NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH421

COURSE TITLE: ORDINARY DIFFERENTIAL EQUATIONS

MTH 421

ORDINARY DIFFERENTIAL EQUATIONS

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Course Materials

These include:

1. Course Guide
2. Study Units
3. Recommended Texts
4. Tutor Marked Assignments.
5. Presentation Schedule

Study Units

There are thirteen study units in this course:

MODULE ONE:

UNIT 1: Ordinary Differential Equations

UNIT 2: Existence and Uniqueness Theorems

UNIT 3: Properties of Solutions

MODULE TWO:

UNIT 4: Linear Systems

UNIT 5: Adjoint Systems

MODULE THREE

UNIT 6: Stum-Liouville Boundary Value Problems

UNIT 7: Linear Operator on Hilbert Spaces

UNIT 8: Stability Theory

MODULE 1

UNIT 1: ORDINARY DIFFERENTIAL EQUATIONS

Contents

1 Introduction

(1)

(2)

(3)

order

2 Objectives

3 Main Content

3.1 First Order ODEs

- - /

3.1.1 Concept of Solution

solution

Example 3.1 / -

Solution.

- -

- -

Example 3.2 —

general solution
particular solution

3.1.2 Initial Value Problem (IVP)

inital condition

initial value problem (IVP)

Example 3.3

—

Solution.

3.1.3 Separable ODEs.

method of separating variables
separable equation

Example 3.4

Solution.

Example 3.5

Solution.

Reduction to Separable Form

Example 3.6

Solution.

3.1.4 Exact ODEs.

$$\frac{\partial}{\partial} \qquad \frac{\partial}{\partial}$$

$$\frac{}{} \quad - \quad \frac{}{}$$

exact differential equation **exact**

$$\frac{\partial}{\partial} \qquad \frac{\partial}{\partial}$$

implicit solution, ϕ

(a) $\frac{\partial}{\partial}$ **(b)** $\frac{\partial}{\partial}$

$$\frac{\partial}{\partial} \quad \frac{\partial}{\partial\ \partial} \qquad \frac{\partial}{\partial} \quad \frac{\partial}{\partial\ \partial}$$

$$\frac{\partial}{\partial} \quad \frac{\partial}{\partial}$$

$\partial \quad \partial$

$\partial \quad \partial$

Example 3.7

Solution.

Step 1. Test for exactness.

$$\frac{\partial}{\partial} \quad - \qquad \frac{\partial}{\partial} \quad -$$

Step 2. Implicit general solution.

$$\frac{\partial}{} \qquad \frac{}{} \qquad \partial$$

Step 3. Checking an implicit solution.

$$\frac{\partial}{\partial} \qquad \frac{\partial}{\partial}$$

Example 3.8 WARNING! Breakdown in the Case of Nonexactness

- - ∂ ∂ -
∂ ∂

- $\frac{\partial}{\partial}$ -

∂ ∂

Reduction to Exact Form. Integrating Factors

-

integrating factor of (20).

Example 3.9 Integrating Factor

-

Solution.

Step 1. Nonexactness.

$$\frac{\partial}{\partial} \quad \frac{\partial}{\partial} \qquad \frac{\partial}{\partial} \quad \frac{\partial}{\partial} \quad -$$

Step 2. Integrating factor. General solution.

$$\frac{}{} \left(\frac{\partial}{\partial} - \frac{\partial}{\partial} \right) \quad \frac{}{-} \quad -$$

$$\frac{}{} \left(\frac{\partial}{\partial} - \frac{\partial}{\partial} \right) \quad \frac{}{-} \quad - \quad - \quad - \quad -$$

$$- \ ^{-}$$

$$\frac{\partial}{\partial} \quad \frac{}{} \quad -\ ^{-} \quad \frac{}{} \quad -\ ^{-} \quad ^{-}$$

$$^{-}$$

Step 3. Particular solution. $-$ $^{-}$

Step 4. Checking.

3.1.5 Linear ODEs. Bernoulli Equation.

linear

any

"standard form" (1)

Homogeneous Linear ODE.

≡ /

homogeneous.

— - | | -

- ± ⩾

trivial solution

Nonhomogeneous Linear ODE.

/

nonhomogeneous.

-

-

| |

Example 3.11 First-Order ODE, General Solution

Solution.

Example 3.12 First-Order ODE, Initial Value Problem.

Reduction to Linear Form. Bernoulli Equation

Bernoulli equation.

Example 3.13 Logistic Equation.

logistic equation **Verhulst equation**

Solution.

≡

3.2 Second Order Linear ODEs

linear

nonlinear

standard form (37)

≡

homogeneous. / **nonhomogeneous.**

coefficients **Solutions**

Homogeneous Linear ODEs: Superposition Principle

Example 3.14 Homogeneous Linear ODEs: Superposition of Solutions

-

-

-

- - - -

linear combination

superposition principle **linearity principle.**

Theorem 3.3 Fundamental Theorem for the Homogeneous Linear ODE (38)

Proof.

. . .

■

Remark 3.1

does not hold

Example 3.15 A Nonhomogeneous Linear ODE

Example 3.16 A Nonlinear ODE

-

- -

Initial Value Problem. Basis. General Solution

initial value problem
initial conditions

general solution

particular soluion

Example 3.17 Initial Value Problem

-

Solution.

Step 1. General solution.

Step 2. Particular solution.

-

-

Definition 3.1 **General Solutin, Basis, Particular Solution**
general solution
basis
fundamental system
particular solution

(a) **(b)**

/

linearly independent

linearly dependent
/ /

- — - —

Definition 3.2 **Basis (Reformulated)**
basis

Example 3.18

3.2 Second Order Linear ODEs

3.2.1 Homogeneous Linear ODEs with Constant Coefficients

-

λ

λ λ λ λ

λ λ λ

λ **characteristic equation**

λ λ

λ – - $\sqrt{\quad}$ - λ – - - $\sqrt{\quad}$ -

λ λ

-

(Case I) -

(Case II) -

(Case III) -

Case I. Two Distinct Real Roots λ and λ

λ λ

λ λ

Example 3.20 -
λ - λ λ - -

-

Example 3.21

- -

Solution.

Step 1. General solution.

λ λ -

λ – - √ λ – - - √ -

-

Step 2. Particular solution. - -

- -

-

Case II. Real Double Root λ -

- λ λ λ

-

-

- - -

- -

-

Example 3.22 - - λ λ λ
λ -
-

Example 3.23

-

Solution.

λ λ λ λ -

-

- - -

- - -

- -

Case III. Complex Roots - − ω **and** - − - ω

-

- ω ω ω

ω - −

ω

ω ω

Example 3.24

Solution.

Step 1. General solution. λ λ

- $\pm$ ω -

Step 2. Particular solution.

-

- - -

-

Example 3.25 Complex Roots

ω ω

ω ω

ω

3.2.2 Nonhomogeneous ODEs

Method of Undetermined Coefficients

/

Definition 3.3 General Solution, Particular Solution
general solution

Particular solution

Theorem 3.4 Relations of Solutions of (53) to Those of (54)

(a)

(b)

Proof.

(a)

(b) - - -

■

Theorem 3.5 A General Solution of a Nonhomogeneous ODE includes All Solutions

Proof.

-

- -

■

Method of Undetermined Coefficients

method of undetermined coefficients.

constant coefficients a and b

Method of Undetermined Coeffients

y . . . ω ω α ω α ω	y - - . . . ω ω α ω ω

Example 3.26 Application of the Basic Rule (a)

Solution.

Step 1. General solution of the homogeneous ODE.

Step 2. Solution of the nonhomogeneous ODE.

- - -

-

Step 3. Solution of the initial value problem.

-

-

Example 3.27 Application of the Modification Rule (b)

- -

Solution.

Step 1. General solution of the homogeneous ODE.

λ λ λ

-

Step. 2 Solution of the nonhomogeneous ODE. -

-
-
- -
-

-

- - -

- -
- -

- - -

Step 3. Solution of the initial value problem.

- - - - - -

-

- - - - -

Example 3.28 Application of the Sum Rule (c)

-

Solution.

Step 1. General solution of the homogeneous ODE.

λ λ λ λ -

-

Step 2. Solution of the nonhomogeneous ODE.

Step 3. Solution of the initial value problem.

3.2.3 Solution by Variation of Parameters

method of variation of parameters

-

Example 3.29 Method of Variation of Parameters

Solution.

- -

-

| |

| |

-

3.3 Higher Order Linear ODEs

th order

linear

- - . . .

coefficients -

standard form.

nonlinear.

≡

- - . . .

homogeneous. **nonhomogeneous.**

solution

4 Conclusion

first order, second order of higher order .

5 Summary

first order

general solution,

initial condition

initial value problem

particular solution

exact ODE

differential

integrating factor

Linear ODEs

Bernoulli equation

second-order **linear**

homogeneous
≡

/ **nonhomogeneous**

by superposition principle

a basis
fundamental system
general solution **partic6(r)-3ular solution**

method of undetermined coefficients.

λ λ

λ λ

	λ λ - – - – ± ω	λ λ - - ω ω

6 Tutor Marked Assignments(TMAs)

Exercise 6.1 First-Order ODEs

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-

–

√________
-

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-

√_______
-

-

-

Exercise 6.2 Second Order ODEs

Exercise 6.3 Higher-Order ODEs

6 Tutor Marked Assignments(TMAs)

UNIT 2: EXISTENCE AND UNIQUENESS THEOREM

Contents

1 Introduction

2 Objectives

3 Main Content

3.1 Existence and Uniqueness Theorem of First-Order Equations

3.1.1 Some Concepts from Real Function Theory

Uniform Convergence

Definition 3.1 (Convergent Sequence of Real Numbers) { }

| - |

→ ∞

Definition 3.2 (Pointwise Convergence) { } →
≥ →

| - | ≤

≥

Definition 3.3 (Uniform Convergence) { } →
≥ →

| - | ≤

≥

Example 3.1 { } ≤ ≤

——— ≤ ≤

Solution.

{ } →∞
≤

→∞

{ } ≤ ≤
≤ ≤ ≤ ≤

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≤ ≤ ≤ ≤

Theorem 3.1 { }
≤ ≤ ≤ ≤
≤ ≤

Example 3.2 { }
≤ ≤

≤ ≤
≤ ≤
≤ ≤
≤ ≤ ≤ ≤

Theorem 3.2 { }
≤ ≤ ≤ ≤ α
β ≤ α β ≤
β β
→∞ α α →∞

Example 3.3 { }

≤ ≤

≤ ≤

→ ∞ →

$\to \infty$

$\to \infty$

$\to \infty$ $\to \infty$

series

$\leq$ $\leq$

∞

Definition 3.4

$\leq$ $\leq$ $\{ \}$

$\cdots\cdots\cdot$

$\cdots$

$\cdots\cdots\cdot$

∞

$\leq$ $\leq$

$\{ \}$ $\leq$ $\leq$

Theorem 3.3 (Weierstrass M-Test) $\{ \}$

∞ ∞

$| \quad | \leq$

∞

$\leq$ $\leq$ $\leq$ $\leq$

Example 3.4 ∞ ≤ ≤

∞

| |

≤ ≤ ∞

≤ ≤

Functions of Two Real Variables; the Lipschitz Condition.

Definition 3.5

connected

open

domain.

boundary point

closed domain.

Example 3.5

≤

Definition 3.6

δ

| - |

| - | δ | - | δ

Definition 3.7

bounded

| | ≤

Theorem 3.4

≤ ≤ ≤ ≤

Example 3.6 ≤

≤ ≤ ≤ | |

| | ≤

Definition 3.8

Lipschitz Condition

| - | ≤ | - |

Lipschitz Constant.

Theorem 3.5 ∂ / ∂

∂ / ∂

Example 3.7 | | ≤

| | ≤ ∂ / ∂ ∂ / ∂

| - | | - | | || - | ≤ | -

|

Example 3.8 | |

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| - | | | | - | || ≤ | || - | ≤ | -

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$\frac{\partial}{\partial}$ I

3.1.2 Existence and Uniqueness of Solutions

The Basic Problem and a Basic Lemma

$\alpha \le \quad \le \beta$ α β

φ α β

φ φ α β

φ φ φ α β

φ

φ α β

α β

φ α β φ

φ^1 α β

Lemma 3.1 φ

$\alpha \le \quad \le \beta$ φ α β

α β φ α β

φ φ

φ φ

α β

Proof. φ α β φ φ

α β

φ φ

φ φ α β

φ α β

φ φ

α β φ α β

φ ■

The Existence and Uniqueness Theorem.

Theorem 3.6

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φ

| - | ≤ φ

| - | ≤

φ φ

φ

Remark 3.1

φ

| - | ≤

φ | - | ≤

| - | ≤ | - |

Method of Proof

| - | ≤

φ φ φ φ

successive approximations *(Picards Iterants)*

φ

φ φ

φ φ

φ φ -

{φ }
|φ - | ≤ | - | ≤ φ

{φ }

|φ - φ - | — · ——— | - | ≤

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→ ∞ {φ } φ
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φ — | - | ≤

φ

φ | - | ≤

— φ

Remarks and Examples

φ — | - | ≤ φ

φ

Example 3.9

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— ⁻ | | ≤

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δ $-\ \delta$

$$\frac{\delta^{-}\ -\ \ -\ \delta^{-}}{\delta\ -\ \ -\ \delta} \quad \frac{}{\delta^{-}}$$

δ

φ φ φ

⁻

— ≥

φ

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$-\ \infty$ ∞

φ

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| - | ≤ φ

Example 3.10

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∂ / ∂

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$| \; - \; | \le \; -$

$- \; -$

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Theorem 3.7 $- \infty \quad \infty$

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$\varphi \quad \varphi$

$- \infty \quad \infty \quad \varphi$

$- \infty \quad \infty$

$- \infty$ ∞ $- \infty$ ∞

3.1.3 Dependence of Solutions on Initial Condition and on the Function

φ | - | ≤

| - | ≤ | - |

| - | ≤ | - | ≤ | - | ≤

| - |

≤

ψ | - | ≤

| |

δ

| - | ≤ δ φ | - | ≤

Theorem 3.8

δ

| - | ≤ δ

φ | - | ≤

φ φ

| - | δ ≤ δ

|φ - φ | ≤ δ | - | ≤

φ

φ φ $| - | \leq$

Theorem 3.9

$| - | \leq$

φ

ψ

φ ψ $| - | \leq$

$|\varphi - \psi | \leq - | - | \leq$

φ
ψ $| - | \leq$

Example 3.11

-

$| - | | |$

3.2 Existence and Uniqueness Theorem of Linear Differential Equations

3.2.1 The Basic Existence Theorem

Definition 3.9

Theorem 3.10 -

| - | ≤ | - | ≤ | - | ≤

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| | ≤

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| - | ≤ | - | | - | . . . | - |

φ φ φ

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φ φ φ

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Outline of Proof.

φ

φ

φ φ - φ -

φ

|φ - φ - | — - | - |

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|φ - φ - | ≤ —

{φ }

φ φ φ - φ -

φ φ

φ φ φ

| - | ≤

φ φ φ

| - | ≤ φ

1 -

Definition 3.10

1

- 1 -

solution φ

≤ ≤

φ φ[1] φ -

φ φ φ[1] φ - ≤ ≤

Theorem 3.11

1

- 1 - -

φ

φ φ[1] φ - -

| - | ≤

3.2.2 Basic Existence and Uniqueness theorems on Linear Systems

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1 . . .

1 . . .

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Lemma 3.2 ≤
≤

. . .

≤ ≤ - ∞ ∞

| - | ≤ | - | | - | . . . | -
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≤ ≤

Proof. ≤ ≤
| | ≤
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| - - . . . - |

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≤ | - | | - | . . . | - |

■

Theorem 3.12
≤ ≤
≤ ≤

φ φ φ

φ φ φ
≤ ≤

Outline of Proof.

φ

φ

φ

φ

-

. .

.

φ

-

≤ ≤

φ ≤ ≤

| . . . | ≤ ≤ ≤

. . .

≤ ≤

|φ - φ - | - | - |

≤

≤ ≤

|φ - φ - | ≤

≤ ≤ | - | | - |

≤ ≤

Theorem 3.13 -

≤ ≤ *l* ≤ ≤

≤ ≤ -

φ

φ φ[1] φ - -

≤ ≤

Proof.

- -

1

1

1 -

1 - - - . . . - -

φ φ φ φ
φ φ φ φ^1 φ φ -

φ φ φ -

φ φ φ φ

φ φ $\leq$ $\leq$
φ φ

■

Example 3.12

- - — — ——— -

-
- ∞ ∞ - -
- $\leq$
$\leq$ -
$\leq$ $\leq$

- ... -

Corollary 3.1 φ

φ φ^1 φ -

$\leq$ $\leq$

l

φ $\leq$ $\leq$

Proof. φ φ

$\leq$ $\leq$

■

4 Conclusion

5 Summary

6 Tutor Marked Assigments(TMAs)

Exercise 6.1

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√ _

| | ≤ —

{ } ≤ ≤

≤ ≤

≤ ≤

- ≤ ≤

≤ ≤

≤ ≤

| | ≤ | | ≤

φ

φ φ^{1} - φ^{11}

-

1 11 111 -

∞ - -

-

1

- ∞ ∞ - ∞

∞

- ∞ ∞ /

1

1 /

-

1

1

UNIT 3: PROPERTIES OF SOLUTIONS OF LINEAR DIFFEREINTIAL EQUATIONS

Contents

1 Introduction

2 Objectives

3 Main Content

3.1 Basic Theory of Linear Differential Equations

3.1.1 Definition and Basic Existence Theorem

Definition 3.1 *Linear differential equation of order n*

– –

≤ ≤ ≤ ≤

– –

Example 3.1

Example 3.2 –

Theorem 3.1

– – **1**

≤ ≤

≤ ≤

≤ ≤ –

– –

≤ ≤

Example 3.3

$-\infty$ ∞
$-$
$-\infty$ ∞

Example 3.4

$-$

$-$ $-$

$-$
$-\infty$ ∞
$-$ $-$
$-\infty$ ∞

Corollary 3.1 *homogeneous*

$-$ $-$

$-$ $\leq$ $\leq$

$\leq$ $\leq$

Example 3.5

3.1.2 The Homogeneous Equations

Theorem 3.2 Basic Theorem on Linear Homogeneous Differential Equations

Definition 3.2

linear combination

Theorem 3.3 (Theorem 3.2 restated)

Example 3.6

Example 3.7

$-$

$-$ $-$

$-$

$-$ $-$ $-$

Definition 3.3 *linearly dependent* $\leq$ $\leq$

$\leq$ $\leq$

$\leq$ $\leq$

$\leq$ $\leq$

Example 3.8 ≤ ≤

− ≤ ≤ −

Example 3.9 − − ≤
≤

−

− ≤ ≤

Definition 3.4 *linearly independent* ≤ ≤
linearly independent ≤ ≤

≤ ≤

Example 3.10 ≤ ≤
≤ ≤

Theorem 3.4

Definition 3.5 (General Solution)
≤ ≤

≤ ≤

general solution ≤ ≤

Example 3.11 $-\infty$ ∞

Example 3.12 –

$-$ $-$

$-\infty$ ∞

–

Definition 3.6 $-$

$\leq$ $\leq$

– – –

Wronskian

$\leq$ $\leq$

Theorem 3.5

$\leq$ $\leq$

$\leq$ $\leq$

Theorem 3.6 $\leq$ $\leq$

$\leq$ $\leq$

$-$

Example 3.13

Example 3.14

3.1.3 Nonhomogeneous Equation

(1)

Theorem 3.7

(2)

Example 3.15

Definition 3.7

$$- \quad - \tag{1}$$

$$- \quad - \tag{2}$$

Example 3.16

3.2.1 Homogeneous Linear Equations with Constant Coefficents.

– –

–

λ

λ λ λ – λ – λ λ λ

λ λ λ – – λ

λ λ

λ λ – – λ

auxilliary equation ***characteristic equation***

λ λ

λ

λ

3.2.2 Case I. Distinct Real Roots

λ λ λ

λ λ λ

Theorem 3.8

λ λ λ

λ λ λ

Example 3.17 –

λ – λ

λ – λ – λ λ

Example 3.18 −

$$\lambda - \lambda \quad \lambda$$

λ −

$$\lambda \quad \lambda - \lambda$$

$$\lambda \quad \lambda - \quad \lambda -$$

$$\lambda \quad - \lambda \quad \lambda$$

$$-$$

3.2.3 Case II. Repeated Real Roots

Example 3.19 An introductory Example.

$$-$$

$$\lambda - \lambda$$

$$\lambda -$$

$$\lambda \quad \lambda$$

λ λ

$-$

λ λ λ

λ $-$

λ λ λ $-$

λ λ λ λ λ $-$

λ λ λ λ λ $-$

λ λ λ λ $-$

λ

$\lambda \quad \lambda \quad \lambda$

λ

Theorem 3.9

λ

$- \quad \lambda$

$\lambda \quad \lambda$

$- \quad \lambda \quad \lambda \quad \lambda$

λ

Example 3.20

$- \quad -$

$\lambda - \lambda - \lambda$

$-$

$-$

$-$

Example 3.21

$- \quad -$

$\lambda - \lambda \quad \lambda \quad \lambda -$

$-$

$-$

$-$

$-$

3.2.4 Case III. Conjugate Complex Roots

θ θ θ

θ

Theorem 3.10

Example 3.22

λ λ

Example 3.23

$-$

$\lambda \quad - \quad \lambda$

$\lambda \qquad \frac{\sqrt{\quad - \quad}}{\qquad} \qquad \frac{\quad}{\quad}$

Example 3.24

$- \qquad -$

$- \qquad -$

$- \qquad -$

3.2.5 An Initial-Value Problem

Example 3.25

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$-\infty \qquad \infty$

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$-$ $\sqrt{}$

$-$

$\sqrt{}$

$\sqrt{}$ $-\sqrt{}$

$\sqrt{}$

φ

φ

φ $-\sqrt{}$

φ $\sqrt{}$

3.2.6 The Method of Undetermined Coefficients

$-$ $-$

Definition 3.8

Definition 3.9

Example 3.26

Examples

Example 3.27

– – –

– –

–

–

Example 3.28 Initial-Value Problem

$-\infty$ ∞

3.2.7 Variation Of Paramenters

The Method

Examples

Example 3.29

4 Conclusion

5 Summary

6 Tutor Marked Assignments(TMAs)

Exercise 6.1

(a) (b)

$-$

$-\infty$

∞

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∞

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$-\infty$ ∞

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$-\infty$

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$-\infty$ ∞

Exercise 6.2

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$-$ $-$

Exercise 6.3

MODULE 2

UNIT 4: LINEAR SYSTEM

Contents

1 Introduction

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + a_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \cdots + \frac{dx}{dt} + a_n(t)x = f(t)$$

$$\dot{x} = Ax + F(t) = f(t, x)$$

×
→ × →

Linear systems

2 Objectives

homogeneous *nonhomogeneous*

3 Linear System

3.1 Properties of Solution of Homogeneous Linear System

Definition 3.1 **linear homogeneous system** **linear nonhomogeneous system.** **order**

Definition 3.2 Φ → **solution**

→ × Φ
Φ
Φ

Definition 3.3 Φ →

Φ

| | ∞

.

Theorem 3.1

Φ Φ Φ ≡

Φ Φ Φ Φ

Proof.

≡

Φ Φ Φ

Φ− Φ Φ− Φ . . . Φ− Φ

— Φ Φ

Φ Φ Φ . . . Φ Φ

$\dot{\Phi}$ — Φ — Φ Φ Φ Φ

■

Theorem 3.2 -
solution space.

Proof. ≡

-

Φ Φ
Φ

Φ

Φ

Φ

Φ

Φ

Φ Φ

Φ

■

Definition 3.4 Φ Φ Φ Φ Φ Φ

Φ Φ Φ Φ Φ Φ

Φ Φ · · · Φ

Φ Φ · · · Φ

Φ Φ · · · Φ

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Theorem 3.3

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Lemma 3.1

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Proof of Lemma

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■

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Proof of Theorem1.3

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■

Remark 3.1

. .

. .

Theorem 3.4 Φ Φ
$\times$ Φ

Proof. Φ

$\dot{\phi}$ Φ

— Φ $\dot{\phi}$ Φ

Φ

Φ

Φ Φ /

Φ Φ

Φ Φ

Φ Φ^{-} Φ Φ Ψ Φ

$\dot{\phi}$ $\dot{\phi}$ Ψ ϕ $\dot{\Psi}$

Φ $\dot{\Phi}$ Ψ Φ $\dot{\Psi}$ Φ $\dot{\Psi}$

$\dot{\psi}$ Φ

∎

3.2 The Adoint of the System (3)

Φ Φ^{-}

$\Phi\Phi^{-}$

$\times$

$\dot{\phi}\Phi^{-}$ $\Phi\dot{\phi}^{-}$ $\Phi\Phi^{-}$ $\Phi\dot{\phi}^{-}$

$\dot{\Phi}^{-}$ - Φ^{-}

$\dot{\Phi}^{-}$ Φ^{-}

Φ^{-}

. -

adjoint

. -

Theorem 3.5 Φ Ψ

Ψ Φ

$\times$

Proof. Φ Φ^{-}
Ψ Φ^{-} Φ^{-}

Ψ Φ^{-}

$\dot{\Psi}$ Φ^{-} Ψ Φ

■

3.3 Linear nonhomogeneous System

\ \ ∞

3.3.1 Variation of Parameter Technique

Φ

Φ

Φ

Φ

· ϕ̇ Φ ·
Φ Φ ·

Φ

Φ

· Φ⁻

Φ⁻ τ τ τ

Φ τ Φ⁻ τ τ τ

Φ Φ⁻ Φ τ Φ⁻ τ τ τ

Theorem 3.6 × -
Φ Φ

Φ Φ⁻ Φ⁻ τ τ τ

Proof. Φ

Φ

· ϕ̇ Φ ·

Φ ·

· Φ⁻

$\Phi^{-}\ \tau\quad \tau\quad \tau$

$\Phi\quad \Phi\quad \Phi^{-}\ \tau$

$\Phi\quad \Phi^{-}\quad \Phi\quad \Phi^{-}\ \tau\quad \tau\quad \tau$

$\Phi\quad \Phi^{-}\quad \Phi^{-}\ \tau\quad \tau\quad \tau$

■

Theorem 3.7 $\Phi\quad \Phi$

$\Phi\ ^{-}\ \Phi\quad \Phi\ \tau\quad \tau\quad \tau$

Proof. Φ

Φ

$\cdot\quad \dot{\Phi}\quad \Phi\quad \cdot$

$\cdot\quad -\quad \Phi\quad \Phi\quad \cdot\quad \Phi$

$\Phi\quad -\ \Phi\quad \Phi\ \tau\quad \tau\quad \tau$

$\Phi\quad \Phi\quad \Phi\ \tau\quad \tau\quad \tau$

$\Phi\ ^{-}\ \Phi\quad \Phi\ \tau\quad \tau\quad \tau$

■

3.4 Linear Homogenous Equation with constant Coefficients

.

×

c λ

. λc λ c λ

- λ

c λ

solution λ c

Theorem 3.8 λ λ λ

λ

Proof.

–

λ · · ·
λ · · ·
λ · · ·

· · · λ –
· · · λ

–

. – . – –

. λ

λ

.

λ

λ

$\ldots$

λ

λ $\ldots$

λ $\ldots$

λ $\ldots$

λ λ λ

$\ldots$ λ –

$\ldots$ λ

■

Example 3.1

.

Solution.

$- \lambda$

$- \lambda$ $- \lambda$ $- \lambda$ $-$ λ $-$ λ $-$

λ $\pm \sqrt{}$ λ $\sqrt{}$ λ $- \sqrt{}$

λ $\sqrt{}$

$-$ $-$ $\sqrt{}$ $-$ $-$ $\sqrt{}$

$-$ $\sqrt{}$ $-$ $\sqrt{}$

$\sqrt{}$

$\lambda \quad - \quad \sqrt{}$

$- \quad \sqrt{} \qquad - \quad \sqrt{}$

$\sqrt{} \qquad \sqrt{}$

$\sqrt{}$

$-$

$\sqrt{} \quad \sqrt{} \qquad \sqrt{} \quad - \quad - \quad \sqrt{}$

.

Φ

$\sqrt{} \quad - \quad \sqrt{} \quad - \qquad \sqrt{} \quad - \quad \sqrt{} \quad - \quad -$

$-$

$-$

$\sqrt{} \qquad \sqrt{} \quad - \quad \sqrt{}$

–

– ■

–√

Example 3.2

- - .

.

Solution.

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-

- λ

- λ λ - λ - λ

- - λ

λ λ - λ

λ

- -

- -

- -

λ

λ -

-

-

-

λ

-

λ

-

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-

-

λ

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-

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- –

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- –   –   –

- –   –   –

- –   - –   –   –

- –   –   –

Example 3.3

.

Solution.

λ λ - λ $\pm$

.   -

.

Example 3.4

.

Solution.

.

$| \quad - \lambda |$

$\lambda \quad - \qquad \lambda \quad \pm$

λ

$-$

$-$

$\lambda \quad -$

$- \qquad -$

$-$

$.$

$\Phi \quad \Phi$

$-$

Φ

$- \quad -$

$- \qquad -$

$\Phi^{-} \qquad -$

$- \quad -$

$\Phi \quad \Phi^{-} \qquad \Phi \qquad \Phi^{-} \; \tau \quad \tau \quad \tau$

$- \qquad - \qquad \tau \quad -\tau$

$- \qquad - \qquad \tau$

$- \; - \qquad - \qquad - \; - \qquad \tau \; - \; \imath \qquad \tau$

Example 3.5

.

$- \omega$

Solution.

.

$- \omega$

$\bar{\omega}$ ω ω

ω $- \omega$ ω

ω $\bar{\omega}$ ω

Φ

$- \omega$ ω ω

.

$- \omega$

Φ Φ^{-} Φ Φ^{-} τ τ τ

ω $\bar{\omega}$ ω ω $\bar{\omega}$ ω

$- \omega$ ω ω $- \omega$ ω ω

$\omega\tau$ $\bar{\omega}$ $\omega\tau$

τ

$- \omega$ $\omega\tau$ $\omega\tau$ τ

Example 3.6

.

- - _ _

$- \quad - \quad \tau \quad -\tau$

τ

$- \quad - \quad - \quad - \quad - \quad \tau \quad - \quad \imath \quad \tau$

| - λ |

λ - λ ±

λ

λ -

Φ Φ

Φ

Φ^{-}

Φ Φ^{-} Φ Φ^{-} τ τ τ

Φ Φ^{-}

Φ Φ^{-} Φ Φ^{-} τ τ τ

3.5 Characterization of Fundamental Matrix in terms of exponential functions

$\times$

∞

$\leq$

∞

$\leq$

τ $-\tau$

τ τ

—

Theorem 3.9

Φ $|\;|$ ∞

τ ξ

$^{-\tau}\,\xi$

Φ

Proposition 3.1 $\times$

$\cdots$

$\cdots$

$\cdots$

$\cdots$

$\cdots$

λ λ λ

λ $\cdots$

λ $\cdots$

λ $\cdots$

$\cdots$ λ

$\cdots$ λ

λ λ

λ $\cdots$

λ $\cdots$

λ $\cdots$

$\cdots$ λ

λ

$\cdots$
$\cdots$
$\cdots$

$\cdots$

λ

$-$

$\times$

$-$ $-$

$-$ $-$

Proof.

$-$ $-$

$\dots$

$\dots$

$\dots$

$\dots$

∞

∞ $\dots$

∞ $\dots$

∞ $\dots$

$\dots$ ∞

$\dots$

$\dots$

$\dots$

$\dots$

λ $\dots$

λ $\dots$

λ $\dots$

$\ldots \quad \lambda$

λ

λ

$\cdots$

$\cdots$

λ $\cdots$

$\cdots$

$\times$ $\cdots$

■

3.6 Linear Nonhomogeneous System

.

$\times$ $\to$

$| \quad | \quad \infty$

.

Φ

$\Phi \quad \Phi^{-}$

Theorem 3.10 Φ Φ

.

Φ Φ

Φ Φ $-$ Φ Φ τ τ τ

Φ

Proof. Φ Φ

— Φ $\dot{\Phi}$ Φ ·
$\dot{\Phi}$ Φ { }

$\dot{\Phi}$ - Φ

— Φ - Φ Φ Φ Φ

Φ - Φ Φ τ τ τ

Φ Φ Φ τ τ τ

Φ ‾ Φ Φ τ τ τ

■

Example 3.7

3.7 Linear Homogeneous System with Periodic Coefficients

×

ω

ω

Theorem 3.11 Φ Φ Ψ

Ψ Φ ω ω

Φ ω

Φ

Proof. Φ Φ

— Φ Φ

– Φ ω ω Φ ω

Φ ω

Φ ω

Φ ω Φ ω /

×

Φ ω Φ

ω

Φ ω Φ ω

ω Φ ω – ω

Φ ω . – ω

Φ –

ω Φ – ■

Φ

Φ

Φ ω Φ Φ ω

Φ - ω Φ– ω

Φ ρω Φρ ω ρ

Φ ω Φ

×

Φ ω

Φ ω Φ Φ ω - ω

Φ Φ - ω Φ ω

Φ - ω Φ– ω

ω

Φ ω Φ ω Φ ω

Φ ω Φ ω

Φ ρω Φ$^{\rho}$ ω ρ

≡ Φ
Φ

Φ Φ Φ

Φ - Φ Φ - Φ Φ^{-}

Φ Φ

Φ ω - ρΦ

ρ Φ Φ

Φ Φ

Φ

Φ ρ

Φ ρ

$^{\omega}$ - ρ

ρ $^{\omega}$ Φ

$^{\omega}$

τ

ρ $^{\tau\omega}$

4 Conclusion

5 Summary

.

Φ →

Φ Φ

Φ

solution space

linear combination

fundamental matrix

′

(method of variation of constants)

–

–

×

adjoint

6 Tutor Marked Assignments(TMAs)

Exercise 6.1

-

-

- λ - ± λ

-

- -

- - λ λ λ -

- - -

-

λ λ

-

UNIT 5: ADJOINT SYSTEMS

Contents

1 Introduction

2 Objectives

3 The Adjoint Equation

3.1 Definitions and Examples

Definition 3.1

adjoint operator

adjoint equation

$\leq \quad \leq \qquad \leq \quad \leq$

$\leq \quad \leq$

$- \qquad - \qquad \leq \quad \leq$

3.1.1 The Second-Order Case

Example 3.1

3.1.2 Lagrange Identity

Theorem 3.1 Langrange Identity.

$$\leq \quad \leq$$

$$\leq \quad \leq$$

Proof.

■

Corollary 3.1 Green's formula

≤ ≤

Proof. ■

The Second-Order Case.

Theorem 3.2

Proof.

■

The Second Order Case

Example 3.2

Solution.

Theorem 3.3

Proof.

■

Example 3.3

3.2 Self Adjointness

Definition 3.2 *self adjoint*

Theorem 3.4

$\leq$ $\leq$

$\leq$ $\leq$

Proof.

■

Corollary 3.2

Proof.

■

Example 3.4 (Legendre's Equation)

Theorem 3.5

$\leq \quad \leq \qquad \leq \quad \leq$

$\leq \quad \leq$

Proof.

Example 3.5

$$-$$

$$- \qquad -$$

$$\leq \quad \leq$$

$$-$$

$$-$$

4 Conclusion

5 Summary

6 Tutor Marked Assignments(TMAs)

Exercise 6.1

4 Conclusion

MODULE 3

UNIT 6: STURM-LIOUVILLE BOUDARY VALUE PROBLEMS

Contents

1 Introduction

2 Objectives

3 Main Content

3.1 Sturm-Liouville Problems

3.1.1 Definition and Examples

Definition 3.1

— — λ

≤ ≤ λ

Example 3.1

— λ

π

— — λ

Example 3.2

λ

φ φ

λ

Example 3.3

λ

π

Solution.

λ λ λ

Case I: λ

π π

λ

Case II: λ λ

$\sqrt{-\lambda}$ λ $\sqrt{-\lambda}$ α

λ

α $-\alpha$

π

$\alpha\pi$ $-\alpha\pi$

$\alpha\pi$ $-\alpha\pi$

$\alpha\pi$ $-\alpha\pi$ α

α α $\sqrt{-\lambda}$ λ

λ λ **Case**

III: λ λ $\sqrt{-\lambda}$

$\sqrt{\lambda}$

$\sqrt{\lambda}$ $\sqrt{\lambda}$

π

$\sqrt{\lambda}\pi$ $\sqrt{\lambda}\pi$

$\sqrt{\lambda}\pi$

$\sqrt{\lambda}\pi$

$\sqrt{\lambda}\pi$

π

λ

λ

λ

λ λ

λ

λ

3.1.2 Characteristic Values and Characteristic Functions

λ

Definition 3.2

λ

Characteristic values

characteristic functions

Example 3.4

λ

π

λ

λ

Example 3.5

λ

π

λ

Solution.

λ λ λ

λ

λ

λ

λ

λ

$\sqrt{\lambda}$ $\sqrt{\lambda}$

λ

$\sqrt{\lambda}$ $\sqrt{\lambda}$

$\sqrt{\lambda}$ $\sqrt{\lambda}$ $-$ $\sqrt{\lambda}$ $\sqrt{\lambda}$

$\sqrt{\lambda}$ $\sqrt{\lambda}$ $-$ $\sqrt{\lambda}$ $\sqrt{\lambda}$

$\sqrt{\lambda}$

π

$\sqrt{\lambda}$ $-\pi$ $\sqrt{\lambda}$ π $-$ $\sqrt{\lambda}$ $-\pi$ $\sqrt{\lambda}$ π

π π

$\sqrt{\lambda}$ $-\pi$ $\pi\sqrt{\lambda}$

$\pi\sqrt{\lambda}$

$\pi\sqrt{\lambda}$ π

λ

λ

λ

$\geq$

$\geq$

$\lambda \quad \lambda \quad \lambda$

$\lambda \to \infty \quad \to \infty$

$\lambda \to \infty \quad \to \infty$

Theorem 3.1

λ

$\leq \quad \leq \quad \lambda$

λ

$\lambda \quad \lambda \quad \lambda$

$\lambda \rightarrow \infty \quad \rightarrow \infty$

λ

$\varphi \quad \leq \quad \leq$

$\varphi \quad \lambda$

$-$

Example 3.6

λ

π

λ

λ

λ

$- \quad < \quad \pi \quad \pi$

π

6 Tutor Marked Assignments(TMAs)

Exercise 6.1

— λ

π

— λ

π

— λ

— λ

— λ

π − π

— λ

−

π − π

− — λ

π

λ

π

λ

λ

π

UNIT 7: NONLINEAR EQUATIONS

Contents

1 Introduction

2 Objectives

3 Main Content

3.1 Phase Plane, Paths, and Critical Points

3.1.1 Basic Concepts and Definitions

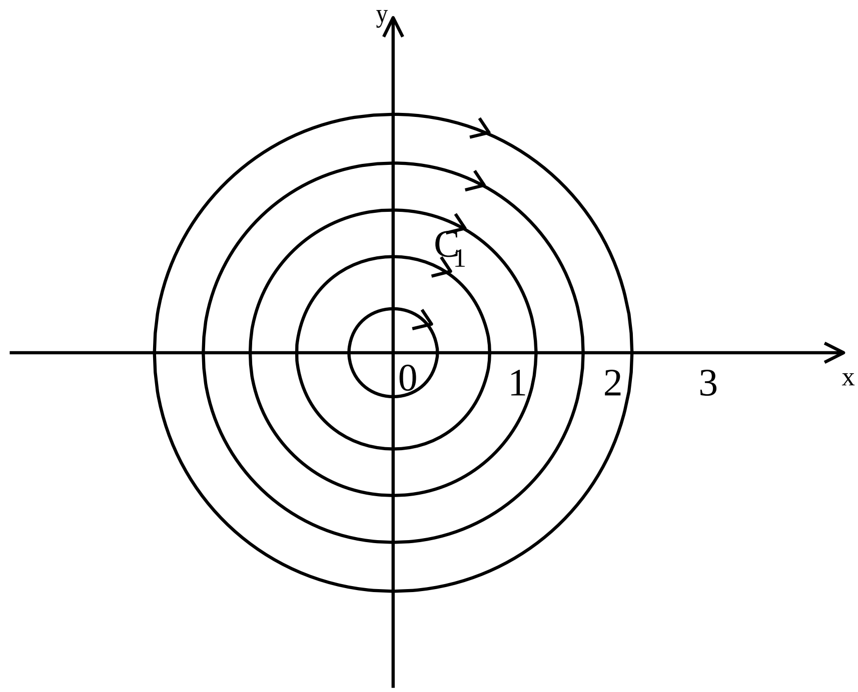
Definition 3.1

critical point

Example 3.1

$-\pi$

$-\pi$



Definition 3.2 *isolated*

Definition 3.3

$\to \infty$

$\to \infty$ $\to \infty$

Definition 3.4 $\to \infty$

$\to \infty$

$$\lim_{\to \infty} \frac{-}{-}$$

$\to \infty$

3.1.2 Types of Critical Points

Center:

$\to \infty$ $\to -\infty$

Saddle point

spiral point

$\to \infty$ $\to -\infty$

node

$\to \infty$ $\to -\infty$

3.1.3 Stability

Definition 3.5

$$\frac{}{- \quad -}$$

stable δ

δ

$\geq$

$\leq \infty$

3.2 Critical Points and Paths of Linear Systems

3.2.1 Basic Theorems

.

.

λ

λ

λ

λ – λ –

λ λ

λ λ

λ λ

λ λ

λ λ

λ λ

λ λ

Theorem 3.1 λ λ

node

saddle point

node

spiral point.

center

Theorem 3.2

.

.

3.2.2 Examples and Applications

Example 3.2

$$\cdot \quad -$$

$$\cdot \quad -$$

Solution.

$$- \qquad -$$

$$\lambda \quad \lambda$$

$$\lambda \quad - \quad \lambda \quad -$$

Example 3.3

$$\cdot$$

$$\cdot \quad -$$

Solution. $-$

$$\lambda - \lambda$$

3.3 Critical Points and Paths of Nonlinear Systems

3.3.1 Basic Theorems on Nonlinear Systems

$$\cdot$$

$$\cdot$$

Example 3.4

Theorem 3.3

.

.

.

.

λ λ

λ − λ −

λ λ

λ λ

λ λ

λ λ

λ λ

λ λ

Theorem 3.4

λ λ
stable *stable*

λ λ *unstable*
unstable

Example 3.5

λ $-$

λ λ $-$

Example 3.6

$$\lambda \quad \lambda$$

$$\lambda \quad - \quad \lambda \quad -$$

Example 3.7

$$\lambda$$

4 Conclusion

5 Summary

6 Tutor Marked Assignments(TMAs)

Exercise 6.1

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3 Stability

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η η
η

- η

η

δ δ δ - δ
-
δ

Definition 3.2
δ

- - δ
→∞

Example 3.1

Solution.

.

. - -

-

≡

| | | |

| | | |

| | | - |
≤ | | | |
δ -

Example 3.2

Solution.

Example 3.3

Proof.

λ

λ

$\leq \quad -$

$\alpha \quad \lambda \quad \alpha \quad \beta\, \alpha \quad \alpha \quad \cdots \quad \alpha$

$- \quad \lambda \qquad \ldots$

$\lambda \quad \lambda \quad \lambda \quad \ldots \quad - \quad \lambda$

$- \quad \lambda \quad \leq \quad | \; - \; | \; \frac{\alpha}{\quad} \; . \; \beta \quad \alpha$

$\leq \quad | \; - \; | \frac{-\;\alpha}{\quad}$

$\leq \quad - \quad \to \quad \to \infty$

3.2 Quasilinear System

Theorem 3.1

.

×

$- \alpha - \beta \quad \leq$

$\leq \beta \qquad \delta \quad \frac{-}{\beta}$

$\leq \beta \qquad - \alpha - \beta$

$\to\infty \qquad \leq \quad {}_{\to\infty}\beta \qquad - \alpha - \beta$

■

3.3 Lyapunov Second Method

.

$\to$

$\to \qquad \frac{\partial}{\partial}$

Ω

$\Omega \quad \{ \qquad \}$

Definitions

$\Omega \to$

Ω

$\Omega \to \quad \Omega$

$\geq \qquad \leq \qquad \Omega$

Example 3.4

-

$$\cdot \quad — \quad \frac{\partial}{\partial} \cdot \quad \frac{\partial}{\partial}$$

Theorem 3.2

.

→

≡

Theorem 3.3 Lyapunov

→

Theorem 3.4 →

Theorem 3.5 (Cêtaev) On Instability

.

→

≡

Example 3.5

Solution.

≡

Example 3.6

Solution.

≥

Example 3.7

Solution.

≥ ≥

$\frac{\partial}{\partial}$ $\frac{\partial}{\partial}$

Example 3.8

→

Solution.

4 Conclusion

5 Summary

6 Tutor Marked Assignments(TMAs)

Exercise 6.1

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. .

≥ ≥ _

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