B are given by.

a. $x = \text{a Cos}wt + \text{a Cos}(\theta + wt)$

$y = \text{a Sin}wt + \text{a Sin}(\theta + wt)$

Hence, we see that there is only one generalised coordinate, namelyθ.

Consequently, from (a) and (b) alone, we have that the kinetic energy *T* of the bead is given by

c. $$T = \frac{1}{2}ma^2\left[w^2 + (\dot{\theta} + w)^2 + 2w(\dot{\theta} + w)\text{Cos}(\dot{\theta} + wt).\right]$$

The potential energy of the system is clearly zero here. This implies that the generalised force is zero since the system is conservative.

From (c), we have:

$$\frac{\delta T}{\delta \dot{\theta}} = ma^2(\dot{\theta} = w + wCos\theta)$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\theta}}\right) = +ma^2(\ddot{\theta} - w + w\dot{\theta}Sin\theta)$$

$$\frac{\delta T}{\delta \theta} = -ma^2\left[w(\dot{\theta} + w)Sin\theta\right]$$

Hence, we have:

$$ma^2(\ddot{\theta} - w\dot{\theta}Sin\theta) + ma^2 w(\dot{\theta} = w)Sin\theta = 0$$

(d) Implies

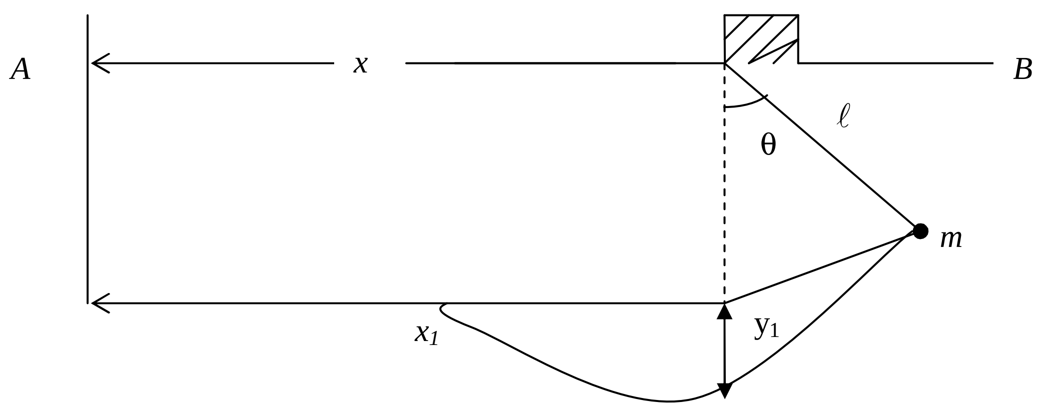
$$\ddot{\theta} + w^2 Sin\theta = 0$$

SELF ASSESSMENT EXERCISE 3

Compare $\ddot{\theta} + w^2 Sin\theta = 0$ with the equation of motion of a simple pendulum of length ℓ given by: $\ddot{\theta} + \frac{g}{L}Sin\theta = 0$

Here, we see that the bead β oscillates about the line OA as if it were a simple pendulum of length $\ell = \frac{g}{w^2}$.

SELF ASSESSMENT EXERCISE 4



In the diagram, M is a mass constraint to slide on the smooth track AB.

A particle of mass M is connected to M by a mass less inextensible string of length ℓ. Find the equation of motion leading to small oscillations of the system.

Solution

Let x denote the position of m at time t and let (x_1, y_1) be the position of m also at time t. Notice that here, y, is measured from the equilibrium position of m. Let θ be the angular displacement of the string connecting m to m. θ is assumed increasing the counter clockwise direction. It is clear at t = 0.

Next, we seek T and ø, the kinetic energy and potential energy of this system of two particles, we have

3a. $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x}_i + \dot{y}_i)$

and

3b. $ø = mgy_1$

Next, x_1 and y_1 may be expressed in terms of x and θ.

Thus,

3c. $x_1 = x + \ell \mathrm{Sin}\theta$

3d. $y_1 = x + \ell(1 - \mathrm{Cos}\theta)$

Using (3c.) and (3d.) in (3b.), we have:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + \ell\dot{\theta}\mathrm{Cos}\theta)^2 + \ell^2\dot{\theta}^2\mathrm{Sin}^2\theta\right]$$

$$ø = mg\ell(1 - \mathrm{Cos}\theta).$$

From the expressions for T and β which involves only two coordinates x, and θ the generalised coordinates for this problem, we conclude that the system under consideration has two degrees of freedom.

Next, we have:

$$L = T - \theta = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[+ (\dot{x}^2\ell\dot{\theta}\mathrm{Cos}\theta)^2 + \ell^2\dot{\theta}^2\mathrm{Sin}^2\theta\right] - mg\ell(1 - \mathrm{Cos}\theta)$$

$$\frac{\partial L}{\partial \theta} = m(\dot{x} + \ell\dot{\theta}\text{Cos}\theta)\,\ell\cos\theta + m\,\ell^2\dot{\theta}\text{Sin}^2\theta\,.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = m(\ddot{x} + \ell\ddot{\theta}\text{Cos}\theta - \ell\dot{\theta}^2\text{S}in\,\theta\,)\,\ell\,\text{Sin}\theta - m(\dot{x} + \ell\dot{\theta}\text{Cos}\theta) + \ell\dot{\theta}\text{Sin}\theta$$

$$+ 2m\,\ell^2\dot{\theta}^2\text{Sin}\theta\,\text{Cos}\theta + m\ell^2\ddot{\theta}\text{Sin}^2\theta$$

$$\frac{\partial L}{\partial \theta} = -m(\dot{x}\ell\dot{\theta}\text{Cos}\theta)\,\ell\dot{\theta}\text{Sin}\theta + \frac{1}{2}mx2\ell^2\dot{\theta}^2\text{Sin}\theta\,Cos\theta + -\,\ell mg\ Sin\,\theta$$

Hence, the first Lagrangian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0\,.$$

SELF ASSESSMENT EXERCISE 5

$$m(\dot{x} + \ell\ddot{\theta}\text{Cos}\theta - \ell\dot{\theta}^2\text{Sin}\theta)(\text{Cos}\theta - m(x + \ell\dot{\theta}\text{Cos}\theta)$$

$$\ell\dot{\theta}\text{Sin}\theta + \frac{1}{2}2m\ell^2\dot{\theta}^2\text{Sin}\theta\,Cos\theta + m\ell^2\ddot{\theta}\text{Sin}^2\theta - mg\ell\,\text{Sin}\theta.$$

Since we are only interested in small oscillations of the system, we have the approximations.

$\text{Sin}\theta \approx \theta$ and
$\text{Cos}\theta \approx 1.$

$\therefore\ \ddot{x} + \ell\ddot{\theta} + g\theta = 0$ Neglecting terms involving

θ^2, $\dot{\theta}^2$ and $\dot{\theta}$ and higher other terms

Terms

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + m(\dot{x} + \ell\dot{\theta}\text{Cos}\theta)$$

$$\frac{\partial L}{\partial \dot{x}} = 0$$

$$\frac{d}{dt}(m\dot{x} + m\dot{x} + m\ell\dot{\theta}Cos\theta) = 0$$

$$(m + m)\ddot{x} + m\ell\ddot{\theta}Cos\theta - m\ell\dot{\theta}^2 Sin\theta = 0$$

Neglecting terms involved θ^2, $\dot{\theta}^2$, we get

$$m\ddot{x} + m\ddot{x} + m\ell\ddot{\theta} = 0$$

$$(m + m)\ddot{x} + m\ell\ddot{\theta} = 0$$

4.0 CONCLUSION

We have discussed the Lagrange's equation in general and for Holonomic and Non-holonomic constraints with related examples as shown above.

5.0 SUMMARY

In summary, we have assumed that from the general equation:

- $r_i = \underset{\sim}{r_i}(q_1, q_2, \ldots\ldots, q_n, t)$
- $V_i = \dot{r}_i = \sum_{j=1}^{N} . \frac{\delta \underset{\sim}{r_i}}{\delta q_j} . \dot{q}_j + \frac{\delta \underset{\sim}{r_i}}{\delta q_i}$
- $\delta \underset{\sim}{r_i} = \sum_{j=1}^{N} . \frac{\delta \underset{\sim}{r_i}}{\delta q_j} . \dot{q}_j$ being the virtual displacement.

  By using equation of D' Alembert Principle we compute that

- $Q_j = \sum_{i=j}^{N} F_i(\text{actual}) . \frac{\delta r_i}{\delta q_j}$

  This is called the component of the generalised force.

Similarly

$$Q_j = \sum_{i=1}^{N} \dot{P}_i \delta \underset{\sim}{r_i} = \sum_{j=1}^{N} \sum_{j=1}^{N} \delta q_j \left[\frac{d}{dt}\left(m_i\, r_i . \frac{\delta \dot{r}_i}{\delta q_j} \right) - m_i\, r_i . \frac{d}{dt} . \left(\frac{\delta r_i}{\delta q_j} \right) \right]$$

- Also, $\frac{d}{dt}.\left(\frac{\delta \underset{\sim}{r_i}}{\delta q_j}\right) = \frac{\delta \underset{\sim}{V_i}}{\delta q_j}$, and

- $\frac{\delta \underset{\sim}{V_i}}{\delta \dot{q}_j} = \frac{\delta \underset{\sim}{r_i}}{\delta q_j}$

Consequently, from (5) and (6) can conclude thus,

- $$\sum_{i=1}^{N} \dot{P}_i \delta \underset{\sim}{r_i} = \sum_{j=1}^{N} \left[\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{d}{dt}.\right]\delta q_j$$

  Where

- $T = \sum_{i=1}^{N} \frac{1}{2} m_i \underset{\sim}{V}^2{}_i$ which is the total kinetic energy of the system.

  From D'Alambert's Principle, we have that

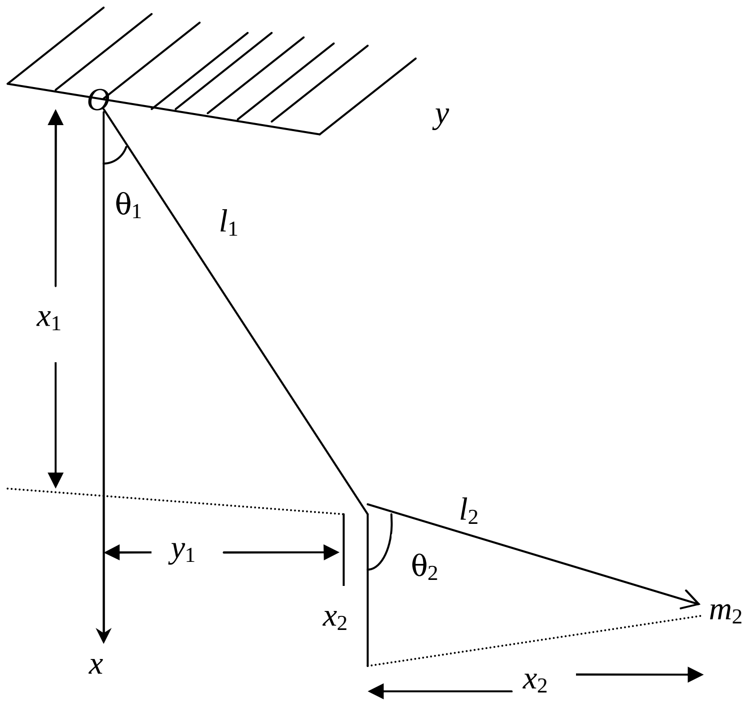
- $$\sum_{j=1}^{N} \left[Q_j - \left\{\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j}\right\}\right]\delta q_j = 0$$

  Hence,

- $Q_j = \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta \dot{q}_j}$ for $j = 1, 2\ldots, n$. are called at Lagrangian equation.

6.0 TUTOR-MARKED ASSIGNMENT

1. A double pendulum vibrates in a vertical plane, write the Lagrangian of the system and hence obtain the equation of motion.



2. Use Lagrange's equations to describe the motion of a particle of mass *m* down a frictionless inclined plane of angle $\propto$.

3(a) Set up the Lagrangian for a one dimension harmonic oscillator; and

(b) Write out the Lagrange's equation.

7.0 REFERENCES/FURTHER READING

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MODULE 3

UNIT 1 IMPULSIVE MOTION

CONTENTS

1.0 INTRODUCTION

Phenomena of an impulsive nature, such as the action of very large forces (or voltages) over very short internal of time, are of great practical interest, since, they arise in various applications. This situation occurs, for instance, when a tennis ball is hit, a system is hit by a hammer, an air plane makes a "hard" landing, a ship is hit by a high simple wave, and so on. In this unit, we shall be interested in the change of momentum produced by variable force ℓ acting from time t such that $t_1 < t < t_2$.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define impulsive motion
- identify equations of motion for impulsive forces
- define conservative force fields.

3.0 MAIN CONTENT

3.1 Impulsive Motion of Particles

Definition
The state of rest or motion of a body sometimes undergoes an apparently instantaneous change owing to the sudden application of a force which acts for a very short time only.

For example, a ball stuck by a bat or a collision of two billiard balls. In such cases it is not possible to measure the rate of change of momentum because a finite change of momentum takes place in an infinitesimal interval of time.

We know that the change in momentum produced by a variable force ℓ acting from time

t = t, and t = t_2 ~ is

$$\int \frac{t_2}{t_1} Ldt$$

Of course, it is possible for the force to increase and at the same time the interval $t_2 - t_1$ to decrease in such a way that the integral tends to a finite limit although, we gave no means of measuring the exact value of L at any instant during the interval. Thus, this sort of force is measured by the change of momentum it produces. Any motion resulting from this impulsive force is called an impulsive motion.

The equations of motion for a system of particles acted upon by finite forces are known to be given as:

$$\sum m\ddot{x} = \sum X$$

$$\sum m\ddot{y} = \sum Y$$

$$\sum m\ddot{z} = \sum Z$$

$$and\ \frac{d}{dt} \sum m(x\dot{y} - y\dot{x}) = \sum (xy - yx)$$

By integrating these equations w.r.t. t through an interval from 0 to t, we get

$$\sum m\dot{x} - \sum m\dot{x}_0 = \sum \int_{t_0}^{t} X\, dt$$

$$\sum m\dot{y} - \sum m\dot{y}_0 = \sum \int_{t_0}^{t} Y\, dt$$

$$\sum m\dot{z} - \sum m\dot{z}_0 = \sum \int_{t_0}^{t} Z\, dt$$

In particular if t – to is so small that changes in x, y, are neglected.

Then:

$$\sum m(x\dot{y} - y\dot{x}) - \sum m(x\dot{y}_0 - y\dot{x}_0)$$

$$= \sum \left\{ x\int_{t_0}^{t} y\, dt - y\int_{t_0}^{t} x\, dt \right\}$$

Where $\dot{x}_0, \hat{y}_0$ denote the values of $\dot{x}$, $\hat{y}$ at time to.

By concerning ourselves with the above definition of impulsive forces then,

$$\int_{t_0}^{t} x\, dt, \int_{t_0}^{t} y\, dt \; and \int_{t_0}^{t} y\, dt \; and \int_{t_0}^{t} z\, dt \; are$$

The measures of the components of the impulse which maybe denoted by F, Q, R, respectively which may be rewritten as

$$\sum \mathrm{m}\dot{\mathrm{x}} - \sum \mathrm{m}\dot{\mathrm{x}}_0 = \sum \mathrm{F}$$

$$\sum \mathrm{m}\dot{\mathrm{y}} - \sum \mathrm{m}\dot{\mathrm{y}}_0 = \sum \mathrm{Q}$$

$$\sum \mathrm{m}\dot{\mathrm{z}} - \sum \mathrm{m}\dot{\mathrm{z}}_0 = \sum \mathrm{R}$$

and $\sum m(x\dot{y} - y\dot{x}) - \sum m(x\hat{y}_0 - y\dot{x}_0) = \sum (XQ - YF)$.

Thus, these equations revealed the fact that the instantaneous increase in the linear momentum in any direction is equal to the sum of the externally applied impulsive forces in that direction.

Note, that if M is the total mass of the system of particles and $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the centre of gravity, then

$$m(\dot{x} - \dot{x}_0) = \sum F$$

$$m(\dot{y} - \dot{y}_0) = \sum Q$$

$$m(\dot{z} - \dot{z}_0) = \sum R$$

Also, the equations confirm the principle of conservation of linear and angular momentum, in that if there be a direction in which the external impulsive forces have zero components, there is no change of momentum in that direction.

The Problem Involving Impact of Two Forces

Definition

The coefficient of restitution (e) is the ratio of the relative velocity of the bodies, along their line of centres, after impact to the relative velocity before impact. For example, for a hard substance like steel, (e) is nearly unit, but for a soft substance, it is small. When a substance is perfectly elastic e = 1 and when, t is inelastic e = o.

Let m_1, m_2 be the masses of the two spheres, U_1, U_2 their velocities before impact and u_1, v_2 their velocities after impact and let the motion be along the line of centres.

The momentum in the line of motion is unaltered by the impact so that

$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2.$

By Newton's rule,

$v_1 - v_2 = - e (u_1 - u_2)$

These equations determine the velocities after impact namely,

$$v_1 = \frac{m_1 u_1 + m_2 u_2 - e\, m_2 (v_1 - v_2)}{m_1 + m_2}$$

$$v_1 = \frac{m_1 u_1 + m_2 u_2 - e\, m_2 (v_1 - v_2)}{m_1 + m_2}$$

the impulse of the spheres which reduces the velocity of the first from u_1 to v_1 is $m_1 (u_1 - v_1)$, which is equal to

$$\frac{(l + e)\, m_1 m_2 (u_1 - u_2)}{m_1 + m_2}$$

It is easy to show that the loss in

$$\text{K. E.} = \frac{1}{2}\frac{m_1 m_2 (1-e)(u_1 - u_2)^2}{m_1 + m_2}$$

3.2 Conservative Force Fields

Definition

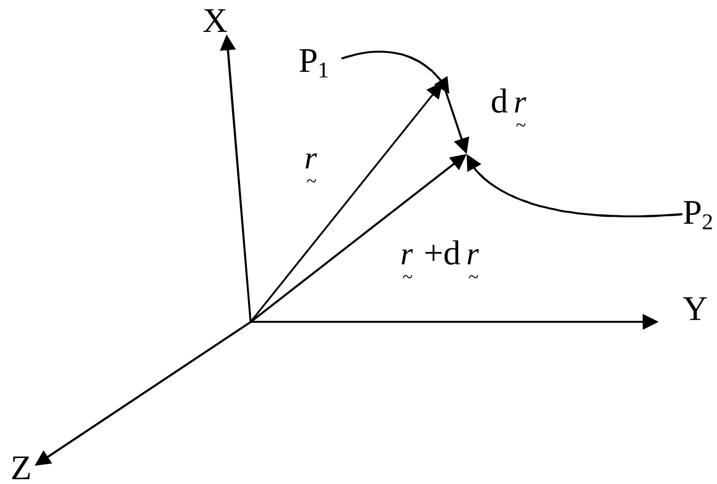
Let V be a scalar function and F the force (extended) acting on a particle of mass M, such that

$F = - \nabla V$

Theorem 1

The total work done in moving the particle along the curve C from P_1 to P_2 is

$$W = \int_{p_1}^{p_2} \underset{\sim}{F} \cdot dx = V(P_1) - V(P_2)$$



In such case the work done is independent of the path C ` joining the points P_1 and P_2. If the work-done by a force field in moving a particle from one point to another point is independent of the path joining the points, then the force field is said to be conservative.

The following theorems are valid:

Theorem 2

A force field $\underset{\sim}{F}$ is conservative if there exists a continuously differentiable scalar field V such that

$F = -\nabla V$ or

Equivalently, if $\nabla \wedge \underset{\sim}{F} = O$ identically.

Theorem 3

A continuously differentiable force field F is conservative for any closed non-intersecting curve C (simply closed curve).

$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = O$$

i.e. the total work-done in moving a particle around any closed path is zero.

Remarks

The scalar V such that $\underset{\sim}{F} = -\nabla V$ is called the potential energy or scalar potential of the particle in the conservative force field $\underset{\sim}{F}$. In this case, the total work-done from P_1 to P_2 along C = potential energy at P_1 minus potential energy at P_2.

i.e. $W = V_1 - V_2$, $V_1 = V(P_1)$, $V_2 = V(P_2)$

3.3 Conservative Forces

Definition

If there is no scalar function V such that $F = -\nabla V$ i.e.

$$\nabla \wedge \underset{\sim}{F} \neq O,$$

Then $\underset{\sim}{F}$ is said to be a non–conservative force field.

SELF ASSESSMENT EXERCISE 1

Show that $\underset{\sim}{F} = x^2y\,z\underset{\sim}{i} - xyx^2\underset{\sim}{k}$ is non – conservative.

Solution

$$\nabla \wedge \underset{\sim}{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & o & -xyz^2 \end{vmatrix}$$

$$= -xz^2 \underset{\sim}{i} + (x^2y + yz^2)\underset{\sim}{j} - x^2z\underset{\sim}{k}$$

$$But\ \nabla \wedge F \neq O$$

Therefore, the field is non-conservative.

4.0 CONCLUSION

The instantaneous increase in the linear momentum in any direction is equal to the sum of the externally applied impulsive forces in that direction.

5.0 SUMMARY

The change in momentum produced by a variable force L acting from time t_1 to t_2 is defined as

$$\int_{t_1}^{t_2} L \quad dt$$

This is called an impulsive force. Any motion resulting from this impulsive force is called an impulsive motion.

The measure of the components of the impulsive force denoted by F, Q and R are:

$$\left.\begin{aligned} \sum m\hat{x} \quad - \sum m\hat{x}_0 \quad &= \sum F \\ \sum m\dot{y} \quad - \sum m\dot{y}_0 \quad &= \sum Q \\ \sum m\dot{z} \quad - \sum m\dot{z}_0 \quad &= \sum R \end{aligned}\right\} \text{———} (2)$$

$$and \sum m(x\dot{y} - y\hat{x}) - \sum m(\dot{x}y - y\dot{x}_0) = \sum (XQ - YF) \text{———} (3)$$

From the above equation, it can be summarised that the instantaneous increase in the linear momentum in any direction is equal to the sum of the externally applied impulsive forces – that direction.

Consequently, if m is the total mass of the system of particles and $\bar{x}, \bar{y}\, and\, \bar{z}$ are the coordinate of the centre of gravity, then,

$$m(\dot{x} - x_0) = \sum F$$
$$m(\dot{y} - y_0) = \sum Q$$
$$m(z - z_0) = \sum R$$

We further showed the loss:

$$\text{K. E.} = \frac{1}{2}\; m_1\, m_2 \frac{(1-e)\;(u_1 - u_2)^2}{m_1 + m_2}$$

Theorem 1

The total work-done: moving the particle along the curve C from p_1 to p_2 is given as

$$w = \int_{P_1}^{P_2} \underset{\sim}{F} \cdot dx = \; V(P_1) - V(P_2)$$

Theorem 2

A force field $\underset{\approx}{F}$ is conservative if there exists a continuously differentiable scalar field V such that

$F = - \nabla V$ consequently,

$\nabla \wedge \underset{\approx}{F} = O$

Theorem 3

A continuously differentiable force field F is conservative if for any closed non-intersecting curve C

$$\oint \underset{\sim}{F} \cdot d\, \underset{\sim}{r} = O.$$

That is, the total work-done in moving a particle around any closed path is zero.

7.0 TUTOR-MARKED ASSIGNMENT

1a. When is a force field said to be conservative?
b. Define the following:

i. impulse
ii. impulsive force and
iii. impulsive motion.

2. Show that the force field F defined by F = $(y^2z^2 - 6xz_2)\,i + 2\,x\,y\,z^3{}_j + (3xy^2z^2\ 6x^2z)\,k$ is a conservative force field.
3. Prove (Theorem 1) that if the force acting on a particle is given by F = ∇V. Then the total work-done in moving the particle along a curve C from P_1 to P_2 is $w = \int_{P_1}^{P_2} F \cdot dr = V(P_1) - V(P_2) = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2$ where $V_A and V_B$ velocities at points $p_1 and p_2$ respectively.
4. A mass of 5000kg moves on a straight line from a speed of 540km/hr to 720km/hr in 2 minutes. What is the impulse developed at this time?
5. Show that the force field given by:

   F = $x^2yz\,i - xyz^2k$ is non-conservative

7.0 REFERENCES/FURTHER READING

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MODULE 4

UNIT 1 SIMPLE HARMONIC MOTION

CONTENTS

1.0 INTRODUCTION

If a small body or a particle vibrates or moves to and fro along a straight line under the influence of a force that its acceleration towards a fixed point (or its equilibrium position) is proportional to its distance or displacement from that point, the body is said to have a simple harmonic motion.

The term simple harmonic motion is not limited to motion in a straight line, and can be applied to the variation of any variable quantity which satisfies a differential relation of the type considered. Examples include simple pendulum, loaded test-tube in a liquid, mass on a spring.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

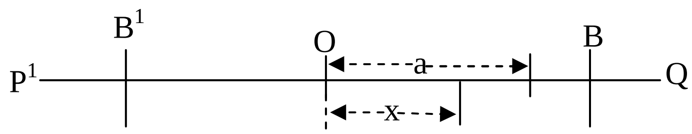
- define simple harmonic motion
- identify the forces causing simple harmonic motion
- define conical pendulum.

3.0 MAIN CONTENT

3.1 Simple Harmonic Motion

If a particle moved in a straight line in such a way that its acceleration is always directed toward a fixed point of the line and it is proportional to its distance from that point, the particle is said to move with simple harmonic motion.

Let O be the fixed point on the line (straight) POQ, and let x be the distance of the particle from O at time t, x is +ve to right and –ve to left. The acceleration of the particle can be taken along OQ as $-w^2x$, where w^2 is a constant which is +ve. If x is +ve, this acceleration is directed towards O, and if –ve, it is directed towards O



We obtain the differential relation

$$V\frac{dv}{dx} = -w^2x$$

i.e. $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -w^2x$

$$\therefore \frac{1}{2}v^2 = -\frac{1}{2}w^2x^2 + c$$

Where v = velocity, t = time, acceleration = $V\frac{dv}{dx}$ and c is constant.

So if t = 0, x = a and v = 0 (where a = distance)

Then, $0 = \frac{1}{2}w^2a^2 + c$,

$$V^2 = w^2\left(a^2 - x^2\right)$$

$$V = \pm w\sqrt{\left(a^2 - x^2\right)}$$

If the initial stage of the motion V is –ve as the particle is moving towards O.

Therefore, $\frac{dx}{dt} = -w\sqrt{(a^2 - x^2)}$

$\frac{dt}{dx} = -\frac{1}{w\sqrt{(a^2 - x^2)}}$

wt $= \cos^{-1}\frac{x}{a} + d$ (where d is a constant)

When t = 0, x = a, and $\cos^{-1}1 = 0$, and d = 0 and we get

$Wt = \cos^{-1}\frac{x}{a}$

X = acos (wt)

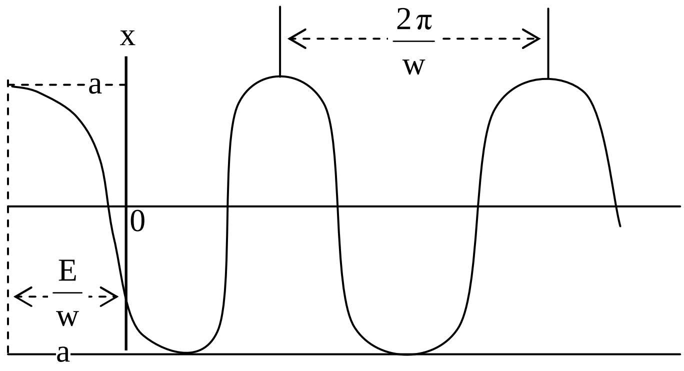
V = (-aw) sinwt

Definition

If a small body or a particle vibrates or moves to and fro along a straight line under the influence of a force that its acceleration towards a fixed point (or its equilibrium position) is proportional to its distance or displacement from that point, the body is said to have a simple harmonic motion.

The term simple harmonic motion is not limited to motion in a straight line, and can be applied to the variation of any variable quantity which satisfies a differential relation of the type considered. Examples include simple pendulum, loaded test-tube in a liquid, mass on a spring.

Note: When wt = $\pi/2w$ with velocity –aw. It continues along the straight line, and its velocity is zero when wt = π and x = -a. It then returns towards 0, arriving at 0 when wt = $\frac{3\pi}{w}$ with zero velocity. The motion is then repeated, and continues indefinitely unless it is destroyed by a frictional force of some kind. Below is the graph of distance against time.



a = length is called amplitude of the motion and is the distance of the extreme points from the centre of the oscillation.

Time $\frac{2\pi}{w}$ is called the period of the oscillation and is the time of a complete oscillation from one extreme point to the other and back again.

The frequency n = $\frac{w}{2\pi}$.

SELF ASSESSMENT EXERCISE 1

A particle is moving with simple harmonic motion of period 4π about a centre O, it passes through a point distance 4m from O with the velocity 4m/sec away from O. Find the time which elapses before it next passes through this point.

Solution

Since period = $\frac{2\pi}{w}$ $\therefore w = \frac{1}{2}$

From $v^2 = w^2 (a^2 - x^2)$,

$4^2 = ¼ (a^2 - 4^2)$,

A = $4\sqrt{5m}$

Therefore we have that

x = a cos wt

x = $4\sqrt{5}\cos\frac{t}{2}$

and when x = 4, $\cos \frac{t}{2} = \frac{1}{\sqrt{5}}$

$= 2n\pi \pm \cos^{-1}(\frac{1}{\sqrt{5}})$

When n = 0, 1, 2… etc.

So the shortest distance between instants when the particles is in this position is given by $\frac{t}{2} = 2\cos^{-1}\frac{1}{\sqrt{5}}$, $t = 4\cos^{-1}\frac{1}{\sqrt{5}}$.

3.2 Forces Causing Simple Harmonic Motion

The force acting on a particle earlier discussed is given as mw^2x which is proportional to the distance from O. Thus, SHM is caused by forces whose magnitude varies with distance. Examples are forces in a spring, which is proportional to the increase or decrease of its length from its natural length and the force of tension in an elastic string.

A particle with mass m, length l, modulus λ on a smooth horizontal table displayed a distance x from equilibrium position O will have a force $\frac{\lambda x}{l}$ towards O. Then

$$m\frac{d^2x}{dt^2} = \frac{-\lambda x}{l} = -\left(\frac{\lambda}{ml}\right)x$$

Its period = $\frac{2\pi}{w}$ where $w^2 = \frac{\lambda}{ml}$.

Its extension c is given as $mg = \frac{\lambda}{l}c$

3.3 Suspension by an Elastic String

A particle suspended by an elastic string also experiences SHM provided. The string does not return to its natural length during the motion; if this happens, the string will become slack and the particle begin to move freely under gravity.

Thus, we have $mg = \frac{\lambda}{l}a$ where a is the distance

$$\frac{\lambda}{l} = \frac{mg}{a}$$

If the particle of mass M hangs freely at the end of the string and given a small vertical displacement x then

$$M\frac{d^2x}{dt^2} = \frac{mg}{M\,a}x,$$

$$\frac{d^2x}{dt^2} = \frac{-mg}{ma}x,$$

Then, the period is $2\pi\sqrt{\left(\frac{Ma}{ma}\right)}$

SELF ASSESSMENT EXERCISE 2

One end of an elastic string of length 24 cm is fixed ended and to the other suspended end, a mass of 5 kg is attached, which when in equilibrium stretches the string 4 cm. The mass is pulled down at a distance of 3cm below its equilibrium position and then released. Find the period of oscillation and the maximum kinetic energy of the mass.

Solution

Let λ be the modulus of elasticity then from mg = $\frac{\lambda}{l}a$

$$5g = \frac{\lambda x\,4}{24}$$

$$\lambda = 30g \text{ N.}$$

For a displacement xm from the equilibrium position, we have

$$5\frac{d^2x}{dt^2} = \frac{30g}{0.24}\text{ N}$$

$$\frac{d^2x}{dt^2} = -25g \text{ N}$$

Hence, $w^2 = 25g$, w = 15.66, and the period is 0.45.

The amplitude is 3cm and force the epoch is zero,

$$x = 0.03 \cos 15.66t$$

$$\frac{dx}{dt} = v = -0.47 \sin 15.66t$$

So the max velocity is therefore 0.47m/sec and the corresponding kinetic energy is

$\frac{1}{2}$ x 5 x 0.47^2

= 0.552J.

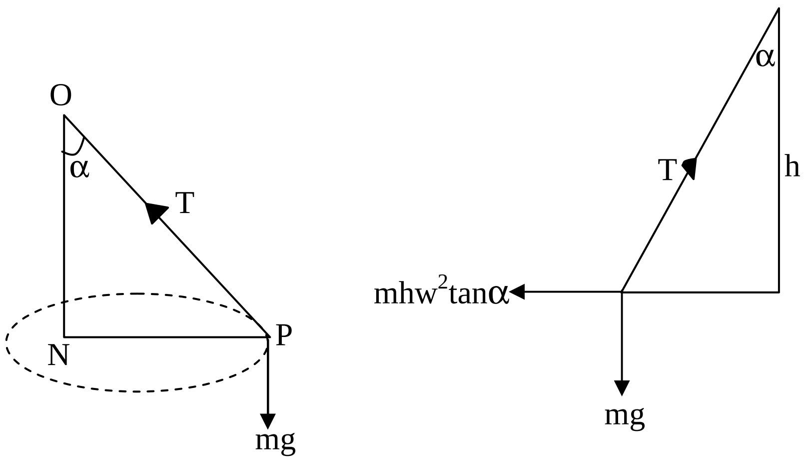
SELF ASSESSMENT EXERCISE 3

A spiral spring supports a carrier weighing 2 kg, and when a 10 kg weight is placed in the carrier the spring extends 5cm. The carrier with its load is then pulled down another 7.5 cm and let go. How high does it rise and what is the period of its oscillation?

3.4 Conical Pendulum

An arrangement by which a particle tied by a string to a fixed point O, and move in a horizontal circle, so that the string describe a cone whose axis is vertical through O is called conical pendulum.

Let the mass be m, height of the cone be h and its semi-circle angle be α. Therefore, the speed of resolution and the tension in the string may be found in terms of h and
α.



If w = angular velocity, T = tension in the string, the radius of the circle is h tan α and the reversed effective force is mhw^2 tan α. The particle may be considered as in equilibrium under the forces T, mg and mhw^2 tan $\propto$.

So $\quad T \sin \alpha = mhw^2 \tan \alpha, \; T \sin \alpha = mh \sin \alpha \, w^2,$

$$\therefore T = mhw^2 = 4\pi^2 n^2 mh$$

$T \cos \alpha = mg$

$\therefore\ w^2 = g/h,$

$T = mg \sec \alpha$

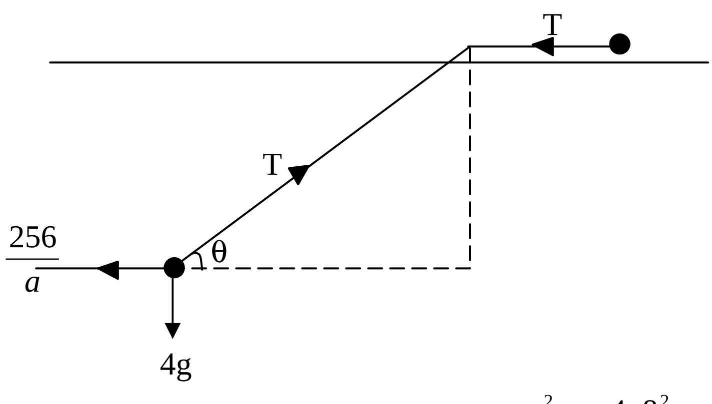
If w is m rad/sec, the time of a complete resolution is $\frac{2\pi}{w}$ sec, and the number of r.p.m. is $\frac{30w}{\pi} = \frac{30}{\pi}\sqrt{g/h}$

SELF ASSESSMENT EXERCISE 4

A mass of 10g rests on a rough horizontal table with coefficient of function ½. It is attached to one end of a light inextensible string which passes through a smooth hole in a mass of 4 kg at its free end. If the mass 4 g describes a horizontal circle with a velocity of 8 m/sec and the mass on the table is on the point of shipping, find the radius of the circle and the length of string below the table.

Solution

Since the 10 kg mass is on the point of slipping the tension in the string must be 5 gN. Let a m be the radius on the circle and θ the angle made by the string with the horizontal.



The reversed effective force (T) is given as $\frac{mv^2}{r} = \frac{4x8^2}{a}$

4 kg mass may be considered as in equilibrium under the forces 5 g, 4 g and $\frac{256}{a}N$.

Then, $5g \cos \theta = \frac{256}{a}$

$5g \sin \theta = 4h$

$\sin \theta = \frac{4}{5}$

$\cos \theta = \frac{3}{5} = \frac{256}{5x9.81xa'}$

$a = \frac{256}{29.43} = 8.7$

the length of string below the table is

$a \sec \theta = \frac{5}{3}a = 14.5\text{m}$

4.0 CONCLUSION

If a small body or a particle vibrates or moves to and fro along a straight line under the influence of a force that its acceleration towards a fixed point (or its equilibrium position) is proportional to its distance or displacement from that point, the body is said to have a simple harmonic motion.

Simple harmonic motion is not limited to motion in a straight line, and can be applied to the variation of any variable quantity which satisfies a differential relation of the type considered.

5.0 SUMMARY

In summary, we explained the term SHM which is concluded as not being limited to motion in a straight line.it was also discussed in this unit that SHM is caused by forces whose magnitude varies with distance. It was also noted that a particle suspended by an elastic string also experiences SHM, provided the string does not return to its natural length during the motion. Lastly, conical pendulum was also discussed and some examples are solved in all these.

8.0 TUTOR-MARKED ASSIGNMENT

1. A particle is moving with simple harmonic motion of period 4π about a centre O, it passes through a point distance 8 m from O with the velocity 4.5m/sec away from O. Find the time which elapses before it next passes through this point.

2. A mass of 12 g rests on a rough horizontal table with coefficient of function 0.3. It is attached to one end of a light inextensible string which passes through a smooth hole in a mass of 6.5 kg at its free end. If the mass 6.5 g describes a horizontal circle with a confirm velocity of 3 m/sec and the mass on the table is on the point of slipping, find the radius of the circle and the length of string below the table.

7.0 REFERENCES/FURTHER READING

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UNIT 2 COLLISION OF SMOOTH SPHERES

CONTENTS

1.0 INTRODUCTION

An object remains in a state of rest until an external force impacts on it. The ratio of the momentum after impact to the momentum before impact is called the coefficient of restitution (or elasticity). This momentum is denoted by the symbol e. If the net external force acting on a particle is zero, its momentum will remain unchanged. That is mv = constant.

The relative velocity of the spheres along the line of centres immediately after impact is – e times the relative velocity before impact.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define and explain, collision of smooth spheres
- identify the forces causing collision of smooth spheres.

3.0 MAIN CONTENT

3.1 Collision of Smooth Spheres

When a body strikes a fixed surface, the impact produced causes the momentum of the body to be destroyed where it undergoes compression thereby altered, its shape. This is followed by a period known as restitution in which the body regains its shape and momentum. The ratio of the momentum after impact to the momentum before impact is called coefficient of restitution (or elasticity) denoted by e.

i.e.

$$e = \frac{\text{momentum after impact}}{\text{momentum before impact}} \quad --- \quad --- \quad --- \quad (i)$$

$$\Rightarrow \; -e = \frac{\text{velocity after impact}}{\text{velocity before impact}} \quad --- \quad --- \quad --- \quad (ii)$$

If velocity is measured in (ii)

NOTE: In the above, when e = 0 we have inelastic body and when e = 1, we have perfectly elastic body.

This important theory is mostly applicable to the impact of spheres on smooth surfaces or on each other, so that the impulse during compression and restitution is normal to the surface.

3.2 Law for the Impact of Spheres

"The relative velocity of the spheres along the line of centres immediately after impact is –e times the relative velocity before impact."

3.2.1 Direct Impact

The impact is direct when it is normal to the surface and we have that

$$V = -eu$$

Where v = velocity after impact, u = velocity before impact and
e = coefficient or restitution.

So, if h is, the height when a sphere falls to a smooth plane, the velocity before impact is

$$U = \sqrt{(2gh)}$$

The velocity after impact is (upward)

$$eu = \sqrt{(2ghe^2)}$$

The velocity is destroyed by gravity when the sphere has risen to a height.

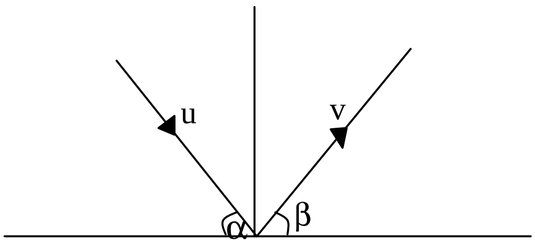
$\therefore \sqrt{(2gh^1)} = \sqrt{2ghe^2}$

$\frac{h^1}{h} = e^2$

$e = \sqrt{\frac{h^1}{h}}$

3.2.2 Indirect Impact

If the surface diagram of the impact is of the form



The velocity before impact parallel to the surface is u cos α and velocity after impact is v.

$\therefore$ u cos α = v cos β

$\Rightarrow$ v sin β = eu sin α

Hence, $v^2\cos^2\beta + v^2\sin^2\beta = u^2 (\cos^2\alpha + e^2\sin^2\alpha)$

$v = u\sqrt{(\cos^2\alpha + e^2\sin^2\alpha)}$

$\tan\beta = e \tan\alpha$

$\Rightarrow u^2 - v^2 = u^2 (1-e^2)\sin^2\alpha$

and loss of kinetic energy due to impact is

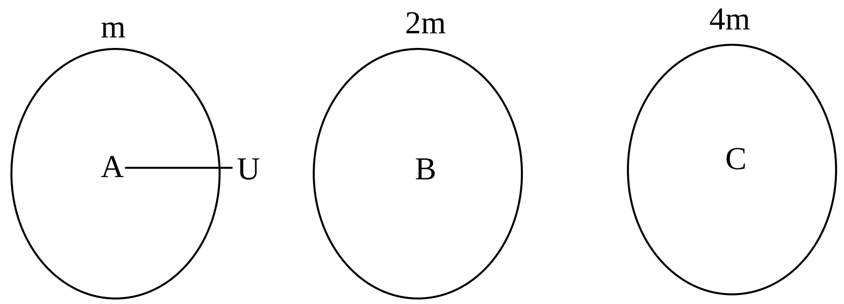
$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2(1-e^2)sm^2 \propto$$

$\therefore$ the proportional loss of kinetic energy is $(1-e^2)\sin^2\propto$

SELF ASSESSMENT EXERCISE 1

Three smooth spheres, A, B, C of masses m, 2m and 4m respectively rest on a smooth plane (horizontal) with their centres collinear, and B lies between A and C. The coefficients of restitution between any two pairs are equal. If A is projected towards B with velocity U and C moves with velocity $\frac{u}{4}$ after it has been struck by B, find the common coefficient of restitution and subsequent velocities of A and B.

Solution



Let V_1, V_2 be velocities of A and B respectively after 1st collision and let e be the common coefficient of restitution between any two spheres. Then momentum conservation and Newton's Law give

$$mV_1 + 2Mv_2 = mu \quad \text{--------------------} \ (1)$$

$$V_1 - v_2 = -e(u-0) = -eu \quad \text{----------------} \ (2)$$

$$(1) \text{ and } (2) \Rightarrow 3v_2 = u(1+e)$$

$$\therefore v_2 = u\frac{(1+e)}{3}$$

Thus, $V_1 = u\frac{(1+e)}{3} - eu = u\frac{(1-2e)}{3}$

B then moves faster ahead of A to strike C at rest causing second collision.

Let V_3, V_4 be velocities of B and C respectively after the second collision,

Then,

$$2mv_3 + 4mv_4 = 2m\,u\frac{(1+e)}{3} \quad \text{----------} \ (3)$$

$$V_3 - V_4 = -eu\frac{(1+e)}{3} \text{ -------------------- (4)}$$

$$(2) \text{ and } (4) \Rightarrow 3v_4 = u\frac{(1+e)}{3} + eu\frac{(1+e)}{3}$$

Therefore, $V_4 = \frac{u}{9}(1 + e + e + e^2) = \frac{u}{9}(1+e)^2$

But, $V_4 = \frac{u}{4}$,

Thus, $\frac{u}{9}(1+e)^2 = \frac{u}{4}$ or $1+ e = \frac{3}{2}$

$\Rightarrow e = \frac{1}{2}$

Velocity of A after 1st collision is

$$V_1 = u\frac{(1-2x\frac{1}{2})}{3} = 0$$

Velocity of B after 2nd collision is

$$V_3 = V_4 - eu\frac{(1+e)}{3} = \frac{u}{4} - u\frac{(1+\frac{1}{2})}{6}$$

$$= \frac{u}{4} - \frac{u}{4} = 0$$

Therefore, subsequently, A and B are put to rest after first and second collision respectively.

SELF ASSESSMENT EXERCISE 2

A particle falls from a height h upon a fixed horizontal plane; if e be the coefficients of restitution on, show that the whole distance described before the particle has finished rebounding is $\frac{(1\_e^2)h}{(1-e^2)}$ and that the whole tune taken is $\frac{1+e}{1-e}$ x $\sqrt{\frac{2h}{g}}$

Solution

(a) Let u be the velocity on first hitting the plane so that

$$U^2 = 2gh$$

Then the particle rebounds with velocity eu. The velocity when it hits the plane the second time is eu and the velocity after the second rebound is e^2u. Similarly, the velocity after the third, fourth etc.; rebounds are e^3u, e^4u etc.

(b) The height to which the particle rises after the first rebound is

$$\frac{(eu)^2}{2g}$$

And after the second $\frac{(e^2u)^2}{2g}$ and so on.

(c) Since $u^2 = 2gh \therefore$ the distances are e^2h, e^4h, etc.

Hence, the whole distance described is

$$H + 2(e^2h + e^4 + \ldots\text{to infinity})$$

$$= h + 2h\left(\frac{e^2}{1-e^2}\right) = h\left(\frac{1+e^2}{1-e^2}\right)$$

(d) The time of flight after the first impact is 2eu/g, after the second $2e^2u/g$, and so on, and the time of falling originally is

$$\sqrt{\left(\frac{2h}{g}\right)}.$$

Hence, the whole time of motion $= \sqrt{\left(\frac{2h}{g}\right)} + \frac{2u}{g}(e + e^2 + e^3 + \ldots\text{to infinity})$

$$= \sqrt{\left(\frac{2h}{g}\right)} + 2\sqrt{\left(\frac{2h}{g}\right)}.(e + e^2 + \ldots)$$

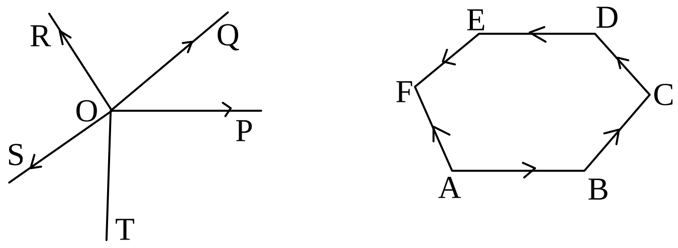
$$= \sqrt{\left(\frac{2h}{g}\right)}(1+2\frac{e}{1-e})$$

$$= \sqrt{\left(\frac{2h}{g}\right)} \text{ x } \left(\frac{1+e}{1-e}\right)$$

3.3 Resultant of any Number of Forces Acting on a Particle

The resultant force is that single force which acting alone will have the same effect in magnitude and direction as two or more forces acting together. Also, the equilibrant of two or more forces is that single fore which will balance all the other forces taken together. Note that the equilibrant force is equal in magnitude but opposite in direction to the resultant force.

If we have a number of forces acting on a particle at O, and we draw a polygon with its sides proportional and parallel to these forces and close the polygon then we know that the system is in equilibrium or if not close, the resultant of the forces is represented by the straight lines. i.e.

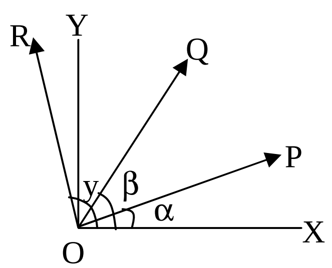


(a) (b)

In vector notation, the resultant of the false forces ABCDEF is a force acting at O. It is represented in magnitude and direction by AF. Thus, AF = AB + BC + CD + EF

3.3.1 Method of Finding the Resultant of any Number of Forces in One Plane Acting on a Particle

Consider the figure below,



Let the forces P, Q, R etc. act upon a particle at O.

Let the forces P, Q, R …… makes angle α, β, y ….. with OX.

The components of P in the directions OX and OY are:

$P\cos\alpha$ and $P \sin \alpha$ respectively.

Similarly, the components of Q are:

$Q\cos\beta$ and $Q\sin\beta$ and so on.

Hence, the forces are equivalent to a component

$P\cos\alpha + Q\cos\beta + R\cos Y$……along OX

and a component

$P\sin\alpha + Q\sin\beta + R\sin Y$……. Along OY

If the components be X and Y respectively and F be their resultant and θ its inclination to OX,

Then, $F\cos\theta = X$

$F\sin\theta = Y$

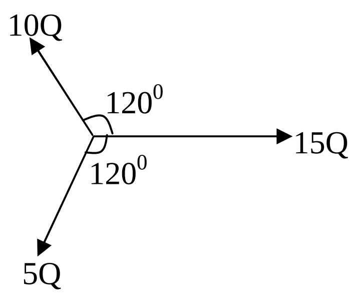
$F^2 = X^2 + Y^2$

and $\tan\theta = \frac{X}{Y}$

SELF ASSESSMENT EXERCISE 3

Three forces of magnitude 15Q, 10Q, 5Q act on a particle in directions which make 120^0 with one another. Find their resultant.

Solution



Since forces 5Q, 10Q, 15Q in the directions indicated are in equilibrium, they can be represented in magnitude and direction by the sides of an equilateral triangle.

Hence, the three given forces are equivalent to forces 10Q and 5Q inclined at an angle 120^0, of which the resultant R is given by

$$R^2 = (10Q)^2 + (5Q)^2 + 2 \text{ x } 10Q \text{ x } 5Q \text{ x } \cos 12^0$$

$$= 100Q^2 + 25Q^2 - 50Q^2$$

$$= 75Q^2$$

$$R = 5\sqrt{3}\, Q$$

The angle θ the resultant makes with the direction of the force 15P is given by

$$\tan\theta = \frac{5\sin 60^0}{10 - 5\cos 60^0} = \frac{5\sqrt{3}}{15}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\therefore\ \theta = 30^0$$

SELF ASSESSMENT EXERCISE 4

A particle is acted on by forces of 1N, 2N, 3N, and 4N, the angles between them being 60^0, 30^0, 60^0 respectively, find the magnitude and direction of the resultant.

4.0 CONCLUSION

Conclusively, having gone through this unit you should be able to define the following terms: coefficient of restitution, direct and indirect impacts. Also, you should be able to solve simple problems on resultants of forces acting on a particle.

5.0 SUMMARY

In this unit, we studied that the ratio of the momentum after impact to the momentum before impact is called coefficient of restitution (or elasticity) denoted by e.

i.e.

$$e = \frac{\text{momentum after impact}}{\text{momentum before impact}}$$ and

$$\Rightarrow -e = \frac{\text{velocity after impact}}{\text{velocity before impact}}$$

We remark here that when e = 0 we have inelastic body and when e = 1, we have perfectly elastic body.

It was also discussed that the relative velocity of the spheres along the line of centres immediately after impact is –e times the relative velocity before impact. Direct impact and indirect impact were treated as well in this unit. However, the impact is direct when it is normal to the surface and we have that

$$V = -eu$$

Where v = velocity after impact
u = velocity before impact and
e = coefficient or restitution.

So, if h is, the height when a sphere falls to a smooth plane, the velocity before impact is

$$u = \sqrt{(2gh)}$$

The velocity after impact is (upward)

$$eu = \sqrt{(2ghe^2)}$$

The velocity is destroyed by gravity when the sphere has risen to a height.

$$\therefore \sqrt{(2gh^1)} = \sqrt{2ghe^2}$$

$$\frac{h^1}{h} = e^2$$

$$e = \sqrt{\frac{h^1}{h}}$$

while in indirect impact the velocity before impact parallel to the surface is u cos α and velocity after impact is u.

$$\therefore u \cos\alpha = v \cos\beta$$

$$\Rightarrow v \sin\beta = eu \sin\alpha$$

and from above it was shown that the loss of kinetic energy due to impact is expressed mathematically as

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2(1-e^2)s\,m^2 \propto$$

where $((1-e^2)\sin^2\propto)$is the proportional loss of kinetic energy of the system of particle.

Remarks

1. Suppose two bodies of masses m_1 and m_2 moving with velocity u_1 and u_2 respectively collide directly. If v_1 and v_2 are the velocities after impact the principle of momentum gives us the equation

   $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$

2. $\frac{v_1 - v_2}{u_1 - u_2} = -e$

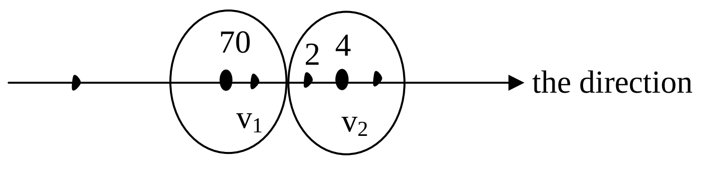
   then, $v_1 - v_2 = -e(u_1-u_2)$

3. From (1) & (2) above,
   $(m_1 + m_2)\, v_1 = (m_1 - em_2)\, u_1 + m_2\,(1+e)u_2$
   and $((m_1 + m_2)\, v_2 = m_1(1+e)\, u_1 + (m_2 - em_1)\, u_2$.

SELF ASSESSMENT EXERCISE 5

1. A ball of mass 10kg, moving at 5m/s, overtakes another of mass 4kg, moving at 2m/s in the same direction. If $\in = \frac{1}{2}$, find the velocities after impact.

Solution



Let v_1, v_2 m/s be the velocities of 10kg and 4kg spheres respectively after impact. By the principle of momentum

$10v_1 + 4v_2 = 10 \times 5 + 4 \times 2 = 58$

and by Newton's law

$$v_1 - v_2 = -\frac{1}{2}(5 - 2) = \frac{-3}{2}$$

$$\therefore\ 14v_1 = 52 \text{ or } v_1 = 3\frac{5}{7}$$

$$\text{and } 14v_2 = 73 \text{ or } v_2 = 5\frac{3}{14}.$$

SELF ASSESSMENT EXERCISE 6

Two masses m_1 and m_2 travelling in the same straight line collide
Find

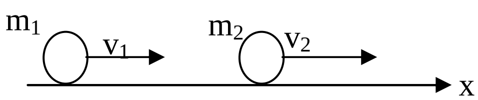
(a) velocities of the particles after collision in terms of the velocities before collision
(b) Briefly discuss:

(i) a perfectly inelastic collision and
(ii) a perfectly elastic collision for the two masses m_1 and m_2.

(c) Show that for a perfectly elastic collision of the particle m_1 and m_2, the total kinetic energy before collision equals the total kinetic energy after collision.

Solution

(a)



Assume that the straight line is taken to be the x axis and that the velocities of the particles before and after collisions are v_1, v_2 and v_1, v_2 respectively,

By Newton's collision rule,

$$v_1 - v_2 = \epsilon(u_2 - u_1) \text{ ----------------- } 1$$

By principle of consideration of momentum, total momentum after collision = total momentum before collision.

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \text{ --------------- } 2$$

$m_1v_1 + m_2v_2 = mu_1 + mu_2$

$10v_1 + 4v_2 = 10 \times 5 + 4 \times 2 = 58$ ---------------- (i)

By Newton's law,

$v_1 – v_2 = -€(u_2 – u_1)$ ----------------…..- (ii)

$10v_1 + 4v_2 = 58$

and from (ii) $v_1 – v_2 = \frac{-3}{2}$

$v_2 = \frac{3}{2} + v_1$

Put in (i)

$10v_1 + 4(\frac{3}{2} + v_1) = 58$

$10v_1 + 6 + 4v_1 = 58$

$14v_1 = 52$

$v_1 = 3\frac{5}{7}$

also

$10(3\frac{5}{7}) + 4\,(v_2)\ 58$

$v_2 = 5\frac{3}{14}$.

Solving equations (1) and (2) simultaneously,

$$v_1 = \frac{(m_1 - \varepsilon m_2)u_1 + m_2(1+\varepsilon)u_2}{m_1 + m_2} \text{ ---------------------- } 3$$

$$v_2 = \frac{m_1(1+\varepsilon)u_1 + (m_2 - \varepsilon m_1)u_2}{m_1 + m_2} \text{ ---------------------- } 4$$

(b) (i) Here we put $\varepsilon = 0$ in (3) and (4) above to obtain

$$v_1 = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}, \quad v_2 = \frac{m_1u_1 + m_2u_2}{m_1 + m_2},$$

Thus, after collision the two particles move with the same velocity i.e. they move as if they were stuck together as a single particle.

(ii) Here we put $\varepsilon = 1$ in (3) and (4) above to obtain

$$v_1 = \frac{(m_1 - m_2)u_1 + 2m_2u_2}{m_1 + m_2}, \quad v_2 = \frac{2m_1v_1 + (m_2 - m_1)u_2}{m_1 + m_2}$$

Then velocities are not the same

(c) Total kinetic energy after collision = $\frac{1}{2}m_2\left\{\frac{2m_1u_1 + (m_2 - m_1)u_2}{m_1 + m_2}\right\}^2$

$$= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m2u_2^2$$

= total kinetic energy before collision

6.0 TUTOR-MARKED ASSIGNMENT

1. Three forces of magnitude 20Q, 15Q, 10Q act on a particle in directions which make 150^0 with one another. Find their resultant.
2. A particle is acted on by forces of 5N, 2N, 1.5N, and 8N, the angles between them being 60^0, 45^0, 90^0 respectively, find the magnitude and direction of the resultant.

7.0 REFERENCES/FURTHER READING

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MODULE 5 NEWTON'S LAW OF MOTION AND APPLICATIONS TO SIMPLE PROBLEMS

UNIT 1 NEWTON'S LAW OF MOTION

CONTENTS

1.0 INTRODUCTION

Isaac Newton, the great scientist considered the following:

(i) how a body A will move when left to itself
(ii) how the motion is affected by the action of an external force
(iii) if this external force is due to another body B_1, how the action of B on A is related to the reaction of A on B.

He then gave three laws which we perfectly call Newton's laws of motion in mechanics.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- state Newton's laws of motion
- define impulsive forces and make some simple applications.

3.0 MAIN CONTENT

3.1 Newton's Law of Motion

Isaac Newton considered the following:

(i) how a body A will move when left to itself
(ii) how the motion is affected by the action of an external force
(iii) if this external force is due to another body B_1, how the action of B on A is related to the reaction of A on B.

He then gave three laws which we considered as the axioms of mechanics which shall form the bases of the unit:

(1) Every object (particle) continues in a state of rest or of uniform motion in a straight line (i.e. with constant velocity) unless acted upon by a force.

(2) If F is the force acting on a particle of mass m which as a consequence is moving with velocity v, then

$$F = \frac{d}{dt}(mv) \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{d}{dt}P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

where P = mv is called the momentum. If m is independent of time t this becomes

$$F = m\frac{dv}{dt} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$= ma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

where a is the acceleration of the particle

(3) If particle A acts on particle B with a force F_{AB} in a direction along the line joining the particles, while particle B acts on particle A with a force F_{BA}, then $F_{BA} = F_{AB}$. In other words, to every action there is an equal and opposite reaction.

3.2 Force

Force is defined as a measure of the "push or pull on an object". The unit of force is Newton (N).

SELF ASSESSMENT EXERCISE 1

Due to a field, a particle of mass 3 units moves along a space curve whose position vector is given as a function of time t by

$$\underline{r} = (4t^2 + 2t)\underline{i} + (t^3 - t^2 + 10)\underline{j} - 6t^3\underline{k}$$

Find (a) the velocity, (b) the momentum (c) the acceleration and (d) the force field at any time t.

Solution

(a) Velocity = V = $\frac{dr}{dt} = (8t + 2)\underline{i} + (3t^2 - 2t)\underline{j} - 18t^2\underline{k}$

(b) Momentum = P = mv = $(24t + 6)\underline{j} + (9t^2 - 6t)\underline{j} - 54t^2\underline{k}$

(c) Acceleration = a = $\frac{dv}{dt} = \frac{d^2r}{dt^2} = 8\underline{i} + (6t - 2)\underline{j} - 36t\underline{k}$

(d) Force = F = $\frac{dp}{dt} = m = 24\underline{i} + (18t - 6)\underline{j} - 48t\underline{k}$

SELF ASSESSMENT EXERCISE 2

A particle of mass 4 moves in a force field depending on time t given by

$$F = 12t\overset{2}{\underline{i}} + (16t - 8)\underline{j} - 20t\underline{k}$$

Assuming that at t = 0 the particle is located at $r_0 = 2i - j + 6k$ and has velocity $V_o = 3\underline{i} + 7\underline{j} - 4\underline{k}$, find (a) the velocity and (b) the position at any time t.

Solution

(a) By Newton's second law, apply equation (iii) above

$$4\frac{dv}{dt} = 12t\overset{2}{\underline{i}} + (16t - 8)\underline{j} - 20t\underline{k}$$

$$\frac{dv}{dt} = 3t\overset{2}{\underline{i}}\,0 + (4t - 2)\underline{j} - 50t\underline{k}$$

Integrating with respect to t and calling c, the constant of integration,

we have $\int\frac{dv}{dt} = \int\ 3t\overset{2}{\underline{i}} + (44 - 2) - 5\underline{k}$

$$V = t^{3}\underline{i} + (t^2 - 2t)\,\underline{j} - \frac{5t^2}{2}t\,\underline{k} + G$$

Since

$v = v_o = 3\underline{i} + 7\underline{j} - 4\underline{k}$ at $t = 0$, we have $c_1 = 3\underline{i} + 7\underline{j} - 4\underline{k}$

and so $v = (t^3 + 3)\underline{i} + (t\,\hat{2}^{-2t} + 7)\,\underline{j} - (\frac{5t^2}{2} - 4)\,\underline{k}$

(b) Since $v = \frac{dr}{dt}$, we have by part (a)

$$\frac{dr}{dt} = (t^3 + 3)\underline{i} + (t^2 - 2t + 7)\,\underline{j} - (\frac{5t^2}{2} - 4)\,\underline{k}$$

Integrating with respect to t we have

$$r = \left(\frac{t^4}{4} + 3t\right)\underline{i} + \left(\frac{t^2}{2} - t^2 + 7t\right)\underline{j} - \left(\frac{5}{4}t^3 - 4t\right)\underline{k} + c_2$$

where c_2 is the constant of integration. Since

$r = r_o = 2i - j + 6k$ at $t = 0$,
we have $c_2 = 2\underline{i} - \underline{j} + 6\underline{k}$

and so

$$r = \left(\frac{t^4}{4} + 3t + 2\right)\underline{i} + \left(\frac{t^2}{2} - t^2 + 7t - 1\right)\underline{j} - \left(\frac{5}{4}t^3 - 4t + 6\right)\underline{k}$$

SELF ASSESSMENT EXERCISE 3

A particle of mass m moves in the xy plane so that its position vector is $r = a\cos wt\,\underline{i} + b\sin wt\,\underline{j}$

where a, b and w are positive constants and $a>b$

(a) Show that the particle moves in an ellipse
(b) Show that the force acting on the particle is always directed toward the origin.

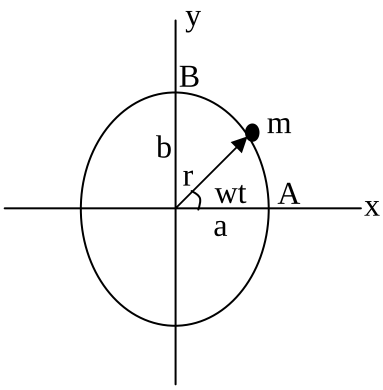
Solution

The position vector is

r = xi + yi

= a coswt i + bsinwtj

and so x = acoswt, y = bsinwt which are the parametric equations of an ellipse having semi-major and semi-minor axes of lengths a and b respectively.



Since $\left(\frac{x}{b}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 wt + \sin^2 wt = 1$

the ellipse is also given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(b) Assuming the particle has constant mass m, the force acting on it is

$$F = \frac{mdv}{dt} = m\frac{d^2r}{dt^2} = \frac{md^2}{dt^2}\left((a\cos wt)\underline{i} + (b\sin wt)\underline{j}\right)$$

$$= m[-w^2 a\cos wt\ i - w^2 b\sin wt)j)]$$

$$= -mw^2[a\cos wt\ \underline{i} + b\sin wt\ \underline{j}]$$

which shows that the force is always directed toward the origin.

4.0 CONCLUSION

Having concluded this unit, you should be able to state promptly the three (**Newton's law of motion)** that is, every object (particle) continues in a state of rest or of uniform motion in a straight line (i.e. with constant velocity) unless acted upon by a force. Secondly, rate of change of momentum is proportional to the applied force and takes place in the direction of the applied force.

Mathematically, F = ma. Lastly, the third law states that to every action there is an equal and opposite reaction.

5.0 SUMMARY

In summary, the first, second and third laws of Isaac Newton were defined and extensively explained coupled with some simple problems on forces that were attended to.

9.0 TUTOR-MARKED ASSIGNMENT

1. State Newton's law of motion.
2. In a force field, a particle of mass 10 units moves along a space curve whose position vector is given as a function of time t by $\underline{r} = (5t^2 + t)\underline{i} + (t^3 - t^2 + 3)\underline{j} - 7t^3\underline{k}$

   Find (a) the velocity, (b) the momentum (c) the acceleration and (d) the force field at any time t.
3. A particle of mass 7kg moves in a force field depending on time t given by $F = 15t^2\underline{i} + (10t + 3)\underline{j} - 12t\underline{k}$

Assuming that at t = 0 the particle is located at $r_0 = i - j + 3k$ and has velocity $V_o = 2\underline{i} + \underline{j} + 4\underline{k}$, find (a) the velocity and (b) the position at any time t.

7.0 REFERENCES/FURTHER READING

Ajibola, S.T. (2006). *Vector Analysis and Mathematical Method.*

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UNIT 2 WORK, POWER AND ENERGY

CONTENTS

1.0 INTRODUCTION

According to Oxford English dictionary, work is said to be the use of bodily or mental power in other to do or make something. While power is said to be the ability to do something/perform work. Consequently, energy is said to be the strength and vitality needed for vigorous activities. In other word, energy is the ability to do work. Energy can be expressed in two forms namely: kinetic and potential energies respectively.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define work, power and energy
- state the principles of linear momentum and some simple applications
- state the principles of angular momentum and some simple applications
- state the principles of conservation of energy and some simple applications.

3.0 MAIN CONTENT

3.1 Work, Power and Energy

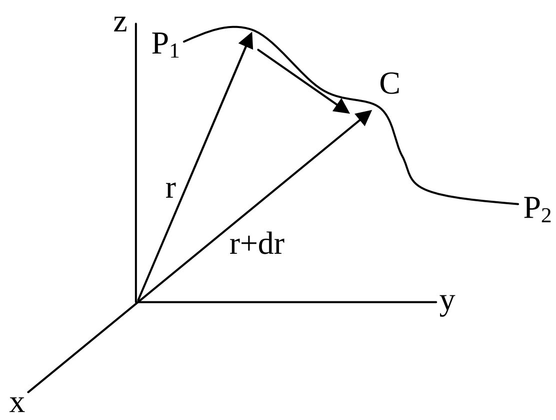
3.1.1 Work

If a force F acting on a particle gives it a displacement dr, then the work done by the force on the particle is defined as

dw = F.dr.

Since only the component of F in the direction of dr is effective in producing the motion.

The total work done by a force field (vector field) F in moving the particle from point P_1 to point P_2 along the curve C of the figure below is given by the line integral.



Where r_1 and r_2 are the position vectors of p_1 and p_2 respectively.

3.1.2 Power

If the particle in work above has constant mass that at times t_1 and t_2 it is located at p_1 and p_2 and moving with velocities $V_1 = \frac{dr_1}{dt}$ and $V_2 = \frac{dr_2}{dt}$ respectively.

Theorems on Power

Theorem 1

The total work done in moving the particle along C from p_1 to p_2 is given by:

$$W = \int_c F dr = \frac{1}{2}m(v_2^2 - v_1^2)$$

3.1.3 Kinetic Energy

If we call the quantity $T = \frac{1}{2}mv^2$, the kinetic energy of the particle, then the theorem 1 above is equivalent to the statement.

Total work done from p_1 to p_2 along C.

= Kinetic energy at p_2 – Kinetic energy at P_1

$W = T_2 - T_1$

where $T_1 = \frac{1}{2}m v_1^2$ and $T_2 = \frac{1}{2}m v_2^2$

SELF ASSESSMENT EXERCISE 1

A particle of constant mass m moves in space under the influence of a force field F. Assuming that at times t_1 and t_2 the velocity is v_1 and v_2 respectively, prove that the work done is the change in kinetic energy, i.e. $\int_{t_1}^{t_2} F.dr = \frac{1}{2}mv_2^2 - \frac{1}{2}m v_1^2$

Solution

$$\text{Work done} = \int_{t_1}^{t_2} F.dr\ dt = \int_{t_1}^{t_2} F.vdt$$

$$= \int_{t_1}^{t_2} m\frac{dv}{dt}.vdt$$

$$= m\int_{t_1}^{t_2} v.dv$$

$$= \frac{1}{2}m \int_{t_1}^{t_2} d(v.v)$$

$$= \frac{1}{2}m v^2 \Big|_{t_1}^{t_2}$$

$$= \frac{1}{2}m v_2^2 - \frac{1}{2}m v_1^2$$

SELF ASSESSMENT EXERCISE 2

Prove that if F is the force acting on a particle and v is the velocity of the particle, then the power applied to the particle is given by P = F-V

Solution

By definition the work done by a force F in giving a particle a displacement dr is

$$dw = F.dr$$

then the power is given by

$$P = \frac{dw}{dt}$$

$$= F.\frac{dr}{dt}$$

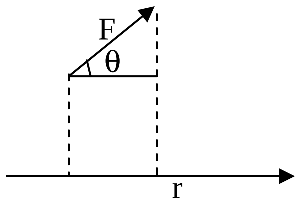
$$P = F.v$$

SELF ASSESSMENT EXERCISE 3

Find the work done in moving an object along a vector

$r = 6i + 4j – 10k$ if the applied force is $F = 4i – 2j – 2k$

Solution



work done = <u>magnitude of force in direction of motion</u> x distance moved

$= F\cos\theta \times r$
$= F.r$
$= (6i + 4j – 10k). (4i -2j – 2k)$
$= 24 – 8 + 20$
$= 36.$

3.2 Principle of Linear Momentum

If $\underline{r}$ is the position vector of the centre of mass of a rigid body relative to an origin O, then

$$\frac{d}{dt} m\dot{\underline{r}} = m\ddot{\underline{r}} = \underline{F}$$

where m is the total mass, and F is the net external force acting on the body.

3.3 Principle of Angular Momentum

If $\int c$ is the moment of inertia of the rigid body about the centre of mass, w is the angular velocity and $\wedge_C$ is the torque or total moment of the external forces about the centre of mass, then

$$\wedge_C = \frac{d}{dt}(\int c\, w)$$
$$= \int c\, \dot{w}$$

3.4 Principle of Conservation of Energy

If the net external forces are conservative so that the potential energy of the rigid body is V, then

$$T + V = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\int c\, w^2 + v$$
$$= E$$
$$= \text{constant}$$

Note: $\frac{1}{2}$ mr^2 is the kinetic energy of translation and $\frac{1}{2}\int^V cw^2$ is the kinetic energy of rotation of the rigid body about the centre of mass.

SELF ASSESSMENT EXERCISE 4

Show that the total momentum of a system of particles can be found by multiplying the total mass m of the system by the external force acting on the system.

Proof

The centre of mass is by definition,

$$\bar{r} = \frac{\Sigma_r r_r}{m}$$

The total momentum is

$$P = \Sigma m_r v_r$$

$$= \Sigma m_r \dot{r}_r = m\frac{d\dot{r}}{at}$$
$$= m\bar{v} \quad \text{Q.E.D.}$$

SELF ASSESSMENT EXERCISE 5

Explain why the ejection of gases at high velocity from the rear of a rocket will move the rocket forward.

Solution

Since the gas particles move backward with high velocity and since the centre of mass does not move, the rocket must move forward.

SELF ASSESSMENT EXERCISE 6

A solid cylinder of radius a and mass m rolls without slipping down on an inclined plane of angleα. Show that the acceleration is constant and equal to $\frac{2}{3g}\sin\alpha$.

Solution

The potential energy is composed of the P.E due to the external forces i.e. gravity the P.E due to internal forces which can be neglected.

Taking the reference level as the base plane and assuming that the height of the centre of mass above this plane initially is H and the height at anytime t is h, we have

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\int c\, w^2 + mgh = MgH \quad (1)$$

or using

$$H - h = x\sin\alpha \quad (2)$$

and $\dot{\underline{r}} = x^2 + y^2$

$= x^2$ .... (3)

i.e. $y^2 = 0$

Substituting (2) and (3) into (1) we have

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\int c\, w^2 = mg\, x\sin\alpha \quad (4)$$

But $w = \theta = \frac{x}{a}$ .... (5)

and $\int_C = \frac{1}{2}ma^2$ .... (6)

Substituting (5) and (6) into (4) we obtain

$$\dot{x}^2 = \frac{4}{3} g \dot{x} \sin\alpha$$

Differentiating w.r.t, we have

$$2\dot{x}\ddot{x} = \frac{4}{3} g \dot{x} \sin\alpha$$

$$\ddot{x} = \frac{2}{3} g \sin\alpha$$

3.4.1 Angular Momentum (Ω)

In $r \times F = \frac{d}{dt}[m(r \times v)]$

then the quantity

$$\Omega = m(r \times v)$$
$$\Omega \; r \times p$$

is called the angular momentum or moment of momentum about O

3.4.2 Conservation of Momentum

If we let F = 0 in Newton's second law, we find

$$\frac{d}{dt}(mv) = 0$$

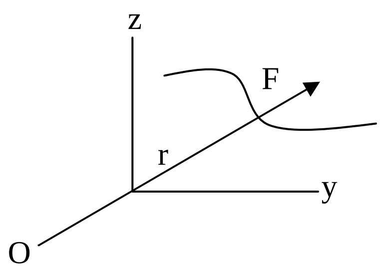
Or mv = constant

Remark

If the net external force acting on a particle is zero, its momentum will remain unchanged.

Torque

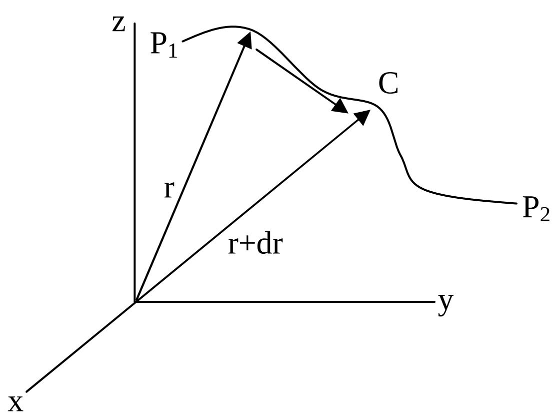
If a particle with position vector r moves in a force field F



we define $\wedge = r \times F$. As the torque or moment of the force F about O.

Impulse

Suppose that the particle in the figure below



is located at $p_1 and p_2$ at times $t_1 and t_2$ where it has velocities $V_1 and V_2$ respectively. The time integral of the force F given by

$$\int_{t_1}^{t_2} F dt$$

is called the impulse of the force F.

SELF ASSESSMENT EXERCISE 7

Determine the torque and the angular momentum about the origin for the particle of mass 4 moves in a force field depending on time t given by F = $12t^2$ i+ (16t – 8)j – 20tk. Assuming that at t = 0, the particle is located at r_0 = 2i – j + 6k and has velocity V_0 = 7j – 4k.

Solution

Hint: Torque $\wedge$ = r x F

4.0 CONCLUSION

If a force F acting on a particle gives it a displacement dr, then the work done by the force on the particle is defined as dw = F.dr. If the net external force acting on a particle is zero, its momentum will remain unchanged.

5.0 SUMMARY

Work is defined as work done by the force on the particle thus
$dw = F.dr$. Power is defined as the total work done in moving the particle along the path C from p_1 to p_2 above is given by

$W = \int_c F - dr = \frac{1}{2}m(v_2^2 - v_1^2)$. The kinetic energy of the particle is given as $T = \frac{1}{2}mv^2$. We further discussed the **principle of conservation of energy as** E = constant = T + V = $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\int c\, w^2 + v$.

Furthermore, the angular momentum or moment of momentum about O is given as Ω = m(r x v).

It is remarked here that if the net external force acting on a particle is zero, its momentum will remain unchanged. That is
mv = constant. Each sub section is followed by an example for better understanding of the unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Explain the following terms and express their mathematical formulae: (a) Work (b) Power (c) Energy
2. Determine the torque and the angular momentum about the origin for the particle of mass 4kg moves in a force field depending on time t given by F = 32t i+ (32t – 8)j – 5tk. Assuming that at t = 0, the particle is located at r_0 = i –3 j + 6k and has velocity V_0 = 3i – 4k.

7.0 REFERENCES/FURTHER READING

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UNIT 3 RECTILINEAR MOTION

CONTENTS

1.0 INTRODUCTION

A force field which has constant magnitude and direction is called a uniform constant force field.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define rectilinear motion
- define weight and acceleration due to gravity.

3.0 MAIN CONTENT

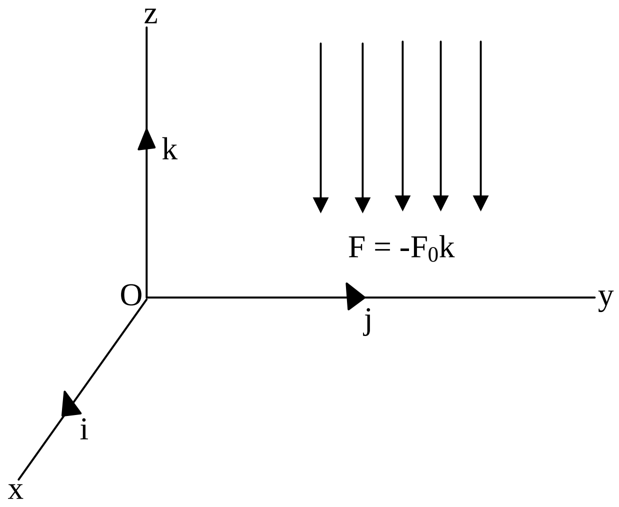
3.1 Rectilinear Motion

3.1.1 Uniform Force Fields

Definition

A force field which has constant magnitude and direction is called a uniform constant force field.

Consider the diagram below, if the direction of the field is taken as the negative z direction as indicated below and the magnitude is the constant $F_0 > 0$,



then the force field is given by $F = -F_0k$ ------------- (a)

3.1.2 Uniformly Accelerated Motion

If a particle of constant mass m moves in a uniform force field, then its acceleration is uniform or constant. The motion is then described as uniformly accelerated motion.

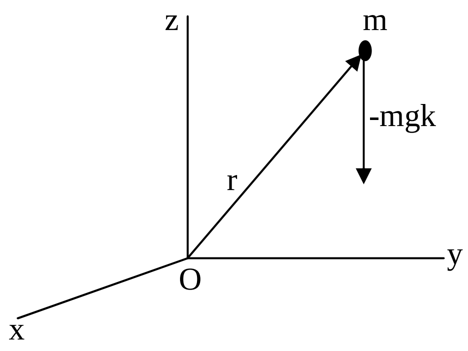
Using our known F = main (a) above, the acceleration of a particle of mass m moving in a uniform force field (a) is given by

$$a = \frac{F_0k}{m}$$

3.2 Weight and Acceleration due to Gravity

By experiment, objects fall near the earth's surface with a vertical acceleration which is constant unless air resistance is negligible.

This acceleration is denoted by g and is called the acceleration due to gravity or the gravitational acceleration. The approximate magnitude of g is $9.8m/sec^2$.



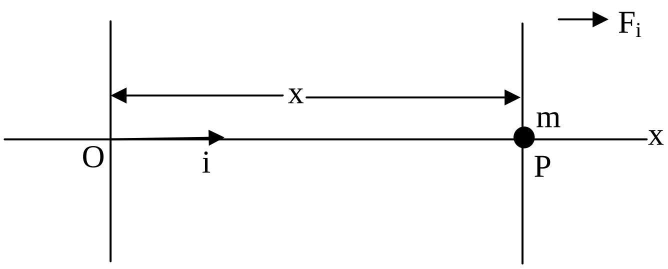
Assuming the surface of the earth is represented by the xy plane in the figure above, the force acting on a particle of mass m is given by
w = -mgk.

This force, which is called the weight of the particle, has magnitude w = mg.

SELF ASSESSMENT EXERCISE 1

A particle of mass m moves along a straight line under the influence of a constant force of magnitude F. If its initial speed is V_0, find (i) the speed, (ii) the velocity and (iii) the distance travelled after time t.

Solution



(i) Let's assume that the straight line along which the particle P moves is the x-axis and suppose that at time t, the particle is at a distance x from origin O. If I is a unit vector in the direction OP and V is the speed at time t, then velocity is V_i.

By Newton's second law we have

$$\frac{d}{dt}(mvi) = Fi \text{ or } m\frac{dv}{dt} = F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\text{Thus, } dv = \frac{F}{m}dt \text{ or } \int dv = \frac{F}{m}dt$$

$$\text{i.e. } V = \frac{F}{m}t + C_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where C_1 is a constant of integration. To find C_1 we note the initial condition that $V = V_0$ at $t = 0$ so that from (2) , $C_1 = V_0$ and

$$V = \frac{F}{m}t + V_0 \text{ or } V = V_0 + \frac{F}{m}t \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

(ii) From (3) the velocity at time t is

$Vi = V_{0i} + \frac{F}{m} ti$ or $V = V_0 + \frac{F}{m} t$

where $V = Vi$, $V_0 = V_{0i}$ and $F = Fi$.

(iii) Since $V = \frac{dx}{dt}$ we have from (3),

$\frac{dx}{dt} = V_0 + \frac{F}{m} t$ or $dx = (V_0 + \frac{F}{m} t)\, dt$

then on integrating, assuming C_2 to be the constant of integration, we have

$$x = V_0 t + \left(\frac{F}{2m}\right) t^2 + C_2$$

Since x = 0 at t = 0, we find $C_2 = 0$

Thus,

$$x = V_0 t + \left(\frac{F}{2m}\right) t^2 \text{ -------------------------- } (4)$$

SELF ASSESSMENT EXERCISE 2

From example (1), show that the speed of the particle at any position x is given by $V = \sqrt{V_0^{\,2} + \left(\frac{2F}{m}\right) x}$

Solution

From example 1,

$$m \frac{dv}{dt} = F$$

$$\frac{dv}{dt} = \frac{F}{m}, \text{ i.e. } \frac{dv}{dx} . \frac{dx}{dt} = \frac{F}{m}$$

or since $V = \frac{dx}{dt}$,

$$V \frac{dv}{dx} = \frac{F}{m}, \text{ i.e. } Vdv = \frac{F}{m} dx$$

Integrating, $\frac{v^2}{2} = \frac{F}{m}x + C_3$

Since V = V_0 when x = 0, we find $C_3 = \frac{V_0^{\ 2}}{2}$ and hence

$$V = \sqrt{V_0^{\ 2} + \left(\frac{2F}{m}\right)x}$$

4.0 CONCLUSION

This unit is concluded thus, going by example 1 in section 3.2 above, you will observed that use is made of Newton's second law which has been discussed earlier in unit 1 of this module. This is to advise the reader that the units are interrelated.

5.0 SUMMARY

Uniform force fields which have constant magnitude and direction are defined as uniform constant force field. Also **uniformly accelerated motion** is defined as motion of a constant mass m in which its acceleration is uniform or constant. Using Newton's second law we have,

$$a = \frac{F_0 k}{m}.$$

Lastly, we show that the approximate magnitude of acceleration due to gravitational force as 9.8m/sec^2.

Objects fall near the earth's surface with a vertical acceleration which is constant unless air resistance is negligible.

This acceleration is denoted by g and is called the acceleration due to gravity or the gravitational acceleration. The approximate magnitude of g is 9.8m/sec^2.

6.0 TUTOR-MARKED ASSIGNMENT

1. State Newton's laws and give their mathematical formulae where they are applicable.
2. An object of mass m is thrown vertically upward from the earth's surface with speed V_0.
3. Find (i) the position at anytime t (ii) the time taken to reach the highest point and (iii) the maximum height reached.

7.0 REFERENCES/FURTHER READING

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MODULE 6

UNIT 1 REDUCTION OF COPLANAR FORCES ACTING ON A RIGID BODY TO A FORCE AND A COUPLE

CONTENTS

1.0 INTRODUCTION

Coplanar forces are forces that lie in the same plane. A rigid body is a body that is made up of many particles which are at fixed distances from each other. Examples include, metre rule, turning fork, writing desks, towers supporting a suspended bridge, landing gear of an aircraft etc.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- prove that any system of coplanar forces acting on a rigid body can be reduced to single force on a single couple
- state that the sum of the moment of the forces of the system about any point on the line of action of the resultant will be zero.

3.0 MAIN CONTENT

3.1 Reduction of Coplanar Forces Acting on a Rigid Body to a Force

Theorem 1

Any system of coplanar forces acting on a rigid body can be reduced to single force on a single couple.

Proof

By the parallelogram of forces, or theorems on resultant parallel forces, the resultant of any two forces P.Q may be found, this continues with a third force R will determine the resultant of the first (P).

Or

We can reduce any three forces P, Q and R to two.We cannot compound P with either Q or R, unless P forms a couple with each of them. In this case, Q and R are equal, parallel, and like forces (for each is in the opposite direction to P), and therefore Q and R can be compounded.

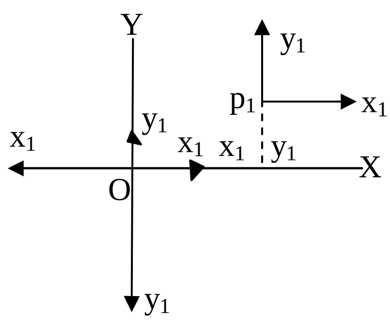
By taking another force of the system with the two forces obtained and by repeating this process we shall obviously reduce the system to two forces which, if not in equilibrium must either form a couple or have a single resultant.

3.2 Analytical Representation

With reference to rectangular axes through a given point O, let the forces of the system in figure below be P_1, P_2, P_3…..

Let the forces act at the points (x_1, y_1), (x_2, y_2), x_3, y_3) etc. respectively.

Let P_1, P_2, P_3……parallel to the axes be x_1 and y_1, x_2 and y_2, x_3 and y_3…. Then, introducing at O the equal and opposite forces x_1 parallel to OX and two equal and opposite forces parallel to OY. Then we have forces x_1 and y_1 acting at O and two couples whose moments are x_1 y_1 and $-y_1$ x_1.



If the resultant of all forces acting at O has components X and Y parallel to the axes, then

$$X = x_1 + x_2 + x_3 + \ldots\ldots\ldots.. = \Sigma x_1,$$
$$Y = y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots = \Sigma y_1.$$

Also, compounding the couple into a single couple G whose moment is the sum of the moment of all the couples, that is,

$$G = [x_1\, y_1 - y_1\, x_1] + [x_2\, y_2 - y_2\, x_2] + [x_3\, y_3 - y_3\, x_3] + \ldots\ldots\ldots$$
$$= \Sigma(x_1\, y_1 - y_1\, x_1].$$

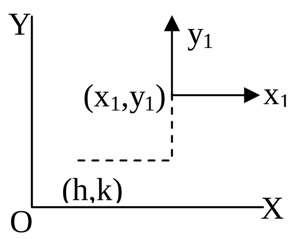
The forces X and Y are equivalent to a single force parting at O, where $P = \sqrt{(x^2 + y^2)}$. Hence, the system reduces to a single force Pat O and a couple G.

3.2.1 Theorem

If the system of forces reduces to a single force its line of action may be found.

The sum of the moment of the forces of the system about any point on the line of action of the resultant will be zero.

The moment of P_1 at $(x_1\ y_1)$ about the point (h,k) is



$(x_1\text{-}h) - (y_1\text{-}k)x_1$

$= (x_1y_1 - y_1x_1) - hy_1 + kx_1$

Hence, the sum of the moments of all the forces about the point (h,k) is $\Sigma(x_1y_1 - y_1x_1) - h\Sigma y_1 + k\Sigma x_1$

$= G - hy + kx$

If the point (h,k) lies on the resultant, we have

$G - hy + kx = 0$
and finally,

$G - xY - yX = 0$

This equation determines a definite straight line unless the coefficient of x and y are both zero, and this is the case where the system reduces to a couple.

3.3 Centre of Mass of Simple Bodies

Definition

Let r_1, r_2……..R_N be the position vectors of a system of N particles of masses.

m_1, m_2….m_N respectively. The centre of mass of the system of particles is defined as that point C having position vector

$$\bar{r} = \frac{m_1 r_1 + m_2 r_2 + \ldots\ldots + m_N r_N}{m_1 + m_2 + \ldots\ldots + m_N}$$

$$\bar{r} = \frac{1}{M} \sum_{i=1}^{N} m_i r_i$$

where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system.

Definition

The continuous systems of particles occupying a region R of space in which the volume density is σ, the centre of mass can be written

$$\bar{r} \; \frac{\int_R \sigma r dr}{\int_R \sigma dr} \text{ --------------- } (1)$$

where the integral is taken over the entire region R.

If we write that $\bar{r} = \bar{x} i + \bar{y} i + \bar{z} k$, rv = xvi + yvi + zvk

then (1) above can equivalently be written as

$$\bar{x} = \frac{\Sigma mvxv}{m}, \quad \bar{y} = \frac{\Sigma mvyr}{m}, \quad \bar{z} = \frac{\Sigma mv\,z\,r}{m} \text{ ------- (2)}$$

and $\bar{x} = \dfrac{\int_R \sigma x dr}{m}, \quad \bar{y} = \dfrac{\int_R \sigma y dr}{m}, \quad \bar{z} = \dfrac{\int_R \sigma z dr}{m}$, ------- (3)

where total mass is given by $m = \Sigma mv$ or $m = \int_R \sigma dr$ ------ (4)

Remarks

(a) The integrals in (1), (3) or (4) can be single, double or triple integrals.

(b) If a system of particles is in a uniform gravitational field, the centre of mass is sometimes called the ***Centre of Gravity.***

3.4 Motion of the Centre of Mass

Suppose that the internal forces between any two particles of the system obey Newton's third law. Then, if F is the resultant external force acting on the system, we have

$$F = \frac{d}{dt}P = m\frac{d^2\bar{r}}{dt^2} = m\frac{dv}{dt}$$

where P is the linear momentum.

Remarks

(a) The acceleration of the centre of mass of a system of particles is the same as that of a single particle having a mass equal to the total mass of the system and acted upon by the sum of the external forces.

(b) If we consider as an example, a swarm of particles moving in a uniform gravitational field. Then

$$\sum F_i = \sum m_i g$$
$$= M\,g$$

4.0 CONCLUSION

This unit deals with the Reduction of Coplanar Forces Acting on a Rigid Body to a Force.

5.0 SUMMARY

In the prove of the first theorem above, we remark that by taking another force of the system with the two forces obtained and by repeating this process we shall obviously reduce the system to two forces which, if not in equilibrium must either form a couple or have a single resultant. Also, this remark is made in respect of the theorem 3.2.1. Thus, the sum of the moment of the forces of the system about any point on the line of action of the resultant will be zero.
More so, the centre of mass of the system of particles is defined as that point C having position vector

$$\bar{r} = \frac{m_1 r_1 + m_2 r_2 + ... + m_N r_N}{m_1 + m_2 + ... + m_N}$$

$$\bar{r} = \frac{1}{M} \sum_{i=1}^{N} m_i r_i$$

where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system.

10.0 TUTOR-MARKED ASSIGNMENT

Differentiate between centre of mass and centre of gravity.

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UNIT 2 MOMENT OF A FORCE

CONTENTS

1.0 INTRODUCTION

The moment of a force about a given point is the product of the force and the perpendicular drawn from the given point to the line of action of the force.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define Moment of a Force
- define the term, "Couples"
- state the Moment of a Couple
- prove the Equilibrium of a Particle.

3.0 MAIN CONTENT

3.1 Moment of a Force

When a number of forces are acting on a body, the algebraic sum of their moments is obtained by giving the value of the moment of each force its proper sign and adding them together. Moment of a force is a vector quantity as it has both magnitude and direction. Its unit is NM.

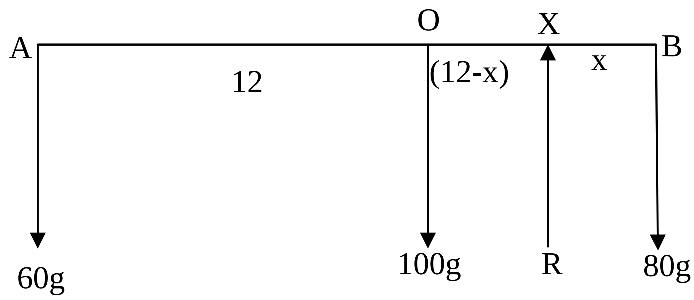
The Principle of Moments can be verified experimentally by applying known forces to a rigid body, such as bar, and making appropriate measurements. Therefore, principle of moments can be used as the basis of static.

SELF ASSESSMENT EXERCISE 1

A uniform beam is 24m long and has a mass 100kg and masses of 60kg and 80 kg are suspended from its ends; at what point must the beam be supported so that it may rest horizontally?

Solution

Let AB be the beam, O its centre of gravity



We need a point about which the moments of the three weights balances and let that point be X. Let R be the supporting force R acting on the beam at X.

$\Rightarrow$ R = (60 + 100 + 80)g
= 240gN

Let BX = xm, then if we take moment about X,
80x = 100 (12-x) + 60 (24-x)
80x = 1200 – 100x + 1440 – 60x
80x = 2640 – 160x
240x = 2640
X = 11

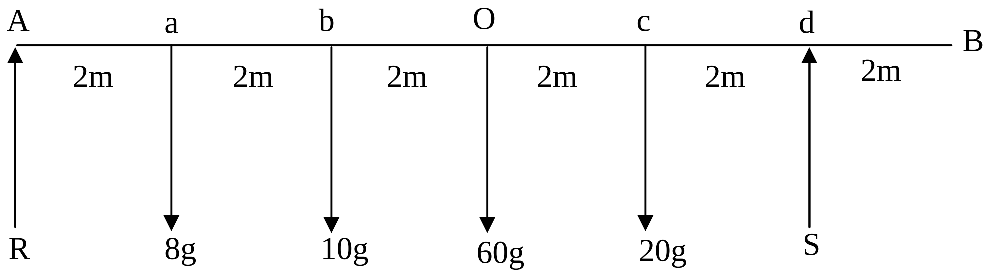
Alternatively, the position of X can be obtained by taking moments of all the forces about one end of the rod.

Hence, taking moments about B,
RX = 100g x 12 + 60g x 24
= (1200 + 1440)g
240X = 2640
X= 11

SELF ASSESSMENT EXERCISE 2

A uniform rod AB, of length 12m, and of mass 60kg, rests or two supports, one at A and the other 2m from B. Masses of 8, 10 and 20kg are attached at points 2m, 4,, and 8m respectively from A. Find the thrust on the supports.

Solution



Let c = the position of the other support
Let o = the centre of gravity of the rod
Let d, e, f = the points where the masses are attached

Then R and S are the reactions at A and B in Newton's respectively.

Taking moments about A,

20s = 8g x 2 + 10g x 4 + 60g x 6 + 20g x 8
= (16 + 40 + 360 + 160)g
20S = 576
S = 28.8

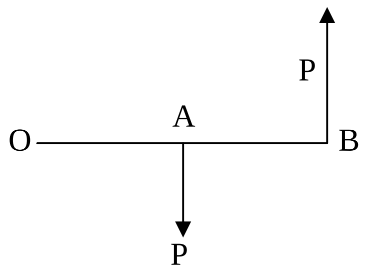
∴ taking moments about C,

20R = 20g x 2 + 60g x 4 + 10g x 6 + 8g x 8
= (40 + 240 + 60 + 64)
20R = 404
R = 20.2

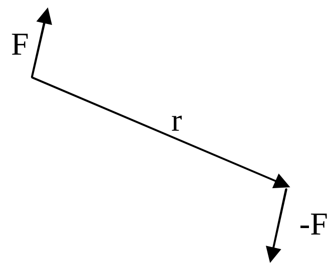
3.2 Couples

Couple is a term used to denote two equal unlike paralleled forces whose line of action are not the same.

Consider the figure below,



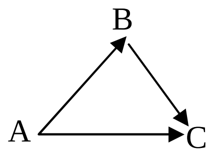
OR



0. Couple consists two unlike equal lines

1. Different sense
2. Different directions
3. The same magnitude

Let P, P be the forces acting as shown above
Let O be any point interior plane
Draw OAB perpendicular to the forces to meet their lines of action in A and B the sum of the moment about O is



AB = AC = BA
AB = CB

P x OB – P x OA (i.e. clockwise and anticlockwise)

= P(OB – OA)


P x AB
and it is independent of the position of O.

3.3 Moment of a Couple

The moment of a couple about any point in the plane of the forces is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces.

In above, the product PXAP, whose P is the magnitude of either of the forces of the couple, and AB is the perpendicular distance between the forces, is called the moment of the couple. This may be positive or negative, depending on the sense of rotation of the couple.

Remarks

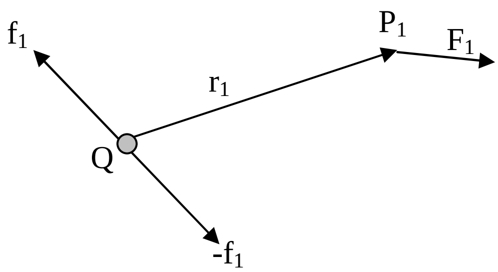
A couple has turning point effect.

Theorem

A force acting at a point of a rigid body can be equivalently replaced by a single force acting at some specified point together with a suitable couple.

Proof

Let the force be F_1 acting at Point P_1 as in the figure below:



If Q is any specified point, it is seen that the effect of f_1 alone is the same if we apply two forces f_1 and $-f_1$ at Q.

In particular if we choose $f_1 = -f_1$ and if F_1 has the same magnitude as F_1 but is opposite in direction, we see that the effect of F1 alone is the same as effect of the couple formed by F_1 and $f_1 = -f_1$ (which has moment r_1 x F_1) together with the force $-f_1 = f_1$.

3.4 Equilibrium of a Particle

Conditions of Equilibrium of any Number of Forces Acting on a Particle

If we resolve the forces in any two directions at right angles and the sums of the components in these directions be X and Y, the resultant F is given by

$$F^2 = X^2 + Y^2$$

But if the forces are in equilibrium (see backed) F must be zero.

Now, it must be noted that the sum of the squares of two real quantities cannot be zero, unless each quantity is separately zero, Therefore,

$$X = 0 \text{ and } Y = 0$$

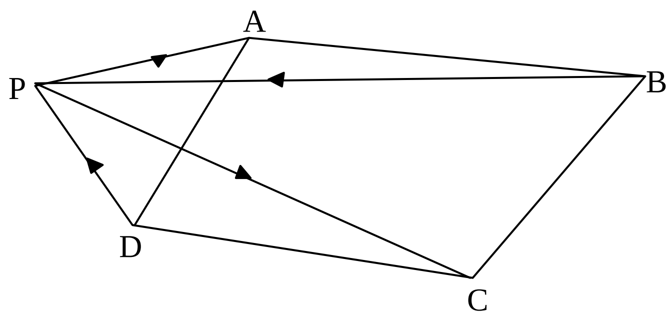
Then we conclude that if any numbers of forces acting on a particle are in equilibrium the algebraic sums of their components in any two directions at right angles must separately vanish.

Conversely, if the sums of their components in two directions at right angles are both zero the forces are in equilibrium then for both X and Y zero, therefore F is zero.

SELF ASSESSMENT EXERCISE 3

ABCD is a parallelogram and P is any point. Prove that the system of forces represented by PA, BP, PC, DP is in equilibrium.

Solution



In above diagram, the resultant (a single force which replaces 2 or more forces) of the forces represented by BP, PA is represented in magnitude and direction by BA. This resultant acts at P.

Again, the resultant of the forces represented by DP, PC is represented in magnitude and direction by DC and act at P.

Now, AB is equal and parallel to DC, so that the resultants are equal in magnitude, and as they act at the same point P, they are in the same straight line. Since their directions are opposite, they will balance and the system is in equilibrium.

Vectorially, we can write that the vector sum of the forces

$$\begin{aligned} &= (BP + PA) + (DP + PC) \\ &= BA + DC \\ &= O \end{aligned}$$

(This is because AB and DC are equal and parallel and the two vectors are oppositely directed).

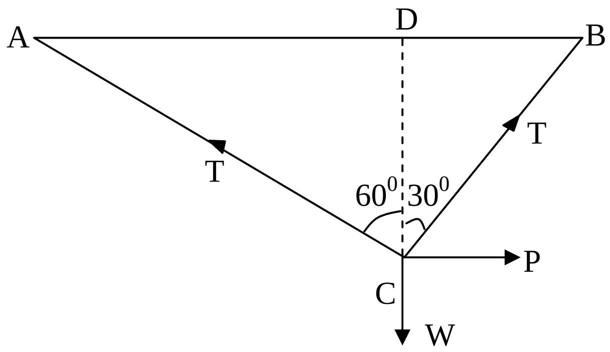
$\therefore$ Since the forces act at a point and their vector-sum is zero, they are in equilibrium.

Q.E.D.

SELF ASSESSMENT EXERCISE 4

A string is tied to two points at the same level, and a smooth ring of weight W which can slide freely along the string is pulled by a horizontal force P. If in the position of equilibrium, the portions of the string are inclined at angles 60^0 and 30^0 to the vertical, find the value of P and the tension in the string.

Solution



AB are the 2 points, C is the position of the ring CD perpendicular to AB Tension T is smooth.

Resolving vertically,

T cos 30^0 + T cos 60^0 = W

$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right) T = W$$

$$T = \frac{2W}{\sqrt{3}+1}$$

Rationalize to give T = W $(\sqrt{3}-1)$.

Resolving horizontally,

P + T sin 300 = T sin 600

$$P = \frac{\sqrt{3}}{2}T - \frac{1}{2}T = \frac{T}{2}(\sqrt{3}-1)$$

$P = W(2-\sqrt{3})$

SELF ASSESSMENT EXERCISE 5

E is the mid-point of the side CD of a square ABCD. Forces 16, 20, P, QN act along AB, AD, EA, CA in the directions indicated by the order of the letters. Find P and Q, if the forces are in equilibrium.

Solution

$\lfloor 12\sqrt{5},\ 4\sqrt{2} \rfloor$.

Remarks

An important special case of motion of a particle occurs when the particle is, or appears to be, at rest or in equilibrium with respect to an initial co-ordinate system or frame of reference.
A necessary and sufficient condition for this is, from Newton's second Law, that F = 0

i.e. the net (external) force acting on the particle is zero.

Also $\wedge = 0$ i.e. external torque on the particle is zero.

SELF ASSESSMENT EXERCISE 6

A particle moves along the x axis in a force field having potential
$V = \frac{1}{2}kx^2$, $k > 0$. Determine the point of equilibrium.

Solution

Equilibrium points occur where

$\nabla V = 0$ or in this case

$$\frac{dv}{dx} = kx = 0$$

or $x = 0$

Thus there is only one equilibrium point, at $x = 0$

3.5 Linear Motion

In addition to our earlier study, we consider a particle moving in a straight line so that its distance from a fixed point 0 in the straight line is x after time t, we than have velocity V at time t as $\frac{dx}{dt}$ along that straight line away from 0.

Again its acceleration at time t in the same direction is $\frac{dv}{dt}$, thus

$$V = \frac{dx}{dt}, \; a = \frac{dv}{dt} = \frac{d.}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

Therefore, it is possible to find the acceleration in terms of x or t by differentiation if the distance x is given as a function of the time, or the velocity V as a function of the distance or time.

3.6 Motion in a Straight Line

Let S be the distance moved in time t, then if the acceleration has a constant value a, we obtain

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = V\frac{dv}{ds} = a$$

Integrate, with respect to t, we have

$$V = \frac{ds}{dt} = at + \text{ constant}$$

If u be the value of the velocity at time t = 0, we have

$V = u + at$ .... (1)

Writing $\frac{ds}{dt}$ for V and integrating again w.r.t. t,

$S = ut + ½ at^2 + \text{constant}$ .... (2)

Also if we write $V\frac{dv}{ds} = a$, we integrate w.r.t.s to obtain $½V^2$ + a s t constant

If u is initial velocity when s = 0, then our constant = $½ u^2$

Then $V^2 = u^2 + 2aS$ .... (3)

Equation (1) to (3) above determines the motion of particle moving in straight line with CONSTANT acceleration. But if the acceleration is VARIABLE, the equation does not hold. Therefore when the acceleration is given as a function of the time, velocity expressions for the acceleration and finds the velocity and the distance by integration.

SELF ASSESSMENT EXERCISE 7

A train moving with constant acceleration passes three posts, A, B, C on a straight road. The distance from A to B is 15m, and from B to C 20m. The train takes 6 sec to go from A to B and 5 sec to go from B to C. Find the acceleration of the train and its distance from A when its speed is 5.5 m/sec.

Solution

Assume Vm/sec be initial velocity of the train and $\propto$m/sec^2 its acceleration. We have for the two stages A to B and A to C:

A to B	A to C
u = Vm/sec	u = Vm/sec
v = ?	v = ?
a = $\propto$m/s^2	a = $\propto$m/s^2
s = 15m	s = 35m
t = 16sec	t = 11sec

Applying our formula $s = ut - ½ at^2$ to each of the stages A to B and B to C we have

$15 = 6v + 18\propto$
$35 = 11v + 60.5\propto$

Solving simultaneously, we have that

$\propto = 3/11$ m/s^2 and $V = 1^{15}/_{22}$m/s.

If the speed be 5.5m/sec at a point X, we have for the stage

A to X, u = 37/22m/s, v = 5.5m/s, a = 3/11m/sec^2
s = ? and t = ?

Applying formula $V^2 = u^2 + 2as$, we have

$$2 \text{ X } 3/11 \text{ x } s = (5.5)^2 - (1.682)^2$$

Hence, the speed is 5.5m/sec at 50.27m from A.

4.0 CONCLUSION

Moment of a force is a vector quantity as it has both magnitude and direction. Its unit is NM. The Principle of Moments can be verified experimentally by applying known forces to a rigid body, such as bar, and making appropriate measurements.

5.0 SUMMARY

In summary**,** the Principle of Moments can be verified experimentally by applying known forces to a rigid body, such as bar, and making appropriate measurements. Therefore, principle of moments can be used as the basis of static.

Couple is a term used to denote two equal unlike paralleled forces whose line of action are not the same.

The moment of a couple about any point in the plane of the forces is equal to the product of one of the forces and the perpendicular distance between the lines of action of the forces. it is remarked further that A couple has turning point effect as it is proved in the foregoing unit.

THEOREM: Force acting at a point of a rigid body can be equivalently replaced by single force acting at some specified point together with a suitable couple.

6.0 TUTOR-MARKED ASSIGNMENT

1. An electric train starts from rest at a station and come to rest at the next station, one kilometre away, in 3 min. It has first a uniform acceleration for 40 sec, then a constant speed and it is brought to rest by a constant retardation for 20 sec. Find the maximum speed of the train and retardation when coming to rest.
2. A train moving with constant acceleration passes three posts, A, B, and C on a straight road. The distance from A to B is 35m, and from B to C 30m. The train takes 8 sec to go from A to B and 5 sec to go from B to C. Find the acceleration of the train and its distance from A when its speed is 10 m/sec.

7.0 REFERENCES/FURTHER READING

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Vladinirou, U.S. *Generalised Function Mathematical Physics.*

MODULE 7 HAMILTONIAN THEORY

UNIT 1 THE HAMILTONIAN

CONTENTS

1.0 INTRODUCTION

We investigated a formulation of mechanics due to Lagrange. In this unit, we shall investigate a formulation due to Hamilton known collectively as Hamiltonian methods or Hamiltonian theory. Although such theory can be used to solve specific problems in mechanics, it develops that it is more useful in supplying fundamental postulates in such fields as quantum mechanics, statistical mechanics and celestial mechanics.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define the Hamiltonian
- state the Hamiltonian for Conservation Systems
- define Ignorable or Cyclic Coordinates
- define phase space
- state Liouville's theorem.

3.0 MAIN CONTENT

3.1 The Hamiltonian

Just as the Lagrangian function, or briefly the Lagrangian, is fundamental to the previous module so the Hamiltonian function, or briefly the Hamiltonian, is fundamental to this unit.

The Hamiltonian, symbolised by *H*, is defined in terms of the Lagrangian *L* as

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \tag{1}$$

It must be expressed as a function of the generalised coordinate's q_α and generalised moment p_α. To accomplish this the generalised velocities $\dot{q}_\alpha$ Must be eliminated from (1) by using Lagrange's equations. In such case the function H can be written

$$H(p_1, \ldots\ldots, p_n, \quad q_1, \ldots\ldots, q_n, t) \tag{2}$$

Or
briefly, $H(p_\alpha, q_\alpha\ t)$, and is also called the Hamiltonian of the system.

3.1.1 The Hamilton's Equations

In terms of the Hamiltonian, the equations of motion of the system can be written in the symmetrical form

$$\left.\begin{aligned} \dot{p}_\alpha &= -\frac{\partial H}{\partial q_\alpha} \\ \dot{q}_\alpha &= \frac{\partial H}{\partial p_\alpha} \end{aligned}\right\} \tag{3}$$

These are called Hamilton's canonical equations, or briefly Hamilton's equations. The equations serve to indicate that the p_α and q_α play similar roles in a general formulation of mechanical principles.

The Hamiltonian for Conservation Systems

If a system is conservative, the Hamiltonian H can be interpreted as the total energy (kinetic and potential) of the system i.e.,

$$H = T + V \tag{4}$$

Often this provides an easy way for setting up the Hamiltonian of a system.

3.1.2 Ignorable or Cyclic Coordinates

A coordinate q_α which does not appear explicitly in the Lagrangian is called an *ignorable or cyclic coordinate*. In such case

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} = 0 \qquad (5)$$

So that pα is a constant, often called a *constant of the motion*.

In such case we also have $\partial H / \partial q_\alpha = 0$.

3.1.3 Phase Space

The Hamiltonian formulation provides an obvious symmetry between the p_α and q_α which we call *momentum* and *position coordinates* respectively. It is often useful to imagine a space of 2n dimensions in which a *representative point* is indicated by the 2n coordinates

$$H(p_1, \ldots\ldots, p_n, \quad q_1, \ldots\ldots, q_n) = \text{constant} = E \qquad (6)$$

Such a space is called a 2n dimensional *phase space* or a *pq* phase space.

Whenever we know the state of a mechanical system at time *t*, i.e. we know all position and momentum coordinates, then this corresponds to a particular point in phase space. Conversely, a point in phase specifies the state of the mechanical system. While the mechanical system moves in the physical 3 dimensional space, the representative point describes some path in the phase space in accordance with equations (3).

3.1.4 Liouville's Theorem

Let us consider a very large collection of conservative mechanical systems having the same Hamiltonian. In such case the Hamiltonian is the total energy and is constant, i.e.,

$$H(p_1, \ldots, p_n, q_1, \ldots q_n) = \text{constant} = E \qquad (7)$$

Which can be represented by a surface in phase space.

Let us suppose that the total energies of all these systems lie between E_1 and E_2. Then the paths of all these systems in phase space will lie between the two surfaces $H = E_1$ and $H = E_2$ as indicated schematically in the figure below.

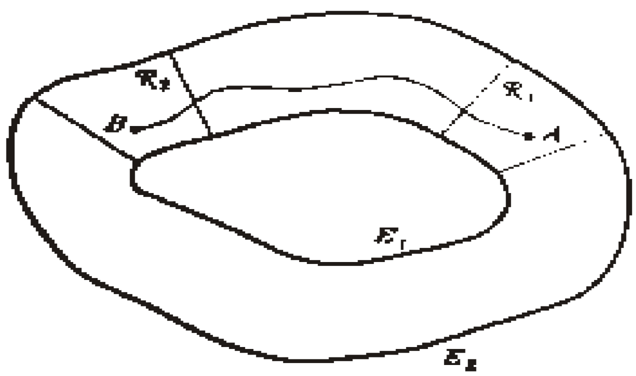

Fig. 12.1

Since the systems have different initial conditions, they will move along different paths in the phase space. Let us imagine that the initial points are contained in region $\mathcal{R}_1$ of Fig. 12 – 1 and that after time t these points occupy region$\mathcal{R}_2$. For example, the representative point corresponding to one particular system moves from point A to point B. From the choice $\mathcal{R}_1$ and $\mathcal{R}_2$ it is clear that the number of representative points in them is the same. What is not so obvious is the following theorem called Liouville’s theorem.

Liouville’s Theorem
The 2n dimensional volumes of $\mathcal{R}_1$ and $\mathcal{R}_2$ are the same, or if we define the number of points per unit volume as the density then the density is constant.

We can think of the points of $\mathcal{R}_1$ as particles of an incompressible fluid which move from $\mathcal{R}_1$ to $\mathcal{R}_2$ in time t.

4.0 CONCLUSION

The Hamiltonian formulation provides an obvious symmetry between the p_α and q_α which we call *momentum* and *position coordinates* respectively. It is often useful to imagine a space of 2n dimensions in which a *representative point* is indicated by the 2n coordinates

$H\,(p_1,\ldots\ldots, p_n,\quad q_1, \ldots\ldots, q_n,) = \text{constant} = E$

Such a space is called a 2n dimensional *phase space* or a *pq* phase space.

5.0 SUMMARY

The Hamiltonian, symbolised by *H*, is defined in terms of the Lagrangian *L* as

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \tag{1}$$

And must be expressed as a function of the generalised coordinate's q_α and generalised moment p_α. To accomplish this the generalised velocities $\dot{q}_\alpha$

Must be eliminated from (1) by using Lagrange's equations. In such case the function H can be written

$$H(p_1, \ldots\ldots, p_n, \quad q_1, \ldots\ldots, q_n, t) \tag{2}$$

$$\left.\begin{aligned} \dot{p}_\alpha &= -\frac{\partial H}{\partial q_\alpha} \\ \dot{q}_\alpha &= \frac{\partial H}{\partial p_\alpha} \end{aligned}\right\} \tag{3}$$

These equations above are called Hamilton's canonical equations, or briefly Hamilton's equations.

Liouville's Theorem

The 2n dimensional volumes of $\mathcal{R}_1$ and $\mathcal{R}_2$ are the same, or if we define the number of points per unit volume as the density then the density is constant.

11.0 TUTOR-MARKED ASSIGNMENT

1. Using surface in phase space represent Liouville's Theorem.
2. Can Hamilton be interpreted as a total energy? Give reasons for your answer.
3. Define 2n dimensional *phase space* or a *pq* phase space.

7.0 REFERENCES/FURTHER READING

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UNIT 2 THE CALCULUS OF VARIATION

CONTENTS

1.0 INTRODUCTION

A problem which often arises in mathematics is that of finding a curve y = Y (x) joining the points where $x = a$ and $x = b$ such that the integral.

$$\int_a^b F(x, y, y')dx \qquad \text{(i)}$$

Where $y' = dy/dx$, is a maximum or minimum, also called an extremum or extreme value. The curve itself is often called an external. It can be shown that a necessary condition for (i) to have an extremum is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \qquad \text{(ii)}$$

which is often called Euler's equation. This and similar problems are considered in a branch of mathematics called the calculus of variations.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- define the calculus of variation
- state the Hamilton's principle
- define canonical or contact transformations
- prove that a transformation can be canonical.

3.0 MAIN CONTENT

3.1 Calculus of Variation

An important class of problems involves the determination of one of more function subject to certain conditions, so as to maximise or minimise certain definite integral, whose integrand depends upon the unknown function(s) and /or certain of their derivatives.

For example, to find the equation y = u(x) of the curve along which the distance from (0,0) to (1,1) in the xy plane at least, we would seek u(x) such that

$$I = \int_0^1 \sqrt{\left(1 + (u')^2\right)}dx = \min$$

With u(0)=0, u(1)=1.

This section presents a brief treatment of some of the simpler aspects of such problems.

We consider first the case when we attempt to minimise or maximise an integral of the form

$$I = \int_0^1 F(x, u, u')dx \qquad 1$$

Subject to the conditions

u(o)=A and u(1)= B 2

where a, b, A and B are given constants to be determined. We suppose that F has continuous second-order derivatives with respect to its three arguments and require that the unknown function u(x) possesses two derivatives everywhere in (a,b). To fix ideas, we suppose that I is to be maximised.

We thus visualise a competition, to which only functions which have two derivatives in (a,b) and which take on the prescribed end values are admissible.

The problem is that of selecting from all admissible competing functions, the function(s) for which I is largest.

Under the assumption that there is indeed a function u(x) having this property, we next consider a one-parameter family of admissible functions which includes u(x), namely, the set of all functions of the form $u(x)+\in\eta(x)$, where $\eta(x)$ is any arbitrary chosen twice differentiable function which vanishes at the end points of the interval (a,b)

$$\eta(a)=\eta(b)=0 \qquad 3$$

and where $\in$ is a parameter which is constant for any one function in the set but which varies from one function to another.

The increment $\in\eta(x)$, representing the difference between the varied function and the actual solution function, is often called a variation of u(x).

If the result of replacing $u(x)+\in\eta(x)$ in I is denoted by

$$I(\in)=\int_a^b F(x,u+\in\eta,u')+\in\eta' dx \qquad 4$$

It then follows that $I(\in)$ takes on its maximum value when $\in$ =0, that is, when the variation of u is zero. Hence, it must follow that

$$\frac{dI(\in)}{d\in}=0 \text{ , when } \in=0 \qquad 5$$

The assumed continuity of the partial derivatives of F with respect to its three arguments implies the continuity of $\frac{dF}{d\in}$, so that we may differentiate $I(\in)$ under the integral sign to obtain

$$I'(0)=\int_a^b\left[\frac{\partial F}{\partial u}\eta(x)+\frac{\partial F}{\partial u'}\eta'(x)\right]dx=0 \qquad 6$$

Here we write, $F\equiv F(x,u,u')$, noticing that the partial derivatives $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial u'}$ have been formed with x, u and u' treated as independent variables.

The next step consists of transforming the integral of the second product (6) by integration by parts, to give

$$\int_a^b\frac{\partial F}{\partial u'}\eta'(x)dx=\left[\frac{\partial F}{\partial u'}\eta(x)\right]_a^b-\int_a^b\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right)\eta(x)dx$$

$$=-\int_a^b \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right)\eta(x)dx=0$$

in consequence of (3). Hence, equation (6) becomes

$$\int_a^b \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right)-\frac{\partial F}{\partial u}\right]\eta(x)dx=0 \qquad 7$$

It is possible to prove rigorously that since (7) is true for any function $\eta(x)$ which is twice differentiable in (a,b) and zero at the ends of that interval; consequently, the coefficient of $\eta(x)$. In the integrand must be zero everywhere in (a,b), so that the condition

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right)-\frac{\partial F}{\partial u}=0 \qquad 8$$

must be satisfied. This is called Euler's equation.

SELF ASSESSMENT EXERCISE 1

We seek to minimise the integral

$$I=\int_0^{\frac{\pi}{2}}\left[\left(\frac{dy}{dt}\right)^2-y^2+2ty\right]dt \qquad 9$$

With y(0)=0 and $y(\pi/2)=0$.

The Euler's equation $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right)-\frac{\partial F}{\partial u}=0$ 10

with u and x replaced by y and t, respectively, becomes

$$\frac{d}{dt}\left(2\frac{dy}{dt}\right)-(-2y+2t)=0 \qquad 11$$

Or

$$\frac{d^2y}{dt^2}+y=t, \qquad 12$$

from which there follows $y=c_1\cos t+c_2\sin t+t$ 13

The end conditions then gives

$y = t - (\pi/2)\sin t.$ in correspondence with which 14

$$I_{\min} = -\frac{\pi}{2}\left(1 - \frac{\pi^2}{1^2}\right). \qquad 15$$

Generalisations in which more dependent and/or independent variables are involved or which involves other modifications, as well as formulations of sufficiency conditions, may be found in the literature.

Two such generalisations, which are particularly straightforward, may be described here:

(a) If the equation

$I = \int_a^b F(x, u, u')dx$ is replaced by the integral 16

$$I = \int_a^b F(x, u_1, u_2, u_3, \ldots\ldots\ldots u_n; u_1', u_2', u_3' \ldots\ldots u_n')dx \qquad 17$$

Where values of the n independent unknown functions $u_1(x), u_2(x), u_3(x), \ldots\ldots\ldots u_n(x)$ are each given at the end points x=a and x=b, we obtain an Euler's equation similar to

$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0$. In correspondence with each u_r 18

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_r'}\right) - \frac{\partial F}{\partial u_r} = 0, \quad \text{where } r = 1, 2, \ldots n.$$
19

SELF ASSESSMENT EXERCISE 2

The Euler's equation associated with b the integral

$$\int_a^b \left(u_1'^2, u_2'^2 - 2u_1u_2 + 2xu_1\right)dx \qquad 20$$

are obtained by use of equation(19) in the form.

$$\frac{d}{dx}(2u_1') - (-2u_2 + 2x) = 0 \qquad 21$$

and

$$\frac{d}{dx}(2u_1') - (-2u_1) = 0 \qquad 22$$

Or

$$u_1'' + u_2 = x, and. u_2'' + u_1 = 0. \qquad 23$$

(b) suppose that we are to maximise or minimise

$$I = \int_a^b F(x, u, u')dx, \qquad 24$$

$$I = \int_a^b F(x, u, u')dx = \text{maximise or minimise.} \qquad 25$$

Where u(x) is to satisfy the prescribed end conditions u(a)=A and u(b)=B as before, but that also a constraint condition is imposed in the form

$$\int_a^b G(x, u, u')dx = k. \qquad 26$$

where k is a prescribed constant. In this case, the appropriate Euler's equation is found to be the result of neglecting F in equation (19) by auxiliary function

$$H = F + \lambda G. \qquad 27$$

Where, λ is an unknown constant. This constant, which is of the nature of Lagrange multiplier, this generally will appear in the Euler's equation and in its solution and is to be determined together with the two constants of integration such a way that the three conditions of u(a)=A, u(b)=B and $\int_a^b G(x, u, u')dx = k$ are satisfied.

SELF ASSESSMENT EXERCISE 3

To minimise the integral $\int_0^1 y^2 dx$

subject to the end conditions y(0)=0, y(1)=0 28

And also to the constant $\int_0^1 y dx = 1$. 29

We write $H = y'^2 + y$, in correspondence with which the Euler's equation is

$$2y'' - \lambda = 0 \qquad 30$$

Hence, y must be of the form

$$y = \frac{1}{4}\lambda x^2 + c_1 x + c_2 \qquad 31$$

The end conditions and the constraints condition yields $c_1 = 6, c_2 = 0. and. \lambda = 2x$,

and hence there follows y=6x(1-x). 32

3.2 Hamilton's Principle

The obvious similarity of (9) of Lagrange's equations leads one to consider the problem of determining the externals of

$$\int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)\, dt \qquad 1$$

or briefly

$$\int_{t_1}^{t_2} L\, dt \qquad 2$$

Where L = T – V is the Lagrangian of a system.

We can show that a necessary condition for an external is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \qquad 3$$

which are precisely Lagrange's equations. The result led Hamilton to formulate a general variational principle known as **Hamilton's Principle**. A conservative mechanical system moves from time t_1 to time t_2 in such a way that

$$\int_{t_1}^{t_2} L\, dt \qquad 4$$

sometimes called the action integral, has an extreme value.

Because the extreme value of (4) is often a minimum, the principle is sometimes referred to as Hamilton's principle of least action.

The fact that the integral (4) is an extremum is often symbolised by stating that

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad 5$$

Where δ is the variation symbol.

3.3 Canonical or Contact Transformations

The ease in solution of many problems in mechanics often hinges on the particular generalised coordinates used. Consequently, it is desirable to examine transformations from one set of position and momentum coordinates to another. For example, if we call qα and pα the old position and momentum coordinates while Q_α and P_α are the new position and momentum coordinates, the transformation is

$$P_\alpha = P_\alpha\,(p_1, \ldots, p_n, q_1, \ldots, q_n, t),\;\; Q_\alpha = Q_\alpha\,(p_1, \ldots, p_n,\; q_1, \ldots, q_n, t) \qquad 6$$

denoted briefly by

$$P_\alpha \;= P_\alpha\,)p_\alpha, q_\alpha, t),\;\; Q_\alpha = Q_\alpha\,(p_\alpha, q_\alpha, t) \qquad 7$$

We restrict ourselves to transformations called canonical or contact transformations for which there exists a function $\mathcal{H}$ called the Hamiltonian in the new coordinates such that

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha},\;\; \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \qquad 8$$

In such case we often refer to Q_α and P_α as canonical coordinates.

The Lagrangian's in the old and new coordinates are L (p_α, q_α, t) and $\mathcal{L}(P_\alpha, Q_\alpha, t)$ respectively. They are related to the Hamiltonians H (p_α, q_α, t) and $\mathcal{H}(P_\alpha, Q_\alpha, t)$ by the equations

$$H = \sum p_\alpha \dot{q}_\alpha - L,\;\; \mathcal{H} = \sum P_\alpha \dot{Q}_\alpha - \mathcal{L} \qquad 9$$

Where the summations extend from α = 1 to *n*.

3.4 Condition that a Transformation be Canonical

The following theorem is of interest.

Theorem 12.2. The transformation

$$P_\alpha = P_\alpha(p_\alpha, q_\alpha, t), \quad Q_\alpha = Q_\alpha(p_\alpha, q_\alpha, t) \qquad 10$$

is canonical if $\sum p_\alpha \, dq_\alpha - \sum P_\alpha \, dQ_\alpha$ 11

in an exact differential.

3.5 Generating Functions

By Hamilton's principle the canonical transformation (5) or (6) must satisfy the conditions that

$\int_{t_1}^{t_2} L\,dt$ and $\int_{t_1}^{t_2} \mathcal{L}\,dt$ are both extrema, i.e. we must simultaneously have

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \text{ and } \delta \int_{t_1}^{t_2} \mathcal{L}\,dt = 0 \qquad 12$$

These will be satisfied if there is a function $\mathcal{G}$ such that

$$\frac{d\mathcal{G}}{dt} = L - \mathcal{L} \qquad 13$$

We call $\mathcal{G}$ a generating function.

By assuming that $\mathcal{G}$ is a function, which we shall denote by δ, of the old position co-ordinates q_α and the new momentum coordinates P_α as well as the time *t*, i.e.

$$\mathcal{G} = \delta(q_{\alpha,} P_\alpha, t) \qquad 14$$

We can prove

$$p_\alpha = \frac{\partial \delta}{\partial q_\alpha}, Q_\alpha = \frac{\partial \delta}{\partial P_\alpha}, \mathcal{H} = \frac{\partial \delta}{\partial t} + H \qquad 15$$

Where $\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha}$ 16

Similar results hold if the generating function is a function of other coordinates.

Examples of Hamilton's Principle

1. Prove that a necessary condition for $I = \int_a^b F(x, y, y')\, dx$ to be an extremum [maximum or minimum] is $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$.

Suppose that the curve which makes I an extremum is given by

$$y = Y(x), \quad a \leqq x \leqq b \qquad 17$$

Then $\quad y = Y(x) + \epsilon\, \eta\, (x) = Y + \epsilon\, \eta$
(18)

Where ϵ is independent of x , is a neighbouring curve through $x = a$ and $x = b$ if we choose

$$\eta(a) = \eta(b) = 0 \qquad 19$$

The value of I for the neighbouring curve is

$$I(\epsilon) = \int_a^b F(x, Y + \epsilon\eta, Y' + \epsilon\eta')\, dx \qquad 20$$

This is an extremum for $\epsilon = 0$. A necessary condition that this be so is that $\left.\frac{dI}{d\epsilon}\right|_{\epsilon = 0} = 0$.

But by differentiation under the integral sign, assuming this is valid, we find

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon = 0} = \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx = 0$$

Which can be written on integrating by parts as

$$\int_a^b \frac{\partial F'}{\partial x}\eta\, dx + \frac{\partial F}{\partial y'}\eta \Big|_a^b - \int_a^b \eta\, \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$= \int_a^b \eta \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\} dx = 0$$

Where we have used (19), since η is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad or \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

This is called Euler's or Lagrange's equation. The result is easily extended to the integral

$$\int_a^b F'\left(x, y'_{1,} y_2, y'_{2'} \dots y_n, y'_n\right) dx$$

and leads to the Euler's or Lagrange's equations.

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'_\alpha}\right) - \frac{\partial F}{\partial y_\alpha} = 0 \quad \alpha = 1,2, \dots, n$$

By using a Taylor's series expansion we find from (20) that

$I(\epsilon) - I(0) = \epsilon \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx$ + Higher order terms in ϵ^2, ϵ^3, etc. 21

The coefficient of ϵ in (20) is often called the variation of the integral and is denoted by

$$\delta \int_a^b F(x, y, y')\, dx$$

The fact that

$\int_a^b F(x, y, y')\, dx$ is an extremum is thus indicated by

$$\delta \int_a^b F(x, y, y')\, dx = 0$$

4.0 CONCLUSION

The ease in solution of many problems in mechanics often hinges on the particular generalised coordinates used. Consequently, it is desirable to examine transformations from one set of position and momentum coordinates to another. For example, if we call qα and pα the old position and momentum coordinates while Q_α and P_α are the new position and momentum coordinates, the transformation is

$P_\alpha = P_\alpha (p_1, \dots, p_n, q_1, \dots, q_n, t)$, $Q_\alpha = Q_\alpha (p_1, \dots, p_n, q_1, \dots, q_n, t)$

denoted briefly by $P_\alpha = P_\alpha)p_\alpha, q_\alpha, t)$, $Q_\alpha = Q_\alpha (p_\alpha, q_\alpha, t)$

5.0 SUMMARY

Obviously, similarity of equation (9) above of Lagrange's equations leads one to consider the problem of determining the extremes of

$$\int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)\, dt$$

or briefly

$$\int_{t_1}^{t_2} L\, dt$$

Where L = T – V is the Lagrangian of a system.

We can show that a necessary condition for an external is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

Which are precisely Lagrange's equations. The result led Hamilton to formulate a general variation principle known as **Hamilton's Principle**. A conservative mechanical system moves from time t_1 to time t_2 in such a way that

$$\int_{t_1}^{t_2} L\, dt$$

Sometimes called the action integral, has an extreme value.

Because the extreme value of (12) is often a minimum, the principle is sometimes referred to as Hamilton's principle of least action.

The fact that the integral (12) is an extremum is often symbolised by stating that

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

Where δ is the variation symbol?

Canonical or Contact Transformations was also discussed as the ease in solution of many problems in mechanics often hinges on the particular generalised coordinates used. Consequently, it is desirable to examine transformations from one set of position and momentum coordinates to another. For example, if we call $q\alpha$ and $p\alpha$ the old position and momentum coordinates while Q_α and P_α are the new position and momentum coordinates, the transformation is

$P_\alpha = P_\alpha\,(p_1, \ldots, p_n, q_1, \ldots, q_n, t),\; Q_\alpha = Q_\alpha\,(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$

denoted briefly by

$$P_\alpha = P_\alpha\,)p_\alpha, q_\alpha, t),\; Q_\alpha = Q_\alpha\,(p_\alpha, q_\alpha, t)$$

We restrict ourselves to transformations called canonical or contact transformations for which there exists a function $\mathcal{H}$ called the Hamiltonian in the new coordinates such that

$$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha},\; \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha}$$

In such case we often refer to Q_α and P_α as canonical coordinates.

The Lagrangians in the old and new coordinates are L (p_α, q_α, t) and $\mathcal{L}(P_\alpha, Q_\alpha, t)$ respectively. They are related to the Hamiltonians H (p_α, q_α, t) and $\mathcal{H}(P_\alpha, Q_\alpha, t)$ by the equations

$$H = \sum p_\alpha \dot{q}_\alpha - L,\; \mathcal{H} = \sum P_\alpha \dot{Q}_\alpha - \mathcal{L}$$

Where the summations extend from $\alpha = 1$ to n.

6.0 TUTOR-MARKED ASSIGNMENT

1. State the Hamilton's principle.
2. Under what condition can a transformation be canonical?
3. What do you understand by the term 'generating function'?

7.0 REFERENCES/FURTHER READING

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UNIT 3 THE HAMILTON-JACOBI EQUATION

CONTENTS

1.0 INTRODUCTION

If we can find a canonical transformation leading to $\mathcal{H} \equiv 0$, then we see from (24) that P_α and Q_α will be constants [i.e., P_α and Q_α will be ignorable coordinates]. Thus, by means of the transformation we are able to find p_α and q_α and thereby determine the motion of the system.

2.0 OBJECTIVES

By the end of this unit, you should be able to:

- state the Hamilton-Jacobi equation
- state a case where Hamiltonian is independent of time
- define phase integrals, action and angle variables.

3.0 MAIN CONTENT

3.1 The Hamilton-Jacobi Equation

The procedure hinges on findings the right generating function. From the third equation of (23) we see by putting $\mathcal{H} \equiv 0$, that this generating function must satisfy the partial differential equation

$$\frac{\partial \delta}{\partial t} + H(p_\alpha, q_\alpha, t) = 0 \qquad (1)$$

Or

$$\frac{\partial \delta}{\partial t} + H\left(\frac{\partial \delta}{\partial q_\alpha} q_\alpha, t\right) = 0 \qquad (2)$$

This is called the Hamilton-Jacobi equation.

3.1.1 Solution of the Hamilton-Jacobi Equation

To accomplish our aims we need to find a suitable solution of the Hamilton-Jacobi equation. Now since this equation contains a total of $n + 1$ independent variables, i.e. $q_1, q_2, \ldots, q_n$ and t, one such solution called the complete solution, will involve $n + 1$ constants. Omitting an arbitrary additive constant and denoting the remaining n constant by $\beta_1, \beta_2, \ldots, \beta_n$ [none of which is additive] this solution can be written

$$\delta = \delta(q_1, q_2, \ldots, q_n, \beta_1, \beta_2, \ldots, \beta_n, t) \tag{3}$$

When this solution is obtained we can then determine the old momentum coordinates by

$$p_\alpha = \frac{\partial \delta}{\partial q_\alpha} \tag{4}$$

Also, if we identify the new momentum coordinates P_α with the constants β_α, then

$$Q_\alpha = \frac{\partial \delta}{\partial \beta_\alpha} = \gamma_\alpha \tag{5}$$

Where $\gamma_\alpha, \alpha = 1, \ldots, n$ are constants.

Using these we can then find q_α as functions of $\beta_\alpha, \gamma_\alpha$ and t, which gives the motion of the system.

3.2 Case Where Hamiltonian is Independent of Time

In obtaining the complete solution of the Hamilton-Jacobi equation, it is often useful to consider the equation

$$\delta = S_1(q_1) + S_2(q_2) + \cdots + S_n(q_n) + F(t) \tag{6}$$

where each function on the tight depends on only one variable. This method, often called the method of *separation of variables*, is especially useful when the Hamiltonian does not depend explicitly on time. We then find that $F(t) = -Et$, and if the time independent part of δ is denoted by

$$S = S_1(q_1) + S_2(q_2) + \cdots + S_n(q_n) \tag{7}$$

The Hamilton-Jacobi equation (26) reduces to

$$H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = E \quad (8)$$

where **E** is a constant representing the total energy of the system.

The equation (8) can also be obtained directly by assuming a generating function *S* which is independent of time. In such case equations (8) and (1) are replaced by

$$p_\alpha = \frac{\partial S}{\partial q_\alpha}, Q_\alpha = \frac{\partial S}{\partial P_\alpha}, \mathcal{H} = H = E \quad (9)$$

where $$\dot{P}_\alpha = -\frac{\partial \mathcal{H}}{\partial Q_\alpha}, \dot{Q}_\alpha = \frac{\partial \mathcal{H}}{\partial P_\alpha} \quad (10)$$

3.3 Phase Integrals, Action and Angle Variables

Hamiltonian methods are useful in the investigation of mechanical systems which are periodic. In such case, the projections of the motion of the representative point in phase space on any $p_\alpha q_\alpha$ plane will be closed curves C_α. The line integral

$$J_\alpha = \oint_{C_\alpha} p_\alpha \, dq_\alpha \quad (11)$$

is called a phase integral or action variable.

We can show that

$$S = S(q_1, \dots, q_n, J_1, \dots, J_n) \quad (12)$$

where $$p_\alpha = \frac{\partial S}{\partial q_\alpha}, Q_\alpha = \frac{\partial S}{\partial J_\alpha} \quad (13)$$

It is customary to denote the new coordinates Q_α by ω_α so that equations (13) are replaced by

$$p_\alpha = \frac{\partial S}{\partial q_\alpha}, \omega_\alpha = \frac{\partial S}{\partial J_\alpha} \quad (14)$$

Thus, Hamilton's equations become [see equations (13) and (14)]

$$\dot{J}_\alpha = -\frac{\partial \mathcal{H}}{\partial \omega_\partial}, \dot{\omega}_\partial = \frac{\partial \mathcal{H}}{\partial J_\alpha} \quad (15)$$

Where $\mathcal{H} = E$ in this case depends only on the J_α. Then from the second equation in (15),

$$\omega_\alpha = f\alpha^t + c_\alpha \qquad (16)$$

Where f_α and c_α are constants. We call w_α angle variables. The frequencies f_α are given by

$$f_\alpha = \frac{\partial \mathcal{H}}{\partial J_\alpha} \qquad (17)$$

4.0 CONCLUSION

The Hamilton-Jacobi equation (26) reduces to

$$H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = E$$

where **E** is a constant representing the total energy of the system.

5.0 SUMMARY

The Hamilton-Jacobi equation is:

$$\frac{\partial \delta}{\partial t} + H(p_\alpha, q_\alpha, t) = 0$$

Or $$\frac{\partial \delta}{\partial t} + H\left(\frac{\partial \delta}{\partial q_\alpha} q_\alpha, t\right) = 0$$

It was also found that Hamiltonian methods are useful in the investigation of mechanical systems which are periodic. In such cases the projections of the motion of the representative point in phase space on any $p_\alpha q_\alpha$ plane will be closed curves C_α. Hence, the line integral

$$J_\alpha = \oint_{C_\alpha} p_\alpha \, dq_\alpha$$

is called a phase integral or action variable.

Lastly, the issues of phase integrals, action and angle variables were discussed extensively.

6.0 TUTOR-MARKED ASSIGNMENT

1. Define phase integrals.
2. What happens when Hamiltonian is independent of time?
3. How would you express angle variables with respect to Hamilton-Jacobi equation?

7.0 REFERENCES/FURTHER READING

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