NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH382

COURSE TITLE: MATHEMATICAL METHODS IV

COURSE GUIDE

MTH382
MATHEMATICAL METHODS IV

Course Team Dr. Bankole Abiola (Writer) - NOUN
Dr. S.O. Ajibola (Programme Leader/
Editor/Coordinator) - NOUN

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
5, Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e-mail: centralinfo@noun.edu.ng
URL: www.noun.edu.ng

Published By:
National Open University of Nigeria

First Printed 2012

ISBN: 978-058-407-2

CONTENTS **PAGE**

Introduction

Mathematical Methods IV is a continuation of MTH281, MTH381 and MTH302. It is a three -credit course. It is a compulsory course for any student majoring in mathematics at undergraduate level or B.Sc. (Education) Mathematics. It is also available to students offering Bachelor of Science (B.Sc.) Computer and Information Communication Technology. Any student with sufficient background in mathematics can also offer the course if he/she so wishes although it may not count as a credit towards graduation if it is not a required course in his/her field of study.

The course is divided into three modules as listed below:

Study Unit

Module 1

Unit 1 Ordinary Differential Equation
Unit 2 The Fixed Point Theorem
Unit 3 The Method of Successive Approximation

Module 2

Unit 1 Special Functions
Unit 2 Hyper Geometric Function
Unit 3 Bessel Function

Module 3

Unit 1 Legendry Function
Unit 2 Some Examples of Partial Different Equations

What You Will Learn in This Course

This Course Guide tells you briefly what the course is about, what course materials you will be using and how you can work with these materials. In addition, it advocates some general guidelines for the amount of time you are likely to spend on each unit of the course in order to complete it successfully. It gives you guidance in respect of your Tutor-Marked Assignment which will be made available in the assignment file. There will be regular tutorial classes that are related to the course. It is advisable for you to attend these tutorial sessions. The course will prepare you for the challenges you will meet in Mathematical Methods IV.

Course Aim

The aim of the course is to provide you with an understanding of Mathematical Methods IV. It also aims to give you a modern way of solving complex problems in Mathematics, make clear distinctions on the ways we handle problems in Real Analysis, and provide you with solutions to some problems that may arise in Engineering and other areas of human endevour.

Course Objectives

To achieve the aims set out, the course has a set of objectives. Each unit has specific objectives which are included at the beginning of the unit. You should read these objectives before you study the unit. You may wish to refer to them during your study to check on your progress. You should always look at the unit objectives after completion of each unit. By doing so, you would have followed the instructions in the unit.

Below are comprehensive objectives of the course as a whole. By meeting these objectives, you would have achieved the aims of the course as a whole. In addition to the aims above, this course sets to achieve some objectives. Thus, after going through the course, you should be able to:

- determine Existence and Uniqueness of Solutions
- solve Special Functions such as Gama Functions, Beta Functions, and Legendry Functions etc
- solve Partial Differential Equations

Working through This Course

To complete this course, you are required to read each study unit, read the textbooks and read other materials which may be provided by the National Open University of Nigeria.

Each unit contains Self-Assessment Exercises and at certain points in the course, you would be required to submit assignments for assessment purposes. At the end of the course there is a final examination. The course should take you about a total of 17 weeks to complete. Below you will find listed all the components of the course, what you have to do and how you should allocate your time to each unit in order to complete the course on time and successfully.

This course entails that you spend a lot of time to read. I would advice that you avail yourself of the opportunities of the tutorial classes provided by the University.

Presentation Schedule

Your course materials have important dates for the early and timely completion and submission of your TMAs and attending tutorials. You should remember that you are required to submit all your assignments by the stipulated time and date. You should guard against falling behind in your work.

Assessment

There are three aspects to the assessment of the course. The first is made up of Self-Assessment Exercises, the second consists of the Tutor-Marked Assignments and third is the written examination/end of course examination.

You are advised to do the exercises. In tackling the assignments, you are expected to apply information, knowledge and technique you gathered during the course. The assignments must be submitted to your facilitator for formal assessment in accordance with deadlines stated in the presentation schedule and the assignment file. The work you submit to your tutor for assessment will count for 30% of your total course work. At the end of the course you will need to sit for a final or end of course examination of about three hours duration. This examination will count for 70% of your total course mark.

Tutor-Marked Assignment

The TMA is a continuous assessment component of your course. It accounts for 30% of the total score. You will be given four (4) TMAs to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMA would be given to you by your facilitator and returned after you have done the assignment. Assignment questions for the units in this course are contained in your reading references and study units. However, it is desirable in all degree level of education to demonstrate that you have read and researched more into your references, which will give you a wider view point and may provide you with a deeper understanding of the subject.

Make sure that each assignment reaches your facilitator on or before the deadline given in the presentation schedule and assignment file. If for any reason you can not complete your work on time, contact your facilitator before the assignment is due to discuss the possibility of an extension. Extension will not be granted after the due date.

Final Examination and Grading

The end of course examination for Mathematical Methods IV is about 3 hour and has a value of 70% of the total course work. The examination will consist of questions, which will reflect the type of self-testing, practice exercise and tutor-marked assignment problems you have previously encountered. All areas of the course will be assessed.

Use the time between finishing the last unit and sitting for the examination, to revise the whole course. You might find it useful to review your self-test, TMAs and comments on them before the examination.

Course Marking Scheme

Assignment	**Marks**
Assignment 1 - 4	Four assignments, best three marks of the four count at 10% each – 30% of course marks
End of course examination	70% overall course marks.
Total	100%

Facilitators/Tutors and Tutorials

There are 16 hours of tutorials provided in support of this course. You will be notified of the dates, times and location of these tutorials as well as the name and phone number of your facilitator, as soon as you are allocated a tutorial group.

Your facilitator will mark and comment on your assignments, keep a close watch on you progress and any difficulties you might face and provide assistance to you during the course. You are expected to mail your Tutor-Marked Assignment to your facilitator before the schedule date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not delay to contact your facilitator by telephone or e-mail if you need assistance.

The following might be circumstances in which you would find assistance necessary, hence you would have to contact your facilitator if:

- You do not understand any part of the study or the assigned readings
- You have difficulty with the self-tests
- You have a question or problem with an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only chance to have face to face contact with your course facilitator and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study.

To gain much benefit from course tutorials, prepare a question list before attending them. You will learn a lot from participating actively in discussions.

Summary

MTH382: Mathematical Method IV is a course that intends to provide solutions to problems normally encountered by engineers and mathematicians in the course doing their normal jobs. It also enables mathematicians to widen the frontiers of their analytical concerns to issues that have significant mathematical implications.

Course Code	MTH382
Course Title	Mathematical Methods IV

Course Team Dr. Bankole Abiola (Writer) - NOUN
Dr. S.O. Ajibola (Programme Leader/
Editor/Coordinator) - NOUN

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
5, Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja

e-mail: centralinfo@noun.edu.ng
URL: www.noun.edu.ng

Published By:
National Open University of Nigeria

First Printed 2012

ISBN: 978-058-407-2

CONTENTS **PAGE**

MODULE 1 EXISTENCE AND UNIQUENESS OF SOLUTIONS

UNIT 1 ORDINARY DIFFERENTIAL EQUATION

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study the theory of ordinary differential equations with a discussion on existence and uniqueness theorems which cover various types of equations. A differential equation is a functional equation where the unknown function or functions are present as derivatives with respect to a single variable in the case of an ordinary differential equation. The order of the highest derivative is called the order of the equation. Derivatives in a differential equation can occur in various ways and we do not admit equations where the unknown is subjected to other operations than algebraic and differential equations.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- classify various types of differential equation
- answer correctly exercises on differential equations.

3.0 MAIN CONTENT

3.1 Definitions and Examples

A differential equation is a functional equation where the unknown function or functions are present as derivatives with respect to single variables in the case of an ordinary differential equation. Consider the following six examples of functional equations involving derivatives. Some are bona fide equations while some are not:

Example (1): $f'(x) = f(x)$

Example (2): $f'(x) = f(x+1)$

Example (3): $f'(x) = a_0(x) + a_{(1}(x) f(x) + a_2(x)[f(x)]^2$

Example (4): $f''(x) = 6x + [f(x)]^2$

Example (5): $f'(x) = \int_0^x \{1 + [f(s)]^2\}^{1/2} ds$

Example (6): $f(x) = \int_0^1 \{[f'(s)]^2 + [f(x)]^2\}^{1/2} ds$

Examples 1 and 3 are ordinary and first order differential equations, while example 2 is a difference differential equation, not a differential equation in the usual sense. Example 4 is a second order differential equation. Example 5 is not a differential equation as it stands but on differentiating will yield
$f''(x) = \{1 + [f(x)]^2\}^{1/2}$ which is a second order differential equation equivalent to example 4. Finally, example 6 is not a differential equation and is not reducible to such an equation by elementary means.

The normal form of a first order differential equation is given as

$$y' = F(x, y) \quad . . . (1)$$

In the simplest case, x and y are real variables and $F(x, y)$ is a function on
R^2 to R^1. We can also allow x and y to be complex variables and F to be a function on C^2 to C^1.

We can also let

$$y = (y_1, y_2, y_3, y_4, ..., y_n) \text{ and } F = (F_1, F_2, F_3, ..., F_n) \quad .... \quad (2)$$

Where y and F are functions on R^{n+1} to R^1. We then define the derivative of a vector as the vector of the derivatives:

$$y' = (y_1', y_2', y_3', ..., y_n') \quad ...... \quad (3)$$

With this notation, equation (1) becomes a condensed convenient way of writing a system of first order differential equations:

$$y'_j(x) = F_j(x, y_1, y_2, y_3, ..., y_n),\ j = 1,2,3,......,n \qquad ...\ (4)$$

Conversely, every such system can be writing as a first order vector differential equation. The generalisation has the advantage of covering **nth** order equations. To convert nth order differential equation in y to a first order vector equation in y, we set

$$y = (y, y', y'', ..., y^{(n-1)}) \quad .... \qquad (5)$$

We can consider differential equations in more general spaces than the Euclidean. Here, the interpretation of the derivatives becomes a matter of concern, and convergence questions also arise if the space is of infinite dimension.
Differential equations normally have infinite number of solutions. In order to find a particular one we have to impose some special conditions on the solution, usually an initial condition. The intent of an existence theorem is to show that there exists a function which satisfies the equation in some neighborhood of point (x_0, y_0). A uniqueness theorem asserts that there is only one such function. We can, however, assert the existence of solution under much more general conditions than those which guarantee uniqueness. This is beyond the scope of this course.

4.0 CONCLUSION

We have examined differential equations in a general setting in this unit. This unit is important to the understanding of other units that would follow subsequently.

5.0 SUMMARY

In this unit, we have a general introduction to various forms of differential equations. This unit must be read carefully before proceeding to the other units.

6.0 TUTOR-MARKED ASSIGNMENT

1. If $f(x)$ satisfies the integral equation

$$f(x) = y_0 + \int_{x_0}^{x_1} F[s, f(s)]ds,$$

Find a differential satisfied by $f(x)$. What initial condition does $f(x)$ satisfy?

2. Transform $f(x) = \int_0^x [f(s)]^2 ds$ into differential equation. Here $f(x) = 0$ is obviously a solution. Are there other solutions of the functional equation?
3. The functional equation $f(x) = 1 + \int_0^x f(s) ds$ implies that f satisfies a differential equation. Find the latter and find the common solution.

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

UNIT 2 THE FIXED POINT METHOD

CONTENTS

1.0 INTRODUCTION

In this unit, we shall use a topological method based on the contraction fixed point theorem. To apply this theorem successfully we have to replace the differential equation by an equivalent integral equation that can be used to define a contraction operator on a suitably chosen metric space.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- apply the contraction fixed point theorem
- determine the existence of solutions for a given differential Equation
- solve correctly the tutor-marked assignment that follows.

3.0 MAIN CONTENT

3.1 The Fixed Point Method

Consider the following differential equation defined by

$$f'(x) = F[x, f(x)],\ f(x_0) = y_0 \qquad \dots (1)$$

Here $F = (F_1, F_2, ..., F_n)$ is a vector valued function defined and continuous in $B : |x - x_0| < a, \quad \|y - y_0\| < b$

We may define the norm on R^n as follows:

$$\left[\sum_1^n (y_j - y_{j0})\right]^{1/2} \quad \text{or} \quad \max|y_j - y_{j0}|$$

We impose two further conditions on F :

$$\|F(x,y)\| \le M \qquad \dots\ (2)$$

$$\|F(x,y_1) - F(x,y_2)\| \le K\|y_1 - y_2\| \qquad \dots\ (3)$$

Conditions (1) and (2) are called boundedness and Lipschitz conditions respectively.

We now replace the vector differential equation by a vector integral equation defined as:

$$f(x) = y_0 + \int_{x_0}^{x} F[s, f(s)]ds \qquad \dots\ (4)$$

We again impose the following property which follows from the definitions of integrals by Riemann as:

$$\left\|\int_{x_0}^{x} Fds\right\| \le \int_{x_0}^{x} \|F\|ds, x_0 < x \qquad \dots\ (5)$$

Theorem (1): Under the stated assumptions on F, the equation (1) has a unique solution defined in the interval ($x_0 - r, x_0 + r$) where

$$r < \min(a, \frac{b}{M}, \frac{1}{K})$$

Proof: We consider the space N of all functions $g(x)$ on R^1 *to* R^n continuous in x $(x_0 - r, x_0 + r)$ such that $g(x_0) = y_0$ and $\|g - y_0\|_N \le b$ where

$\|g - y_0\|_N = \sup_x \|g(x) - y_0\|$.. For such, a g(x) the function $F[x, g(x)]$ exists and is continuous. Furthermore, its N-norm does not exceed M. We now define the transformation:

$$T : g(x) \to y_0 + \int_{x_0}^{x} F[s, g(s)]ds, \quad -r < x - x_0 < r \qquad \dots\ (6)$$

Here, $T[g](x)$ is continuous, $T[g](x_0) = y_0$ and

$$\|T[g](x) - y_0\| < Mr < b \qquad ...(7)$$

(by the choice of r). It follows that $T[g] \in N$. We next observe that

$$\|T[g_1](x) - T[g_2](x)\| = \left\|\int_{x_0}^{x} \{F[s, g_1(s)] - F[s, g_2(s)]\}ds\right\| < K\left|\int_{x_0}^{x} \|g_1(s) - g_2(s)\|ds\right|$$

This shows that

$$\|T[g_1] - T[g_2]\|_N \leq Kr\|g_1 - g_2\|_N = k\|g_1 - g_2\|_N$$

Where $Kr = k < 1$ by choice of r. Hence T is a contraction. This implies that there exist one and only one function $f(x) \in N$ such that

$f(x) = y_0 + \int_{x_0}^{x} F[s, f(s)]ds, f(x_0) = y_0$ is the unique solution of the differential equation (1) with the stated initial condition.

4.0 CONCLUSION

We have shown that we can apply the fixed point theorem to establish the existence of solution to the differential equation stated in (1). You are supposed to master the concept developed in this unit before proceeding to the next unit.

5.0 SUMMARY

The contraction fixed point theorem applied in this unit enables us to develop a unique solution to the differential equation stated in (1). It is one of the most powerful theorems in mathematical analysis. It can be extended to spaces of infinitely in many dimensions. However, this is beyond the scope of this unit.

6.0 TUTOR-MARKED ASSIGNMENT

Determine an interval $(x_0 - r, x_0 + r)$ where the existence of solution to the following differential equations is guaranteed:

i. $y' = y, y(0) = 1$

ii. $y' = y^3, y(0) = 2$

iii. $y' = xy + y^2, y(0) = 0$

iv. $y_1' = y_1 + y_2, y_1(0) = -1, y_2(0) = 1$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

UNIT 3 THE METHOD OF SUCCESSIVE APPROXIMATIONS

CONTENTS

1.0 INTRODUCTION

The method of successive approximations is a refinement of the old device of trial and error. What has been added is control of the limiting process. We know how often the process must be repeated to bring the result with the desired limit of tolerance. The method of trial and error can be traced back to Isaac Newton who was the first to be concerned with approximate solution of algebraic equation. An infinite iteration process for the positive solution of the transcendental equation defined as:

$$x = \alpha \arctan x, \quad 1 < \alpha \quad \ldots\ldots.. \qquad \text{(A)}$$

was given by Joseph Fourier in his *Theorie Analytique de la Chaleur* (1822). Fourier's argument is geometrical and highly intuitive. It is not difficult to give a strict analytic convergence proof.

The method of successive approximation was given by Emile Picard for differential equation in 1891. This method soon became the standard method for proving existence and uniqueness theorems for all sorts of functional equations.

2.0 OBJECTIVES

At end of this unit, you should be able to:

- apply the method of successive approximation to
- determine existence and uniqueness of differential
- equation.

3.0 MAIN CONTENT

3.1 The Method of Successive Approximations

Let us consider a vector differential equation defined by

$y' = F(x, y), y(x_0) = y_0$ . . .
(1)

$F(x, y)$ is defined and continuous in:
$B : |x - x_0| < a$ $\|y - y_0\| < b$, $\|F(x, y)\| \le M$
(2)

$\|F(x, y_1) - F(x, y_2)\| \le K\|y_1 - y_2\|$
(3)

We shall state the following theorem:

Theorem (1): There exists a unique function $f(x), on, R^1 to, R^n$ defined for
$|x - x_0| < r$, where
$r < \min(a, \frac{b}{M})$
(4)

Proof: We replace the differential equation with the initial by the equivalent integral equation:

$$f(x) = y_0 + \int_{x_0}^{x} F[s, f(s)]ds \qquad (5)$$

$$f_0(x) = y_0$$

Now define

$$f_m(x) = y_0 + \int_{x_0}^{x} F[s, f_{m-1}(s)]ds, m = 1,2,3,.... \qquad (6)$$

For these functions to be well defined, we restrict x to the interval $(x_0 - r, x_0 + r)$. Suppose it is known that for some value of *m*, the function $f_{m-1}(x)$ is well defined in this interval. It is obvious that $f_{m-1}(x) = y_0$, but the induction hypothesis must also include that $f_{m-1}(x)$ is continuous and

$\|f_{m-1}(s) - y_0\| < b.$ We then see that $F[s, f_{m-1}(s)]$ is well defined and continuous. Furthermore:

$\|F[s, f_{m-1}(s)]\| \le M$,
Hence
$\int_{x_0}^{x} F[s, f_{m-1}(s)]ds$, exist as a continuous function of x and its norm does not exceed $M|x - x_0| < Mr < b$ by the choice of r.
This implies that $f_m(x)$ is also continuous and satisfies $f_m(x_0) = y_0, \|f_m(x) - y_0\| < b$

It follows that the approximation are well defined for all m. To prove the existence of $\lim f_m(x)$, we resort to the Lipschitz condition. We have

$$\|f_m(x) - f_{m-1}(x)\| = \left\|\int_{x_0}^{x} \{F[s, f_{m-1}(s)] - F[s, f_{m-2}(s)]\}ds\right\| \le K\left|\int_{x_0}^{x} \|f_{m-1}(s) - f_{m-2}(s)\|ds\right|$$

We know that for some m we have the estimate

$$\|f_{m-1}(s) - f_{m-2}(s)\| \le \frac{K^{m-2}}{(m-1)!} M|s - x_0|^{m-1}, |s - x_0| < r$$
(7)

This estimate is certainly very true for $m = 2.$ We then get

$$\|f_m(x) - f_{m-1}(x)\| \le \frac{K^{m-1}}{(m-1)!} M\left|\int_{x_0}^{x} |s - x_0|^{m-1} ds\right| = \frac{K^{m-1}}{m!} M|x - x_0|^m .$$ Therefore the estimate is true for all m

Hence the series
$$f_0(x) + \sum_{n=1}^{\infty} [f_n(x) - f_{n-1}(x)]$$
(8)
Whose partial sum is $f_m(x)$, converges in norm for $|x - x_0| < r$ uniformly in $x.Hence, it, sum, f(x),$ is a continuous function on $R^1 \longrightarrow R^n$.

The strong uniform convergence of the vector series (8) obviously implies the absolute and uniform convergence of the n component series to continuous functions on R^1, to, R^n . The estimate (7) obviously implies that

$$\|f(x)-f_m(x)\| \leq \frac{K^m}{m!}M\exp(K|x-x_0|)|x-x_0|^m \ldots\ldots \quad (9)$$

It is an easy matter to observe that if $|x-x_0|$ is not large, $f_m(x)$ converges rapidly to its limit $f(x)$. Therefore, from the uniform convergence of $f_m(x)$ to $f(x)$ it follows that $F[s, f_{m-1}(s)]$ converges to uniformly to $F[s, f(s)]$ and

$\int_{x_0}^{x} F[s, f_{m-1}(s)]ds \to \int_{x_0}^{x} F[s, f(s)]ds$ uniformly in x. From (6) it follows that $f(x)$ satisfies (5) and consequently, the differential equation and the initial condition. That this is the only solution also follows from the Lipschitz condition. So to prove uniqueness we may suppose that $g(x)$ is a solution defined in some interval (x_0-r_1, x_0+r_1). Then

$g(x) = y_0 + \int_{x_0}^{x} F[s, g(s)]ds$, and if $|x-x_0| < \min(r, r_1)$ we have

$$\|f(x)-g(x)\| = \left\|\int_{x_0}^{x}\{F[s, f(s)] - F[s, g(s)]\}ds\right\| \leq K\left|\int_{x_0}^{x}\|f(s)-g(s)\|ds\right|$$

Set $h(x) = \|h(x)-g(x)\|$, then $h(x)$ is a continuous non-negative function that satisfies, $0 \leq h(x) \leq K\left|\int_{x_0}^{x} h(s)ds\right|$. Hence $h(x)$ is identically 0. Therefore, $f(x)$ is the only solution of (1) with $f(x_0) = y_0$

4.0 CONCLUSION

Various questions arise when we want to use theorem (1) above. The first of these concerns the effective determination of a, b and M and the verification of the Lipschitz condition. We leave this for future considerations. We have justified the existence of solution to functional differential equations. We have also proved the uniqueness of this solution. You are required to read carefully before proceeding to the next unit.

5.0 SUMMARY

We have proved the existence of functional differential equations by successive approximation methods. Successive approximation method is essentially an iterative method that needs to be carefully designed to give a solution to the differential equation under consideration. Once the equivalent integral equation of the given differential equation is known, then it is just an easy matter to design the appropriate iterative scheme for the equation, which will eventually converge to the solution of the equation.

6.0 TUTOR-MARKED ASSIGNMENT

1. Solve $y' = y + x, y(0) = C,$ by method of successive approximation
2. If $y(x)$ is a solution of
   $y'' - x^2 y = 0, y(0) = y_{01}, y'(0) = y_{02},$
   Show that
   $y(x) = y_{01} + y_{02}x + \int_0^x (x-s)s^2 y(s)ds.$
   Use the method of successive approximations to find $y(s)$ in the special case $y_{01} = 1, y_{02} = 0,$ Take $f(x) \equiv 1$
3. The Thomas-Fermi equation defined by
   $x^{1/2} y'' = y^{3/2}$
   arises in nuclear physics. Show that it has a solution of the form Cx^{α}. Show also that it can be transformed into a system to which method of successive approximation can be applied so that its solution in some interval [0,r] satisfies an initial condition of the form $y(0) = a > 0, y'(0) = b$.

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall .

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

MODULE 2 SPECIAL FUNCTIONS

UNIT 1 SPECIAL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall examine some special functions such as Beta function, Gamma function and Factorial function. These functions are of very useful mathematical importance in solving differential equations and other applied mathematics problems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define beta function, gamma function, and factorial notations
- apply these functions to solve mathematical problems.

3.0 MAIN CONTENT

3.1 Special Functions

Below are some of the special functions worthy of note.

3.1.1 Gamma Functions

One of the most important functions is the gamma function, written and defined by the integral

(1) $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt$

(More generally, if we consider also complex values, for those αt whose real part is positive). By integration by parts, we find
$\Gamma(\alpha + 1) = \int_0^\infty e^{-t} t^{\alpha} dt = -e^{-t} t^{\alpha} \Big\|_0^\infty + \alpha \int_0^\infty e^{-t} t^{\beta - 1} dt = \alpha \Gamma(\alpha)$
Thus we obtain the important functional relation of the gamma function

(2) $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$

Let us suppose that the $\alpha + ve$ integer, say, n. Then repeated application of (2) yields

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$= n(n-1)...............3.2 \text{ r } (1)$$

Now $\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t} \Big|_0^1 = 1$

(3) $\therefore \Gamma(n+1) = n1 \ \therefore$

Hence gamma function can be regarded as a generalisation of the eliminating fractional function.

By repeated application of (2)

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + 1)}{(\alpha)} = \frac{\Gamma(\alpha + 2)}{(\alpha)(\alpha + 1)} = ...\frac{\Gamma(\alpha + k + 1)}{(\alpha)(\alpha + 1).....(\alpha + k)}$$

Thus we obtain the relation

(5) $\Gamma(\alpha) = \lim \frac{\Gamma(\alpha + k + 1)}{(\alpha)(\alpha + 1)....(\alpha + k).} (\alpha \neq 0, -1, -2.........)$

Gauss defined Gamma function as follows

(6) $\Gamma(\alpha) = \frac{\lim}{n \to \infty} \frac{n \ln}{\alpha(\alpha + 1)........(\alpha + n)}$

Where

Problem $I, y\alpha > 0$ and n is a tve integer, then

$$\Gamma(\alpha) = \frac{\lim}{n \to \infty} \int_0^n \{ 1 - \frac{t}{n})^n t^{\alpha - 1} dt.$$

Proof: Now consider the integral

$$\int_0^n \{ 1 - \frac{t}{n})^n t^{\alpha - 1} dt.$$

Substitute $t = n^x$ in the integral, we obtain

$$\int_0^n \{ 1 - \frac{t}{n})^n t^{\alpha - 1} d = n^\alpha \int_0^1 (1 - x)^n x^{\alpha - 1} dx$$

By integrating by parts gives the formula

$$\int_0^n (1 - x) x^{\alpha - 1} dx = \frac{n}{\alpha} \int_0^1 (1 - x)^{n-1} x^\alpha dx$$

Repeating integration by parts, we get

$$\int_0^n (1 - x)^n x^{\alpha - 1} dx = \frac{n(n-1)(n-2).......1}{\alpha(\alpha+1)....(\alpha+n-1)} \int_0^1 x^{\alpha + n - 1} dx$$

Thus

$$\int_0^n (1 - \frac{t}{n}) t^{\alpha - 1} dt = \frac{n1n}{\alpha(\alpha +)...(\alpha + n - 1)}$$

$$\therefore dt \int_0^n \{ \{ 1 - \frac{t}{n} \}^n t^{\alpha - 1} dt = \frac{dt}{n \to \infty} \frac{nn\alpha i}{\alpha(\alpha+1)(\alpha+2)....(\alpha+n)} = \Gamma(\alpha)$$

Lemma1. If $0 \le \alpha < 1, \quad 1 + \alpha \le \exp \alpha \le (1 - \alpha)^{-1}$, compare the three series.

$$(1 + \alpha)^{-1} = 1 + \alpha, \quad \exp (\alpha) = 1 + \alpha + \sum_{N=2}^{\infty} \frac{\alpha^N}{n^1}$$

$$(1 + \alpha)^{-1=} = 1 + \alpha + \sum_{N=2}^{\infty} a^n$$

Lemma2. If $0 \le \alpha < 1,$, $(1 - \alpha)^n \ge 1 - n^\alpha$, for a position integer

Proof: For n = 1, $1 - \alpha = 1 - \alpha,$ as derived.

Assume that

$(1-\alpha)\beta \geq 1-\beta\alpha,$

Multiply each member by $1-\alpha,$ to obtain

$(1-\alpha)^{\beta+1} \geq (1-\alpha)(1-\beta\alpha) = 1-(b+1)\alpha + b\alpha^2$

So that

$(1-\alpha)^{\beta+1} \geq 1-(b+1) = 1-(b+1)\alpha + b\alpha^2$

Lemma 2. Follows by induction

Lemma 3. If $0 \leq t < n.$, n a positive integer

$0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n1}$

Proof: In Lemma I, put $\alpha = \frac{t}{n}$, we get

$\left(1+\frac{t}{n}\right) \leq e\frac{t}{n} \leq \left(1-\frac{t}{n}\right\}^{-1}$

From which

(a) $\left(1+\frac{t}{n}\right)^n \leq e^{-t}\frac{t}{n} \leq \left(1-\frac{t}{n}\right\}^{-n}$

Or

$\left(1+\frac{t}{n}\right)^n \geq e^{-t}\frac{t}{n} \geq \left(1-\frac{t}{n}\right\}^{-n}$

So that

$e^t - \left(1-\frac{t}{n}\right\}^n \geq 0$

$\therefore e^t - \left(1-\frac{t}{n}\right\}^n \leq e^t\left[1-\left(1-\frac{t^2}{n^2}\right)\right.$

But by (a)

$e^t \geq \left(1-\frac{t}{n}\right\}^n$

$\therefore e^t - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\left[1-\frac{t^2}{n^2}\right\}^2\Big]$

In Lemma 2, we have shown that

$(1-\alpha)^n \geq 1-n\alpha.$

$\therefore \left(1-\frac{t^2}{n^2}\right)^n \geq 1-n\alpha$

$\therefore e^t - \left(1-\frac{t}{n}\right)^n \leq \frac{e^{-t}t^2}{n}$

Problem 2. Show that the two definitions of gamma function are equivalent.

Proof: By using Gauss's definition, we proved that $\Gamma(z) = \frac{\dim}{n \to w} \int_0^n \left(1 - \frac{t}{n)} \right)^n t^{z-1} dt$

Now

$= \frac{Lim}{n \to w} [\int_0^n [e^{-t} - (1 - \frac{t}{n}]^n] t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dx$

From the convergence of the integral

$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z)$

It follows

$= \frac{Lim}{n \to w} \int_0^n e^{-t} - t^{z-1} dt = 0$

Hence

$\int_0^\infty e^{-t} t^{z-1} dt - r(z) + \frac{Lim}{n \to w} \int_o^n [e^{-t-} (1 - \frac{t}{n}) t^{z-1} dt$

Now

$\int_0^\infty e^{-t} t^{z-1} dt$ Converges, so $\int_0^n e^{-t} t^{z-1} dx$ is bounded.

Thus

$\lim_{n \to \infty} \int_0^n [e^t - (1 - \frac{t}{n}) t^{z-1} dt = 0$

$\int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z)$

Problem 3. Show that

$\Gamma(z) r(1-z) = \frac{\pi}{Sin \pi z}$ $\qquad (z \neq 0. \neq 1, \neq 2, ...$

Proof: using Gauss definition of gamma function

$\Gamma(z) = \lim_{n \to \infty} \frac{n1}{z(z+1)(z+2)...........(z+n)}$

$z \neq \frac{\infty}{\prod} (1 - \frac{z^2}{s^2}$

$= \frac{Sin \pi z}{\pi}$

Note if we put z = 1/2 , we get

$\frac{1}{[\Gamma(1/2)]^2} = \frac{1}{\pi}$

or

$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

Problem 4. Show that

$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}) \qquad (2z = 0, -1, -2............)$

Proof :

$\frac{2^{2z} \Gamma(z) \Gamma(z + \frac{1}{2})}{\Gamma(2z)}$

$$= \lim_{n\to\infty} \{ \frac{2^{2z} nin^z nin^2 + \frac{1}{2}}{z(z+1)...(z+n)(z+\frac{1}{2}(2+\frac{3}{2....}(z+n+\frac{1}{2}}$$

$$= \lim_{n\to\infty} \left[\frac{(ni)^2 2^{2n+1}}{(2n)1\sqrt{n}} \right]$$

The last quantity is independent of z and must be finite since the left side exists.

$$\therefore \frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} = A$$

$\therefore$

Put z = ½

We have

$A = 2\sqrt{\pi}$

$$\therefore \Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}}{\sqrt{\pi}} \therefore$$

3.1.2 Beta-Function

We define Beta-function $B(p,q)$ by

(1) $B(p,q) = \int_0^1 t^{p-1}(1-t,^{q-1} dx, R(c) > 0, R(q) > 0$

Another useful form of this function can be obtained by putting t = $t = Sin\theta^2$, thus arriving at

(2) $B(p,q) = 2\int_0^{\frac{\pi}{l}} Sin^{2p-1}\theta Cos^{2q-1}\theta d\theta, R(p) > 0, R(q) > 0$

Next we establish the relation between gamma and beta-functions

Problem: If $R(p) > 0, R(q) > 0.$ Then

$$B(p,q) = \frac{r(p)r(q)}{r(p+q)}$$

Proof: $r(p)r(q) = \int_0^\infty e^{-t}t^{P-1}dt \int_0^\infty e^{-u}u^{q-1}du$

Substituting $t = x^2 and U = y^2$ it gives $r(p)r(q) = 4\int_0^\infty e^{-x2}x^{2p-1}dx \int_0^\infty e^{-y2}y^{2q-1}dy$

$r(p)r(q) = 4\int_0^\infty \int_0^\infty \exp(-x^2 - y^2)x^{2p-1}y^{2q-1}dxdy$

Next, turn to polar co-ordinate for the iterated integration over the first quadrant in xy-plan. $r(p)r(q) = 4\int_0^\infty \int_0^{\frac{\pi}{2}} \exp(-2^2)r^{2p+2q-2}Cos^{2p-1}\theta Sin^{2q-1} rd\theta d\theta$

$$2\int_0^\infty \exp(-r^2)r^{2p=2q-1}dr2\int^{\frac{\pi}{l}} Cos^{2p-1}\theta Sin^{2q-1}\theta d\theta$$

Take $r^{2=t} and \theta = \frac{1}{2}\pi - \theta,$ we obtain

$$r(p)r(q) = \int_0^\infty \exp(e^{-t}t^{p+q-1}dt2\int_0^{\frac{1\beta\theta}{2}} Sin^{2p-1}\theta Cos^{2q-1}d\theta$$

$$= r(p+q)B(p,q)$$

$$\therefore B(p,q) = \frac{r(p)r(q)}{r(p+q)}.$$

3.1.3 Factorial Notations

(1) $(\alpha)_n = \frac{n}{ll}(\alpha + k - 1)$

$$= \alpha(\alpha+1)......(\alpha+n-1)$$

$$(\alpha)_0 = 1, \alpha \neq 0$$

The function $(\alpha)_0$ is called the factorial notation

Problem: Show that

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

Proof:- $(\alpha)_{2n} = (\alpha)(\alpha+1)(\alpha+2)(\alpha+3)......(\alpha+2n-1)$

$$= \left[(\alpha)(\alpha+2)......(\alpha+2n-2)\right]\left[(\alpha+1)(\alpha+3).......(\alpha+2n-1)\right]$$

$$= 2^{2n}\left[\left(\frac{\alpha}{2}\right)\left(\frac{\alpha+2}{2}\right)\left(\frac{\alpha+4}{2}\right)..........\left(\frac{\alpha+2n-2}{2}\right)\right]$$

$$\left[\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}+\right)...........\left(\frac{\alpha+1}{2}+n-1\right)\right]$$

$$2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$$

Similarly, we can show that

$$(\alpha)_{kn} = K^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)..........\left(\frac{\alpha+k-1}{k}\right)_n$$

Problem: show that

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

Proof:

$$\Gamma(\alpha+n) = (\alpha+n-1)(\alpha+n-2).........\alpha\Gamma(\alpha)$$
$$= (\alpha)(\alpha+1).........(\alpha+n-1)\Gamma(\alpha)$$
$$\Gamma(\alpha+n) = (\alpha)_n\Gamma(\alpha)$$
$$\therefore (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

4.0 CONCLUSION

In this unit, we have studied Gamma function, Beta function and Factorial notations. You are required to study these functions because you would be required to apply them in future.

5.0 SUMMARY

The study of special functions in mathematics is of significant importance. Study this area properly before moving to the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

(1) The Beta function of p, and, q is defined by the integral

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, (p,q>0).$$

By writing $t = \sin^2\theta$ obtain the equivalent form

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta d\theta, (p,q>0)$$

(2) Show that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

(3) By writing $t = x/(x+a)$ in the definition of Beta function, show that

$$\int_0^\infty \frac{x^{p-1}dx}{(x+a)^{p+q}} = a^{-q}B(p,q)$$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

UNIT 2 HYPER GEOMETRIC FUNCTION

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider a class of function usually referred to as hyper-geometric functions. The series solution of the associated differential equation usually takes the form of a geometric series. Most often, hyper-geometric equation has $x = 0, x = 1$ and $x = \infty$ as regular points and ordinary point elsewhere.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- determine the differential equations that can give rise tohyper-geometric functions
- explain the properties of this functions
- apply this function where necessary.

3.0 MAIN CONTENT

3.1 Hyper-Geometric Function

The solutions of the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + [\ c-(a+b+1)x)\frac{dy}{dx-ab}y = 0 \qquad (1)$$

are generally called Hyper-geometric functions.

Note that a, b, and c are fixed parameters.

We solve this equation (1) about the regular singular point $\underline{x = 0}$
Shifting the index

$$\sum_{n=0}^{\infty} n(n+c-1)x^{n-1} - \sum_{n=1}^{\infty} (v+a-1)(n+b-1)e_{n-1}x^{n-1} = 0$$

For $n \geq 1$

$$e_n = (\frac{a(a+a)(n+b-1)}{n(n+c-1)} e_{n-1}$$

$$e_n = (\frac{a(a+a)(a+2)...(a+n-1).b(b+1)(b+2)...(b+n-1)e_0}{n1.c(c+1)(c+2)(c+2).....(c+n-1)}$$

Using factorial notation, we have

$$e_n = \frac{(a)_n (b)_n}{n1.(c)_n} e_0$$

Let us choose $e_0 = 1$

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n1(c)_n} x^n$$

$$= = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n1} x^n$$

We have the symbol

$2F_1(a,b,c,x)$ to represent solution

$$y_1 = \sum_{n=0}^{\infty} \frac{(a)(b)_n}{(c)_n n1} x^n$$

$\therefore$

$$2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n1} x^{n.}$$

The solution is valid in $0 < | \ x < 1.|$. The other root of the indicial equation is

(i-c). We may put $y = \sum_{n=1}^{\infty} f_n{}^n x^{n+1-c}$

For the moment, let c be not an integer for (1), the indicial equation has root zero and i-c. Let $y = \sum_{n=1}^{\infty} e_n x^{n+r}$

$$\sum_{n=1}^{\infty} e_n x^{n+b}(n+b-1)x^{n+b} - \sum_{n=0}^{\infty} e_n (n+b)(n+b-1)x^{n+b} + c\sum_{n=0}^{\infty} e_n (n+b)(n+b-1)x^{n+b} + c\sum_{n=0}^{\infty} e_n (n+b)x^{n+b-1}$$

$$-(a+b+1)\sum_{n=0}^{\infty} e_n x^{n+b=0}$$

or

$$\sum_{n=0}^{\infty} e_n (n+b)(n+b-1+c)x^{n+b-1}$$

$$\sum_{n=0}^{\infty} e_n [\ ab+1)(a+b+1)(n+b)(n+b)(n+b-1)\ ]x^{n+b} = 0$$

The indicial equation is

$$e_n (b)(b-1+c) = 0$$

(Note c is not an integer).

Corresponding to $b = 0$,

$$\sum_{n=1}^{\infty} n(n+c-1)e_n = \sum_{n=0}^{\infty} (n+a)(n+b)e_n x^n = 0$$

Problem 1: If $R(c-a-b)>0$ and if c is neither zero nor a negative integer,

$$2F_1(a,b,c,1)=\frac{r(c)r(c-a-b)}{r(c-a)r(c-b)}$$

Proof

$$2F_1(a,b,c,1)$$
$$=\frac{r(c)}{r(b)r(c-b)}$$
$$=\frac{r(c)}{r(b)r(c-b)}\frac{r(b)r(c-a-b)}{r(c-a)}$$

Problem 2: Show that

(a) $2F\ (\alpha,\beta,\beta,x)=(1-x)^{-\alpha}$

(b) $x2F(1;1;2;-x)=Log(1+x)$

Solution:

(a) $2F\ (\alpha,\beta,\beta,x)$

$$\sum_{n=0}^{\infty}\frac{(x)_n}{n1}x^n$$
$$=1+\alpha x+\frac{(\alpha+1)}{21}x^2+......\frac{\alpha(\alpha+1)(\alpha+1)}{31}x^3+......=(1-x)^{-\alpha}$$

(b) $x2F(1;1;2;-x)=Log(1+x)$

$$x[\ 1+\frac{1.1}{1.2}(-x)+\frac{1.2.1.2}{1.2.2.3}(-x)^2+\frac{1.2.3.1.2.3}{1.2.3.2.34}(x)^3$$
$$y_2=x^{1-c}2F_1(a+1-c,b+1-c;2-c;x)$$

Problem: If $1\le 1\le 1, and$ if $R(c)>R(b)>0,$

$$2F_1(a;b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{=a}\,dt$$

Proof

Beta-function now

$$\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}=\int_0^1 t^{b-1}(1-t)^{c-b-1}\,dt$$

Also

$$\frac{(b)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{(b+n)(\Gamma(c-b)}{\Gamma(c+n)}$$

Thus

$$2F_1(a;b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\sum_{n=0}^{\infty} \frac{(a)z_n}{N1} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} \sum_{n=0}^{\infty} \frac{(a)(zt)^n\, dt}{n1}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1-1}(1-zt)^a\, dt$$

Where

$$(1-z)^a = \sum_{n=0}^{\infty} \frac{(-\alpha)(-\alpha-1)\ldots(-\alpha-n+1)(-1)^n y^n}{n1}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha+1)\ldots(\alpha+n-1)y^n}{n1}$$

$$= \sum_{n=0}^{\infty} \frac{a(\alpha+1)\ldots(\alpha+n-1)y^n}{n1}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n y^n}{n1}$$

4.0 CONCLUSION

You have learnt in this unit some properties of hyper-geometric functions. You are requested to study this unit properly before going to the next unit.

5.0 SUMMARY

Recall that you learnt about the class of differential equation, which usually give rise to hyper-geometric functions. You also learnt about the relations of this function to Gamma and Beta functions. Study this unit properly before going to the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. If R(c – a – b) >0 and if c is neither zero nor a negative integer show that

$$2F_1(a;b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

2. Show that
   (a) $2F_1(\partial;\beta;\beta;x) = (1-x)^{-\alpha}$
   (b) $x^2F_1(1;1;2;-x) = \log(1+x)$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

UNIT 3 BESSEL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In solving differential equation, we often come across some problems which exhibit some characteristic which needed to be studied further. Such equations are Legendry equation and Bessel equations. We shall study in detail in this unit the Bessel equation which gives rise to Bessel functions. This is because of the wide applicability of this function in physics and applied mathematics.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- identify Bessel functions correctly
- solve problems related to Bessel functions.

3.0 MAIN CONTENT

3.1 Bessel Function

The equation

(1) $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-v^{2)y}=0$$

is called Bessel's equation of index $v.$

(i) x = 0 is the regular Singular point of the equation (1) in the finite plane

(ii) Assume that $v.$ is not integer.

$$y=\sum_{n=0}^{\infty} c^m x^{m+r}$$

Substituting this expression and its first and second derivatives into Bessel equation, we obtain

$$=\sum_{n=0}^{\infty}(m+z)(m+z-1)c_m x^{m+r}+\sum_{m=0}^{\infty}(m+r)c_m x^{m+r}+\sum_{m=0}^{\infty}c_m x^{m+r+2}-V^2\sum_{m=0}^{\infty}c_m x^{m+z}=0$$

(a) $r(r-1)c_0+rc_0-v^2c_0=0 \qquad (m=0)$

(b) $(r+1)(r)c_1+(r+1)c_1-v^2c_1=0 \quad (m=1)$

(c) $(m+r)(m+r-1)c_m+c_{m-2}-vc_m=0 \qquad (m=2,3,....$

Now $c_0\neq 0,$ thus the indicial equation from (a) $(r+v)(r-v)=0$

The roots are $r=v-v=0$

$2_1-r_2=2v$

$v\neq 0,$

$v\neq \text{int}\,eger$

$2v$ integral multiply of $2v$, i.e. v is zero or $+ve$ integer

Now we obtain the solution corresponding to the value $r=v.$

From (b) we obtain $c_1=0$ (c) can be written

$(m+r-v)(m+r+v)c_m+c_{m-2}=0$

Since $c_1=0$, it follows that $c_3=c_5=c_7=.......0.$ Thus we put replace m by $2m.$

$(2m+r-v)(2m+r+v)c_m+c_{2m-2}=0$

Now $r=v$

$(2m+2v)(2m)c_{2m}+c_{2m-2}=0$

$\therefore c_{2m=}-\dfrac{c_{2m-z}}{2^2(v+m)m}$ (but v is not integer) $(m=1,2,....)$

Assume

$$c_0=\frac{1}{v_2r(v+1)}$$

$$c_2=\frac{c_0}{2^2(v+1)}=\frac{1}{2^{v+2}11r(v+2)}$$

$$c_4=\frac{c_2}{2.2^2(v+2)}=\frac{1}{2^{v+2}21r(v+3)}$$

$$c_{2m}=-\frac{{}^{\alpha}(-1)^m}{2^{2m+v}m1r(v+m+1)}$$

Thus, the solution is

$$y=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+v}m1r(v+m)}$$

We denote this solution by the notation

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} r(m+v+1)m1} \quad (2)$$

$Jv(x)$ is called the Bessel Function of the first kind of order v.

By Ratio test we know that the series converges for all values of x
Replacing v by $-v$, we have

$$J_{-v}(x) = x^{-v} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-v} r(m-v+1=1)m1} \quad (3)$$

(2) and (3) are the independent solutions.

Thus $y = c_1 J_v(x) + c_2 J_{-v}(x)$

(i) If $v = 0,$ then the solution $J_v(x)$ and $J_{-v}(x)$ are identical. One can verify from (2) and (3)

(ii) If v is $+ve$ integer, then the second solution $J_{-v}(x)$ is not independent of $J_v(x)$

Say $v = n$ then the factor

$\frac{1}{r(m-n+1} = \frac{1}{m-n1}$ in (3) is zero

When $m < n$ hence (3) is equivalent to

Replace m by $m+n$ in 5, we get change the index

$$J_{-v}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+m} \left(\frac{x}{2}\right)^{2m+n}}{m1m+n1} \quad (6)$$

From (2), when $v = n$ integer, thus

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+n}}{m1, m+n1} \quad (7)$$

From (6) and (7), we get

$$J_{-v}(x) = (-1)^n J_n(x) \quad (8)$$

Further properties of Bessel functions of first bind
From (2)

$$x^v J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v}}{2^{2m+v} r(m+v+1)m1}$$

Now we use the formula

$\alpha r(\alpha) = r(\alpha + 1)$

$$x^{v-1} x^v = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2v-1}}{2^{2m+v} r(m+v)m1}$$

$$= x^{v-1}x^{v} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v-1} m1r(m+v)}$$

$$= x^{v} J_{v-1}(x)$$

Thus we obtain

$$\frac{d}{dx}\left[x^{v} J_{v}(x) \right] = x^{v} J_{v-1}(x) \qquad (9)$$

Similarly, we can show that

$$\frac{d}{dx}\left[x^{-v} J_{v}(x) \right] = (-1)x^{-v} J_{v+1}(x) \qquad (10)$$

(9) can also use written

$$vx^{v-1} J_{v}(x) + x^{v} Jv'(x) = x^{v} J_{v-1}(x) \qquad (11)$$

(10) Can also be written

$$-vx^{-v-1} J_{v}(x) + x^{-v} Jv'(x) = -x^{-v} J_{v+1}(x) \qquad (12)$$

Multiplying (12) by x^{2v} and subtracting from (11), we have

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_{v}(x) \qquad (13)$$

Multiplying (12) by x^{2v} and adding with (11), we get

$$J_{v-1}(x) - J_{v+1}(x) = 2J_{v}'(x) \qquad (14)$$

(i) we know that

$$Sinx = x - \frac{x^3}{31} + \frac{x^5}{51} - \frac{x^7}{7} + \ldots\ldots\ldots.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{k+11}}$$

Now $2k+11 = r(2k+2) = (2)_{2k}$

$$2^{2k} + (1) = (2)_{2k}$$

$$2^{2k} + k1\left(\frac{3}{2} \right)k$$

$$2^{2k} + k1\left(\frac{3}{2} \right)k$$

$$\frac{2^{2k} k1r\left(k+1+\frac{1}{2} \right)}{r\left(\frac{3}{2} \right)}$$

But $r(\frac{3}{2}) = \frac{1}{2} r(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

$$\therefore 2k+11 = \frac{2^{2k+1} k1r(k+1+\frac{1}{2})}{\sqrt{\pi}}$$

$$\therefore Sinx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}\sqrt{\pi}}{2^{2k=1} k1r(k+1+\frac{1}{2}}$$

(15)

If we take $V = \frac{1}{2}$, then from (2), we have

$$Jv(x) = x\frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\frac{1}{2}} r(k+\frac{1}{2})}$$

(16)

From (15) and (16), we have

$$Jv(x) = (\frac{2}{\pi x}) \sin x$$

In similar manner, by considering the expansion $\cos z \sum_{n=0}^{\infty} \frac{(-1)z^{2n}}{(2n)}!$, we obtain

The formula

$$J_{-v2}(x) = (\frac{2}{\pi x}) \cos x$$

Problems

(i) $J_o(x) = -1J_1(x)$

(ii) $J_n'' v = \frac{1}{4}(J_{n-2} - 2J_n + J_n + 2)$

(iii) $J_1(x) = J_0(x) - \frac{1}{x} J_1(x)$

(iv) $J_2(x) = (1 - \frac{4}{x^2})J_1(x) - \frac{2}{x} J_0(x)$

(v) $\int x^m J_n(x)dx = x^m J_{n+1}(x) - (m-n-1)\int x^{m-1} J_{n+1} dx(x)$

Solution

$$\int x^m J_n(x)dx = x^{m-n-1}[x^{n+1} J_n(x)]dx$$

$$\int x^{m-n-1} \frac{d}{dx}[x^{n+1} J_{n+1}(x)]dx$$

Integrating by parts, we have
$= x^m J_{n+1}(x) - (m-n-1)\int x^{m-1} J_{n+1}(x)dx$

This proves the result
Prove that:

(vi) $\int J_n(x)dx = -x^m - J_{n-1} + (m+n-1)\int x^{m-1} J_{n-1}(x)dx$
(vii) $\int J_{v+1}(x)dx = \int J_{v-1}(x)dx - 2J_v(x)$

It immediately follows from the identity $2J'_v(x) = J_{v-1}(x) - J_{v+1}(x)$
(viii) $\int J_{v-1}(x)dx = -x^{1-v} J_{v-1}(x) + c$
(ix) $\int x^{1-v} J_{v1}(x)dx = -x^{1-v} J_{v-1}(x) + c$
(x) $\int x^3 J_0(x)dx = -x^3 J_1(x) - 2^2 x J_2(x) + c$

Problem: Defining the Bessel function $J_n(x)$ by means of the general function

Example $\exp\{\frac{1}{2}x(t-t^{-1})\} = \sum_{n-\infty}^{\infty} J_n(x)$ show that,
If n is an integer

(a) $J_n(x) = (\frac{1}{2}x)^n \sum_{r=0}^{\infty} \frac{(-x\frac{x^2}{4})^r}{r1(n+r)1}$
(b) $J_{-n}(x) = (-1)J_n(x)$
(c) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$
(d) $J_{n-1}(x) + J_{n+1}(x) = 2J'_n(x)$

Solution

(a) Replace t by $-\frac{1}{t}$ in the definition

$$\exp\{\frac{1}{2}x(t-\frac{1}{t}) = \sum_{n=\infty}^{\infty} (-1)^n J_n(x)$$
$$= \sum_{n=\infty}^{\infty} (-1)^{-n} t^n J_{-n}(x)$$
$$= \sum_{n=\infty}^{\infty} t^n J_n(x)$$

Thus we get
$J_n(x) + (-1)^n J_{n+1} = J_{-n}(x)$

The exponential on the left can be expressed as a product of two exponential

$$\exp[\frac{x}{2}(t-t^{-1}) = \sum_{n=0}^{\infty} \frac{1}{n1}(\frac{x}{2})^n t^{-n}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m1}(-\frac{x}{2})t^{m-m}$$

Every product of a term of the first series by a term of the second contains a factor t^{n-m}. Let's associate with each term of the second series the term of the first series corresponding to $n = p + m(p > 0)$. The product contains the factors t^p. Therefore the series expansions of the coefficient $j_p(x)$ with $+ve$ p follow.

By associating with each term of the first series the term of the second series which corresponds to $m = p + n(p > 0$, the product contains now the factor t^{-p}. Thus $j_p(x)$ is obtained.

Problem: Prove that

$$j_o(z) = \frac{1}{2}\int_0^{2\pi} \cos(z\cos\theta)d0$$

Proof: We know that

$$\exp[\frac{z}{2}(t-t^{-1}) = \sum_{n\to\infty}^{\infty} j_n(z)t^n$$

Put

$t = ie^{-i\theta}$, take real parts of both sides and integrate between 0 and 2 ………..

$$\exp[\frac{z}{2}ie^{-i\theta} + ie^{+i\theta})] = \sum_{n=\infty}^{\infty} j_n(z)i^n(\cos\theta + \tan\theta)$$

$$j_0(z) + \sum_{n=1}^{\infty} j_n(z)i^n(\cos\theta + \tan\theta)$$

$$+\sum_{n=1}^{\infty} j_n(z)i^n(\cos\theta + \tan\theta)$$

$$\exp[z\cos\theta) + J_0(z)+"+"$$

Problem: Prove that

$$\int_0^{\frac{\pi}{2}} J_0(z\cos\theta)\cos\theta d\theta = \frac{\sin z}{z}$$

Solution

$$j_p(z) = (\frac{z}{2})^b \sum_{m=1}^{\infty} \frac{(-1)^m}{m1m+p1}(\frac{z}{2})^{2m}$$

Put $p = 0$

$$j_0(z) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m1m1}(\frac{z}{2})^{2m}$$

Replace 2 by $z\cos\theta$ multiply both sides of integrate between 0 to $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} j_0(z\cos\theta)\cos\theta d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)z^{2m}}{m1m12^{2m}} \int_0^{\frac{\pi}{2}} j_0(\cos\theta)^{2m} d\theta$$

Now $\int_0^{\frac{\pi}{2}} (\cos\theta)d\theta = \frac{2^{2m}(m1)^2}{(2m+1)1}$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m z^{2m}}{(2m+1)1}$$

Now we know that

$$\sin\theta = \theta - \frac{\theta^3}{13} + \frac{\theta^5}{15}............$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{13} + \frac{\theta^4}{15}............$$

$$= \frac{\sin\theta}{z}$$

3.1.1 Bessel Functions of the First Kind

In the definition of Bessel function $j_p(z)$ put $z = iy$, then (p integer)

$$j_p(iy) = e^{+-i\frac{p\pi}{2}I_p}(y)$$

$$j_p(iy) = e^{+-i\frac{p\pi}{2}I_p}(\frac{iyi}{2})^p \sum_{m=0}^{\infty} \frac{1}{m1m+p_1}(\frac{y}{2})^{2m}$$

(2) Bessel function of the second kind

Solution

$$\int_0^2 tJ_n(at)J_n(bt)dt$$

$$\frac{z\{aJ_n(bZ)J_n{'}(az) - bJ_n(az)J_n'(bz)\}}{b^2 - al}$$

Solution

$$2^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0$$
$$y_1 = J_n(at), y_2 = J_n(bt)$$

(i) $t^2 y'' + ty_1' + (a^2t^2 - n^2)y_1 = 0$
$t^2 y_2'' + ty_1' + (b^2t^2 - n^2)y_2 = 0$

by y_2 and (2) by y_1, and subtracts, we find

$$(y_2 y_2{}'') + t(y_2 y_1 - y_1 y_2'(b^2 - a^2)t^2 y_1 y_2$$

or

$$\frac{d}{dt}\{t(y_2 y_1{}'' - y_1 y_2') + t(y_2 y_1 - y_1 y_2') = (b^2 - a^2)ty_1 y_2'$$

or

$$\frac{d}{dt}\{t(y_2 y_1{}'' - y_1 y_2') = (b^2 - a^2)ty_1 y_2$$

Integrating with subtract to t from o to z yield

$$(b^2 - a^2)\int_0^2 t(y_2 y_1 dt = t(y_2 y_1 - y_1 y_2')$$

4.0 CONCLUSION

We have considered Bessel function in its general setting in this unit . You are required to read this unit carefully before going to the next unit.

5.0 SUMMARY

Recall that Bessel functions are usually associated with a class of equations called Bessel equations. They are usually denoted by the notation:

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2^{m+v}} r(m+v+1)m1}$$

We gave some examples to enable you understand the contents of this unit. We also examined another type of Bessel function usually referred to as Bessel Function of the First Kind. However, you are to master this unit properly before going into the next unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Given that

$$e^{\frac{x}{2}(r-\frac{1}{r})} = \sum_{n=-\infty}^{\infty} r^n J_n(x)$$

Deduce that $(n+1)J_{n+1}(x) = \frac{x}{2}[J_n(x) + J_{n+2}(x)]$

2. Obtain the general solution of each of the following equations in terms of Bessel functions, or if possible in terms of elementary functions.

(a) $x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + xy = 0$ (b) $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0$ (c) $x^4\frac{d^2y}{dx^2} + a^2y = 0$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison – Wesley Publishing Company.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

MODULE 3 SPECIAL FUNCTIONS AND PARTIAL DIFFERENTIAL EQUATION

UNIT 1 LEGENDRY FUNCTION

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider another class of special functions which has wide application in physical problems. This class of functions has orthogonality properties. The functions are legendry functions.

2.0 OBJECTIVES

At the end this unit, you should able to:

- identify legendry functions and legendry polynomial
- solve problems relating to legendry functions
- determine the properties of legendry functions and legendry polynomial.

3.0 MAIN CONTENT

3.1 Legendry Functions

The Legend differential equation of order *n* is given by:

$$(1-x^2)\frac{d^2y}{dx_2} - 2x\frac{dy}{dx} + \frac{dy}{dx} + p(p+1)y = 0$$

The solution of this equation is known as Legendry function

$$(1-x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} c_n n x^{n-1} + p(p-1)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} \{(n+2)(n+1)c_{n+2} - c_n[n(n+1) - p(p+1)]\}x^n = 0$$

Recurrence relation

$$(n+2)(n+1)c_{n+2} = c_2(n^2 + n - p^2 - p) \qquad (2)$$

Thus

$$c_{n+2} \frac{(p-n)(p+n+1)}{(n+2)(n+1)} c_n$$

Therefore

$$c_2 - \frac{p(p+1)}{21} c_o \; c_2 - \frac{p(p+1)}{21} c_o$$

$$c_3 - \frac{p(p+1)}{31} c_o$$

$$c_4 - \frac{p(p-2)(P+3)}{21} c_2$$

$$\frac{(p-2)(p)(p+1)(P+3)}{4.1} c_0$$

$$c_5 - \frac{(p-3)(P+4)}{5.4} c_2$$

$$\frac{(p-3)(p-1)(p+2)(P+4)}{4.1} c_0 etc$$

$$y_1 = 1 - p(p+1)\frac{x^2}{21} + (p-2)p(p+1)(p+3)\frac{x^4}{41} - - - - - - - -$$

$$y_2 = x - (p-1)(p+2)\frac{x^3}{31} + (p-3)(p-1)(p+2)(p+4)\frac{x^5}{51} - - - - - - - -$$

$$P_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2n-2k)1}{2^n k1n - k1n - 2k1}$$

Where m is the largest integer and runs greater than $\frac{n}{2}$.

In particular

$p_o(x) = 1$ $\qquad p_1(x) = \frac{1}{2}(3x^2 - 1)$

$$p_4(x) = \frac{1}{8}(35x^2 - 30x^2 - 3)$$

$p_1(x) = x$

$$\int_{-1}^{1} p_m(z)p_n(z)dz = 0 \text{ if } m \neq n$$

Solution:

$$(1-z^2)p_n p_m'' - p_m p_m''] - qz\{p_n p_m' - p_m p_n'$$
$$= [b(n+1) - m+1)P_m P_n = 0$$

and subtracting, we have

$$(1-z^2)\frac{d}{dz} p_n p_m'' - p_m p_m''] - 2z\{p_n p_m' - p_m p_n'$$
$$= [n(n+1) - m(m+1 p_n p_n$$
$$(1-z^2)\frac{d}{dz} p_n p_m'' - p_m p_n'] - 2z\{p_n p_m' - p_m p_n'$$

Integrate from -1 to 1 we have

$$[n(n+1) - m(m+1)]\int_0^1 p_m p_n d_z$$
$$(1-z^2)(p_n p_m' - p_n p_m')1_{-1}^{1} = 0$$

ii $\int_{-`}^{1} p_m(z)p_n(z)dz = \frac{2}{2n+n}$, if $m = w$

Solution:

$$\frac{1}{\sqrt{1-2zt+t^t}} = \sum_{n=0}^{\infty} p_n(z)t^n$$

Square is

$$\frac{1}{\sqrt{1-2zt+t^t}} = \sum_{n=0}^{\infty} p_n(z)t^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_m(z)p_n t^{m+n}$$

Integrating from -1 to 1

$$\int_{-1}^{1} \frac{dz}{1-2zt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \{\int_{-1}^{1} p_m(2)p_n(z)dz\}t^{m+n}$$

$$\sum_{n=0}^{\infty} \{\int_{-1}^{+1} p_n(2)[p_n(z)]^2 dz\}t^{2n}$$

L.H.S $= -\frac{1}{2^t}\log(1-2zt+t^2)1_{-1}^{1} = \frac{1}{t} \text{ in } \frac{1+t}{t-t}$

$$\sum_{n=0}^{\infty} \{\frac{2}{2n+1}\}t^{2n}$$

Equating the coefficients, we have

(iii) $(n+1)p_{n+1}(2)-(2n+1)zp_n+np_{n-1}(z)=0$

Solution: Differentiating with respect to both sides of the identity

$$\frac{1}{\sqrt{1-2zt+t^2}}=\sum_{n=0}^{\infty} pn(z)t^n$$

Multiply by $1-2z+t^2$, we have

$$(z-t)\sum_{n=0}^{\infty} pn(z)t^n=(1-2z+t^2)\sum_{n=0}^{\infty} npn(z)t^{n-1}$$

Or

$$\sum_{n=0}^{\infty} zpn(z)t^n-\sum_{n=0}^{\infty} \quad =npn(z)t^{n-1}-\sum_{n=0}^{\infty} 2nzp_n(z)t^n+\sum_{n=0}^{\infty} mp_n(z)t^{n+1}$$

Equating the coefficient problem: Show that

$$(1-2xz+z^2)^{-y2}=p_0(x)+p_1(x)z+p_2(x)z^2+..........\sum_{n=0}^{\infty} pn(x)z^n$$

Proof:-

$$(1-2xz+z^2)^{\frac{1}{2}}=1+\frac{1}{2}(2xz-z^2)+$$

$$\frac{\frac{1}{2}\frac{3}{2}(2xz-z^2)^2}{21+.......}+(2xz-z^2)+\frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})....[(2p-1)12]}{P1}(2x2-z^2)^B$$

The power of z^P can only occur in the term going from the pth term $(2xz-z^2)^p$ $[=z^p(2x-z)$ down. Thus, expanding the various powers of $(2x-z)$, we find that the Coefficient of z^p is

$$\frac{(\frac{1}{2})(\frac{3}{2})......[(2p-1)12]}{p1}(2x)^p$$

Prove that

$$p_n(z)=\frac{1}{2^n n1}\frac{d^{\lambda}}{dz^n}(z^2-1)^n$$

$$p_n(z)=\sum_{r=0}^{p} \frac{(-1)^r(2n-2r)}{2^n r1(n-r)1(n-2r)1}z^{n-2r}$$

Where p is $p \ \frac{1}{2}n$ or $\frac{1}{2}(n-1)$.

$$=\frac{1}{2^n n1}\frac{d^n}{dz^n}\sum_{r=0}^{\infty} \frac{(-1)^2 n1}{r1n-r1}z^{2n-2r}$$

$$= \frac{1}{2^n n1} \frac{d^n}{dz^n} \sum_{r=0}^{\infty} \frac{(-1)^2 n1}{r1n - r1} z^{2n-2r}$$
$$= \frac{1}{2^n n1} \frac{d^n}{dz^n} (z^2 - 1)^n$$

3.1.1 Legendry Polynomial

The equation $(1 - x^2)\frac{d^2 y}{dx} - 2x\frac{dy}{dx} + n(n+1)y = 0$
is called Legendry equation.

(i) $x = +1$ are the regular singular points of the equation. We solve the equation with the singular point $x = 1$, we put $x = 1 = u$ and obtain the transformed equation.

(ii) $u(u+2)\frac{d^2 y}{du^2} + n - (n+1)y = 0$ is the regular singular point.

We assume the solution point

$$y = \sum_{k=0}^{\infty} a_k u^{k+c}$$
$$\frac{dy}{du} = \sum_{k=0}^{\infty} a_k u^{k+c} u^{k+c1}$$
$$\frac{d^2 y}{du^2} = \sum_{k=0}^{\infty} (k+c)(k+c-1) a_k u^{k+c} u^{k+c-2}$$

The roots of the indicial equations are $c = 0,0\,0$. Hence one solution is logarithmic. We are only interested here in the non-logarithmic solution.
Hence

$$y = \sum_{k=0}^{\infty} a_k u^k$$

We assume a_0 is non-zero arbitrary constant, and

$$a_{k\frac{-(k-n-1)(k+n)}{2k^2}} a_{k-1}$$

Solving the recurrence solution, we have

$$a_k = \frac{(-1)^k (-n)_k (1+n)_k a_0}{2^k (k1)^2}$$

Thus the solution is

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (-n)_k (n+1)_k (x-1)^k}{2^k (k1)^2}$$

Where $a_0 = 1$

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{(-1)_k (n+1)_k}{(1)_k (k1)} \left(\frac{1-k}{2} \right)^k$$

$$y_1 = 2F_1(-n, n+1;1;(\frac{1-x}{2})$$

$$= P_n(x).$$

$P_n(x)$ is called the Legendry Polynomials

It is customary to take

$$c_n = \frac{2n1}{2^n (n1)^2}, \; n = 0,1,2,...$$

But from (3)

$$c_{n-2} = -\frac{n(n-1)}{(2)(2n-1)}, \qquad c_{n-2} \text{ ,or}$$

$$c_n = -\frac{(n)(2n-1)}{n(n-1)}, \quad c_{n=} - \frac{(2n-2)!}{2^n (n-1)!(n-2)}$$

$$c_{n-4} = \frac{(2n-4)}{2n2!(n-2)(n-4)}$$

or

$$c_n - 2k = \frac{(2n-2k)!(-1)^k}{2nk!(n-k)(n-2k)!}$$

Then the legendry Polynomials of degree n is given by

$$P_n(x) = \sum_{K=0}^{M} \frac{(-1)^k (2n-2k)!}{2^n k! n-k! n-k! n-2k!} x^{n-2k}$$

integer not greater that $\frac{n}{2}$.

$$(1)\, P_n(x) = \sum_{K=0}^{M} \frac{(-1)^k}{2^n k!(n-k)!} \frac{d}{dx^{n-1}} (x^{2n-2k-1})$$

Since

$$\frac{d}{dx^n}(x^{2n-2k}) = (2n-2k)\frac{d}{dx^{n-1}}(^{2n-2k-1})$$
$$(2n-2k)(2n-2k-1).......(n-2k+1)x^{n-2k}$$
$$\frac{2n-2k!}{n-2k!}x^{n-2k}$$

Hence
$$P_n(x) = \sum_{K=0}^{M} \frac{1}{2^n n1}\frac{d^n}{dx^n}\sum_{K=0}^{M} \frac{(-1)^k n1(x^2)^{n-k}}{k1(n-k)1}$$
We may now extend the range of this sum by taking k range from 0 to *n*. This extension will not affect the result, since the added terms are a polynomial of degree less than n and the nth derivative will vanish.
$$P_n(x) = \frac{1}{2^n n1}\frac{d^n}{dx^n}\sum_{K=0}^{M} \frac{(-1)^n n1(x^2)^{n-k}}{kn-k1}$$

and by binomial theories, we have
$$P_n(x) = \frac{1}{2^n n1}\frac{d^n}{dx^n}(x^2-1)^{n,} \qquad n = 0,1,2,.........$$

This is known as Rodrigues formula

Example: Show that
$$P_2(x) = \frac{1}{2}(3x^{2,}-1)$$
Solution: By Rodrigues' formula
$$P_2(x) = \frac{1}{421}\frac{d^2}{dx}(x^{2,}-1) = \frac{1}{8}\frac{d^2}{d_2}(x^4-2n^2+1)$$
$$= \qquad \frac{1}{2}(3x^2-1)$$

Problem: Show that

(i) $P_n' + 1(x) = (2n+1)P_n(x) + P_{n-1}'(x).\ n = 1,2,........(1)$

(ii) $P_n' + 1(x) = xP_n'(x) + (n+1)P_n(x).$ (2)

Solution

(i) $P_{n+1}'(x) = [\ \frac{d}{dx}\frac{1}{2^n n1}\frac{d^n}{dx^{n+1}}(x^2-1)^{n+1}.\ ]$

$$= \frac{d}{dx}[\frac{1}{2^n n1}\frac{d^n}{dx^n}[x(x^2-1)^n]] \quad (n+1)(n^2-1)^n 2x$$

$$= \frac{1}{2^{nn1}} = \frac{d^{n+1}}{dx^{n+1}}[x(x^2-1)^n]$$
$$= \frac{1}{2^{nn1}} = \frac{d^{n}}{dx^{n+1}}[x(x^2-)^n]$$
$$= \frac{1}{2^{nn1}} = \frac{d^{n}}{dx^{n}}[(2n+1)(x^2-1)^n + 2n(x^2-1)^{n-1}]$$
$$= [(2n+1)P_n(x)-1) + P'_{n-1}(x)$$

Solution (ii)

Now we have that

$$\frac{d}{dx}[f(x)(x) + f'(x)$$
$$\frac{d^2}{dx^2}[xf(x)] = f''(x) + 2f'(x)$$

and in general

$$\frac{d^{p+1}}{dx^{p+1}}[xf(x)] = (x)\frac{d^{p+1}}{dx^{p+1}}[x(x^2-1)^n$$

Now

$$p'_{n+1}(x)\frac{1}{2} + \frac{d^{n+1}}{dx^{p+1}}\frac{d^{n+1}}{dx^{n+1}}[x(x^2-1)^n$$
$$\frac{1}{2^n n1}[x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + (n+1)\frac{d^n}{dx^n} + (n+1\frac{d^n}{dx^n}(x^2-1)^n]$$
$$= xp'_{n+1} + (n=+1)p_n(x)$$

Eliminating p'_{n+1}, we have $np(x) = p'_n - p'(x) - p'_{n-1,} n = 1,2,........(3)$

Finally

$$(n+1)p_{n+1}(x) - (2_{n+1)} xp_n(x) + p_{n-1}(x)$$
$$= \frac{xp'_{n+1}(x) - p'_n(x)}{(3)} - \frac{x(p'_{n+1}(x) - p'_{n+1}(x)}{(1)}$$

Thus

$$-(n+1)p_{n+1}(x) + np_{n-1}(x) = (2n+1)xp_n(x)$$

$$py'' + \theta y' + Ry = 0 \qquad (A)$$
$$p\mu + (2p' - \theta)\mu' + (p'' - \theta' + R)\mu = 0 \qquad (B)$$
$$p\mu'' + (2p' - 2p' - \theta)\mu' + (p'' - 2p'' - \theta') + p'' - \theta + R)\mu = 0$$

4.0 CONCLUSION

You have learnt about legendry polynomial and legendry functions in this unit. Read this unit properly before going to the next unit.

5.0 SUMMARY

You will recall that the legendry polynomial is defined as:

$$P_n(x) = \sum_{K=0}^{M} \frac{(-1)^k (2n-2k)!}{2^n k! n-k! n-k! n-2k!} x^{n-2k}$$

This polynomial has Orthogonality property which we have mentioned in this unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that the substitution $t = 1 - x$ transform Legendre's equation to the form:

$$t(2-t)\frac{d^2y}{dt^2} + 2(1-t)\frac{dy}{dt} + p(p+1)y = 0$$

2. Problem: Show that
   a. $P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x).\ n = 1,2,........$
   b. $P'_{n+1} + (x) = xP'_n(x) + (n+1)P_n(x).$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.

UNIT 2 SOME EXAMPLES OF PARTIAL DIFFERENTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

A partial differential equation is an equation that contains one or more partial derivatives. Such equations occur frequently in application of mathematics. We shall only discuss certain partial differential equations which are used frequently in applied mathematics. In fact, we are going to discuss a kind of boundary value problems which enters modern applied mathematics at every turn.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- recognise partial differential equations by type and character
- explain the methods of solving partial differential equations
- apply the knowledge in some other related field.

3.0 MAIN CONTENT

3.1 Some Examples of Partial Differential Equations in Applied Mathematics

Many linear problems in applied mathematics involve the solution of an equation obtained by specialising the form.

$$\Delta^2\theta + f = \lambda \frac{d^2\theta}{dt^2} + \mu \frac{d\theta}{dt} \qquad (1)$$

Where f is a specified function of position and λ and μ are certain specified physical constant. Here, Δ^2 is the Laplacian operator in one, two or dimension under consideration and is of the form.

$$\Delta^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \qquad (2)$$

In rectangular co-coordinator of three space, the unknown function ϕ is the function of the position co-ordinates (x, y, z) and the tine t.

(i) Laplace Equation

$$\Delta^2\theta = 0 \qquad (3)$$

It is satisfied by the velocity potential in an ideal incompressible fluid without vertical or continuously distributed sources; and by gravitational potential in free space; electrostatic potential in the steady flow of electric currents in solid conductors, and by the steady-state temperature distribution in solids.

(ii) Poisson's Equation

$$\Delta^2\theta + f = 0 \qquad (4)$$

is satisfied, for example, by the velocity potential of an incompressible, irrotational, ideal fluid with continuously distributed sources or by steady temperature distribution due to distributed heat sources, and by a 'sheds function' involved in the elastic torsion of prismatic bars, with a suitably prescribed function f.

(iii) Wave Equation

$$\Delta^2\theta = \frac{1}{c}\frac{\partial^2\theta}{\partial t^2} \qquad (5)$$

This arises in the study of propagation of waves with velocity *c*, independent of the wave length. In particular, it is satisfied by the components of the electric or magnetic vector in electromagnetic theory, by suitably chosen component of displacement, in the theory of elastic vibrations, and by the velocity potential in the theory of sound (acoustics) for a perfect gas.

(iv) The Equation of Heat Conduction

$$\Delta^2\theta = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t} \qquad (6)$$

This is satisfied, for example, by the temperature at a point of a homogeneous body and by the concentration of a diffused substance in the theory of diffusion, with a suitably prescribed constant θ.

(v) The Telegraphic Equation

$$\frac{\partial^2\theta}{\partial x^2}+\lambda\frac{\partial\theta}{\partial t}+\mu\frac{\partial\theta}{\partial t} \qquad (7)$$

This is one dimensional specialisation of (1), and is satisfied by the potential in a telegraph cable, where $\lambda = Lc$ and $\mu = Rc,$ if the Leakage is neglected (L is inductance, c capacity and R resistance per unit length).

(vi) Differential equation of higher order, involving the operator Δ^2, are rather frequently encountered, in particular, the bi-Laplacian equation in two dimensions.

$$\Delta^4\theta = \Delta^2\Delta^2\theta = \frac{\partial^u\theta}{\partial x^u}+2\frac{\partial^2\theta}{\partial x^2\partial y^2}+\frac{\partial\theta}{\partial y^u}=0 \qquad (8)$$

is involved in many two dimensional problem of the theory of elasticity.

The solution of a given problem must satisfy the proper differential equation, together with similarly prescribed boundary condition or initial conditions (2 f time is involved).

The above equation can be changed to cylindrical co-ordinates $r\theta z,$ related to x, y and z by the equations

$$x = rCos\theta, y = rSin\theta, z = z$$

$$\Delta^2\theta = \frac{\partial^2\theta}{\partial r^2}+\frac{1}{2}\frac{\partial\theta}{\partial r}+\frac{1}{2}\frac{\partial\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\theta}{\partial\theta^2}+\frac{\partial^2\theta}{\partial z^2}=0 \qquad (9)$$

In spherical co-ordinates P,θ,θ related to x, y, z by the equations $x = PSin\theta Cos\theta, y = PSin\theta, z = PCos\theta$. Laplace' equation is

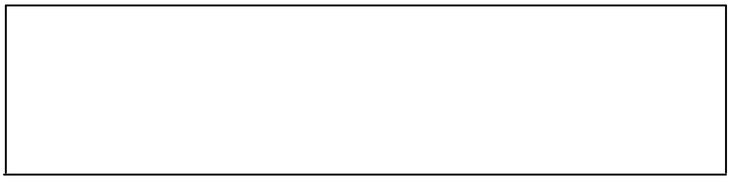
$$\frac{\partial^2 v}{dp^2}+\frac{2}{p}\frac{\partial v}{\partial e}+\frac{\partial^2 v}{\partial\theta}+\frac{\cot\theta}{p^2}\frac{\partial v}{\partial\theta}+\frac{Cos^2\theta}{p^2}\frac{\partial^2 v}{\partial\theta^2}=0. \qquad (10)$$

In what now follows we shall solution methods of partial differential equations:

Method of separation of variables
Consider the equation

$$\alpha^2\frac{\partial^2 v}{dx^2}=\frac{\partial u}{dt}, 0 < x < l, t > 0 \qquad (a)$$

This is called the heat conduction equation



This is a straight bar of uniform cross section and homogenous material. The temperature v can be considered constant on any given cross section.

$v = U(x,t).$

α^2 is a constant known as $v = U(x,t).$ In addition, we shall assume that the ends $v = U(x,t).$of the bar are held at temperature zero: thus $v = 0$ when and $x = e.$

$$\therefore u(o,t) = 0, \quad u(l,t) = 0, \qquad t > 0, \qquad (1)$$

Finally, the initial distribution of temperature in the bar is assumed to be given thus $U(x,o) = h(x) \qquad 0 \le x \le l$ (2)

(1) and (2) are called boundary conditions
We assume that

$$u(x,t) = f(x)g(t) \qquad (3)$$

Substituting equation (3) for $u(x,t)$ in (1) yields

$$\alpha^2 f''(x)g(t) = f(x)g'(t) \qquad (4)$$

or

$$\alpha^2 \frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)} \qquad (5)$$

Now equation (5) is said to have its variable separated; that is, the left member of equation (5) is a function of x alone and the right member of equation (5) is a function of x (5) alone.

Since x and t are independent variables, the only way in which a function of x alone can equal to function of t alone is for each function to be constant.

$$\therefore \frac{f''(x)}{f(x)} = b^{\theta} \qquad (6)$$

$$\alpha^2 \frac{g'(x)}{g(x)} = b \qquad (7)$$

In which b is arbitrary

The partial differential equation (1) has now been replaced by two ordinary differential equations. This is the essence of the **method of separation of variables.**

Boundary Conditions

$$xv(o,t) = f(o)g(t) = 0 \qquad (8)$$

by (1), if $g(t) = 0$ then ux,t will be identically zero. It is not acceptable because it does not satisfy the equation (2). Thus it must satisfy the condition

$$f(o) = 0 \qquad (9)$$

Similarly, the boundary condition at $x(l)$ $U(l,t) = 0$ requires

$$f(l) = 0 \qquad (10)$$

There are two possible values of the constant k i.e. $k = 0$ or $k \neq 0$.
Values of the constant k:

(i) $k = 0$, then the general solution of equation (6) is

$$f(x) = c_1 + c_2 \qquad (11)$$

(11) Must satisfy the boundary value conditions (9) and (10). In order to satisfy (9)

$$f(o) = c_1 + c_2 \Longrightarrow c_2 = 0 \qquad (12)$$

It is also satisfies the equation (10)

$\therefore f(l) = c_1 l = c_1 = 0$ Since $l \neq o.$

$$\therefore c_1 = 0 \qquad (13)$$

Hence, $f(x)$ is identically zero, and therefore $U(x,t0$ is also identical zero

(ii) $k \neq 0$, we take $k = -\lambda^2$, where λ is a new parameter. Thus, the equation (6) becomes

$$f''(x) + \lambda^2 f(x) = 0 \qquad (14)$$

and its general solution is

$$f''b_1 e^{i\lambda x} + b_2 e^{i\lambda x} \qquad (15)$$

Applying the boundary condition (9) and (10), we have

$$\left.\begin{aligned} &b_1 + b_2 = 0 \\ &f''b_1 e^{i\lambda x} + b_2 e^{i\lambda x} = 0 \end{aligned}\right\} \qquad (16)$$

The system (16) has a non-trivial solution $k_1 = 0$ and $k_2 = 0$ always, but it is not acceptable $u(x,t)$ is identically zero. Non-trivial solution exists if and only the determinant.

$$\left| \, i_{e^{i\lambda e}} i_{e^{i\lambda e}} \, \right| = 0 \qquad (17)$$

If we write, $\lambda = \mu + iv$ then

$$e^{-vie} e^{ve} - e^{i\mu e} e^{-ve} = 0 \text{ or}$$

$$e^{-ve}(\cos \mu e - 1\sin \mu e) - e^{-ve}(\cos \mu + \sin \mu e) = 0$$

$$\left.\begin{aligned} &\cos \mu e(e^{ve} - e^{-ve}) = 0 \\ &\sin \mu e(e^{ve} - e^{-ve}) = 0 \end{aligned}\right\} \qquad (18)$$

Now $\cos \mu e(e^{ve} - e^{-ve} > 0$ for v and l, thus $\sin \mu e = 0 \Rightarrow v = 0$ (19) must be so chosen that

$$\mu \frac{n\pi}{l}, \qquad (20)$$

where n is a non-zero integer. From (16) $k_1 = -k_2$ (21)

From (15), we have

$$f(x) = b_1(e^{-\mu ie} e^{ve} - e^{i\mu e}) = \frac{b_i}{2}(e\frac{^{-\mu ie} e^{i\mu e}}{2})$$

Thus, $f(x)$ is proportional to $\sin \mu x$ (23)

$$k = -\lambda^2 = -\frac{n^2\pi^2}{l^2} \qquad (24)$$

Where n is an integer

From (7), we have (25)

Hence the function

$$u_n(x,t) = c_n \exp[-\frac{n^2\pi^2\alpha^2 t}{l^2}]\sin\frac{n\pi x}{l} \qquad (26)$$

$n = 1,2,3,\ldots\ldots$ where c_n is an arbitrary constant, satisfies the boundary conditions $2,9,10$ as well as the differential equation (1). The functions u_n are sometimes called fundamental solution of the heat conduction problem (a) (1) and (2).

By the boundary condition (2) we get from (26).

$$u_n(x,o) = c_n \sin\frac{n\pi x}{l} \qquad (27)$$

For $n = 1,2,\ldots\ldots\ldots$

Each solution given by (27) satisfies the differential equation and the boundary condition. Since partial differential equation involved is linear and homogeneous in u and its derivatives, a sum of solution are also a solution. From the known solutions, $u_{1,}u_2\ldots\ldots\ldots u_n,\ldots\ldots\ldots$ we may thus construct others with sufficiently strong convergence condition. It is true that even the infinite series

$$u = \sum_{n=1}^{\infty} u_n \text{ or}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\Big(-\frac{n^2\pi^2\alpha^2 t}{l^2}]\sin\frac{n\pi x}{l} \qquad (28)$$

is a solution of the differential equation. In order to satisfy the initial condition (2) we must have

$$u(x,o) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} h(x) \qquad (29)$$

Now let us suppose that it is possible to express $h(x)$ by means of an infinite series forms

$$h(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (30)$$

We know how to compute b_n i.e

We can satisfy the equation (29) by choosing $c_n = b_n$ for each n. With the coefficient selected in this manner, equation (28) gives the solution of the boundary value problem (a) (1) and (2)

Thus, we have solved the problem consisting of the heat condition equation.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \; 0 < x < l, \qquad t > o \tag{1}$$

The boundary condition

$$u(o,t) = 0, \qquad u(l,t) = 0, \qquad t > o \tag{2}$$

and the initial condition

$$u(x,o) = h(x), \qquad 0 \le x \le l \tag{3}$$

we found the solution to be

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp[\frac{n\pi^2 \alpha^2 t}{l^2}] \sin \frac{n\pi x}{l} \tag{4}$$

With the coefficients b_n are the same as in the series

$$h(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{5}$$

Where

$$b_n = \frac{2}{l} \int_0^l h(x) \sin \frac{n\pi x}{l} dx \tag{6}$$

The series in equation (5) is just the Fourier

Example 2: If we consider the problem of the heat conduction equation of boundary conditions and the initial condition, the boundary conditions are known as non-homogeneous boundary condition.

Solution: If we shall reduce the present problem to one having homogeneous boundary condition, we use the physical argument. After a long time, i.e. ……………, we anticipate a steady state temperature distribution ……………….. will be reached, and must satisfy difficulties (1), then (which is independent of time t and initial condition).

…………. (4)

and it satisfied the boundary condition

…………. (5)

Which apply even as …………….. The solution (4) with condition (5)

………….. (6)

Hence the steady state temperature is a linear function of x.

We shall express $U(x,t)$ as the sum of the steady state temperature and another distribution w(x, t).

$$\therefore U(x,t) = U(x) + (x,t) \quad (7)$$

(7) satisfies (1), we have

$$\alpha^2 (u+w)xx = (u+w)_t \quad (8)$$

It follows that

$$\alpha^2 W_{xx} = W_t \quad (9)$$

Now boundary condition

$$w(o,t) = u(o,t) - u(o) = T_1 - T_1 = 0 \quad (10)$$

$$w(l,t) = u(l,t) - u(l) = T_2 - T_2 = 0 \quad (11)$$

The initial condition

$$w(x,o) = (u(x,o) - u(x) = f(x) - u(x) \quad (12)$$

Where u(x) is given by (6)

The problem now becomes precisely the previous one and we have the solution

$$W(x,t) = \sum_{n=1}^{\infty} b_n \exp\left[-\frac{n^2\pi^2\alpha^2 t}{l} \right] Sin \frac{n\pi x}{l} \quad (13)$$

Where

$$b_n = \frac{2}{l}\int_0^l W(x,0) Sin \frac{n\pi x}{l} dx \quad (14)$$

Where

$$U(x,t) = (T_2 - T_1)\frac{x}{l} + T_1 + \sum_{n=1}^{\infty} b_n \exp\left[-\frac{n^2\pi^2\alpha^2 t}{l} \right] Sin \frac{n\pi x}{l} \quad (15)$$

Where

$$b_n = \frac{2}{l}\int_0^l \left[f(x) - (T_2 - T_1)\frac{x}{l} - T_1 \right] Sin\frac{n\pi x}{l}dx$$

(16)

Example 3: Now we consider the problem of the heat conduction equation with the boundary condition

$$\alpha^2 U_{xx} = U_x \qquad (1)$$

$$V_x(0,t) = 0 \ , \quad V_x(l,t) = 0 \quad , \qquad t > 0 \qquad (2)$$

and the initial condition

$$v(x,0) = f(x) \qquad (3)$$

Solution: We solve the equation by the method of separation of variables. [when the ends of the bar are insulated so that there is no passage of heat through them].

$$V(x,t) = h(x)g(t) \qquad (4)$$

(4) satisfies (1), we have

$$\frac{h'(x)}{h(x)} = \frac{1}{\partial^2}\frac{g'(t)}{g(t)} - \alpha \qquad (5)$$

We assume that α is real, we consider three cases α = 0 and –ve.

(i) If α = 0 then equation (5) given

$$V(x,t) = K_1 x + K_2 \qquad (6)$$

Applying boundary condition (2), we get

$K_1 = 0.$ Hence corresponding

$h'(x) = 0$

$h(x) = C_1 + C_2$

$$g'(t) = 0 \rightarrow g(t) = C_B \qquad (7)$$

Solution is

(ii) If $\alpha = \lambda^2$ where λ is real and +ve

$$\therefore h''(x) - \lambda\, h(x) = 0 \qquad (8)$$

$$g'(t) - \lambda^{2=} h(x) = 0 \qquad (9)$$

From (8) and (9), we have

$$U(x,t) =_{ex} \alpha^2 \lambda^{2^t} (k_1 Sinh\lambda x + k_2 Cosh\lambda x). \qquad (10)$$

Now we apply the boundary condition (2)

$U_x(x,t) = e^{\alpha 2\lambda 2}(k_1\lambda Cos\lambda x + K_2 Six\lambda x)$
$U_x(o,t) = 0$ and $U_x(l,t) = 0$
$K_2\lambda Sin\lambda l = 0 \Rightarrow K_{21}0$
$K_2\lambda Sin\lambda l = 0 \Rightarrow K_{21}0$
$\rightarrow K_1 = 0$ and $K_2 = 0$

This is not acceptable, because it does not satisfy the initial condition of some examples of partial differential equation, hence σ, can not be positive.

(iii) $\sigma = -\lambda^2$, where λ is +ve and real. From (8) and (9), we obtain

$$U_x(x,t) = e^{\alpha 2\lambda 2}(k_1\lambda Cos\lambda x + K_2 Cos\lambda x) \quad (11)$$

Now we apply boundary condition, we get (x=0) $K_1 = 0$ and (x=e) $\lambda \frac{\lambda}{e}$ for n=1,2,…… $Sin\theta = 0, \theta = n\pi)$

$$\therefore \sigma = -(\frac{n^2\pi^2}{e^2}), \text{ where n is +ve integer} \quad (12)$$

Combining the solution, we have

$$U_o(x,t) = \frac{1}{2}\text{Co} \quad (13)$$

$$U_n(x,t) = C_n \exp[\ -\frac{n^2\pi^2\alpha^2 t}{e^2}\ ]Cos\frac{n\pi x}{e}, n = 1,2,..... \quad (14)$$

These solution functions satisfy the differential equation (1) and bounding conditions (2) for any value of the constant C_n.Both differential equation and boundary values are linear and homogeneous, any finite sum of the fundamental solutions will also satisfy them. We will assume that this is also line for convergent infinite sums of fundamental solution as well.

Thus

$$U(x,t) = \frac{1}{2}Co + \sum_{n=1}^{\infty} c_n \exp[\ -\frac{n^2\pi\ \alpha^2 t}{e^2}\ ]Cos\frac{n\pi x}{e}$$
(15)

Where C_n are determined by the initial requirement that

$$U(x,o) = \frac{1}{2}Co + \sum_{n=1}^{\infty} c_n Cos\frac{n\pi x}{e} = f(x) \qquad (16)$$

Thus, the unknown coefficients in equation (15) must be coefficients in the Fourier Cosine series of period 2e for f. Hence

$$U_n \frac{e}{e}\int_o^l f(x)Cos\frac{n\pi x}{e}dx, n = 0, \quad 1, 2, \ldots\ldots\ldots\ldots$$

With this choice of the coefficients, (15) provides the solution of the equation.

Example: Elastic string with non-zero initial displacement

First, suppose that string is displaced from its equilibrium position, and then released with zero velocity at time t = 0 to vibrate freely. Then in vertical displacement U(x,t) must satisfy the wave equation.

$$X^2U_{xx} = U_{tt} \qquad 0<x<l,\ t>0 \qquad (1)$$

The boundary conditions are

$$U(o,t) = 0, \quad U(l,t) = 0, \quad t \geq 0 \qquad (2)$$

and the initial conditions

$$U(x,o) = f(x), U_t(x,0) = 0, \quad 0 \leq x \leq l \qquad (3)$$

Where f is given function describing the configuration of the string at t = 0

Solution: We use the equation (1) by the method of separation of variables.

Assuming that

$$U(x,t) = X(x) \quad T(t) \qquad (4)$$

Substituting u in (1), we get

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{Tt}{T} = \sigma$$

(4)

We assume that $\underline{\sigma \text{ is real}}$ (we shall prove it somewhere else. We consider these cases σ = 0, -ve and +ve.

(i) $\sigma = 0$, then $X'' = 0$, and $X(x) = K_1 x + K_2$ (5)

(ii) If $\sigma > 0$, then $X'' - \lambda^2 x = 0$ and $X(x) = K_1 Sinh\lambda x + K_2 Cosh\lambda x$ (6)

Where $\lambda = \sqrt{\sigma}$

Consider the solution given by (5)

$U(x,t) = (K_1 x + K_2)T(t)$

By boundary conditions $U(o,t) = 0$

$U(o,t) = (0 + K_2)T(t) = 0,$ $T(t)$ can not be zero, because U(x,t) will be identically zero. Thus, $K_2 = 0$. Next consider the second boundary condition $U(l,t) = 0$ then $U(l,t) = (K_1 l)T(t) = 0 \Rightarrow K_1 = 0$

Thus X(x) = 0, it is not acceptable.

(iii) Similarly, we can show that for (6) under boundary condition $K_1 = 0 = K_2$.

Thus, $\sigma = 0$ and $\sigma = +ve$ real number are not acceptable. We now consider the last cast

(ii) $\sigma = -\lambda x = 0$
(7)
$X'' + \lambda X = 0$
(8)
$T'' + \lambda^2 \alpha^2 T = 0$

$\therefore X(x) = K_1 Sin\lambda + K_2 Cos\lambda x$ (9)

$T(t) = K_2 Sin\lambda\alpha t + K_u Cos\lambda\alpha t$ (10)

Thus

$U(x,t) = (K_1 Sin\lambda x + K\ Cos\lambda\alpha t)(K_3 Sin\lambda\alpha t + K_4 Cos\lambda\alpha t)$ (11)

Satisfies (1) for all values of K_1, K_2, K_3, K_4 and for and $\lambda > 0$.

Now we impose the boundary conditions

$U(o,t) = 0,$ Thus

$U(o,t) = K_2(K_3 Sin\lambda\alpha t + K_4 Cos\lambda\alpha t) \Rightarrow K_2 = 0$ (12)

Secondly, boundary condition $U(l,t) = 0,$

$$U(l,t) = (K_1 Sin\lambda l)(K_3 Sin\lambda\alpha t + K_u Cos\lambda\alpha t) = 0 \qquad (13)$$

If $K_1 = 0,$ then U (x,t) is zero identically, thus for non-trivial solution.

$$Sin\lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}, n = 1,2 .... \qquad (14)$$

Hence the functions which satisfy the equation (1) and boundary condition (2) are of the form.

$$U_n(x,t) = Sin\frac{n\pi x}{l}(C_n Sin\frac{n\pi\alpha t}{l} + K_n Cos\frac{n\pi\alpha t}{l}) \qquad (15)$$

Where n=1,2,…….. C_n and K_n are arbitrary constants. Now we apply the principle of super position of solution and assume that

$$U_n(x,t) = \sum_{n=1}^{\infty} U_n(r,t)$$

$$\sum_{n=1}^{\infty} Sin\frac{n\pi x}{l}(C_n Sin\frac{n\pi\alpha t}{l} + K_n Cos\frac{n\pi\alpha t}{l})$$

(16)

Further, we assume that (16) can be differentiated term by term with respect to t

$U_t(x,o) = 0$ yields

$$U_n(x,o) = \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} Sin\frac{n\pi x}{l} = 0$$

(17)

$\Rightarrow C_n = 0$ for all values of n

$\Rightarrow$ The other condition $U(x,o) = f(x)$

Given

$$U_n(x,o) = \sum_{n=1}^{\infty} K_n Sin\frac{n\pi a}{l} = f(x)$$

(18)

Consequently, K_n must be line coefficients in the Fourier Series of period 2 l for f and are given by

$$K_n \frac{2}{l}\int_0^l f(x) Sin \frac{n\pi a}{l} dx, n = 1,2,.... \qquad (19)$$

Thus, the formal solution of the problem (1) with condition (2) and (3) is

$$U_n(x,t) = \sum_{n=1}^{\infty} K_n Sin \frac{n\pi a}{l} Cos \frac{n\pi\alpha x}{l} \qquad (20)$$

Where the coefficients K_n are given by (19).

For a fixed value of n the function

$Sin \frac{n\pi a}{l} Cos \frac{n\pi\alpha x}{l}$ is periodic in time t with the period $\frac{2l}{n\alpha}$; it therefore represents a vibratory motion of the string having this period or having the frequency $\frac{n\pi\alpha x}{l}$. The quantities $\lambda a = \frac{n\pi a}{l}$ for n=1, 2,…… are the natural frequencies of the string. The factor $K_n Sin \frac{n\pi a}{l}$ represents the displacement pattern occurring in the string, when it is executing vibrations of the given frequency.

In the case of heat conduction problem, it is attempting to try to show this by directly substituting equation (20) for U(x,t) in (1), (2) and (3), we compute.

$$U_{xx}(x,t) = -\sum_{n=1}^{\infty} K_n (\frac{n\pi}{l})^2 Sin \frac{n\pi x}{l} Cos \frac{n\pi at}{l} \qquad (21)$$

Due to the presence of the factor n^2 in the numerator, this series may not converge. It is not possible to justify directly with respect to either variable in 0,l and t>0, provided h is twice continuously differentials on $(-\omega, \omega)$. This require f' and f'' are continuous on [o,l]. Furthermore, since h'', we must have

$f''(o) = f''(l) = 0$

Example: General problem for inelastic string.

Consider the equation

$\alpha^2 U_{xx} - U_{tt} = 0$, o,<x,l, t>o; (1)

The boundary condition

U (o,t) = 0, U(l,t) = 0 (2)

and the initial conditions

$$U(x,o) = f(x), Ut(x,o) = g(x), 0 \le x \le 0 \qquad (3)$$

Solution: As we have done in the previous case, we obtain the solution

$$U_{xx}(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} Sin\frac{n\pi x}{l} C_n Sin\frac{n\pi\alpha x}{l} + K_n Cos\frac{n\pi\alpha t}{l} \qquad (4)$$

Applying the initial condition U(x,o) = f(x) yields

$$U(x,o) = -\sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l} = f(x) \qquad (5)$$

Where the coefficients K_n are given in the Fourier Sine Series of period 2l for f and are given

$$K_n \frac{2}{l}\int_0^l f(x) Sin\frac{n\pi a}{l} dx, n = 1,2,.... \qquad (6)$$

Differentiate (4) with respect to t and putting substitution. We establish the validity in a different way. We show

$$U(x,t) = \frac{1}{2}[\ h(x-\alpha t) + h(x+\alpha t)\ ] \qquad (22)$$

Where h is function obtained by extending the initial data f(x) into (-l,o) as an odd function, and other values of x as a periodic function on period 2l.

That is

$h(x+2l) = h(x)$.

Now

$$h(x) = \sum_{n=1}^{\infty} K_n Sin\frac{n\pi x}{l}$$

Then

$$h(x-\alpha t) = \sum_{n=1}^{\infty} K_n (Sin\frac{n\pi x}{l} Cos\frac{n\pi\alpha t}{l} - Cos\frac{n\pi\alpha t}{l} Sin\frac{n\pi\alpha t}{l})$$

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} Sin\frac{n\pi x}{l} C_n Sin\frac{n\pi\alpha x}{l} + K_n Cos\frac{n\pi\alpha t}{l}$$

Equation h is the function (20) follows on adding the two equations.

Note: [If $f(x)$ has a Fourier series, then it must be periodic and continuous].

1. If $U(x,t)$ is continuous for $0 < x < l$ and $t > 0$ provided that h is requires that f is continuous on line interval $(-\infty,\infty)$. This requires that f is continuous on the interval (o,l) and, since h is odd periodic extension of f, that f be zero at x=0 and x=p.
2. U is twice continuously differentiable t=0, we get

$$U_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{e} C_n Sin \frac{n\pi a}{l} = g(x) \qquad (7)$$

Hence the coefficients $\vartheta(\frac{n\pi a}{l})C_n$ are the coefficients in the Fourier Sine series of period 2l for g: Thus

$$\frac{n\pi a}{l} = C_n \frac{2}{l}\int_0^l \left[\ g(x) Sin \frac{n\pi x}{l} dx, \quad \text{n=1,2.........} \qquad (8)$$

Thus, the equation (4) with the equation (6) and (8) constitutes the formal solution of the equation (1).

Example: Laplace equation: One of the most important of all the partial differential equations occurring in applied mathematics is associated with the name of Laplace. Here is Laplace equation in two dimensions.

$$U_{xx} + U_{yy} = 0 \qquad (1)$$

and in dimension

$$U_{xx} + U_{yy} = 0 \qquad (2)$$

Now solve (1) under the boundary condition. The problem of finding a solution of Laplace equation which takes on given boundary values is known as *Dirichilet* problem.

Problem I: Solve the Laplace equation

$$U_{xx} + U_{yy} = 0 \qquad (1)$$

In the rectangle $0 < x < \alpha, \quad 0 < y < b,$ and which satisfies the boundary condition

$$U(x,0) = 0, \qquad U(x,b) = 0, \qquad 0 < x < a, \qquad (2)$$
$$U(o,y) = 0, \qquad U(a,y) = f(y), \qquad 0 \le y \le b$$

Where f is given function on $0 \le y \le b$

Solution:

$U(x, y) = X(x)Y(y)$ (3)

Substituting $U(x, y)$ in (1), we get

$\frac{X''}{X} = -\frac{Y''}{Y} = K$ (4)

We assume that k is real.

(i) If K=0 then $X'' = 0$ and $Y'' = 0$, and $U(x, y) = (K_{1x} + K_2)(C_1 y + C_2)$
(5)

The homogeneous boundary conditions y=0 and y=b can be satisfied y $C_1 = C_2 = 0 \Rightarrow U(x, y),$ is identically zero. Hence K=0 is acceptable.

(ii) $K = \lambda^2, \lambda > 0,$ then

$X'' = \lambda^2 X = 0$

$Y'' = \lambda^2 Y = 0$

and thus

$U(x, y) = (K_1 Sinh\lambda x + K_2 Cos\lambda x)(C_1 Sin\lambda y + C_2 Cos\lambda y)$
(6)

In order to satisfy the boundary conditions $x = 0$ and $y = 0 \Rightarrow K_2 = 0 = C_2$

The condition at y=b becomes

$K_1 C_1 Sin\lambda x Sin\lambda b = 0$
(7)

$\Rightarrow Sin\lambda b = 0$

It follows that

$\lambda b = n\pi,$ =1,2,3,……….. (8)

Thus, the solution of the differential equation must be of the forms.

$U_n(x, y) = C_n Sinh\frac{n\pi x}{b} Sin\frac{n\pi x}{b,}$, n=1,2,3,…. (9)

There functions are the fundamental solution of the present problems. We assume

$U_{xx}(x, y) = \sum_{n=1}^{\infty} U_n(x, y) = \sum_{n=1}^{\infty} C_n Sin\frac{n\pi x}{l} C_n Sinh\frac{n\pi\alpha x}{b} + Sin\frac{n\pi\alpha t}{b}$ (10)

Now the last boundary conditions

$$U_{xx}(a,y) = f(y) = \sum_{n=1}^{\infty} C_n Sinh\frac{n\pi a}{b} Sinh\frac{n\pi x}{b} Sin\frac{n\pi y}{b} \qquad (11)$$

Thus, the coefficients $C_n Sinh\frac{n\pi a}{b}$ must be the coefficients in the Fourier Sine series of period 2b for f(y) and are given by

$$C_n Sinh\frac{n\pi a}{b}\frac{2}{b}\int_o^b f(y) Sin\frac{n\pi y}{b}dy \qquad (12)$$

Thus, (10) is the solution of the equation (1) satisfying the boundary condition (2) and coefficients $C_n Sinh\frac{n\pi a}{b}$ are computed from (12).

(iv) If $K = -\lambda^2{}_n$ then

$X'' + \lambda^2 X = 0$

$X'' + \lambda^2 Y = 0$

and

$U(x,y) = (K_1 Sinh\lambda x + K_2 Cos\lambda x)$

$C_1(x,y)Sinh\lambda y + C_2 Cosh\lambda y)$

(13)

Again, the boundary condition at y=0 and y=b lead to $C_1 = C_2 = 0$, so again U(x,y) is zero, everywhere. Hence $K = \lambda^2$ is not acceptable.

Problem: Dirichilet problem for a circle

Consider the Laplace equation in polar co-ordinates

$$U_{zz} + \frac{1}{z}U_2\frac{1}{2} + U_2\frac{1}{z^2}U\theta\theta \qquad (1)$$

With boundary condition

$U(a,\theta) = (f(\theta)$ (2)

f is a given function on $0 \le \theta \le \theta 2\pi$.

Moreover, in order that U(z, o) the single valued, it is necessary that, as a function of θ, U must be periodic with period 2π.

Solution: Let $U(z,o) = R(z)\theta(\theta)$ (3)

We substitute (3) in (1)

$$r''\theta + \frac{1}{2}R'\theta + \frac{1}{r^2}R\theta'' = 0$$

Or

$$r^2\frac{R''}{R} + z\frac{R'}{R} = -\frac{\theta''}{\theta} = K \qquad (4)$$

Again, we assume that the separation constant must be real.

(1) Suppose K=0

$$r^2R'' + rR'\theta = 0$$

$$\theta'' = 0$$

$$\therefore U(r,\theta) = (K_1 + K_2 Log 2$$

$$\theta(\theta) = C_1 + C_2\theta$$

$$\therefore U(r,\theta) = (K_1 + K_2 Log 2(C_1 + C_2) \qquad (5)$$

Since equation is periodic in θ, thus $C_2 = 0$.
Further $r \to 0$ the term log r is unbounded. This behaviour is unacceptable. Thus, we impose the condition that U (z, o) remains finite at all points of the circle and hence we must take

$K_2 = 0$

$$\therefore U_0(r,o) = \text{Constant} = \frac{1}{2}c_0 \text{ say} \qquad (6)$$

(ii) If $K = \lambda^2$ then

$$\theta'' - \lambda^2\theta = 0 \qquad (7)$$

$$\therefore \theta(\theta) = C_1^{\lambda\theta} + C_2e^{-\lambda\theta} \qquad (8)$$

The function $U(r,o)$ is periodic thus $C_1 + C_2 = 0$.
This makes $U(r,0)$ identically zero. This is not acceptable.

(iii) Finally, $K = \lambda^2, \lambda > 0,$ yields

$$r^2R'' + rR'' - \lambda^2R = 0 \qquad (9)$$

And

$$\theta'' + \lambda^2\theta = 0 \qquad (10)$$

$$\therefore R(r) = K_1r^{\lambda} + K_2r^{-\lambda}$$

$$\theta(\theta) = C_1 Sin\lambda\theta + C_2 Cos\lambda\theta \qquad (11)$$

In order that θ be periodic with period 2π, it is necessary that λ be a positive integer.

Moreover, the solution $r^{-\lambda}$ of (a) be discarded, since it becomes unbounded as $r \to o$. Consequently, $K_2 = 0$. Hence the solutions (1) are
$U_n(r,o) = r^n(C_n Cosn\theta + K_n Sinn\theta)$, n=1,2,…………

These functions, together with that of equations (6), serve as fundamental solutions of the present problem. Thus

$$U(r,\theta) = \frac{1}{2}Co + \sum_{n=1}^{\infty} r^n(C_n Cosn\theta + K_n Sinn\theta) \qquad (13)$$

The boundary condition (2) then requires that

$$f(\theta) = U(a,\theta) = \frac{1}{2}Co + \sum_{n=1}^{\infty} a^n(C_n Cosn\theta + K_n Sinn\theta)$$

(14)

for $0 \le \theta \le 2\pi$.

The function $f(\theta)$ may be extended outside the interval. So also it is periodic of period 2π, and has a Fourier series of the function (14).

$$a^c C^n = \frac{1}{2}\int_0^{2\pi} f(\theta)Cosn\theta d\theta \qquad (15)$$

$$a^c K_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)Sinn\theta d\theta \qquad (16)$$

With this choice of coefficients (13) represents the solutions of the boundary value problem of equations (1) and (2).

(1) The heat conduction equation in two space dimension may be expressed in terms of polar co-ordinates as

$$\alpha^2(U_{zz} + \frac{1}{2}U_2 + \frac{1}{2^2}U\theta\theta) = U_t$$

Assuming that

$U(r,\theta,t) = R(r)\theta(\theta)T(t)$. Find ordinary equation satisfied by $R(r),\theta(\theta)T(t)$

The integrand is an even function.

$$= \int_0^C (1 - Cos\frac{2n\pi x}{c})dx$$

$$= [\ x - \frac{C}{2n\pi}Sin\frac{2n\pi x}{c}\ ]_0^c = C. \qquad \text{n=1,2,…………….}$$

Similarly, we can obtain

$I_5 = \int_{-c}^{C} Cos^2 \frac{n\pi x}{c} dx = C$ for n=1,2,3…………

$= 2C$ for n=0.

Similarly, we can show that

$I_4 = \int_{-c}^{C} Sin^2 \frac{n\pi x}{c} Sin \frac{k\pi x}{c} dx = C$ if $k = n$

$I_5 = \int_{-c}^{C} Cos \frac{n\pi x}{c} Cos \frac{k\pi x}{c} dx = C$ if $k = n$

Periodic Function: A function f is said to be periodic with period T if the domain of f contains $x+T$ whenever it contains x, and y.

$f(x+T) = f(x)$ for every value of x.

$f(x+T) = f(x)$

With fundamental period $T = 21/m$, every such function has the period (2l)

The function $Sin^2 \frac{m\pi x}{l} and Cos \frac{m\pi x}{l}, n = 1,2,\ldots\ldots\ldots$ are periodic.

3. Fourier Series

We assume that there exists a series expansion of the type

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [\ a_n Cos \frac{n\pi x}{C} b_n Sin \frac{n\pi x}{C}\ )$

(1)

Valid in the interval $-C \le x \le C$

$j_p(z) = (\frac{z}{2})^b \sum_{m=1}^{\infty} \frac{(-1)^m}{m1m+p1} (\frac{z}{2})^{2m}$

(1) is called the Fourier series corresponding to f(x), $a_n and b_n$.

Multiply (1) by $Sin(\frac{K\pi x}{C})dx,$ where k is a *tve* integer, and then integrate each term from –c to c, thus arriving at

$\int_{-c}^{C} f(x) Sin \frac{b\pi x}{c} dx = \frac{1}{2} ao \int_{-c}^{c} \frac{n\pi x}{C} dx + \sum_{n=1}^{\infty} [\ a_n \int_{-c}^{c} = Cos \frac{n\pi x}{C} Sin \frac{k\pi x}{C} dx\ ]$

(2)

As seen earlier

$\int_{-c}^{C} Cos \frac{n\pi x}{C} Sin \frac{k\pi x}{C} dx = 0$ for all k and n

And

$\int_{-c}^{C} Sin \frac{n\pi x}{c} Sin \frac{k\pi x}{C} dx = 0$ if $k \ne n$

=c if $k \ne n$

Using (2), we have

$\int_{-c}^{C} f(x) Sin \frac{b\pi x}{C} dx = \frac{cbk}{e},$ K= 1,2,3 ,,,,,,,,,,,,,,,,,,, or

$$b_n \frac{1}{c} ao \int_{-c}^{c} f(x) Sin \frac{n\pi x}{C} dx,$$ n=1,2, ………..

Let us now evaluate the coefficients $a_{n.}$ using the multiplies $Cos \frac{k\pi x}{C} dx$ throughout equation (1) and then integrating term by term for –c to c, we get

$$\int_{-c}^{C} f(x) Cos \frac{k\pi x}{C} dx = \frac{1}{2} a_o \int_{-c}^{c} Cos \frac{k\pi x}{C} dx + \sum_{n=1}^{\infty} [a_n \int_{-c} Cos \frac{n\pi x}{C} Cos \frac{k\pi x}{C} dx + b_n \int_{-c}^{c} Sin \frac{n\pi x}{c} Cos \frac{k\pi x}{c} dx$$

(4)

Now we know that

$$\int_{-c}^{C} Cos \frac{n\pi x}{C} Cos \frac{k\pi x}{C} dx = 0 \quad \text{for } n \neq k$$

$$= c \quad \text{for } n \neq k$$

If $k \neq 0$, (4) reduces to

$$\int_{-c}^{C} f(x) Cos \frac{k\pi x}{C} dx = ea_k$$

or

$$a_n \int_{-c}^{=c} Cos f(x) Cos \frac{n\pi x}{c} dx \qquad (5)$$

Next we determine the coefficient a_n. Suppose $K = 0$ in (4)

$$\int_{-c}^{C} f(x) dx = \frac{1}{2} a_o \int_{-c}^{c} f(x) dx + \sum_{n=1}^{\infty} [[a_n \int_{-c} Cos \frac{n\pi x}{C} dx + b_n \int_{-c}^{c} Sin \frac{n\pi x}{c} dx \qquad (6)$$

Thus we have

$$\int_{-c}^{C} f(x) dx = \frac{1}{2} a_0 (2c)$$

or $$\int_{-c}^{C} a_0 = \frac{1}{c} f(x) dx \qquad (7)$$

Thus we write the formal expansion as follows

$$f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} [a_n Cos \frac{n\pi x}{C} + b_n Sin \frac{n\pi x}{c} dx) \qquad (8)$$

With

$$a_n = \frac{1}{c} \int_{-c} f(x) Cos \frac{n\pi x}{C} dx, n = 0, 1 \ldots\ldots\ldots\ldots \qquad (9)$$

$$b_n = \frac{1}{c} \int_{-c} f(x) Sin \frac{n\pi x}{C} dx, n = 1, 2, \ldots\ldots\ldots \qquad (10)$$

Note that the formulae (9) and (10) depend only upon the values of f(x) in the interval $-c \leq x \leq c.$ Since each of the terms in the Fourier series (8) is periodic with period 2c, the series converges for all x whenever it converges in $-c \leq x \leq c.$, and its sum is also a periodic function with period 2c. Hence f(x) is determined for all x by its values in the interval $-c \leq x \leq c.$.

4.0 CONCLUSION

You have been introduced to partial differential equation in this unit. The attempts here are just introductory. You are required to study this unit properly because you will refer to it in your subsequent courses in mathematics.

5.0 SUMMARY

In this unit, various forms and types of partial differential equations were studied. These include (1) Wave equation (2) Laplace equation and (3) Heat equation. We also proposed various methods of solving these equations which include method of separation of variables and Fourier series applications. You are required to study this unit properly and attempt all the exercises at the end of the unit.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that the boundary-value problem
   $\frac{d^2y}{dx^2} - k^2 y = 0, y(0) = y(l) = 0$ cannot have a nontrivial solution for real values of k
2. Determine those values of k for which the partial differential equation
   $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ possesses nontrivial solutions of the form $T(x, y) = f(x)\sinh ky$ which vanish when $x = 0, and, when, x = l$
3. By considering the characteristic functions of the problem
   $$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0,$$
   Show that $\int_{-1}^{1} P_r(x)P_s(x)dx = 0$

7.0 REFERENCES/FURTHER READING

Earl, A. Coddington (nd). *An Introduction to Ordinary Differential Equations*. India: Prentice-Hall.

Einar, Hille (nd). *Lectures on Ordinary Differential Equations*. London: Addison-Wesley Publishing Company.

Francis, B. Hildebrand (nd). *Advanced Calculus for Applications*. New Jersey: Prentice-Hall.