NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: PHY203

COURSE TITLE: OSCILLATIONS AND WAVES

Course Code	PHY 203
Course Title	OSCILLATIONS AND WAVES
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OSCILLATIONS AND WAVES

Course Introduction

The phenomena we normally observe in nature can be broadly classified into two categories: those concerned with matter and those concerned with waves. Physics courses usually begin with discussion of phenomena dealing with matter mechanics and properties of matter. Next comes the phenomena of waves. Out of our five senses (touch, taste, smell, hearing and seeing), two deal with the waves—hearing and seeing. Our contact with the external world is mainly through these two senses. Sound and light, though of entirely different nature, have many properties in common. In this course, you will learn about waves in general. This unified approach to wave motion is meant to bring out the underlying similarity between apparently widely differing phenomena. Even our understanding of modern physics, particularly quantum mechanics, depends upon a clear understanding of this course.

Before coming to wave motion it is essential to understand the physics of oscillations of an isolated body as well as of two or more bodies coupled together. The first part of this course deals with the study of oscillations of an isolated system such as a pendulum and two or more bodies coupled together, under different conditions. In particular, the effect of damping and an external harmonic force are discussed in detail. The second part deals with wave motion. The basics of progressive waves, their reflection, transmission and refraction are discussed in detail. Superposition of waves can give rise to beats, stationary waves, interference and diffraction. These have been discussed with particular emphasis on sound waves.

In Unit 1 of this course we have developed the mathematical theory of simple harmonic motion. It is used to study oscillations, by analogy, of entirely different systems from different branches of physics. Unit 2 deals with the superposition of two or more collinear or orthogonal harmonic oscillations of same/different frequencies.

In nature, most oscillations left to themselves die down gradually. This happens because of damping. The effect of damping on harmonic oscillations is discussed in Unit 3. In Unit 4 you will learn about the motion of a damped harmonic oscillator on which a periodic harmonic force is acting. This leads to the spectacular phenomenon of resonance. Unit 5 deals with the analysis of coupled oscillations. You will learn that in the limit of their number becoming very large, we are lead to the phenomenon of wave motion.

In Unit 6, you will learn the basic concepts and vocabulary of wave motion as well as wave propagation in one and more dimensions. The wave equation for one dimensional progressive waves in a stretched string as well as fluids (gases and liquids) are established. Its connection with wave impedance presented by a medium is also discussed for transverse as well as longitudinal waves.

In Unit 7, you will learn the changes a wave undergoes at the interface of two different media, using Huygen’s construction and the concept of wave impedance. Expressions for reflection and transmission amplitude and energy coefficients are derived. In this unit we have also discussed Doppler’s effect and shock waves.

In Units 8 and 9, you will learn about the superposition of waves. You will study superposition of waves under different conditions. You will see that the superposition of two waves which have the same amplitude, frequency and wavelength but are moving in opposite directions result in the formation of stationary waves. These are basically responsible for production of music. Two waves having slightly different frequencies but traveling in the same direction give rise to a wave group and beats. In Unit 9

In each unit, we have given many SAQ's and TQ's to' fix-up your ideas. If you are not able to solve them yourself, you can look for solutions at the end of each unit.

We hope that after studying this Block you will realise the wide applicability of simple harmonic motion and its connection with wave motion. You are, therefore, expected to master the mathematical technique needed to study SHM under different conditions.

We wish you success.

UNIT 1 SIMPLE HARMONIC MOTION

Structure

1.1 INTRODUCTION

In your school science courses you must have learnt about different types of motions. You are familiar with the motion of falling bodies, planets and satellites. A body 1 released from rest and falling freely (under the action of gravity) moves along a straight line. But an object dropped from an aeroplane or a ball thrown up in the air follows a curved path (except when it is thrown exactly vertically). You must have also observed the motion of the pendulum of a wall clock and vibrating string of a violin or some other string instrument. These are examples of oscillatory motion. The simplest kind of oscillatory motion which can be analysed mathematically is the Simple Harmonic Motion (SHM). We can analyse oscillatory motions of systems of entirely different physical nature in terms of SHM. For example, the equation of motion that we derive for a pendulum will be similar to the equation of motion of a charge in a circuit containing an inductor and a capacitor. The form of solutions of these equations and the time variation of energy in these systems show remarkable similarities. However, there are many important phenomena which arise due to superposition of two or more harmonic oscillations. For example, our eardrum vibrates under a complex combination of harmonic vibrations. But we shall discuss this aspect in the next unit.

In this unit we will study oscillatory systems using simple mathematical techniques. Our emphasis would be on highlighting the similarities between different systems.

Objectives

After studying this unit you should be able to:

- state the basic criteria for the simple harmonic motion of a system
- establish the differential equation for a system executing SHM and solve it
- define the terms amplitude, phase and time period
- compute potential, kinetic and total energies of a body executing SHM
- deduce expressions for average potential and average kinetic energies and discuss their significance

- write down the equation of motion and expressions for displacement, time period and energy of
- simple physical systems executing SHM
- identify similarities between different oscillating systems.
- define wave motion an state its characteristics
- distinguish between longitudinal and transverse waves
- represent graphically waves at a fixed position or at a fixed time
- relate wavelength, frequency and speed of a wave
- establish wave equations for longitudinal and transverse waves
- compute the energy transported by a progressive wave
- derive expressions for velocities of longitudinal and transverse waves
- derive expressions for characteristic impedance and acoustic impedance
- write two and three dimensional wave equations

1.2 SIMPLE HARMONIC MOTION: BASIC CHARACTERISTICS

You all know that each hand of a clock comes back to a given position after the lapse of a certain time. This is a familiar example *of periodic motion.* When a body in periodic motion moves to-and-fro (or back and forth) about its position, the motion is called vibratory or *oscillatory.* Oscillatory motion is a common phenomenon. Well known examples of oscillatory motion are: the oscillating bob of a pendulum clock, the piston of an engine, the vibrating strings of a musical instrument, the oscillating uranium nucleus before it fissions, Even large scale buildings and bridges may at times undergo oscillatory motion. Many stars exhibit periodic variations in brightness). You must have observed that normally such oscillations, left to themselves, do not continue indefinitely, i.e., they gradually die down due to various damping factors like friction and air resistance, etc. Thus, in actual practice, the oscillatory motion may be quite complex, as for instance, the vibrations of a violin string. We begin our study with the discussion of the essential features of SHM. For this we consider an idealised model of a spring-mass system, as an example of a *simple harmonic oscillator.*

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1.2.1 Oscillations of a Spring-mass System

A spring-mass system consists of a spring of negligible mass whose one end is fixed to a rigid support and the other end carries a block of mass *m* which lies flat on a horizontal frictionless table (Fig. 1.1a). Let us take the x-axis to be along the length of the spring. When the mass is at rest, we mark a point on it and we define the origin of the axis by this point. That is, at equilibrium the mark lies at $x = 0$.

If the spring is stretched by pulling the mass longitudinally, due to elasticity a restoring force comes into play which tends to bring the mass back towards the equilibrium position (Fig l.lb). If the spring were compressed the restoring force would tend to extend the spring and restore the mass to its equilibrium position (Fig. l.lc).

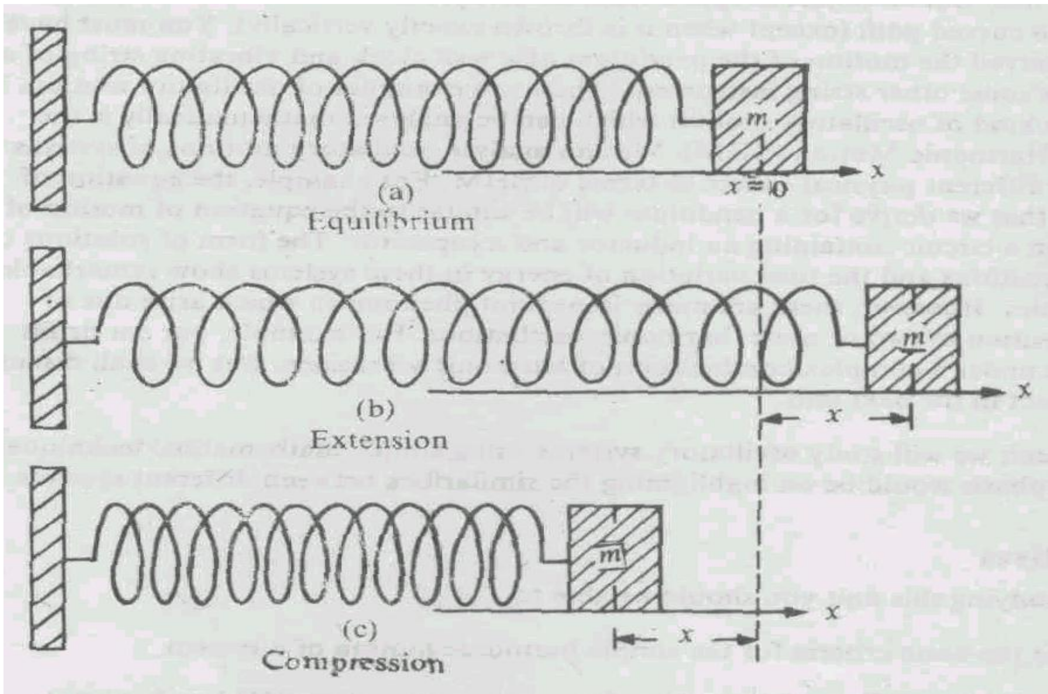


Fig. 1.1 A Spring-mass System as an ideal oscillator (a) The equilibrium configuration, (b) An extended configuration, (c) A compressed configuration.

The more you stretch/compress the spring, the more will be the restoring force. *So the direction of the restoring force is always opposite to the displacement.* If total change in the length is small compared to the original length, then the magnitude of restoring force is linearly proportional to the displacement. Mathematically, we can write

$$F = -kx \tag{1.1}$$

The negative sign signifies that the restoring force opposes the displacement. The quantity k is called the *spring constant* or *the force constant* of the spring. It is numerically equal to the magnitude of restoring force exerted by the spring for unit extension. Its SI unit is Nm^{-1}.

SAQ 1
The spring in Fig. 1.1 a is stretched by 5 cm when a force of 2 N is applied. Calculate the spring constant. How much will this spring be compressed by a force of *2.5* N?

How does the spring-mass system oscillate? To answer this question, we note that when we pull the mass, the spring is stretched. The restoring force tends to bring the mass back to its equilibrium position ($x = 0$). Therefore, on being released, the mass moves towards the equilibrium position. In this process it acquires kinetic energy and overshoots the equilibrium position. Do you know why? It is because of inertia. Once it overshoots and moves to the other side, the spring is compressed and the mass is acted upon by a restoring force but in the opposite direction. Thus we can say that oscillatory motion results from two intrinsic properties of the system: (i) elasticity and (ii) inertia.

What is the direction of the restoring force vis-a-vis the equilibrium position of an oscillating body?

The restoring force is always directed towards the equilibrium position of the oscillating body.
In discussing the spring-mass system we observed two important points:

(i) The restoring force is linearly proportional to the displacement of mass from its equilibrium position.

(ii) The restoring force is always directed towards the equilibrium position.

Any oscillatory motion which satisfies both these conditions is called simple harmonic motion. The study of SHM is important because, as you will see, oscillatory motion of systems of entirely different physical nature can be analysed in terms of it.

Let us now study the effect of gravity on oscillations of spring-mass system. Consider a spring of negligible mass suspended from a rigid support with a mass *m* attached to its lower end (Fig.1.2).

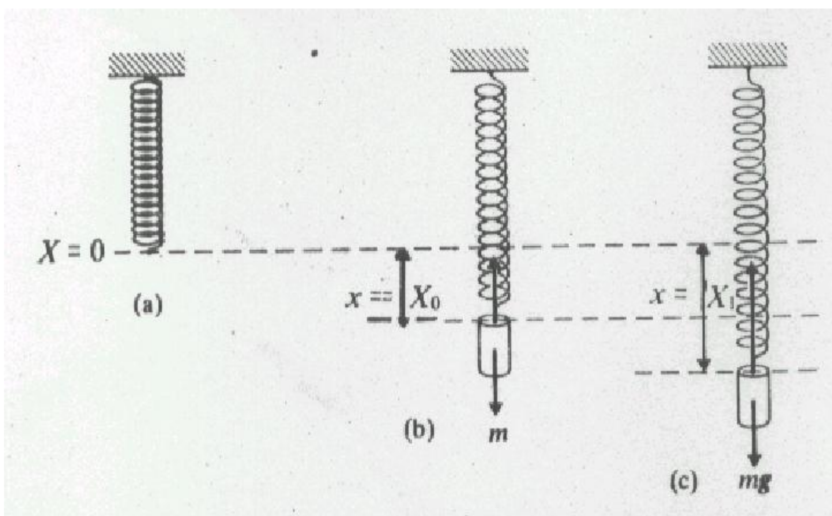


Fig 1.2: A vertically hanging spring-mats system, (a) The sprint with no object suspended from it, (b) The spring in equilibrium with mass *m* suspended, (c) Spring-mass system displaced from equilibrium position.

Let us choose the *X*-axis along the length of the spring. We take the bottom of the spring as our reference point, $X = 0$, when no wejght is attached to it (Fig. l.2a). When a mass *m* is suspended from the spring, let the reference point move to $X=X_0$ (Fig. 1.2b). At equilibrium, the weight, *mg*, balances the spring force, kX_0.. Since the net force is zero, we have

$$mg - kX_0 = 0$$

or

$$mg = kX_0 \qquad (1.2)$$

Now if the mass is pulled downwards so that the reference mark shifts to X_1 (Fig 1.2c), then the total restoring force will be kX_1 and points in the upward direction. The net downward force will therefore be (using Eq. (1.2)),

$$mg - kX_1 = k(X_0 - X_1) \equiv -kx$$

where $x = X_1 - X_0$.

Thus, the resulting restoring force on the mass is

$$F = -kx$$

where x is its displacement from the equilibrium position, X_0. This result is of the same form as Eq. (1.1) for the horizontal arrangement. It is thus clear that gravity has no effect on the frequency of oscillations of a mass hanging vertically from a spring; it only displaces the equilibrium.

1.3 DIFFERENTIAL EQUATION OF SIMPLE HARMONIC MOTION

Let us now find the differential equation which describes the oscillatory motion of a spring-mass system. The equation of motion of such a system is given by equating the two forces acting on the mass:

mass × acceleration = restoring force

or

$$m\frac{d^2x}{dt^2} = -kx$$

where $\frac{d^2x}{dt^2}$ is the acceleration of the body.

It is important to note that in this equation, the equilibrium position of the body is taken as the origin, $x = 0$.

You will note that the quantity k/m has units of $Nm^{-1}kg^{-1} = (kg.mg^{-2})kg^{-1}m^{-1} = s^{-2}$. Hence we can replace k/m by $\omega_0{}^2$ where ω_0 is called the *angular frequency* of the oscillatory motion. Then the above equation takes the form

$$m\frac{d^2x}{dt^2} + \omega_0{}^2 = 0 \qquad (1.3)$$

It may be remarked here that Eq. (1.3) is the differentia] form of Eq. (1.1) and describes simple harmonic motion in one dimension.

A differential equation having terms involving only the first power of the variable and its derivatives is known as a linear differential equation. If such an equation contains no term independent of the variable it is said to be homogeneous. We may, therefore, say that Eq. (1.3) is a second order linear homogeneous equation. Its solution will contain two arbitrary constants.

1.4 SOLUTION OF THE DIFFERENTIAL EQUATION FOR SHM

To find the displacement of the mass at any time *t*, we have to solve Eq. (1.3) subject to given initial conditions. A close inspection of Eq. (1.3) shows that *x* should be such a function that its second derivative with respect to time is the negative of the function itself, except for a multiplying factor o>o. From elementary calculus, we know that sine and cosine functions have this property.
You can check that this property does not change even if sine and cosine functions have a constant multiplying factor.

A general solution for *x (t)* can thus be expressed as a linear combination of both sine and cosine terms, i.e.,

$$x(t) = A_1 \cos\alpha t + A_2 \sin\alpha t \tag{1.4}$$

$$\frac{d(\sin\alpha t)}{dt} = \alpha\cos\alpha t \quad \text{and} \quad \frac{d^2(\sin\alpha t)}{dt^2} = -\alpha^2 \sin\omega t$$

Similarly,

$$\frac{d(\cos\alpha t)}{dt} = -\alpha\sin\alpha t \quad \text{and} \quad \frac{d^2(\cos\alpha t)}{dt^2} = -\alpha^2 \cos\omega t$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$

Putting $A_1 = A\cos\phi$ and $A_2 = -A\sin\phi$, we get

$$x(t) = A\cos(\alpha t + \phi)$$

Differentiating this equation twice with respect to time and comparing the resultant expression with Eq. (1.3), we obtain $\alpha = \pm\omega_0$. The negative sign is dropped as it gives negative frequency which is a physically absurd quantity.

Substituting $\alpha = \omega_0$ in the above equation, we get

$$x(t) = A\cos(\omega_0 t + \phi) \tag{1.5}$$

The constants *A* and ϕ occurring in Eq. *(1.5)* are determined using the initial conditions on displacement *(x)* and velocity $\frac{dx}{dt}$.

Let us assume that the mass is held steady at some distance a from the equilibrium position and then released at $t = 0$. Thus the initial conditions are: at $t = 0$, $x = a$

and $\frac{dx}{dt} = 0$. Then, from Eq. (1.5) we would have

$$x\,(\text{at } t = 0) = A\cos\theta = a$$

and

$$\frac{dx}{dt}(\text{at } t = 0) = -A\omega_0 \sin\phi = 0$$

These conditions are sufficient to fix A and ϕ. The second condition tells us that θ is either zero or $n\pi$ (n = 1, 2, ...). We reject the second option because the first condition requires cos ϕ to be positive. Thus with the above initial conditions, Eq. (1.5) has the simple form

$$x = a\cos\omega t \qquad (1.6)$$

SAQ2
Take $A_1 = B\sin\theta$ and $A_2 = B\cos\theta$ in Eq.(1.4). In this case show that the solution is

We therefore observe that both cosine and sine forms are valid solutions of Eq. (1.3). If you plot Eq. (1.5), the graph will be a cosine curve with a definite initial phase (Fig. 1.3).

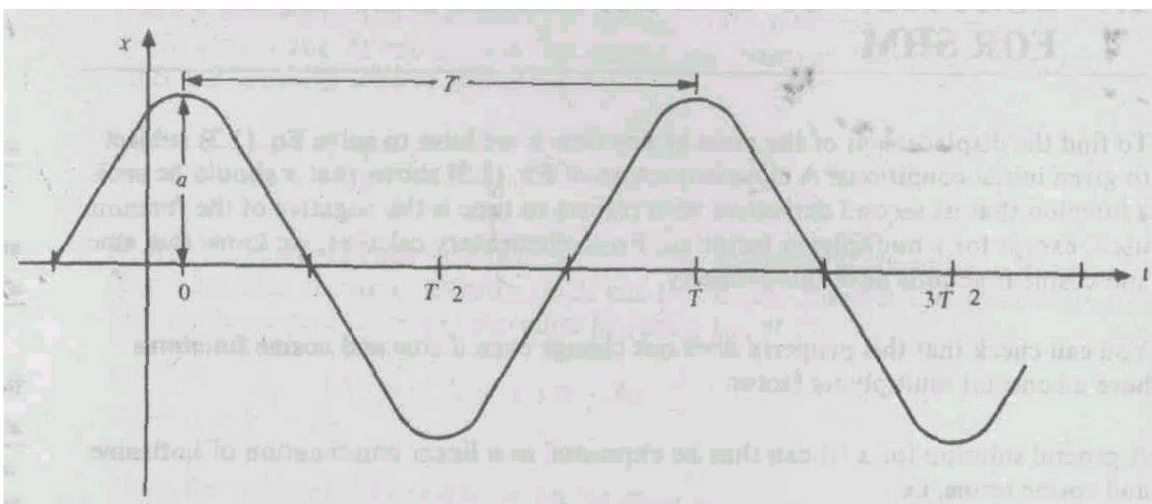


Fig. 1.3: Displacement-time graph of simple harmonic motion with an initial phase ϕ

1.4.1 Phase and Amplitude

The quantity $(\omega_0 t + \phi)$, occurring in Eq. (1.5) is called the *phase angle* or the *phase angle* of the system at $t = 0$, also called the initial phase we start measuring the displacement. If at $x = x_0$, then, from Eq. (1.5) it follows that

$$x_0 = a\cos\phi$$

We know that the value of the sine and cosine .functions lies between 1 and -1. When $\cos(\omega_0 t + \phi)$ = 1 or -1, the displacement has the maximum value. Let us denote it by a or $-a$. The quantity a is called the *amplitude* of oscillation.

We can, therefore, rewrite Eq. (1.5) as

$$x(t) = \cos(\omega_0 t + \phi) \tag{1.7}$$

The displacement-time graphs for ϕ = 0, $\pi/2$ and π are shown in Fig.1.4. In all the cases, the graphs have exactly the same shape if we shift the origin along the time axis. When the phase difference is π two oscillations are said to be in opposite phase or out of phase by π.

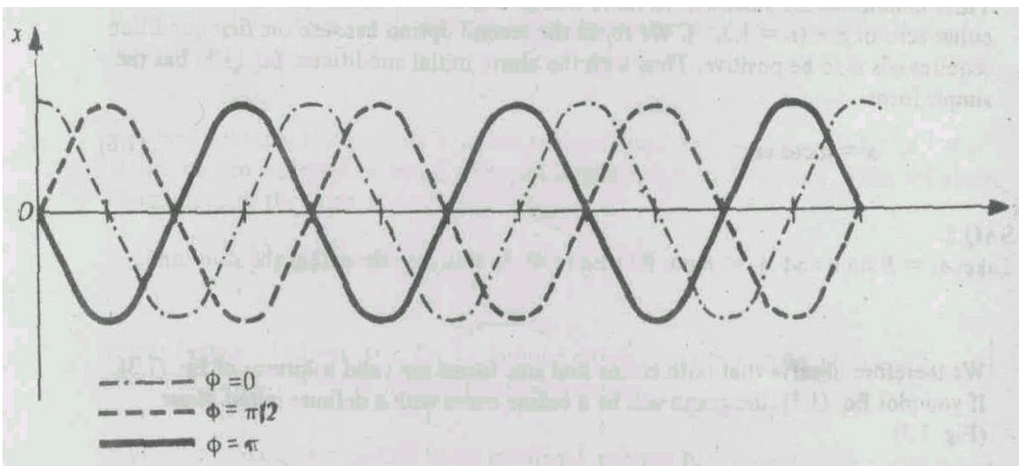


Fig. 1.4: Plot of Eq. (1.7) for ϕ = 0, $\pi/2$ and π

The mass in Fig .1.1 oscillates with an amplitude a. If the time is measured from the instant when it is at (i) x = + a, (ii) x = $-a$, and (iii) $x = a/\sqrt{2}$, calculate the phase constant for the equations (a) $x = a\sin(\omega_0 t + \phi)$ and (b) $x = a\cos(\omega_0 t + \phi)$.

1.4.2 Time Period and Frequency

If we put $t = t + (2\pi/\omega_0)$ in Eq. (1.7), we obtain

$$\begin{aligned} x(t) &= a\cos[\omega_0(t + 2\pi/\omega_0) + \phi] \\ &= a\cos[\omega_0 t + 2\pi + \phi] \\ &= a\cos(\omega_0 t + \phi) \end{aligned}$$

That is, the displacement of the particle repeats itself after an interval of time $2\pi/\omega_0$. In other words, the oscillating particle completes one vibration in time $2\pi/\omega_0$. This time is called the *period of vibration* or the *time period.* We denote it by T :

$$T = 2\pi / \omega_0 \tag{1.8}$$

For a spring-mass system, $\omega_0^2 = k/m$, so that

$$T = 2\pi\sqrt{m/k} \tag{1.9}$$

The number of vibrations executed bv the oscillator per second is called the *frequency*. The unit of frequency is Hertz (Hz). Denoting it ν_0, we have for a spring-mass system

$$\nu_0 = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.10}$$

This means that stiffer the spring, higher will be the frequency of vibration.

1.4.3 Velocity and Acceleration

We know that the displacement of a mass executing simple harmonic motion is given by

$$x = a\cos(\omega_0 t + \phi)$$

Therefore, the instantaneous velocity, which is the lust time derivative of displacement, is given by

$$v = \frac{dx}{dt} = -\omega_0 a \sin(\omega_0 t + \phi) \tag{1.11}$$

We can rewrite it as

$$v = \omega_0 a \cos(\pi/2 + \omega_0 t + \phi) \tag{1.12a}$$

You may also like to know the value of v at any point .v. To this end, we rewrite Eq. (1.11) as

$$v = -\omega_0 [a^2 - a^2 \cos(\omega_0 t + \phi)]^{1/2} \text{ for } -a \le x \le a \tag{1.12b}$$

We also know that acceleration is the first time derivative of velocity. From Eq. (1.11)

$$\boxed{\sin\theta = \sqrt{1-\cos^2\theta}\,;\ \cos(90^0 + \theta) = -\sin\theta}$$

$$\begin{aligned} \frac{dv}{dt} &= -\omega_0^2 a\cos(\omega_0 t + \phi) \\ &= \omega_0^2 a\cos(\pi + \omega_0 t + \phi) \end{aligned} \tag{1.13a}$$

Obviously, in terms of displacement,

$$\frac{dv}{dt} = -\omega_0^2 x \tag{1.13b}$$

If you compare Eqs. (1.7), (1.12 a) and (1.13 a), you will note that (i) $\omega_0 a$ is the velocity amplitude and $\omega_0{}^2 a$ is the acceleration amplitude, and (ii) velocity is ahead of displacement by $\pi/2$ and acceleration is ahead of velocity by $\pi/2$.

If you plot displacement, velocity, and acceleration as functions of time, you will get graphs as shown in Fig.1.5.

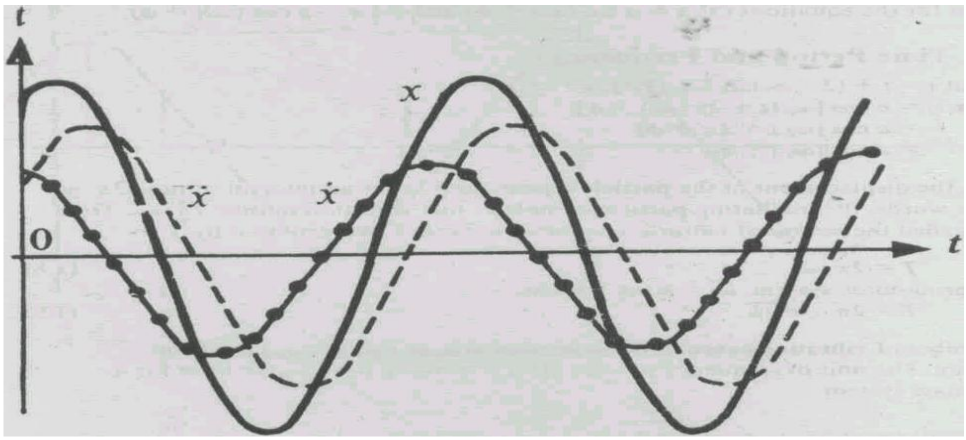


Fig. 1.5: Time variation of displacement, velocity and acceleration of a body executing SHM (ϕ = 0)

SAQ4

The displacement of a particle executing simple harmonic motion is given hy $x = 0.01\cos 4\pi(t + 0.0625)$ metre. Deduce (i) the amplitude, (ii) the time-period, (iii) maximum speed, (iv) maximum acceleration and (v) initial displacement.

1.5 TRANSFORMATION OF ENERGY IN OSCILLATING SYSTEMS: POTENTIAL AND KINETIC ENERGIES

Consider the spring-mass system shown in Fig 1.1. When the mass is pulled, the spring is elongated. The amount of energy required to elongate the spring through a distance *dx* is equal to the work done in bringing about this change. It is given by $dW - dU = F_0 dx$, where F_0 is the applied force (such as by hand). This force is balanced by the restoring force. That is, its magnitude is same as that of F and we can write $F_0 = kx$. Therefore, the energy required to elongate the spring through a distance *x* is

$$U = \int_0^x F_0 dx = k\int_0^x x dx = \frac{1}{2}kx^2 \qquad (1.14)$$

This energy is stored in the spring in the form of potential energy and is responsible for oscillations of the spring-mass system.

On substituting for the displacement from Eq. (1.7) in Eq. (1.14), we get

$$U = \frac{1}{2}ka^2 \cos(\omega_0 t + \phi) \qquad (1.15)$$

Note that at t = 0, the potential energy is

$$U_0 = \frac{1}{2}ka^2 \cos^2 \phi \qquad (1.16)$$

As the mass is released, it moves towards the equilibrium position and the potential energy starts changing into kinetic energy *(K.E)*. The kinetic energy at any time / is given by $K.E. = \frac{1}{2}mv^2$. Using Eq. (1.11), we get

$$K.E. = \frac{1}{2}m\omega_0{}^2 \sin^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}ka^2 \sin^2(\omega_0 t + \phi) \qquad (1.17)$$

since $\omega_0{}^2 = k/m$.

One can also express *K.E.* in terms of the displacement by writing $K.E.$ in terms of the displacement by writing

$$K.E. = \frac{1}{2}ka^2[1 - \cos^2(\omega_0 t + \phi)]$$

$$= \frac{1}{2}ka^2 - \frac{1}{2}ka^2 \cos^2(\omega_0 t + \phi) \qquad (1.17)$$

$$= \frac{1}{2}ka^2 - \frac{1}{2}kx^2 = \frac{1}{2}k(a^2 - x^2) \qquad (1.18)$$

This shows that when an oscillating body passes through the equilibrium position $(x = 0)$, its kinetic energy is maximum and equal to $\frac{1}{2}ka^2$.

SAQ5
Show that the periods of potential and kinetic energies are one-half of the period of vibration.

It is thus clear from the explicit time dependence of Eqs. (1.15) and (1.17) that in a spring-mass system the mass and the spring alternately exchange energy. Let us consider that the initial phase ϕ = 0. At t = 0, the potential energy stored in the spring is maximum and *K.E.* of the mass is zero. At t = T/4, the potential energy is zero and *K. E.* is maximum. As the mass oscillates, energy oscillates from kinetic form to potential form and vice versa. At any instant, the total energy, E, of the oscillator will be sum of both these energies. Hence, from Eqs. (1.15) and (1.17), we can write

$$E = U + K.E. = \frac{1}{2}ka^2 \cos^2(\omega_0 t + \phi) + \frac{1}{2}ka^2 \sin^2(\omega_0 t + \phi) = \frac{1}{2}ka^2 \qquad (1.19)$$

This means that the total energy remains constant (independent of time) and is proportional to the square of the amplitude. As long as there are no dissipative forces like friction, the total mechanical energy will be conserved.

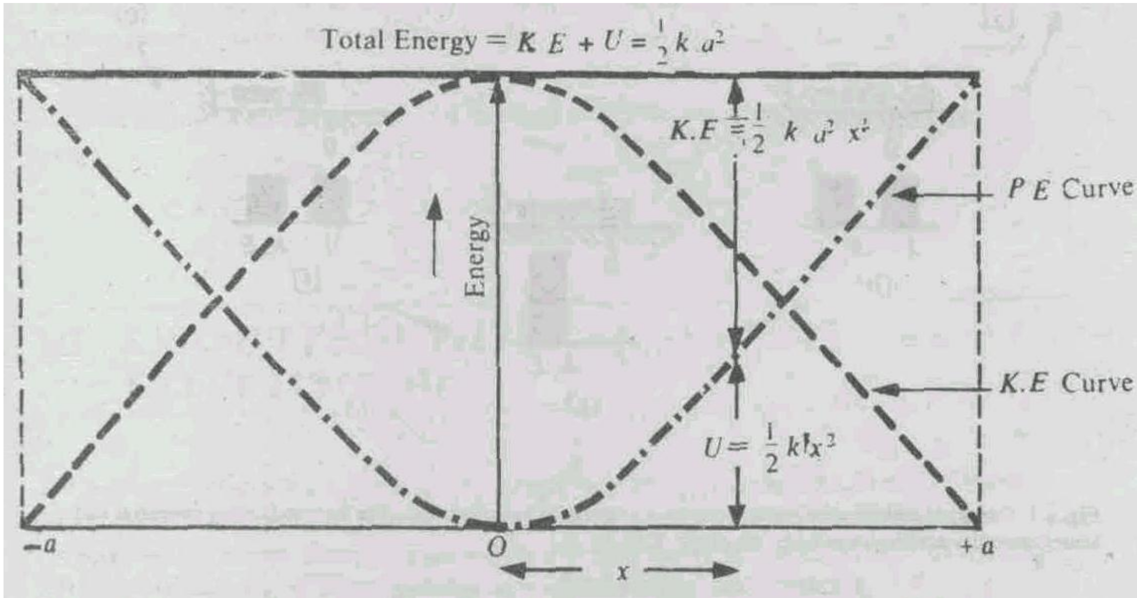


Fig. 1.6 Variation of potential energy (U), kinetic energy *(K.E)* and total energy (E) with displacement according to Eqs. (1.14), (1.18) and (1.19)

The plots of U and *K.E.* as a function of x as obtained from Eqs. (1.14) and (1.18) are shown in Fig 1.6. You will note that

(i) the shape of these curves is parabolic, (ii) the shape is symmetric about the origin, and (iii) the potential and kinetic energy curves are inverted with respect to one another. Why? This is due to the phase difference of $\pi / 2$ between the displacement and velocity of a harmonic oscillator. At any value of x, the total energy is the sum of kinetic and potential energies and is equal to $\frac{1}{2}ka^2$. This is represented by the horizontal line.

The points where this horizontal line intersects the potential energy curve are called the 'turning points.' The oscillating particle cannot go beyond these and turns back towards the equilibrium position At these points, the total energy of the oscillator is entirely potential $(E = U = \frac{1}{2}ka^2)$ and *K.E.* is zero. At the equilibrium position *(x* = 0) the energy is entirely kinetic $(K.E. = E = \frac{1}{2}ka^2)$ so that the maximum speed, $v_{\max}$ is given by the relation $\frac{1}{2}mv_{\max}^2 = E$, i.e., $v_{\max} = \sqrt{2E/m}$.

At any intermediate position, energy is partly kinetic and partly potential, but the total energy always remains the same. The transformation of energy in a spring-mass system is shown in Fig 1.7.

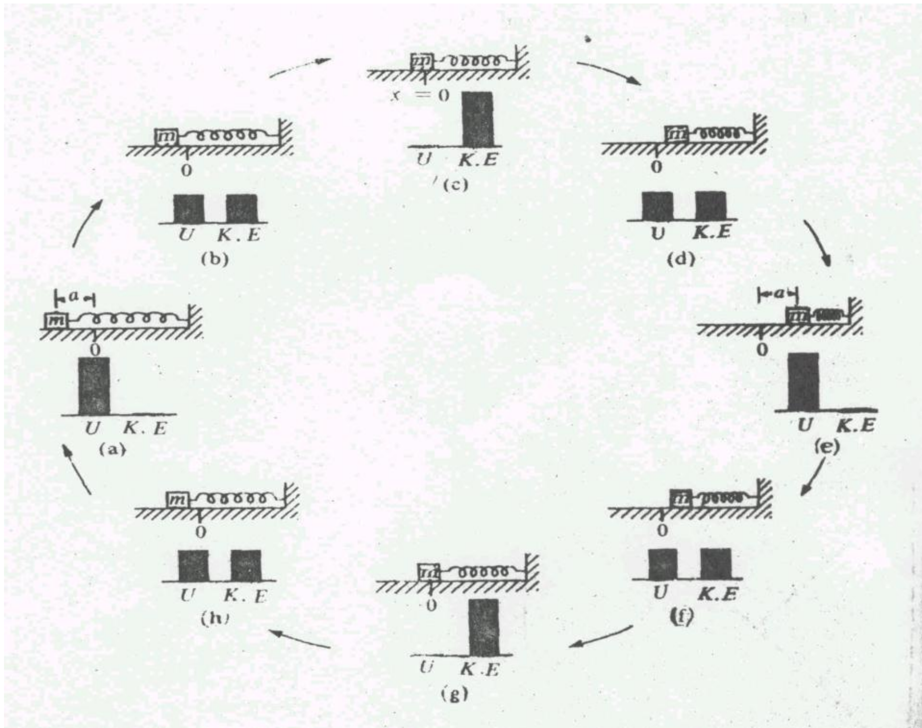


Fig 1.7 Energy transformation in a spring-mass system at various times. The bars indicate potential and kinetic energies are shown at intervals of $t = T/8$.

Do you know why the point of minimum potential energy is regarded as the position of stable equilibrium? This is because there is no net force acting on the system in this position.

SAQ 6

A body of mass *m* fell from height h onto the pan of a spring balance. The masses of the pan and the spring are negligible. The stiffness constant of the spring is *k.* Having stuck to the pan, the body executes harmonic oscillations in the vertical direction. Find the amplitude and the energy of oscillation.

1.6 CALCULATION OF AVERAGE VALUES OF QUANTITIES ASSOCIATED WITH SHM

In Fig. 1.5 we have plotted displacement, velocity and acceleration as a function of time. You will note that for any complete cycle in each case, the area under the curve for the first half is exactly equal to the area under the curve in the second half and the two are opposite in sign. Thus over one complete cycle the algebraic sum of these areas is zero. This means that the average values of displacement, velocity and acceleration over one complete cycle are zero. If we plot x^2 (or v^2) versus t, the curves would lie in the upper half only so that the total area will be positive during one complete cycle. This suggests that we can talk about average values of kinetic and potential energies.

The time average of kinetic energy over one complete cycle is defined as

$$< K.E. > = \frac{\int_0^T K.E.dt}{T} \tag{l.20a}$$

On substituting for *K.E* from Eq. (1.17), we get

$$< K.E. > = \frac{ka^2}{2T}\int_0^T \sin^2(\omega_0 t + \phi)dt \tag{1.20b}$$

On solving the integral in Eq. (l.20b) you will find that its value is $T/2$. So, the expression for average kinetic energy reduces to

$$< K.E. > = \frac{ka^2}{4} \tag{1.21}$$

Similarly, one can show that the average value of potential energy over one cycle is

$$< U > = \frac{ka^2}{4} \tag{1.22}$$

That is, the average kinetic energy of a harmonic oscillator is equal to the average potential energy over one complete period.

Thus the sum of average kinetic and average potential energies is equal to the total energy :

$$< K.E. > + < U > = \frac{1}{4}ka^2 + \frac{1}{4}ka^2$$

$$= \frac{1}{2}ka^2 = E$$

1.7 EXAMPLES OF PHYSICAL SYSTEMS EXECUTING SHM

We have seen that for a system to execute simple harmonic motion, i»must have two parts: one which can store potential energy (like spring) and the other capable of storing kinetic energy (such as mass). We will now study physical systems executing SHM using techniques developed for our model spring-mass system.

1.7.1 Simple Pendulum

A simple pendulum is an idealized system consisting of a point mass (bob) suspended by an inextensible, weightless string. As the bob of mass m is displaced by an angle θ from its equilibrium position, the restoring force is provided by the tangential component of the weight mg along the arc (Fig. 1.8). It is given by

$$F = -mg\sin\theta$$

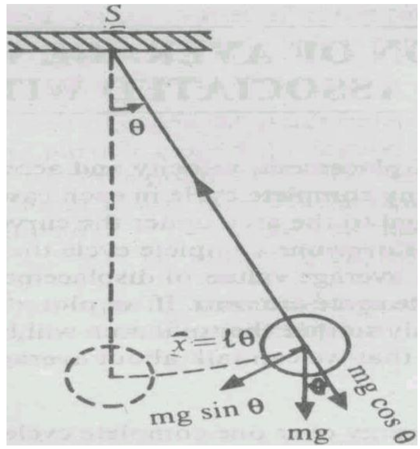


Fig. 1.8 A simple pendulum

The equation of motion of the bob is, therefore,

$$m\frac{d^2x}{dt^2} = -mg\sin\theta \tag{1.23}$$

The bob is moving along the arc whose length at any instant is given by *x*. If the corresponding angular displacement from the equilibrium position is θ, then the length of arc is

$$x = l\theta \tag{1.24}$$

where l is length of the string by which the bob is suspended. Differentiating Eq. (1.24) twice with respect to *t*, and substituting the result in Eq. (1.23), we get

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \tag{1.25}$$

For small angular displacements, sin θ may be approximated to θ. In this approximation, Eq. (1.25) takes the form

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta = 0 \tag{1.26}$$

where $\omega_0 = \sqrt{g/l}$.

Eq. (1.26) is exactly of the standard form (1.3) showing that pendulum executes simple harmonic motion. The time period of oscillation is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{l/g} \tag{1.27}$$

By analogy, we can write the general solution of the Eq. (1.26) as

$$\theta = \theta_m \cos(\omega_0 t + \phi) \tag{1.28}$$

where is the maximum angular displacement.

From Eq. (1.27) you will note that for small angular displacements, the frequency of oscillation of a simple pendulum depends on g and but not on the mass of the bob. The appearance of the factor g in Eq. (1.27) implies that a pendulum clock will move slower near the equator than at the poles. Do you know why? This is because the value of g varies with latitude. For the same reason, the period of a pendulum will be different on moons and planets.

When the amplitude of oscillation is not small, we are required to solve the general Eq. (1.25). The time period, which can be expressed in the form of a series involving the maximum angular displacement θ_m is given by

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{2^2}\sin^2\frac{\theta_m}{2} + \frac{1}{2^2}\cdot\frac{3^2}{4^2}\sin^4\frac{\theta_m}{2} + \ldots\right)$$

You can check the accuracy of Eq. (1.27) by comparing the value of T obtained from Eq. (1.29). For example, you will find that when θ_m is 15° (corresponding to a total to **and** fro angular displacement of 30°), the actual value of time period differs from that given by Eq. (1.27) by less than 0.5%.

SAQ7

Use the principle of conservation of energy to show that the angular speed of a simple pendulum is given by,

$$\dot{\theta} = \left[\frac{2}{ml^2}[E - mgl(1 - \cos\theta)]\right]^{1/2}$$

where the symbols have the usual meanings.

1.7.2 Compound Pendulum

A compound pendulum is a rigid body capable of oscillating freely about a horizontal axis passing through it (Fig 1.9). At equilibrium position, the centre of gravity G lies vertically below the point of suspension S. Let the distance SG be l. If the pendulum is given a small angular displacement θ at any instant, it oscillates over the same path. Is its motion simple harmonic? To answer this question we note that the restoring torque about S is $-mgl\sin\theta$ and it tends to bring the pendulum towards the equilibrium position.

If I is the moment of inertia of the body about the horizontal axis passing through S, the restoring torque equals $Id^2\theta/dt^2$. Hence, the equation of motion can be written as

> The moment of inertia is the ratio of the torque of a body rotating about a given axis to the angular acceleration about that axis. Note that moment of inertia always refers to a definite axis of rotation.

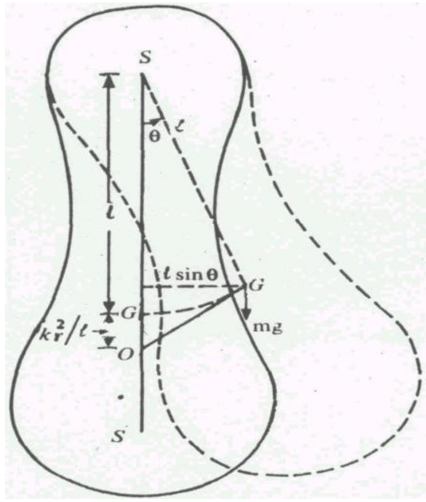


Fig. 1.9 A rigid body oscillating about a horizontal axis: Compound pendulum.

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta \tag{1.30}$$

For small angular displacement, $\sin\theta \approx \theta$ and Eq. (1.30) takes the form

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0 \tag{1.31}$$

This equation shows that a compound pendulum executes SHM and the time period is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{I/mgl} \tag{1.32}$$

There is a very useful and important theorem of parallel axes in the study of moment of inertia. According to this theorem, the moment of inertia I of a body about any axis and its inertia I_g about a parallel axis passing through its center of gravity are connected by the relation

$$I = I_g + ml^2 \tag{1.33}$$

where l is the distance between the two axes and $I_g = mk_r^2$. The quantity k_r is the radius of gyration of the body about the axis passing through G. It is the radial distance at which the whole mass of the body could be placed without any change in the moment of inertia of the body about that axis.

On substituting the expression for I from Eq. (1.33) in Eq. (1.32), we obtain

$$T = 2\pi\sqrt{\frac{k_r^2 + l^2}{gl}} \tag{1.34}$$

On comparing this expression for T with that given by Eq. (1.27) for a simple pendulum, you will note that two periods become equal if l in Eq. (1.27) is replaced by $L = \sqrt{(k_r^2 / l) + l}$. This is called the length of an equivalent simple pendulum. If we produce the line SG and take a point O on it such that $SO = \dfrac{k_r^2}{l} + l$, then O is called the centre of oscillation.

1.7.3 Torsional Systems

If one end of a long thin wire is clamped to a rigid support and the other end is fixed to the centre of a massive body such as a disc, cylinder, sphere or rod, then the arrangement is called a *torsional pendulum* (Fig 1.10). You will come across many instruments in your physics laboratory which execute torsional oscillations. The most familiar of these is the inertia table. It is commonly used to determine the moment of inertia of regular as well as irregular bodies. Ammeters, voltmeters and moving coil galvanometers are other measuring devices where restoring torque is provided by spiral springs or suspension fibres.

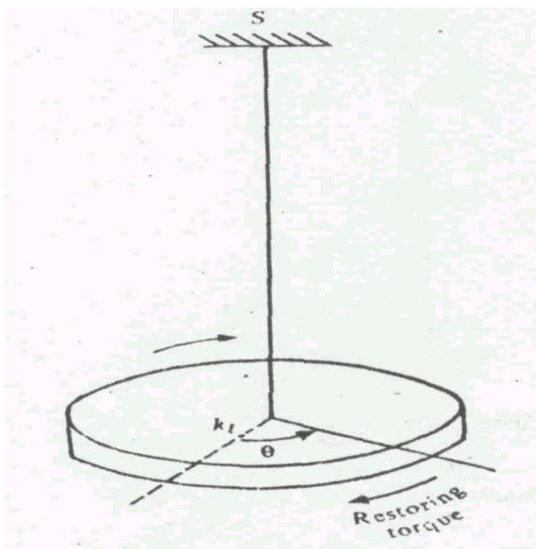


Fig. 1.10 Torsional Pendulum: Restoring torque is opposite to θ.

When a torsional system is twisted and then left free, it executes torsional oscillations in a horizontal plane. For an angular displacement θ, the restoring torque is $-k_t\theta$. Here, k_t is a constant which depends on the properties of the wire.

If I is the moment of inertia of the system about the axis of rotation and $d^2\theta/dt^2$ is the angular acceleration, then the equation of motion is

$$I\frac{d^2\theta}{dt^2} = -k_t\theta$$

or
$$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta \qquad (1.35)$$

where $\omega_0 = \sqrt{k_t / l}$. This is exactly of the standard form (1.3). Hence, the motion is SHM and the period of oscillation is

$$T = 2\pi\sqrt{I / k_t} \tag{1.35a}$$

You will note that this expression for T contains no approximation. This means that the time period for large amplitude oscillations will also remain the same, provided the elastic limit of the suspension wire is not exceeded. The solution of Eq. (1.35) is given by Eq. (1.28).

SAQ8

A solid sphere of mass 3 kg and radius 0.01 m suspended from a wire. Find the period of oscillations, if the torque required to twist the wire is 0.04 N-m rad^{-1}. The moment of inertia of a sphere about an axis passing through its centre is given by

$$I = \frac{2}{5} mr^2$$

1.7.4 An L-C Circuit

So far we have discussed oscillations of mechanical systems. We will now discuss harmonic oscillations of charge in an ideal *(R* = 0) *L-C* circuit depicted in Fig. 1.11. As we know, an L-C circuit has no moving parts, but the electric and magnetic energies in such a circuit play roles analogous to potential and kinetic energies respectively for a spring-mass system. For simplicity, we assume that the inductor has no resistance.

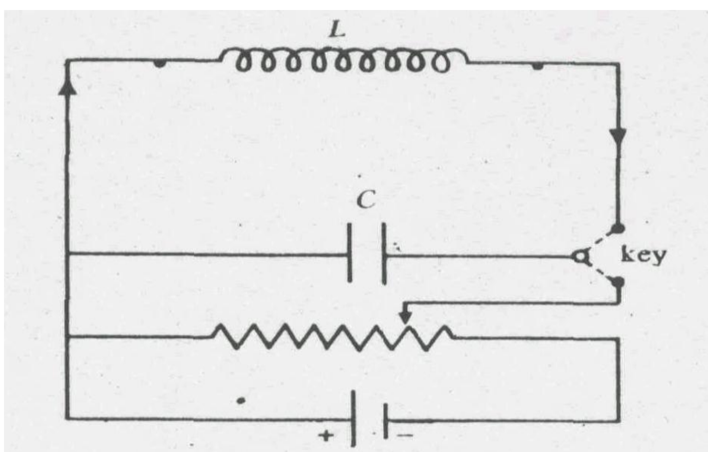


Fig. 1.11 An ideal L-C circuit.

In a pendulum, the mean position is taken as the equilibrium state. What is the equilibrium state in an L-C circuit? It corresponds to the state when there is no current in the circuit. It may be disturbed by charging or discharging the capacitor. Let the capacitor be given a charge Q_0 coulomb. Then the voltage across the capacitor plates will be Q_o/C. Now if the circuit is disconnected, the capacitor discharges through the inductor. As a result current starts building up in the circuit gradually and the charge on the plates of the capacitor decreases. At any time t, let the current in the circuit be q and the charge on capacitor plates be q. Then the voltage drop across the inductor will be

$$V_L = -L\frac{dI}{dt}$$

This must be equal to the voltage $V_c = q/C$ across the capacitor plates at that time. Thus, we can write

$$V_c = V_L$$

or $$\frac{q}{C} = -L\frac{dI}{dt} \qquad (1.36)$$

Since $I = dq/dt$ and $\frac{dI}{dt} = \frac{d^2q}{dt^2}$, Eq. (1.36) takes the form

$$\frac{d^2q}{dt^2} + \omega_0^2 q = 0 \qquad (1.37)$$

where $\omega_0^2 = \frac{1}{LC}$. This means that one can have a wide range of frequencies by changing the values of *L* and C. That is how you tune different stations in your radio sets.

Eq. (1.37) represents SHM and has the solution

$$q = q_0 \cos(\omega_0 t + \phi) \qquad (1.38)$$

This shows that charge oscillates harmonically with the period

$$T = 2\pi\sqrt{LC} \qquad (1.39)$$

Differentiating Eq.(1.38) with respect to time, we get the instantaneous current

$$\begin{aligned} I &= -Q_0\omega_0 \sin(\omega_0 t + \phi) \\ &= I_0 \cos(\omega_0 t + \phi + \pi/2) \end{aligned}$$

where $I_0 = Q_0\omega_0$.

Thus the current leads the charge in phase by $\pi/2$. In practice, you will always find that an inductor offers some resistance in an *L*-*C* circuit. Its effect on charge oscillations will be discussed in Unit 3.

Let us now calculate the energy stored in the inductor *L* and the capacitor *C* at any instant *t*. As the current rises from zero to *I* in time *t*, the energy stored in the inductor, E_L, is obtained by integrating the instantaneous power with respect to time,
i.e.

$$E_L = -\int_0^t IV_L dt$$

The negative sign implies that work is done against, rather than by the emf. On substituting for *VL*, we get

$$E_L = L\int_0^I \frac{dI}{dt} Idt = \frac{1}{2}LI^2$$

The energy stored in the capacitor at time *t* is

$$E_C = q^2 / 2C$$

Thus the total energy

$$E = E_L + E_C = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C} \qquad (1.40)$$

This expression, for total energy is similar to the one for mechanical oscillator $(E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2)$. As q and I vary with time, the inductor and capacitor exchange energy periodically. This is similar to the energy exchange in the spring-mass system. Further, the mass and inductor play analogous roles in mechanical and electrical systems, respectively.

SAQ 9

Calculate the frequency of electrical oscillations when an inductor of 20 mH is connected with a capacitor of 1 μF. If the maximum potential difference across the capacitor is 10 V, calculate the energy of oscillation.

1.7.5 An Acoustic Oscillator

Consider a flask of volume V with a narrow neck of length l and area of cross-section A, such that $V >> lA$ (Fig. 1.12). Such a system is also called *Helmholiz resonator* because the system can resonate when the frequency of sound incident on it coincides with its natural frequency. We will here calculate the expression for the natural frequency of the resonator.

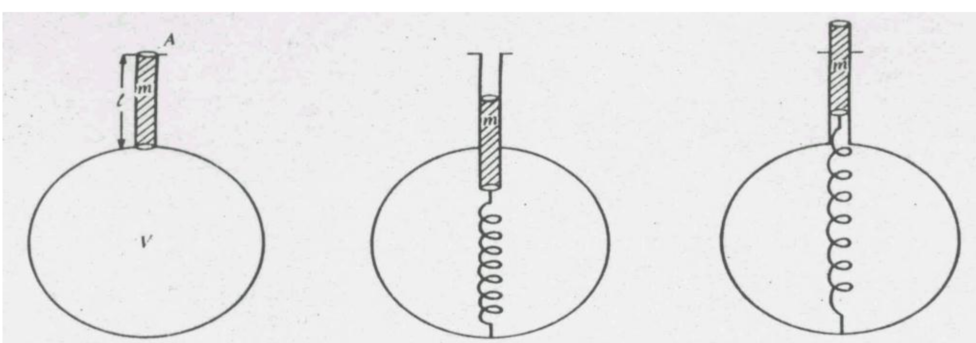


Fig. 1.12 (a) An Acoustic oscillator, (b) As air in the neck is pushed, air in the flask is compressed, and (c) Due to elasticity, air in the flask exerts a restoring force on the air in the neck.

We consider free vibrations of air in the neck of the flask. As the air in the neck moves in, the air in the flask is compressed. If air in the neck goes out, the air in the flask is rarefied. So the air in the neck behaves like the mass and the air in the flask behaves like the spring in a mechanical oscillator.

Suppose that the air in the neck moves inward through a distance *x* from the equilibrium position. The change in the volume of the air in the bulb $\Delta V = x A$. Let the increase in pressure over the atmospheric pressure be Δp. We know that the volume of a gas depends on the pressure as well as the temperature. Therefore, the pressure changes in acoustic vibrations should alternately heat and cool the air in the flask as it gets compressed and rarefied. We assume that the pressure changes are so rapid that they do not permit any exchange of heat. That is, the process is adiabatic. Hence, we can write

$$\Delta p = -E_\gamma \frac{\Delta V}{V} = -E_\gamma \frac{Ax}{V} \tag{1.41}$$

where E_γ is the *adiabatic elasticity* of the gas. It is defined as the ratio of the stress to volume strain. Numerically, stress is same as pressure. So we can write

$$E_\gamma = -\frac{\Delta p}{(\Delta V / V)}$$

The negative sign signifies the fact that as pressure increases, volume decreases and vice-versa.

This excess pressure Δp of air inside the bulb provides the restoring force *F*, which acts upward. We can therefore write

$$F = \Delta p A = -\frac{E_\gamma A^2}{V} x$$

If ρ is the density of air, the mass of the air in the neck $m = lA\rho$. Hence, the equation of motion of air in the neck can be written as

$$lA\rho \frac{d^2x}{dt^2} = -\frac{E_\gamma A^2}{V} x$$

or
$$\frac{d^2x}{dt^2} + \frac{E_\gamma A}{Vl\rho} x = 0 \tag{1.42}$$

This equation has the standard form for simple harmonic motion. Hence, the frequency of oscillation of air in the neck is

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{E_\gamma A}{vl\rho}} = \frac{v_s}{2\pi}\sqrt{\frac{A}{vl}}$$

where $v_s = \sqrt{E_\gamma / \rho}$ is the speed of sound. We know that v_s is proportional to the square root of temperature. So the frequency of vibration of air in a flask is also proportional to the square root of the temperature.

SAQ 10

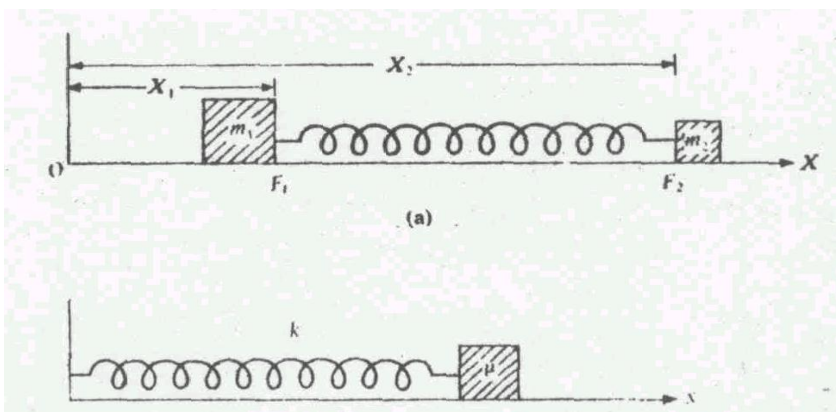
A flask has a neck of radius 1 cm and length 10 cm. If the capacity of the flask is 2 litres, determine the frequency at which the system will resonate (speed of sound in air = 350 ms^{-1}).

1.7.6 A Diatomic Molecule: Two-Body Oscillations

A diatomic molecule like *HCl* is an example of a two-body system which can oscillate along the line joining the two atoms. The atoms of a diatomic molecule are coupled through forces which have electrostatic origin. The bonding between them may be likened to a spring. Thus we may consider a diatomic molecule as a system of two masses connected by a spring. We will now consider the oscillations of such a system.

Suppose that two masses m_1 and m_2 are connected by a spring of force constant *k*. The masses are constrained to oscillate along the axis of the spring (Fig. 1.13a). Let r_0 be the normal length of the spring. We choose X -axis along the line joining the two masses. If X_1 and X_2 are the coordinates of the two ends of the spring at time *t*, the change in length is given by

$$x = (X_2 - X_1) - r_0 \tag{1.44}$$



(b) **Fig. 1.13** (a) A two-body oscillator (b) An equivalent one-body oscillator.

For $x > 0$, $x = 0$ and $x < 0$, the spring is extended, normal and compressed respectively. Suppose that at a given instant of time the spring is extended, i.e. $x > 0$. Though the spring exerts the same force *(kx)* on the two masses, the force F_1 (= *kx)* acting on m_1 opposes the force $F_2 (= -kx)$ on m_2, i.e.,

$$F_1 = kx \text{ and } F_2 = -kx$$

According to Newton's second law, above equation can be written as

$$m_1 \frac{d^2 X_1}{dt^2} = kx$$

and

$$m_2 \frac{d^2 X_2}{dt^2} = -kx$$

On rearranging terms, we obtain

$$\frac{d^2 X_1}{dt^2} = \frac{kx}{m_1} \tag{1.45a}$$

and

$$\frac{d^2 X_1}{dt^2} = \frac{kx}{m_1} \tag{1.45b}$$

On subtracting one from the other, we get

$$\frac{d^2 (X_2 - X_1)}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) kx$$

Since r_0 denotes a constant length of the spring, Eq. (1.44) tells us that

$$\frac{d^2 x}{dt^2} + \frac{k}{\mu} x = 0 \tag{1.46}$$

where $\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1 + m_2}$ is called the *reduced mass* of the system.

Eq. (1.46) describes simple harmonic oscillation of frequency

$$\nu_0 = \frac{1}{2\pi} \sqrt{k / \mu} \tag{1.47}$$

This means that a diatomic molecule behaves as a single object of mass μ, connected by a spring of force constant *k* (Fig. 1.13b).

SAQ 11

For an *HCl* molecule, r_0 = 1.3Å. Find the value of the force constant and the frequency of oscillation. Given: $m_H = 1.67 \times 10^{-27} kg$ and $m_{cl} = 35 m_H$. Use $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \, Nm^2 C^{-2}$.

1.7 SUMMARY

1. Simple Harmonic Motion: An oscillatory motion is said to be simple harmonic when the acceleration is proportional to the displacement and is always directed against it.

We can also say that in SHM the restoring force is linearly proportional to the displacement and acts against it.

2. Differential Equation of SHM is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \text{ where } \omega_0 = \sqrt{k/m}$$

3. The most general solution of the differential equation of SHM is

$$x = a\cos(\omega_0 t + \phi)$$

4. The period and frequency characterizing a SHM are represented by the relations:

$$T = \frac{2\pi}{\omega_0} \text{ and } \nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{T}$$

5. Total Energy of oscillation

$$E = U + K.E.$$

$$= \frac{1}{2}ka^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}ka^2\sin^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}ka^2 = \frac{1}{2}mv_{\max}^2$$

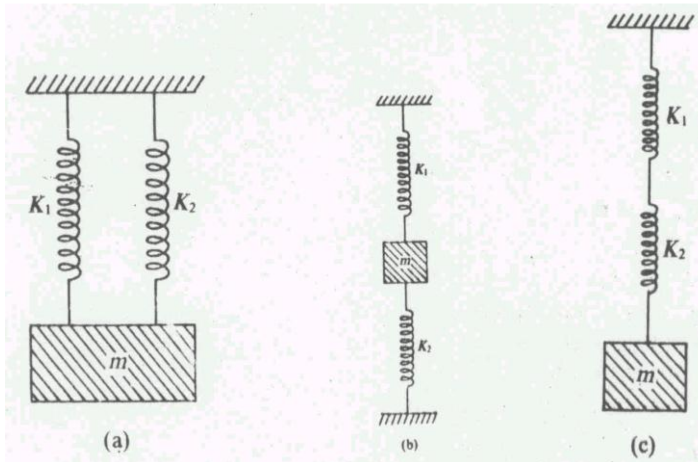
The time averaged kinetic energy and potential energy are the same; equal to $\frac{1}{4}ka^2$.

System	Differential equation	Inertial factor	Spring factor	ω_0	Period of oscillation
Spring-mass system	$m\ddot{x} + kx = 0$	m	k	$\sqrt{k/m}$	$2\pi\sqrt{m/k}$
Simple pendulum	$m\ddot{\theta} + \frac{mg}{l}\theta = 0$	m	$\frac{mg}{l}$	$\sqrt{g/l}$	$2\pi\sqrt{l/g}$
Compound pendulum	$I\ddot{\theta} + mgl\theta = 0$	I	mgl	$\sqrt{mgl/I}$	$2\pi\sqrt{I/mgl}$
Torsional pendulum	$I\ddot{\theta} + k_t\theta = 0$	I	k_t	$\sqrt{k_t/I}$	$2\pi\sqrt{I/k_t}$
L-C circuit	$L\ddot{q} + \frac{1}{C}q = 0$	L	$1/C$	$1/\sqrt{LC}$	$2\pi\sqrt{LC}$
Acoustic resonator	$lA\rho\,\ddot{x} + \frac{E_\gamma A^2}{V}x = 0$	$lA\rho$	$\frac{E_\gamma A^2}{V}$	$\sqrt{\frac{E_\gamma A^2}{Vl\rho}}$	$2\pi\sqrt{\frac{Vl\rho}{E_\gamma A}}$
Two-body oscillator	$\mu\ddot{x} + kx = 0$	$\mu = \frac{m_1 m_2}{m_1 + m_2}$	k	$\sqrt{k/\mu}$	$2\pi\sqrt{\mu/k}$

1.8 TERMINAL QUESTIONS

1. In Figs. 1.14a, b, and c, three combinations of two springs of force constants k_1 and k_2 are given. Show that the periods of oscillation in the three cases are:

(a) $2\pi\sqrt{m/(k_1+k_2)}$
(b) $2\pi\sqrt{m/(k_1+k_2)}$
(c) $2\pi\sqrt{m/(1/k_1+1/k_2)}$



2. A smooth tunnel is bored through the earth along one of its diameters and a ball is dropped into it. Show that the ball will execute simple harmonic motion with period $T = 2\pi\sqrt{R/g}$ where R is the radius of the earth and g is the acceleration due to gravity at the surface of the earth. Assume the earth to be a homogeneous sphere of uniform density.

3. Find the angular frequency and the amplitude of harmonic oscillations of a particle if at distances x_1 and x_2 from the equilibrium position its velocity equals v_1 and v_2 respectively.

4. Show that the centers of suspension and oscillations in a compound pendulum are mutually interchangeable.

5. The potential energy of a diatomic molecule at a separation r of its atoms is represented as

$$U(r) = -\frac{e^2}{4\pi r\varepsilon_0} + \frac{c}{r^9}$$

The first term represents the attractive part and the second term represents the repulsive part. Show that the force constant is $2e^2/\pi\varepsilon_0 r_0{}^3$, where r_0 is the equilibrium separation.

Hint: $F = -kr$ and $F = -\dfrac{dU}{dr}$.

1.9 SOLUTIONS

SAQ 1

Force constant $k = \dfrac{Force}{Displacement} = \dfrac{2.0N}{5.0\times10^{-2}\,m} = 40Nm^{-1}$

Compressed length = Force / Force constant = $\dfrac{2.5N}{40Nm^{-1}} = 6.3\times10^{-3}\,m$

SAQ 2

Putting $A_1 = B\sin\theta$ and $A_2 = B\cos\theta$ in equation (1.4), we get $x(t) = B\sin(\omega t+\theta)$ since $\sin(A+B) = \sin A\cos B + \sin B\cos A$

SAQ 3

(a) $x = a\sin(\omega_0 t+\phi)$

Since time is measured from the instant $x = a$, we get

(i) $x = a = a\sin\phi$, i.e., $\sin\phi = 1$ or $\phi = \pi/2$

Similarly,

(ii) $x = -a = +a\sin\phi$, i.e., $\sin\phi = -1$ or $\phi = -\pi/2$

and

(iii) $x = \dfrac{a}{\sqrt{2}} = a\sin\phi$, i.e., $\sin\phi = \dfrac{1}{\sqrt{2}}$ or $\phi = \pi/4$

(b) $x = a\cos(\omega_0 t+\phi)$

At $t = 0$,

(i) $x = a = a\cos\phi$, i.e., $\cos\phi = 1$ or $\phi = 0$

(ii) $x = -a = a\cos\phi$, i.e., $\cos\phi = -1$ or $\phi = \pi$

(iii) $x = \dfrac{a}{\sqrt{2}} = a\cos\phi$, i.e., $\cos\phi = \dfrac{1}{\sqrt{2}}$ or $\phi = \pi/4$

SAQ 4

$x = 0.01\cos 4\pi(t+0.0625)$

Compare it with the standard equation $x = a\cos(\omega_0 t+\phi)$. We can write

(i) Amplitude $a = 0.01\ m$

(ii) Period $T = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{4\pi} = 0.5\ s$

(iii) Maximum speed = $\omega_0 a = 4\pi\, s^{-1} \times 0.01 m = 0.13 ms^{-1}$

(iv) Maximum acceleration = $\omega_0^2 a = (4\pi)^2 s^{-2} \times 0.01 m = 1.6 m s^{-2}$

(v) Displacement at $t = 0$ is $x_0 = 0.01 \cos 4\pi \times 0.0625 m$

$$= 0.01 \times \frac{1}{\sqrt{2}} m = 7.1 \times 10^{-3} m$$

SAQ 5

The graphs for the variation of *U*, *K.E* and *E* with time are shown in Fig. 1.6.

Since $U(t + \pi/\omega_0) = (1/2)ka^2 \cos^2[\omega_0(t + \pi/\omega_0) + \phi]$

$$= (1/2)ka^2 \cos^2(\omega_0 t + \phi + \pi)$$
$$= (1/2)ka^2 \cos^2(\omega_0 t + \phi)$$
$$= U(t)$$

This means that the period of oscillation of potential energy is π/ω_0, i.e., one half of that of vibration.

Similarly, you can show that the period of oscillation of kinetic energy is π/ω_0. This means that in each cycle, fixed amount of energy is transferred from the mass to the spring and back again twice.

SAQ 6

Potential energy (P.E.) = mgh and maximum kinetic energy $(K.E.)_{\max} = (1/2)ka^2$. By equating these expressions we can calculate the amplitude of the oscillation, i.e.,

$$a = \left(\frac{2mgh}{k}\right)^{1/2}$$

SAQ 7

$$U = mg(l - l\cos\theta);\ K.E. = (1/2)m(l\dot{\theta})^2$$
$$E = K.E. + U = mgl(1 - \cos\theta) + (1/2)m(l\dot{\theta})^2$$
$$\dot{\theta} = [2/ml^2\{E - mgl(1 - \cos\theta)\}]^{1/2}$$

SAQ 8

$T = 2\pi\sqrt{I/k_t}$; k_t is the torque producing unit angular displacement.

For a sphere, $I = (2/5)mR^2$

$$T = 2\pi\sqrt{\frac{2mR^2}{5k_t}}$$

$$= 2\pi\sqrt{\frac{2\times 3kg\times(0.01)^2 m^2}{5\times 0.04 Nm\,rad^{-1}}}$$
$= 0.34\ s$

SAQ 9

$$\nu_0 = \frac{1}{2\pi\sqrt{LC}} = (2\pi\sqrt{20\times 10^{-3} mH}\times 10^{-6} F)^{-1} = 1125 \text{ Hz}$$
$$E = (1/2)CV^2 = (1/2)\times 10^{-6} F\times 10^2 V^2 = 5\times 10^{-5} J$$

SAQ 10

$$\nu_0 = \frac{v_s}{2\pi}\sqrt{\frac{A}{Vl}}$$

Since V = 2 litres = 2000 c.c. = $2000(10^{-2})^3 m^3 = 2\times 10^{-3} m^3$, we get

$$\nu_0 = \frac{350ms^{-1}}{2\pi}\sqrt{\frac{\pi\times(.01)^2 m^2}{2\times 10^{-3} m\times 10^{-1} m}}$$
= 69.8 Hz

An audible note of about this frequency can be heard when an empty flask of this size is suddenly uncorked.

SAQ 11

The force constant of a diatomic molecule is given by

$$k = \frac{2e^2}{4\pi\varepsilon_0 r^3} = \frac{2(1.6\times 10^{-19} C)^2\times 9\times 10^9 Nm^2 C^{-2}}{(1.3\times 10^{-10})^3 m^3}$$
$= 209.7\ Nm^{-1}$

Since

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = \frac{35\times 1.67\times 10^{-27}}{36} kg$$
$$\nu_0 = \frac{1}{2\pi}\sqrt{k/\mu} = \frac{1}{2\pi}\sqrt{\frac{209.7\,Nm^{-1}\times 36}{35\times 1.67\times 10^{-27} kg}}$$
$= 5.7\times 10^{13}$Hz

Terminal Questions

1a. In this arrangement, both springs will be extended by the same length x. The restoring force

$$F = -k_1 x - k_2 x$$

or

$$m\frac{d^2x}{dt^2}+(k_1+k_2)x=0$$

Hence, the period of the system is given by

$$T=2\pi\sqrt{\frac{m}{k_1+k_2}}$$

b. In this arrangement, if the mass is displaced up or down by x, the restoring forces are

$$F_1=-k_1x \text{ and } F_2=-k_2x$$

Hence,

$$F=-k_1x-k_2x$$

i.e.,

$$m\frac{d^2x}{dt^2}+(k_1+k_2)x=0$$

so that the period $T=2\pi\sqrt{\dfrac{m}{k_1+k_2}}$.

c. Here the two springs are connected in series. When the mass is displaced by x, the same restoring force will act in the springs, extending them by x_1 and x_2 due to their different force constants. Thus,

$$F=-k_1x_1=-k_2x_2$$

and

$$x=x_1+x_2=-F/k_1-F/k_2$$

i.e.,

$$x=-(1/k_1+1/k_2)F$$

or $$F=-\frac{1}{\left(\frac{1}{k_1}+\frac{1}{k_2}\right)}x$$

Therefore time period $T=2\pi\sqrt{\dfrac{m}{\frac{1}{k_1}+\frac{1}{k_2}}}$.

2. The force on a mass m at the surface of the earth is

$$mg=\frac{GmM}{R^2}\text{, i.e., } g=\frac{GM}{R^2} \qquad \text{(i)}$$

Now, $M=(4/3)\pi R^3\rho$ assuming the earth to be a sphere of radius R and of uniform density ρ. Therefore,

$$Rg=\frac{4}{3}\pi\frac{R^3\rho G}{R^2}=\frac{4}{3}\pi R\rho G \qquad \text{(ii)}$$

If g' is the acceleration due to gravity at a depth d below the surface of the earth, then

$$g'=(4/3)\pi\rho G(R-d) \qquad \text{(iii)}$$

Dividing Eq. (iii) by Eq. (ii), we get

$$g'/g = (R-d)/R \qquad \text{(iv)}$$

If the distance to be measured from the center of the earth, let us put $R-d = x$. Then, Eq. (iv) can be rewritten as

$$g' = \frac{d^2x}{dt^2} = -\frac{g}{R}x$$

where the negative sign shows that acceleration is directed towards the center of the earth. Thus,

$$\frac{d^2x}{dt^2} + \frac{g}{R}x = 0$$

This equation represents SHM whose period is

$$T = 2\pi\sqrt{R/g}$$

3. $x = a\cos(\omega_0 t + \phi)$

$$v = \frac{dx}{dt} = -a\omega_0 \sin(\omega_0 t + \phi)$$

$$= -a\omega_0\sqrt{1-\frac{x^2}{a^2}}$$

Hence, we can write

$$\frac{v_1}{a\omega_0} = -\sqrt{1-\frac{x_1^2}{a^2}}$$

and $$\frac{v_2}{a\omega_0} = -\sqrt{1-\frac{x_2^2}{a^2}}$$

On squaring these expressions, we get

$$\left(\frac{v_1}{a\omega_0}\right)^2 = 1-\frac{x_1^2}{a^2}$$

$$\left(\frac{v_2}{a\omega_0}\right)^2 = 1-\frac{x_2^2}{a^2}$$

These equations may be combined to give

$$\omega_0 = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$$

Using this result in Eq. (i), we get on simplification

$$a = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$$

4. For a compound pendulum, time period

$$T = 2\pi\sqrt{\frac{(k_r^2/l)+l}{g}} = 2\pi\sqrt{\frac{l+l'}{g}} \qquad \text{(i)}$$

Suppose T' is the time period, when the pendulum is suspended from the center of oscillation. Then we can write

$$T' = 2\pi\sqrt{\frac{(k_r^2/l')+l'}{g}} \qquad \text{(ii)}$$

The distance between the center of oscillation and centre of gravity $l' = k_r^2/l$, i.e., $k_r^2 = ll'$. Using this in Eq. (ii), we get

$$T' = 2\pi\sqrt{\frac{(k_r^2/l')+l'}{g}} = T$$

That is, the periods about the centre of suspension and centre of oscillation are equal. This property of a compound pendulum is called mutual interchangeability of the centres of suspension and oscillation. The mutual interchangeability of centres of suspension and oscillation of a compound pendulum arises because the periods of oscillation about S and O are equal. In all, there are four points (S, O´, O, S´) on the line SS´(so that GS = GS´ and GO = GO´) about which the periods of oscillation are the same.

5. $$U(r) = -\frac{e^2}{4\pi\varepsilon_0 r} + \frac{c}{r^9}$$

$$F = -\frac{dU}{dr} = \frac{e^2}{4\pi\varepsilon_0}\,\frac{1}{r^2} - \frac{9c}{r^{10}}$$

At the equilibrium separation, $r = r_0$, force vanishes, i.e.,

$$\frac{e^2}{4\pi\varepsilon_0}\,\frac{1}{r_0^2} - \frac{9c}{r_0^{10}} = 0$$

or $$c = \frac{e^2 r_0^8}{36\pi\varepsilon_0}$$

Now, $$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = -\frac{e^2}{4\pi\varepsilon_0}\left.\frac{2}{r^3}\right|_{r=r_0} + \left.\frac{90c}{r^{11}}\right|_{r=r_0}$$

$$= -\frac{e^2}{2\pi\varepsilon_0}\,\frac{1}{r_0^3} + \frac{5}{r_0^3}\,\frac{e^2}{2\pi\varepsilon_0}$$

$$= \frac{2e^2}{\pi\varepsilon_0 r_0^3}$$

which is positive. Hence, $r = r_0$ is the separation at stable equilibrium, and Force constant

$$k = \frac{2e^2}{\pi\varepsilon_0 r_0^3}$$

SUPERPOSITION OF HARMONIC OSCILLATIONS

Structure

2.1 INTRODUCTION

In Unit 1, we studied simple harmonic motion and considered a number of examples from different areas of physics. We found that in each case the motion is governed by a homogeneous second order differential equation. The solution of this equation gives us information regarding displacement of the body as a function of time. In many situations, one has to deal with a combination of two or more simple harmonic oscillations. Do you know that our eardrums vibrate under a complex combination of harmonic vibrations? The resultant effect is given by the principle of superposition. You must have observed that oscillations of a swing gradually die out, when left to itself. This is due to factors like friction and air resistance. The system loses energy and its motion is said to be damped. We will discuss damped harmonic oscillations in the next unit.

In this unit we first discuss the principle of superposition. Then you will learn to apply this principle to situations where two (or more) harmonic oscillations are superposed, either along the same line or in perpendicular directions.

Objectives

After studying this unit you should be able to

- state the principle of superposition
- apply the principle of superposition to two harmonic oscillations of (a) the same frequency and (b) different frequencies along the same line
- apply the methods of vector addition and complex numbers for superposition of many simple harmonic oscillations, and
- apply the principle of superposition to two mutually perpendicular harmonic oscillations of different frequencies/phases and describe the formation of Lissajous figures.

2.2 PRINCIPLE OF SUPERPOSITION

We know that for small oscillations, a simple pendulum executes simple harmonic motion. Let us reconsider this motion and release the bob at the instant $t = 0$ when it has initial displacement a_1. Let the displacement at a subsequent time t be x_1. Let us repeat the experiment with an initial displacement a_2. Let the displacement after the same interval of time t be x_2. Now if we take the initial displacement to be the sum of the earlier displacements, viz., $a_1 + a_2$, then according to the superposition principle,, the displacement x_3 after the same interval of time t will be

$$x_3 = x_1 + x_2$$

You can perform this activity by taking three identical simple pendulums. Release all three bobs simultaneously such that their initial velocities are zero and initial displacements of the first, second and the third pendulum are a_1, a_2 and $a_1 + a_2$, respectively. You will find that at any time the displacement x_3 of the third pendulum will be the algebraic sum of the displacements of the other two. In general, the initial velocities may be non-zero. Thus, the principle of superposition can be stated as follows:

Superposition of Simple Harmonic Oscillations

When we superpose the initial conditions corresponding to velocities and amplitudes, the resultant displacement of two (or more) harmonic displacements will be simply the algebraic sum of the individual displacements at all subsequent times.

You will note that the principle of superposition holds for any number of simple harmonic oscillations. These may be in the same or mutually perpendicular directions, i.e. in two dimensions.

In Unit 1, we observed that Eq. (1.3) describes SHM:

$$\frac{d^2x}{dt^2} = -\omega_0^2 x \tag{2.1}$$

This is a linear homogeneous equation of second order.

Such an equation has an important property that the sum of its two linearly independent solutions is itself a solution. We have already used this property in Unit I while writing Eq. (1.4).

Let $x_1(t)$ and $x_2(t)$ respectively satisfy equations

$$\frac{d^2x_1}{dt^2} = -\omega_0^2 x_1 \tag{2.2}$$

$$\frac{d^2x_2}{dt^2} = -\omega_0^2 x_2 \tag{2.3}$$

Then by adding Eqs. (2.2) and (2.3), we get

$$\frac{d^2(x_1 + x_2)}{dt^2} = -\omega_0^2 (x_1 + x_2) \tag{2.4}$$

According to the principle of superposition, the sum of two displacements given by

$$x(t) = x_1(t) + x_2(t) \tag{2.5}$$

also satisfies Eq. (2.1). In other words, the superposition of two displacements satisfies the same linear homogeneous differential equation which is satisfied individually by x_1 and x_2.

SAQ 1
For a simple pendulum we know that the equation of motion is

$$\frac{d^2\theta}{dt^2} = -\omega_0 \sin\theta$$

A linear differential equation has terms involving only the first power of the variable and its derivatives. A homogeneous equation contains no term independent of the variable. Let there be a set of functions $x_1,\ x_2,\ \ldots,\ x_n$. If their linear combination $c_1x_1 + c_2x_2 + \ldots + c_nx_n$ vanishes only when $c_1 = c_2 = \ldots = c_n = 0$, $x_1,\ x_2,\ \ldots,\ x_n$ are said to be linearly independent.

If in this equation you use the expansion

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

will it remain linear in θ? If you retain the first two terms and consider the resulting equation for the two displacements θ_1 and θ_2, will the principle of superposition still hold? If not, why?

You will find that in the case of a simple pendulum you can apply the principle of superposition only for small oscillations, i.e. when $\sin\theta \approx \theta$. Here we shall study only those oscillations for which the displacement satisfies linear homogeneous differential equations.

2.3 SUPERPOSITION OF TWO HARMONIC OSCILLATIONS OF THE SAME FREQUENCY ALONG THE SAME LINE

Let us superpose two collinear (along the same line) harmonic oscillations of amplitudes a_1 and a_2 having frequency ω_0 and a phase difference of π. The displacements of these oscillations are given by

$$x_1 = a_1 \cos\omega_0 t \tag{2.6}$$

and

$$x_2 = a_2 \cos(\omega_0 t + \pi)$$
$$= -a_2 \cos\omega_0 t \tag{2.7}$$

According to the principle of superposition, the resultant displacement is given by

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ &= a_1 \cos\omega_0 t - a_2 \cos\omega_0 t \\ &= (a_1 - a_2)\cos\omega_0 t \end{aligned} \tag{2.8}$$

This represents a simple harmonic motion of amplitude $(a_1 - a_2)$. In particular, if two amplitudes are equal, i.e. $a_1 = a_2$, the resultant displacement will be zero at all times. The displacement-time graph depicting this situation is shown in Fig. 2.1.

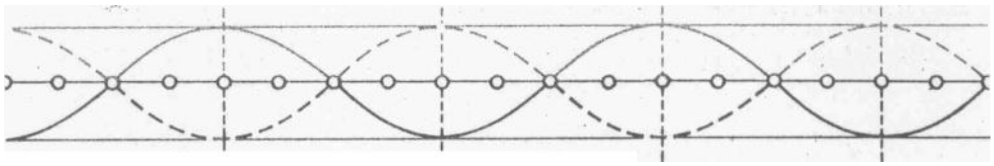


Fig. **2.1** Superposition of two collinear harmonic oscillations of equal amplitude but out of phase by π

SAQ 2

Two harmonic oscillations of amplitudes a_1 and a_2 have the same frequency ω_0 and are in phase. Show that their superposition gives a harmonic oscillation of amplitude $a_1 + a_2$.

We will now discuss the general case of superposition of two harmonic oscillations. Let one of these be characterised by amplitude a_1 and inital phase ϕ_1 and the other with amplitude a_2 and phase ϕ_2. Both oscillations have frequency ω_0 and are collinear, i.e. they are along the same line. Then, we can write

$$x_1 = a_1 \cos(\omega_0 t + \phi_1) \tag{2.9}$$

and

$$x_2 = a_2 \cos(\omega_0 t + \phi_2) \tag{2.10}$$

According to the principle of superposition, the resultant displacement is the sum of x_1 and x_2 and we have

$$x(t) = x_1(t) + x_2(t) = a_1 \cos(\omega_0 t + \phi_1) + a_2 \cos(\omega_0 t + \phi_2)$$

Using the expression for the cosine of the sum of two angles, this can be written as

$$\begin{aligned} x(t) &= a_1 \cos\omega_0 t \cos\phi_1 - a_1 \sin\omega_0 t \sin\phi_1 \\ &\quad + a_2 \cos\omega_0 t \cos\phi_2 + a_2 \sin\omega_0 t \sin\phi_2 \end{aligned}$$

Collecting the coefficients of $\cos\omega_0 t$, and $\sin\omega_0 t$, we get

$$\begin{aligned} x(t) &= (a_1 \cos\phi_2 + a_2 \cos\phi_2)\cos\omega_0 t \\ &\quad - (a_1 \sin\phi_1 + a_2 \sin\phi_2)\sin\omega_0 t \end{aligned} \tag{2.11}$$

$$\boxed{\cos(A+B) = \cos A \cos B - \sin A \sin B}$$

Since a_1, a_2, ϕ_1 and ϕ_2 are constant, we can set

$$a\cos\phi = a_1 \cos\phi_1 + a_2 \cos\phi_2 \tag{2.12}$$

$$a\sin\phi = a_1 \sin\phi_1 + a_2 \sin\phi_2 \tag{2.13}$$

where a and ϕ have to be determined. Then, we can rewrite Eq. (2.11) in the form

$$x(t) = a\cos\phi\cos\omega_0 t - a\sin\phi\sin\omega_0 t$$

It has the form of the cosine of the sum of two angles and can be expressed as

$$x(t) = a\cos(\omega_0 t + \phi) \tag{2.14}$$

This equation has the same form as either of our original equations for separate harmonic oscillations. Hence, we have the important result that the sum of two collinear harmonic oscillations of the same frequency is also a harmonic oscillation of the same frequency and along the same line. But it has a new amplitude and a new phase constant ϕ. The amplitude can easily be calculated by squaring Eqs. (2.12) and (2.13) and adding the resultant expressions. On simplification we have

$$a^2 = a_1^2 + a_2^2 + 2a_1 a_2\cos(\phi_1 - \phi_2) \tag{2.15}$$

Similarly, the phase ϕ is determined by dividing Eq. (2.13) by Eq. (2.12):

$$\phi = \tan^{-1}\left[\frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2}\right] \tag{2.16}$$

SAQ 3

Two harmonic oscillations of frequency ω_0 have initial phases ϕ_1 and ϕ_2 and amplitudes a_1 and a_2. Their resultant has the phase

(a) $\phi_1 - \phi_2 = 2n\pi$

and (b) $\phi_1 - \phi_2 = (2n+1)\pi$

where n is an integer. Using Eq. (2.15), show that the amplitudes of the resultant oscillations are equal to $(a_1 + a_2)$ and $(a_1 - a_2)$, respectively.

SAQ 4

Two harmonic oscillations of frequency ω_0 having an amplitude 1 cm and initial phases zero and $\pi/2$, respectively, are superposed. Calculate the amplitude and the phase of the resultant vibration.

2.4 SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS OF DIFFERENT FREQUENCIES

In a number of cases, we have to deal with superposition of two or more harmonic oscillations having different angular frequencies. A microphone diaphragm and human eardrums are simultaneously subjected to various vibrations. For simplicity, we shall first consider superposition of two harmonic oscillations having the same amplitude a but slightly different frequencies ω_1 and ω_2 such that $\omega_1 > \omega_2$:

$$x_1 = a\cos(\omega_1 t + \phi_1)$$
$$x_2 = a\cos(\omega_0 t + \phi_2)$$

We note that the phase difference between these two harmonic vibrations is

$$\phi = (\omega_1 - \omega_2)t + (\phi_1 - \phi_2)$$

The first term $(\omega_1 - \omega_2)t$ changes continuously with time. But the second term $(\phi_1 - \phi_2)$ is constant in time and as such it does not play any significant role here. Therefore, we may assume that the initial phase of two oscillations are zero. Then, two harmonic oscillations can be written as

$$x_1(t) = a\cos\omega_1 t$$

and $$x_2(t) = a\cos\omega_2 t \tag{2.17}$$

The superposition of two oscillations gives the resultant

$$x(t) = x_1(t) + x_2(t) = a\cos(\omega_1 t + \cos\omega_2 t) \tag{2.18}$$

This equation can be rewritten in a particularly simple form using the formula

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$x(t) = 2a\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t\cos\left(\frac{\omega_1 + \omega_2}{2}\right)t \tag{2.19}$$

This is an oscillatory motion with angular frequency $\left(\frac{\omega_1 + \omega_2}{2}\right)$ and amplitude $2a\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t$

Let us define an average angular frequency

$$\omega_{av} = \frac{\omega_1 + \omega_2}{2} \tag{2.20a}$$

and a modulated angular frequency

$$\omega_{\text{mod}} = (\omega_1 - \omega_2)/2$$
(2.20b)

Then we find that the amplitude

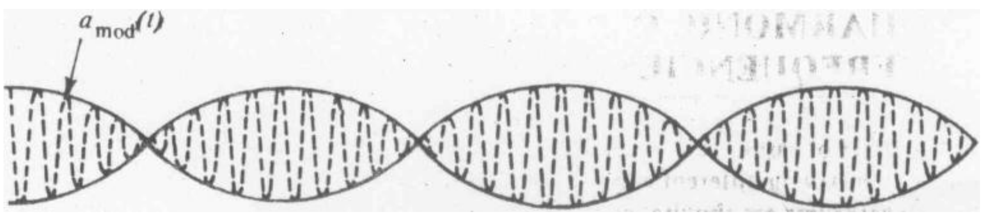
$$a_{\text{mod}}(t) = 2a\cos\omega_{\text{mod}}t$$
(2.20c)

varies with a frequency $\dfrac{\omega_{\text{mod}}}{2\pi} = \dfrac{\omega_1 - \omega_2}{4\pi}$

This also implies that in one complete cycle the modulated amplitude takes values of $2a$, 0, $-2a$, 0 and $2a$ for $\omega_{\text{mod}}t$ = 0, $\pi/2$, π, $3\pi/2$ and 2π, respectively. The resultant oscillation can be written as

$$x(t) = a_{\text{mod}}(t)\cos\omega_{av}t$$
(2.21)

This equation resembles the equation of SHM. But this resemblance is misleading because its amplitude varies with



time.

Fig. 2.2 Plot of Eq. (2.21)

The displacement-time graph depicting the resultant of two collinear harmonic oscillations of different frequencies is shown in Fig. 2.2. You will note that individual oscillations are harmonic but their superposition shows variation with time; it is periodic but not simple harmonic.

In the general case, we consider two harmonic oscillations having amplitudes a_1 and a_2 and angular frequencies ω_1 and ω_2. If their initial phases are zero, the resultant oscillation can be written as

$$x(t) = a_{\text{mod}}(t)\cos(\omega_{av}t + \theta_{\text{mod}})$$
(2.22)

The modulated amplitude and phase constant are respectively given by

$$a_{\text{mod}}(t) = [a_1^{\,2} + a_2^{\,2} + 2a_1 a_2 \cos(2\omega_{\text{mod}} t)]^{1/2} \qquad (2.23)$$

and $$\theta_{\text{mod}} = \left[\frac{(a_1 - a_2)\sin\omega_{\text{mod}} t}{(a_1 + a_2)\cos\omega_{\text{mod}} t}\right] \qquad (2.24)$$

For $a_1 = a_2$ you will note that the expression for $a_{\text{mod}}(t)$ reduces Eq. (2.20c) and $\theta_{\text{mod}} = 0$.

If ω_1 and ω_2 are nearly equal, ω_{mod} would be much less than ω_{av} and the modulated amplitude will vary very slowly with time. That is, for $\omega_{\text{mod}} << \omega_{av}$ one can regard $a_{\text{mod}}(t)$ as essentially constant over the period $(2\pi / \omega_{av})$. Then, Eq. (2.22) will represent an almost harmonic oscillation of angular frequency ω_{av}.

The amplitude of the resulting motion is maximum $(= a_1 + a_2)$ when $\cos 2\omega_{\text{mod}} t = 1$. This means that

$$2\omega_{\text{mod}} t = 2n\pi \qquad n = 0, 1, 2, \ldots$$

or $$(\omega_1 - \omega_2)t = 2n\pi \qquad n = 0, 1, 2, \ldots$$

or $$t = 0\,,\ \frac{1}{(\nu_1 - \nu_2)}, \ \frac{2}{(\nu_1 - \nu_2)}, \ \ldots, \ \frac{n}{(\nu_1 - \nu_2)}$$

where $\nu_1 = (\omega_1 / 2\pi)$ and $\nu_2 = (\omega_2 / 2\pi)$ are the frequencies of two harmonic oscillations.

Similarly, you will note that the amplitude of the resultant oscillation attains a minimum value ($a_1 - a_2$) when

$$\cos 2\omega_{\text{mod}} t = -1$$

That is, when

$$t = \frac{1}{2(\nu_1 - \nu_2)}, \ \frac{3}{2(\nu_1 - \nu_2)}, \ \frac{5}{2(\nu_1 - \nu_2)}, \ \ldots$$

2.5 SUPERPOSITION OF MANY HARMONIC OSCILLATIONS OF THE SAME FREQUENCY

In the preceding sections, we considered superposition of two collinear harmonic oscillations. How will you calculate the resultant of a number of harmonic oscillations of the same frequency? You may suggest that an obvious way is to extend the procedure outlined in Sec. 2.3. But we find that the mathematical analysis, though simple, becomes unwieldy. A convenient way out in such a case is to use either the method of vector analysis or complex numbers. We will now discuss these in turn.

2.5.1 Method of Vector Addition

This method is based on the fact that the displacement of a harmonic oscillation is the projection of a uniform circular motion on the diameter of the circle. Therefore, it is important to understand the connection between SHM and uniform circular motion.

Uniform Circular Motion and SHM

Let us suppose that a particle moves in a circle with constant speed V (Fig. 2.3). The radius vector joining the centre of the circle and position of the particle on the circumference will rotate with a constant angular frequency. We take the x-axis to be along the direction of the radius vector at time t = 0. Then the angle made by the radius vector with the x-axis at any time t will be given by

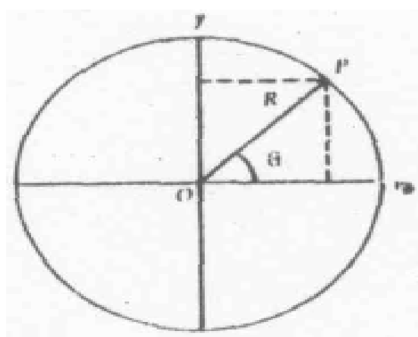


Fig. 2.3 Uniform circular motion and its connection with SHM

$$\theta = \frac{\text{length of arc}}{\text{radius of the circle}} = \frac{Vt}{R}$$

The x- and y- components of the position of the particle at time t are

$$x = R\cos\theta$$

and $$y = R\sin\theta$$

Thus,

$$\frac{dx}{dt} = -R\sin\theta\frac{d\theta}{dt}$$

$$= -\omega_0 R \sin\theta$$

since $\dfrac{d\theta}{dt} = \omega_0 = \dfrac{V}{R}$

Similarly, you can write

$$\frac{dy}{dt} = \omega_0 R \cos\theta$$

Differentiating again with respect to time, we get

$$\frac{d^2 x}{dt^2} = -\omega_0^2 x$$

and $$\frac{d^2 y}{dt^2} = -\omega_0^2 y$$

These expressions show that when a particle moves uniformly in a circle, its projections along x- and y- axes execute SHM. In other words, *a simple harmonic motion may be viewed as a projection of a uniformly rotating vector on a reference axis*.

Suppose that the vector **OP'** with |**OP'**| = $\mathbf{a}_0$ is rotating with an angular frequency ω_0 in an anticlockwise direction, as shown in Fig. 2.4. Let *P* be the foot of the perpendicular drawn from *P'* on x-axis. Then *OP* = *x* is the projection of **OP'** on the x-axis. As vector **OP'** rotates at constant speed, the point P executes simple harmonic motion along the x-axis. Its period of oscillation is equal to the period of the rotating vector **OP'**. Let $OP_0^{'}$ be the initial position of a rotating vector. Its projection OP_0 on the x-axis is $a_0 \cos\phi$. If this rotating vector moves from $OP_0^{'}$ to OP' in time t, then $\angle P'OP_0^{'} = \omega_0 t$ and $\angle P'O = (\omega_0 t + \phi)$. Then we can write

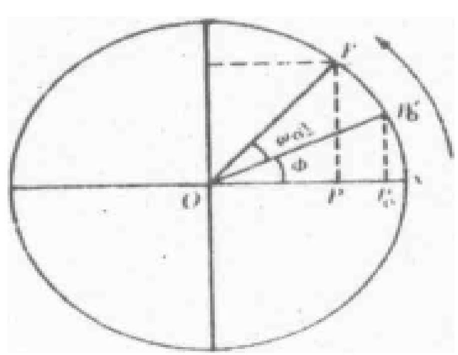


Fig. 2.4a Projections of a rotating vector along the diameter of a circle

$$OP = OP' \cos \angle P'Ox$$

or $$x = a_0 \cos(\omega_0 t + \phi) \tag{2.28a}$$

Thus, point P executes simple harmonic motion along x-axis.

If you project *OP'* on the y-axis, you will find that the point corresponding to the foot of the normal satisfies the equation

$$y = a_0 \sin(\omega_0 t + \phi) \tag{2.28b}$$

This means that a rotating vector can, in general, be resolved into two orthogonal components, and we can write

$$r = x_x + y_y \tag{2.29}$$

where x_x and y_y are unit vectors along the x- and the y-axes, respectively.

Vector Addition

Let us now consider the superposition of n harmonic oscillations, all having the same amplitude a_0 and angular frequency ω_0. The initial phases of successive oscillations differ by ϕ_0. Let the first of these oscillations be described by the equation

$$x_1(t) = a_0 \cos \omega_0 t$$

Then, other oscillations are given by

$$\begin{aligned} x_2(t) &= a_0 \cos(\omega_0 t + \phi_0) \\ x_3(t) &= a_0 \cos(\omega_0 t + 2\phi_0) \\ &. \\ x_n(t) &= a_0 \cos[\omega_0 t + (n-1)\phi_0] \end{aligned} \tag{2.30}$$

From the principle of superposition, the resultant oscillation is given by

$$\begin{aligned} x(t) = a_0[\cos \omega_0 t + \cos(\omega_0 t + \phi_0) + \cos(\omega_0 t + 2\phi_0) + ... \\ + \cos\{\omega_0 t + (n-1)\phi_0\}] \end{aligned} \tag{2.31}$$

Let us denote the harmonic oscillations given in Eq. (2.31) as projections of rotating vectors $\mathbf{OP}_1^{'}$, $\mathbf{OP}_2^{'}$, $\mathbf{OP}_3^{'}$, ... (Fig. 2.4b).

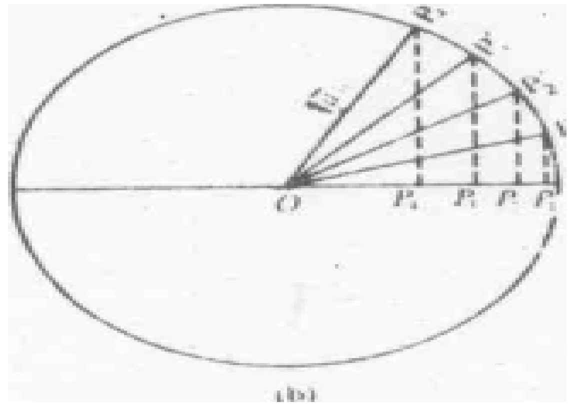


Fig. 2.4b Projections of rotating vectors $\mathbf{OP}_1^{'}$, $\mathbf{OP}_2^{'}$, … on the diameter of a circle

When a vector is translated parallel to itself, it remains unaltered.

To find the resultant of these vectors, we translate them parallel to themselves so that the head of the first coincides with the tail of the second and so on. You will observe that

(i) combining vectors form successive sides of an incomplete n-sided polygon (Fig. 2.4c)

(ii) $\mathbf{OP}_1^{'} \parallel \mathbf{OP}_1^{'}$, $\mathbf{P}_1^{'}\mathbf{P}_2^{'} \parallel \mathbf{OP}_2^{'}$, $\mathbf{P}_2^{'}\mathbf{P}_3^{'} \parallel \mathbf{OP}_3^{'}$ and so on.

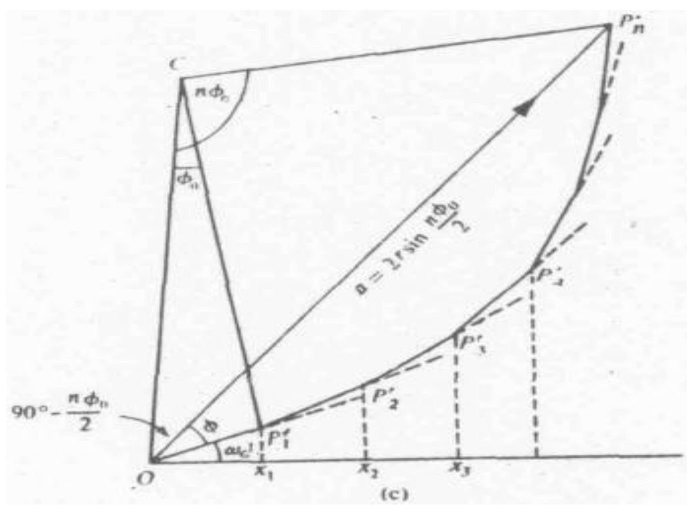


Fig. 2.4c: Superposition of a large number of harmonic oscillations of equal amplitude a_0 and having successive phase difference ϕ_0

Let us now project each of these vectors along the x-axis. Then, we get

$$x_1 = \text{Proj}\,(\mathbf{OP}_1^{'})_x = a_0 \cos \omega_0 t$$

$$x_2 = \text{Proj}\,(\mathbf{P}_1^{'}\mathbf{P}_2^{'})_x = a_0 \cos(\omega_0 t + \phi_0)$$

x_3 = Proj $(\mathbf{P}_2'\mathbf{P}_3')_x$ = $a_0\cos(\omega_0 t + 2\phi_0)$

.

.

x_n = Proj $(\mathbf{P}_{n-1}'\mathbf{P}_n')_x$ = $a_0\cos(\omega_0 t + (n-1)\phi_0)$

The law of vector addition implies that the resultant of $\mathbf{OP}_1'$, $\mathbf{P}_1'\mathbf{P}_2'$, $\mathbf{P}_2'\mathbf{P}_3'$, … is given by the vector $\mathbf{OP}_n'$, i.e.,

$$\mathbf{OP}_n' = \mathbf{OP}_1' + \mathbf{P}_1'\mathbf{P}_2' + \mathbf{P}_2'\mathbf{P}_3' + ... + \mathbf{P}_{n-1}'$$

Thus, Proj $(\mathbf{OP}_n')$ = Proj $(\mathbf{OP}_1')_x$ + Proj $(\mathbf{P}_1'\mathbf{P}_2')_x$ + …

or $\quad x(t) = x_1(t) + x_2(t) + x_3(t) + ...$

Let the length of OP_n' be a and its phase with respect to the first vector be ϕ. Then the projection of OP_n' along the x-axis is given by

$$x(t) = \text{Proj}\,(\mathbf{OP}_n')_x = a\cos(\omega_0 t + \phi)$$

Hence, the sum in Eq. (2.31) reduces to calculating a and ϕ characterising the resultant vector $\mathbf{OP}_n'$. To this end, we recall that any regular polygon will lie on a circle of radius r, as shown in Fig. 2.4b. The angle subtended at the centre C of the circle by individual vectors will be equal to ϕ_0. Hence, the total angle subtended at C by the resultant vector $\mathbf{OP}_n'$ will be $n\phi_0$.

A circle is an infinite-sided regular polygon

From the triangle OCP_n we note that

$$a = \sqrt{2r^2 - 2r^2\cos n\phi_0}$$

Using the trigonometric relation $\cos 2\theta = 1 - 2\sin^2\theta$ and simplifying the resultant expression, we get

$$a = 2r\sin(n\phi_0/2) \tag{2.33a}$$

Similarly, we can show that

$$a_0 = 2r\sin(\phi_0/2) \tag{2.33b}$$

On combining Eqs. (2.33a) and (2.33b), we obtain the amplitude of the resultant vector $\mathbf{OP}_n'$:

$$a = a_0\frac{\sin(n\phi_0/2)}{\sin(\phi_0/2)} \tag{2.34}$$

The phase difference ϕ of the resultant oscillation relative to the first oscillation is given by

$$\phi = \angle COP_1' - \angle COP_n' \tag{2.35}$$

In the isosceles $\Delta\ COP_1'$, $\angle OCP_1' = \phi_0$ and $\angle OP_1'C = \pi/2$.

Since the sum of the angles of a triangle is equal to π, we can write

$$
\begin{aligned}
COP_1' &= \pi - \angle COP_1' - \angle OCP_1' \\
&= \pi - \frac{\pi}{2} - \phi_0 \\
&= \frac{\pi}{2} - \phi_0 \qquad (2.36a)
\end{aligned}
$$

Similarly, in the isosceles $\Delta\ COP_n'$, $\angle OCP_n' = n\phi_0$ and $\angle COP_n' = \angle CP_n'O$.

Therefore, $2\angle COP_n' = \pi - n\phi_0$

Or $$\angle COP_n' = \frac{\pi}{2} - n\frac{\phi_0}{2} \qquad (2.36b)$$

Hence, by combining Eqs. (2.35) and (2.36), we get

$$\phi = \left(\frac{\pi}{2} - \phi_0\right) - \left(\frac{\pi}{2} - n\frac{\phi_0}{2}\right) = (n-1)\frac{\phi_0}{2} \qquad (2.37)$$

That is, the initial phase of the resultant oscillation is equal to half the phase difference between the nth and the first oscillations. Hence,

$$x(t) = a_0 \frac{\sin\left(n\frac{\phi_0}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} \cos\left[\omega_0 t + (n-1)\frac{\phi_0}{2}\right] \qquad (2.38)$$

We shall obtain the same result in the next subsection using the method of complex numbers. For the moment, let us examine the behaviour of the amplitude of the resultant oscillation defined by Eq. (2.38):

$$a = a_0 \frac{\sin\left(n\frac{\phi_0}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$$

You will notice that the value of a depends on the value of ϕ_0. When n is very large, ϕ_0 becomes very small. Then, using Eq. (2.37), we can write

$$\phi = (n-1)\frac{\phi_0}{2} \approx \frac{n\phi_0}{2}$$

so that $\sin\frac{\phi_0}{2} \approx \frac{\phi_0}{2} = \frac{\phi}{n}$

Hence, for large n, we have

$$a = a_0 \frac{\sin\phi}{(\phi/n)} = na_0 \frac{\sin\phi}{\phi}$$
$$= A\frac{\sin\phi}{\phi}$$

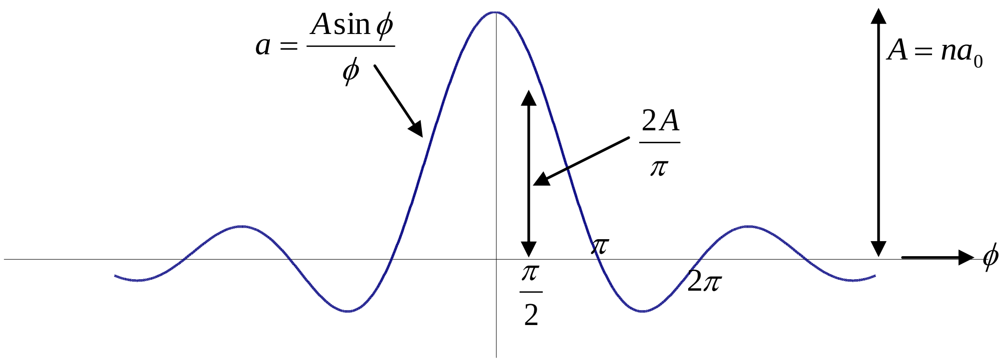


Fig. 2.5 Plot of $\frac{A\sin\phi}{\phi}$ as a function of ϕ

That is, the polygon becomes an arc of the circle with centre at *O* and length $na_0 = A$ with *a* as chord. The plot of $A\frac{\sin\phi}{\phi}$ for different values of ϕ is shown in Fig. 2.5.

The pattern is symmetric about ϕ = 0 and is zero for $\sin\phi = n\pi$ (n = 1, 2,...) except at ϕ = 0. When ϕ = 0, ϕ_0 = 0 and the resultant of *n* oscillations (vectors) is a straight line of length *A*. As ϕ increases, *A* becomes the arc of the circle until at $\phi = \pi/2$ the last and first contributions are out of phase and the arc *A* becomes a semi-circle whose diameter is the resultant *a*. A further increase in ϕ curls the length *A* into the circumference of a circle $(\phi = \pi)$ with a zero resultant and so on.

SAQ5

Three collinear harmonic oscillations, represented by $x_1 = 4\cos 20\pi t$, $x_2 = 4\cos(20\pi t + \pi/3)$, $x_3 = 4\cos(20\pi t + 2\pi/3)$ are superposed. Determine the amplitude and phase of the resultant vibration.

2.5.2 Method of Complex Numbers

In the preceding section we used a geometrical method of vector addition to calculate the resultant of *n* superposed harmonic oscillations. The same result can be obtained in a very convenient and compact form using the method of complex numbers. In fact, as you proceed you will observe that the use of complex numbers simplifies mathematical steps very much. We know that in complex number notation, a vector can be represented as $z = a\exp[i(\omega_0 t + \phi)]$. The complex exponential $\exp(i\theta)$ is given by

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

with $\cos\theta = \text{Re}[\exp(i\theta)]$

$$\sin\theta = \mathrm{Im}[\exp(i\theta)] =$$

Let us now see how this technique of complex numbers is used to obtain the resultant of *n* harmonic oscillations given by Eq. (2.30). In the complex exponential notation, we can write

$$\begin{aligned} Z_1 &= a_0 \exp(i\omega_0 t) \\ Z_2 &= a_0 \exp[i(\omega_0 t + \phi_0) \\ Z_3 &= a_0 \exp[i(\omega_0 + 2\phi_0)] \end{aligned} \tag{2.40}$$

The principle of superposition implies that the resultant, Z, is given by

$$\begin{aligned} Z &= a_0 e^{i\omega_0 t} + a_0 e^{i(\omega_0 t + \phi_0)} + \ldots + a_0 e^{i(\omega_0 t + (n-1)\phi_0)} \\ &= a_0 e^{i\omega_0 t}[1 + e^{i\phi_0} + e^{2i\phi_0} + \ldots + e^{i(n-1)\phi_0}] \end{aligned}$$

This series is in geometric progression with common ratio $e^{i\phi_0}$. Its sum is given by

$$\begin{aligned} Z &= a_0 \exp(i\omega_0 t)\frac{1 - e^{in\phi_0}}{1 - e^{i\phi_0}} \\ &= a_0 \exp(i\omega_0 t)\frac{e^{in\phi_0/2}}{e^{i\phi_0/2}}\left(\frac{e^{-in\phi_0/2} - e^{in\phi_0/2}}{e^{-i\phi_0/2} - e^{i\phi_0/2}}\right) \end{aligned}$$

Using the relation

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

we get

$$\begin{aligned} Z &= a_0 \exp(i\omega_0 t)\frac{\sin\left(n\dfrac{\phi_0}{2}\right)}{\sin\left(\dfrac{\phi_0}{2}\right)}\exp\left(\frac{in\phi_0}{2}\right)\exp\left(-\frac{i\phi_0}{2}\right) \\ &= a_0 \exp(i\omega_0 t)\frac{\sin(n\phi_0/2)}{\sin(\phi_0/2)}\exp\left[\frac{i(n-1)\phi_0}{2}\right] \end{aligned} \tag{2.42}$$

Since $Z = a\exp[i(\omega_0 t + \phi)]$, we find that the amplitude and phase of the resultant vibration are the same as given by Eqs. (2.34) and (2.37), respectively.

The cosine form of the resultant oscillation is obtained by taking the real part of Eq. (2.42).

2.6 OSCILLATIONS IN TWO DIMENSIONS

So far we have confined our discussion to harmonic oscillations in one dimension. But oscillatory motion in two dimensions is also possible. Most familiar example is the motion of a simple pendulum whose bob is free to swing in any direction in the *x*-*y* plane. (We call this arrangement a *spherical pendulum*.) We displace the pendulum in the *x*-direction and as we release it, we give it an impulse in the *y*-direction. What happens when such a pendulum oscillates? The result is a composite motion whose maximum *x*-displacement occurs when *y*-displacement is zero and *y*-

velocity is maximum and vice versa. Remember that since the time period of the pendulum depends only on the acceleration due to gravity and the length of the cord, the frequency of the superposed SHM's will be the same. The result is a curved path, in general, an ellipse.

We now apply the principle of superposition to the case where two harmonic oscillations are perpendicular to each other.

2.6.1 Superposition of Two Mutually Perpendicular Harmonic Oscillations of the Same Frequency

Consider two mutually perpendicular oscillations having amplitudes a_1 and a_2, such that $a_1 > a_2$ and angular frequency ω_0. These are described by equations

$$x_1 = a_1 \cos \omega_0 t \tag{2.43}$$

and
$$x_2 = a_2 \cos(\omega_0 t + \phi) \tag{2.44}$$

Here we have taken the initial phase of the vibrations along the *x* and the *y*-axes to be zero and ϕ respectively. That is, ϕ is the phase difference between the two vibrations.

We shall first find out the resultant oscillation for a few particular values of phase difference ϕ.

Case 1. $\phi = 0$ or π

For $\phi = 0$

$$x = a_1 \cos \omega_0 t$$

and

$$y = a_2 \cos \omega_0 t$$

Therefore

$$y / x = a_2 / a_1$$

or

$$y = (a_2 / a_1) x \tag{2.45}$$

Similarly, for $\phi = \pi$,

$$x = a_1 \cos \omega_0 t$$
$$y = -a_2 \cos \omega_0 t$$

So that

$$y = -(a_2 / a_1) x \tag{2.46}$$

Eqs. (2.45) and (2.46) describe straight lines passing through the origin. This means that the resultant motion of the particle is along a straight line. However, for $\phi = 0$ the motion is along one diagonal (PR in Fig. 2.6a) but when $\phi = \pi$ the motion is along the other diagonal (QS in Fig. 2.6b).

The equation of a straight line is $y = mx + c$, where m is the slope and c is the intercept on the y-axis.

Case II. $\phi = \pi/2$

In this case the two vibrations are given by

$$x = a_1 \cos \omega_0 t$$
$$y = a_2 \cos(\omega_0 t + \pi/2) = -a_2 \sin \omega_0 t$$

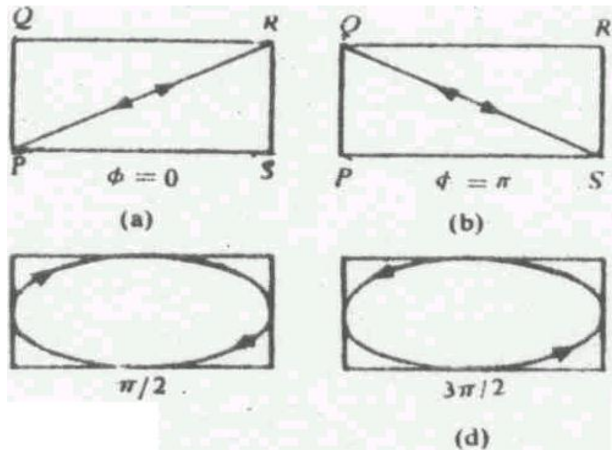


Fig. 2.6 Superposition of two mutually perpendicular harmonic oscillations having the same frequency different phases

On squaring these expressions and adding the resultant expressions, we get

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = \cos^2\theta + \sin^2\theta = 1 \tag{2.47}$$

This is the equation of an ellipse. Thus the resultant motion of the particle is along an ellipse whose principal axes lie along the *x*- and the *y*-axes. The semi-major and semi-minor axes of the ellipse are a_1 and a_2. Note that as time increases *x* decreases from its maximum positive value but y becomes more and more negative. Thus, the ellipse is described in the clockwise direction as shown in Fig. 2.6c. If you analyse the case when $\phi = 3\pi/2$ or $\phi = -\pi/2$, you will obtain the same ellipse, but the motion will be in the anticlockwise direction (Fig. 2.6d).

When amplitudes a_1 and a_2 are equal, $a_1 = a_2 = a$, Eq. (2.47) reduces to

$$x^2 + y^2 = a^2$$

This equation represents a circle of radius a. This means that the ellipse reduces to a circle.

General case

We will now consider the general case for any arbitrary value of ϕ. Let the two SHM's given by Eqs. (2.43) and (2.44) be superposed. To find the resultant oscillation, we write Eq. (2.44) as

$$\frac{y}{a_2} = \cos(\omega_0 t + \phi) = \cos\omega_0 t \cos\phi - \sin\omega_0 t \sin\phi \quad (2.48)$$

From Eq. (2.43),

$$\cos\omega_0 t = x / a_1$$

so that $\sin\omega_0 t = \sqrt{1-(x^2/a_1^2)}$

Substituting these values of $\cos\omega_0 t$ and $\sin\omega_0 t$ in Eq. (2.48), we have

$$\frac{y}{a_2} = \frac{x\cos\phi}{a_1} - \sqrt{1-(x^2/a_1^2)}\sin\phi$$

or $$\frac{x}{a_1}\cos\phi - \frac{y}{a_2} = \sqrt{1-(x^2/a_1^2)}\sin\phi$$

Squaring both sides and arranging terms, we get

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\frac{xy}{a_1 a_2}\cos\phi = \sin^2\phi \quad (2.49)$$

as the equation of the resultant path. This describes an ellipse whose axes are inclined to the coordinate axes.

For some typical values of ϕ lying between 0 and π, the resultant paths traced out by the resultant oscillation when two mutually perpendicular SHM's of equal frequency are superposed are shown in Fig. 2.7. These can be most easily demonstrated on a cathode ray oscilloscope.

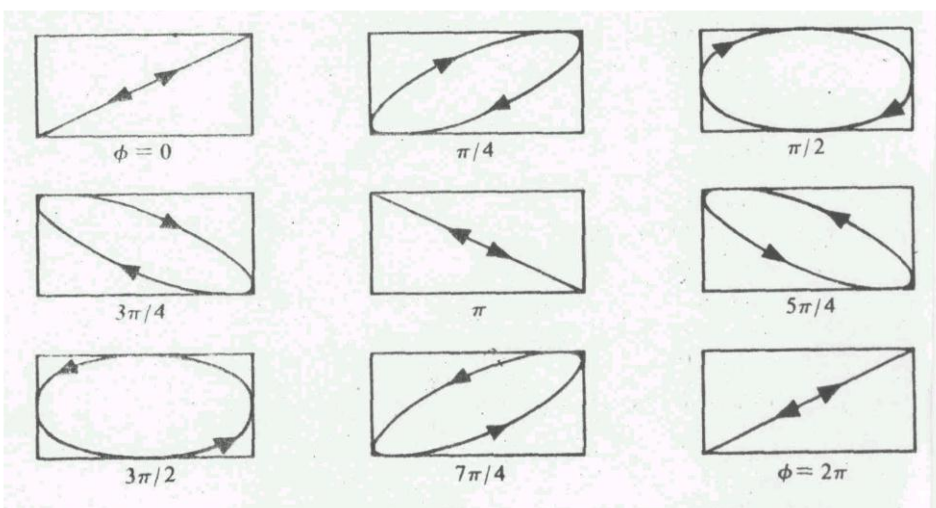


Fig. 2.7 Superposition of two mutually perpendicular harmonic oscillations of the same frequency and having values of ϕ lying between 0 and 2π

We may thus conclude that an elliptical motion is a combination of two mutually perpendicular linear harmonic oscillations of unequal amplitudes and having a phase difference ϕ. A circular motion is a combination of two harmonic oscillations of equal amplitudes

SAQ6
In a cathode ray oscilloscope, the deflection of electrons by two mutually perpendicular electric fields is given by

$$x = 4\cos 2\pi\nu t$$
$$y = 4\cos(2\pi\nu t + \pi/6)$$

What will be the resultant path of electrons?

2.6.2 Superposition of Two Rectangular Harmonic Oscillations of Nearly Equal Frequencies: Lissajous Figures

We now know that when two orthonormal vibrations have exactly the same frequency, the shape of the curve traced out by the resultant oscillation depends on the phase difference between component vibrations. For a few values of the phase difference ϕ in the range 0 to 2π radians, these curves are shown in Fig. 2.6. When the two individual rectangular vibrations are of slightly different frequencies, the resulting motion is more complex. This is because the relative phase $[\phi = \omega_2 t + \phi_0 - \omega_1 t = (\omega_2 - \omega_1)t + \phi_0]$ of the two vibrations gradually changes with time. This makes the shape of the figure to undergo a slow change. If the amplitudes of vibrations are a_1 and a_2, respectively, then the resulting figure always lies in a rectangle of sides $2a_1$ and $2a_2$. The patterns which are traced out are called Lissajous figures. When the two vibrations are in the same phase, i.e. ϕ = 0, the Lissajous figure reduces to a straight line and coincides with the diagonal $y = (a_2/a_1)x$ of the rectangle. As ϕ changes from 0 to $\pi/2$, the Lissajous figure is an ellipse and passes through oblique positions in the rectangle. When ϕ increases from $\pi/2$ to π, the ellipse closes into a straight line which coincides with the (other) diagonal $y = -(a_2/a_1)x$ of the rectangle. Further, as ϕ changes from π to 2π, the series of changes mentioned above take place in the reverse order. In general, the shape of curve depends on the amplitudes, frequencies and the phase difference. All these changes are shown'in Fig 2.7. The phase ϕ changes by 2π in the time interval $2\pi/(\omega_2 - \omega_1)$. Therefore, the period of the complete cycle of changes is $2\pi/(\omega_2 - \omega_1)$ and its frequency is $\dfrac{\omega_2 - \omega_1}{2\pi} = \nu_1 - \nu_2$, i.e., equal to the difference of the frequencies of the individual vibrations.

Lissajous figures can be illustrated easily by means of a cathode ray oscilloscope (CRO). Different alternating sinusoidal voltages are applied at *XX* and *YY* deflection plates of the CRO. The electron beam traces the resultant effect on the fluorescent screen. When the applied voltages have the same frequency, we can obtain various curves of Fig. 2.7 by adjusting the phases and amplitudes.

If the frequencies of individual perpendicular vibrations are in the ratio 2:1, the Lissajous figures are relatively complex. It has the shape of parabola for ϕ = 0 or π and for $\phi = \pi/2$ its shape is that of figure '8'. To clarify this let us study the following example:

Two rectangular harmonic vibrations having frequencies in the ratio 2:1 are represented as follows:

$$x = a_1 \cos(2\omega_0 t + \phi) \tag{2.50}$$

and $$y = a_2 \cos \omega_0 t \tag{2.51}$$

We will calculate the resultant motion for ϕ = 0, $\pi / 2$ and π.

(i) When $\phi = 0$, $x = a_1 \cos 2\omega_0 t = a_1(2\cos^2 \omega_0 t - 1)$
and

$$y = a_2 \cos \omega_0 t$$

Since $\frac{y}{a_2} = \cos \omega_0 t$, we can rewrite the above equation as

$$\frac{x}{a_1} = \frac{2y^2}{a_2^2} - 1$$

On rearranging terms, we get

$$y^2 = \frac{a_2^2}{2a_1}(x + a_1) \tag{2.52}$$

This equation represents a parabola (Fig. 2.8a).

(ii) When $\phi = \frac{\pi}{2}$

$$x = -a_1 \sin 2\omega_0 t$$

or $$-\frac{x}{a_1} = 2\sin \omega_0 t \cos \omega_0 t$$

and $$y = a_2 \cos \omega_0 t$$

Since we can write

$$\cos \omega_0 t = \frac{y}{a_2}$$

and $$\sin \omega_0 t = \sqrt{1 - \frac{y^2}{a_2^2}}$$

The first of these equations reduces to

$$-\frac{x}{a_1} = \frac{2y}{a_2}\sqrt{1 - \frac{y^2}{a_2^2}}$$

On squaring and rearranging terms, we get

$$\frac{4y^2}{a_2^2}\left(\frac{y^2}{a_2^2}-1\right)+\frac{x^2}{a_1^2}=0 \qquad (2.8b)$$

which represents figure ‘8’ in shape (Fig. 2.8b).

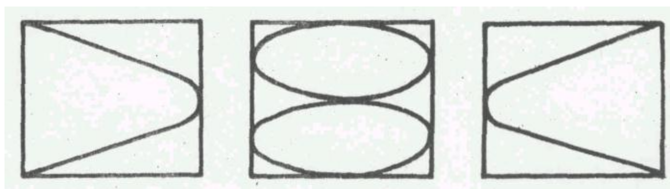
(iii) When $\phi = \pi$

$$x = -a_1 \cos 2\omega_0 t$$

or $$-\frac{x}{a_1} = 2\cos^2 \omega_0 - 1$$

and $$y = a_2 \cos \omega_0 t$$

On combining these equations, we get



$\phi = 0$ $\phi = \pi / 2$ $\phi = \pi$

Fig 2.8 Superposition of two harmonic oscillations having frequencies in the ratio 2:1 and phase difference ϕ = (i) 0, (ii) $\pi / 2$ and π, respectively

$$\frac{2y^2}{a_2^2} = -\frac{x}{a_1} + 1$$

or $$y^2 = -\frac{a_2^2}{2a_1}(x - a_1) \qquad (2.54)$$

This represents a parabola which is oppositely directed to the case when $\phi = 0$ (Fig. 2.8c).

2.7 SUMMARY

The principle of superposition states that when we superpose the initial conditions corresponding to velocities and amplitudes, the resultant displacement of two (or more) harmonic displacements will be simply the algebraic sum of the individual displacements at all subsequent times.

$$x(t) = x_1(t) + x_2(t)$$

When two collinear harmonic oscillations of the same frequency, given by

$$x_1 = a_1 \cos(\omega_0 t + \phi_1)$$

and $$x_2 = a_2 \cos(\omega_0 t + \phi_2)$$

are superposed, the resultant is given by

$$x(t) = a\cos(\omega_0 t + \phi)$$

where $a = [a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)]^{1/2}$

and $\phi = \tan^{-1}\left[\dfrac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}\right]$

When two collinear harmonic oscillations of different frequencies are superposed, the modulated oscillation is represented by

$$x(t) = a_{\text{mod}}(t)\cos\omega_{av} t$$

where $a_{\text{mod}}(t) = 2a\cos\omega_{\text{mod}} t$

with $\omega_{\text{mod}} = \dfrac{\omega_1 - \omega_2}{2}$

and $\omega_{av} = \dfrac{\omega_1 + \omega_2}{2}$

Superposition of n harmonic collinear oscillations of the same amplitude (a_0) and frequency (ω_0) but having a constant phase difference (ϕ_0) between successive oscillations yields a harmonic oscillation. It is given by

$$x(t) = a\cos(\omega_0 t + \phi)$$

where $a = \dfrac{a_0 \sin\left(\dfrac{n\phi_0}{2}\right)}{\sin\left(\dfrac{\phi_0}{2}\right)}$

and $\phi = (n-1)\dfrac{\phi_0}{2}$

When two mutually perpendicular harmonic oscillations are superpose, the resultant form traces out different curves. If the oscillations have equal frequencies, the shape of the curve depends on the phase difference. In general, the curve is elliptical but for certain phases, it closes into a straight line. When the frequencies are nearly equal, the curves are termed Lissajous figures.

2.8 TERMINAL QUESTIONS

1. The motion of a simple pendulum is described by the differential equation

$$\frac{d^2x}{dt^2} + 4x = 0$$

Solve it for the following initial conditions: (i) at $t = 0$, $x = 3$ cm and $\dfrac{dx}{dt} = 0$

(ii) at $t = 0$, $x = 2$ cm and $\dfrac{dx}{dt} = 4\,\text{cm}\ s^{-1}$. Denote these two solutions by x_1 and x_2.

Show that for a new displacement $x_3 = x_1 + x_2$, the initial conditions on the bob are the superposition of the initial conditions of x_1 and x_2.

2. Two simple harmonic vibrations are represented by
$x_1 = 3\sin(20\pi t + \pi/6)$
and
$x_2 = 4\sin(20\pi t + \pi/3)$.
Find the amplitude, phase constant and the period of resultant vibration.

3. Consider the following two simple harmonic oscillations
$x_1 = a_1 \cos\omega_1 t$
and
$x_2 = a_2 \cos\omega_2 t$
Use complex number analysis to obtain the following expressions of the amplitude for the resultant motion:
$a = [a_1^2 + a_2^2 + 2a_1a_2\cos(\omega_1 - \omega_2)t]^{1/2}$
Show that the resultant amplitude oscillates between the values $a_1 + a_2$ and $a_1 - a_2$.

4. Two tuning forks A and B of frequencies close to each other are used to obtain Lissajous figures and it is observed that the figure goes through a cycle of changes in 20 s. Now if A is loaded slightly with wax, the figure goes through a cycle of changes in 10 s. If the frequency of B is 300 Hz, what is the frequency of A before and after loading?

2.9 SOLUTIONS

SAQ

1. On using the given expansion, we get

$$\frac{d^2\theta}{dt^2} + \omega_0\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right] = 0$$

Since this equation contains terms of power higher than θ, it is not linear.

Even if we retain the first two terms in the expansion, the resulting equation will not be linear and hence the principle of superposition will not hold.

2. $x_1 = a_1 \cos\omega_0 t$
$x_2 = a_2 \cos\omega_0 t$
According to the principle of superposition
$x = x_1 + x_2 = (a_1 + a_2)\cos\omega_0 t$
Since the cosine function varies between + 1 and -1, the amplitude of the resultant oscillation is $|a_1 + a_2|$.

3. The resultant of two harmonic oscillations having amplitudes a_1 and a_2 and initial phases ϕ_1 and ϕ_2 is given by

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2) \qquad \text{(i)}$$

(a) When $\phi_1 - \phi_2 = 2n\pi$, $\cos(\phi_1 - \phi_2) = 1$ and Eq. (i) reduces to

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2$$
$$= (a_1 + a_2)^2$$

so that

$$a = (a_1 + a_2)$$

The negative sign is dropped as it will be physically absurd.

(b) When $\phi_1 - \phi_2 = (2n+1)\pi$, $\cos(\phi_1 - \phi_2) = -1$

Then Eq. (i) reduces to

$$a^2 = a_1^2 + a_2^2 - 2a_1a_2$$
$$= (a_1 - a_2)^2$$

so that

$$a = (a_1 - a_2)$$

As before, the negative sign is dropped.

4. From Eq. (2.15), we get for $a_1 = a_2$

$$a = \sqrt{2}a_1[1 + \cos(\phi_1 - \phi_2)]^{1/2}$$

Since $\phi_1 - \phi_2 = \pi/2$, this expression reduces to $a = \sqrt{2}a_1 = \sqrt{2}$ cm since $a_1 = 1$ cm.
Similarly, from Eq. (2.16), we get
$\tan\phi = 1$ or $\phi = \pi/4$.

5. Here, $n = 3$, $a_0 = 4$ and $\phi_0 = \pi/3$ rad. From Eq. (2.34) we note that the amplitude of the resultant oscillation is given by

$$a = a_0 \frac{\sin\left(\frac{n\phi_0}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$$

$$= a_0 \frac{\sin\left(\frac{3\times\pi}{2\times3}\right)}{\sin\left(\frac{\pi}{2\times3}\right)}$$

$$= a_0 \frac{\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{6}\right)}$$

Since $\sin\frac{\pi}{2} = 1$ and $\sin\frac{\pi}{6} = \frac{1}{2}$, we get

$a = 2a_0 = 8$ units

The phase of the resultant oscillation is given by Eq. (2.37):

$$\phi = (n-1)\frac{\phi_0}{2}$$
$$= 2 \times \frac{\pi}{6}$$
$$= \frac{\pi}{3} \text{ rad.}$$

6. Using Eq. (2.49), we have

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} - \frac{2xy}{4 \times 4}\cos\frac{\pi}{6} = \sin^2\frac{\pi}{6}$$

or $$\frac{x^2}{16} + \frac{y^2}{16} - \frac{2xy}{16}\frac{\sqrt{3}}{2} = \frac{1}{4}$$

or $$x^2 + y^2 - \sqrt{3}xy - 4 = 0$$

The resultant path is an ellipse.

Terminal Questions

1. $$\frac{d^2x}{dt^2} + 4x = 0 \qquad \text{(i)}$$

Comparing it with the standard equation for SHM $\frac{d^2x}{dt^2} + \omega_0{}^2 x = 0$,

We find that the solution of this equation is

$$x = a\cos(2t + \phi) \qquad \text{(ii)}$$

Differentiating Eq. (ii) with respect to t, we get

$$\frac{dx}{dt} = -2a\sin(2t + \phi) \qquad \text{(iii)}$$

(1) Since at $t = 0$, $x = 3$ cm and $\frac{dx}{dt} = 0$, from Eqs. (ii) and (iii), we obtain

$$3 \text{ cm} = a\cos\phi$$

and $$0 = -2a\sin\phi$$

The latter of these two relations implies that $\phi = 0$. Using this in the former, we get

$$a = 3 \text{ cm}$$

Therefore, the complete solution is

$$x(t) = 3\cos 2t \text{ cm} \qquad \text{(iv)}$$

(2) Again if at $t = 0$, $x = 2$ cm and $\frac{dx}{dt} = 4 \text{ cm}\, s^{-1}$, we find

$2\text{ cm} = a\cos\phi$

and $4\text{ cm}\,s^{-1} = -2a\sin\phi$

or $2\text{ cm}\,s^{-1} = -a\sin\phi$

On dividing one by the other, we get

$$\tan\phi = -1 \text{ or } \phi = -\frac{\pi}{4}. \text{ Hence, } a = 2\sqrt{2}\text{ cm}$$

Therefore, the solution in the second case is

$$x_2 = 2\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)\text{cm} \qquad \text{(v)}$$

Since superposition of x_1 and x_2 yields x_3, from Eqs. (iv) and (v), we get

$$x_3 = x_1 + x_2 = 3\cos 2t\,cm + 2\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)cm$$

$$= 3\cos 2t\,cm + 2\sqrt{2}[\cos 2t\cos\pi/4 + \sin 2t\sin\pi/4]cm$$

$$= 3\cos 2t\,cm + 2\sqrt{2}[\cos 2t\cos\pi/4 + \sin 2t\sin\pi/4]cm$$

$$= 3\cos 2t\,cm + 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}}\sin 2t\right)cm$$

$$= 5\cos 2t\,cm + 2\sin 2t\,cm$$

Now, if we superpose the initial conditions of x_1 and x_2, we have

at $t = 0$, $x = 5$ cm and $\dfrac{dx}{dt} = 4\,cm\,s^{-1}$,

$\therefore$ 5 cm = $a\cos\phi$

and 4 cm = $-2a\sin\phi$

Hence, $\tan\phi = -\dfrac{2}{5}$

$\therefore$ $\sin\phi = -\dfrac{2}{\sqrt{29}}$

$\cos\phi = \dfrac{5}{\sqrt{29}}$

and $a = \sqrt{29}$ cm

Therefore, the solution obtained on superposing initial conditions is

$x_3 = \sqrt{29}\cos(2t+\phi)$ cm = $\sqrt{29}[\cos 2t\cos\phi - \sin 2t\sin\phi]$ cm

On substituting for $\cos\phi$ and $\sin\phi$, we get

$x_3 = 5\cos 2t\,cm + 2\sin 2t\,cm$

This is the same as given by Eq. (vi) and obtained by the superposition of x_1 and x_2.

2. $x_1 = 3\cos(20\pi t + \pi/6 - \pi/2)$ cm

and $x_2 = 4\cos(20\pi t + \pi/3 - \pi/2)$ cm

or $x_1 = 3\cos(20\pi t - \pi/3)$ cm

and $x_2 = 4\cos(20\pi t - \pi/6)$ cm

Hence, the resulting vibration is defined by

$x = a\cos(20\pi t + \phi)$ cm

where $a = (3^2 + 4^2 + 2\times 3\times 4\cos\pi/6)^{1/2}$ cm

$= (9 + 16 + 12\sqrt{3})^{1/2}$ cm

$= 6.77$ cm

and $\phi = \tan^{-1}\left(\dfrac{3\sin\pi/3 + 4\sin\pi/6}{3\cos\pi/3 + 4\cos\pi/6}\right) = \tan^{-1}\left(\dfrac{3\sqrt{3}+4}{3+4\sqrt{3}}\right)$

$= -0.24\pi$

3. $z = a_1 \exp(i\omega_1 t) + a_2 \exp(i\omega_2 t)$

$a^2 = (zz^*) = (a_1 e^{-i\omega_1 t} + a_2 e^{-i\omega_2 t}) \times (a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t})$

$= a_1^2 + a_2^2 + a_1 a_2 \exp[i(\omega_1 - \omega_2)t] + a_1 a_2 \exp[-i(\omega_1 + \omega_2)t]$

On taking the real part, we get

$a = [a_1^2 + a_2^2 + 2a_1 a_2 \cos(\omega_1 - \omega_2)t]^{1/2}$

When $(\omega_1 - \omega_2)t = \pi$ or $(2n+1)\pi$, $a_{\min} = a_1 - a_2$.

When $(\omega_1 - \omega_2)t = 0$, or $n\pi$, $a_{\max} = a_1 + a_2$.

Hence, the resultant amplitude oscillates between the values $a_1 + a_2$ and $a_1 - a_2$.

4. $\nu_A - \nu_B = \pm 0.05$ Hz

Now on loading the prong of the tuning fork A with wax, the frequency of A will decrease. However, now the cycle of changes of figures is completed in 10 s and hence the frequency difference increases to 0.1. This means that the frequencies of A before and after loading are, respectively, (300-0.05) Hz = 299.95 Hz and (300 – 0.1) Hz = 299.9 Hz.

UNIT 3 DAMPED HARMONIC MOTION

Structure

3.1 INTRODUCTION

In Unit 1 you learnt that SHM is a universal phenomenon. Now you also know that in the ideal case the total energy of a harmonic oscillator remains constant in time and the displacement follows a sine curve. This implies that once such a system is set in motion it will continue to oscillate forever. Such oscillations are said to *be free* or *undamped.* Do you know of any physical system in the real world which experiences no damping? Probably there is none. You must have observed that oscillations of a swing, a simple or torsional pendulum and a spring-mass system when left to themselves, die down gradually. Similarly, the amplitude of oscillation of charge in an LCR circuit or of the coil in a suspended type galvanometer becomes smaller and smaller. This implies that every oscillating system loses some energy as time elapses. The question now arises: Where does this energy go? To answer this, we note that when a body oscillates in a medium it experiences resistance to its motion. This means that damping force comes into play. Damping force can arise within the body itself, as well as due to the surrounding medium (air or liquid). The work done by the oscillating system against the damping forces leads to dissipation of energy of the system. That is, the energy of an oscillating body is used up in overcoming damping. But in some engineering systems we knowingly introduce damping. A familiar example is that of brakes – we increase friction to reduce the speed of a vehicle in a short time. In general, damping causes wasteful loss of energy. Therefore, we invariably try to minimise it.

Many a time it is desirable to maintain the oscillations of a system. For this we have to feed energy from an outside agency to make up for the energy losses due to damping. Such oscillations are called *forced oscillations.* You will learn various aspects of such oscillations in the next unit.

In this unit you will learn to establish and solve the equation of motion of a damped harmonic oscillator. Damping may be quantified in terms of logarithmic decrement, relaxation time and quality factor. You will also learn to compute expressions for the logarithmic decrement, power dissipated in one cycle and the quality factor.

Objectives

After going through this unit you will be able to:

- establish the differential equation for a damped harmonic oscillator and solve it
- analyse the effect of damping on amplitude, energy and period of oscillation
- highlight differences between weakly damped, critically damped and over-damped systems
- derive expressions foe power dissipated in one oscillation
- compute relaxation time and quality factor of a damped oscillator, and
- draw analogies between different physical- systems.

3.2 DIFFERENTIAL EQUATION OF A DAMPED OSCILLATOR

While considering the motion of a damped oscillator, some of the questions that come to our mind are: Will Eq.(1.2) still hold? If not, what modification is necessary? How to describe damped motion quantitatively? To answer these questions we again consider the spring-mass system of Unit 1. Let us imagine that the mass moves horizontally in a viscous medium, say inside a lubricated cylinder, as shown in Fig.3.1. As the mass moves, it will experience a drag, which we denote by F_d. The question now arises: How to predict the magnitude of this damping force? Usually, it is difficult to quantify it exactly. However, we can make a reasonable estimate based on our experience. For oscillations of sufficiently small amplitude, it is fairly reasonable to model the damping force after Stokes' law. That is, we take F_d to be proportional to velocity and write

$$F_d = -\gamma v \tag{3.1}$$

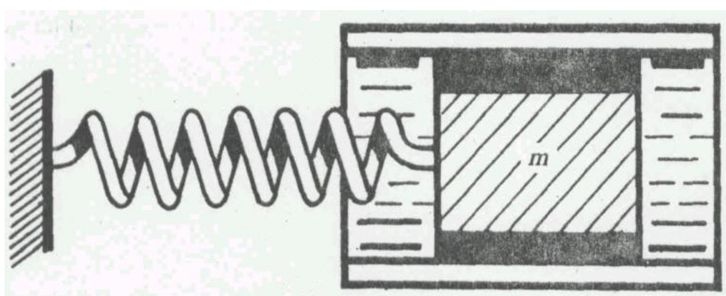


Fig. 3.1 A damped spring-mass system

> The force experienced by a body falling freely in a viscous medium is given by $F_d = 6\pi\eta r v$. This is known as Stokes' law. Here η is the coefficient of viscosity of the medium and *r* is radius of body – assumed to be spherical, and v is its velocity.

The negative sign signifies that the damping force opposes motion. The constant of proportionality γ is called the *damping coefficient*. Numerically, it is equal to force per unit velocity and is measured in $\dfrac{N}{ms^{-1}} = \dfrac{kgms^{-2}}{ms^{-1}} = kg\,s^{-1}$.

We will now establish the differential equation which describes the oscillatory motion of a damped harmonic oscillator. Let us take the x-axis to be along the length of the spring. We define the origin of the axis (x = 0) as the equilibrium position of the mass. Imagine that the mass (in the spring-mass system) is pulled longitudinally and then released. It gets displaced from its equilibrium position. At any instant, the forces acting on the spring-mass system are:

(i) *a restoring force:* $-kx$ where *k* is the spring factor, and

(ii) *a damping force:* $-\gamma v$. where $v = \dfrac{dx}{dt}$ is the instantaneous velocity of the oscillator.

This means that for a damped harmonic oscillator, the equation of motion must include the restoring force as well as the damping force. Hence, in this case Eq. (1.2) is modified to .

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} \tag{3.2}$$

After rearranging terms and dividing throughout by *m*, the equation of motion of a damped oscillator takes the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \tag{3.3}$$

where $\omega_0^2 = k/m$. (You will note that a factor of 2 has been introduced in the damping term as it helps us to obtain a neat expression for the solution of this equation.) The constant *b* has the dimensions of

$$\frac{\text{force}}{\text{velocity} \times \text{mass}} = \frac{MLT^{-2}}{LT^{-1}M} = T^{-1}$$

Hence, its unit is s^{-1}, which is the same as that of ω_0.

You will note that like Eq. (1.3), Eq. (3.3) is a linear second order homogeneous differential equation with constant coefficients. If there were no damping, the second term in Eq. (3.3) will be zero and the general solution of the resulting equation will be given by Eq. (1.5), i.e. $x = A\cos(\omega_0 t + \phi)$. On the other hand, if there is damping and no restoring force, the third term in Eq.(3.3) will be zero. Then the general solution of the resulting equation is given by $x(t) = Ce^{-2bt} + D$, where *C* and *D* are constants. (You can show this by substituting the assumed solution in Eqs. (3.3).) This means that the displacement will decrease exponentially in the absence of any restoring force. Thus we expect that the general solution of Eq. (3.3) will represent an oscillatory motion whose amplitude decreases with time.

3.3 SOLUTIONS OF THE DIFFERENTIAL EQUATION

How does damping influence the amplitude of oscillation? To discover this we have to solve Eq. (3.3) when both the restoring force and the damping force are present. The general solution, as discussed above, should involve both exponential and harmonic terms. Let us therefore take a solution of the form

$$x(t) = a\exp(\alpha t) \tag{3.4}$$

where a and α are unknown constants.

Differentiating Eq.(3.4) twice with respect to time, we get

$$\frac{dx}{dt} = a\alpha \exp(\alpha t)$$

and $$\frac{d^2x}{dt^2} = a\alpha^2 \exp(\alpha t)$$

Substituting these expressions in Eq. (3.3), we get

$$(\alpha^2 + 2b\alpha + \omega_0^2)a \exp(\alpha t) = 0 \tag{3.5}$$

For this equation to hold at all times, we should either have

$$a \quad = 0$$

which is trivial, or

$$\alpha^2 + 2b\alpha + \omega_0^2 = 0 \tag{3.6}$$

This equation is quadratic in α. Let us call the two roots α_1 and α_2

The roots of the equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\alpha_1 = -b + (b^2 - \omega_0^2)^{1/2} \tag{3.7a}$$

and $$\alpha_2 = -b - (b^2 - \omega_0^2)^{1/2} \tag{3.7b}$$

These roots determine the motion of the oscillator. Obviously α has dimensions of inverse time. Did you not expect it from the form of $\exp(\alpha t)$?

Thus, the two possible solutions of Eq. (3.3) are

$$x_1(t) = a_1 \exp[-\{b - (b^2 - \omega_0^2)^{1/2}\}t]$$

and $$x_2(t) = a_2 \exp[-\{b + (b^2 - \omega_0^2)^{1/2}\}t] \tag{3.8}$$

Since Eq.(3.3) is linear, the principle .of superposition is applicable. Hence, the general solution is obtained by the superposition of x_1 and x_2:

$$x(t) = \exp(-bt)[a_1 \exp\{(b^2 - \omega_0^2)^{1/2}\}t + a_2 \exp\{\exp\{-(b^2 - \omega_0^2)^{1/2}\}t] \tag{3.9}$$

You will note the quantity $(b^2 - \omega_0^2)$ can be negative, zero or positive respectively, depending on whether b is less than, equal, to or greater than ω_0 respectively. We, therefore, have three possibilities:

(i) If $b > \omega_0$, we say that the system is over damped,

(ii) If $b = \omega_0$, we have a critically damped system,

(iii) If $b > \omega_0$, we have an under-damped system.

Each of these conditions gives a different solution, which describes a particular behaviour.

We will now discuss these solutions in order of their increasing importance.

3.3.1 Heavy Damping

When resistance to motion is very strong, the system is said to be heavily damped. Can you name a heavily damped system of practical interest? Springs joining wagons of a train constitute the most important heavily damped system. In your physics laboratory, vibrations of a pendulum in a viscous medium such as thick oil and motion of the coil of a dead beat galvanometer are heavily damped systems.

Mathematically, a system is said to be heavily damped if $b > \omega_0$. Then the quantity $(b^2 - \omega_0^2)$ is positive definite. If we put

$$\beta = \sqrt{b^2 - \omega_0^2}$$

The general solution for damped oscillator given by Eq. (3.9) reduces to

$$x(t) = \exp(-bt)[a_1 \exp(\beta t)[a_1 \exp(\beta t) + a_2 \exp(-\beta t)] \tag{3.10}$$

This represents non-oscillatory behaviour. Such a motion is called *dead-beat*. The actual displacement will, however, be determined by the initial conditions. Let us suppose that to begin with, the oscillator is at its equilibrium position, i.e $x = 0$ at $t = 0$. Then we give it a sudden kick so that it acquires a velocity $\mathbf{v}_0$, i.e $\mathbf{v} = \mathbf{v}_0$ at $t = 0$. Then from Eq. (3.10) we have

$$a_1 + a_2 = 0$$

and $$-b(a_1 + a_2) + \beta(a_1 - a_2) = v_0$$

These equations may be solved to give

$$a_1 = -a_2 = \frac{v_0}{2\beta}$$

On substituting these results in Eq. (3.10), we can write the solution in compact form:

$$x(t) = \frac{v_0}{2\beta} \exp(-bt)[\exp(\beta t) - \exp(-\beta t)]$$

$$= \frac{v_0}{\beta}\exp(-bt)\sinh\beta t \qquad (3.11)$$

where $\sinh\beta t = (1/2)[\exp(\beta t) - \exp(-\beta t)]$ is the hyperbolic sine function. From Eq. (3.11) it is clear that $x(t)$ will be determined by the interplay of an increasing hyperbolic function and a decaying exponential. These are plotted separately in Fig. 3.2(a). Fig. 3.2(b) shows the plot of Eq. (3.11) for a heavily damped system when it is suddenly disturbed from its equilibrium position. You will note that initially the displacement increases with time. But soon the exponential term becomes important and displacement begins to decrease gradually.

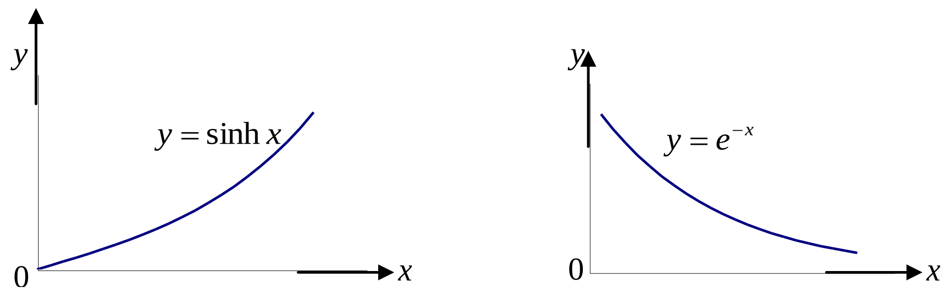


Fig. 3.2a Plot of $\sinh x$ and $\exp(-x)$

Fig. 3.2b Plot of Eq. (3.11) for a heavily damped system

3.3.2 Critical Damping

You may have observed that on hitting an isolated road bump, a car bounces up and down and the occupants feel uncomfortable. To minimise this discomfort, the bouncing caused by the road bumps must be damped very rapidly and the automobile restored to equilibrium quickly. For this we use critically damped shock absorbers. Critical damping is also useful in recording instruments such as a galvanometer (pointer type as well as suspended coil type) which experience sudden impulses. We require the pointer to move to the correct position in minimum time and stay there without executing oscillations. Similarly, a ballistic galvanometer coil is required to return to zero displacement immediately.

Mathematically, we say that a system is critically damped if b is equal to the natural frequency, ω_0, of the system. This means that $b^2 - \omega_0^2 = 0$, so that Eq (3.9) reduces to

$$\begin{aligned} x(t) &= (a_1 + a_2)\exp(-\beta t) \\ &= a\exp(-bt) \qquad (3.12) \end{aligned}$$

where $a = a_1 + a_2$.

Let us pause for a minute and recall that the solution of the differential equation for SHM involves two arbitrary constants which are fixed by giving the initial conditions. But Eq. (3.12) has only one constant. Does this mean that it is not a complete solution? It is important to understand how this happens. The reason is simple: the quadratic equation for α (Eq. 3.6) has equal roots. So, the two terms in Eq. (3.9) give the same time dependence and reduce to one term. It can be easily verified that in this case the general solution of Eq. (3.3) is

$$x(t) = (p + qt)\exp(-bt) \tag{3.13a}$$

where *p* and *q* are constants, *p* has the dimensions of length and *q* those of velocity. These can be determined easily from the initial conditions.

Let us assume that the system is disturbed from its mean equilibrium position by a sudden impulse. (The coil of a suspended type galvanometer receives some electric charge at $t = 0$.) That is, at $t = 0$, $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = v_0$. This gives $p = 0$ and $q = v_0$, so that the complete solution is

$$x(t) = v_0 t \exp(-bt) \tag{3.13b}$$

Fig. 3.3 illustrates the displacement-time graph of a critically damped system described by Eq.(3.13 b). At maximum displacement, $\left.\frac{dx}{dt}\right|_{x=x_{\max}} = 0$ and $\left.\frac{d^2x}{dt^2}\right|_{x=x_{\max}} < 0$.

This occurs at time $t = 1/b$:

$$x_{\max} = v_0 t e^{-1} = 0.368\frac{v_0}{b} = 0.736\frac{mv_0}{\gamma}$$

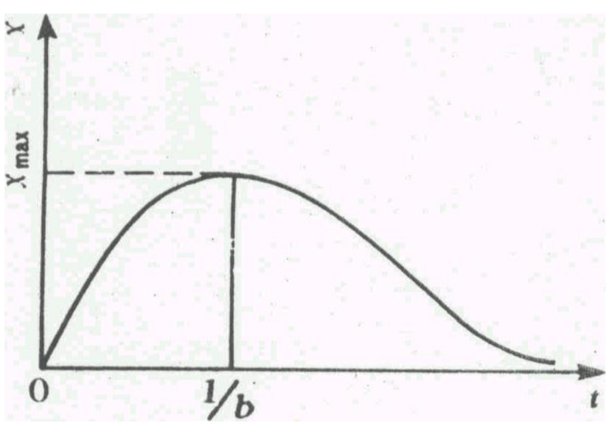


Fig. 3.3 Displacement-time graph for a critically damped system described by Eq. (3.13b)

3.3.3 Weak or light damping

When $b < \omega_0$, we refer to it as a case of weak damping. This implies that $(b^2 - \omega_0^2)$ is a negative quantity, i.e. $(b^2 - \omega_0^2)^{1/2}$ is imaginary. Let us rewrite it as

$$\begin{aligned}(b^2 - \omega_0^2)^{1/2} &= \sqrt{-1}(\omega_0^2 - b^2)^{1/2} \\ &= \pm i\omega_d\end{aligned}$$

where $i = \sqrt{-1}$ and

$$\omega_d = (\omega_0^2 - b^2)^{1/2} = \left[\frac{k}{m} - \frac{\gamma^2}{4m^2}\right]^{1/2} \qquad (3.14)$$

is a real positive quantity. You will note that for no damping $(b = 0)$, ω_d reduces to ω_0, the natural frequency of the oscillator.

On combining Eqs. (3.9) and (3.14) we find that the displacement now has the form

$$x(t) = \exp(-bt)[a_1 \exp(i\omega_d t) + a_2 \exp(-i\omega_d t)] \qquad (3.15)$$

To compare the behaviour of a damped oscillator with that of a free oscillator, we should recast Eq.(3.15) so that the displacement varies sinusoidally. To do this, we write the complex exponential in terms of sine and cosine functions. This gives

$$\boxed{\exp(\pm ix) = \cos x \pm i \sin x}$$

$$x(t) = \exp(-bt)[a_1(\cos\omega_d t + i\sin\omega_d t) - a_2(\cos\omega_d t - i\sin\omega_d t)]$$

On collecting coefficients of $\cos\omega_d t$ and $\sin\omega_d t$, we obtain

$$x(t) = \exp(-bt)[(a_1 + a_2)\cos\omega_d t + i(a_1 - a_2)\sin\omega_d t] \qquad (3.16)$$

Let us now put

$$a_1 + a_2 = a_0 \cos\phi$$

and

$$-i(a_1 - a_2) = a_0 \sin\phi \qquad (3.17)$$

where a_0 and ϕ are arbitrary constants. These are given by

$$a_0 = 2\sqrt{a_1 a_2}$$

and

$$\tan\phi = -i\frac{a_1 - a_2}{a_1 + a_2} \qquad (3.18)$$

From the second of these results we note that tan 0 is a complex quantity. Does this moan that ϕ is also complex? How can we interpret a complex angle? To know this, we use the identity

$$\sec^2\phi = 1 + \tan^2\phi$$

and calculate $\cos\phi$. The result is

$$\cos\phi = \frac{a_1 + a_2}{2\sqrt{a_1 a_2}}$$

This means that $\cos\phi$, and hence ϕ, is real.
Substituting Eq. (3.17) into Eq. (3.16) we find that the expression within the parentheses is the cosine of the sum of two angles. Hence, the general solution of Eq. (3.3) for a weakly damped oscillator ($b < \omega_0$) is

$$x(t) = a_0 \exp(-bt)\cos(\omega_d t + \phi) \tag{3.19}$$

with ω_d as given by Eq. (3.14). You will note that the solution given by Eq. (3.19) describes sinusoidal motion with frequency ω_d which remains the same throughout the motion. This property is crucial for the use of oscillators in accurate timepieces. How is the amplitude modified vis-a-vis an ideal SHM? You will note that the amplitude decreases exponentially with time at a rate governed by *b. So we can say that the motion of a weakly damped system is not simple harmonic.*

The damped oscillatory behaviour described by Eq. (3.19) is plotted in Fig. 3.4 for the particular case of ϕ = 0. Since the cosine function varies between +1 and -1, we observe that the displacement-time curve lies between $a_0\exp(-bt)$ and $-a_0\exp(-bt)$. Thus, we may conclude that *damping results in decrease of amplitude and angular frequency.*

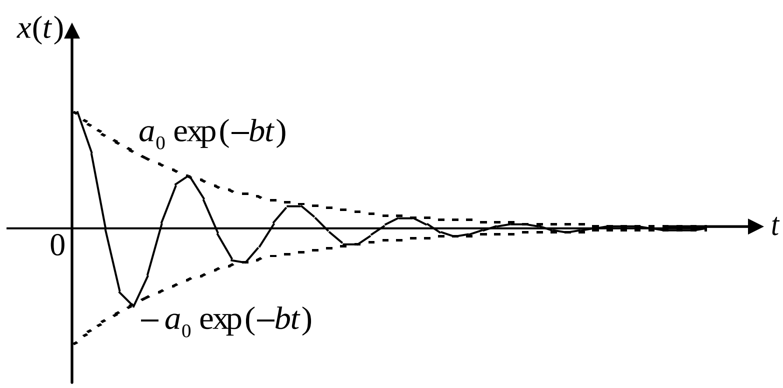


Fig. 3.4 Displacement-time graph for a weakly damped harmonic oscillator

How does damping influence the period of oscillation? You can discover this effect by noting that the period of oscillation is given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}$$

If $b > 0$, $\omega_d < \omega_0$. This means that the period of vibration of a damped oscillator is more than that of an ideal oscillator. Did you not expect it since damping forces resist motion?

SAQ 1

The amplitude of vibration of a damped spring-mass system decreases from 10 cm to 2.5 cm in 200 s. If this oscillator performs 100 oscillations in this time, compare the periods with and without damping.

We have discussed solutions of the differential equation for a damped oscillator for heavy, critical and weak dampings. In the following discussion we shall concentrate only on weakly damped systems.

3.4 AVERAGE ENERGY OF A WEAKLY DAMPED OSCILLATOR

In Unit 1 we calculated the average energy of an undamped oscillator. The question now arises: How does damping influence the average energy of a weakly damped oscillator? To answer this we note that in the presence of damping the amplitude of oscillation decreases with the passage of time. This means that energy is dissipated in overcoming resistance to motion. From Unit I we recall that at any time, the total energy of a harmonic oscillator is made up of kinetic and potential components. We can still use the same definition and write

$$\begin{aligned} E(t) &= K.E.(t) + U(t) \\ &= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 \end{aligned} \tag{3.20}$$

where dx/dt denotes instantaneous velocity.

For a weakly damped harmonic oscillator, the instantaneous displacement is given by Eq.(3.19): $x(t) = a_0 \exp(-bt)\cos(\omega_d t + \phi)$. By differentiating it with respect to time, we get instantaneous velocity:

$$\frac{dx(t)}{dt} = v = -a_0 \exp(-bt)[b\cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)]^2 \tag{3.21}$$

Hence, kinetic energy of the oscillator is

$$\begin{aligned} K.E. &= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}ma_0^2 \exp(-2bt)[b\cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)]^2 \\ &= \frac{1}{2}ma_0^2 \exp(-2bt)[b^2 \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi) \end{aligned}$$

$$+ \ b\omega_d \sin 2(\omega_d t + \phi)] \qquad (3.22a)$$

Similarly, the potential energy of the oscillator is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2$$

since $k = m\omega_0^2$.

On substituting for x, we get

$$U = \frac{1}{2}ma_0^2\omega_0^2 \exp(-2bt)\cos^2(\omega_d t + \phi) \qquad (3.22b)$$

Hence, the total energy of the oscillator at any time t is given by

$$E(t) = \frac{1}{2}ma_0^2 \exp(-2bt)[(b^2 + \omega_0^2)\cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi)$$

$$+ \ b\omega_d \sin 2(\omega_d t + \phi)] \qquad (3.23)$$

When damping is small, the amplitude of oscillation does not change much over one oscillation. So we may take the factor $\exp(-2bt)$ as essentially constant. Further, since $< \sin^2(\omega_d t + \phi) > = < \cos^2(\omega_d t + \phi) > = 1/2$ and $< \sin(\omega_d t + \phi) > = 0$, the energy of a weakly damped oscillator when averaged over one cycle is given by

$$< E > = \frac{1}{2}ma_0^2 \exp(-2bt) < [(b^2 + \omega_0^2)\cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi)$$

$$+ \ b\omega_d \sin 2(\omega_d t + \phi)] >$$

$$= \frac{1}{2}ma_0^2 \exp(-2bt)\left[\frac{b^2 + \omega_0^2}{2} + \frac{\omega_d^2}{2}\right]$$

$$= \frac{1}{2}ma_0^2\omega_0^2 \exp(-2bt) \qquad (3.24a)$$

From Unit 1 we recall that $\frac{1}{2}ma_0^2\omega_0^2$ is the total energy of an undamped oscillator. Hence, we can write

$$< E > = E_0 \exp(-2bt) \qquad (3.24b)$$

This shows that the *average energy of a weakly damped oscillator decreases exponentially with time.* This is illustrated in Fig. 3.5. From Eq.(3.24 b) you will also observe that the rate of decay of energy depends on the value of *b;* larger the value of *b,* faster will be the decay.

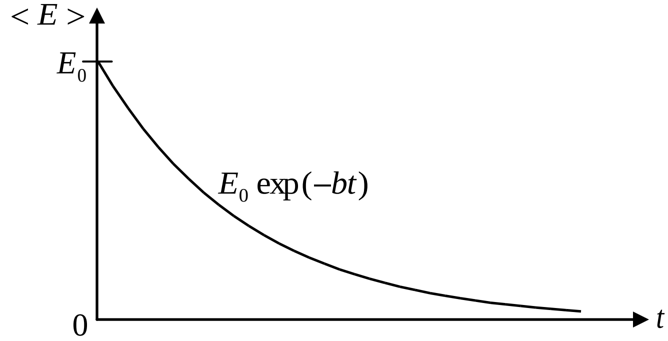


Fig. 3.5 Time variation of average energy for a weakly damped system

3.4.1 Average Power Dissipated Over One Cycle

Since energy of a damped oscillator does not remain constant in time, $\left(\frac{dE}{dt}\right)$ is not zero. In fact, it is negative. The rate of loss of energy at any time gives instantaneous power dissipated. From Eq. (3.20) we can write

$$\frac{dE}{dt} = P(t) = \left[m\frac{d^2x}{dt^2} + kx \right]\frac{dx}{dt}$$

On combining this result with Eq. (3.2) we find that power dissipated by a damped oscillator is given by

$$P(t) = -\gamma\left(\frac{dx}{dt}\right)^2$$

This relation shows that the rate of doing work against the frictional force is directly proportional to the square of instantaneous velocity. On substituting for $\left(\frac{dx}{dt}\right)$ from Eq. (3.21), we obtain

$$P(t) = -\gamma\, a_0^2 \exp(-2bt)[b^2 \cos^2(\omega_d t + \phi) + \omega_d^2 \sin^2(\omega_d t + \phi) + b\omega_d \sin(2\omega_d t + \phi)]$$

Hence, the average power dissipated over one cycle is given by

$$\begin{aligned} < P > &= \frac{1}{2}\gamma\, a_0^2 \omega_0^2 \exp(-2bt) \\ &= -\frac{\gamma}{m} < E > \\ &= -2b < E > \end{aligned}$$

The negative sign here signifies that power is dissipated.

3.5 METHODS OF CHARACTERISING DAMPED SYSTEMS

We now know that in the viscous damping model, a damped oscillator is characterised by γ and ω_0. We also know that this model applies to vastly different physical systems. Therefore, you may ask: Are there other ways of characterizing damped oscillations? Experience tells us that in certain cases it is more convenient to use other parameters to characterise damped motion. In all cases we can relate these to *y* and coo. We will now discuss these briefly.

3.5.1 Logarithmic Decrement

The most convenient way to determine the amount of damping present in a system is to measure the rate at which *amplitude* of oscillation dies away. Let us consider the damped vibration shown graphically in Fig. 3.6. Let a_0 and a_1 be the first two successive amplitudes of oscillation separated by one period.

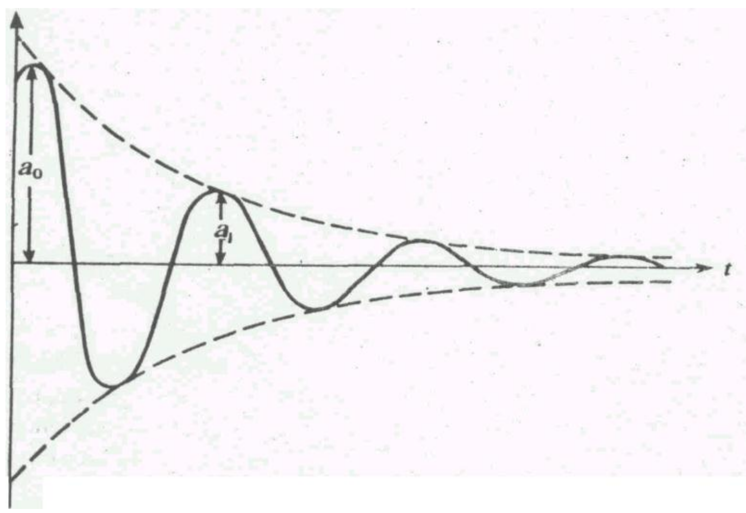


Fig. 3.6 A damped oscillation. The first two amplitudes are a_0 and a_1

You will note that these amplitudes lie in the same direction/quadrant. If T is the period of oscillation, then using Eq. (3.19) for a weakly damped oscillator, we can write

$$a_1 = a_0 \exp(-bT)$$

so that $$\frac{a_0}{a_1} = \exp(bT) = \exp(\gamma T / 2m) \tag{3.26}$$

You will note that in the ratio a_0 / a_1, the larger amplitude is in the numerator. That is why this ratio is called the *decrement.* It is denoted by the symbol *d.* You may now ask: Is the decrement same for *any* two consecutive amplitudes? The answer is: yes, it is. To show this let us consider the ratio of the second and the third amplitudes. These are observed for *t* = Tand *t* = *2T,* respectively in Eq. (3.19). Then, we can write

$$\frac{a_1}{a_2} = \frac{a_0 \exp(-bT)}{a_0 \exp(-2bT)} = \exp(bT)$$

So, we may conclude that for any two consecutive amplitudes separated by one period, we have

$$\frac{a_{n-1}}{a_n} = \exp(bT) = d \tag{3.27}$$

That is, *decrement is the same for two successive amplitudes* and we can write

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = ... = \frac{a_{n-1}}{a_n} = d \tag{3.28}$$

The logarithm of the ratio of successive amplitudes of oscillation separated by one period is called the logarithmic decrement. It is usually denoted by the symbol λ:

$$\lambda = \ln\left(\frac{a_{n-1}}{a_n}\right) = \frac{\gamma T}{2m} \tag{3.29a}$$

This equation shows that we can measure λ by knowing two successive amplitudes. But from an experimental point of view it is more convenient and accurate to compare amplitudes of oscillations separated by *n* periods. That is, we measure a_0 / a_n. To compute this ratio, we first invert Eq. (3.29 a) to write

$$\frac{a_{n-1}}{a_n} = \exp(\lambda) \tag{3.29b}$$

The ratio a_0 / a_n can now be written as

$$\frac{a_0}{a_n} = \left(\frac{a_0}{a_1}\right)\left(\frac{a_1}{a_2}\right)\left(\frac{a_2}{a_3}\right)...\left(\frac{a_{n-1}}{a_n}\right) = [\exp(\lambda)]^n \tag{3.30}$$

since the ratio of any two consecutive amplitudes is the same.

Taking the log of both sides, we get the required result:

$$\lambda = \frac{1}{n}\ln\left(\frac{a_0}{a_n}\right) \tag{3.31}$$

This shows that if we plot $\ln(a_0 / a_n)$ versus *n* for different values of *n*, we will obtain a straight line. The slope of the line gives us λ.

SAQ2

A damped harmonic oscillator has the first amplitude of 20 cm. It reduces to 2 cm after 100 oscillations, each of period 4.6 s. Calculate the logarithmic decrement and damping constant. Compute the number of oscillations in which the amplitude drops by 50%.

3.5.2 Relaxation Time

In physics we often measure decay of a quantity in terms of the fraction e^{-1} of the initial value. This gives us another way of expressing the damping effect by means of the time taken by the amplitude to decay to e^{-1} = 0.368 of its original value. This time is called the *relaxation time.* To understand this, we recall that the amplitude of a damped oscillation is given by

$$a(t) = a_0 \exp(-bt)$$

If we denote the amplitude of oscillation after an interval of time *r* by *a (t + T),* we can write

$$a(t+\tau) = a_0 \exp[-b(t+\tau)]$$

By taking the ratio $a(t+\tau)/a(t)$, we obtain

$$\frac{a(t+\tau)}{a(t)} = \exp(-b\tau)$$

$$= \frac{1}{e} \text{ for } b\tau = 1 \qquad (3.32)$$

This shows that for $b = \tau^{-1}$ the amplitude drops to $1/e$ = 0.368 of its initial value. Using this result in Eq. (3.25), we get

$$< P > = \frac{2 < E >}{\tau}$$

The relaxation time, τ, is therefore a measure of the rapidity with which motion is damped. (You will note that the negative sign occurring in Eq. (3.25) has been dropped here.)

3.5.3 The Quality Factor

Yet another way of expressing the damping effect is by means of the rate of decay of energy. From Eq. (3.24 b) we note that the average energy of a weakly damped oscillator decays to $E_0 e^{-1}$ in time $t = \frac{1}{2b} = \frac{m}{\gamma}$ seconds. If ω_d is its angular frequency, then in this time the oscillator will vibrate through $\omega_d m/\gamma$ radians. *The number of radians through which a weakly damped system oscillates as its average energy decays to* $E_0 e^{-1}$ *is a measure of the quality factor, Q :*;

$$Q = \frac{\omega_d m}{\gamma} = \frac{\omega_d}{2b} = \frac{\omega_d \tau}{2} \qquad (3.33)$$

You will note that Q is only a number and has no dimensions. In general, γ is small so that Q is very large. A tuning fork has Q of a thousand or so, whereas a rubber band exhibits a much lower (~10) Q. This is due to the internal friction generated by the coiling of the long chain of molecules in a rubber band. An undamped oscillator $(\gamma = 0)$ has an infinite quality factor.

For a weakly damped mechanical oscillator, the quality factor can be expressed in terms of the spring factor and damping constant. For weak damping,

$$\omega_d \approx \omega_0 = \sqrt{k/m}$$

Hence, $Q = \sqrt{km/\gamma^2}$

That is, the quality factor of a weakly damped oscillator is directly proportional to the square root of k and inversely proportional to γ.

We can rewrite Eq. (3.33) in a more physically meaningful form using Eq. (3.25):

$$\begin{aligned} Q &= \frac{\omega_d}{2b} = \frac{2\pi}{T_d} \times \frac{<E>}{<P>} \\ &= 2\pi \frac{\text{average energy stored in the system in one cycle}}{\text{average energy lost in one cycle}} \end{aligned} \tag{3.34}$$

The quality factor is related to the fractional change in the frequency of an undamped oscillator. To establish this relation, we note that,

$$\omega_d = \sqrt{\omega_0^2 - b^2}$$

or

$$\begin{aligned} \frac{\omega_d^2}{\omega_0^2} &= 1 - \frac{b^2}{\omega_0^2} \\ &= 1 - \frac{1}{4Q^2} \end{aligned}$$

where we have used Eq. (3.33). This result can be rewritten as

$$\begin{aligned} \frac{\omega_d}{\omega_0} &= \left(1 - \frac{1}{4Q^2}\right)^{1/2} \\ &= 1 - \frac{1}{8Q^2} \end{aligned}$$

where in the binomial expansion we have retained terms up to first order in Q^2. Hence, the fractional change in ω_0 is $1/(8Q^2)$.

SAQ4
The quality factor of a tuning fork of frequency 256 Hz is 10^3. Calculate the time in which its energy becomes 10% of its initial value.

3.6 EXAMPLES OF DAMPED SYSTEMS

You know that all harmonic oscillators in nature have some damping, which in general, is quite small. To enable you to appreciate the effect of damping, we will consider two specific cases: (i) Oscillations of charge in an *LCR* circuit, and (ii) motion of the coil in a suspension type galvanometer. These are of particular interest to us as the former has wide applications in radio engineering and the latter is used in the physics laboratory.

3.6.1 An LCR Circuit

In Unit 1 we observed that in an ideal LC circuit, charge excites SHM. Do you expect any change in this behaviour when a resistor is added? To answer this question we consider Fig. 3.7. If a current I flows through the circuit due to discharging/charging of the capacitor, the voltage drop across the resistor is RI. Thus Eq. (1.36) Now modifies to -

$$\frac{q}{C} = -L\frac{dI}{dt} - \frac{dq}{dt}$$

Eq. (3.35) may be rewritten as

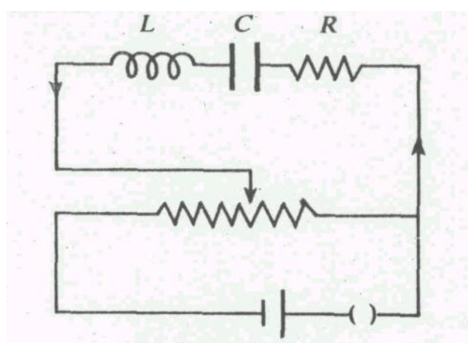


Fig. 3.7 An LCR circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad (3.36)$$

Comparing it with Eq. (3.2) we find that *L*, *R* and 1/C are respectively analogous to m, γ and *k*. This means that a resistor in an electric circuit has an *exactly* analogous effect as that of the viscous force in a mechanical system.

To proceed further, we divide Eq. (3.36) throughout by *L* obtaining

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 \qquad (3.37)$$

In this form, Eq. (3.37) is analogous to Eq. (3.3) and the two may be compared directly. This gives

$$\omega_0^2 = \frac{1}{LC}$$

and $$b = \frac{R}{2L} \tag{3.38}$$

We know that b has dimensions of time inverse. This means that *R/L* has the unit of s^{-1}, same as that of ω_0. That is why $\omega_0 L$ is measured in ohm.

With these analogies all the results of Section 3.3 apply to Eq. (3.37). For a weakly damped circuit, the charge on the capacitor plates at time *t* is

$$q(t) = q_0 \exp\left(-\frac{R}{2L}t\right)\cos(\omega_d t + \phi) \tag{3.39a}$$

with angular frequency

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{3.39b}$$

Eq. (3.39 a) shows that the charge amplitude $q_0 \exp\left(-\frac{R}{2L}t\right)$ will decay at a rate which depends on the resistance. Thus in an *LCR* circuit, resistance is the only dissipative element; an increase in *R* increases the rate of decay of the charge and decreases the frequency of oscillations.

Since $\omega_0 L$ is measured in ohms, $1/\omega_0 C$ is also measured in ohms. These are respectively referred to as *inductive reactance* and *capacitive reactance.*

For $R = 0$, Eq (3.39 a) reduces to Eq. (1.38) and $\omega_d = \omega_0$. The *Q* value of a weakly damped *LCR* circuit is

$$Q = \frac{\omega_d}{2b} \approx \omega_0 \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{3.40}$$

This equation shows that for a purely inductive circuit ($R = 0$), the quality factor will be infinite.

SAQ 5

In an -*LCR* circuit, *L*= 2mH and *C*= $5\,\mu F$. If *R* =$1\,\Omega$, $40\,\Omega$ and I00$\,\Omega$, calculate the frequency of oscillation and the quality factor when the discharge is oscillatory.

3.6.2 A Suspension Type Galvanometer

A suspension type galvanometer consists of a current carrying coil suspended in a magnetic field. The field is produced by a horseshoe magnet. The magnet is shaped so that the coil is aligned always along the magnetic lines of force. To ensure uniform strength, an iron cylinder is suspended between the poles of the magnet, as shown in Fig. (3.8). When we pass Charge through the galvanometer coil, it rotates through some angle θ. Since the coil is mechanically a torsional pendulum, it experiences a restoring couple $-k_t\theta$ and a damping couple $-\gamma \frac{d\theta}{dt}$. Do you know how damping creeps in, in this case? It has origin in air friction and electromagnetic induction.

Part of the damping arises from the viscous drag of air. In general, it is small. As the galvanometer coil rotates in the magnetic field, an induced e.m.f. is produced, which opposes its motion in accordance with Lenz's law. This so-called electromagnetic damping controls the motion of the coil when galvanometer is in use.

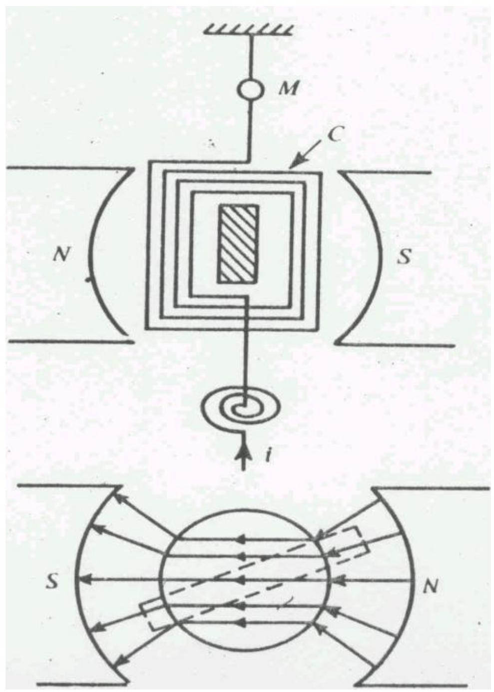


Fig. 3.8 A schematic representation of a suspension type galvanometer.

Hence, for the motion of the coil, Eq.(1.35) modifies to

$$I\frac{d^2\theta}{dt^2} = -k_t\theta - \gamma\frac{d\theta}{dt} \tag{3.41}$$

where I is the moment of inertia of the coil about the axis of suspension. Comparing it with Eq. (3.2) we find that I and k_t are analogous to *m* and *k* respectively.

Dividing throughout by I and defining

$$2b = \gamma / I$$

and $$\omega_0^2 = k_t / I \qquad (3.42)$$

we get $$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + \omega_0^2\theta = 0 \qquad (3.43)$$

This equation is of the same form as Eq. (3.3). Hence, all results deduced earlier will apply to the motion of the coil described by Eq. (3.43).

For low damping, the solution of Eq. (3.43) is

$$\theta = \theta_0 \exp(-bt)\cos(\omega_0 t + \phi) \qquad (3.44)$$

where $\theta_0 \exp(-bt)$ is the amplitude of oscillation. Eq. (3.44) describes oscillatory motion with the period of oscillation T given by

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\left[\frac{k_t}{I} - \frac{\gamma^2}{4I^2}\right]^{1/2}} \qquad (3.45)$$

This explains why a weakly damped suspension type galvanometer is called a *ballistic galvanometer*. You will note that for damping to be small, we must decrease γ and increase I. The question now arises: How can we reduce γ? As mentioned earlier, air damping is usually small. Nevertheless, it will always be present. To reduce electromagnetic damping, we must minimise the induced emf. To ensure this, we wind the coil over a non-conducting bamboo or ivory frame. If the frame is metallic, it is cut at one place, so that no current can flow through it. The quality factor of a ballistic galvanometer is

$$Q = \frac{\omega_d}{2b} = \frac{I}{\gamma}\sqrt{\frac{k_t}{I} - \frac{\gamma^2}{4I^2}} \qquad (3.46a)$$

If $\frac{k_t}{I} >> \frac{\gamma^2}{4I^2}$, this expression reduces to

$$Q = \sqrt{\frac{k_t I}{\gamma^2}}$$

This relation shows that a lightly damped suspension type galvanometer will have high quality factor.

SAQ6

The period of vibration of a galvanometer coil is 4 s. The amplitude of its vibration decreases to one-tenth of its original value in 46 s. Calculate the damping constant γ and the quality factor.

3.7 SUMMARY

1. The differential equation of a damped harmonic oscillator is

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

where $2b = \gamma / m$ and $\omega_0^2 = k / m$. The solution of the equation for heavy damping is

$$x(t) = \exp(-bt)[a_1 \exp(\beta t) + a_2 \exp(-\beta t)]$$

where $\beta = \sqrt{b^2 - \omega_0^2}$

For critical damping

$$x(t) = (p + qt)\exp(-bt)$$

and in the case of weak damping,

$$x(t) = a_0 e^{-bt} \cos(\omega_0 t + \phi)$$

2. The amplitude and average energy of a weakly damped oscillator decrease exponentially with time:

$$a = a_0 e^{-bt}$$

and $< E >= E_0 \exp(-2bt)$

where a_0 is the initial amplitude and E_0 is the total initial energy.

3. The period of a weakly damped system is given by

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{(\omega_0^2 - b^2)^{1/2}} = \frac{2\pi}{\left(\frac{k}{m} - \frac{\gamma}{4m^2}\right)^{1/2}}$$

4. The logarithmic decrement is defined as the logarithm of the ratio of successive amplitudes separated by one period. It is given by

$$\lambda = \ln\left(\frac{a_{n-1}}{a_n}\right) = bT$$

5. The rate of loss of energy or power dissipated by a weakly damped harmonic oscillator over one cycle is

$$< P >= 2 < E > / \tau$$

6. The Q-factor of a weakly damped oscillator is given by

$$Q = \omega_d \tau / 2$$

7. The differential equation describing flow of charge in a LCR circuit is

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

The effect of L, R and 1/C in an LCR circuit is respectively analogous to those of m, γ and k in a mechanical oscillator. In a weakly damped circuit, the charge oscillates harmonically:

$$q(t) = q_0 \exp\left(-\frac{R}{2L}t\right)\cos(\omega_d t + \phi)$$

and the frequency of oscillation is given by

$$\nu_d = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

For low R circuit

$$Q = \omega_0 L / R = \frac{1}{R}\sqrt{L/C}$$

8. The differential equation of a damped suspension type galvanometer is

$$I\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + k_t\theta = 0$$

For weak damping, it describes ballistic motion given by

$$\theta = \theta_0 \exp(-bt)\cos(\omega_d t + \phi)$$

where $\omega_d = \sqrt{\dfrac{k_t}{I} - \dfrac{\gamma^2}{4I^2}}$

3.7 TERMINAL QUESTIONS

1. A simple pendulum has a period of 2 s and an amplitude of 5^0. After 20 complete oscillations, its amplitude is reduced to 4^0. Find the damping constant and the time constant.

2. The quality factor of a sonometer wire is 4,000. The wire vibrates at a frequency of 300 Hz. Find the time in which the amplitude decreases to half of its original value.

3. A box of mass 0.2 kg is attached to one end of a spring whose other end is fixed to a rigid support. When a mass of 0.8 kg is placed inside the box, the system performs 4 oscillations per second and the amplitude falls from 2 cm to 1 cm in 30 s. Calculate (i) the force constant, (ii) the relaxation time and (iii) the Q-factor.

4. In an LCR circuit L = 5 mH, C = 2μF and R = 0.2Ω. Will the discharge be oscillatory? If so, calculate the frequency and quality factor of the circuit. How long does charge oscillation take to decay to half? What value of R will make the discharge just non-oscillatory?

5. The quality factor of a tuning fork of frequency 512 Hz is 6×10^4. Calculate the time in which its energy is reduced to e^{-1} of its energy in the absence of damping. How many oscillations will the tuning fork make in this time?

3.9 SOLUTIONS

SAQ's

1. $T_d = \dfrac{200s}{100} = 2\ s$

Now, $T_d = 2s = \dfrac{2\pi}{(\omega_0^2 - b^2)^{1/2}}$

so that $\omega_0^2 = \pi^2 + b^2$

Hence, $T_0 = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{(\pi^2 + b^2)^{1/2}}$

To compute b, we use the relation

$$a = a_0 \exp(-bt)$$

This may be rewritten as

$$b = \frac{1}{t}\ln\left(\frac{a_0}{a}\right)$$

$$= \frac{1}{200\,s}\ln\left(\frac{10cm}{2.5cm}\right)$$

$$= \frac{2.3}{200\,s}\log_{10} 4$$

$$= 6.9\times10^{-3}\,s^{-1}$$

Substituting this value in (i), we get

$$T_0 = \frac{2\pi}{[\pi^2 + (0.015942)^2]^{1/2}} \approx 2\text{ s} \approx T_d$$

This means that the system is weakly damped.

2. We know that

$$\lambda = \frac{1}{n}\ln\left(\frac{a_0}{a_n}\right)$$

$$= \frac{1}{100}\ln 10$$

$$= \frac{2.3}{100}\log_{10} 10 = 2.3\times10^{-2}$$

Since $b = \dfrac{\lambda}{T}$

we get

$$b = \frac{2.3\times10^{-2}}{4.6s}$$

$$= 5.0\times10^{-3}\,s^{-1}$$

Further, to calculate n for which the amplitude drops by 50%, we invert (i) to write

$$n = \frac{1}{\lambda}\ln\left(\frac{a_0}{a_n}\right)$$

$$= \frac{\ln 2}{2.3\times10^{-2}} = \frac{2.3\log_{10}2}{2.3\times10^{-2}}$$
$$= 30$$

3. Since

$$\lambda = bT = \frac{1}{n}\ln\left(\frac{a_0}{a_n}\right)$$

we can write

$$b = \frac{1}{nT}\ln(a_0/a_n)$$
$$= \frac{1}{200\,s}\ln 4$$
$$= \frac{2.3\times0.6010}{200}s^{-1}$$

Hence,

$$\tau = \frac{1}{b} = \frac{200}{2.3\times0.6010\,s^{-1}} = 145\,s$$

4. $Q = 10^3$ and $\nu = 256$ Hz

We know that for weak damping, Q is given by

$$Q = \frac{\omega_0\tau}{2} = \pi\nu\tau$$

On inverting this relation, we get

$$\tau = \frac{Q}{\pi\nu} = \frac{10^3}{256\pi\,s^{-1}} = 1.24\,s$$

Since $E = E_0\exp(-2bt) = E_0\exp(-2t/\tau)$, we get for $E/E_0 = 1/10$

$$\frac{1}{10} = \exp\left(-\frac{2t}{\tau}\right)$$

Hence,

$$t = \frac{\tau}{2}\ln 10$$
$$= \frac{1.24}{2}\times 2.3$$
$$= 1.4\ s$$

5. $L = 2\times10^{-3}$ H and $C = 5\times10^{-6}$ F

$$\therefore \qquad \frac{1}{LC} = \frac{1}{2\times10^{-3}H\times5\times10^{-4}F} = 10^8 s^2$$

Case I: R = 1 Ω

$$\therefore \qquad \frac{R^2}{4L^2} = \frac{1\Omega^2}{4\times(2\times10^{-3})^2 H^2} = 6\times10^4 \frac{\Omega^2}{H^2}$$

Thus,

$\frac{1}{LC} >> \frac{R^2}{4L^2}$ so that the discharge is oscillatory. The frequency of oscillation

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= 1.6 \text{ kHz}$$

and the quality factor of the circuit

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi\times1.6\times10^3 s^{-1}\times2\times10^{-3} H}{1\Omega}$$

$$= 20$$

Case II: R = 40 Ω

In this case,

$$\frac{R^2}{4L^2} = \frac{40\times40\Omega^2}{4\times(2\times10^{-3})^2 H^2} = 10^8 \frac{\Omega^2}{H^2}$$

Hence, $\frac{1}{LC} = \frac{R^2}{4L^2}$ and this is the case of critical damping.

Case III: R = 100 Ω

Here,

$$\frac{R^2}{4L^2} = \frac{100^2\Omega^2}{4\times(2\times10^{-3})^2 H^2} = 6\times10^8 \frac{\Omega^2}{H^2}$$

That is, $\frac{R^2}{4L^2} > \frac{1}{LC}$. This corresponds to dead beat motion.

You will note that increasing resistance in the circuit increases damping.

6. $$T = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = 4\ s$$

or

$$\omega_0^2 - b^2 = \frac{\pi^2}{4}$$

Also,

$$\ln\left(\frac{a_0}{a_n}\right) = \ln 10 = bt$$

or

$$b = \frac{1}{t}\ln 10$$

$$= \frac{2.3}{46\,s}\log_{10}10 = 0.05\ s^{-1}$$

Hence,

$$\omega_0^2 = (0.0025 + 2.4649)\,s^{-2}$$

$$= 2.467\ s^{-2}$$

$$\Rightarrow \qquad \omega_0 = 1.57\,s^{-1}$$

and

$$Q = \frac{\omega_0\tau}{2} = \frac{\omega_0}{2b} = \frac{1.57}{0.1} = 15.7$$

Terminal Questions

1. Since $\theta = \theta_0 e^{-bt}$, we can write

$$b = \frac{1}{t}\ln\left(\frac{\theta_0}{\theta}\right)$$

Substituting the given data, we get

$$b = \frac{1}{40}\ln\frac{5}{4}$$

$$= 5.57\times10^{-3}\,s^{-1}$$

and

$$\tau = \frac{1}{b} = 179.5\ s^{-1}$$

2. Since $Q = \dfrac{\omega_0\tau}{2}$, we can write $\tau = \dfrac{2Q}{\omega_0} = \dfrac{2\times4000}{2\pi\times300} = 4.24\ s$

Now, $a = a_0 e^{-bt} = a_0 e^{-t/\tau}$

$$\therefore \qquad t = \tau\ln\frac{a_0}{a} = 4.24\ s\times\ln 2 = 2.94\,s$$

3. (i) Here, $\omega_0 = 2\pi\nu = 2\times3.14\,rad\times4\,s^{-1} = 25\,rad\,s^{-1}$

But $\omega_0 = \sqrt{k/m}$ or $k = m\omega_0^2 = 1kg\times25^2\,s^{-2} = 625Nm^{-1}$

(ii) $a = a_0 e^{-bt}$ or 0.01 m = 0.02 m e^{-30b}

$$\therefore \qquad b = \frac{\ln 2}{30} = 2.3\times10^{-2}\,s^{-1}$$

Hence, relaxation time $\tau = \dfrac{1}{b} = \dfrac{1}{2.3\times10^{-2}\,s^{-1}} = 43.5\ s$

(iii) For a weakly damped system, $Q = \dfrac{\omega_0\tau}{2} = 25\times43.5 = 1088$

Here, $\dfrac{1}{LC} = \dfrac{1}{5\times10^{-3}\,H\times 2\times10^{-6}} = 10^8\,s^{-2}$

And $\dfrac{R^2}{4L^2} = \dfrac{(0.2)^2\Omega^2}{4\times(5\times10^{-3})^2 H^2} = 400\dfrac{\Omega^2}{H^2}$

Since $\dfrac{1}{LC} > \dfrac{R^2}{4L^2}$, the discharge is oscillatory and has frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 1.59\times10^3\,\text{Hz}$$

The quality factor of the circuit is

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi\times1.59\times10^3 s^{-1}\times5\times10^{-3}}{0.2\,\Omega} = 250$$

Also,

$$t = \frac{R}{2L}\ln\left(\frac{q_0}{q}\right) = \frac{0.2\,\Omega}{2\times5\times10^{-3}H}\ln 2 = 14\ s$$

The discharge will be just non-oscillatory when

$$\frac{1}{LC} = \frac{R^2}{4L^2} \text{ or } R^2 = \frac{4L}{C} = \frac{4\times5\times10^{-3}H}{2\times10^{-6}} = 10^4\,HF^{-1} \text{ or}$$
$$R = 100\,\Omega$$

5. The average energy of a damped harmonic oscillator at any time t is given by

$$\begin{aligned} <E> &= E_0\exp(-2bt) \\ &= E_0\exp(-2t/\tau) \end{aligned}$$

where $\tau = b^{-1}$ is the relaxation time.

When $t = \tau/2$, $<E> = \dfrac{E_0}{e}$

Also, $Q = \dfrac{\omega_d \tau}{2}$

Hence,

$$\tau = \frac{2Q}{\omega_d} = \frac{2\times6\times10^4}{2\pi\times512\,s^{-1}} = 37.3\ s$$

Thus, energy will reduce to $1/e$ of its initial value in 18.7 *s*.

The number of oscillations made by the tuning fork in this time is given by

$$\begin{aligned} n &= \nu_d\times t \\ &= 512\times18.7 \\ &= 95.7\times10^2 \end{aligned}$$

UNIT 4 FORCED OSCILLATIONS AND RESONANCE

Structure

4.1 INTRODUCTION

In the previous unit we studied how the presence of damping affects the amplitude and the frequency of oscillation of a system. However, in systems, such as a wall clock or an ideal *LC* circuit, oscillations do not seem to die out. To maintain oscillations we have to feed energy to the system from an external agent called a *driver*. In general, the frequencies of the driver and the driven system may not match. But in steady-state, irrespective of its natural frequency, the system oscillates with the frequency of the applied periodic force. Such oscillations are *called forced oscillations*. However, when the frequency of the driving force exactly matches the natural frequency of the vibrating system a spectacular effect is observed; the amplitude of forced oscillations becomes very large and we say that *resonance* occurs. Do you know that Galileo was the first physicist who understood how and why resonance occurs?

Resonances are desirable in many mechanical and molecular phenomena. But resonance can be disastrous also; it can literally break an oscillating system apart. For instance, fast blowing wind may set a suspension bridge in oscillation. If the frequency of the fluctuating force produced by the wind matches the natural frequency of the bridge, it gains in amplitude and may ultimately collapse. In 1940, the Tacoma Narrows bridge in Washington State collapsed within 4 months of its being opened. Similarly, when the army marches on a suspension bridge, soldiers are instructed to break step to avoid resonant vibrations. In practice, isolated systems are rare. In solid state and molecular physics, two or more systems are coupled through interatomic forces. In an electric circuit we have inductive and capacitative couplings. The oscillations of such systems will be studied in the next unit.

In this unit we shall study, in detail, the response of a system when it is driven by an external harmonic force.

Objectives

After studying this unit you should be able to:

- establish the differential equation of a system driven by a harmonic force and solve it
- analyse the response of the oscillator at different frequencies
- compute resonance width and the quality factor of a forced oscillator, and
- establish the differential equation for an *LCR* circuit under the influence of harmonic emf and write its solution by drawing an analogy with a mechanical system.

4.2 DIFFERENTIAL EQUATION FOR A WEAKLY DAMPED FORCED OSCILLATOR

To establish the differential equation of a forced weakly damped harmonic oscillator, let us again consider the spring-mass system of Unit 3. It is now also subjected to an external driving force, $F(t)$. That is, instead of. allowing the model oscillator to oscillate at its natural frequency, we push it back and forth periodically at a frequency ω (Fig. 4.1). We can write the driving force as

$$F(t) = F_0 \cos \omega t \qquad (4.1)$$

where F_0 is a constant.

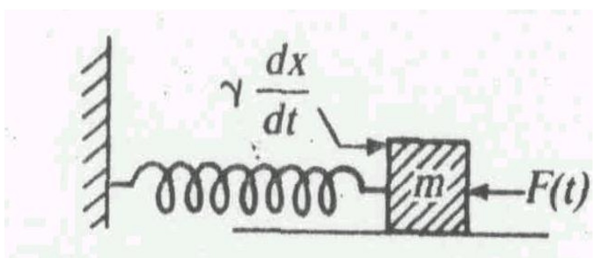


Fig. 4.1 A weakly damped forced spring-mass system.

Let the mass be displaced from its equilibrium position and then released. At any instant, it is subject to (i) a restoring force, $-kx$, (ii) a damping force, $-\gamma \frac{dx}{dt}$ and (iii) a driving force, $F(t) = F_0 \cos \omega t$.

So for a forced oscillator Eq. (3.2) is modified to

$$m\frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} + F_0 \cos \omega t \qquad (4.2)$$

Dividing by *m* and rearranging terms, the equation of motion of a forced oscillator takes the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \cos \omega t \qquad (4.3)$$

where $2b = \gamma / m$, $\omega_0^2 = k / m$ and $f_0 = F_0 / m$ is a measure of the driving force.

You may now ask: Does this equation apply only to a mass on a spring? No, it applies to any oscillator whose natural frequency is ω_0 and is subject to a harmonic driving force.

You will note that Eq. (4.3) is an inhomogeneous second order linear differential equation with constant coefficients. We will now solve this equation to learn about the motion of a forced oscillator.

4.3 SOLUTIONS OF THE DIFFERENTIAL EQUATION

Before we solve Eq. (4.3), let us analyse the situation physically. From the previous unit you will recall that when there is no applied force, a weakly damped system $(b < \omega_0)$ oscillates harmonically with angular frequency $\omega_d = \sqrt{\omega_0^2 - b^2}$. But when a driving force of angular frequency ω is applied, it imposes its own frequency on the oscillator. Thus, we expect that the actual motion will be a result of superposition of two oscillations; one of frequency ω_d (of damped oscillations) and the other of frequency ω (of the driving force). Thus, when $\omega \neq \omega_d$ the general solution of Eq. (4.3) can be written as.

$$x(t) = x_1(t) + x_2(t)$$

where $x_1(t)$ is a solution of the equation obtained by replacing the RHS of Eq. (4.3) by zero.

On substituting this result in Eq. (4.3), you will find that $x_2(t)$ satisfies the equation

$$\frac{d^2 x_2}{dt^2} + 2b\frac{dx_2}{dt} + \omega_0^2 x_2 = f_0 \cos \omega t$$

It is thus clear that $(x_1 + x_2)$ is the complete solution of Eq. (4.3). In your course on differential equations you must have learnt that x_1 is called the *complementary function* and $x_2(t)$ is called the *particular integral.*

You may recall that when there is no driving force, the displacement of a weakly damped ($b < \omega_0$) system at any instant is given by Eq. (3.19):

$$x_1(t) = a_0 e^{-bt} \cos(\omega_d t + \phi)$$

Obviously this complementary function decays exponentially and after some time it will disappear. That is why it is also referred to as the *transient solution.* In the transient state, the system oscillates with some frequency which is other than its natural frequency or the frequency of the driving force.

After a sufficiently long time $(t >> \tau)$, natural oscillations of the spring-mass system will disappear due to damping. However, we know that the general solution of Eq. (4.3) will not decay with time. That is, the system will oscillate with the frequency of the driving force. The system is then said to be in the *steady-state.* We will now obtain the steady-state solution of Eq. (4.3).

4.3.1 Steady-state Solution

To obtain the steady state solution of Eq. (4.3), let us suppose that the displacement of the forced oscillator is given by

$$x_2(t) = -a\omega \cos(\omega t - \theta) \tag{4.4}$$

where a and θ are unknown constants. By comparing Eqs. (4. 1) and (4.4) you will note that the driving force leads the displacement in phase by an angle θ.

To determine a and θ we differentiate Eq. (4.4) twice with respect to time. This gives,

$$\frac{dx_2}{dt} = -a\omega \sin(\omega t - \theta)$$

and $$\frac{d^2 x_2}{dt^2} = -a\omega^2 \cos(\omega t - \theta)$$

Substituting these results back in Eq. (4.3), we get

$$(\omega_0^2 - \omega^2)a\cos(\omega t - \theta) - 2ab\omega \sin(\omega t - \theta) = f_0 \cos \omega t$$

Using, the formulae

$\cos(\omega t - \theta) = \cos \omega t \cos\theta + \sin\omega t \sin\theta$ and $\sin(\omega t - \theta) = \sin\omega t \cos\theta - sos\omega t \sin\theta$

and rearranging terms, we get

$$\begin{aligned}&[(\omega_0^2 - \omega^2)a\cos\theta + 2ab\omega\sin\theta - f_0]\cos\omega t \\ &\qquad + [(\omega_0^2 - \omega^2)a\sin\theta - 2ab\omega\cos]\sin\omega t = 0\end{aligned} \tag{4.5}$$

We know that both $\cos\omega t$ and $\sin\omega t$ never simultaneously become zero; when one vanishes, the other takes a maximum value. Therefore, Eq. (4.5) can be satisfied only when both terms within the square brackets become zero separately, i.e.

$$(\omega_0^2 - \omega^2)a\cos\theta + 2ab\omega\sin\theta = f_0 \tag{4.6a}$$

$$(\omega_0^2 - \omega^2)a\sin\theta - 2ab\omega\cos = 0 \tag{4.6b}$$

Eq. (4.6b) readily gives the phase by which the driving force leads the displacement:

$$\theta = \tan^{-1}\frac{2b\omega}{\omega_0^2 - \omega^2} \tag{4.7a}$$

The amplitude of steady-state displacement can be determined from Eq. (4.6a) once we know the values of $\sin\theta$ and $\cos\theta$. To get these values we construct the so-called acoustic impedance triangle, as shown in Fig. 4.2. We can readily write

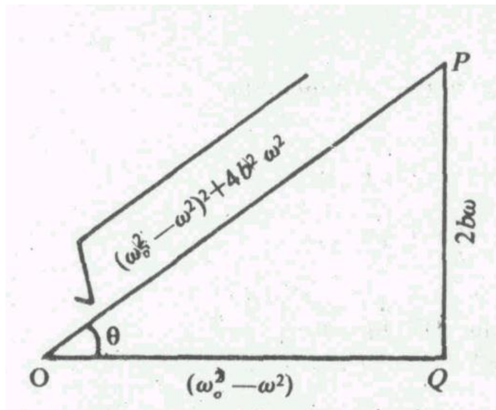


Fig. 4.2 An acoustic impedance triangle

$$\sin\theta = \frac{2b\omega}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}$$

and $$\cos\theta = \frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}$$

Using these values of $\sin\theta$ and $\cos\theta$ in Eq. (4.6a) and rearranging terms, we get

$$a = \frac{f_0}{[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}} = \frac{F_0}{m[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}} \tag{4.7b}$$

Thus, we find that the steady-state amplitude of forced oscillations depends on (i) amplitude and angular frequency of the driving force, (ii) mass and the natural angular frequency of the oscillating system and (iii) the damping constant.

Putting this value of a in Eq. (4.4) we can write the steady-state solution of Eq.(4.3) as

$$x_2(t) = \frac{F_0}{m[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}\cos(\omega t-\theta) \tag{4.8}$$

The important point to note here is that the steady-state solution has the frequency of the driving force and its amplitude is constant. Moreover, its phase is also defined completely with respect to the driving force. Therefore, it does not depend on the initial conditions. In other words, the motion of a driven system in steady-state is independent of the way we start the oscillation.

The transient solution, steady-state solution and their sum,

$$x(t) = a_0 e^{-bt}\cos(\omega_d t+\phi) + \frac{F_0\cos(\omega t-\theta)}{m[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{1/2}}$$

i.e. the complete general solution of Eq. (4.3) are shown in Fig. 4.3. The contribution of the transient part diminishes with time and ultimately disappears completely. The time for which transients persist is determined by *b* and hence by the damping factor γ. The greater the value of *b*, the more quickly do the transients die out.

For an undamped system, the steady-state solution is obtained by putting $b = 0$ in Eqs. (4.7a) and (4.8). This gives

$$\theta = 0$$

and $$x_2(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \tag{4.9}$$

That is, the driving force and the displacement are in phase $(\theta = 0)$. From this we may conclude that phase lag is essentially a consequence of damping. We further note that if the frequency of the driving force equals the frequency of the undamped oscillator, its amplitude will become infinitely large. Then *resonance* is said to occur.

You may now ask: Do we observe infinitely large amplitude in practice? No, the amplitude is finite since some damping is always present in every system.

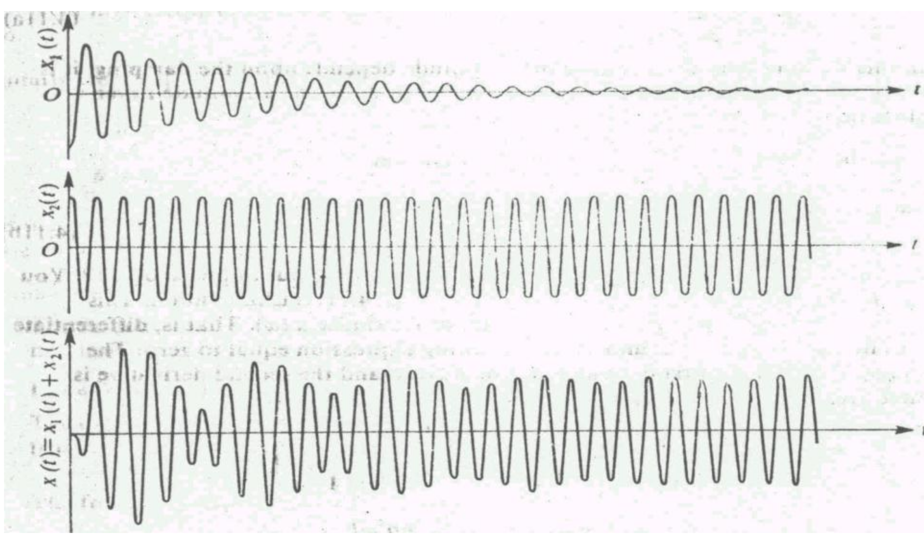


Fig. 4.3 Time variation of the transient solution, steady-state solution and the general solution of Eq. (4.3) for a weakly damped system.

4.4 EFFECT OF THE FREQUENCY OF THE DRIVING FORCE ON THE AMPLITUDE AND PHASE OF STEADY-STATE FORCED OSCILLATIONS

We know that the variation with the frequency of the driving force of the steady-state amplitude $a(\omega)$ of a forced system is given by Eq. (4.7b). Depending on the relative magnitudes of the natural and the driving frequencies, three cases arise. We will now discuss these separately in detail.

4.4.1 Low Driving Frequency $(\omega << \omega_0)$

To know the behaviour of $a(\omega)$ at low driving frequencies, we first rewrite Eq. (4.7 b) as

$$a(\omega) = \frac{f_0}{\omega_0^2\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{4b^2\omega^2}{\omega_0^2}\right]^{1/2}}$$

For $\omega << \omega_0$, we note that the ratio ω^2/ω_0^2 will be much less than one. So, we neglect terms containing ω^2/ω_0^2. This gives

$$a(\omega) = \frac{f_0}{\omega_0^2} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \qquad (4.10a)$$

Thus, at very low driving frequencies, the steady-state amplitude of the oscillation is controlled by the stiffness constant and the magnitude of the driving force.

Under this condition Eq. (4.7 a) yields

$$\tan\theta = \frac{2b\omega}{\omega_0^2 - \omega^2} \;\; \rightarrow \;\; 0 \;\; \text{for} \;\; \frac{\omega}{\omega_0} << 1 \qquad (4.10b)$$

That is, the driving force and the steady-state displacement are in phase.

4.4.2 Resonance Frequency $(\omega = \omega_0)$

To calculate the value of $a(\omega)$ at resonance, we set $\omega = \omega_0$ in Eq. (4.7b). The first term in the denominator vanishes and the amplitude is given by

$$a(\omega_0) = \frac{f_0}{2b\omega_0} \qquad (4.11a)$$

From this we note that at resonance the amplitude depends upon the damping; it is inversely proportional to *b*. That is why in actual practice the amplitude never becomes infinite.

Similarly by setting $\omega = \omega_0$ in Eq. (4.7a) we find that

$$\tan\theta \;\rightarrow\; \infty$$

so that

$$\theta = \pi/2 \qquad (4.11b)$$

This means that the driving force and the displacement are out of phase by $\pi/2$. You may be thinking that the value of $a(\omega)$ given by Eq. (4.11a) is maximum. This however is not true. Why? To answer this, let us maximise $a(\omega)$. That is, differentiate Eq. (4.7b) with respect to ω

and set the resulting expression equal to zero. The frequency at which the first derivative becomes zero and the second derivative is negative gives the correct answer:

$$\frac{da(\omega)}{d\omega} = \frac{d}{d\omega}\left[\frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}\right]$$
$$= -\frac{f_0[-4\omega(\omega_0^2 - \omega^2) + 8b^2\omega}{2[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$
$$= -\frac{f_0[-4\omega(\omega_0^2 - \omega^2) + 8b^2\omega]}{2[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} = 0 \text{ for } \omega = \omega_r$$

This equality will hold only when the numerator vanishes identically, i.e. $-4\omega(\omega_0^2 - \omega^2) + 8b^2\omega = 0$ for $\omega = \omega_r$.

We ignore the root $\omega_r = 0$, which is trivial. Then we must have

$$\omega_r^2 - \omega_0^2 + 2b^2 = 0$$

This equation is quadratic in ω_r, and the acceptable root is

$$\omega_r = (\omega_0^2 - 2b^2)^{1/2} \tag{4.12}$$

The root corresponding to the negative sign is physically meaningless and is ignored.

For $a(\omega)$ to be maximum, its second derivative with respect to ω should be negative. You can easily verify that at $\omega_r = (\omega_0^2 - 2b^2)^{1/2}$ $\left.\frac{d^2a}{d\omega^2}\right|_{\omega=\omega_r}$ is negative. Thus, we can conclude that the peak value of amplitude is attained at a frequency slightly below ω_0. The shift is caused due to damping. We can visualize it as follows: When the driver imparts maximum push, the driven system does not accept it instantly due to a finite phase difference between $x(t)$ and $F(t)$.

On substituting for ω_0 from Eq. (4.12) in Eq. (4.7b) and simplifying the resulting expression, we get the peak value of steady-state amplitude:

$$a_{\max} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} \tag{4.13}$$

When at a particular frequency, the amplitude of the driven system becomes maximum, we say that *amplitude resonance occurs*. The frequency ω_r is referred to as the *resonance frequency*. It is instructive to note that ω_r is less than ω_0 as well as $\omega_d = \sqrt{\omega_0^2 - b^2}$.

4.43 High Driving Frequency

For $\omega >> \omega_0$ we rewrite Eq. (4.7b) as

$$a(\omega) = \frac{f_0}{\omega^2\left[\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 + \frac{4b^2}{\omega^2}\right]^{1/2}}$$

and neglect terms containing ω_0^2 / ω^2 as well as $(2b/\omega)^2$, as they are both much smaller than unity. Then the amplitude of resulting vibration is given by

$$a(\omega) = \frac{f_0}{\omega^2} \tag{4.14a}$$

That is, at high frequencies the amplitude decreases as $1/\omega^2$ and ultimately becomes zero.

Similarly from Eq. (4.7a), the phase is given by

$$\tan\theta = \frac{2b\omega}{(\omega_0^2 - \omega^2)} \approx -\frac{2b}{\omega} \xrightarrow[\omega\to\infty]{} 0$$

or $$\theta = \pi \tag{4.14b}$$

This means that at high frequencies the driving force and displacement are out of phase by π.

We may thus conclude that

(i) The amplitude of oscillation in steady-state varies with frequency. It becomes maximum at $\omega_r = \sqrt{(\omega_0^2 - 2b^2)}$ and has value $f_0 / 2b\sqrt{(\omega_0^2 - b^2)}$. For $\omega > \omega_r$, $a(\omega)$ decreases as ω^{-2}.

(ii) The displacement lags behind the driving force by an angle θ, which increases from zero at $\omega = 0$ to π at extremely high frequencies. At $\omega = \omega_0$, $\theta = \pi / 2$.

The frequency dependence of $a(\omega)$ and $\theta(\omega)$ is shown in Fig. 4.4.

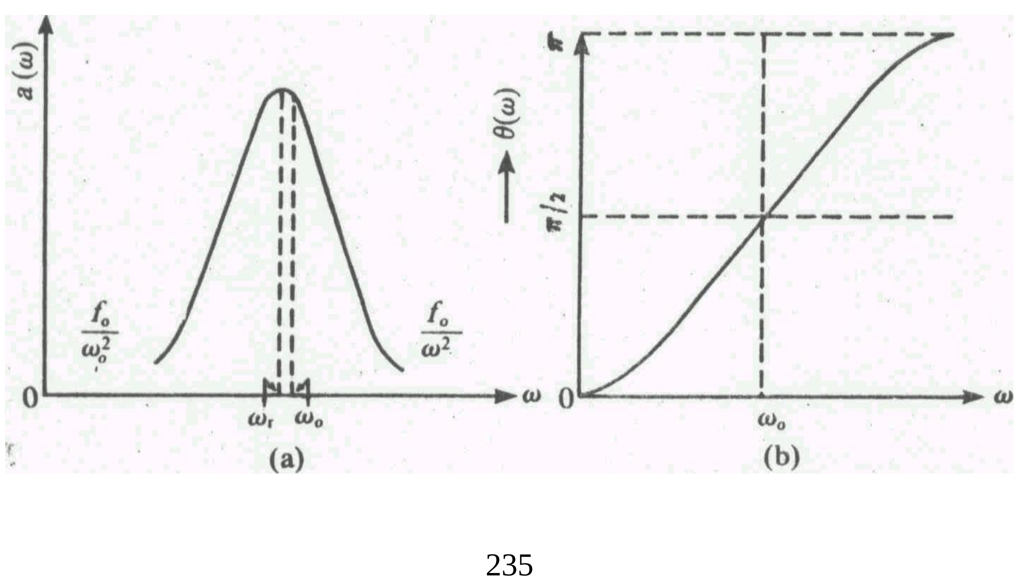


Fig. 4.4 Frequency variation of (a) steady-state amplitude, and (b) phase of a forced oscillator

4.5 POWER ABSORBED BY A FORCED OSCILLATOR

You now know that every oscillating system loses energy in doing work against damping. But oscillations of a forced oscillator are maintained by the energy supplied by the driving force. It is, therefore, important to know the average rate at which energy must be supplied to the system to sustain steady-state oscillations. So, we now calculate the average power absorbed by the oscillating system.

By definition, the instantaneous power is given by

$$P(t) = \text{force} \times \text{velocity}$$
$$= F(t) \times v$$

Differentiating Eq. (4.8) with respect to time, we get

$$v = \frac{dx_2(t)}{dt} = -\frac{F_0\omega}{m[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$
$$= -v_0 \sin(\omega t - \theta) = v_0 \cos(\omega t - \phi) \qquad (4.15)$$

where

$$v_0 = \frac{f_0\omega}{m[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} \qquad (4.15a)$$

is the velocity amplitude and

$$\phi = \theta - \pi/2 \qquad (4.15b)$$

is the phase difference between velocity and the applied force. On substituting for $F(t)$ and v from Eqs. (4.1) and (4.15), respectively we find that the instantaneous power absorbed by the oscillator is given by

$$P(t) = F_0 v_0 \cos\omega t \cos(\omega t - \phi)$$

Since $\cos(\omega t - \phi) = \cos\omega t \cos\phi + \sin\omega t \sin\phi$, we can rewrite the expression for instantaneous power as

$$P(t) = F_0 v_0 [\cos^2\omega t \cos\phi + \cos\omega t \sin\omega t \sin\phi]$$

From this we can easily calculate the average power absorbed over one cycle:

$$< P > = \frac{1}{2} F_0 v_0 \sin\phi < \sin 2\omega t > + F_0 v_0 \cos\phi < \cos^2\omega t > \qquad (4.16)$$

From Unit 1 you may recall that $< \sin 2\omega t > = 0$ so that the first term on the RHS of Eq. (4.16) drops out. Also $< \cos^2\omega t > = 1/2$. Then Eq. (4.16) reduces to

$$< P > = \frac{1}{2} F_0 v_0 \cos\phi = \frac{1}{2} F_0 v_0 \sin\theta \tag{4.17}$$

On substituting for $\sin\theta$ from Fig. 4.2 and v_0 from Eq. (4.15a) in Eq. (4.17), we get

$$< P > = \left(\frac{bF_0^2}{m}\right) \frac{\omega^2}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]} \tag{4.18}$$

From Eq. (4.17) we note that the average power absorbed by a forced oscillator will be maximum when $\sin\theta = 1 = \cos\phi$, i.e., $\phi = \pi/2 \quad (\phi = 0)$. This happens for $\omega = \omega_0$. This happens for $\omega = \omega_0$. Using this result in Eq. (4.18), we get

$$< P >_{\max} = \frac{1}{4bm} F_0^2 \tag{4.19}$$

That is, the peak value of average power absorbed by a maintained system is determined by damping and the amplitude of the driving force. The frequency variation of $< P >$ is shown in Fig. 4.5

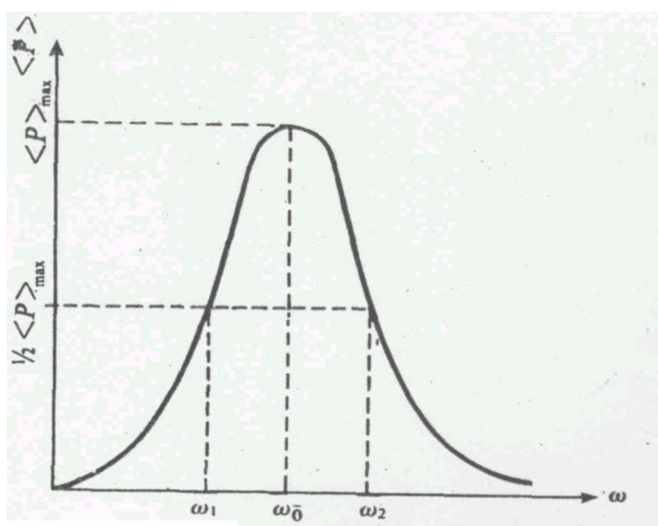


Fig. 4.5. Frequency variation of average power, a., and $a>_{:}$ correspond to hair-power points.

It is important to note that unlike the case of amplitude resonance, maximum average Forced Oscillations and Resonance power is transferred at the natural frequency of the system. This arises because velocity and driving force are in phase.

4.6 QUALITY FACTOR

In Unit 3, we defined the quality factor of a damped oscillator as

$$Q = 2\pi \frac{\text{average energy stored in one cycle}}{\text{average energy dissipated in one cycle}}$$

You can use the same definition to calculate Q of a forced oscillator once you know $< E >$ and $< P >$.

SAQ1
Show that the average energy of a forced oscillator is

$$< E > = \frac{1}{4} m(\omega^2 + \omega_0^2) a^2$$

and the quality factor is given by

$$Q = \frac{\omega^2 + \omega_0^2}{4b\omega}$$

Another equivalent and more useful interpretation of the quality factor is in terms of amplitudes. The Q factor is defined as the ratio of the amplitude at resonance to the amplitude at low frequencies $(\omega \to 0)$. Using this definition, the value of the quality factor can be calculated rather easily on dividing Eq. (4.13) by Eq. (4. l0a).

$$Q = \frac{a_{\max}}{a(\omega \to 0)} = \frac{f_0}{2b(\omega_0^2 - b^2)^{1/2}} \times \frac{\omega_0^2}{f_0} = \frac{\omega_0^2}{2b(\omega_0^2 - b^2)^{1/2}} \tag{4.20a}$$

If damping is small, $b^2 << \omega_0^2$ and the expression for the quality factor reduces to

$$Q = \frac{\omega_0}{2b} = \frac{\omega_0 \tau}{2} \tag{4.20b}$$

which is the same as Eq. (3.33) with $b = 0$.

SAQ 2
Using Eq. (4.20b), show that the amplitude and phase of a weakly damped forced oscillator can be expressed as

$$a(\omega) = a_0 \frac{\omega_0 / \omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

and $$\tan\theta = \frac{1/Q}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)}$$

where $a_0 = F_0 / m\omega_0^2$.

For different values of Q frequency variation of $a(\omega)$ and $\theta(\omega)$ based on these equations is shown in Fig.4.6. We observe that as Q increases (i.e., damping decreases), the value of $a(\omega)$ increases.

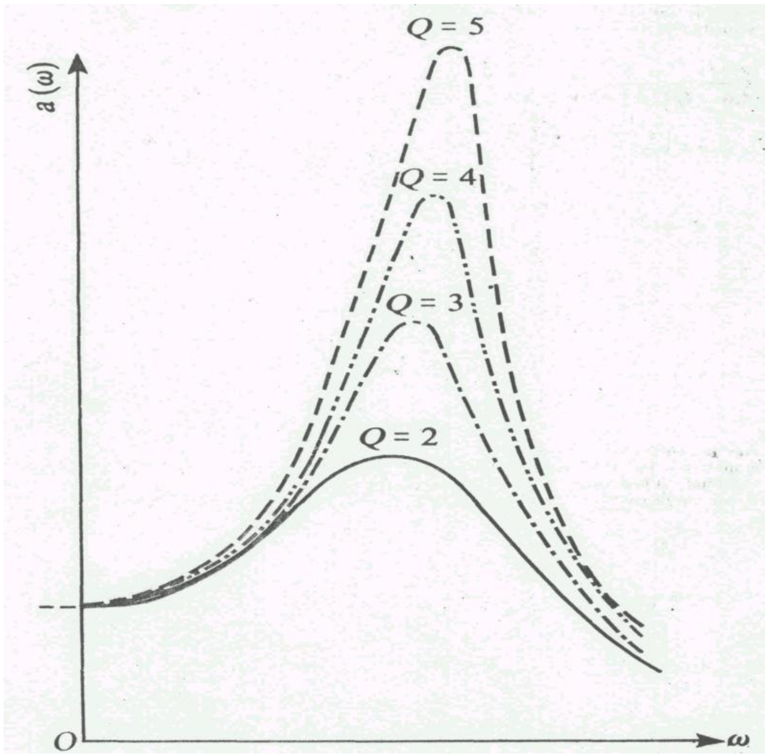


Fig. 4.6 (a) Amplitude as a function of driving frequency for different values of Q,
(b) Phase difference θ as a function of driving frequency for different values of Q.

SAQ3

Express $<P>$ in terms of Q and show that $<P>=\frac{1}{2}\frac{F_0^2\omega_0}{k}Q$

4.6.1 *Q* in Terms of Band Width: Sharpness of a Resonance

The Q of a system can also be defined as

$$Q=\frac{\text{Frequency at which power resonance occurs}}{\text{Full width at half}-\text{power points}} \tag{4.21}$$

To calculate the frequency at which average power drops to half its maximum value we can write from SAQ 3

$$\frac{1}{2}\frac{F_0^2\omega_0}{kQ}\frac{\omega_0^2\omega^2}{\left[(\omega_0^2-\omega^2)^2+\frac{\omega_0^2\omega^2}{Q^2}\right]}=\frac{1}{4}\frac{F_0^2\omega_0 Q}{k}$$

On simplification we can write

$$(\omega_0^2 - \omega^2) = \frac{\omega^2 \omega_0^2}{Q^2}$$

so that

$$(\omega_0^2 - \omega^2) = \pm \frac{\omega^2 \omega_0^2}{Q^2}$$

This equation has 4 roots. Of these, two roots correspond to negative frequencies and are physically unacceptable. The other two acceptable roots are

$$\omega_1 = -\frac{\omega_0}{2Q} + \omega_0 \left(1 + \frac{1}{4Q^2}\right)^{1/2}$$

and $$\omega_2 = \frac{\omega_0}{2Q} + \omega_0 \left(1 + \frac{1}{4Q^2}\right)^{1/2} \qquad (4.22)$$

Obviously, the second of these roots is greater than ω_0 and the other root is smaller than ω_0. This is illustrated in Fig. 4.5.

The frequency interval between two half-power points is

$$\omega_2 - \omega_1 = 2\Delta\omega = \frac{\omega_0}{Q}$$

From Eq. (4.23) it is clear that a high Q system has small bandwidth and the resonance is said to be sharp. On the other hand, a low Q system has a large bandwidth and the resonance is said to be flat. This is illustrated in Fig. 4.6. Thus, the sharpness of resonance refers to the rapid rate of the fall of power with frequency on either side of resonance. We measure it in terms of the Q-value of the system. The Q factor has its greatest importance in reference to electrical circuits which we will discuss now.

SAQ 4

Calculate the energy stored in a mass of 0.1 kg attached to a spring. The mass is oscillating with an amplitude of 5 cm and is in resonance with a driving force of frequency 30 Hz. If the Q factor is 100, calculate the power loss.

4.7 AN LCR CIRCUIT

We have so far discussed the resonant behaviour of a simple mechanical system subject to a periodic force. Another physical system which also exhibits resonant behaviour is a series *LCR* circuit containing a source of alternating e.m.f. We will discuss the behaviour of this system by drawing similarities with a mechanical system.

From Unit 3 we know that in an *LCR* circuit, charge oscillations die out because of power losses in the resistance. What changes do you expect in this behaviour when a source of alternating e.m.f. of frequency ω is introduced? To answer this question, let us consider Fig, 4.7. Let I be the current in the circuit at a given time. Then, the applied emf is equal to the sum of the potential differences across the capacitor, resistor and the inductor. Then Eq. (3.35) modifies to

$$\frac{q}{C} + RI + L\frac{dI}{dt} = E_0 \cos \omega t \tag{4.24}$$

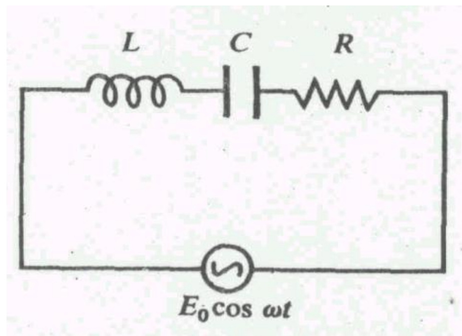


Fig. 4.7 A harmonically driven LCR circuit

Since $I = \frac{dq}{dt}$, this equation can be rewritten as

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 \cos \omega t \tag{4.25}$$

Dividing through by L, we get

$$\frac{d^2 q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E_0}{L} \cos \omega t \tag{4.26}$$

In this form Eq. (4.26) is similar to Eq. (4.3). Hence its steady-state solution can be written by analogy. For a weakly damped system, the charge on capacitor plates at any instant of time is given by

$$q = \frac{E_0 / L}{\left[\left(\frac{1}{LC} - \omega^2\right) + \left(\frac{\omega R}{L}\right)^2\right]^{1/2}} \cos(\omega t - \theta) \tag{4.27}$$

where $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ is the angular frequency of oscillation and

$$\tan\theta = \frac{(\omega R / L)}{\frac{1}{LC} - \omega^2} \tag{4.28}$$

defines the phase with respect to applied emf.

The current in the circuit is obtained by differentiating Eq. (4.27) with respect to t. The result is

$$I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \phi) \tag{4.29}$$

where $\phi = \theta - \pi/2$ is the phase difference between E_0 and I. Since

$$\tan\phi = -\cot\theta = \frac{\frac{1}{LC} - \omega^2}{\omega R / L}$$

we find that

$$\phi = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R} \tag{4.30}$$

From Eq. (4.29) we note that current in a LCR circuit is a function of the frequency.

When $\omega L << \frac{1}{\omega C}$, the circuit is capacitive in nature and we can write

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = \frac{1}{\omega^2 C^2}$$

Thus, if we are working at low frequencies and R is also small, the current amplitude will be small. What will be its magnitude for $\omega \to 0$? In this limit $I \to 0$ and leads the applied emf by $\pi/2$.

As the driving frequency increases, the reactance $\left(\omega L - \frac{1}{\omega C}\right)$ decreases and current amplitude increases. When

$$\omega L = \frac{1}{\omega C} \tag{4.31}$$

the term under the radical sign in Eq. (4.29) becomes minimum; equal to R. Then the current attains its peak value $I_0 = E_0 / R$ and the circuit is said to resonate with frequency

$$\nu_r = \frac{1}{2\pi\sqrt{LC}} \tag{4.32}$$

At resonance, the current and applied emf are in phase. When the driving frequency is high, the circuit will be inductive and the current lags behind emf by $\pi/2$.

For different values of R, the frequency variation of peak current and phase is shown in Fig. 4.8. You will observe that the lower the resistance, the higher the peak value of the current and the sharper the resonance.

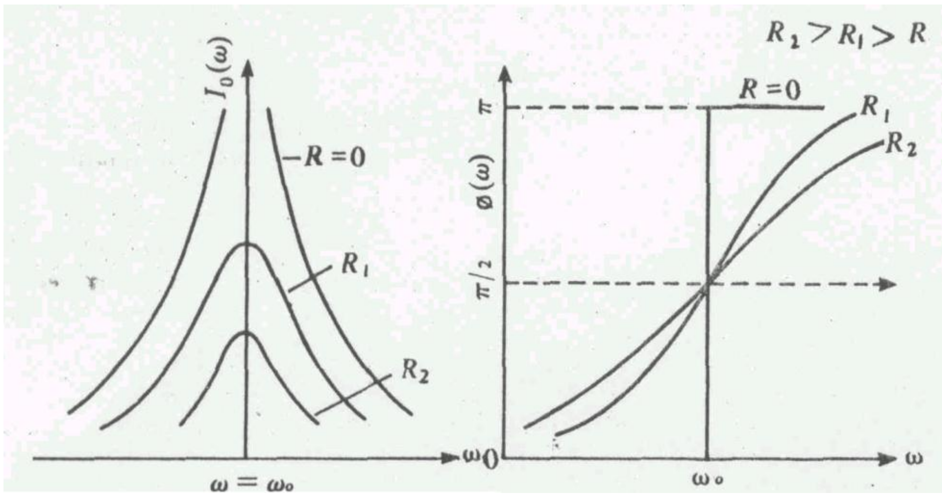


Fig. 4.8 Frequency variation of peak current and phase for different values of *R* in a driven *LCR* circuit

The power in an electric circuit is defined as the product of current and the emf. For an *LCR* circuit, we can write

$$P = EI = E_0 I_0 \cos\omega t \cos(\omega t - \phi)$$

where $I_0 = E_0 / \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

Using the formula $2\cos A \cos B = \cos(A+B) + \cos(A-B)$, we can rewrite the above expression for power as

$$P = \frac{E_0 I_0}{2}[\cos\phi + \cos(2\omega t - \phi)]$$

Power averaged over one complete cycle is obtained by noting that $<\cos(2\omega t - \phi)> = 0$. Hence

$$<P> = \frac{E_0 I_0}{2}\cos\phi$$
$$= E_{rms} I_{rms} \cos\phi$$

where $E_{rms} = E_0 / \sqrt{2}$ and $I_{rms} = I_0 / \sqrt{2}$ are, respectively the root mean square values of emf and current. Since $< P >$ varies with $\cos\phi$, it is customary to call $\cos\phi$ the power factor.

The quality factor of an LCR circuit is given by

$$A = \frac{\omega_0 L}{R} \tag{4.34}$$

where $\omega_0 = 1/\sqrt{LC}$

You can verify that the bandwidth of power resonance curve for an LCR circuit is given by

$$\omega_2 - \omega_1 = \frac{2}{\tau} = \frac{R}{L} = \frac{\omega_0}{Q} \tag{4.35}$$

so that

$$Q = \frac{\text{Frequency at resonance}}{\text{Full width at half} - \text{power points}}$$

The Q of a circuit determines its ability to select a narrow band of frequencies from a wide range of input frequencies. This, therefore, acquires particular importance in relation to radio receivers. Signals of various frequencies from all stations are present around the antenna. But the receiver selects just one particular station to which we wish to tune and discards others. Normally, radio receivers operating in MHz region have Q values of the order of 10^2 to 10^3. Microwave cavities have Q values of the order of 10^5.

4.8 SUMMARY

- When a harmonic force $F = F_0 \cos\omega t$ is impressed upon a damped harmonic oscillator, the oscillator executes forced oscillations. The differential equation of motion of a driven oscillator is

  $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = F_0 \cos\omega t$$

  where $2b = \frac{\gamma}{m}$, $\omega_0 = \sqrt{\frac{k}{m}}$ and $f_0 = \frac{F_0}{m}$

- The general solution of the differential equation of a driven oscillator is

  $$x(t) = a\cos(\omega t - \theta) + a_0 e^{-bt} \cos(\omega_d t + \theta)$$

  where

  $$a = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$

  and

  $$\theta = \tan^{-1} \frac{2b\omega}{\omega_0^2 - \omega^2}$$

are steady-state amplitude and phase respectively.

- Amplitude resonance occurs at a frequency

$$\omega_r = \sqrt{\omega_0^2 - 2b^2} = \omega_0 \sqrt{1 - (1/2Q^2)}$$

At resonance frequency

$$a_{\max} = \frac{f_0}{2b(\omega_0^2 - b^2)^{1/2}}$$

- The average power absorbed by a forced oscillator is given by

$$< P > = \frac{1}{2} F_0 v_0 \sin\theta$$

It is a maximum when $\theta = \pi/2$.

- The quality factor of a forced oscillator can be interpreted as amplitude Forced Oscillations and Resonance amplification. It is related to full width at half maximum by the relation

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

- The differential equation of a driven *LCR* circuit is

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E_0}{L}\cos\omega t$$

Its steady-state solution is given by

$$q = \frac{E_0/L}{\left[\left(\frac{1}{LC} - \omega^2\right) + \left(\frac{\omega R}{L}\right)^2\right]^{1/2}}\cos(\omega t - \theta)$$

with $\tan\theta = \dfrac{\omega R/L}{\dfrac{1}{LC} - \omega^2}$

4.9 TERMINAL QUESTIONS

1. A body of mass 0.1 kg is suspended from a spring of force constant 100 Nm^{-1}. The frictional force acting on the body F_d = $5v$ N. Set up the differential equation of motion and find the period of free oscillations. Now a harmonic force $F = 2\cos 20t$ is applied. Calculate the amplitude of forced oscillations and phase lag in the steady-state.
2. For a high Q-system, show that the width of the amplitude resonance curve is nearly $\sqrt{3}b$, where the full width is measured between those frequencies where $a = a_{\max}/2$.

3. An alternating potential of frequency 10^5 Hz and amplitude 1.2 V is applied to a series *LCR* circuit. If L = 0.5 mH and R = $40\,\Omega$, find the value of the capacitance C to get resonance. Also calculate the rms value of this current.

4.10 SOLUTIONS

SAQ 1

$$< K.E. > = (1/2)m < v^2 >$$
$$= (1/2)m\omega^2 a^2 < \sin^2(\omega t - \theta) > = (1/4)m\omega^2 a^2$$
$$< U > \quad = (1/2)k < x^2 >$$
$$= (1/2)m\omega_0^2 a^2 < \cos^2(\omega t - \theta) > = (1/4)m\omega_0^2 a^2$$

Therefore, the time-averaged energy is

$$< E > \quad = (1/4)m(\omega^2 + \omega_0^2)a^2$$

The average energy dissipated per second is $< \gamma v^2 > = mb\omega^2 a^2$.

By definition,

$$Q = 2\pi \frac{\text{average energy stored in one cycle}}{\text{average energy dissipated in one cycle}}$$
$$= 2\pi \frac{\text{average energy stored in one cycle}}{\text{time period} \times \text{average energy dissipated per second}}$$
$$= 2\pi \frac{m(\omega_0^2 + \omega^2)\,a^2}{4Tmb\omega^2 a^2}$$
$$= \frac{\omega_0^2 + \omega^2}{4\omega b}$$

SAQ 2

From Eq. (4.7b) we recall that the amplitude of a weakly damped forced oscillator is given by

$$a = \frac{F_0}{m[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$
$$= \frac{F_0 / m}{\omega\,\omega_0\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{4b^2}{\omega_0^2}\right]^{1/2}}$$

If we put $a_0 = \dfrac{F_0}{m\omega_0^2}$ and use Eq. (4.20), we get the required result:

$$a(\omega) = a_0 \frac{\omega_0 / \omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Similarly, from Eq. (4.7a), we recall that

$$\tan\theta = \frac{2b\omega}{\omega_0^2 - \omega^2}$$

$$= \frac{2b\omega}{\omega\,\omega_0\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)}$$

or $$\tan\theta = \frac{1/Q}{\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)}$$

SAQ 3

From Eq. (4.18)

$$< P >= \frac{bF_0^2}{m}\,\frac{\omega^2}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]}$$

Putting $Q = \dfrac{\omega_0}{2b}$, we get

$$< P >= \frac{\omega_0 F_0^2}{2mQ}\,\frac{\omega^2}{[(\omega_0^2 - \omega^2)^2 + (\omega^2\omega_0^2 / Q^2)]}$$

At $\omega = \omega_0$, the denominator in the parenthesis will become minimum and the average power absorbed by the oscillator becomes maximum:

$$< P >_{\max} = \frac{F_0^2 Q}{2m\omega_0}$$

$$= \frac{1}{2}\frac{F_0^2\omega_0 Q}{m\omega_0^2}$$

$$= \frac{1}{2}\frac{F_0^2\omega_0 Q}{k}$$

SAQ 4

From SAQ 1, you would recall that the average energy of a weakly damped oscillator is given by

$$< E >= \frac{m}{4}(\omega_0^2 + \omega^2)a^2$$

At resonance, $\omega = \omega_0$, and the expression for average energy reduces to

$$< E >= (1/2)m\omega_0^2 a^2$$

$$= (1/2)\times 0.1\,kg \times (2\pi \times 30\,s^{-1})^2 \times (5\times 10^{-2})\,m)^2$$

$$= 4.44 \text{ J}$$

Now, $$Q = 2\pi\,\frac{\text{average energy stored in one cycle}}{\text{average energy dissipated in one cycle}}$$

Period $T = (1/30)\,s$,

Average energy dissipated in $(1/30)\,s = \dfrac{2\pi \times 4.44}{100}$

= 0.28 J

Average energy dissipated per second = 30×0.28 J = 8.4 J.

Terminal Questions

1. For free oscillations, the differential equation is

$$m\ddot{x} = -kx - \gamma\,\dot{x}$$

Substituting m = 0.1 kg, k = 100 Nm^{-1} and γ = 5 $N\,m\,s^{-1}$ we get

$$0.1\frac{d^2x}{dt^2} = -100x - 5\frac{dx}{dt}$$

or
$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 1000x = 0$$

Period $T = \dfrac{2\pi}{[\omega_0^2 - b^2]^{1/2}} = \dfrac{2\pi}{[1000-625]^{1/2}}\,s = 0.32\,s$.

On the application of the harmonic force, the equation of motion becomes

$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 1000x = 20\cos 20t$$

Here,
$$a = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$

$$= \frac{20\,N/kg}{[(1000-400)^2 + 2500\times 400]^{1/2}\,s^{-2}}$$

= 1.7×10^{-3} m

$$\tan\theta = \frac{2b\omega}{\omega_0^2 - \omega^2} = \frac{50\times 20}{1000-400}$$

$\tan\theta$ = 1.67

or $\theta = 59.1^0$

2. Let ω_r be the value of the angular frequency when $a = \dfrac{a_{\max}}{2}$.

Using Eq. (4.7b) and (4.13), we get

$$\frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} = \frac{f_0}{4b(\omega_0^2 - b^2)^{1/2}}$$

On cross-multiplying and squaring both sides, we get

$$(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2 = 16b^2(\omega_0^2 - b^2)$$

Since $\omega_r^2 = \omega_0^2 - 2b^2$, we can rewrite this as

$$(\omega_r^2 - \omega^2 + 2b^2)^2 + 4b^2\omega^2 = 16b^2(\omega_r^2 + b^2)$$

For low damping, we can write this as

$$(\omega_r^2 - \omega^2)^2 + 4b^2\omega^2 = 16b^2\omega_r^2$$

or

$$(\omega_r^2 - \omega^2)^2 = 12b^2\omega_r^2$$

On taking square roots, we get

$$\omega_r^2 - \omega^2 = \pm 2\sqrt{3b}\omega_r$$

or

$$\omega_r - \omega^2 = \frac{\pm 2\sqrt{3b}\omega_r}{\omega_r + \omega} = \pm\sqrt{3}b$$

$\therefore$ Half bandwidth $\Delta\omega = |\omega_r - \omega| = \sqrt{3}b$

and Full bandwidth $2\Delta\omega = 2\,|\omega_r - \omega| = 2\sqrt{3}b = \sqrt{3}\ \gamma / m$.

3. At resonance, capacitance is given by

$$C = \frac{1}{4\pi^2\nu_r^2} = \frac{1}{4(3.14)^2 \times (10^5)^2 s^{-2} \times 0.5 \times 10^{-3} H}$$

$$= 5.1\times 10^{-9}\text{ F}$$

$$I_{rms} = \frac{E_{rms}}{R} = \frac{E_0}{R\sqrt{2}} = \frac{1.2V}{4\sqrt{2}\,\Omega}$$

= 0.21 A (at resonance $Z = R$)

Peak-potential difference across the capacitor

= Peak current $\times$ reactance offered by the capacitor

$$= \frac{E_0}{R} \times \frac{1}{\omega C}$$

$$= \frac{1.2V}{4\Omega} \times \frac{1}{2\pi \times 10^5 s^{-1} \times 5 \times 10^{-9} F}$$

= 95.5 V

UNIT 5 COUPLED OSCILLATIONS

Structure

5.1 INTRODUCTION

In this block so far you have studied isolated (single) oscillating systems such as a spring-mass system, a pendulum or a torsional oscillator. In nature we also come across many examples of coupled oscillators. We know that atoms in a solid are coupled by interatomic forces. In molecules, say the water molecule, two hydrogen *Moms* are coupled to an oxygen atom while in a carbon dioxide molecule oxygen atoms are coupled to one carbon atom. In all these cases, oscillations of one atom are affected by the presence of other atom(s). In radio and TV transmission, we use electrical circuits with inductive/ capacitative couplings. Therefore, it is important to extend our study of preceding units to cases where such simple systems are coupled.

We begin this unit with a study of longitudinal oscillations of coupled masses. Do you expect the motion of these masses to be simple harmonic? You will learn that their motion is not simple harmonic. But it is possible to analyse it in terms of normal modes, each of which has a definite frequency and represents SHM. The presence of coupling leads to exchange of energy between two masses. To illustrate this further, we will determine, by analogy, normal mode frequencies of two coupled pendulums and two inductively coupled LC circuits. This analysis will then be extended to N coupled oscillators. When N becomes very large, i.e. we have a homogeneous medium, exchange of energy leads to the phenomenon of wave«motion.

In the next unit you will learn the details of wave propagation with particular reference to waves in strings, liquids and gases.

Objectives

After studying this unit you should be able to:

- describe the effect of coupling on the oscillations of individual oscillators
- establish the equation of motion of a coupled system executing longitudinal oscillations
- define normal modes and analyse the motion of two coupled oscillators in terms of normal modes
- compute, by analogy, the normal mode frequencies for a physical system of interest, and derive the wave equation.

5.2 OSCILLATIONS OF TWO COUPLED MASSES

To analyse the effect of coupling we start again with the model spring-mass system. We consider two such identical systems connected (coupled) by a spring, as shown in Fig. 5.la. In this system we have two equal masses attached to springs of stiffness constant *k'* and coupled to each other by a spring of stiffness constant *k*. In the equilibrium position, springs do not exert any force on either mass. The motion of this system will depend on the initial conditions. That is, the motion may be transverse or longitudinal depending on how the masses are disturbed. For simplicity, we first consider longitudinal motion of these two coupled masses.

We pull one of the masses longitudinally and then release H. The restoring force will tend to bring it back to its equilibrium position. As it overshoots the equilibrium . mark, the coupling spring will pull the other mass. As a result both masses start oscillating longitudinally. This means that motion imparted to one of the two coupled masses is not confined to it only; it is transmitted to the other mass as well. We now establish the equation of motion of these masses.

5.2.1 The Differential Equation

We choose x-axis along the length of the spring with O as the origin (Fig. 5.la).

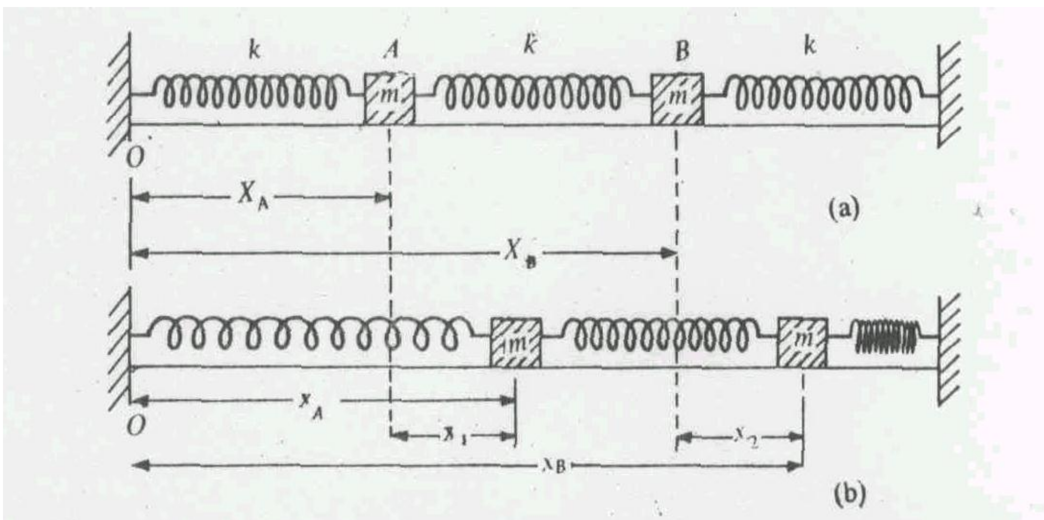


Fig. 5.1 Longitudinal oscillations of two coupled masses, (a) Equilibrium configuration (b) Configuration at time *t*.

Let X_A and X_B be the coordinates of the centre of the masses *A* and *B* respectively. When mass *B* is displaced towards the right and then released, mass *A* will also get pulled towards the right due to the coupling spring. The coupled system would then start oscillating. Suppose X_A and X_B are the instantaneous positions of masses *A* and *B* respectively. Then their displacements from their respective equilibrium positions are given by

$$x_2 = x_B - X_B \text{ and } x_1 = x_A - X_A$$

Now at any instant of time during oscillation, the forces acting on mass A are

(i) restoring force : $-k'(x_A - X_A) = -k'x_1$; and

(ii) a coupling force : $k(x_B - x_A) - (X_B - X_A) = k(x_2 - x_1)$

We are here assuming that the masses are moving on a frictionless surface. By Newton's second law, the equation of motion of mass A is thus given by

$$m\frac{d^2x_A}{dt^2} = -k'(x_A - X_A) + k[x_B - X_B - x_A - X_A]$$

or $$m\frac{d^2(x_A - X_A)}{dt^2} = m\frac{d^2x_1}{dt^2} = -k'x_1 + k(x_2 - x_1) \tag{5.1}$$

since $\frac{dX_A}{dt} = 0$.

Dividing through by m and rearranging terms, we get

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 - \omega_s^2(x_2 - x_1) = 0 \tag{5.2}$$

where $\omega_0^2 = \frac{k'}{m}$ and $\omega_s^2 = \frac{k}{m}$

Similarly, the equation of motion of the mass B is

$$m\frac{d^2x_2}{dt^2} = -k'x_2 - k(x_2 - x_1) \tag{5.3}$$

This can also be written as

$$\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 + \omega_s^2(x_2 - x_1) = 0 \tag{5.4}$$

Let us pause for a minute and ask: Do Eqs. (5.2) and (5.4) represent simple harmonic motion? No, we cannot, in general, identify the motion described by these equations as simple harmonic because of the presence of the coupling term $\omega_s^2(x_2 - x_1)$. This means that the analysis of previous units will not work since these equations are coupled in x_1 and x_2. The question now arises: How to solve these equations? These equations will have to be solved simultaneously. For this purpose we first add Eqs. (5.2) and (5.4) to obtain

$$\frac{d^2}{dt^2}(x_1 + x_2) + \omega_0^2(x_1 + x_2) = 0 \tag{5.5a}$$

Next we subtract Eq. (5.4) from Eq. (5.2) and rearrange terms. This gives

$$\frac{d^2}{dt^2}(x_1 - x_2) + (\omega_0^2 + 2\omega_s^2)(x_1 - x_2) = 0 \tag{5.5b}$$

By looking at Eqs. (5.5a) and (5.5b) you will recognise that these are standard equations for SHM. This suggests that if we introduce two new variables defined as

$$\xi_1 = x_1 + x_2 \quad (5.6a)$$

and

$$\xi_2 = x_1 - x_2 \quad (5.6b)$$

the motion of a coupled system can be described in terms of two uncoupled and independent equations:

$$\frac{d^2\xi_1}{dt^2} + \omega_1^2 \xi_1 = 0 \quad (5.7)$$

and

$$\frac{d^2\xi_2}{dt^2} + \omega_2^2 \xi_2 = 0 \quad (5.8)$$

where we have put

$$\omega_1^2 = \omega_0^2 = k'/m \quad (5.9)$$

and

$$\omega_2^2 = \omega_0^2 + 2\omega_s^2 = \frac{k'+2k}{m} \quad (5.10)$$

We therefore find that new co-ordinates ξ_1 and ξ_2 have decoupled Eqs, (5.2) and (5.4) into two independent equations which describe simple harmonic motions of frequencies ω_1 and ω_2 and $\omega_2 > \omega_1$. The new coordinates are referred to as *normal coordinates* and simple harmonic motion associated with each coordinate is called a *normal mode*. Each normal mode has its own characteristic frequency called the *normal mode frequency*.

5.2.2 Normal Coordinates and Normal Modes

The normal coordinates ξ_1 and ξ_2 are not a measure of displacement like ordinary co-ordinates x_1 and x_2. Yet they specify the configuration of a coupled system at any instant of time. Using the analysis of Unit I, you can readily write the general solution of Eqs. (5.7) and (5.8) as

$$\xi_1(t) = a_1 \cos(\omega_1 t + \phi_1) \quad (5.11)$$

and

$$\xi_2(t) = a_2 \cos(\omega_2 t + \phi_2) \quad (5.12)$$

where a_1 and a_2 are the amplitudes of normal modes and, ϕ_1 and ϕ_2 are their initial phases. Since $x_1(t) = (\xi_1 + \xi_2)/2$, we can write the displacement of mass *A* as

$$x_1(t) = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2)] \quad (5.13)$$

Similarly, we can write the displacement of the mass *B* as

$$x_2(t) = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) - a_2 \cos(\omega_2 t + \phi_2)] \tag{5.13}$$

The constants a_1, a_2, ϕ_1, ϕ_2 are fixed by the initial conditions. Once we know these, we can completely determine the motion of the coupled masses.

SAQ 1

Solve Eqs. (5.13) and (5.14) subject to the following initial conditions:

(A) $x_1(0) = a$, $x_2(0) = a$, $\left.\frac{dx_1}{dt}\right|_{t=0} = 0$ and $\left.\frac{dx_2}{dt}\right|_{t=0} = 0$

(B) $x_1(0) = a$, $x_2(0) = -a$, $\left.\frac{dx_1}{dt}\right|_{t=0} = 0$ and $\left.\frac{dx_2}{dt}\right|_{t=0} = 0$

On solving this SAQ you will observe that when both masses are initially given the same displacement to the right and then released, their displacements are equal, i.e. $x_1(t) = x_2(t)$, or $\xi_2 = 0$ at all times. The motion is described by Eq. (5.7) and the normal mode frequency is the same as that of the uncoupled masses. This means that coupling has no influence and both masses oscillate in phase. In this mode of vibration, the coupling spring is neither stretched nor compressed (and is as good as not being there), as shown in Fig. 5.2a.

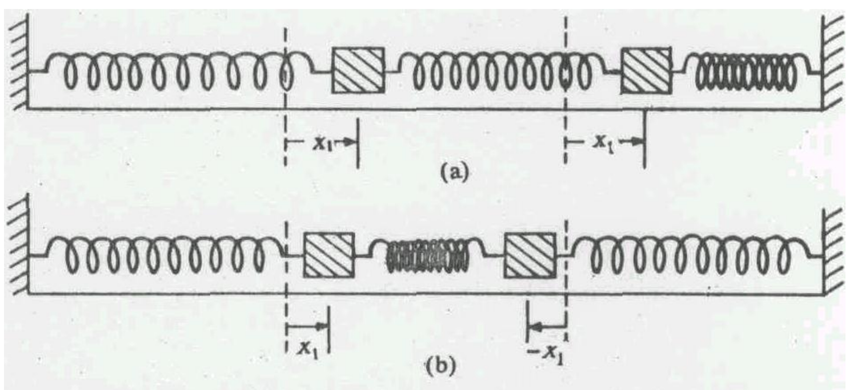


Fig. 5.2. Normal modes (a) When two coupled masses are given equal displacement in the same direction, (b)When two coupled masses are pulled together equally.

When two coupled masses are initially pulled, equally towards each other and then released, the displacements are equal but out of phase by π, i.e., $x_1 = -x_2$ or $\xi_1 = 0$ (Fig 5.2b). The normal mode frequency will be higher than that of the uncoupled masses $(\omega_2 > \omega_1)$. This means that the coupling spring is either compressed or stretched and we say that coupling is effective. We thus conclude that normal coordinates allow us to write the equation of motion of a coupled system into a set of linear differential equations with constant coefficients. Each equation contains only one dependent variable. Moreover, the motion of a coupled system may be regarded as a superposition of its possible normal modes.

5.2.3 Modulation of Coupled Oscillations

In the above discussion we assumed that the two coupled masses are pulled equally in the same direction or in opposite directions. What will happen if only one of them is pulled and then released? To understand this we have to solve Eqs. (5.13) and (5.14). Suppose the initial condition is as follows:

$$x_1(0) = 2a\,,\ x_2(0) = 0\,,\ \left.\frac{dx_1}{dt}\right|_{t=0} = 0 \text{ and } \left.\frac{dx_2}{dt}\right|_{t=0} = 0 \tag{5.15}$$

You will find that the displacements of two coupled masses are given by

$$x_1(t) = a(\cos\omega_1 t + \cos\omega_2 t) \tag{5.16a}$$

and

$$x_2(t) = a(\cos\omega_1 t - \cos\omega_2 t) \tag{5.16b}$$

Expressing the sum (difference) of two cosine functions into their product, these equations can be rewritten in a physically more familiar form:

$$x_1(t) = 2a\cos\left(\frac{\omega_2 - \omega_1}{2}\right)t\cos\left(\frac{\omega_2 + \omega_1}{2}\right)t \tag{5.17}$$

and

$$x_2(t) = 2a\sin\left(\frac{\omega_2 - \omega_1}{2}\right)t\sin\left(\frac{\omega_2 + \omega_1}{2}\right)t \tag{5.18}$$

You would recall that Eq. (5.17) is essentially the same as Eq. (2.19) obtained for modulated oscillations. As before, we define $\omega_{av} = (\omega_1 + \omega_2)/2$ as the average angular frequency and $\omega_{\text{mod}} = (\omega_2 - \omega_1)/2$ as the modulated angular frequency.

Then Eqs. (5.17) and (5.18) represent modulated oscillations respectively defined by

$$x_1(t) = a_{\text{mod}}(t)\cos\omega_{av}t \tag{5.19}$$

and

$$x_2(t) = b_{\text{mod}}(t)\sin\omega_{av}t \tag{5.20}$$

where

$$a_{\text{mod}} = 2a\cos\omega_{\text{mod}}t \tag{5.21}$$

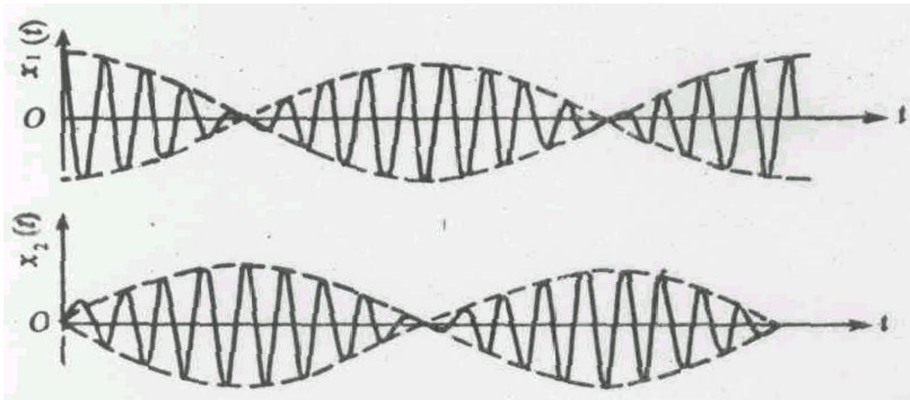
and

$$b_{\text{mod}} = 2a\sin\omega_{\text{mod}}t \tag{5.22}$$

are modulated amplitudes.

What is the phase difference between the displacements of the two masses? Since sine and cosine functions differ by $\pi/2$, the phase difference between the displacements of the coupled masses is $\pi/2$. The same is true of modulated amplitudes as well.

The displacement-time graphs for the two masses are shown in Fig. 5.3. We observe at $t = 0$, the amplitude of A is maximum while that of the mass at B is zero. With time, the amplitude of A decreases and becomes zero at $t = T/4$ while that of B becomes a maximum. After $t = T/4$, this trend is reversed for the succeeding quarter of the period. This process will repeat itself indefinitely provided damping is not present.



5.2.4 Energy of Two Coupled Masses

If the coupling between two masses is weak, ω_2 will be only slightly different from ω_1, so that ω_{mod} will be very small. Consequently a_{mod} and b_{mod} will take quite some time to show an observable change. That is, a_{mod} and b_{mod} will be practically constant over a cycle of angular frequency ω_{av}. Then Eqs. (5.19) and (5.20) can be regarded as characterising almost simple harmonic motion. Let us now calculate the energies of masses *A* and *B* using these equations.

We know that the energy of an oscillator executing SHM is given by

$$E_1 = \frac{1}{2} m\omega_{av}^2 a_{\text{mod}}^2(t) = 2ma^2\omega_{av}^2 \cos^2 \omega_{\text{mod}} t$$

and

$$E_2 = \frac{1}{2} m\omega_{av}^2 b_{\text{mod}}^2(t) = 2ma^2\omega_{av}^2 \cos^2 \omega_{\text{mod}} t$$

The total energy of two masses coupled through a spring which stores almost no energy is given by

$$E = E_1 + E_2 = 2ma^2\omega_{av}^2 \tag{5.24}$$

which remains constant with time.

Using Eq. (5.24), we can rewrite Eqs. (5.23a) and (5.23b) as

$$E_1 = \frac{E}{2}[1 + \cos(\omega_2 - \omega_1)t] \tag{5.25a}$$

and

$$E_2 = \frac{E}{2}[1 - \cos(\omega_2 - \omega_1)t] \tag{5.25b}$$

These equations show that at $t = 0$, $E_1 = E$ and $E_2 = 0$. That is, to begin with, mass at A possesses all the energy. As time passes, energy of mass at A starts decreasing. But mass at B begins to gain energy such that the total energy of the system remains constant.

When $(\omega_2 - \omega_1)t = \pi/2$, the two masses share energy equally. When $(\omega_2 - \omega_1)t = \pi$, $E_1 = 0$ and $E_2 = E$, i.e. mass *B* possesses all the energy. As time passes, the energy exchange process continues. That is, the total energy flows back and forth twice between two masses in time *T*, given by

$$T = 2\pi/(\omega_2 - \omega_1)$$

5.2.5 General Procedure for Calculating Normal Mode Frequencies

In most physical situations of interest, coupled masses may not be equal. Then the above analysis is not of much use; it has to be modified. To calculate normal mode frequencies in such cases, we follow the procedure outlined below:

(i) Write down the equation of motion of coupled masses;
(ii) Assume a normal mode solution;
(iii) Substitute it in the equations of motion and compare the ratios of normal mode amplitudes; and
(iv) Solve the resultant equation.

We now illustrate this procedure for two unequal masses m_1 and m_2 coupled through a spring of force constant *k*. The equations of motion of two coupled masses are

$$m_1\ddot{x}_1 = -k' x_1 + k(x_2 - x_1) \tag{5.26a}$$

and

$$m_2\ddot{x}_2 = -k' x_2 - k(x_2 - x_1) \tag{5.26b}$$

Let us assume solutions of the form

$$x_1 = a_1 \cos(\omega t + \phi)$$

and

$$x_2 = a_2 \cos(\omega t + \phi)$$

where ω is the angular frequency and ϕ is the initial phase.
Then

$$\ddot{x}_1 = -\omega^2 x_1$$

and

$$\ddot{x}_2 = -\omega^2 x_2$$

On substituting for x_1 and x_2 in Eqs.(5.26a) and (5.26b), we get

$$\left(\omega_0^2 + \frac{k}{m_1} - \omega^2\right)x_1 = \frac{k}{m}x_2 \tag{5.27a}$$

and

$$\left(\omega_0^2 + \frac{k}{m_2} - \omega^2\right) x_2 = \frac{k}{m} x_1 \tag{5.27b}$$

From Eq. (5.27a) we can write

$$\frac{x_1}{x_2} = \frac{k / m_1}{\left(\omega_0^2 + \dfrac{k}{m_1} - \omega^2\right)}$$

and from Eq. (5.27b), we have

$$\frac{x_1}{x_2} = \frac{\left(\omega_0^2 + \dfrac{k}{m_2} - \omega^2\right)}{k / m_2}$$

For non-zero values of x_1 and x_2, we can equate these values of x_1 / x_2 to obtain

$$\frac{k / m_1}{\left(\omega_0^2 + \dfrac{k}{m_1} - \omega^2\right)} = \frac{\left(\omega_0^2 + \dfrac{k}{m_2} - \omega^2\right)}{k / m_2}$$

On cross-multiplying, we get

$$\omega^4 - \left(2\omega_0^2 + \frac{k}{m_1} + \frac{k}{m_2}\right)\omega^2 + \left(\omega_0^2 + \frac{k}{m_1} + \frac{k}{m_2}\right)\omega_0^2 = 0$$

This equation is quadratic in ω^2 and has roots

$$\omega_1^2 = \omega_0^2 \tag{5.28a}$$

and

$$\omega_2^2 = \omega_0^2 + k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \tag{5.28b}$$

You will note that for $m_1 = m_2$, these results reproduce Eqs. (5.9) and (5.10).

5.3 NORMAL MODE ANALYSIS OF OTHER COUPLED SYSTEMS

So far we have analysed the motion of two coupled masses. This analysis can readily be extended to other systems of entirely different physical nature. We will first compute normal mode frequencies of two coupled simple pendulums by drawing analogies from the preceding analysis.

5.3.1 Two Coupled Pendulums

Let us consider two identical simple pendulums *A* and *B* having bobs of equal mass, m, and suspended by strings of equal length l, as shown in Fig. 5.4 (a). The bobs of the two pendulums are connected by a weightless, spring of force constant *k*. In the equilibrium position, the distance between the bobs is equal to the length of the unstretched spring.

Suppose that both bobs are displaced to the right from their respective equilibrium positions. Let $x_1(t)$ and $x_2(t)$ be the displacements of these bobs at time t, as shown in Fig. 5.4 (b). The tension in the coupling spring will be $k(x_1 - x_2)$. It opposes the acceleration of A but will support the acceleration of *B*. For small amplitude approximation, we recall from Unit 1 that the equation of motion of a simple pendulum is

$$m\frac{d^2x}{dt^2} = -\frac{mg}{l}x$$

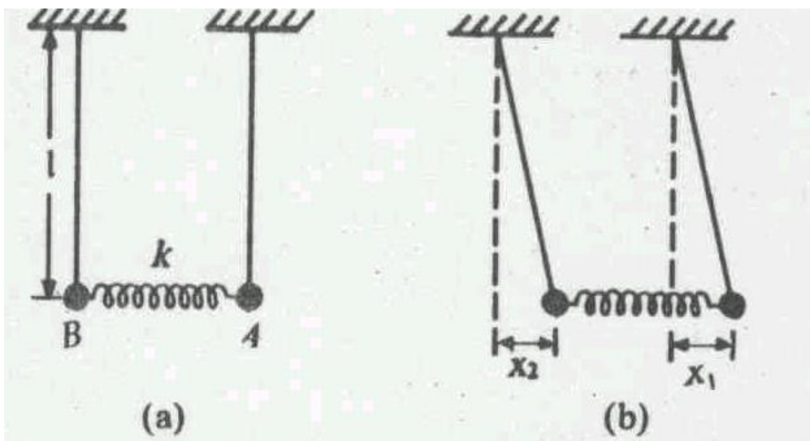


Fig. 5.4 Two identical pendulums simple together (a) Equilibrium configuration (b) Instantaneous configuration

In the present case, the equations of motion of bobs *A* and *B* are

$$m\frac{d^2x_1}{dt^2} = -\left(\frac{mg}{l}\right)x_1 - k(x_1 - x_2)$$

and

$$m\frac{d^2x_2}{dt^2} = -\left(\frac{mg}{l}\right)x_2 + k(x_1 - x_2)$$

The term $\pm k(x_1 - x_2)$ arises due to the presence of coupling. Dividing throughout by *m* and rearranging terms, we get

$$\frac{d^2 x_1}{dt^2} + -\omega_0^2\, x_1 - \omega_s^2 (x_1 - x_2) = 0 \tag{5.29a}$$

and

$$\frac{d^2 x_2}{dt^2} + -\omega_0^2\, x_2 - \omega_s^2 (x_1 - x_2) = 0 \tag{5.29b}$$

where we have substituted $\omega_0^2 = g/l$ and $\omega_s^2 = k/m$.

You will recognize that these equations are respectively identical to Eqs. (5.2) and (5.4). Thus the entire analysis of preceding sections applies and we can describe the motion of coupled pendulums by drawing analogies. The normal modes of this system are shown in Fig. 5.5. In mode 1 $(x_1 = x_2)$, the bobs are in phase and oscillate with frequency $\omega_1 = \omega_0 = \sqrt{g/l}$. But in mode 2 $(x_1 = -x_2$ or $x_2 = -x_1)$, the bobs are in opposite phase and oscillate with frequency $\omega_2 = [\omega_0^2 + 2\omega_s^2]^{1/2} = [(g/l) + 2(k/m)]^{1/2}$.

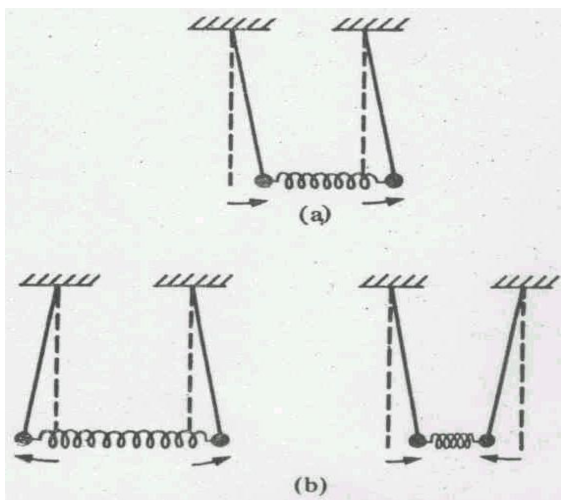


Fig. 5.5 Normal modes of a coupled pendulum (a) In-phase normal mode (b) Out-of-phase normal mode.

SAQ 2

The kinetic and potential energies of two identical coupled pendulums are given by

$$K.E. = \frac{m}{2}[(\dot{x}_1)^2 + (\dot{x}_2)^2]$$

and

$$U = \frac{1}{2}\left(\frac{mg}{l}\right)(x_1^2 + x_2^2) + \frac{1}{2}k(x_1 - x_2)^2$$

where $\dot{x}_i = \dfrac{dx_i}{dt}$ (i = 1,2). Express these in terms of normal coordinates.

On solving this SAQ, you will observe that

$$K.E. = \frac{m}{4}(\xi_1)^2 + \frac{m}{4}(\xi_2)^2$$

and

$$U = \frac{1}{4}(m\omega_1^2\xi_1^2 + m\omega_2^2\xi_2^2)$$

We can rewrite these expressions in a more elegant form by defining normal coordinates as

$$\xi_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \text{ and } \xi_2 = \sqrt{\frac{m}{2}}(x_1 - x_2) \qquad (5.30)$$

SAQ 3

Using the definition given in Eq. (5.30), calculate the total energy of a system of two coupled pendulums in terms of normal coordinates. At any instant, $\xi_1 = 1.5\times10^{-3} m\sqrt{kg}$ and $\xi_2 = 0.5\times10^{-3} m\sqrt{kg}$. Calculate $x_1(t)$ and $x_2(t)$ at the same instant. Given *m* = 0.1 kg.

In the above discussion you have learnt to calculate normal mode frequencies of two pendulums whose bobs are coupled. Will these frequencies remain the same if the strings of these pendulums were coupled, as shown in Fig. 5.6 (a). To discover the answer to this question, we consider the configuration shown in Fig 5.6 (b). At any time t, let the change in the length of the spring be $x_2' - x_1' = (d/l)(x_2 - x_1)$, where x_1 and x_2 are displacements of the bob from their equilibrium positions and d is the distance between the points of suspension and coupling. Hence, the restoring force in the spring is given by

$$F = k(x_2' - x_1') = \frac{kd}{l}(x_2 - x_1)$$

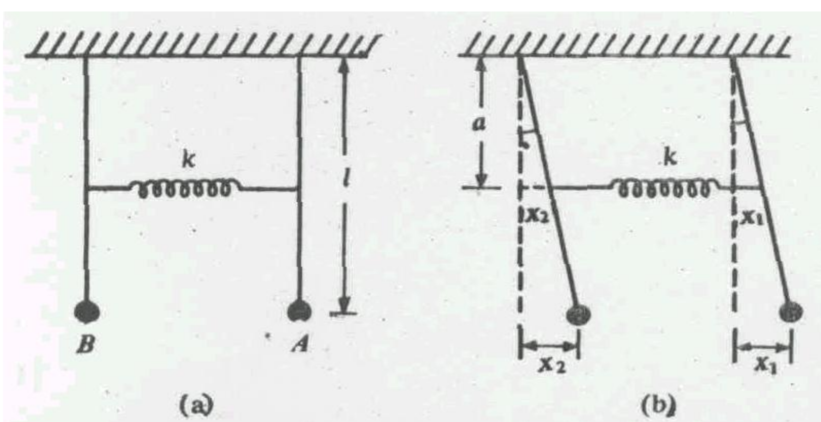


Fig. 5.6 Two identical simple pendulums whose strings are coupled by a spring (a) Equilibrium configuration (b) Instantaneous configuration

SAQ 4

Write down the equations of motion of two pendulums coupled at a distance *d* from the point of suspension. Compute the normal mode frequencies by analogy. You will find that the frequencies of the normal modes are given by

$$\omega_1 = \sqrt{\frac{g}{l}}$$

and

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2kd^2}{ml^2}}$$

This shows that ω_2 depends on the distance between the points of suspension and coupling. Obviously, for $d = l$, the expression for ω_2 reduces to that for coupled bobs.

5.3.2 Inductively Coupled LC circuits

In Unit 1 we learnt that in an LC-circuit, charge oscillates back and forth with a frequency *vo* — 1 / *(2ir \^LC)*. The form of energy repeatedly changes from electric to magnetic and vice versa. If two such circuits are coupled, we expect that some energy will foe exchanged between them. This study finds important applications in areas of power transmission and radio reception. Let us therefore consider two LC-circuits, as show in Fig. 5.7. Do you know as to how these circuits are coupled? These circuits are coupled inductively. This coupling forms the basis of operation of a voltage transformer as well as an oscillator.

> Two electrical circuits are said to be inductively coupled when a change in the magnetic flux linked with one circuit induces emf (and hence gives rise to a current) in the other circuit. The coupling coefficient is given by $\mu = M / \sqrt{L_1 L_2}$, where M is mutual inductance and L_1 and L_2 are self-inductances of two coupled circuits.

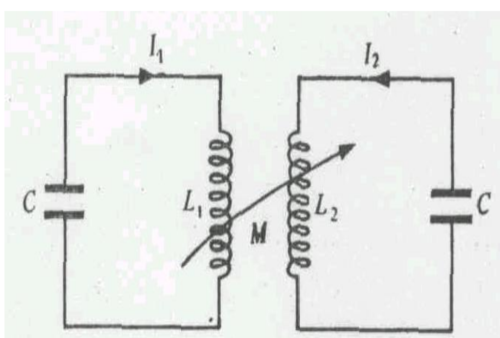


Fig. 5.7 Two inductively coupled identical LC circuits.

Let I and I_2 be the instantaneous values of currents in the two circuits. The equation of motion of charge in circuit 1 is obtained by modifying Eq. (1.36) as

$$\frac{q_1}{C_1} = -L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

where MdI_2 / dt is the emf produced in circuit 1 due to current I_2 in the second circuit. Obviously, it tends to increase I_1. Similarly, for circuit 2 we can write

$$\frac{q_2}{C_2} = -L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}$$

Since $I = \frac{dq}{dt}$ and $\frac{dI}{dt} = \frac{d^2 q}{dt^2}$, we can rewrite these equations as

$$\frac{d^2 q_1}{dt^2} + \omega_p^2 q_1 = \frac{M}{L_1} \frac{d^2 q_2}{dt^2} \tag{5.31a}$$

and

$$\frac{d^2 q_2}{dt^2} + \omega_s^2 q_2 = \frac{M}{L_2} \frac{d^2 q_1}{dt^2} \tag{5.31b}$$

Eqs. (5.31a) and (5.3Ib) are two coupled equations. To find normal mode frequencies, we write

$$q_1 = A\cos(\omega t + \phi)$$

Using these in Eqs. (5.31a) and (5.31b), we get

$$(\omega_p^2 - \omega^2) q_1 = -\frac{M}{L_1} \omega^2 q_2$$

and

$$(\omega_s^2 - \omega^2) q_2 = -\frac{M}{L_2} \omega^2 q_1$$

Equating the values q_1 / q_2 obtained from these equations, we have

$$\frac{M}{L_1} \frac{\omega^2}{(\omega_p^2 - \omega^2)} = \frac{L_2}{M} \frac{(\omega_s^2 - \omega^2)}{\omega^2}$$

This expression may be rearranged as

$$(\omega_p^2 - \omega^2)(\omega_s^2 - \omega^2) = \frac{M^2}{L_1 L_2} \omega^4 = \mu^2 \omega^2 \tag{5.32}$$

where μ is the coupling coefficient.

Eq. (5.32) is quadratic in ω^2; its roots give us normal mode frequencies. For simplicity, we assume that the circuits are identical so that their natural frequencies are equal, i.e. $\omega_p = \omega_s = \omega_0$, say, then

$$(\omega_0^2 - \omega^2)^2 = \mu^2 \omega^4$$

or

$$\omega_0^2 - \omega^2 = \pm \mu \omega^2$$

so that

$$\omega = \pm \frac{\omega_0}{\sqrt{1-\mu}}$$

The acceptable normal mode frequencies are those values of ω which correspond to positive roots and are given by

$$\omega = \frac{\omega_0}{\sqrt{1-\mu}} \tag{5.33a}$$

and

$$\omega = -\frac{\omega_0}{\sqrt{1-\mu}} \tag{5.33b}$$

When coupling is weak $(\mu << 1)$, $\omega_1 \approx \omega_2 \approx \omega_0$ and the two circuits behave as essentially independent. But when coupling is strong, ω_1 and ω_2 will be much different.

SAQ 5
Two identical inductively coupled circuits, each having a natural frequency of 600 Hz, have coupling coefficient 0.44. Calculate the two normal mode frequencies.

5.4 LONGITUDINAL OSCILLATIONS OF N COUPLED MASSES: THE WAVE EQUATION

We know that every fluid or a solid contains more than two coupled atoms held by intermolecular forces. To know normal modes of such a system we have to extend the preceding analysis to three or in general to *N* coupled oscillators, which may not all be of the same mass.

For simplicity, we first consider a system of *N* identical masses held together by $(N+1)$ identical springs, each of force constant *k,* as shown in Fig. 5.8. The free ends of the system are rigidly fixed at $x = 0$ and $x = l$. In the equilibrium state, the masses are situated at $x = a, 2a, ...,$ Na so that $l = (N + 1)\ d$. If ψ_{n-1}, ψ_n and ψ_{n+1} are respective displacements of $(n-1)$th, nth and $(n+1)$th masses from their mean positions, we can write the equation of motion of the nth mass as

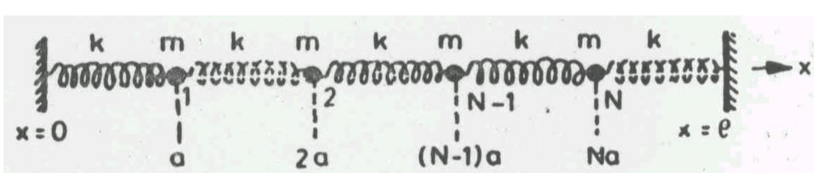


Fig. 5.8 Equilibrium configuration for N coupled masses

$$m\frac{d^2\psi_n}{dt^2} = -k(\psi_n - \psi_{n-1}) - k(\psi_n - \psi_{n-1}) \tag{5.34}$$

From Unit 1 you may recall that the spring constant *k* is defined as restoring force per unit extension. So we can write *k* = *F/d*, where *d* is extension in the spring. Using this relation in Eq. (5.34), we get

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left[\left(\frac{\psi_{n+1} - \psi_n}{d}\right) - \left(\frac{\psi_n - \psi_{n-1}}{d}\right)\right] \tag{5.35}$$

Now let us assume that $N \to \infty$. The separation between any two consecutive masses will become very small. That is, we will have a continuous distribution of masses. Then we can replace *d* by Δx and Eq. (5.35) becomes

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left[\left(\frac{\psi_{n+1} - \psi_n}{\Delta x}\right) - \left(\frac{\psi_n - \psi_{n-1}}{\Delta x}\right)\right]$$

or

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left[\left(\frac{\delta\psi}{\delta x}\right)_{n+1} - \left(\frac{\delta\psi}{\delta x}\right)_n\right] \tag{5.36}$$

If the n th mass is located at a distance *x* from the origin, then in the limit $\Delta x \to 0$, we have

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left[\left(\frac{d\psi}{dx}\right)_{n+1} - \left(\frac{d\psi}{dx}\right)_n\right] \tag{5.37}$$

This means that ψ is now a function of *t* a well as x. We know that any continuous function $f(x + \Delta x)$ can be expressed in terms of the function defined at x and its derivatives using the following expansion:

$$f(x + \Delta x, t) = f(x,t) + \frac{\partial f(x,t)}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 f(x,t)}{\partial x^2}(\Delta x)^2 + ... \tag{5.38}$$

Taking $d\psi / dx$ as f and retaining terms only up to first order in Ax, we can write

$$\left.\frac{\partial\psi}{\partial x}\right|_{x+\Delta x} = \left.\frac{\partial\psi}{\partial x}\right|_x + \frac{\partial^2\psi}{\partial x^2}\Delta x + ... \tag{5.39}$$

so that terms within the square brackets in Eq. (5.37) can be rewritten as

$$\left.\frac{\partial\psi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial\psi}{\partial x}\right|_x = \frac{\partial^2\psi}{\partial x^2}\Delta x \tag{5.40}$$

Using this result in Eq. (5.37) and re-arranging terms, we get

$$\frac{\partial^2\psi}{\partial t^2} = \frac{F}{\rho l}\frac{\partial^2\psi}{\partial x^2}$$

where $\rho l = m/\Delta x$. You will observe that Eq. (5.41) is a partial differential equation. Moreover, the quantity $F/\rho l$ has the dimensions of the square of velocity. For this reason, this equation is referred to as the *wave equation.*

We thus find that longitudinal motion of a large number of coupled masses results in the phenomenon of wave propagation. We obtain a similar equation for a system of large number of coupled masses executing transverse oscillations. These will be discussed in detail later.

5.5 SUMMARY

- The longitudinal oscillations of two (identical) coupled masses are nut simple harmonic. The resultant amplitudes of coupled masses resemble a modulated pattern.

- The displacement of either of the two (identical) coupled masses can be regarded as superposition of normal modes of the system. Each mode represents .an independent SHM. The normal mode frequencies are given by

  $$\omega_1 = \sqrt{\frac{k'}{m}} \text{ and } \omega_2 = \sqrt{\frac{k'+2k}{m}}$$

  where k and k' are spring constants.

- The total energy of two identical coupled masses is given by $E = 2ma^2\omega_{av}^2$. It flows back and forth twice between the masses in time given by

  $$T = \frac{2\pi}{\omega_2 - \omega_1}$$

- The normal mode frequencies of a system of two coupled pendulums are given by

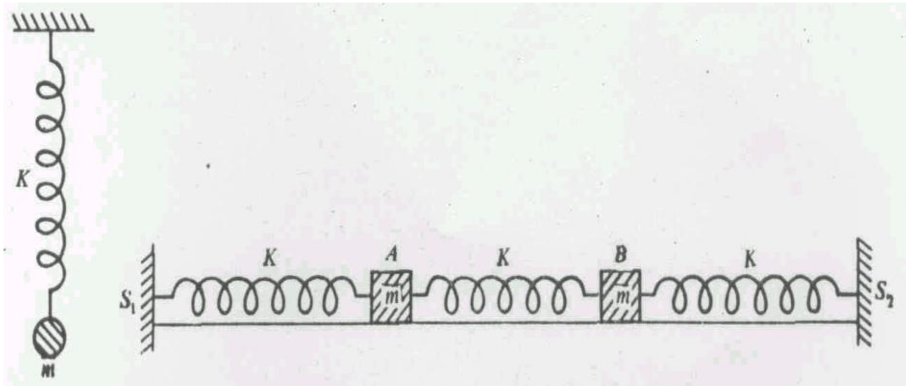
  $$\omega_1 = \sqrt{\frac{g}{l}} \text{ and } \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

- The Energy-exchange during longitudinal oscillations of # coupled masses leads to the propagation of a wave in the limit $N \to \infty$. The wave equation is given by

  $$\frac{\partial^2\psi}{\partial t^2} = \frac{F}{\rho l}\frac{\partial^2\psi}{\partial x^2}$$

5.6 TERMINAL QUESTIONS

1. An object of mass *m* is suspended to a rigid support with the help of a spring of force constant *K*. It vibrates with a frequency 2 Hz (Fig. 5.9a.). Next two identical objects *A* and *B*, each of mass *m*, are joined together by a spring of force constant *K'*. Then these are connected to rigid supports S_1 and S_2 by two identical springs, each of force constant *K* (Fig. 5.9b). Now, if *A* is clamped, *B* vibrates with a frequency 2.5 Hz. Calculate the frequencies of the two modes of vibration.



(a) (b)

Fig. 5.9

2. Two equal masses (m) are connected to each other with the help of a spring of force constant K and then the upper mass is connected to a rigid support by an identical spring as shown in Fig. 5.10. The system is made to oscillate in the vertical direction. Show that the two normal frequencies are given by

$$\omega^2 = (3 \pm \sqrt{5})\frac{K}{2m}$$

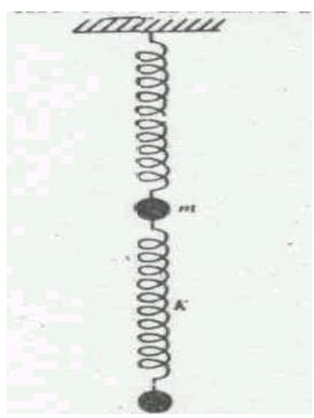


Fig. 5.10

3. Consider two capacitively coupled circuits shown in Fig. 5.11. Write down the equations of motion for current and compute normal mode frequencies.

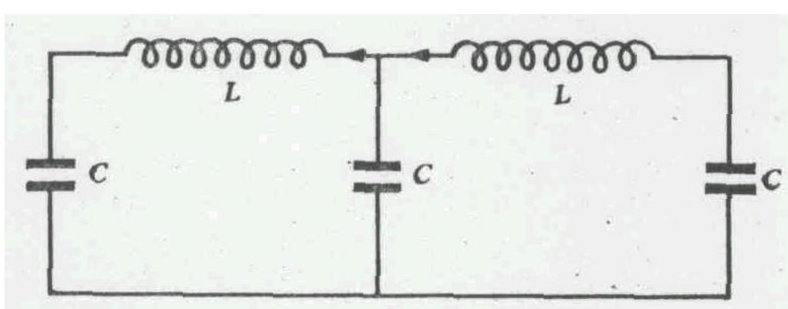


Fig. 5.11

5.7 SOLUTIONS

SAQ 1

$$x_1 = \frac{\xi_1 + \xi_2}{2} = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2)] \qquad \text{(i)}$$

and

$$x_2 = \frac{\xi_1 - \xi_2}{2} = \frac{1}{2}[a_1 \cos(\omega_1 t + \phi_1) - a_2 \cos(\omega_2 t + \phi_2)] \qquad \text{(ii)}$$

$$\frac{dx_1}{dt} = -\frac{1}{2}[a_1\omega_1 \sin(\omega_1 t + \phi_1) + a_2\omega_2 \sin(\omega_2 t + \phi_2)] \qquad \text{(iii)}$$

$$\frac{dx_2}{dt} = \frac{1}{2}[-a_1\omega_1 \sin(\omega_1 t + \phi_1) + a_2\omega_2 \sin(\omega_2 t + \phi_2)] \qquad \text{(iv)}$$

(A) Using the initial conditions, we get

$$2a = a_1 \cos\phi_1 + a_2 \cos\phi_2,\ 2a = a_1 \cos\phi_1 - a_2 \cos\phi_2$$

and

$$0 = a_1\omega_1 \sin\phi_1 + a_2\omega_2 \sin\phi_2,\ 0 = a_1\omega_1 \sin\phi_1 - a_2\omega_2 \sin\phi_2$$

Hence,

$$a_1 \cos\phi_1 = 2a,\ a_2 \cos\phi_2 = 0,\ a_1\omega_1 \sin\phi_1 = 0,\ a_2\omega_2 \sin\phi_2 = 0$$

As a_1, a_2, ω_1, ω_2 are not equal to zero,

$$\phi_1 = \phi_2 = 0,\ a_1 = 2a,\ a_2 = 0$$

$$\therefore \quad x_1 = a\cos\omega_1 t,\ x_2 = a\cos\omega_1 t$$

That is, $\xi_1 = a\cos\omega_1 t$ and $\xi_2 = 2a\cos\omega_2 t$

SAQ 2

At the displacements x_1 and x_2, the speeds of pendulums A and B will be $dx_1/dt = \dot{x}_1$ and $dx_2/dt = \dot{x}_2$ respectively. Therefore,

Kinetic energy $K.E. = (1/2)m(\dot{x}_1)^2 + (1/2)m(\dot{x}_2)^2$ (i)

Potential energy $P.E. = (1/2)m\omega_0^2(x_1^2 + x_2^2) + (1/2)m\omega_s^2(x_1 - x_2)^2$ (ii)

Where $\omega_0^2 = \dfrac{g}{l}$ and $\omega_s^2 = \dfrac{k}{m}$

From Eqs. (5.6a) and (5.6b), we have

$$x_1 = \frac{\xi_1 + \xi_2}{2} \text{ and } x_2 = \frac{\xi_1 - \xi_2}{2}$$

Hence,

$$\dot{x}_1 = \frac{\dot{\xi}_1 + \dot{\xi}_2}{2} \text{ and } \dot{x}_2 = \frac{\dot{\xi}_1 - \dot{\xi}_2}{2}$$

$$\therefore \quad K.E. = \frac{1}{8}m(\dot{\xi}_1 + \dot{\xi}_2)^2 + \frac{1}{8}m(\dot{\xi}_1 - \dot{\xi}_2)^2$$

$$= \frac{m}{4}[(\dot{\xi}_1)^2 + (\dot{\xi}_2)^2] \qquad \text{(iv)}$$

$$\boxed{[(a+b)^2 + (a-b)^2 = 2a(a^2 + b^2)]}$$

and

$$P.E. = \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2) + \frac{1}{2}m\omega_0^2(x_1^2 - x_2^2)$$

$$= \frac{1}{4}m\omega_0^2(\xi_1^2 + \xi_2^2) + \frac{1}{2}m\omega_s^2\xi_2^2$$

$$= \frac{m}{4}(\omega_0^2\xi_1^2 + \omega_0^2\xi_2^2) \qquad \text{(v)}$$

where $\omega_2^2 = \omega_0^2 + 2\omega_s^2$

SAQ 3

If normal coordinates are defined as

$$\xi_1 = \sqrt{\frac{m}{2}}(x_1 + x_2) \text{ and } \xi_2 = \sqrt{\frac{m}{2}}(x_1 - x_2)$$

we can write

$$x_1 = \frac{1}{\sqrt{2m}}(\xi_1 + \xi_2) \text{ and } x_2 = \frac{1}{\sqrt{2m}}(\xi_1 - \xi_2)$$

Substituting in (i) and (ii) of SAQ 2, we get

$$K.E. = \frac{1}{2}[(\dot{\xi}_1)^2 + (\dot{\xi}_2)^2] \qquad \text{(i)}$$

and

$$P.E. = \frac{1}{2}(\omega_1^2\xi_1^2 + \omega_2^2\xi_2^2) \qquad \text{(ii)}$$

Hence, the total energy of the coupled pendulum in terms of normal coordinates is

$$E = \frac{1}{2}[(\dot{\xi}_1)^2 + (\dot{\xi}_2)^2 + \omega_1^2\xi_1^2 + \omega_2^2\xi_2^2]$$

$$x_1 = \sqrt{\frac{1}{0.2kg}}(1.50 + 0.50)\times 10^{-3}\, kg^{1/2} m = 4.5\times 10^{-3}\, m$$

$$x_2 = \sqrt{\frac{1}{0.2kg}}(1.50 - 0.50)\times 10^{-3}\, kg^{1/2} m = 2.2\times 10^{-3}\, m$$

SAQ 4

The equations of motion of the coupled pendulums *A* and *B* are

$$\ddot{x}_1 = -\frac{g}{l}x_1 + \frac{ka^2}{ml^2}(x_2 - x_1)$$

and

$$\ddot{x}_2 = -\frac{g}{l}x_2 - \frac{ka^2}{ml^2}(x_2 - x_1)$$

By comparing these equations with Eqs. (5.29a) and (5.29b), we get

$$\omega_1^2 = \omega_0^2 = \frac{g}{l}$$

and

$$\omega_2^2 = \omega_0^2 + 2\omega_s^2 = \left(\frac{g}{l} + \frac{2ka^2}{ml^2}\right)$$

SAQ 5

From Eqs. (5.33a) and (5.33b), we have

$$\omega_1 = \frac{\omega_0}{\sqrt{1+\mu}}$$

and

$$\omega_2 = \frac{\omega_0}{\sqrt{1-\mu}}$$

Here, ω_0 = 600 Hz and μ = 0.44 so that

$$\omega_1 = \frac{600}{\sqrt{1.44}} \text{ Hz} = 500 \text{ Hz}$$

and

$$\omega_2 = \frac{600}{\sqrt{0.56}} \text{ Hz} = 802 \text{ Hz}$$

Terminal Questions

1. $\nu_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{K}{m}} = \dfrac{\omega_0}{2\pi} = 2$ Hz

$$\therefore \quad \frac{k}{m} = \omega_0^2 = 4\pi^2\nu_0^2 = 4\pi^2 \times 2^2 = 16\pi^2 \,(\text{Hz})^2 \qquad \text{(i)}$$

When A is clamped, the equation of motion of B will be given by

$$m\frac{d^2x_B}{dt^2} = -(K+K')x_B$$

or

$$\frac{d^2x_B}{dt^2} + \left(\frac{K+K'}{m}\right)x_B = 0$$

The frequency of this simple harmonic motion is given by

$$\nu_B = \frac{1}{2\pi}\sqrt{\frac{K}{m}+\frac{K'}{m}} = 2.5 \text{ Hz}$$

$$\therefore \quad \frac{K}{m}+\frac{K'}{m} = 4\pi^2\nu_B^2 = 4\pi^2(2.5)^2\,(\text{Hz})^2 = 25\pi^2\,(\text{Hz})^2 \qquad \text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get

$$\frac{K'}{m} = 9\pi^2\,(\text{Hz})^2$$

Now, the angular frequencies of two normal modes of vibration are given by

$$\omega_1^2 = 4\pi^2\nu_1^2 = \omega_0^2 = \frac{K}{m} = 16\pi^2\,(\text{Hz})^2$$

or ν_1 = 2 Hz

$$\omega_2^2 = 4\pi^2\nu_2^2 = \frac{K}{m} + \frac{2K'}{m} = (16\pi^2 + 18\pi^2)\,(\text{Hz})^2 = 34\pi^2\,(\text{Hz})^2$$

$$\nu^2 = \sqrt{\frac{17}{2}}\ \text{Hz} = 2.9\ \text{Hz}$$

2. Equations of motion of mass *A* and *B* are:

$$m\frac{d^2x_1}{dt^2} = -K(x_1 - x_2)$$

and

$$m\frac{d^2x_2}{dt^2} = -Kx_2 - K(x_2 - x_1)$$

Hence,

$$m\frac{d^2x_2}{dt^2} + \frac{K}{m}x_1 - \frac{K}{m}x_2 = 0 \qquad \text{(i)}$$

and

$$m\frac{d^2x_2}{dt^2} + \frac{K}{m}x_1 + \frac{2K}{m}x_2 = 0 \qquad \text{(ii)}$$

Let us assume that

$$x_1 = A_1\cos(\omega t + \phi)$$

and

$$x_2 = A_2\cos(\omega t + \phi)$$

Then,

$$\ddot{x}_1 = -\omega^2 x_1$$

and

$$\ddot{x}_2 = -\omega^2 x_2$$

Using these results in Eqs. (i) and (ii) we get

$$\left(\frac{K}{m} - \omega^2\right)\ddot{x}_1 - \frac{K}{m}x_2 = 0$$

and

$$\left(\frac{2K}{m} - \omega^2\right)\ddot{x}_2 - \frac{K}{m}x_1 = 0$$

For non-zero values of x_1 and x_2, this set of simultaneous equations can be solved for normal mode frequencies by equating the following determinant to zero:

$$\begin{vmatrix} \frac{K}{m} - \omega^2 & -\frac{K}{m} \\ -\frac{K}{m} & \frac{2K}{m} - \omega^2 \end{vmatrix}$$

Hence,

$$\omega^4 - \frac{3K}{m}\omega^2 + \frac{K^2}{m^2} = 0$$

$$\therefore \quad \omega^2 = \frac{3K}{2m} \pm \frac{\sqrt{5}K}{2m}$$

Thus, $\omega^2 = (3-\sqrt{5})\dfrac{K}{2m}$ for the slower mode.

and

$$\omega^2 = (3+\sqrt{5})\frac{K}{2m} \text{ for the faster mode.}$$

3. Refer to Fig. 5.11. The equations governing the balance of emf's in two circuits are

$$L\frac{di_1}{dt} = -\frac{q_1}{C} + \frac{q_3}{C}$$

and

$$L\frac{di_2}{dt} = -\frac{q_2}{C} + \frac{q_3}{C}$$

Differentiating these equations with respect to time and using the relation $i = dq/dt$, we get

$$L\frac{d^2 i_1}{dt^2} = -\frac{1}{C}i_1 + \frac{1}{C}(i_2 - i_1)$$

and

$$L\frac{d^2 i_1}{dt^2} = -\frac{1}{C}i_2 - \frac{1}{C}(i_2 - i_1)$$

If we replace L by m, $1/C$ by $k' = k$ and i by x, then these equations become identical to Eqs. (5.1) and (5.3). Hence, the two normal frequencies of the system are:

$$\omega_1 = \sqrt{\frac{1}{LC}}$$

and

$$\omega_2 = \sqrt{\frac{3}{LC}}.$$

UNIT 6 WAVE MOTION

Structure

6.1 INTRODUCTION

In Unit 5 you have learnt that when one mass in a system of *N* coupled masses is disturbed, the disturbance is gradually felt by all other masses. You can think of many other similar situations in which oscillations at one place are transmitted to some other place through the intervening medium. When we talk, our vocal cord inside the throat vibrates. It causes air molecules to vibrate and the effect-speech is transmitted. When it makes our ear drum to vibrate, it is heard. Do you know what carries the audio information? The information is carried by a (sound) *wave* which propagates through the medium (air). If you have ever stood at a seashore, you would need no description of waves.

In addition to sound and water waves, other familiar types of waves are: ultrasound waves and electromagnetic waves, which include visible light, radio waves, microwaves and x-rays. Matter waves, shock waves, and seismic waves are other less familiar but important types of waves. You will note that all our communications depend on transmission of signals through waves. The use of x-rays in medical diagnosis is so very well known. Nowadays we also use ultrasound waves – sound waves of frequency greater than 20 kHz – to make images of soft tissues in the interior of humans. Sound waves are used in sound ranging; sonars and prospecting for mineral deposits and oil – commodities governing the economy of nations these days. This means that understanding of the physics of wave motion is of fundamental importance to us. In this unit we will confine to mechanical waves with particular reference to sound waves.

When a progressive wave reaches the boundary of a finite medium or an interface between two media, waves undergo reflection and/or refraction. These will be discussed in detail in the next unit.

You would recall that our discussion of oscillations was simplified because of some basic similarities between different physical systems. Once we understood the behaviour of a model spring-mass system, we could easily draw analogies for others. Exactly the same simplification occurs in the study of waves. The basic description of a wave and the parameters required to quantify this description remain the same when we deal with a one-dimensional (1-D) wave travelling along a string, a 2-D wave on the surface of a liquid or a 3-D sound wave. For this reason, in this unit we shall first consider basic characteristics of wave motion. Then we would calculate the energy transported by progressive waves. The vocabulary language and ideas developed here will then be applied to waves on strings, liquids and gases.

Objectives

After going through this unit, you should be able to:

- define wave motion and state its characteristics
- distinguish between longitudinal and transverse waves
- represent graphically waves at a fixed position or at a fixed time
- relate wavelength, frequency and speed of a wave
- establish wave equations for longitudinal and transverse waves
- compute the energy transported by a progressive wave
- derive expressions for velocities of longitudinal and transverse waves
- derive expressions for characteristic impedance and acoustic impedance
- write two and three dimensional wave equations.

6.2 BASIC CONCEPTS OF WAVE MOTION

You may have enjoyed dropping small pebbles in still water. It will not take you long to convince yourself that water itself does not move with the wave (evidenced as circular disturbance). If you place a paper boat, a flower or a small piece of wood, you will observe that it bounces up and down, without any forward motion. You may ask: Why does the paper boat bounces up and down? It bounces due to the energy imparted by waves. Let us reconsider the motion of a system of *N* coupled masses (Fig. 6.1). If we disturb the first mass from its equilibrium position, individual masses gradually begin to oscillate about their respective equilibrium positions. That is, neither of the masses (or connecting springs) nor the system as a whole moves from its position. What moves instead is a wave, which carries energy. How can you say that? It is evidenced by compression and stretching of springs as the wave propagates. Thus the most important characteristic of wave motion is: *A wave transports energy but not matter.*

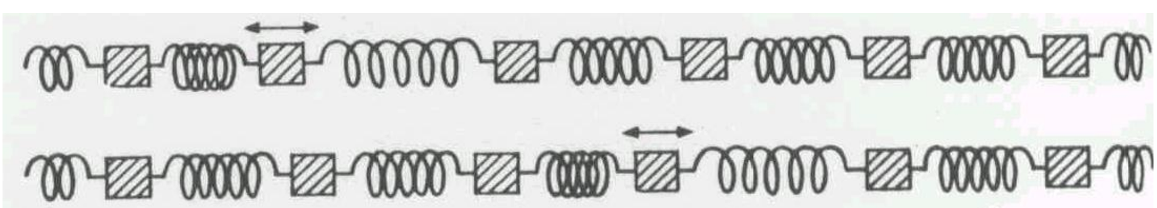


Fig. 6.1 The motion of a disturbed mass in the coupled spring-mass system. The disturbance is eventually communicated to adjacent masses. This results in wave propagation. You would note that regions of compression and elongation move along the system, which is shown here at two different times.

A vivid demonstration of the energy carried by water waves is in damage caused in coastal areas by tidal waves in stormy weather. Do you know that 3m high oceanic wave can lift 30 bags of wheat by about 10 ft.?

Another important characteristic of mechanical waves is their velocity of propagation, referred to as *wave velocity.* It is defined as the distance covered by a wave in unit time. It is different from the particle velocity, i.e. the velocity with which the particles of the medium vibrate to-and-fro about their respective equilibrium positions. Moreover, the wave velocity depends on the nature of the medium in which a wave propagates. A wave has a characteristic amplitude, wavelength and frequency. You must have learnt about these in your earlier classes. We will, however recapitulate these in sub-section 6.2.4.

You can see with unaided eyes the actual propagation of a disturbance in water. Can you see a sound wave propagating in air? Obviously, you cannot. Then you may like to know as to how we detect sound waves. We observe the motion at the source (like a guitar string or drum membrane) or at the receiver (microphone membrane). Another question that comes to our mind is: Are sound waves and water waves similar? If not, how are waves classified? Let us now proceed to know the answer to this question.

6.2.1 Types of Waves

In your school you must have learnt that waves can be classified as transverse or longitudinal depending upon the direction of vibration of particles relative to the direction of propagation of the wave. In fact, we can classify waves in many other ways. For instance, we have mechanical and non-mechanical waves depending on whether a wave needs a medium for propagation or not. Sound waves and water waves are mechanical (or elastic) waves whereas light waves are not. Waves can also be classified as one-, two- and three-dimensional waves, according to the number of dimensions in which they propagate energy. Waves on strings are one-dimensional (1-D). Ripples on water are two-dimensional (2-D). Sound waves and light waves originating from a small source are three-dimensional (3-D). Sometimes we classify waves as *plane waves* or spherical waves depending on the shape of the wavefront. In 2-D, a spherical wave appears circular, as in case of waves on the surface of water.

Waves set up by a single, isolated disturbance are called *pulses.* The dropping of a stone in still water of a pond, the sound produced by clapping of hands, a single word of greeting or command shouted from one person to another belong to this category. When an engine joins the compartments, the jerk produces a disturbance which is carried through as a pulse. But continuous and regular oscillations produce *periodic waves.* This, along with waveforms for sound produced by a violin and a piano, is shown in Fig. 6.2. The simplest type of a periodic wave is a harmonic wave.

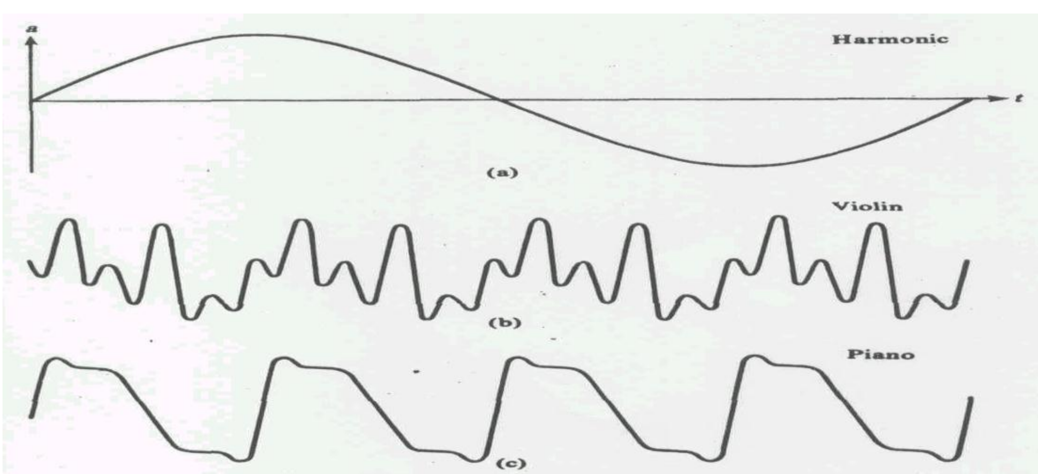


Fig. 6.2 Waveform for (a) harmonic wave (b) the violin and (c) piano

When the motion of particles of the medium is perpendicular to the direction in which the wave propagates, it is called a *transverse wave*. Waves on a string under tension are transverse, as in a violin. You can generate transverse waves on a coupled spring-mass system of Fig. 6.1 by displacing a mass at right angles to the spring as shown in Fig. 6.3a. (Electromagnetic waves are also transverse in nature. But they do not require medium for propagation.)

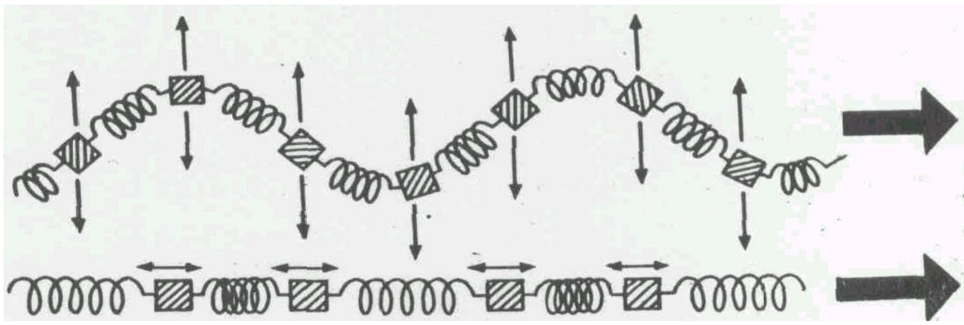


Fig. 6.3 (a) A transverse wave and (b) a longitudinal wave on a coiled spring-mass system. The broad arrow Indicates the direction of wave propagation.

When the motion of particles of the medium is along the direction in which wave propagates, the wave is called a *longitudinal wave*. Sound waves in air are the most familiar example of longitudinal waves. You can generate a longitudinal wave on a coupled spring-mass system of Fig. 6.1 by displacing a mass along the length of the spring (Fig. 6.3b).

In your school you may have been told that water waves, produced by winds or otherwise, are transverse and the motion is confined to the surface. But this is not correct. Strictly speaking, the motion gradually extends with diminishing amplitude to deeper layers. Moreover, oscillations have longitudinal as well as transverse components. That is, water waves are composite; partly transverse and partly longitudinal. This is illustrated in Fig. 6.4. Similar waves can occur at the surfaces of elastic solids. Such waves are called *Rayleigh waves.*

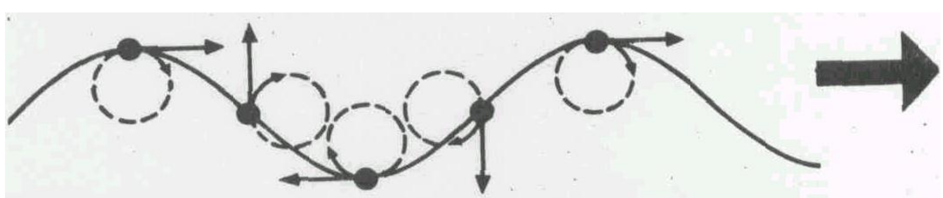


Fig. 6.4 Water waves are composite. The arrows at different points indicate instantaneous velocity of water whereas dotted circles are paths traced out by parcels of water as the wave passes. The direction of wave propagation is shown by the broad arrow.

In general, only longitudinal waves can propagate in gases and liquids but in solids both transverse and longitudinal waves can propagate.

Earlier in this course, you have read about torsional oscillations. When such a disturbance propagates in a medium, we have a *torsional wave*. In the following sections, you will learn about mechanical waves in general and sound waves in particular. But before that you should answer the following SAQ.

SAQ1

(i) The frequency of ultrasound wave is more than ....

(ii) Water waves are .... waves

(iii) Waves transfer.......... not ....

(iv) Light waves require .... medium

(v) Waves on sitar strings are ....

6.2.2 Propagation of Waves

To see how waves propagate in a medium, you can perform the following activity:

Take a long elastic string and fix its one end to a distant wall. Hold the other end tightly. Move your hand up and down. What do you observe? A disturbance travels along the string. This disturbance is due to the up and down motion of the particles of the string about their respective mean positions. When the motion of the arm (hand) is periodic, the disturbance on the string is a wave with a sinusoidal profile. The shape of a portion of the string at intervals of $T/8$ is shown in Fig. 6.5. You will observe that the waveform moves to the right, as shown by the broad arrow. You may ask: Why is the whole string not displaced simultaneously? The time lag between different parts is due to gradual transfer of disturbance between successive particles.

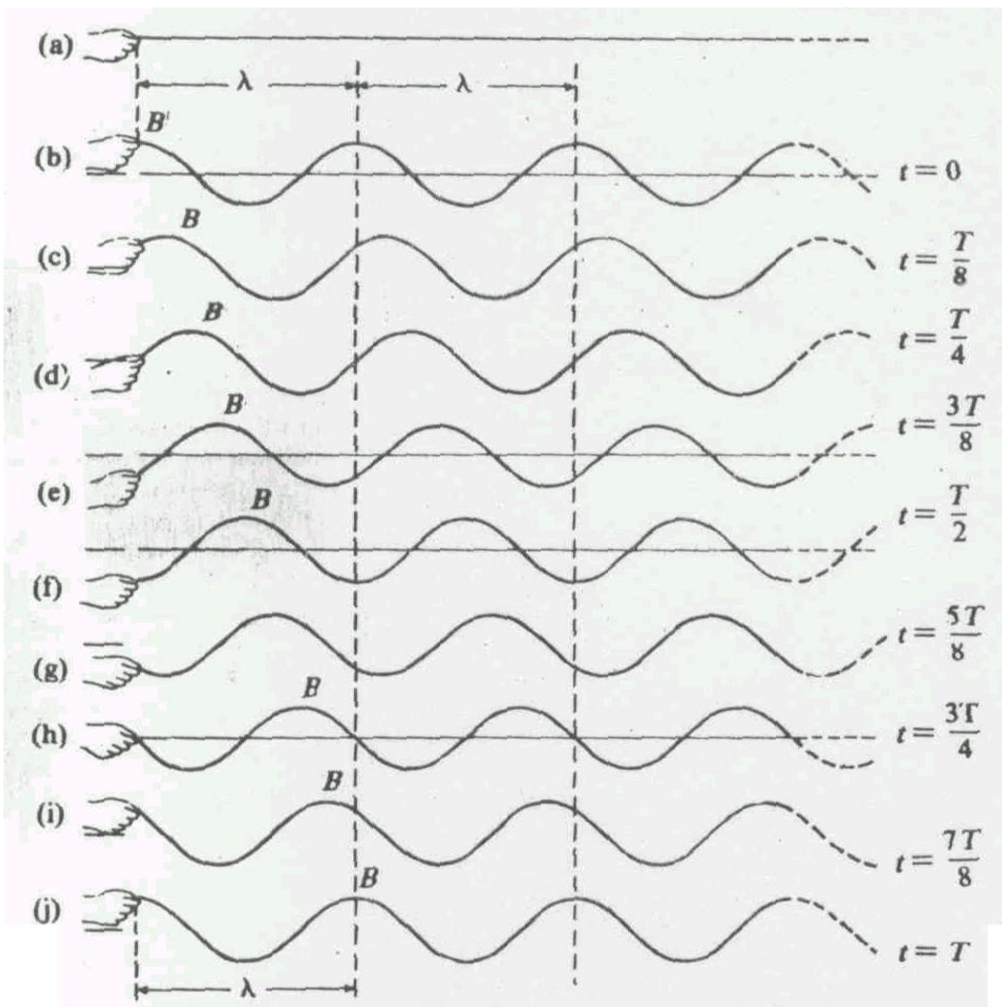


Fig. 6.5 A periodic motion of the hand generates waves with *a* sinusoidal profile. In parts (b)-(j) we have depicted wave profile on the string at intervals of $T/8$

In this connection, it is important to distinguish between the motion of the waveform and the motion of a particle of the string. While the waveform moves with a constant speed, the particles of the string execute SHM. To illustrate this difference clearly, let us mark nine equidistant points on the initial portion of the string. We assume that this string oscillates with a period *T*. Let us tie one end of this string (at mark 1) to a vertically oscillating spring-mass system as shown in Fig. 6.6. As the mass *m* on the spring moves up and down, the particles at the marked positions begin to oscillate one after the other. In time /; the disturbance initiated at the first particle will reach the ninth particle. This means that in the interval $T/8$, the disturbance propagates from the particle at mark 1 to the particle at mark 2. Similarly, in the next 778 interval, the disturbance travels from the panicle at mark 2 to the particle at mark 3 and so on. In parts (a) - (i) of Fig. 6.7 we have shown the instantaneous positions of particles at all nine marked positions at intervals of 778. (The arrows indicate the directions of motion along with which particles at various marks are about to move.) You will observe that

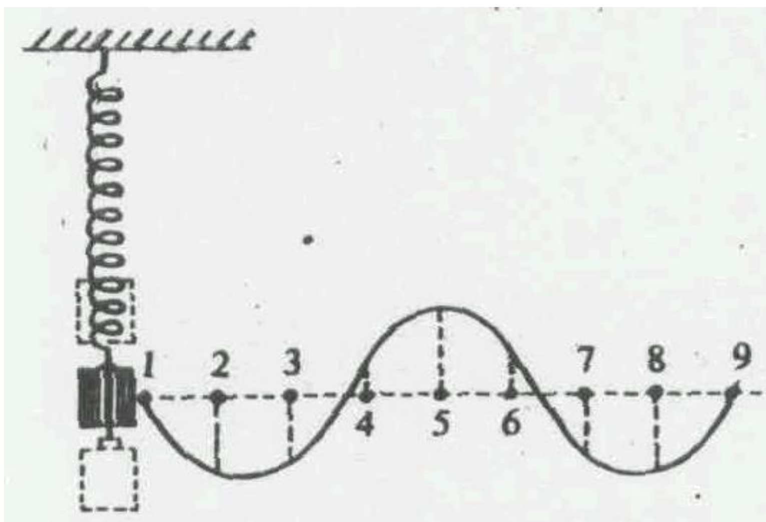


Fig. 6.6 A string fastened to an oscillating spring mass system: Illustration of the difference between motion of the waveform and the motion of particles

(i) At t = 0, all the particles are at their respective mean positions.
(ii) At t = T, the first, fifth and ninth particles are at their respective mean positions. The first and ninth particles are about to move upward whereas the fifth particle is about to move downward. The third and seventh particles are at position of maximum displacement but on opposite sides of the horizontal axis. The envelope joining the instantaneous positions of all the particles at marked positions in Fig. 6.7 (i) are similar to those in Fig. 6.5 and represents a *transverse wave.* The positions of the third and the seventh particles denote a *trough* and & *crest,* respectively.

The important point to note here is that *while the wave moves along the string all panicles of the string are oscillating up and down about their respective equilibrium positions with the same period* (D *and amplitude (a).* This wave remains *progressive* till it reaches the fixed end.

We would now like you to know to represent a wave graphically as well as mathematically. This forms the subject of our discussion in the following sections.

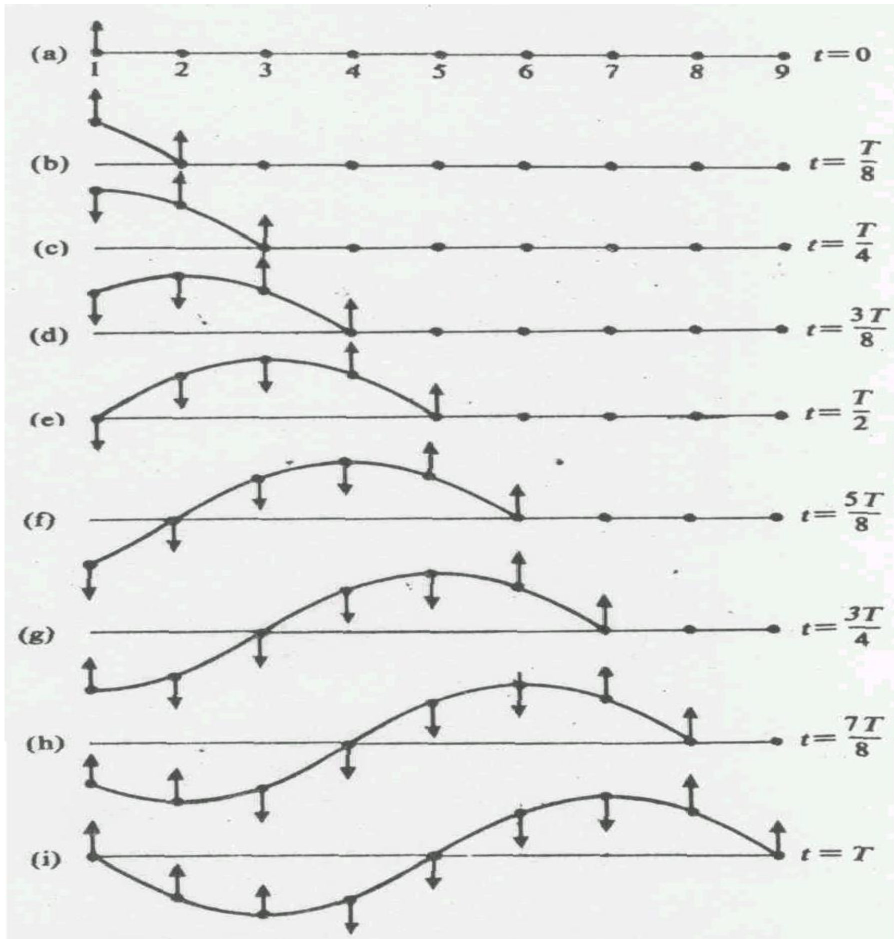


Fig. 6.7 Instantaneous profiles at intervals of $T/8$ when a transverse wave is generated on a string.

6.2.3 Representation of Wave Motion

From the above activity you would recall that when a wave moves along a string/spring, three parameters are involved: particle displacement, its position and time. In a 2-D graph, you can either plot displacement against time (at a given position) as shown in Fig. 6.8a or displacement against position (at a given time) as shown in Fig. 6.8b. You can easily identity that both plots are sinusoidal and have amplitude *a*. We can represent these as

$$y(t) = a \sin 2\pi \left(\frac{t}{T} \right) \text{ and } y(x) = a \sin 2\pi \left(\frac{x}{\lambda} \right)$$

The argument of the sine function ensures that the function repeats itself regularly.

We can draw an analogy between the wavelength and the period. The wavelength is *separation in space* between successive in-phase points on the wave. On the other hand, period is *separation in time* between equivalent instants in successive cycles of vibration. This means that the wavelength and the period are respectively the *spatial* and the *temporal* properties of a wave.

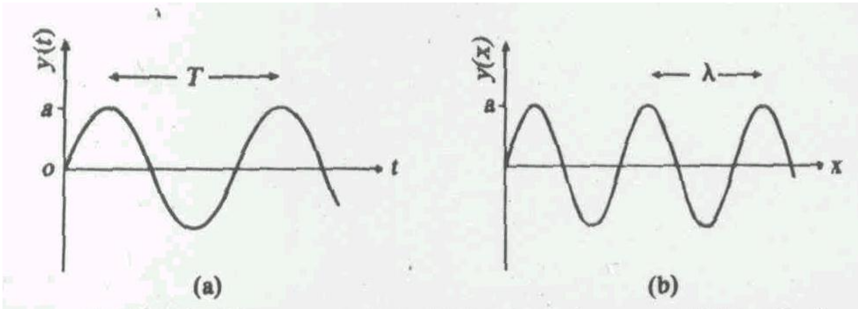


Fig. 6.8 (a) The profile of vibrations at a given position in the path of a wave, and (b) the profile of a wave at a particular Instant. It is snapshot of the wave travelling along the string.

It is important to note that the scales for $y(x)$ and $y(t)$ are different, For sound waves, the displacement amplitudes are a small fraction of 1 mm whereas *x* extends to several metres.

Human ears can hear sound of 1000 Hz quite clearly. The amplitude of the wave corresponding to the faintest sound that a normal human ear can hear is approximately 10^{-11}m. This is smaller than the radius of the atom (= 10^{-10}m). Yet our ears respond to such a small displacement!

Another point about graphical representation is that it can be used for both transverse and longitudinal waves.

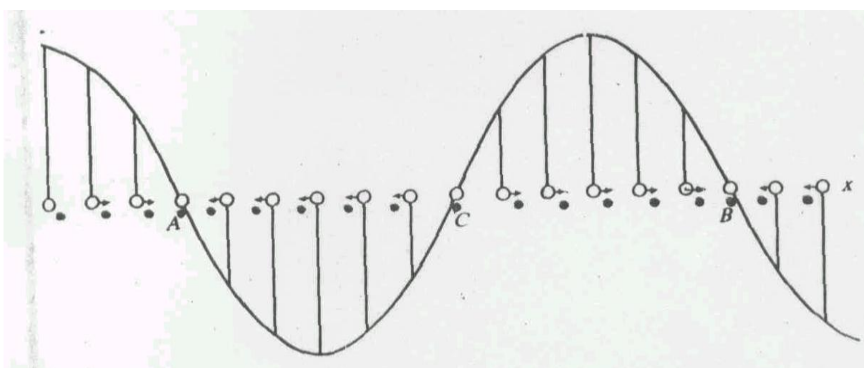


Fig. 6.9 Graphical representation of a longitudinal wave

In longitudinal waves, the displacement of particles is along the direction of wave propagation. In Fig. 6.9, the hollow circles represent the mean positions of equidistant particles in a medium. The arrows show their (rather magnified) longitudinal displacements at a given time. You will observe that the arrows are neither equal in length nor in the same direction. This is clear from the positions of solid circles, which describe instantaneous positions of the particles corresponding to the heads of the arrows. The displacements to the right are shown in the graph towards + y axis and displacements to the left towards the $-y$ axis.

For every arrow directed to the right, we draw a proportionate line upward. Similarly, for every arrow directed to the left, a proportionate line is drawn downward. On drawing a smooth curve through the heads of these lines, we find that the graph resembles the displacement-time curve for a transverse wave. If we look at the solid circles, we note that around the positions *A* and *B*, the particles have crowded together while around the position C, they have separated farther. These represent regions of *compression and rarefaction.* That is, there are alternate regions where density (pressure) are higher and lower than average. A sound wave propagating in air is very similar to the longitudinal waves that you can generate on your string/spring. This similarity is clearly illustrated in Fig. 6.10. A sound wave may be considered either as a pressure wave or as a displacement wave. However, the pressure wave is 90° put of phase with the displacement wave. That is, when displacement from equilibrium at a point is maximum, the excess pressure (over the normal) is zero and vice versa. The variations of pressure and density are represented graphically in. Fig. 6.11. This means that in longitudinal waves, alternate high and low pressures propagate along the wave.

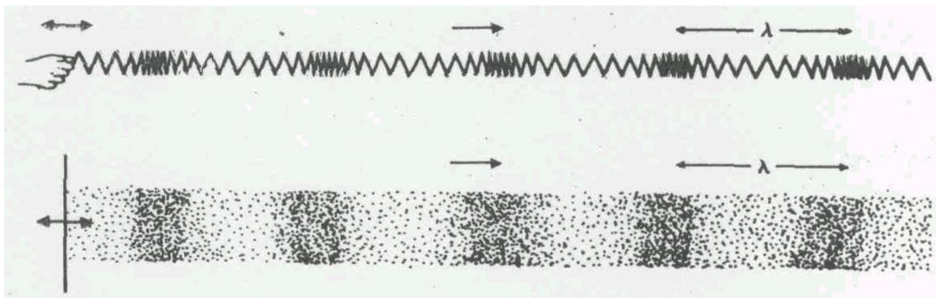


Fig. 6.11 The sound waves can be viewed in terms of changes in pressure or density

6.2.4 Relation between Wave Velocity, Frequency and Wavelength

Refer to Fig. 6.7(i). You will note that the first and ninth particles are in the *same* state of vibration. They are, therefore, said to be *in phase* with each another. *The distance between successive particles vibrating in phase is known as the wavelength.* It is usually denoted by Greek letter λ (lambda). Since the wave moves a distance of one full wavelength in one period, its speed

$$v = \frac{\text{wavelength}}{\text{period}} = \frac{\lambda}{T} \tag{6.3}$$

Since frequency, f, is the reciprocal of the period T, we can also write

$$v = f\lambda$$

That is, the speed of any wave is equal to the product of its frequency and the wavelength. This equation predicts that in a given medium, the speed of a wave of given frequency is constant, This is a very important relation.

You will note that we have derived Eq. (6.4) with reference to a transverse wave in a string. But it holds for all other media like air, water, glass etc. as well as longitudinal waves. At STP, the speeds of sound waves in air, water and steel are 332 $\text{m}\,s^{-1}$, 1500 $\text{m}\,s^{-1}$ and 5100 $\text{m}\,s^{-1}$, respectively. (This explains why the whistle of an approaching train may be heard twice - first as the sound travels through the railroad track and again as it (ravels through the air.) Ripples on the surface of a pond move with a speed of about 0.2 $\text{m}\,s^{-1}$. The seismic waves move with a speed of the order of $6\times10^3\,\text{m}\,s^{-1}$ in the earth's outer crust and light moves with a speed of $3\times10^8\,\text{m}\,s^{-1}$. That is why light that originates on or near the earth reaches us almost instantly.

6.3 MATHEMATICAL DESCRIPTION OF WAVE MOTION

In the preceding section you have learnt that at a particular time, a wave is described by Eq. (6.1). As time passes, the wave propagates along the $+x$ direction. So at a given value of *x*, the displacement of medium particles must change with time. This information is not contained in Eq. (6.1). This means that it is not a complete equation for the wave. You would like to know as to how we can modify Eq. (6.1). To answer this question, let us consider Fig. 6.12, which shows a 'snapshot' of a wave moving with speed v along x-axis. Now imagine two particles, say at *A* ($x = 0$) and at *B* separated by a distance *x*. You can easily convince yourself that a disturbance created at *A* will reach *B* in time t/v. This means that the particle at *B* will have the same displacement as particle at *A* at time $t' = t - x/v$. Mathematically, we can express this as

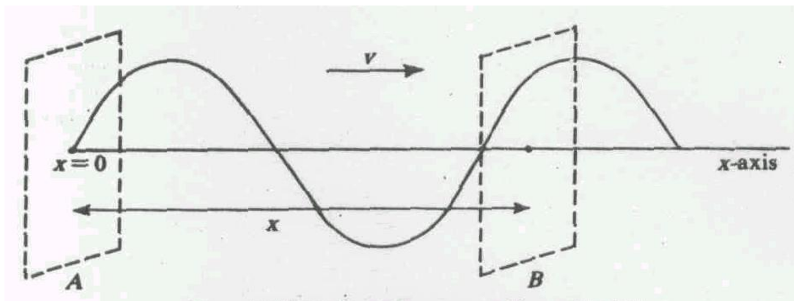


Fig. 6.12 Planes *A* and Bare separated by a distance *x*.

We can obtain $y(x = 0)$ on replacing t in Eq. (6.2) by $t' = t - x/v$. Then it readily follows that when a wave moves along $+x$ direction with speed v, the displacement of medium particles as a function of *x* and *t* is given by the equation

$$y(x,t) = a\sin\left[\frac{2\pi}{T}\left(t - \frac{x}{v}\right)\right] \tag{6.6a}$$

or

$$y(x,t) = a\sin\left[\frac{2\pi}{\lambda}(vt - x)\right] \tag{6.6b}$$

since $v = \lambda/T$. We can also rewrite this equation as

$$y(x,t) = a\sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right] \tag{6.6c}$$

At $t = 0$ (as also when *t/T is* an integer), this equation reduces to Eq. (6.1). [The negative sign implies slight re-adjustment in the phase of the wave at $t = 0$.] Further, you can check that this equation displays periodicity by calculating the displacement at a distance $x + \lambda$ from the origin. To ensure that you have grasped these ideas we would like you to solve the following SAQ.

SAQ3

A 275 Hz sound wave travels with a speed of 340 ms along the *x*-axis. Each point of the medium moves up and down through 5.0 mm. Write down the equation for the wave. Calculate (i) the wavelength of the wave, and (ii) velocity and acceleration of medium particles.

Eqs. (6.6a), (6.6b) and (6.6c) show that if we sit at a fixed position x = 0 say, then the displacement varies sinusoidally in time: $y(x=0,t)=a\sin(2\pi t/T)$. We know that this relation expresses SHM with angular frequency $\omega_0 = 2\pi/T$.

Another equivalent and convenient form for $y(x,t)$ is written in terms of the *wave number, k* defined as the number of radians of wave cycle per unit distance:

$$k=\frac{2\pi}{\lambda}$$

(You should not confuse this *k* with spring constant used earlier.)

You will recognize that the wave number is the spatial analog of the angular frequency. In terms of ω_0 and k, we can also write the wave equation as

$$y(x,t)=a\sin(\omega_0 t-kx) \tag{6.6d}$$

The simple way in which *k* and ω_0 enter this description of the wave explains why these quantities are so often used.

On comparing Eqs. (6.6a) and (6.6d) you will observe that *v* and *k* are connected through the relation

$$v=\frac{\omega_0}{k} \tag{6.7}$$

Eqs. (6.6a - d) provide us equivalent description of a travelling wave moving in the + *x* direction. But the choice of the particular form to be used for a specific problem is a matter of convenience. How would you describe a wave propagating in the negative x-direction? You can easily convince yourself that to describe a wave moving in the negative x-direction, we should replace *x* by $-x$ in Eq. (6.6).

SAQ4
A sinusoidal water wave having a maximum height of 7.4 cm above the equilibrium water level is propagating in the $-x$ direction with a speed of 93 cm s^{-1}. The distance between two successive crests is 55 cm. Write the wave equation in terms of angular frequency and wave number. Also calculate the particle velocity.

6.3.1 Phase and Phase Difference

In a periodic motion, particle displacement, velocity, and acceleration repeatedly undergo a cycle of changes. The different stages in a cycle may be described in terms of the phase angle. The argument of the sine function is called the *phase angle* or simply *phase.* We will denote it by the symbol θ. Thus, the phase at *x* at time *t* in a wave represented by Eq. (6.6d) is given by

$$\theta=\omega_0 t-kx \tag{6.8}$$

You will note that the phase changes both with time and the space coordinate. With time, it changes according to

$$\Delta\theta = \omega_0 \Delta t = 2\pi\ f\Delta t \text{ for fixed } x \qquad (6.9a)$$

and with position according to

$$\Delta\phi = -k\Delta x \text{ for fixed } t \qquad (6.9b)$$

The minus sign in this equation signifies that in a wave moving along $+x$ direction, the forward points lag in phase. That is, they reach the successive stages of vibration later.

6.3.2 Phase Velocity

From our experience we know that water waves travel with constant velocity as long as the properties of the medium remain constant. For harmonic progressive waves, this velocity is called the *phase velocity,* v_p. To show this, let us follow a given wave crest or through as the wave propagates. In order to keep the phase $\phi(x,t)$ defined by Eq. (6.8) constant, we must look for different *x* as *t* changes. Thus by taking the differential of $\phi(x,t)$ and setting the result equal to zero, you can find the relation between *x* and *t* for a point of constant phase. The differential of $\phi(x,t)$ is given by

$$d\phi = \omega_0 dt - kdx$$

It will become zero provided dx and dt are related by

$$v_p = \left(\frac{dx}{dt}\right)_\phi = \frac{\omega_0}{k} \qquad (6.10)$$

On comparing Eqs. (6.7) and (6.10) you will observe that the expression deduced earlier is actually the *phase velocity.*

6.3.3 Energy Transported by Progressive Waves

We now know that the most spectacular characteristic of progressive waves is that they transport energy we will now calculate the energy carried by a wave. To do so, we should know both the kinetic energy and the potential energy. If the instantaneous displacement of a particle is $y(x,t) = a\sin(\omega_0 t - kx)$, then the equation of a wave moving along + *x* direction is

Let us consider a thin layer of thickness and cross-sectional area *A* at a distance *x* from the source. If ρ is the density of the medium, the mass of the layer is $\rho\Delta x A$. Therefore, kinetic energy of the layer

$$K.E.(x,t) = \frac{1}{2}mv^2 = \frac{1}{2}\rho\Delta x A\left[\frac{\partial y(x,t)}{\partial t}\right]^2$$

$$= \frac{1}{2}\rho\Delta x A\omega_0^2 a^2 \cos^2(\omega_0 t - kx)$$

$$= 2\pi^2 f_0^2 \rho A\Delta x a^2 \cos^2(\omega_0 t - kx) \qquad (6.11)$$

This expression implies that kinetic energy oscillates between zero and $\frac{1}{2}\rho\Delta x A\omega_0^2 a^2$. This is because the value of the function $\cos^2(\omega_0 t - kx)$ varies between 0 and 1.

Over one full cycle, the average value of $\cos^2\theta$ is $\frac{1}{2}$. So, the average kinetic energy over a time period is

$$< K.E. >= \frac{1}{4}\rho\omega_0^2 a^2 A\Delta x = \pi^2 a^2 f_0^2 A\rho\Delta x \qquad (6.12)$$

What about the potential energy? From Unit 1 you would recall that in SHM, the average kinetic energy and average potential energy are equal. Is the same true for a harmonic wave as well? Physically, we except so. Let us now compute potential energy analytically.

The layer under consideration will be subject to a force

$$\begin{aligned} F &= \rho A\Delta x \frac{\partial^2 y(x,t)}{\partial t^2} \\ &= -4\pi^2 f_0^2 \rho\Delta x A y(x,t) \end{aligned}$$

We know that the work done by this force, when the layer of interest is displaced through y from its equilibrium position, is stored in the layer as its potential energy. So we can write

$$\begin{aligned} U(x,t) &= -\int_0^y A4\pi^2 f_0^2 \rho\Delta x y' dy' \\ &= -2\pi^2 f_0^2 \rho\Delta x A y^2 \\ &= -2\pi^2 f_0^2 \rho A\Delta x a^2 \sin^2(\omega_0 t - kx) \qquad (6.13) \end{aligned}$$

The minus sign tells us that the work is done on the layer. (This is of no consequence when we calculate total energy of the wave.) The time-averaged potential energy of the wave is

$$< U >= \pi^2 f_0^2 \rho A a^2 \Delta x =< K.E. > \qquad (6.14)$$

On combining Eqs. (6.11) and (6.13), we find that the total energy of the wave is

$$\begin{aligned} E &= K.E. + U \\ &= 2\pi^2 a^2 f_0^2 \rho A\Delta x \\ &= < K.E. > + < U > \qquad (6.15) \end{aligned}$$

This shows that half the energy of the wave is kinetic and the other half is potential.

What happens to this energy? As the layer moves, it pushes the next layer. In the process, it transmits its energy. Now you may like to know: How long does this layer take to give up its energy? Or, what is the average rate of energy flow? To calculate this, we note that if the wave is moving at speed v the energy passes the layer in time $\Delta t = \Delta x / v$. Hence

$$P = \frac{E}{\Delta t} = \frac{2\pi^2 a^2 f_0^2 \rho \Delta x A}{\Delta x / v}$$

$$= 2\pi^2 a^2 f_0^2 \rho v A \qquad (6.16a)$$

This shows that average rate of energy flow, or what we call *power*, is proportional to wave Speed and the square of the amplitude.

6.3.4 Intensity and the Inverse Square Law

From our common experience we know that the chirping of birds, the shout of a person, vehicular noise, sound of crackers or light from a lamp fade out beyond a certain distance. If it were not true, noise pollution would have made life hell on our planet. To understand the principle governing such situations, we note that amplitude of an outward spreading wave decreases as the distance from the source increases. This means that the average rate of energy flow associated with a wave decreases as it spreads out. It is therefore not very useful to talk about the total energy of progressive waves. In general, it makes more sense to describe the strength of a wave by specifying its *intensity*. It is defined as the *energy carried by a wave in unit time across a unit area normal to the direction of motion.*

Using this definition, Eq. (6.16) gives

$$I = 2\pi^2 a^2 f_0^2 \rho v = (1/2) v V_A^2 \rho \qquad (6.16b)$$

where $V_A^2 = 2\pi\, f_0 a$.

The SI units of intensity are $Jm^{-2}s^{-1}$ or Wm^{-2}. From Unit 1, you will recall that total energy is proportional to the square of the amplitude of oscillation. In the same way, the intensity of a wave at a given position is proportional to the square of amplitude at that position. For a second wave, we can write

$$I \propto p_0^2 \qquad (6.18\ a)$$

where p_0 is the maximum change in pressure over normal pressure. Note that when we express intensity in terms of p_0, the frequency does not appear explicitly in the expression. This means that 100 Hz sound wave has the same intensity as a 10 kHz sound wave; both have the same amplitude.

How does the intensity of a wave at a point vary with distance from the source?

We know that the area crossed by a wave increases as it spreads out. If it originates from a point source or the distance from the source is much greater than the size of the source, the area will be almost spherical $(\propto r^2)$. Then principle of conservation of energy demands that $E = 4\pi I r^2$ be constant. So as *r* increases, intensity decreases as $1/r^2$:

$$I \propto \frac{1}{r^2} \tag{6.18b}$$

On combining Eqs. (6.18 a) and (6.18 b), we find that

$$p_0 \propto \frac{1}{r}$$

Since $a \propto p_0$, this relation implies that amplitude of a wave is inversely proportional to the distance from the source. This explains why we can be heard up to a certain distance. (Beyond this the amplitude becomes too small to affect our eardrums.) You must note that these results will hold if the wave is not absorbed or obstructed. In Table 6.1 we have listed intensities of waves generated from different sources.

Table 6.1 Wave Intensities

Source/Wave	Intensity
Sound	
Threshold of hearing	10^{-12}
Rustle of leaves	10^{-11}
Whisper, intensity at eardrum	10^{-10}
Ordinary conversation	3.2×10^{-6}
Street traffic	10^{-5}
Bursting cracker, at 1 m	8×10^{-5}
Jet Taking off, at 30m	5
Electromagnetic Waves	
Radio in home	10^{-8}
TV signal, 5.0 km from 50 kW transmitter	1.6×10^{-4}
Sunlight intensity at earth's orbit	1368
1m from typical camera flash	4000
Inside microwave oven	6000
Target of laser fusion experiment	10^{18}
Seismic wave	
5 km from Richter 7.0 quake	4×10^{4}

SAQ5

At a distance of 1m from a bursting cracker, the intensity of sound is $8\times10^{-5}\ Wm^{-2}$. The threshold of human hearing is about $10^{-12} Wm^{-2}$. If sound waves spread out evenly in all directions, how far from the source could such a sound be heard?

Our ear is sensitive to an extremely large range of intensities. So we can define a logarithmic intensity scale. The intensity level of a sound wave is defined by the equation

$$\beta = 10\ln(I / I_0)$$

where $I_0 (= 10^{-12} Wm^{-2})$ denotes the threshold of hearing. Intensity levels are expressed in decibels, abbreviated db. Our ear can tolerate intensity up to 120 db.

6.4 ONE DIMENSIONAL PROGRESSIVE WAVES: WAVE EQUATION

Do you know how music reaches you? What determines whether or not waves can propagate in a medium and when they move, how fast they do so? Experimental investigations show that the speed of waves does not depend on the wavelength or period. This means that the answer to these questions should lie in the physical properties of the medium. To discover this, now we consider particular physical systems. For simplicity, we first study waves on a stretched string.

6.4.1 Waves on a Stretched String

Consider a uniform string, having mass per unit length *m*, stretched by a force *F*. Let us choose the x-axis along the length of the string in its equilibrium state. Suppose that the string is plucked so that a part of it is normal to the length of the string, i.e. along the y-axis (Fig. 6.13). What happens when the string is released? It results in wave motion. We wish to know the speed of this wave. We expect that the interplay of inertia and elasticity of the medium will determine the wave speed. For a stretched string, the elasticity is measured by the tension in the string and inertia is measured by *m*. Before proceeding further, we would like you to carry out the following exercise.

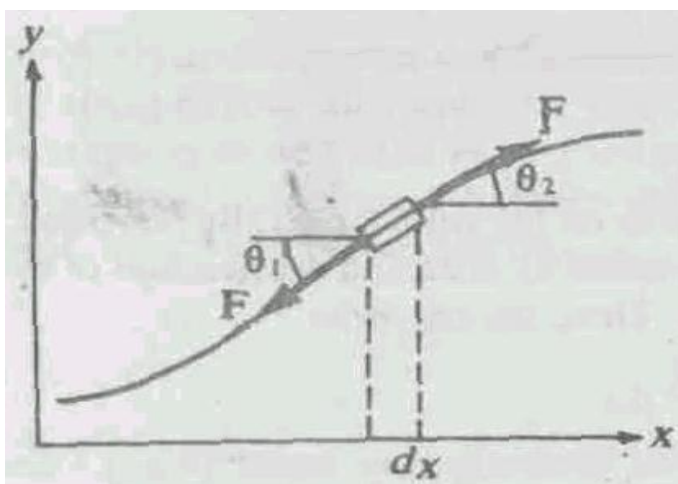


Fig. 6.13: A magnified view of a small element of the stretched string. The net force acting on it is non-zero. The vertical distortion is exaggerated (for clarity)

SAQ6

Using dimensional analysis, show that

$$v = K\sqrt{F / m}$$

where K is some dimensionless constant.

We will now analyse the problem mechanically by considering a small element along the string.

Suppose that the string is distorted slightly so that the magnitude of tension on the string essentially remains unchanged. Fig. 6.13 shows (magnified view of) a small element of the distorted string. You will observe that the direction of the (tension) force varies along the element under consideration. Why? This is because the string is curved. This means that the tension forces pulling at opposite ends of the element, although of the same magnitude, do not exactly cancel out. To calculate the net force along *x* and *y*-axis, we resolve *F* in rectangular components. The difference in the *x* and *y* components of tension between the right and the left ends of the element is respectively given by

$$F_x = F\cos\theta_2 - F\cos\theta_1$$

and $$F_y = F\sin\theta_2 - F\sin\theta_1$$

Where θ_1 and θ_2 are angles which the tangents drawn at the ends make with the horizontal. For small oscillations ($\theta \le 4°$).

$$\cos\theta_1 \approx \cos\theta_2 \approx 1$$

This means that there is no net force in the x-direction, $F_x = 0$, and the string will be very nearly horizontal. This implies that the sines of the angles are very nearly the same as their tangents, i.e.

$$\sin\theta_1 \approx \tan\theta_1$$

and $$\sin\theta_2 \approx \tan\theta_2$$

But the tangents of the two angles are just the slopes of the string - or the derivatives dy/dt at the ends of the element under consideration. Then, the -component of the force on the element is approximately

$$\begin{aligned} F_y &= F(\tan\theta_2 - \tan\theta_1) \\ &= F\left(\left.\frac{dy(x,t)}{dx}\right|_{x+\Delta x} + \left.\frac{dy(x,t)}{dx}\right|_{x}\right) \end{aligned} \qquad (6.20)$$

From the previous unit you may recall that the quantity in parentheses is just the change in the first derivative from one end of the interval Δx to the other. Dividing that change by Δx gives, in the limit $\Delta x \to 0$, the rate of change of the first derivative. But we know that the displacement of the string is a function of position as well as time. If either of these variables changes, the displacement also changes. You will recognise that Eq. (6.20) is valid for the configuration of the string at a particular instant of time. Therefore, the derivative in this equation is to be taken with the time fixed. We call a derivative taken with respect to one variable while other(s) is (are) kept constant a *partial* derivative. We denote partial derivatives with the symbol ∂ in place of the usual symbol *'d'*. Then, Eq. (6.20) can be rewritten as

The Taylor series expansion of a function $f(x+\Delta x)$ about the point x is given by $f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + ...$

$$F_y = F\frac{\partial^2 y(x,t)}{\partial x^2}\Delta x$$

This equation gives the net force on the segment Δx. By Newton's second law of motion, we can equate this force to the product of mass and acceleration of the segment. The mass of the segment of length Δx is $m\Delta x$. Then, we can write

$$m\Delta x\frac{\partial^2 y(x,t)}{\partial t^2} = F\frac{\partial^2 y(x,t)}{\partial t^2}\Delta x$$

Cancelling Δx on both sides, we obtain

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{m}{F}\frac{\partial^2 y(x,t)}{\partial t^2} \tag{6.21}$$

You can check that F/m has dimensions of velocity square.

Eq. (6.21) has been obtained by applying Newton's second law to a small element of a stretched string. Since there is nothing special about this particular element on the string, this equation applies to the whole of it.

Let us pause for a minute and ask: What goals we set for ourselves and how Eq. (6.21) helps us in attaining them? We wish to know what determines the speed of a wave. To know this let us assume that a harmonic wave described by

$$y(x,t) = a\sin(\omega_0 t - kx)$$

moves on the string. If this mathematical form is consistent with Newton's law, then you can be sure that such waves can move on the string. To see this, you should calculate the second partial derivatives of the particle displacement:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 a\sin(\omega_0 t - kx)$$

and

$$\frac{\partial^2 y}{\partial t^2} = \omega_0^2 a\sin(\omega_0 t - kx)$$

Substituting these derivatives in Eq. (6.21), we get, on simplification

$$k^2 = \frac{m}{F}\omega_0^2$$

or

$$\left(\frac{\omega_0}{k}\right)^2 = \frac{F}{m}$$

What is implied by this equality? We know that it has followed from Newton's law of motion applied to a stretched string when a harmonic wave is travelling along it. So, the above relation tells us that only those waves can propagate on the string for which wave properties ω_0 and k are related to F and m through the relation

$$\frac{\omega_0}{k} = \sqrt{\frac{F}{m}}$$

But ω_0 / k is just the wave speed (Eq. 6.7), so that

$$v = \frac{\omega_0}{k} = \sqrt{\frac{F}{m}} \tag{6.22}$$

This relation tells us that the velocity of a (transverse) wave on a stretched string depends on tension as well as mass per unit length of the string, not on wavelength or time period. This means that *v* is not a property of the material of the string. It involves an external factor – tension – which can be adjusted for fine-tuning. This explains why musicians are seen adjusting tension in their stringed instruments. However, no such thing is done in case of a flute or a harmonium.

Using Eq. (6.22), we can write Eq. (6.21) in the form

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{6.23}$$

This result is identical to Eq. (5.41) and expresses one dimensional *wave equation*. It holds so long as the oscillations of the string have small amplitude. You may now ask: Will Eq. (6.23) hold for large amplitude disturbances as well? The answer to this question is that large amplitude oscillations result in a more complicated equation and the wave speed tend to depend on wavelength as well. Before you proceed further, you may like to solve the following

SAQ 7

A 1 m long string having mass 1 g is sketched with a force of 10 N. Calculate the speed of transverse waves.

We now know that the speed of a wave is determined by the interplay of elasticity and inertia of the medium. Elasticity gives rise to the restoring force and inertia tells us how the medium responds to them. Since a fluid (gas or liquid) lacks rigidity, transverse waves can propagate only in solids. However, longitudinal waves can propagate in all phases of matter – plasmas, gases, liquids and solids – in the form of condensations and rarefactions. We will now consider wave propagation in a fluid.

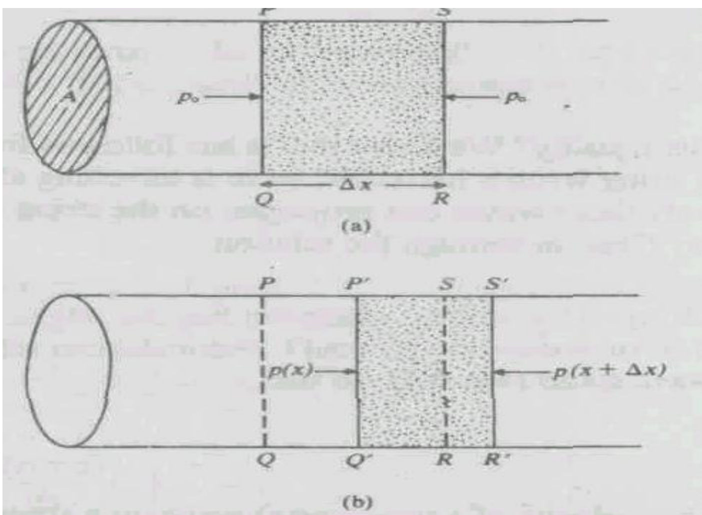
6.4.2 Waves in a Fluid

Let us consider a fixed mass of a fluid of density ρ contained in a long tube of cross sectional area *A* and under pressure p_0. As for a string, we shall consider a small element (column) of the fluid. Let us assume that the column is at rest and is contained in the region *PQRSP* extending between planes at *x* and $x + \Delta x$; (Fig. 6.14a). Then the mass of the column *PQRSP is* $\rho \Delta x A$. How can you generate longitudinal waves in the fluid? You can do so by placing a vibrating tuning fork at its one end or displacing the fluid to the right using a piston. As the wave passes through the column under consideration, its pressure, density and volume change. Let us assume that in time t planes *PQ* and *SR* move to *P'Q'* and *S'R'*, respectively, as shown in Fig. 6.14b. If the planes *PQ* and *SR* are displaced through $\psi(x)$ and $\psi(x+\Delta x)$ the change in thickness, Δl , is

$$\Delta l = \psi(x+\Delta x) - \psi(x) = \frac{\partial \psi}{\partial x}\Delta x$$

In writing this expression we have used Taylor series expansion for $\psi(x+\Delta x)$ about $\psi(x)$. This means that the change in volume ΔV is

$$\Delta V = A\Delta l = A\Delta x \frac{\partial \psi}{\partial x}$$



Fig, 6.14: (a) Equilibrium state of the column *PQRS* of a fluid contained in a long tube of cross sectional area *A*. (b) Displaced position of column under pressure difference

The *volume strain*, which is defined as the change in volume per unit volume, is given by

$$\frac{\Delta V}{V} = -\frac{A\Delta x \frac{\partial \psi}{\partial x}}{A\Delta x} = -\frac{\partial \psi}{\partial x} \tag{6.24}$$

The minus sign signifies that the column is compressed. This happens because the pressure on its two sides does not balance. Let the difference of pressures be $p(x+\Delta x) - p(x)$.

The net force acting on the column is, therefore $A[p(x+\Delta x) - p(x)]$. Using Taylor series and retaining only first order term in Δx , we can write

$$F = A\frac{\partial p(x)}{\partial x}\Delta x$$

$$= A\frac{\partial(p_0 + \Delta p)}{\partial x}\Delta x$$

$$= A\frac{\partial(\Delta p)}{\partial x}\Delta x$$

where p_0 is the equilibrium pressure and Δp is the excess pressure due to the wave.

Hence, the equation of motion for the column under consideration, according to Newton's second law, is

$$\rho \Delta x A\frac{\partial^2 \psi}{\partial t^2} = A\Delta x\frac{\partial(\Delta p)}{\partial x} \tag{6.25}$$

To express this result in a familiar form, we note that Δp and E, the *bulk modulus of elasticity*, are connected by the relation

$$E = \frac{\text{stress}}{\text{volume strain}} = -\frac{\Delta p}{\Delta V / V}$$

The negative sign is included to account for the fact that when pressure increases, volume decreases. This ensures that E is positive.
We can rewrite it as

$$\Delta p = -E\left(\frac{\Delta V}{V}\right)$$

On substituting for $\Delta V / V$ from Eq. (6.24), we get

$$\Delta p = E\frac{\partial \psi}{\partial x}$$

Using this result in Eq. (6.25) we find that

$$\rho\frac{\partial^2 \psi}{\partial x^2} = E\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= E\frac{\partial^2 \psi}{\partial x^2}$$

or

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{6.26}$$

If we identify

$$v = \sqrt{E / \rho} \tag{6.26a}$$

as the speed of longitudinal waves, Eq. (6.26) becomes identical to Eq. (5.41).

You will note that the wave speed is determined only by F and ρ; two properties of the medium through which the wave is propagating. Let us now consider the propagation of sound waves in a gas.

a. Sound Waves in a Gas

For a gaseous medium, such as air, the volume elasticity depends upon the thermodynamic changes arising in the medium, when a longitudinal wave is propagation. These changes can be *isothermal or adiabatic.* For sound waves, Newton assumed that changes in the medium are isothermal. For an isothermal change, you can check using Boyle's law that the volume elasticity equals equilibrium pressure. Then, we can write

$$v = \sqrt{p_0 / \rho} \tag{6.27}$$

This is known as Newton's formula for the velocity of sound.

For air at STP, $\rho = 1.29\, kg\, m^{-3}$ and $p_0 = 1.01 \times 10^5\, N\, m^{-2}$. Then the velocity of sound in air using Newton's formula comes out to be

$$v = \sqrt{\frac{1.01 \times 10^5\, Nm^{-2}}{1.29\, kg\, m^{-3}}} = 280 \text{ m}$$

But experiments show that the velocity of sound in air at STP is 332 m s^{-1}. This gives rise to an interesting question: How could Newton come so close to the correct answer and yet miss it by about 15%? It means that something is wrong with his derivation. You may now ask: How to explain the discrepancy? The problem lies with the use of Boyle's law, which holds only at constant temperature. The discrepancy was solved when Laplace pointed out that when sound waves move in a medium, the particles oscillate very rapidly. In the process, regions of compression are heated up while the regions of rarefaction get cooled. That is, local changes in temperature do occur when sound propagates in air. This effect produces a larger phase velocity. However, the total energy of the system is conserved. This means that adiabatic changes occur in air when sound propagates.

For an adiabatic change, E is γ times the pressure, where γ is the ratio of the specific heats at constant pressure and at constant volume, i.e. $E = \gamma\, p_0$. Then Eq. (6.27) becomes

$$v = \sqrt{\frac{\gamma\, p_0}{\rho}} \tag{6.28}$$

For air, = 1.4. So the velocity of sound in air at STP works out to be 331 m s^{-1}, which is an excellent agreement with the measured value. This shows that Laplace's agreement is correct.

At a given temperature, p_0 / ρ is constant for a gas. So Eq, (6.28) shows that the velocity of a longitudinal wave is independent of pressure.

A process is said to be isothermal if temperature remains constant during the process. In an adiabatic process, the total energy of the system remains constant.

For an isothermal process, Boyle's law states that pV = constant. Any change in p and/or V is related as $V\Delta p + p\Delta V = 0$ or $-\dfrac{\Delta p}{\Delta V / V} \equiv E = p$.

For an adiabatic change, the equation of state is pV^{γ} = constant. The change in p and V are connected through the relation $V^{\gamma}\Delta p + p\gamma V^{\gamma-1}\Delta V = 0$, or $-\dfrac{\Delta p}{\Delta V / V} \equiv E = p$.

You will now like to know why the heat does not have time to flow from a compression to a rarefaction and equalize the temperature everywhere. To attain this condition, heat has to flow a distance of one-half wavelength in a time much shorter than one-half of the period of oscillation. This means that we would need

$$v_{\text{heat}} >> v_{\text{sound}} \tag{6.29}$$

Since heat flow is mostly due to conduction, the speed of air molecules is given by

$$v_{rms} = \sqrt{\frac{k_H T}{M}} \tag{6.30}$$

where M is the mass of air molecules and T is the absolute temperature. We can similarly write

$$v_{\text{sound}} = \sqrt{\frac{\gamma k_H T}{M}} \tag{6.31}$$

Thus, even if air molecules travel a distance of $\lambda/2$, they will not be able to transfer heat in time. In practice they move randomly in zig-zag paths of the order of 10^{-5} cm and as long as $\lambda > 10^{-5} cm$, the adiabatic flow is a very good approximation. The shortest wavelength for audible sound (1.6cm) corresponds to 20 kHz.

The ability to measure the speed of sound has been put to many uses in defence. During World War I, a technique called *sound ranging* was developed to locate the position of enemy guns by using the sound of cannon in action.

b. Sound Waves in a Liquid

Liquids are, in general, almost incompressible. For water, $E = 2.22 \times 10^{9}\, N\, m^{-2}$ and $\rho = 10^{3}$ kg m^{-3}. This gives a wave velocity of about 1500 ms^{-1}. Compare this with the speed of sound in air at STP. Though air is about 10^{-3} times less dense than water, sound propagates faster in water than air. This means that we can send messages from one ship to another faster via water than in air. This has led to the development of Sonar. High frequency sound waves are used in Sonar which can measure the depth of sea bed, detect submarines and enemy torpedoes.

6.4.3 Waves in a Uniform Rod

For a solid elastic rod, changes take place only in length; the volume remaining almost constant. The bulk modulus is replaced by *Young's modulus* defined as

$$Y = \frac{stress}{longitudinal\ strain} = \frac{\Delta p}{\Delta l / l}$$

Then Eq. (6.26) modifies to

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2 \psi}{\partial x^2} \tag{6.32a}$$

which is the equation of wave motion in the rod. The velocity of longitudinal waves is given

$$v = \sqrt{\frac{Y}{\rho}} \tag{6.32b}$$

This shows that v is independent of the cross-sectional area of the rod.

SAQ8

For a steel rod, $Y = 2 \times 10^{11}\, Nm^{-2}$ and $\rho = 7800\, kgm^{-3}$. Compute the speed of sound. On working out this SAQ you will find that $v = 5 \times 10^3\, ms^{-1}$, which shows that longitudinal waves travel faster in a solid than in a gas or a liquid. This means that you can know about a coming train by putting your ear on the rails. However, you are advised never to do so!

6.5 WAVE MOTION AND IMPEDANCE

When a wave travels through a medium, the medium opposes its motion. This resistance to wave motion is called the *wave impedance.* You should not confuse it with the electric impedance in the case of AC circuits where it arises due to resistance offered to the flow of current. The impedance offered to the transverse waves travelling on strings is called the *Characteristic Impedance.* Usually, the impedance offered to the longitudinal (sound) waves in air is called the *'acoustic impedance'.* You may now ask: Why impedance arises and what factors determine it? To discover the answer to this question, we recall that when a wave propagates in a medium, each particle vibrates about its mean position. Moreover, each particle in motion attempts to make the succeeding particle vibrate by transferring energy. Likewise, each particle at rest tends to slow down the neighbouring particle. That is, a vibrating particle experiences a dragging force, which is similar to the viscous force. According to Newton's third law of motion, it will be equal to the driving force *F*. From Unit 3 of this course you would recall that when oscillations of the particles are small, we can model the viscous force on the basis of stokes' law and write

$$\mathbf{F} = Z\mathbf{v}$$

The constant of proportionality Z is called the *wave impedance*. From this equation it is clear that impedance is numerically equal to the driving force which imparts unit velocity to a particle. We will now consider some specific examples.

6.5.1 Impedance Offered by Strings: Transverse Waves

Let us consider a wave travelling on a stretched string. Let us choose the x-axis along the length of the string (see Fig. 6.15). The transverse waves are generated by applying a harmonic force $F = F_0 \cos\omega_0 t$ at the end $x = 0$ of the string. The displacement of the particles of the string at position x and at time t is given by Eq. (6.6d).

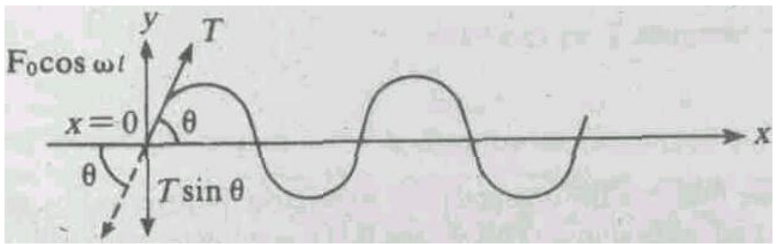


Fig. 6.15 A siring vibrating under the harmonic force

Let T be the tension in the string. The vertical component of tension (T) along the negative y-direction is equal to the applied transverse force (to give zero resultant force at the $x = 0$ end of the string):

$$F_0 \cos\omega_0 t = -T\sin\theta$$

For small values of $\theta\,(< 5°)$, $\sin\theta \approx \tan\theta$, so that we can write

$$F_0 \cos\omega_0 t = -T\tan\theta \tag{6.33}$$

The tangent (or slope) is defined at the $x = 0$ end of the string. Using Eq. (6.6d) we can relate (dy/dx) to (dy/dt):

$$\frac{dy}{dx} = -\frac{k}{\omega_0}\frac{dy}{dt}$$

Inserting this result in Eq. (6.33), we get

$$F_0 \cos\omega_0 t = \frac{kT}{\omega_0}\left(\frac{dy}{dt}\right)_{x=0}$$

Since $\left(\dfrac{dy}{dt}\right)_{x=0} = a\omega_0 \cos\omega_0 t$, this equality becomes

$$F_0 \cos\omega_0 t = \frac{Tk}{\omega_0} a\omega_0 \cos\omega_0 t$$

$$= \frac{T}{v} a\omega_0 \cos \omega_0 t$$

where $v = \omega_0 / k$.

Writing $a\omega_0 = v_0$ as the *velocity amplitude* of the wave, the above equation reduces to

$$F_0 \cos \omega_0 t = \frac{Tv_0}{v} \cos \omega_0 t$$

giving

$$F_0 = \frac{Tv_0}{v}$$

$$\frac{F_0}{v_0} = \frac{T}{v} \tag{6.34}$$

This result specifies the ratio of the amplitude of the applied force to the amplitude of particle velocity for transverse waves in terms of the tension in the string and particle velocity. This result can be used to get an expression for the characteristic impedance (Z) of the string which is defined as:

$$Z = \frac{\text{amplitude of the transverse applied force } (F_0)}{\text{transverse velocity amplitude of the wave } (v_0)}$$

Using Eq. (6.34), we find that

$$Z = \frac{F_0}{v_0} = \frac{T}{v} \tag{6.35}$$

This result shows that the characteristic impedance has units of $Nm^{-1}s$ but its dimensions are MT^{-1}.

From Eq. (6.26a) we recall that $v = \sqrt{T/m}$ where m is the mass per unit length of the string.

Then Eq. (6.36) can be written as:

$$Z = \frac{T}{v} = \sqrt{Tm} \tag{6.36a}$$

Alternatively, if we eliminate T we can write

$$Z = \frac{v^2 m}{v} = mv \tag{6.36b}$$

From Eq. (6.36a) we find that the characteristic impedance is governed by the mass per unit length of the string and the tension in it. This means that a sonometer wire will offer different impedance when it is loaded by different weights. Eq. (6.36b) tells us that since Z is related to the velocity of the wave, it depends on the inertia as well as the elasticity of the medium.

SAQ9

Calculate the characteristic impedance offered by a thin wire of steel stretched by a force of 80 N. It weights 2 g per metre.

6.5.2 Impedance Offered by Gases: Sound Waves

For sound waves propagating in a gas, the role played by excess pressure due to the wave is analogous to that of applied force in case of a transverse wave. So we define the *acoustic impedance* as:

$$Z = \frac{\text{excess pressure due to a sound wave}}{\text{particle velocity}} = \frac{\Delta p}{\partial \psi / \partial t} \tag{6.37}$$

It means that dimensionally Z is the ratio of force per unit area to velocity.

The excess pressure Δp experienced by the medium when a longitudinal wave propagates through it is given by

$$\Delta p = -E \frac{\partial \psi}{\partial x} \tag{6.38}$$

where E is the Bulk modulus of elasticity of the medium. This means that to know Z, we must compute $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi}{\partial x}$. To do so, we recall that the particle displacement for a longitudinal wave travelling in the +ve x-direction is written as

$$\psi(x,t) = a \sin\left[\frac{2\pi}{\lambda}(vt - x)\right]$$

Differentiating it with respect to x and t, we get

$$\frac{\partial \psi}{\partial x} = -a\frac{2\pi}{\lambda}\cos\left[\frac{2\pi}{\lambda}(vt - x)\right] \tag{6.39a}$$

and

$$\frac{\partial \psi}{\partial t} = a\left(\frac{2\pi v}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(vt - x)\right] \tag{6.39b}$$

On combining Eqs. (6.38) and (6.39a), we find that

$$\Delta p = Ea\left(\frac{2\pi}{\lambda}\right)\cos\left[\frac{2\pi}{\lambda}(vt - x)\right] \tag{6.40}$$

On substituting for Δp and $\partial\psi/\partial t$ from Eqs. (6.40) and (6.39b) in Eq. (6.37) we find that the acoustic impedance is given by

$$Z = \frac{\Delta p}{\partial\psi/\partial t} = \frac{Ea(2\pi/\lambda)\cos[2\pi/\lambda(vt-x)]}{a(2\pi v/\lambda)\cos[2\pi/\lambda(vt-x)]} \qquad (6.41)$$

$$= \frac{E}{v}$$

where v is the wave velocity. This result shows that the units of acoustic impedance are $Nm^{-3}s$ and the dimensions are $ML^{-2}T^{-1}$ (You should verify these before proceeding further.)

From Eq. (6.26a) we recall that the wave velocity is given by

$$v = \sqrt{\frac{E}{\rho}}$$

where ρ is the density of the medium. Hence the acoustic impedance *Z* can also be expressed as:

$$Z = \frac{E}{v} = \sqrt{E\rho} = \rho v \qquad (6.42)$$

This result shows that the acoustic impedance Z is given by the product of the density of the medium and the wave velocity. This means that the denser the medium, the greater will be the impedance offered. Yet we know that sound moves faster in solids than gases.

In the next unit, you will apply these results to compute reflection and transmission amplitude and energy coefficients for a wave incident on a boundary separating two media.

SAQ10

Calculate the acoustic impedance of air at standard temperature and pressure. Use = 1.29 $kg\,m^{-3}$ and v = 332 $m\,s^{-1}$. Will this value be more for air or water? Justify your answer.

6.6 WAVES IN TWO AND THREE DIMENSIONS

So far we have confined ourselves to waves propagating along 1-D, as in a stretched string. The waves are constrained to move along the string whereas particles vibrate in perpendicular direction. But all musical instruments are not stringed. What happens when a drum membrane is suddenly disturbed in a direction normal to the plane of the membrane? Particles of the membrane vibrate along the direction of the applied force. But tension in the membrane makes the disturbance to spread over the surface. That is, waves on stretched membranes ate two-dimensional (2-D). Similarly, surface waves or ripples on water, caused by dropping a pebble into a quite pond, are 2-D. In such cases, the displacement is a function of *x*, *y* and *t*, i.e. $\psi = \psi(x, y, t)$. You may now ask: What is the equation of a 2-D wave? Will the preceding analysis as such apply in this case?

We will not go into mathematical details to answer these questions. However, from physical consideration, extension of Eq. (6.23) for 2-D wave is a straightforward exercise. Since forces along x and y axes act independently, each one will contribute analogous term to the wave equation so that Eq. (6.23) modifies to

$$\frac{\partial^2 \psi(x,y,t)}{\partial t^2} = \frac{F}{\rho}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t) \tag{6.43}$$

The solution of this equation is

$$\psi(x,y,t) = a\sin(\omega_0 t - \mathbf{k}\cdot\mathbf{r}) \tag{6.44}$$

where $\mathbf{k}\cdot\mathbf{r} = (k_x\mathbf{i} + k_y\mathbf{j})\cdot(x\mathbf{i} + y\mathbf{j}) = k_x x + k_y y$.

Let us pause for a minute and ask: Do sound and light waves emanate radially from a small two-dimensional source? How can we describe seismic waves or a wave propagating in an elastic solid? These are three-dimensional waves. To analyse 3-D waves we have to extend the preceding arguments. The result is a 3-D wave equation.

SAQ 11

Generalize Eq. (6.43) in three dimensions.

6.7 SUMMARY

- Mechanical (elastic) waves can be transverse as well as longitudinal. In a transverse wave, particles of the medium vibrate normal to the direction in which a wave moves, in a longitudinal wave vibrations of the particles of the medium are along the direction of wave propagation.

  The wave velocity, frequency and wavelength are connected by the relation: $v = f\,\lambda$.

  We can also express v as ratio of the angular frequency and wave number: $v = \dfrac{\omega_0}{k}$.

- A harmonic wave in 1-D is described by the equation

$$\begin{aligned} y(x,t) &= a\sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right] = \\ &= a\sin\left[\frac{2\pi}{\lambda}(vt - x)\right] \\ &= a\sin(\omega_0 t - kx) \end{aligned}$$

  where T is the time period. The phase of a wave, $\phi = \omega_0 t - kx$ varies both with time and space.

- Waves carry energy. The total energy carried by a wave is half kinetic and half potential:

$$\begin{aligned} E &= 2\pi^2 a^2 f_0^2 \rho A\Delta x \\ &= \langle K.E.\rangle + \langle U\rangle \end{aligned}$$

- The average rate of energy flow or average power is proportional to the wave speed and to the square of the wave amplitude:

$$\langle P\rangle = 2\pi a^2 f_0^2 \rho A$$

- For waves propagating in space, intensity is a more useful measure of energy carried by waves. The intensity of a plane wave remains constant as the wave propagates. But for spherical waves, the intensity decreases as the inverse square of the distance from the source.
- A wave propagating along a string (1-D wave) is described by the equation

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

where $\psi(x,t)$ is the displacement and v is the wave speed.
- The speed of a wave on a stretched string is given by

$$v = \sqrt{F/m}$$

where F is the tension in the string and m is the mass per unit length. For a longitudinal wave

$$v = \sqrt{E/\rho}$$

where E is the elasticity and ρ is the density of the medium. For sound waves in air

$$v = \sqrt{\gamma / p_0}$$

where γ is the ratio of specific heats at constant pressure to that at constant volume. For sound waves in solids

$$v = \sqrt{Y/\rho}$$

where Y is the Young's modulus of elasticity and ρ is the density of the medium.
- When a wave travels through a medium, the medium opposes its motion. This resistance to wave motion is referred to as the wave impedance. In case of transverse waves, the characteristic impedance is given by

$$Z = \frac{T}{v} = \sqrt{Tm} = mv$$

For sound waves in air, the acoustic impedance is given by

$$Z = \frac{E}{v} = \sqrt{E\rho} = \rho v$$
- A wave propagating in 2-D is described by the equation

$$\frac{\partial^2 \psi(r,t)}{\partial t^2} = v^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(r,t)$$

The solution of this equation is given by

$$\psi(r,t) = a \sin(\omega_0 t - \mathbf{k} \cdot \mathbf{r})$$

where $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y$ and $k^2 = k_x^2 + k_y^2$.

6.8 TERMINAL QUESTIONS

1. A transverse harmonic wave of amplitude 1 cm is generated at one end $(x = 0)$ of a long string by a tuning fork of frequency 500 Hz. At one instant of time, the displacements of the particles at x = 10 cm is – 0.5 cm and at x = 20 cm is 0.5 cm. Calculate the velocity and the wavelength of the wave. If the wave is travelling along the positive $x-$direction and the end x = 0 is at the equilibrium position at t = 0, write the displacement in terms of the wave velocity.

2. In normal conversation, the intensity (energy flux/area) of sound is $5\times10^{-6}\,Wm^{-2}$. The frequency of normal human voice is about 1000 Hz. Compute the amplitude of sound waves, given that the density of air at STP is 1.29 $kg\,m^{-3}$. Take the velocity of sound at STP as $332\,m\,s^{-1}$.

3. The wavelength of a note of sound of frequency 500 Hz is found to be 0.70 m at room temperature (15°C). Given that the density of air at STP is 1.29 $kg\,m^{-3}$, calculate γ.

4. A longitudinal disturbance generated by an earthquake travels 10^3 km in 2.5 minutes. If the average density of the rock is taken to be $2.7\times10^3\,kgm^{-3}$, calculate the bulk modulus of elasticity of the rock.

6.9 SOLUTIONS

SAQs

1. (i) 20 kHz (ii) composite (iii) no (v) transverse

2. From Eq. (6.4), we have

$$v=\frac{c}{\lambda}$$

For λ = 4000Å = $4\times10^{-7}\ m$, the frequency of visible light is

$$f=\frac{3\times10^8\,ms^{-1}}{4\times10^{-7}\,m}$$
$$=7.5\times10^{14}\,s^{-1}$$

Similarly for λ = 7200Å, we have

$$f=\frac{3\times10^8\,ms^{-1}}{7.2\times10^{-7}\,m}$$
$$=4.1\times10^{14}\,s^{-1}$$

3. The wave equation is

$$y(x,t)=(2.5\times10^{-3}\,m)\sin\left[600\pi\left(t-\frac{x}{340}\right)\right]$$

The wavelength is given by

$$\lambda=\frac{v}{f}=\frac{340ms^{-1}}{275\,s^{-1}}=1.24\ m.$$

The velocity of the medium particles is

$$v=\frac{\partial y}{\partial t}=(2.5\times10^{-3})(600\pi\,s^{-1})\cos\left[600\pi\left(t-\frac{x}{340}\right)\right]$$

and acceleration is

$$a=\frac{\partial^2 y}{\partial t^2}=-(2.5\times10^{-3}\,m)(600\pi\,s^{-1})^2\sin\left[600\pi\left(t-\frac{x}{340}\right)\right]$$

4. A wave moving along the $-x$-direction is described by the equation

$$y(x,t)=a\sin(\omega_0 t+kx)$$

The angular frequency is connected to v and λ by the relation

$$\omega_0 = \frac{2\pi v}{\lambda}$$

Here v = 93 $cm\,s^{-1}$ and λ = 55 cm. Hence,

$$\omega_0 = \frac{2\pi \times 93\,cm\,s^{-1}}{55\,cm}$$

$$= \frac{372}{55} s^{-1}$$

Similarly, the wave number is related to wavelength as

$$k = \frac{2\pi}{\lambda} = \frac{2\times 22}{7\times 55\,cm} = \frac{4}{35} cm^{-1}$$

Hence,

$$y(x,t) = (7.4\,cm)\sin\left(\frac{372}{35}t + \frac{4}{35}x\right)$$

$$= 78.6\,cm\,s^{-1}\cos(10.6t + 0.1x)$$

5. $I = 8\times 10^{-5} Wm^{-2}$

$I_0 = 10^{-12} Wm^{-2}$

The intensity falls off as the inverse square of the distance. If r denotes the distance at which this sound could just be heard, then

$$\frac{8\times 10^{-5} Wm^{-2}}{10^{-12} Wm^{-2}} = \frac{r^2}{(1m)^2}$$

$r = 9\times 10^3$ m = 9 km

But this is not observed in practice. It is because of absorption of energy by the medium.

6. $v = kf(F,m)$

$[v] = [K][F]^a[m]^b$

$[LT^{-1}] = [MLT^{-2}]^A[ML^{-1}]^b$

or

$[LT^{-1}] = [M^{a+b}][L^{a-b}][T^{-2a}]$

On comparing the powers of T, we get

$-1 = -2a$

or

$a = 1/2$

On comparing the powers of L, we have

$a - b = 1$

or

$b = a - 1$

$= -1/2$

Hence,

$v = k\sqrt{F/m}$

7. We know that the wave velocity on a stretched string is given by

$v = \sqrt{F/m}$

On substituting the given values, we get

$$v = \sqrt{\frac{10N}{10^{-3}\,kgm^{-1}}}$$

$= 100\ m\,s^{-1}$

From Eq. (6.32b),

$v = \sqrt{\gamma / \rho}$

On inserting the given values, you will get

$$v = \frac{\sqrt{2\times 10^{11}\,Nm^{-2}}}{7800\,kg\,m^{-3}}$$

$= 5\times 10^{3}\,ms^{-1}$

9. $m = 2.0\,gm^{-1} = 2.0\times 10^{-3}\,kg\,m^{-1}$ and

$T = 80N$

From Eq. (6.36b) we recall that impedance offered by a string is given by

$Z = \sqrt{Tm}$

On substituting the given data, you will get

$$Z = \sqrt{(80N)\times(2.0\times 10^{-3}\,kgm^{-1})}$$

$= 0.4\ Nm^{-1}s$

10. From Eq. (6.41), we have

$Z = \rho v$

Here, $\rho = 1.29\ kgm^{-3}$ and $v = 332\,ms^{-1}$

$$\therefore\ Z = (1.29\,kgm^{-3})\times(332\,ms^{-1})$$
$$= (4.28\times 10^{2})kgm^{-2}s^{-1}$$
$$= 4.28\times 10^{2}\ Nm^{-3}s$$

The impedance offered by water will be more than that offered by air. This is because the density of water is much greater than that for air at STP. Also, the velocity of sound in water is almost five times the velocity of sound in air.

11. In 3-D, the wave equation has the form

$$\frac{\partial^2 \psi(r,t)}{\partial t^2} = v^2 \nabla^2 \psi(r,t)$$

TQs

1. We know that a harmonic wave in 1-D is described by

$$\psi(x,t) = a\sin\left[\frac{2\pi}{\lambda}(vt - x)\right]$$

where $a = 1$ cm

(a) At $x = 10\,cm$, $\psi(x,t) = -0.5\,cm$

$\therefore$ $$-0.5\,cm = (1\,cm)\sin\left[\frac{2\pi}{\lambda}(vt-10)\right]$$

or

$$\sin\left(\frac{2\pi}{\lambda}(vt-10)\right) = -1/2 = \sin\left(\pi + \frac{\pi}{6}\right)$$

This equality implies that

$$\frac{2\pi}{\lambda}(vt-10) = \frac{7\pi}{6}$$

or

$$vt - 10 = \frac{7}{12}\lambda \qquad \text{(i)}$$

(b) At $x = 20\,cm$, $\psi(x,t) = +0.5\,cm$

$\therefore$ $$0.5\,cm = (1\,cm)\sin\left[\frac{2\pi}{\lambda}(vt-20)\right] = \sin\frac{\pi}{6}$$

so that

$$vt - 20 = \frac{\lambda}{12} \qquad \text{(ii)}$$

From (i) and (ii), we get

$$\frac{\lambda}{2} = 10$$

or

$$\lambda = 20\,cm$$
$$= 0.2\ m$$

We know that

$$v = f\,\lambda$$

$\therefore$ $$v = (500\,Hz)\times(0.2\,m) = 100\,ms^{-1}$$

Hence,

$$\psi(x,t) = (0.01\,cm)\sin\left[\frac{2\pi}{0.2}(100t - x)\right] = (0.01\,m)\sin[10\pi(100t - x)]$$

2. The expression for the intensity is

$$I = 2\pi^2 \rho\, f_0^2 a^2 v$$

so that

$$a = \frac{1}{\pi\, f_0}\sqrt{\frac{1}{2\rho v}}$$

Here, $v = 332\,ms^{-1}$, $\rho = 1.29\,kgm^{-3}$, $f_0 = 1000\,Hz$, $I = 5\times10^{-6}\,Wm^{-2}$.

Hence,

$$a = \frac{1}{3.14\times1000Hz}\sqrt{\frac{5\times10^{-6}\,Wm^{-2}}{2\times1.29kgm^{-3}\times332\,ms^{-1}}}$$
$$= 2.4\times10^{8}\ m$$

3. Here we use the expression

$$v_1 = \sqrt{\frac{\gamma\, p}{\rho}}\ ;\ v_0 = \sqrt{\frac{\gamma\, p_0}{\rho_0}}$$

and the gas equation

$$\frac{p_0 V_0}{T_0} = \frac{pV}{T}$$

The gas equation can be rewritten in terms of ρ and ρ_0 as

$$\frac{p_0 \rho}{p \rho_0} = \frac{T_0}{T}$$

since $\rho = m/V$ and $\rho_0 = m/V_0$

Since $v_1 = f\,\lambda$, we find that

$$v_1 = 500\,Hz \times 0.70\,m$$
$$= 350\ ms^{-1}$$

Hence,

$$\frac{v_0}{v_1} = \sqrt{\frac{p_0 \rho}{p \rho_0}} = \sqrt{\frac{T_0}{T}} = \sqrt{\frac{273\,K}{(273+15)K}}$$

so that

$$v_0 = (350\,ms^{-1})\sqrt{\frac{273K}{288K}} = 341 ms^{-1}$$

Hence, γ, the ratio of specific heats at constant pressure to that at constant volume, is given by

$$\gamma = \frac{v_0^2 \rho_0}{p_0} = \frac{(341 ms^{-1})^2 \times 1.29\,kgm^{-3}}{0.76m \times 9.8\,ms^{-2} \times 13.6 \times 10^3\,kg\,m^{-3}}$$
$$= 1.5$$

4. The speed of the seismic wave is

$$v = \frac{10^3 \times 10^3\,m}{2.5 \times 60\,s} = 6.7 \times 10^3\,ms^{-1}$$

Since

$$v = \sqrt{\frac{E}{\rho}}$$

we can write

$$E = v^2 \rho$$

On substituting the given data, we get

$$E = (6.7 \times 10^3\ ms^{-1})^2 \times 2.7 \times 10^3\,kgm^{-3}$$
$$= 12.1 \times 10^{10}\,Nm^{-2}$$

UNIT 7 WAVES AT THE BOUNDARY OF TWO MEDIA

Structure

7.1 INTRODUCTION

In Unit 6 we discussed the basic characteristics of wave motion. The propagation of waves on strings and in fluids was discussed with particular reference to sound. You may now ask: What happens to a wave when it encounters a rigid barrier, as for instance, in the case of a string whose one end is tied to a rigid wall. The wave energy will not flow into the wall. But the wave cannot stop there. Then where will its energy go? What happens is that the wave turns around and bounces back along the string. We say that the wave has been *reflected.*

You must have experienced sound reflection in the form of echoes in large halls or in the neighbourhood of hills. You must have also observed reflection of water (sea) waves from a fixed barrier (sea shore). In the case of light, reflection from silvered surfaces, say in a looking mirror, is the most common optical effect we know. The reflection of ultrasonic (sound) waves forms the operating principle of *sonars* in depth-ranging, navigation, prospecting for oil and mineral deposits. The reflection of e.m. waves governs the working of a radar for detection of aircrafts. Reflection of radiowaves by the ionosphere makes signal transmission from one place to another possible and is so crucial in the area of communications.

You may now like to know as to what would happen to the incident wave. You would agree that the boundary is not very rigid and properties of the medium change suddenly. Now suppose that we connect two strings of different mass per unit lengths. We observe that in such a case energy is partly transmitted into the second string and the rest is reflected back along the first. The phenomenon of partial reflection and transmission at a junction of strings has its analog in the behaviour of all waves at interfaces between two different media. Shallow water waves are partially reflected if water depth changes suddenly. Light incident on our atmosphere undergoes partial reflection because of changes in the density of the medium. Partial reflection of ultrasound waves at the interfaces of body tissues with different densities makes ultrasound a valuable diagnostic tool.

Does this mean that waves never undergo complete refraction? Were this true, we could not explain the working of lenses, which is fundamental to seeing and our contact with the surroundings. You may have seen the sun before actual sunrise and after actual sunset. This is because of refraction of light in the atmosphere.

In Sections 7.2 and 7.3 you will learn, using the concepts of Huygens' construction and the concept of impedance, that when a wave is incident at a boundary separating two media, its wavelength changes but frequency remains constant. But there are many situations where the frequency of a wave also undergoes a change. This effect is known as Doppler Effect. You will learn it in Section 7.5.

After going through this unit, you will be able to:

- define a wavefront
- construct the wave front for a given source
- explain reflection and refraction of waves using Huygens' construction
- compute the reflection and transmission amplitude coefficients
- compute reflection and transmission energy coefficients
- compute the apparent frequency of sound when the source and/or the observer (listener) are in motion.

7.2 THE CONCEPT OF WAVEFRONT AND HUYGENS' CONSTRUCTION

Let us consider the propagation of a wave on the surface of water. If you dip your finger in water repeatedly, a series of crests and troughs travel out. That is, waves set out in all directions. At any instant, a trough or a crest is circular in shape. The locus of points in the same phase at a particular time is called a *wavefront*. The shape of the wavefront depends on the nature of source. In the case of waves from a point source in air, the wavefronts are spherical. (In two dimensions, as on the water surface, the wavefronts are circular.) If the source is a long slit, the wavefront will be cylindrical. At large distances from the source (whether point or slit), the wavefront appears to be a plane. To understand the formation of wavefronts, we use Huygens' construction.
Following Huygens, we make the following assumptions:

i) Each point on a wavefront becomes a fresh source of secondary wavelets, which move out in all directions with the speed of the wave in that medium.
ii) The new wavefront, at any later time, is given by the forward envelope of the secondary wavelets at that time.
iii) In an isotropic medium, the energy carried by waves is transmitted equally in all directions.

If S is the source of sound or light (Fig. 7.la), then after an interval of time t, all particles of the medium lying on the surface *AB* vibrate in the same phase. This is because all particles on the surface *AB* are equidistant from the source. Any disturbance emanating from *S* is handed on to them at the same time.

According to Huygen's construction, surface *AB* is called a primary wavefront. Each point on *AB*, like *a, b, c*, etc., acts as secondary source (derived from the original source S). These secondary sources give out waves (or disturbances) in all directions as demonstrated by drawing circles around the points *a, b, c,* etc. The envelope of all these waves (which acts as a tangent to all of them at any given instant), like the one at *CD*, forms another wavefront, called the *secondary* wavefront. This, in short, means that the source S gives out wavelets in all directions. The envelope of these wavelets acts as a primary wavefront. Each point on this primary wavefront acts as a source for secondary wavelets. An envelope of these secondary wavelets forms a secondary wavefront. Each point on this secondary wavefront gives out further wavelets to form further secondary wavefronts. This process goes on and the wave keeps on spreading in space. The direction SP (Fig. 7.la) in which the disturbance (originating at S) propagates is called a *ray*. A ray is always normal to the expanding wavefront.

To visualise the Huygens' construction in space, you may imagine a point source to be at the centre of a hollow sphere. The outer surface of this sphere then acts as a primary wavefront. If this sphere is further enclosed by another hollow sphere of larger radius, the outer surface of the second hollow sphere will then act as a secondary wavefront. If this sphere is further enclosed by another sphere of still bigger radius, the surface of the outermost sphere becomes the secondary wavefront. For this, the surface of the inner sphere acts as the primary wavefront. In two dimensions, the primary and secondary wavefronts appear to be concentric circles, the parts of which are shown in Figs. 7.la and 7.1 b.

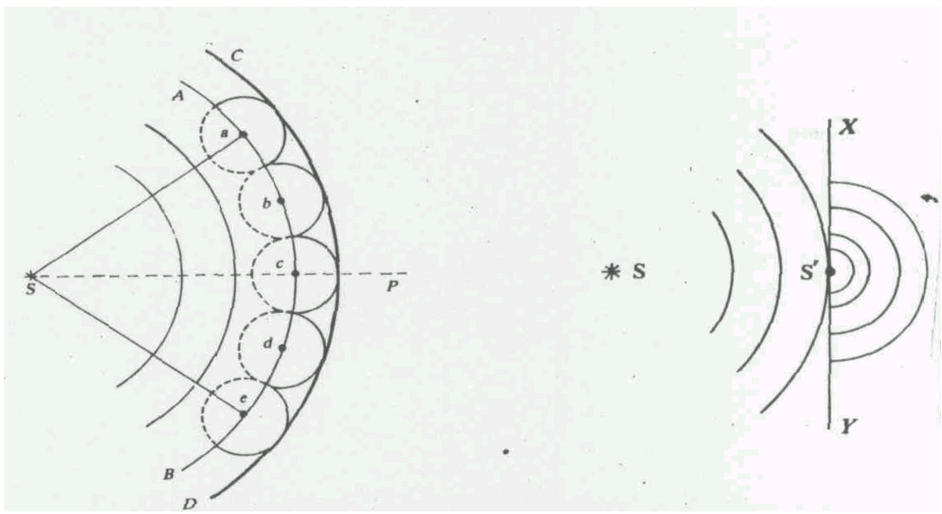


Fig. 7.1 (a) Construction of Huygens' wavefront, (b) Depletion of • Secondary source

The formation of secondary sources as visualised by Huygens can also be understood pictorially through a simple diagram. If we place a screen *XY* with a tiny hole at S' in the path of waves emanating from the source S, S' acts as a secondary source (Fig. 7.1b). This gives out waves on the other side of the screen. These waves spread out from *S' as* if *S'* is an original source itself.

In your school classes you have studied reflection and refraction of waves. We observe these whenever a wave travelling in one medium, say air, meets the boundary of another medium. Suppose we clamp one end of a string to a rigid wall and generate a pulse by moving the other end. You will observe that the pulse is reflected at the fixed end. Similarly, you can study the reflection of ripples in a water basin. You will be surprised to know that the same physical laws govern the reflection (refraction) of all waves, including light. We will now consider *reflection* and *refraction* of waves using Huygens' wave theory.

7.2.1 Reflection of Waves

Refer to Fig. 7.2. LM represents a part of a plane wavefront travelling towards a smooth reflecting surface S_1S_2, It first strikes at *A* and then at successive points towards *D*. If *v* is the wave speed, the point *M* on the wavefront reaches *D* at a time $t = DC/v$ later compared to the point *L*. According to Huygens' Principle, each point on the reflecting surface will give rise to secondary wavelets. In this case we expect that they should constitute the reflected wavefront.

Can you locate the reflected wavefront? To discover this, we note that at the instant D is just disturbed, the wavelet from A has grown for time DC/v and has travelled to E so that the distance AE is equal to DC. We can draw a circle of radius AE (= DC) to represent this wavelet

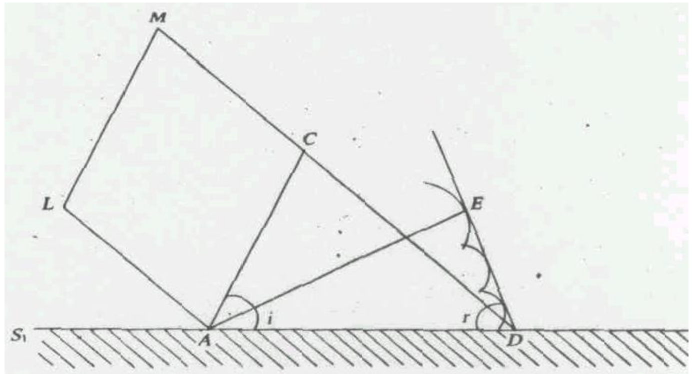


Fig. 7.2 Huygens' construction for reflection of wives

with A as centre. Similarly, we can draw many circles from the intermediate points. The tangent or the envelope to these circles from D defines the reflected wavefront.

From Fig, 7.2 it is clear that Δs ACD and DEA, are congruent. Hence

$$\angle CAD = \angle ADE$$

or

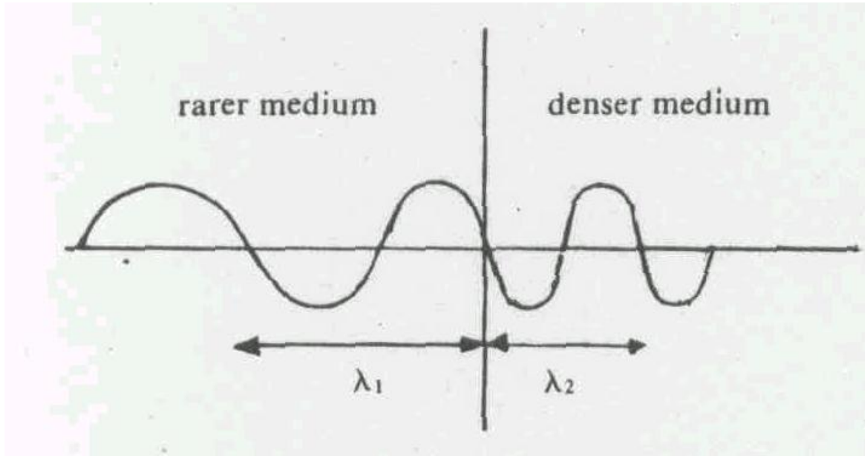
$$\angle i = \angle r \tag{7.1}$$

That is, the angle of incidence is equal to the angle of reflection. Moreover, you will note that the incident ray, the reflected ray and the normal at the point of incidence lie in the plane of the paper.

In this connection it is important to mention here that the reflected wavefront undergoes a phase change of π. In fact, it is true for any wave travelling in a rarer medium (air) and undergoing reflection at the interface with a denser medium (water). However, the reverse is not true.

7.2.2 Refraction of Waves

When a wave reaches the boundary of two different media, it may be partly reflected and partly transmitted. You can study this by joining two strings: one thick and another thin so that their mass per unit lengths are different- In Unit 6, you have learnt that the velocity of a wave is inversely proportional to the density of the medium. This means that when a wave moves from a lighter to a denser medium, its velocity decreases. This results in a change (decrease) in wavelength. But the frequency remains the same. Fig. 7.3 depicts this situation when a wave is refracted (i.e. only transmitted).



$$\lambda_1 > \lambda_2$$
$$v_1 > v_2$$

Fig. 7.3 Refraction of a wave changes its wavelength

Let v_t and λ_t respectively denote the speed and wavelength of a wave of frequency f in a rarer medium. On being refracted at the interface of a denser medium, let its speed and wavelength be v_d and λ_d, respectively. Mathematically, we can connect these quantities through the relation

$$\frac{v_t}{\lambda_t} = \frac{v_d}{\lambda_d} \tag{7.2}$$

since f is the same.

This relation holds for waves in water, air and string alike.

Using Huygens' principle, you can prove the laws of refraction as well (TQ1). But you will agree that Huygens' method is essentially geometrical and can be used when the wave is either reflected or refracted at the interface. You may now ask: Can we apply this method to study partial reflection and refraction, as in the case of two strings having different mass per unit lengths? In principle, we .can do so but it is more convenient to study partial reflection and refraction in terms of impedance offered by a medium. To this end, we normally compute reflection and transmission amplitude coefficients. You will now learn to compute these in the following section.

7.3 REFLECTION AND TRANSMISSION AMPLITUDE COEFFICIENTS

From Unit 6 you would recall that different media offer different impedances to waves travelling through them. These impedances depend on the properties of the medium. You may like to know how waves respond to the abrupt change of impedance at the boundary of the media? We now answer this interesting question by considering transverse waves.

7.3.1 Transverse Waves

Let us reconsider the strings *AO* and *OB* joined together at *O* and kept under the same tension *T*. Let us assume that they offer characteristic impedances of Z_1 and Z_2, respectively. A wave travelling in the positive x-direction (Fig. 7.4) gets partly reflected and partly transmitted at *O*. The particle displacements due to incident, reflected and transmitted waves can be written as:

$$y_i(x,t) = a_i \sin(\omega_0 t - k_1 x) \tag{7.3}$$

$$y_r(x,t) = a_r \sin(\omega_0 t + k_1 x) \tag{7.4}$$

and

$$y_t(x,t) = a_t \sin(\omega_0 t - k_2 x) \tag{7.5}$$

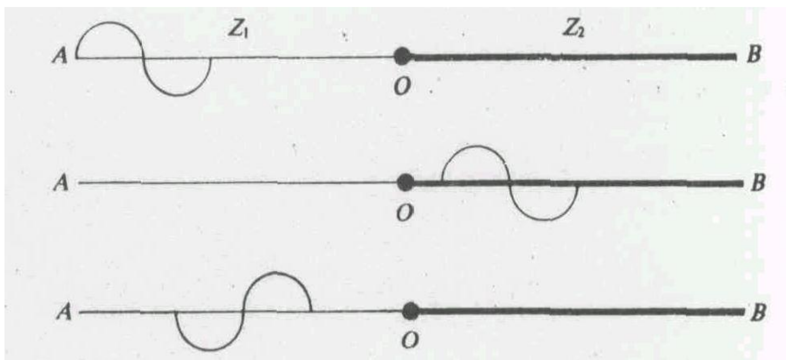


Fig. 7.4 Transverse waves on strings having different mass per unit length

where the subscripts i, r and t on displacements and the amplitudes refer to the incident, reflected and the transmitted waves, respectively. You will note that the angular frequency of these waves remains the same. Moreover the propagation constant for the incident and the reflected waves is the same but differs for the transmitted wave. Do you know why? This is because the wave speed changes as the density of the medium changes. You would also note that for the reflected wave we have used a positive sign before $k_1 x$. This is because it is travelling in the negative x-direction.

To give physical meaning to the reflection and transmission coefficients, we have to consider the *boundary conditions*. The boundary conditions are the conditions which must be satisfied at the interface where the two media meet. Here the total displacement and the total transverse component of tension on one side of the boundary are the result of the combination of incident and reflected waves. So the boundary conditions in this case are:

1. The particle displacements immediately to the left and the right of the boundary (i.e. at x = 0 must also be the same. This implies that the particle velocities $\frac{\partial y(x,t)}{\partial t}$ should also be the same.

2. The transverse components of tension $\left(-T\frac{\partial y(x,t)}{\partial x}\right)$ must also be the same immediately on two sides of the boundary.

These conditions require:

$$y_i(x,t)\big|_{x=0} + y_r(x,t)\big|_{x=0} = y_t(x,t)\big|_{x=0} \tag{7.6}$$

and

$$-T\frac{\partial y_i}{\partial x}\bigg|_{x=0} + T\frac{\partial y_r}{\partial x}\bigg|_{x=0} = -T\frac{\partial y_t}{\partial x}\bigg|_{x=0} \tag{7.7}$$

Using Eqs. (7.3) to (7.5), the condition expressed by Eq. (7.6)

$$a_i \sin\omega_0 t + a_r \sin\omega_0 t = a_t \sin\omega_0 t$$

or

$$a_i + a_r = a_t \tag{7.8}$$

The condition expressed by Eq. (7.7) gives:

$$a_i k_1 T\cos\omega_0 t - a_r k_1 T\cos\omega_0 t = a_1 k_2 T\cos\omega_0 t$$

or

$$k_1 T(a_i - a_r) = k_2 T a_t \tag{7.9}$$

We know that

$$k_1 T = \frac{2\pi}{\lambda_1}T = \frac{2\pi f}{v_1}T = 2\pi\ f m_1 v_1 = 2\pi\ f\ Z_1$$

where Z_1 is impedance offered by the first medium.

In arriving at this result, we have used Eqs. (6.22) and (6.36b). Similarly, you can write

$$k_2 T = 2\pi\ f\ Z_2$$

where Z_2 is the impedance offered by the second medium.

Using these results, we can rewrite Eq. (7.9) as

$$2\pi\ f\ Z_1(a_i - a_r) = 2\pi\ f\ Z_2 a_t$$

or

$$Z_1(a_i - a_r) = Z_2 a_t \tag{7.10}$$

Eqs. (7.8) and (7.10) enable us to calculate the ratios a_r / a_i and a_t / a_i. These ratios give us the fractions of the incident amplitude reflected and transmitted at the boundary. These ratios are usually called the *reflection* and *transmission amplitude coefficients.* We will denote these by the symbols R_{12} and T_{12}*:*

$$R_{12} = \frac{a_r}{a_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{7.11}$$

and

$$T_{12} = \frac{a_t}{a_i} = \frac{2Z_1}{Z_1 + Z_2} \tag{7.12}$$

We note that the reflection *and* transmission amplitude coefficients depend only on the impedances of the two media.

Let us now consider the implications of results arrived at in Eqs. (7.11) and (7.12):

(i) Assume that the string is rigidly fixed to a wall. This means that the second medium is extremely heavy, meaning thereby that $Z_2 = \infty$. In such a case, $R_{12} = -1$ and $T_{12} = 0$. This result implies that $a_r = -a_i$ and $a_t = 0$. That is, the amplitude of reflected wave is equal to the amplitude of incident wave with just a reversal of sign and there is no transmitted wave. This means that the incident wave suffers a change of phase of π on reflection from a denser medium.

(ii) When $Z_2 > Z_1$, i.e. the second string (medium) is denser, R_{12} is still negative, implying a phase change of π on reflection. In this case, however, the incident wave is partly reflected and partly transmitted.

(iii) When $Z_2 < Z_1$, R_{12} is positive, indicating no change of phase on reflection. Both transmitted and reflected waves exist in this case also.

(iv) When $Z_1 = Z_2$, $R_{12} = 0$ showing no reflected wave. In this case $T_{12} = 1$, which gives $a_t = a_i$. This means that the amplitude of a transmitted wave is equal to the amplitude of the incident wave.

The points, (i), (ii) and (iii) above clearly show that if a wave travelling in a medium of lower impedance meets the boundary of a medium of higher impedance (air to water), the reflected wave undergoes a phase change of π. If, however, a wave travelling through a medium of higher impedance meets the boundary of a medium of lower impedance (water to air), no change of phase takes place for the reflected wave. You may also note that T_{12} is always positive, indicating that there is no change of phase for the transmitted wave in any case. These results are depicted in Fig. 7.5.

From Eq. (6.36a, b), you will recall that for a given tension, the wave velocity will be lower in a medium of higher impedance. Using this observation, can you now connect the above discussion with the one given in Sec. 7.2.1? Is there not a one to one correspondence between the two cases? This explains why we expected all waves, whether sound waves, water waves, waves on string or light waves to follow the same laws.

Coming to the point (iv) above, we note that when $Z_1 = Z_2$, the two strings are made up of the same material and there effectively exists no boundary. That is why there is no reflection at all.

SAQ 1

Two strings of linear densities m_1 and m_2 (= $4m_1$) are joined together and stretched with the same tension *T*. For transverse wave, calculate the reflection and transmission amplitude coefficients.

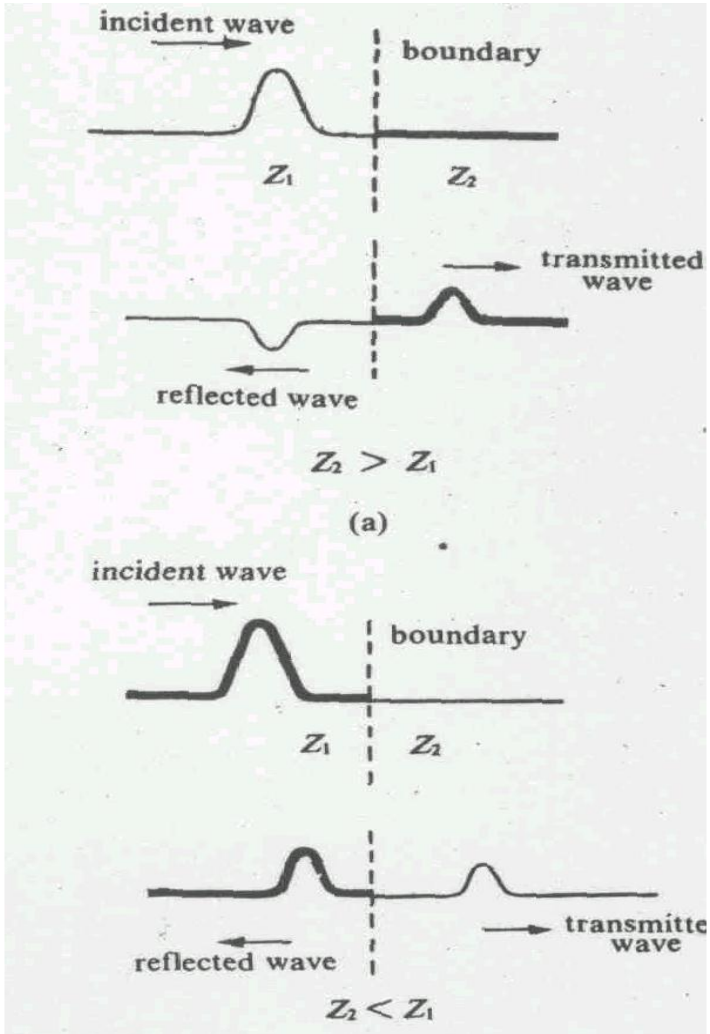


Fig. 7.5 Reflected and transmitted waves when the incident wave (a) travels front a medium of lower impedance to a medium of higher Impedance, and (b) when reverse b the case

7.3.2 Longitudinal Waves

To analyse the reflection and transmission of longitudinal waves, you can follow the same procedure as outlined for transverse waves. Let us consider a wave incident on a boundary at $x = 0$ separating media of acoustic impedances Z_1 and Z_2. As in the case of transverse waves, you can represent the particle displacements for the incident, reflected and transmitted waves by expressions similar to Eqs. (7.3), (7.4) and (7.5).

The boundary conditions in this case are:

(i) The particle displacement $\psi(x,t)$ is continuous at the boundary. That is, it has the same value immediately to the left and right of the boundary at $x = 0$.

(ii) The excess pressure is also the same immediately on the two sides of the boundary.

Using the boundary conditions stated above you can show that the reflected and the transmitted longitudinal waves obey the same characteristics as transverse waves (TQ5).

7.4 REFLECTION AND TRANSMISSION ENERGY COEFFICIENTS

We know that progressive waves are a useful means of transferring energy from one point to another in a medium. It is therefore interesting to consider as to what happens to the energy in a wave when it encounters the boundary between two media of differing impedances. As before, we will consider transverse as well as longitudinal waves.

You have seen in Unit 6 that when a string of mass per unit length *m* vibrates with amplitude a and angular frequency ω_0, the total energy is given by

$$E = \frac{1}{2} m a^2 \omega_0^2 \tag{7.13}$$

Let us assume that the wave is travelling with a speed v. Then the rate at which the energy is carried along the string is obtained by multiplying the expression for energy with the speed v of the wave and is equal to $\frac{1}{2} m a^2 \omega_0^2 v$.

Now refer to the case of the transverse waves discussed in Section 7.3.1. The rate at which the energy reaches the boundary alongwith the incident wave is given by

$$P_i = \frac{1}{2} m_1 a_i^2 \omega_0^2 v = \frac{1}{2} Z_1 \omega_0^2 a_i^2 \tag{7.14}$$

Similarly, the rates at which the energy leaves the boundary alongwith the reflected and the transmitted waves are

$$P_r = \frac{1}{2} Z_1 \omega_0^2 a_r^2 \tag{7.15}$$

and

$$P_t = \frac{1}{2} Z_1 \omega_0^2 a_t^2 \tag{7.16}$$

Using Eqs. (7.8) and (7.10), we can write a_r and a_t in terms of a_i. Substituting the resulting expression in Eqs. (7.15) and (7.16) we find that

$$P_r = \frac{1}{2} Z_1 \omega_0^2 \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2 a_i^2 \tag{7.17}$$

and

$$P_t = \frac{1}{2} Z_1 \omega_0^2 \left(\frac{2Z_1}{Z_1 + Z_2} \right)^2 a_i^2 \quad (7.18)$$

These results can be used to obtain the reflection and transmission energy coefficients R_E and R_E :

$$R_E = \frac{\text{rate at which energy is reflected at the interface}}{\text{rate at which energy is incident at the interface}} = \frac{P_r}{P_i} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2 \quad (7.19)$$

$$T_E = \frac{\text{rate at which energy is transmitted at the interface}}{\text{rate at which energy is incident at the interface}} = \frac{P_t}{P_i} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \quad (7.20)$$

We note from Eq. (7.19) that if $Z_1 = Z_2$: (which is also possible if we have $T_1 m_1 = T_2 m_2$, $R_E = 0$. That is, no energy is reflected back when impedances match. Such an impedance matching plays a very important role in energy transmission. Long distance cables carrying energy need to be matched accurately at all joints; otherwise a lot of energy will be wasted due to reflection. We need impedance matching when we wish to transfer sound energy from air in a loudspeaker to the air of the room. Similarly, when light waves travel from air into glass lens or a slab, we wish not to have reflections (as it will reduce intensity).

SAQ 2
Show that the energy is conserved when a transverse wave meets the boundary between two media of characteristic impedances Z_1 and Z_2.

For longitudinal waves, it is customary to calculate energy transfer in terms of their intensity. From Unit 6 we recall that the intensity of sound waves in a gas is given by

$$\begin{aligned} I &= \frac{1}{2} \rho a^2 \omega_0^2 v \\ &= 2\pi^2 f^2 a^2 Z \end{aligned} \quad (7.21)$$

where Z is the impedance offered by the medium to wave motion. Hence the incident, reflected and transmitted wave intensities can be written as

$$I_i = 2\pi^2 f^2 a_i^2 Z_1 \quad (7.22)$$

$$I_r = 2\pi^2 f^2 a_r^2 Z_1 \quad (7.23)$$

and

$$I_t = 2\pi^2 f^2 a_t^2 Z_2 \quad (7.24)$$

Using these equations, you can easily show that the reflection and transmission energy coefficients are given by

$$R_E = \frac{I_r}{I_i} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2 \quad (7.25)$$

$$T_E = \frac{I_t}{I_i} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \tag{7.26}$$

You will observe that these relations are the same as for transverse waves. This means that the same conclusions hold even for longitudinal waves.

SAQ 3

Sound waves are incident on a water-steel interface. Show that 86% of the energy is reflected back. Impedances of water and steel are respectively $1.43\times10^6\ Nm^{-3}s$ and $3.9\times10^7\ Nm^{-3}s$.

7.5 THE DOPPLER EFFECT

We have so far discussed the situations where the wavelength (or the wave velocity) undergoes a change, but its frequency remains the same. Do you know of any situation where the frequency of a wave changes, or at least appears to change? In this context we are reminded of an anecdote. The famous physicist W.L. Bragg jumped a red-right while driving in London. He was booked for the offence. In the following lines we report the conversation Bragg had with the Magistrate when the latter asked him to appear in his court.

Magistrate: Why did you jump the red light?
Bragg: Sir, I saw it as green light.
Magistrate: At what speed of your vehicle do you see a red light as green?
Bragg: (on some calculation) He could do so if he was driving at about two hundred million kilometers per hour.
Magistrate: O.K. you are now fined for over-speeding.

This dialogue suggests that frequency can change with the speed of the observer or source. You all must have heard the whistle of a moving train. What do you feel when the train approaches you? The pitch of the whistle seems to rise. But when the engine passes by, the pitch appears to decrease. The apparent change of frequency due to the relative motion between the source and the observer (or the listener) is known as the *Doppler Effect.*

In general, when the source approaches the listener or the listener approaches the source, or both approach each other, the apparent frequency is higher than the actual frequency of the sound produced by the source. Similarly when the source moves away from the listener, or when the listener moves away from the source, or when both move away from each other, the apparent frequency is lower than the actual frequency of the sound produced by the source.

Do you know that Doppler shift in ultrasound reflected from moving body tissues allows measurement of blood flow? It is commonly used by obstetricians to detect foetal heartbeat. Do you know how it arises? As the heart muscle pulsates, the reflected ultrasound waves are Doppler shifted from the incident waves. Similarly, a sonar makes use of the Doppler effect in determining the velocity of a submarine relative to a ship.

The electromagnetic waves, including light, are also subject to the Doppler effect. In air navigation, radar works by measuring the Doppler shift of high frequency radio waves reflected from moving aeroplanes. The Doppler shift of starlight allows us to study stellar motion. When we examine light from stars in a spectrograph, we observe several spectral lines. These lines are slightly shifted as compared to the corresponding lines from the same elements on the earth. This shift is generally towards the *red-end* and is attributed to stellar motion. This is illustrated in Fig. 7.6 for hydrogen atoms in a double star system. (The Doppler shift of light from distant galaxies is an evidence that our universe is expanding.)

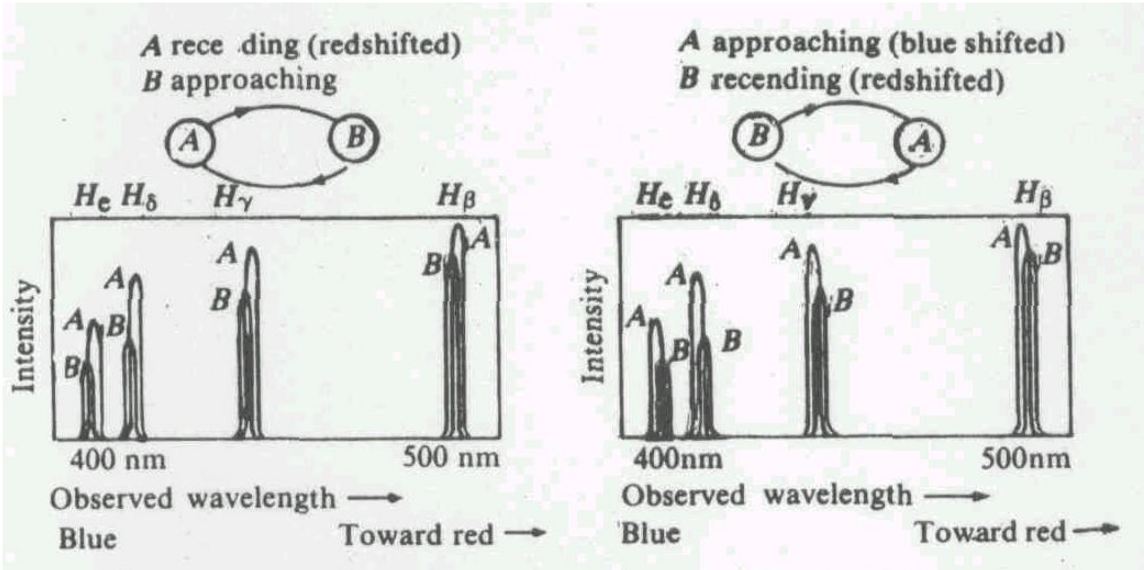


Fig. 7.6 The wavelength of light emitted by hydrogen atoms in a binary star reveals the stellar motion

To study the Doppler effect for sound waves, we have to consider the following situations:

(i) Whether the source is in motion, or the observer is in motion, or both are in motion.

(ii) Whether the motion is along the line joining the source and the observer, or inclined (at an angle) to it.

(iii) Whether the direction of motion of the medium is along or opposite to the direction of propagation of sound.

(iv) Whether the speed of the source is greater or smaller than the speed of sound produced by it.

We will now consider some of these possibilities.

7.5.1 Source in Motion and Observer Stationary

Let us suppose that a source S is producing sound of frequency f, and wavelength λ. The waves emitted by the source spread out as spherical wavefronts of sound. When the velocity of the source is less than the velocity of sound, wavefronts lie inside one another. The distance between successive wavefronts is minimum along the direction of motion and maximum in a direction opposite to it (Fig. 7.7).

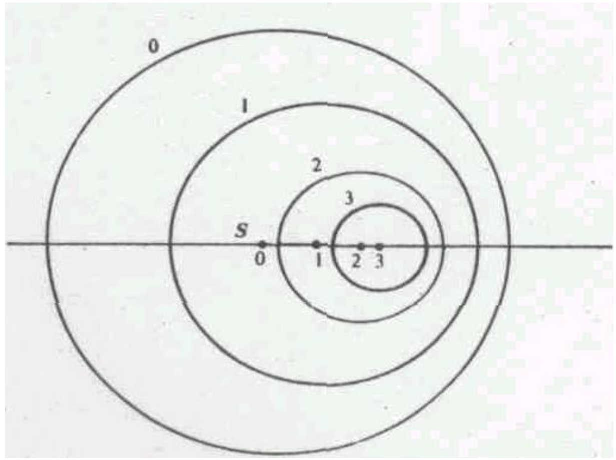


Fig. 7.7 Successive wavefronts emitted by a moving source

Representing the same situation in terms of waves, as shown in Fig. 7.8a, we find that if v the speed of sound produced, f waves occupy a length v in one second, if the source is stationary. After one second, when the source has moved a distance u, towards the listener, the same number of waves get crowded a length $(v-u_s)$ as shown in Fig. 7.8 (b).

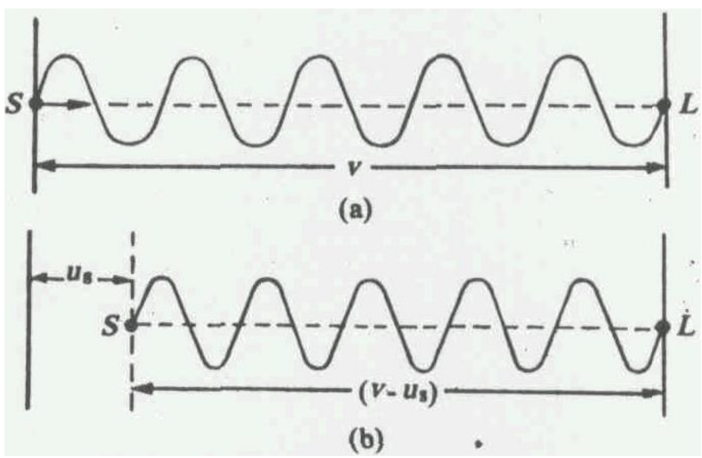


Fig. 7.8 Crowding of waves when source H moving

The reduced wavelength, λ', then becomes

$$\lambda' = \frac{v-u_s}{f}$$

The apparent frequency of sound (heard by the listener) is then

$$f' = \frac{v}{\lambda'} = f\,\frac{v}{v-u_s} \tag{7.27}$$

If, however, the source moves away from the observer (in a direction opposite to sound), u_s, is negative and Eq. (7.27) becomes

$$f' = f\frac{v}{v+u_s} \tag{7.28}$$

To fix up the ideas discussed above, you may now like to solve a SAQ.

SAQ4

A person is standing near a railway track. A train approaches him/her with a speed of 72 kmh^{-1}. The apparent frequency of the whistle heard by the person is 700 Hz. What is the actual frequency of the whistle? Use the speed of sound in air as 350 ms^{-1}.

7.5.2 Source Stationary and Observer in Motion

If the observer is at rest, the length of the block of waves passing him per second is v and contains f waves. However, when the observer moves with speed u_0, he will be at O' after one second and find that only a block of waves with length $(v-u_0)$ passes him in one second. For him the apparent frequency is then

$$f' = \frac{v-u_0}{\lambda} = f\frac{v-u_0}{v} \tag{7.29}$$

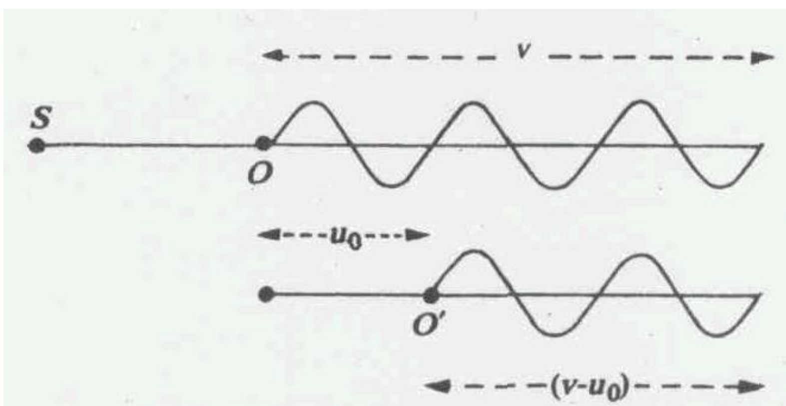


Fig. 7.9 Waves received when listener is moving

If the listener moves towards the source, u_s is negative, and the apparent frequency is given by

$$f' = f\frac{v+u_0}{v} \tag{7.30}$$

7.53 Source and Observer both in Motion

When both source and observer are in motion (and approach each other), we have to combine the results contained in Eqs. (7.27) and (7.29). The source in motion causes a change in wavelength. The listener in motion results in a change of number of waves received. In such a case, apparent frequency f' is given by

$$f' = \frac{\text{length of block of waves received}}{\text{reduced wavelength}}$$

$$= \frac{v - u_0}{v - u_0} \tag{7.31}$$

You may now ask: Is there any difference in the apparent frequency when the source approaches the listener or the listener approaches the source with the same velocity? Eq. (7.31) tells us that the apparent frequency will be different in these cases.

For electromagnetic waves, Eq. (7.31) has to be modified. For sound, u_0 and u_s are measured relative to the medium. This is because the medium determines the wave speed. However, e.m. waves do not require a medium for propagation so that their speed relative to source or the observer is always the same. For these waves we have to consider only the relative motion of the source and the observer. If u_0 is the speed of source relative to observer, and $u_s << v$, we can rewrite Eq. (7.31) as

$$f' = f\frac{u}{v - u_s}$$

$$= f\left(1 - \frac{u_s}{v}\right)^{-1} \tag{7.32}$$

Using binomial expansion and retaining only first order terms in (u_s / u_v), we get

$$f' = f\left(1 + \frac{u_s}{v}\right)$$

In air navigation, we take u_s to be twice the approach velocity of the aeroplane. This is because the radar detects e.m. waves sent by it and reflected back by the aeroplane.

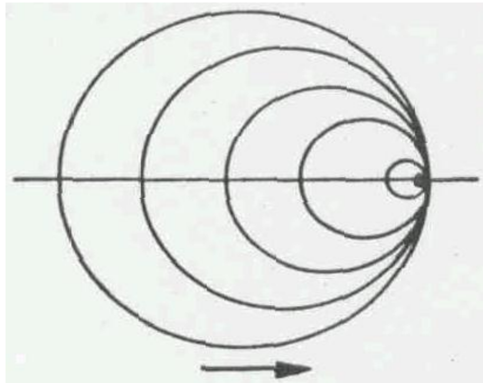
SAQ 5

A stationary observer notes that the spectral line of wavelength 4000 A emitted by a star is shifted towards the red from its normal position by 100 A. Calculate the speed of the star in the line of sight? Speed of light = $3\times10^8\ ms^{-1}$.

7.6 SHOCK WAVES

So far we have considered the cases where the velocity of sound is greater than the velocity of the source. As u_s increases, Eq. (7.31) predicts that Doppler shifted frequency will increase gradually and diverge for $u_s = v$. What does this mean? When the source moves exactly at wave speed, wave crests emitted in the forward direction pile up into a very large amplitude at the front of the source, as shown in Fig. 7.10.

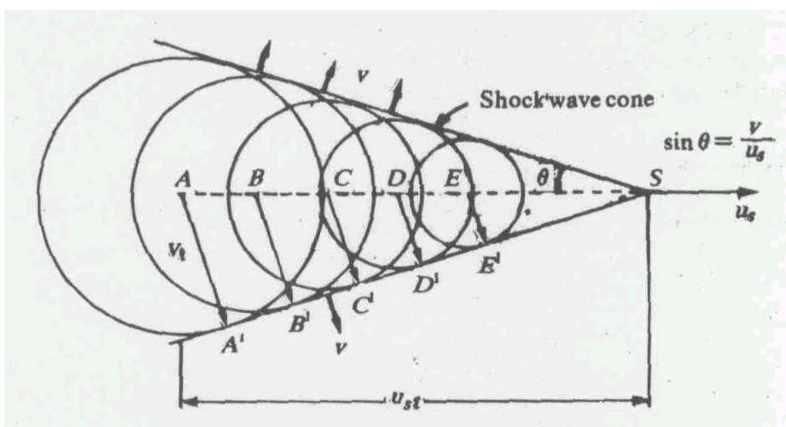
Now you may ask: What happens when the speed of source exceeds the speed of sound waves as for supersonic planes? To discover the answer to this question, let us see if we can draw wave patterns similar to those shown in Fig. 7.7.



$u_s = v$

Fig, 7.10 Schematic representation of piling of waves when the source moves at the wave speed

Let us suppose that the source is at point *A* at $t = 0$. After time t, the waves emitted at A are on a sphere of radius vt. Since $u_s > v$, the distance travelled by the source *AS* = $u_s t$ *is* more than the distance travelled by sound waves. The waves emitted at successive points, *B*,*C*,*D*, *E*.... are on the line *A'S*, where the circles are most crowded. We thus see that sound waves pile up on a cone whose half angle is given by $\sin\theta = v/u_s$ as shown in Fig. 7.11a. No sound waves are present outside this cone. The velocity of sound waves is normal to the surface of the cone. When this cone hits an observer, he detects the sudden arrival of a large amplitude wave, known as a *shock wave*. A supersonic aircraft generates shock waves, also called *sonic booms*, due to the formation of two principal shock fronts; one at its nose and the other at its tail (Fig. 7.11b). A strong boom can break window glasses or cause other damage to buildings.



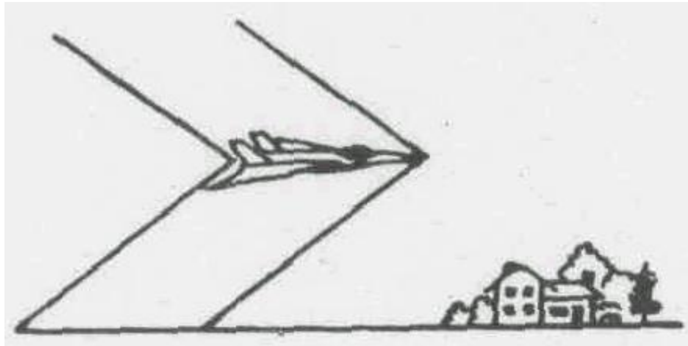


Fig. 7.11 (a) Shock waves created by a sound moving faster than the speed of sound,
(b) Sonic booms produced by a supersonic aircraft

Shock waves are also generated in a ripple tank by a moving source for Mach numbers greater than one. You can also observe that shock waves are formed by a boat moving faster than the speed of water waves.

7.7 SUMMARY

- The locus of points in a given phase is called a wavefront. The shape of a wavefront depends on the nature of the source.
- According to Huygens, each point on a wavefront becomes a fresh source of secondary wavelets, which move out in all directions with the speed of the wave in that medium.
- When waves travelling through one medium meet the boundary of another medium with a different impedance, they are partly reflected and partly transmitted. The reflection and transmission amplitude coefficients are respectively given by

$$R_{12} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

and

$$T_{12} = \frac{2Z_1}{Z_1 + Z_2}$$

- When a wave travelling from a medium of lower impedance is reflected from a medium of higher impedance, a phase change of π takes place.
- Due to the relative motion between the source of sound waves and the observer (listener), the apparent frequency of the sound is different from the actual one. This is known as the Doppler effect. The Doppler shifted frequency (when both approach each other) is given by

$$f' = f\,\frac{v - u_0}{v - v_s}$$

7.8 TERMINAL QUESTION

1. Using Huygens construction, verify that $\mu_{12} = \frac{v_1}{v_2} = \frac{\sin i}{\sin r}$

2. A sound wave travelling through air falls normally on the surface of water. Calculate the ratio

of the amplitude of sound wave that enters the second medium to the amplitude of the incident wave. Use $\rho = 1.29\ kgm^{-3}$. Speeds of sound in air and water are 350 ms^{-1} and 1500 ms^{-1} respectively.

3. A rope is made up of a number of identical strands twisted together. At one point, the rope becomes frayed so that only a single strand continues (Fig. 7.12). The rope is held under tension and a wave of amplitude 1.0 cm is sent from the single strand. The wave reflected back along the single strand has an amplitude of 0.45 cm. How many strands are in the rope?

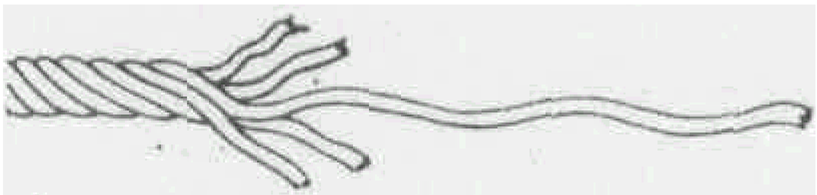


Fig. 7.12 A frayed rope

4. A car moving at a velocity 20 ms^{-1} passes by a stationary source of frequency 500 Hz. The closest distance between them is 20 *m*. Calculate the apparent frequency heard by the driver as a function of distance. Take $v = 340\,ms^{-1}$

5. Using boundary conditions for longitudinal waves, calculate the amplitude reflection and transmission coefficients.

7.9 SOLUTIONS

SAQs

1. We know that impedance is related to tension and mass per unit length through the relation

$$Z = \sqrt{mT}$$

For the given strings

$$Z_1 = \sqrt{m_1 T} \text{ and } Z_2 = \sqrt{m_2 T}$$

$$\therefore \quad \frac{Z_1}{Z_2} = \frac{m_1}{m_2} = \frac{1}{2}$$

From Eqs. (7.11) and (7.12), we note that the reflection and transmission amplitude coefficients are

$$R_{12} = \frac{a_r}{a_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{Z_1/Z_2 - 1}{Z_1/Z_2 + 1} = \frac{1/2 - 1}{1/2 + 1} = -\frac{1}{3}$$

and

$$T_{12} = \frac{a_t}{a_i} = \frac{2Z_1}{Z_1 + Z_2} = \frac{2Z_1/Z_2}{Z_1/Z_2 + 1} = \frac{2}{3}$$

The negative sign in R_{12} implies a phase change of π at the interface.

2. From Eq. (7.14) we know that the rate at which energy reaches the boundary is given by

$$P_1 = \frac{1}{2} Z_1 \omega_0^2 a_i^2$$

Similarly, the rate at which energy leaves the boundary with reflected and transmitted waves is given by

$$P_2 = \frac{1}{2} Z_1 \omega_0^2 a_r^2 + \frac{1}{2} Z_1 \omega_0^2 a_t^2$$

On substituting for a_r and a_t we get

$$P_2 = \frac{1}{2} Z_1 \omega_0^2 \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2 a_i^2 + \frac{1}{2} Z_2 \omega_0^2 \left(\frac{2Z_1}{Z_1 + Z_2} \right)^2 a_i^2$$

$$= \frac{1}{2} Z_1 \omega_0^2 \left[\left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2 + \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \right] a_i^2$$

$$= \frac{1}{2} Z_1 \omega_0^2 a_i^2$$

Since the rate at which energy arrives al the interface is equal to the rate at which energy leaves the interface (with reflected and transmitted waves), we can say that energy is conserved in this process.

3. Reflection energy coefficient

$$R_E = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2 = \left[\frac{(1.43 - 39)10^6\, Nm^{-3}s}{(1.43 + 39)10^6\, Nm^{-3}s} \right]$$

$$= \left(-\frac{37.57}{40.43} \right) = 0.86$$

This means that when sound waves are incident on water-steel interface, only 86% of the energy is reflected back.

4. From Eq. (7.27) we have v

$$f' = f \frac{v}{v - u_s}$$

Rearranging terms, we can write

$$f = f' \frac{v - u_s}{v}$$

Here v = 350 ms^{-1}, f' = 700 Hz and $u_s = 72\, kmh^{-1} = 20\, ms^{-1}$

$$f = \left(\frac{350\, ms^{-1} - 20\, ms^{-1}}{350\, ms^{-1}} \right) \times 700\, Hz = 660 Hz$$

5. Since the wavelength increases, we can say that the star is moving away along the line of sight. This means that frequency decreases. Using Eq. (7.28) for the case of light you can write

$$f' = f \left(\frac{c}{c + u_s} \right)$$

$$= f \left(1 + \frac{u_s}{c} \right)^{-1}$$

$$= f\left(1 - \frac{u_s}{c}\right) \text{ for } u_s << c$$

Since $v = c/\lambda$, you can write

$$\frac{1}{\lambda'} = \frac{1}{\lambda}\left(1 - \frac{u_s}{c}\right)$$

or

$$u_s = \frac{c}{\lambda'}(\lambda' - \lambda)$$

Here λ' = 4100 Å and $c = 3 \times 10^8 ms^{-1}$

Hence,

$$u_s = \frac{3 \times 10^8 \, ms^{-1}}{4000} \times 100$$

$$= 7.3 \times 10^6 ms^{-1}$$

$$= 7.3 \times 10^3 kms^{-1}$$

TQs

1. Refer to Fig 7.13. *AB* represents a part of a wavefront moving towards the interface, S_1S_2 which separates the two media say air and water. Let us assume that wave speeds in medium 1 and medium 2 be v_1 and v_2 respectively.

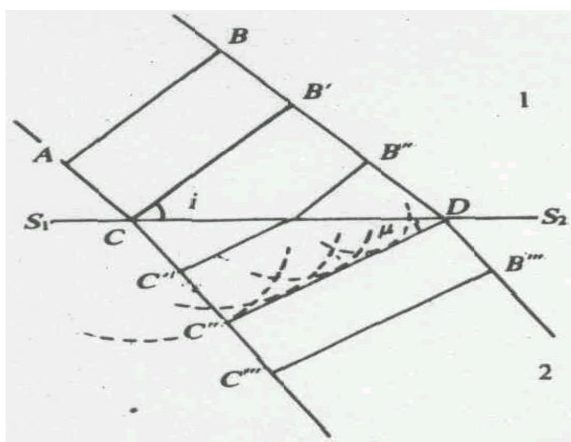


Fig. 7.13: Huygen's construction to deduct the laws of refraction

The wavefront will first strike at *C* and then at successive points towards *D*. The point *B* on the wavefront reaches D at a time $t = B'D/v_1$ later than the point *A* reaches C. From each point on S_1S_2, a secondary wavelet starts growing into the second medium at speed v_2. At the instant when *D* is just disturbed, the wavelet from *C* has grown for time $t(= B'D/v_1)$ and acquired the radius

$$CC' = \frac{B'D}{v_1} v_2 \qquad \text{(i)}$$

You can represent this wavelet by drawing an arc of radius CC' with C as centre. Draw a tangent *DC"* from *D* to this arc If you repeat this process for other intermediate points between C and D you will observe that *DC"* is a common tangent to all of them. Thus, *DC"* represents the refracted wavefront.

From Δs *CB'D* and *CC" D*, you can write

$$\frac{\sin i}{\sin r} = \frac{B'D/CD}{CC''/CD} = \frac{B'D}{CC''} \qquad \text{(ii)}$$

Using the result contained in (i), you would get

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \text{a constant}$$

That is, the sine of the angle of incidence to the sine of angle of refraction of the wavefront is equal to the ratio of the wave speeds and is a constant This constant is known as the *refractive index* of medium 2 with respect to medium I. We denote it by the symbol μ_{12}.

For sound, with medium 1 as air and medium 2 as water

$$\mu_{12} = 0.23$$

and for light

$$\mu_{12} = 1.33$$

2. $$\rho_1 = 1.29\, kgm^{-3}$$
$$\rho_2 = 1000\, kgm^{-3}$$
$$v_1 = 350\, ms^{-1}$$
$$v_2 = 1500\, ms^{-1}$$

Since sound waves are longitudinal, from Eq. (7.12) we have

$$\frac{a_t}{a_i} = \frac{2Z_1}{Z_1 + Z_2} = \frac{2(Z_1/Z_2)}{1 + (Z_1/Z_2)}$$

Since $Z = \rho v$, we can write

$$\frac{Z_1}{Z_2} = \frac{\rho_1 v_1}{\rho_2 v_2} = \frac{1.29\, kgm^{-3} \times 350\, ms^{-1}}{1000\, kgm^{-3} \times 1500\, ms^{-1}}$$

Using this result in (i), we get

$$\frac{a_t}{a_i} = \frac{2(3.01 \times 10^{-4})}{1 + (3.01 \times 10^{-4})} = 6.02 \times 10^{-4}$$

3. From Eq, (7.11) you can write

$$x = \frac{a_t}{a_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$= \frac{(Z_1/Z_2)-1}{(Z_1/Z_2)+1}$$

For a string under tension, $Z \propto \sqrt{m}$, so we can write

$$\frac{Z_1}{Z_2}\sqrt{\frac{m_1}{m_2}}$$

Hence

$$x = \frac{\sqrt{m_1/m_2}-1}{\sqrt{m_1/m_2}+1}$$

Assume that the first portion has n strands. Then

$$x = \frac{\sqrt{n}-1}{\sqrt{n}+1}$$

Solving this for n, we find that

$$\sqrt{n} = \frac{1+x}{1-x} = \frac{1.45}{0.55}$$

Hence,

$$n = \left(\frac{1.45}{0.55}\right)^2$$
$$= 6.76$$
$$\approx 7$$

4. In this case, the velocity of the car is not directed towards the sound source (Fig. 7.14a), and we have to find the component of the velocity vector directed towards the source. Referring to Fig. 7.14b it is given by

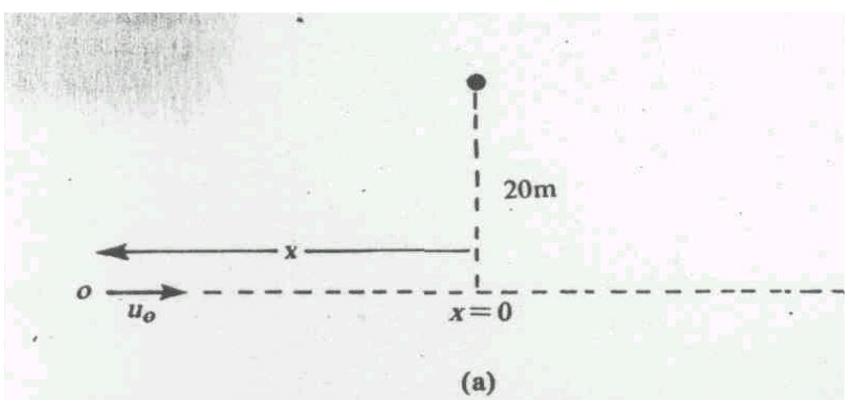
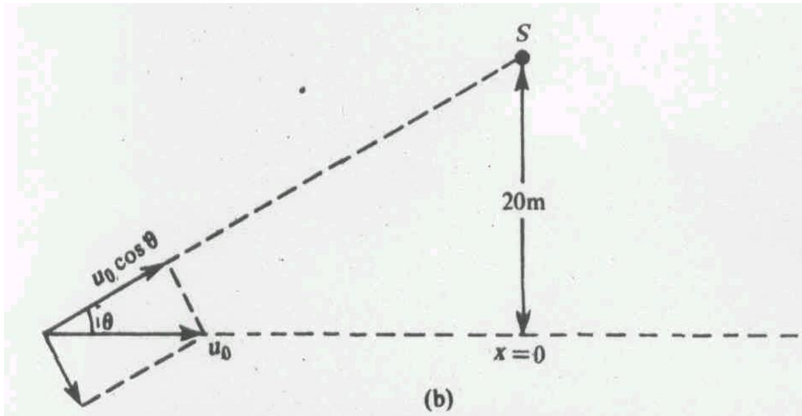


Fig. 7.14 (a) Observer moving along a line not intersecting the line of motion of source



(b) The component of velocity of observer towards the source is responsible for Doppler shift

$$u_0 \cos\theta = u_0 \frac{x}{\sqrt{x^2+20^2}} = 20\ m s^{-1} \frac{x}{\sqrt{x^2+20^2}}$$

Then the space dependence Doppler-shifted frequency is given by

$$v(x) = f_0 \frac{v + u_0 \cos\theta}{v}$$

$$= 500\, Hz\left(1 + 0.06 \frac{x}{\sqrt{x^2+20^2}}\right)$$

You can plot this as a function of x for -100 m < x < 100 m. At x = 0, the car is moving perpendicular to the wave and at the instant when the car passes this point, the driver hears the true frequency, 500 Hz.

5. The particle displacement for the incident, reflected and transmitted waves are

$$\psi_i(x,t) = a_i \sin(\omega_0 t - k_1 x) \quad \text{(i)}$$

$$\psi_r(x,t) = a_r \sin(\omega_0 t + k_1 x) \quad \text{(ii)}$$

and

$$\psi_t(x,t) = a_t \sin(\omega_0 t + k_2 x) \quad \text{(iii)}$$

The boundary conditions in this case are:

1. The particle displacement $\psi(x,t)$ is continuous at the boundary. That is, it has the same value immediately to the left and the right of the boundary at $x = 0$.
2. The excess pressure is the same on the two sides of the boundary.

The first condition implies that

$$a_i + a_r = a_t \quad \text{(iv)}$$

For a longitudinal wave, $\Delta p = -E\dfrac{\partial \psi}{\partial x}$, where E is the elasticity. Since $E = \gamma\, p_0$, where $\gamma = \dfrac{c_p}{c_v}$ and p_0 is the equilibrium pressure, we find that p_0 cancels out on both sides and the second condition implies that

$$\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_r}{\partial x} = \frac{\partial \psi_t}{\partial x} \qquad \text{(v)}$$

Eq. (v) gives:

$$-a_i k_1 \cos\omega_0 t + a_r k_1 \cos\omega_0 t = -a_t k_2 \cos\omega_0 t$$

giving

$$k_1(a_i - a_r) = k_2 a_t \qquad \text{(vi)}$$

We know that

$$k_1 = \frac{\omega_0}{v_1}$$

Multiplying by $\rho_1 v_1$, we get

$$k_1 = \frac{\omega_0}{\rho_1 v_1^2}\rho_1 v_1 = \frac{\omega_0 Z_1}{\gamma\, p_0}$$

Similarly, you can show that

$$k_2 = \frac{\omega_0 Z_2}{\gamma\, p_0}$$

Using these results in (vi), we find that

$$\frac{\omega_0 Z_1}{\gamma\, p_0}(a_i - a_r) = \frac{\omega_0 Z_2}{\gamma\, p_0} a_t$$

or

$$Z_1(a_i - a_r) = Z_2 a_t$$

Since relations (vi) and (vii) connecting the incident, reflected and transmitted amplitudes are exactly the same as in the transverse case, the reflection and transmission amplitude 50 coefficients are also given by the same relations.

UNIT 8 SUPERPOSITION OF WAVES-1

Structure

8.1 INTRODUCTION

You have studied in Unit 2, how a particle acted upon simultaneously by two simple harmonic oscillations gives rise to the formation of Lissajous figures.

You have also read about the general characteristics of waves in Unit 6; and of their behaviour at the interface of two media in Unit 7. In this unit you will study the principle of superposition of waves. Under certain conditions, the superposition of waves leads to some interesting phenomena like the formation of stationary waves, beats, wave groups, interference, diffraction etc. In the present unit you will study the phenomena of stationary waves, wave group and beats. The other two topics, viz. Interference and Diffraction, will be discussed in Unit 9.

In the present unit you will study the basic features, *especially the sound producing part of the woodwind instruments.* There are two basic types of pipes, viz. flute pipe and reed pipe, which you will study in this unit. Stationary waves are formed when two waves of the same angular frequency (i.e., same (ω), same wavelength (i.e., of the same wave vector or propagation constant **k)** and of same amplitude, travelling in opposite directions superpose on each other. On the other hand, if two sound waves of slightly different frequencies are superposed, they produce beats.

Wave groups, sometimes also called the wave packets, are the result of superposition of waves of slightly different frequencies. The concept of wave packet is of great importance in the study of quantum mechanics, which we consider later.

In the next Unit you will study the superposition of two waves, which leads to the phenomena of interference. There you will also study the necessary conditions for the interference of two waves. Towards the end, you will learn about diffraction of waves and some typical cases of diffraction phenomena.

Objectives

- After going through this unit, you will be able to
- Describe the principle of superposition of waves
- Explain the ideas underlying the formation of stationary waves
- Identify the positions of nodes and antinodes on a stationary wave
- List the characteristics of stationary waves
- Describe the formation of wave groups

- Compute the value of group velocity knowing the dependence of wave velocity on wavelength
- Calculate the number of beats produced if the frequencies of two superposing notes are known.

8.2 PRINCIPLE OF SUPERPOSITION OF WAVES

In Unit 2 of this course material, you have studied the superposition of simple harmonic motions. You saw that when two or more simple harmonic motions act simultaneously on a particle, the resultant displacement of the particle at any instant of time is simply given by the algebraic sum of the individual displacements. This can also be extended to the case of waves.

Two or more waves can traverse the same path in a given space, independent of one another. This means that the resultant displacement of a particle at a given time is simply the algebraic sum of the displacements that are given to the particle by the individual waves. In other words, we can say that the resultant displacement of the particle is found simply by adding algebraically the displacements due to the individual waves. This is known as superposition of waves.

An interesting case of superposition of waves is that of radio waves. You know that radio waves of different frequencies are transmitted hy different radio stations to broadcast their programmes. When they fall on the receiving antenna, the resultant electric current set up in the antenna is quite complex because of the superposition of different waves. Nevertheless, we find that we can still tune to a particular station. That is, out of the many, we can still choose and pick up the particular wave we want. In other words, if we have a wave group obtained by the superposition of a large number of individual waves, we can still separate the different waves that were superposed. This is indicative of the individual behaviour of waves, which is the basis of the superposition principle in waves.

Now you can demonstrate the principle of superposition by considering two pulses travelling on a rope in opposite directions as shown in Fig. 8.1. Before and after crossing each other, they act completely independently. At the time of crossing, the resultant displacement is the algebraic sum of the individual displacements.

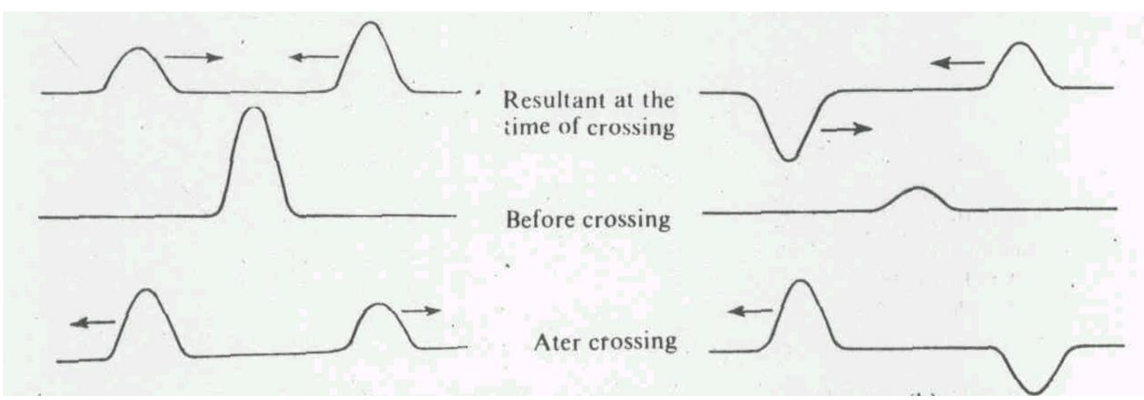


Fig. 8.1 Superposition or two pulses travelling in opposite directions

You have also studied in Unit 2, the mathematical basis for the superposition of oscillations. It lies in the linearity of the equation. Consider two waves acting independently on a particle at any position *x*. Let $y_1(x,t)$ and $y_2(x,t)$ be the displacements of the particle at the instant of time t

due to the two waves. Then the resultant displacement $Y(x,t)$ of the particle is mathematically written as:

$$Y(x,t) = y_1(x,t) + y_2(x,t) \qquad (8.1)$$

You have studied in Unit 6, that a wave is essentially characterised by its amplitude, angular frequency, wave vector and phase. Depending on which of these components are the same or different, you will study the various phenomena in Physics due to the superposition of waves. Let us consider some of these phenomena. For this, you consider the superposition of the following pair of waves.

$$\left.\begin{array}{ll} \text{(i)} & y_1 = a_1 \sin(\omega t - kx) \text{ and } y_2 = a_2 \sin(\omega t - kx) \\ \text{(ii)} & y_1 = a \sin(\omega t - kx) \text{ and } y_2 = a \sin(\omega t - kx + \phi) \\ \text{(iii)} & y_1 = a \sin(\omega_1 t - k_1 x) \text{ and } y_2 = a \sin(\omega_2 t - k_2 x) \\ \text{(iv)} & y_1 = a \sin(\omega t - kx) \text{ and } y_2 = a \sin(\omega t + kx) \end{array}\right\} \qquad (8.2)$$

You will draw the following conclusion from above combination of waves.

(a) In case (i) only the amplitude of two waves differ.
Now let us consider the superposition of two waves of the same angular frequency, wave vector and phase but different amplitude. These two waves are shown in case (i). Now applying Eq. (8.1) we can calculate that the resultant wave is given by

$$\begin{aligned} Y(x,t) &= a_1 \sin(\omega t - kx) + a_2 \sin(\omega t - kx) \\ &= (a_1 + a_2) \sin(\omega t - kx) \end{aligned} \qquad (8.3)$$

Eq. (8.3) implies that the resultant wave has same the frequency and phase and the resultant amplitude is $(a_1 + a_2)$. It is shown in Fig. 8.2.

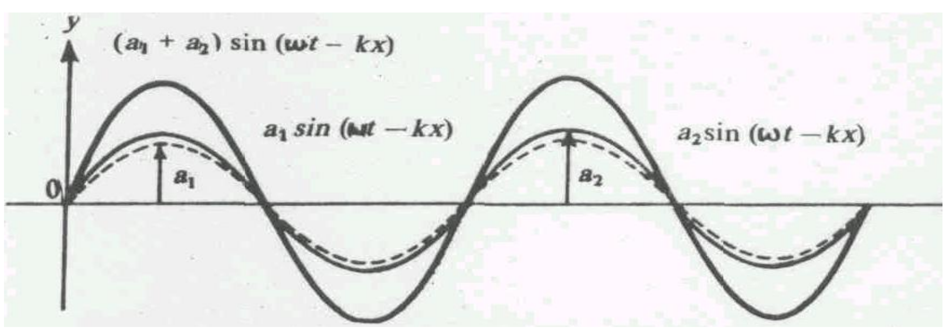


Fig. 8.2 Superposition of two waves of same frequency, wave vector and phase, but differing amplitudes a_1 and a_2

(b) In case (ii) only the phase of two waves differ.
Now you will consider the superposition of two waves which have the same amplitude, frequency and wave vector but differ in phase. When such waves superpose, you will find that the phenomenon of interference will occur. You will study this phenomenon in detail in Unit 9.

(c) In case (iii), the frequency ω and wave vector **k** of the two waves differ.
Now let us consider the case when the frequencies and wave vectors of two waves differ slightly. In such a case, irrespective of phase difference the superposition results in an interesting phenomenon of 'Beats'. If however many waves of slightly different frequencies superpose, then they form a waves group (or a wave packet). This gives rise to group velocity, quite distinct from the wave velocity. You will study group velocity in detail in Section 8.4.

(d) In case (iv), the waves equations have different signs before the wave vector (**k**). In this case, the first wave $y_1(x,t)$ is propagating along the positive direction of x-axis, while the other wave, $y_2(x,t)$, is propagating in negative direction along x-axis. This implies that they are propagating in opposite directions. When such kind of waves superpose then stationary or standing waves are produced. You will study stationary waves in Section 8.3.

8.3 STATIONARY WAVES

You have just learnt in the above section that stationary waves result if two waves of same angular frequency (i.e, ω) and wavelength (i.e. of same wave vector ***k***), and of same amplitude travelling in opposite directions superpose on each other. To realize waves of exactly the same amplitude and wavelength, it is easier to consider one wave as incident wave, and the other as reflected wave from a rigid boundary.

The reflection of the incident wave can take place at a fixed boundary (like that of a string fixed to a wall, or the closed end of an organ pipe) or at a free boundary (like the free end of a string, or the open end of an organ pipe). We have learnt in the last Unit that at a fixed boundary, the displacement $y(x,t)$ stays zero, and the reflected wave changes its sign. At a free boundary, however, the reflected wave has the same sign as the incident wave. In other words, at a fixed boundary, a phase change of π takes place, while at a free boundary, no such change of phase takes place.

Let us consider the case where the reflection is taking place at a free boundary. In this case, the resultant displacement is given by:

$$\begin{aligned} Y(x,t) &= a\sin(\omega t - kx) + a\sin(\omega t + kx) \\ &= 2a\sin\omega t\cos kx \end{aligned} \tag{8.4}$$

This can be written as:

$$Y(x,t) = (2a\cos kx)\sin\omega t \tag{8.5}$$

From Eq. (8.5) you see that the amplitude is given by $(2a\cos kx)$ which is not fixed. It is dependent (or varies harmonically) on the position x of the particle. Further, the resultant motion has the same frequency and the wavelength as the individual waves.

Looking at equation (8.4) we note that the particles distributed along the x-axis execute vibrations perpendicular to the x-axis. The amplitudes with which they execute these vibrations are different at different positions along the x-axis. However, the time period of vibrations of all the particles is same.

We note that Eq. (8.5) does not represent a travelling wave since the argument of the sine function is independent of the space variable x. We thus see that although we started with two waves propagating in opposite directions, we have ended up with something that does not propagate in space. The wave that does not travel (or propagate) is called a stationary (or a standing) wave. Since it does not propagate, it transports no energy *along with it.*
From equation (8.5) it is clear that the displacement $Y(x,t)$ is maximum when

$$\cos kx = \cos\frac{2\pi}{\lambda}x = \pm 1 \tag{8.6}$$

and minimum when

$$\cos kx = \cos\frac{2\pi}{\lambda}x = 0 \tag{8.7}$$

To satisfy Eq. (8.6) we require, $\frac{2\pi}{\lambda}x = m\pi$. Similarly Eq, (8.7) requires $\frac{2\pi}{\lambda}x = (2m+1)\frac{\pi}{2}$ with m = 0, 1, 2, ... These give the points of maximum displacement at x = 0, $\lambda/2$, λ, ..., $m\lambda/2$; and minimum displacement at x = $\lambda/4$, $3\lambda/4$, …, $(2m+1)\lambda/4$.

The points of maximum displacement are called 'Antinodes', while those of minimum displacement are called 'Nodes'. The distance between any two consecutive nodes or antinodes is $\lambda/2$, while that between a node and an antinode is $\lambda/4$ (Fig. 8.3).

From the above discussion you have learnt that a stationary wave results due to the superposition of two identical progressive waves travelling in opposite directions. The result is a non-progressive wave in which the disturbance is not handed over from one particle to the next. The space (or the region) where the two waves superpose gets divided into compartments or segments (Fig. 8.3). Each segment ends with points called the nodes where the displacement of the particles is always zero.

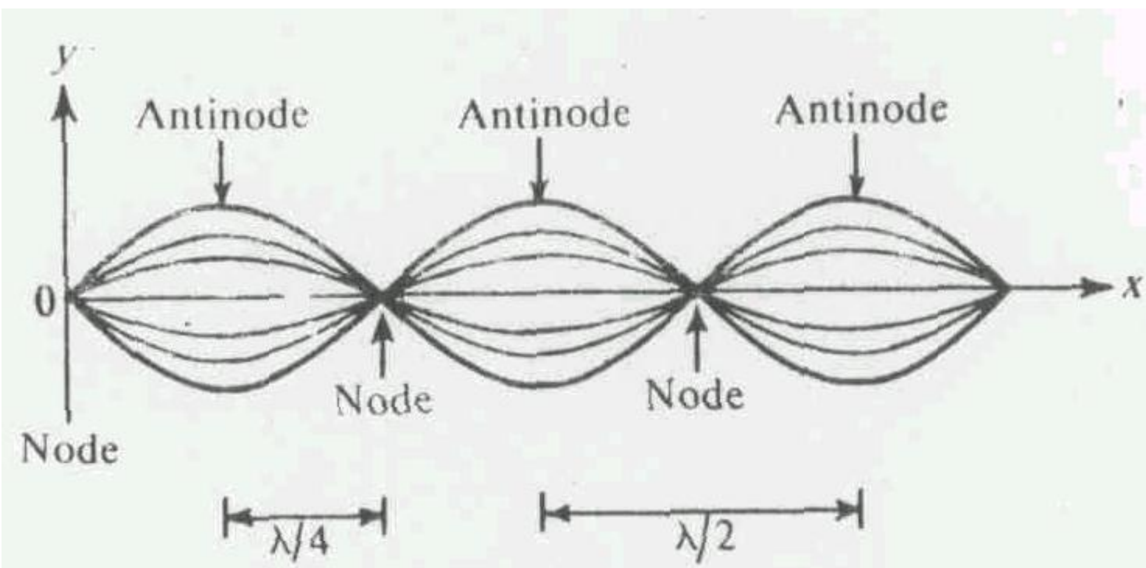


Fig 8.3 The envelope of a standing wave showing the pattern of nodes and antinodes

The particles at the central points of these segments (called the antinodes) execute vibrations with maximum amplitude. The particles lying in-between the nodes and the antinodes execute vibrations with amplitudes lying in between zero and the maximum amplitude. This is shown in Fig. 8.4.

The particle *a*, for example, is always at rest. The particle *b* always executes vibration with maximum amplitude, and the particle *c* always with intermediate amplitude as shown in Fig. 8.4.

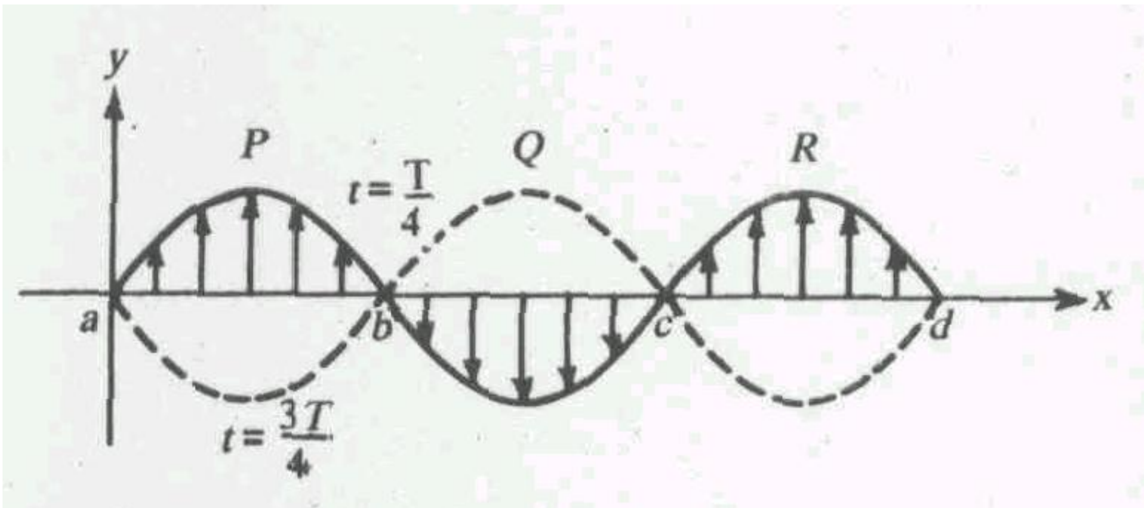


Fig 8.4 Stationary wave, with arrowheads indicating the amplitudes with which various particles vibrate

SAQ 1

Derive equation for the displacement of a particle lying on a standing wave on a string fixed at both ends. Will the fixed end of a string be a node or an antinode? If the standing wave is in an open-ended air pipe, will there be a node or an antinode at the end? How do you explain the absence of energy flow in a standing wave?

8.3.1 Velocity of a Particle and Strain at any Point in a Stationary Wave

You know that the velocity of a particle is defined as the rate of change of displacement with respect to time. The velocity of a particle in a stationary wave is calculated by differentiating the resultant displacement $Y(x,t)$ with respect to time keeping *x* as constant. If we differentiate Eq. (8.5) w.r.t. time, we get

$$\text{velocity} = \frac{\partial Y}{\partial t} = 2a\omega \cos kx \cos \omega t \qquad (8.8)$$

The velocity is maximum when cos $kx = \pm 1$, i.e. at points where $x = 0,\ \lambda/2,\ \lambda,\ ...,\ m\lambda/2$.

(see Eq. (8.6) and the discussion that follows). The velocity is minimum (zero) when cos $kx = 0$, i.e. at points where $x = \lambda/4,\ 3\lambda/4,\ \ldots,\ (2m+1)\lambda/4$. It means that the velocity is maximum at the antinodes where the displacement is also maximum. The velocity is zero at the nodes where the displacement is zero. At points in between the antinodes and nodes, the velocity gradually decreases from maximum at the antinodes to zero at the nodes. The lengths of the arrowheads in Fig. 8.4 may also be taken to represent the velocities of the particles in a stationary wave.

The strain on a particle in a stationary wave can be calculated by differentiating the resultant amplitude i.e. $Y(x,t)$ w.r.t. x keeping t constant. If we differentiate Eq. (8.5) w.r.t. x we get strain

$$\frac{\partial y}{\partial x} = -2ak\sin kx\sin \omega t \tag{8.9}$$

You can show that the strain is maximum for the particles at the nodes where the displacement and the velocity are zero. This can also be visualised from Fig. 8.4. The particles at the nodes are stretched by particles moving in opposite directions. The strain is minimum at the antinodes where the displacement and velocity are maximum. Again referring to Fig. 8.4, we can see that the particles at the antinodes always move along with the particles at their sides, not causing much strain on particles at the antinodes.

In the case of stationary waves, the particles get divided into segments like the P, Q and R in as shown in Fig. 8.4. Particles in one segment always move along in the same direction. When particles in segment P move up, those in Q move down. When those in Q move up, the ones in P move down. That is, in any two adjacent segments, particles move in opposite directions.

All particles in a particular segment reach the extreme positions at the same time, and also pass through the mean positions at the same time. This is shown in Fig. 8.5. All this is possible since all the particles have the same time period T but have different velocities. The particles which have to cover larger distances have greater velocities. Those which have to cover smaller distances, have smaller velocities.

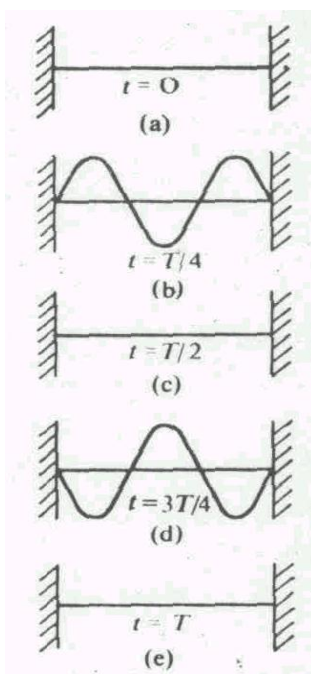


Fig. 8.5 Stationary waves on a string fixed at both ends. Shape of the string at different times during a time period is shown

Now coming to the individual particle, we can see when its velocity is maximum, and when it is zero. Writing Eq. (8.8) as:

$$\begin{aligned}\frac{\partial Y}{\partial t} &= 4\pi af \cos\frac{2\pi}{\lambda}x\cos 2\pi\, f\, t\\ &= \frac{4\pi a}{T}\cos\frac{2\pi}{\lambda}x\cos\frac{2\pi}{T}t\end{aligned}$$

you can see that the particle velocity is zero for $t = T/4$ and $3T/4$, and is maximum for $t = 0$, $T/2$ and T. Thus, during each time period, the particles of the medium have their maximum velocity when they pass through the mean position, and have zero velocity when they are at the extreme positions. Now in the next section you will study the conditions for producing different harmonics in stationary waves.

8.3.2 Harmonics in Stationary Waves

All musical instruments based on strings utilise the stationary wave phenomenon. A string clamped at both ends allows stationary waves with some fixed wavelengths.

If the length of the string is L, the wavelength of the possible stationary waves on this string, starting from the longest wavelength are: $2L$, $\frac{2}{3}L$, $\frac{L}{2}$,..., etc. (See Fig. 8.6).

These wavelengths determine the frequencies of oscillation of the string through the relation $\lambda f = v$. Here v the velocity of the transverse wave on the string. It is given by the relation

$$v = \sqrt{\frac{T}{\mu}}$$

where T is the tension in the string, and μ *is* the linear mass density (mass per unit length) of the string.

The lowest frequency f_0 of vibration is called the fundamental frequency. It is given by:

$$f_0 = \frac{v}{\lambda} = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{8.10}$$

The other frequencies are called the overtones, and are integral multiples of the fundamental frequency f_0 (See Fig. 8.6).

The fundamental frequency is also called the first harmonic. The first overtone, with frequency $f = 2f_0$, is called the second harmonic. The second overtone, with frequency $f = 3f_0$, is called the third harmonic, and so on.

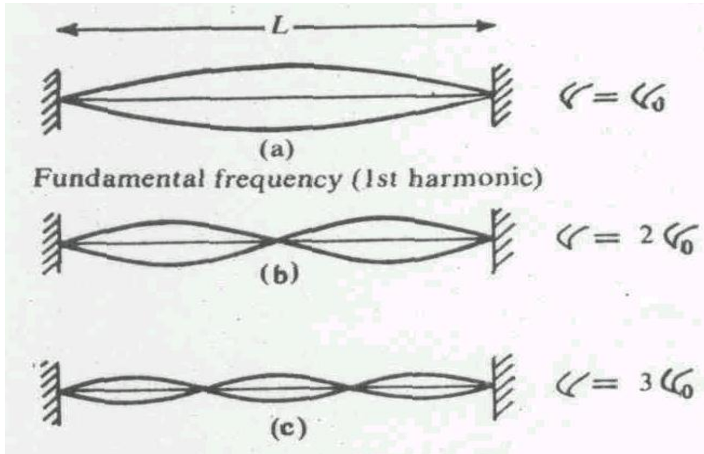


Fig. 8.6 Allowed stationary waves on a string or length L clamped at its ends

The musical instruments based on the principle of standing wave are flute, reed, etc. The primary elements, which determine the tone, quality and overall sound are (1) the source of noise or vibration (2) the size and shape of the bore, and (3) the size and positions of the finger holes. The quality of woodwind tones depends on the combination of physical and *musical experience.* From the physics point of view, the air is stored under pressure in the wind chest. A large reservoir is required to keep the pressure steady, while the various combinations of notes are played with fingers. In the above instruments one end is open, making them open ended organ pipes. The closed end of an organ pipe acts as a fixed boundary, while the open end as a free boundary. At the closed end there is always a node, and at the open end there is always an antinode.

For a pipe having one end closed, the fundamental wavelength is $\lambda = 4L$. This gives the fundamental frequency $f_0 = v/4L$. In such a pipe, the even-numbered harmonics are absent (See Fig. 8.7). For a both ended open pipe, the fundamental wavelength is $\lambda = 2L$, giving fundamental frequency $f_0 = v/2L$.

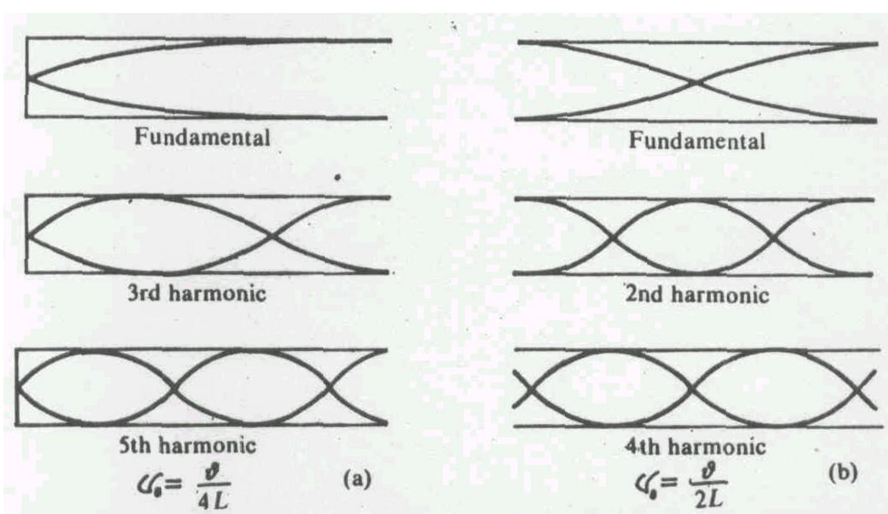


Fig. 8.7 Modes of vibration of longitudinal stationary waves in organ pipes with (a) one end closed, and (b) both ends open

SAQ 2

(a) A piano string of length 1 m (mass per unit length 0.015 kgm^{-1}) fixed at both ends is to be used to strike a fundamental note of frequency f = 220 Hz. Find the tension to be applied to the string.

(b) Estimate the frequency of the fundamental mode in a one end closed organ pipe of length 1.0 m. Use velocity of sound v =350 m/s. What happens to frequency if the pipe is overblown?

8.3.3 Properties of Stationary Waves

The properties of stationary waves that distinguish them from progressive waves have been highlighted in the foregoing discussion. Can you now write down the various points which characterise the stationary waves. After doing this, compare your points with the ones listed below:

(i) Stationary waves are not progressive. In these the disturbance is not handed over from one particle to the next.

(ii) The amplitude of each particle is not the same. It is maximum at the antinodes and zero at the nodes. In between, it gradually decreases from that at the antinode to the one at the node, i.e., zero.

(iii) The distance between two consecutive nodes or two consecutive antinodes is half the wavelength of the stationary wave. The medium splits into segments, with the length of each segment equal to half the wavelength.

(iv) All the particles between two consecutive nodes are in phase, i.e. they reach their maximum and minimum displacement positions (mean positions) at the same time. The phase of particles, in one segment is opposite to that of particles in the adjoining segment.

(v) The velocity of particles at the nodes is zero. The velocity of particles at the antinodes is maximum. For particles in between, the velocity gradually decreases from that at the antinodes to the one at the nodes (i.e. zero).

8.4 WAVE GROUPS AND GROUP VELOCITY

So far we have considered the superposition of two identical waves travelling in opposite directions to give rise to stationary waves. Now let us see what happens when two waves of slightly different angular frequencies ω_1 and ω_2 travelling in the same direction, superpose on each other (Case iii). To avoid unnecessary mathematical complexities, we take the amplitudes of the two waves to be equal. The superposition of such two waves is given by

$$\begin{aligned} Y(x,t) &= a\sin(\omega_1 t - kx) + a\sin(\omega_2 t - kx) \\ &= 2a\sin\left[\frac{(\omega_1+\omega_2)t-(k_1+k_2)x}{2}\right]\cos\left[\frac{(\omega_1-\omega_2)t-(k_1-k_2)x}{2}\right] \end{aligned} \qquad (8.11)$$

If ω_1 and ω_2, and similarly k_1 and k_2, are only slightly different, we can write $\omega_1 - \omega_2 = \Delta\omega$ and $k_1 - k_2 = \Delta k$.

Further, writing

$$\omega_{av} = \frac{\omega_1 + \omega_2}{2} \text{ and } k_{av} = \frac{k_1 + k_2}{2},$$

Eq. (8.11) becomes:

$$Y(x,t) = 2a\sin(\omega_{av}t - k_{av}x)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \tag{8.12}$$

Now let us see what the new wave form represented by Eq. (8.12) looks like. Firstly, its amplitude is twice that of the amplitude of either wave. Secondly, it is made up of two parts. The faster varying part (i.e. sine part) has a frequency which is the mean of the frequencies of the two component waves. The slowly varying part (i.e. cosine part) has a frequency which is half of the difference of the two frequencies. The propagation vector of the slowly varying part of the superposed wave is $\Delta k/2$. It acts as an envelope over the faster varying part as shown in Fig.8.8.

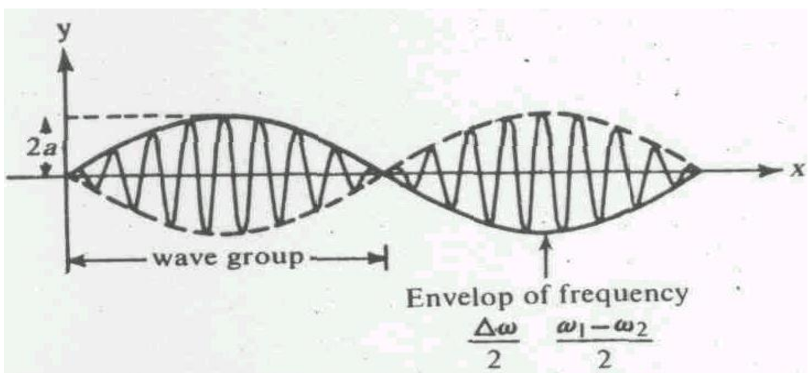


Fig. 8.8 Superposition of two waves of slightly different frequencies ω_1 & ω_2

The superposition, as you can see in Fig. 8.8, results in the formation of groups (or segments) called wave groups (or wave packets). A wave group can travel with a velocity which may be different from that of the individual waves, or of the resultant wave. The velocity of the wave group is called the group velocity. The ratio of angular frequency and wave vector of the slowly moving part of the superposed wave is called group velocity. It is given by the following relation

$$v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k} \tag{8.13}$$

If a group consists of a number of component waves with angular frequencies lying between ω_1 and ω_2 (with $\omega_1 \approx \omega_2$), and similarly in wave vector k_1 and k_2 (with $|k_1| > |k_2|$), the group velocity v_g is then written as :

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \tag{8.14}$$

Here $d\omega$ and dk represent the spreads (gaps between the maximum and the minimum) in angular frequencies and propagation constants of the component waves that go to make a wave group.

The velocity of the resultant superposed wave is called the phase velocity. You can obtain this using Eq. (8. 12), i.e.

$$v_p = \frac{\omega_{av}}{k_{av}}$$

If, however, the individual wave velocities are equal, i.e.

$$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = v \text{ (say)}$$

then, $\quad v_p = \dfrac{\omega_1 + \omega_2}{k_1 + k_2} = \dfrac{k_1 v + k_2 v}{k_1 + k_2} = v$

and $\quad v_g = \dfrac{(\omega_1 - \omega_2)/2}{(k_1 - k_2)/2} = \dfrac{vk_1 - vk_2}{k_1 - k_2} = v$

i.e., the group velocity is equal to the phase velocity.

The group velocity is a more fundamental quantity in physics as the energy is transferred by the wave with the group velocity. The relation between the phase and the group velocities is given by:

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(vk)$$

which on simplification gives

$$v_g = v + k\frac{dv}{dk}$$

If we write

$$k = 2\pi/\lambda \tag{8.15}$$

then, $\quad dk = -\dfrac{2\pi}{\lambda^2}d\lambda$

Inserting this in Eq. (8.15), we get

$$v_g = v + \frac{2\pi}{\lambda}\frac{dv}{(-2\pi/\lambda^2)(d\lambda)}$$

$$= v - \lambda\frac{dv}{d\lambda} \tag{8.16}$$

This gives another relation connecting the phase and the group velocities. The wavelength of the resultant wave is given by

$$\lambda = \frac{2\pi}{k}$$

and that of the enveloping wave by:

$$\lambda_e = \frac{2\pi}{\Delta k/2} = \frac{4\pi}{\Delta k}$$

since Δk is very small compared to k, $\lambda_e >> \lambda$.

If λ_1 and λ_2 represent the wavelengths of the component waves, it can be easily shown that

$$\frac{\lambda_e}{2} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \tag{8.17}$$

This gives the length (or the extent) of the wave group. We can see from Fig. 8.9 that the length of the wave group is half of the wavelength of the enveloping wave, i.e., it is equal to $\lambda_e / 2$.

To illustrate the difference between phase and group velocities, we consider the striking example of waves in deep water – called "gravity waves." These waves are strongly dispersive. For them, the phase velocity is found to be proportional to the square root of the wavelength, i.e.,

$$v_p = C\lambda^{1/2}$$

or

$$v_p = C_1 k^{1/2} \text{ (since } k = 2\pi/\lambda)$$

Here, the new constant, $C_1 = C\sqrt{2\pi}$. $v_p = C_1 k^{1/2}$, therefore, $\omega = C_1 k^{1/2}$. Differentiating ω with respect to k, we get,

$$v_g = \frac{d\omega}{dk} = \frac{1}{2} C_1 k^{-1/2} = \frac{1}{2} v_p$$

That is, the group velocity for gravity waves is just half of the phase velocity. In other words, for these waves, the component wave crests move faster through the group as a whole.

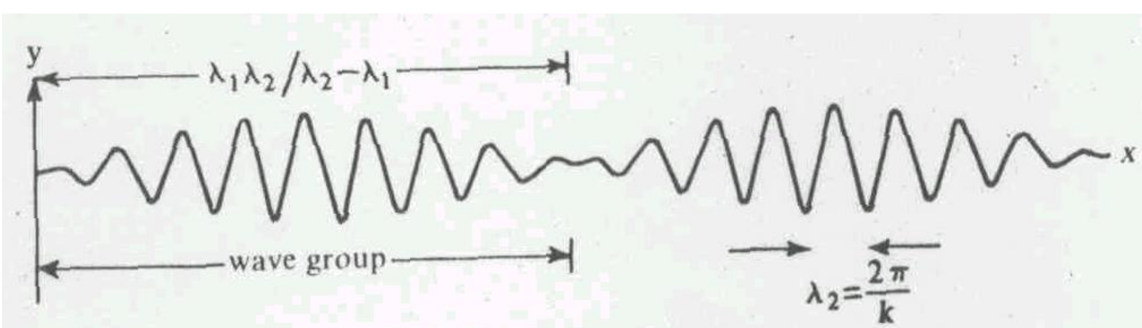


Fig. 8.9 Wave group and its extent

SAQ 3
The phase velocity of a wave in a certain medium is represented by:

$$v = C_1 + C_2\lambda$$

where C_1 and C_2 are constants. What is its group velocity?

8.5 BEATS

We have seen above that the superposition of two waves of slightly different angular frequencies ω_1 and ω_2 leads to the formation of wave groups. You may have noticed from Figs. 8.8 and Fig. 8.9 that we have plotted the resultant displacement $Y(x,t)$ against distance x. This may be called the superposition in space. For this we kept the time t as constant. We may now consider another type of superposition, where we may plot $Y(x,t)$ against t and call it superposition in time. For this we may keep x as constant.

The superposition in time for sound waves leads to the interesting phenomenon of beats. The beats are loud sounds which we hear at regular intervals of time depending on the difference in frequencies of the two superposing waves. The beats are often used by musicians for tuning their instruments.

Let us consider two waves of slightly different angular frequencies ω_1 and ω_2, and of the same amplitude a, proceeding in the same direction, as we have done in the last section. Let us fix the spatial coordinate *x* in Eq. (8.10), say, at *x* = 0. This corresponds to an observer standing at *x* = 0, and watching the waves passing by. He will observe a resultant waveform given by:

$$\begin{aligned} Y(x,t) = Y(0,t) &= a\sin\omega_1 t + a\sin\omega_2 t \\ &= 2a\sin\omega_{av}\cos\frac{\Delta\omega}{2}t \end{aligned} \tag{8.18}$$

Like the earlier case discussed in Section 8.4, Eq. (8.18) indicates that the amplitude of the resultant wave at a given point is not constant, but varies in time. This has an angular frequency $\omega_{av} = \dfrac{\omega_1 + \omega_2}{2}$. Its amplitude varies between $2a$ and zero, because of the presence of the $\cos\left(\dfrac{\Delta\omega}{2}t\right)$ term. This term acts as an envelope on the sine term.

If ω_1 and ω_2 are nearly equal, $\Delta\omega$ is small. In that case the amplitude of the resultant wave varies slowly. The periodic rise and fall of this wave leads to the appearance of beats; or to the hearing of loud sounds at regular intervals of time.

Beat are heard at the maxima of amplitude (See Fig. 8.10). They occur whenever $\cos\dfrac{\Delta\omega}{2}t = \pm 1$. This is because the intensity of sound is directly proportional to the square of the amplitude. The maximum amplitude occurs twice in every time period associated with the angular frequency $\frac{\Delta\omega}{2}$. Thus the frequency of beau is simply the difference of the two component frequencies i.e. $(\omega_1 - \omega_2)$.

In terms of frequencies f_1 and f_2, the beat frequency is $\Delta f = f_1 - f_2 = \frac{\Delta\omega}{2\pi}$. The time elapsed between any two consecutive beats, called the beat period = $\frac{1}{\Delta f}$ (see Fig. 8.10).

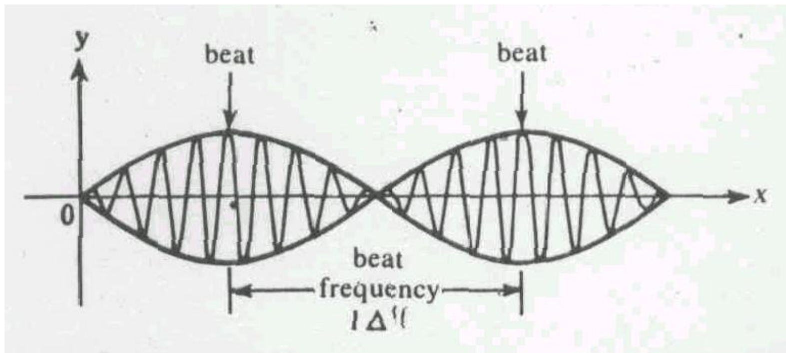


Fig. 8.10 Formation of bead due to superposition of two waves of nearly the same frequency

SAQ 4

When a certain note of a piano is sounded with a tuning fork of frequency 560 Hz, 6 beats are heard every second. Find the frequency of the note.

8.6 SUMMARY

1. When two waves travelling through the same space superpose on each other, the resultant displacement at any point is given by the algebraic sum of the individual displacements.

2. Stationary waves result because of the superposition of two waves of same amplitude, frequency and wavelength travelling in opposite directions and confined between two points.

3. On a stationary wave, nodes and antinodes are points of zero and maximum displacement, respectively. Distance between any two consecutive nodes or antinodes is half the wavelength of the stationary wave.

4. Both transverse and longitudinal waves can have different modes of vibration.

5. Superposition of two waves of slightly different frequencies travelling in the same direction gives rise to a wave group, and beats.

6. The number of beats produced per second is equal to the difference in the frequencies of the two waves.

7. The velocity with which a wave group travels is called the group velocity. It is equal to the wave velocity if the two component waves have the same velocity; otherwise, it is different from the wave velocity.

8. The smaller the difference between the wavelengths of the component waves, the greater the length of the wave group.

8.7 TERMINAL QUESTIONS

1. Two points on a string are observed as a travelling wave passes them. The points are at $x_1 = 0$ and $x_2 = 1$ m. The transverse motion of the two points are found to be as follows:

$$y_1 = 0.2\sin 3\pi t$$
$$y_2 = 0.2\sin(3\pi t + \pi/8)$$

(a) What is the frequency in hertz?
(b) What is the wavelength?
(c) With what speed does the wave travel?

2. Fifty tuning forks are arranged in order of increasing frequency and any two successive forks gives 5 beats per second when sounded together. If the last fork gives the octave of the first, calculate the frequency of the latter. (A note is octave of another nole if its frequency is double that of the other.)

3. A closed pipe, 25cm long, resounds when full of oxygen, to a given tuning fork. Find the length of a closed pipe, full of hydrogen which will resound to the same tuning fork. (Velocity of sound in oxygen = 320 m/s and velocity of sound in hydrogen = 1280 m/s).

4. The phase velocity (v) of transverse wave in a crystal of atomic separation d is given by

$$v = C\frac{\sin(kd/2)}{(kd/2)}$$

where C is a constant. Show that the group velocity is $C\cos(kd/2)$

8.8 SOLUTIONS

SAQ l

Since there is a phase change of π on reflection at the fixed end, the reflected wave is given by:

$$y_2 = -a\sin(\omega t + kx)$$

This leads to the resultant displacement

$$Y(x,t) = a\sin(\omega t - kx) - a\sin(\omega t + kx)$$
$$= -2a\sin kx\cos\omega t$$
$$= A\cos\omega t$$

with $A = -2a\sin kx$

At the fixed end, there is always a node, as the displacement is zero. In open-ended pipes, as shown, there is always an antinode at the end.

A standing wave is formed because of a positive *x* directed incident wave, and a negative *x* directed reflected wave. Each carry the same amount of energy in opposite directions. The net energy flow is thus always zero.

SAQ2

(a) Wavelength of fundamental mode

$$\lambda = 2L = 2\times 1 = 2\,m$$

velocity of wave = $220\,\text{Hz}\times 2\,m = 440\,m/s$

From $v = \sqrt{\dfrac{T}{\mu}}$, $T = v^2\mu$

$= (440\,m/s)^2 \times 0.015 kg/m$

$= 2.9\times 10^3\,N$

(b) Wavelength of fundamental mode

$$\lambda = 4L = 4\times 1m = 4\,m$$

Frequency $f = \dfrac{v}{\lambda} = \dfrac{350\,m/s}{4\,m} = 87.5 \approx 88\,Hz$

By overblowing the pipe, pitch jumps by a factor of 3, giving the next harmonic with frequency $f = 3\times 88 = 264\,Hz$.

SAQ3

We know that

$$v_g = v - \lambda\frac{dv}{d\lambda},$$

For the wave in question, $\dfrac{dv}{d\lambda} = C_2$

Inserting in the above equation,

$$v_g = C_1 + C_2\lambda - \lambda C_2 = C_1$$

SAQ4

Let the frequency of the note be f. Then

6 = [560 − f]

∴ $f = 554\,Hz$ or 566 Hz

In this case, the frequency of the note cannot be found without ambiguity. However, it is either of the above two.

TQs

1(a) f =1.5 Hz

(b) $\lambda = \dfrac{16}{16n-1}m$, $n = 1, 2, 3, \ldots$ for positive moving wave.

$= \dfrac{16}{16n+1}m$, $n = 1, 2, 3, \ldots$ for positive moving wave.

(c) $u = 8/5\,m/s$, etc.

$v = -24\,m/s$, etc.

2. Let the frequency of the first note be *n*.

Then the frequency of the Second fork = n + 5 = n + (2 – 1)5
Frequency of the Third fork = n + 5 + 5 = n + 10 = n + (3 – 1)5
Frequency of the Fourth fork = n + 5 + 5 + 5 = n + 15 – n + (4 – 1)5
Frequency of the Fifth fork = n + 20 = n + (5 – 1)5
Therefore the frequency of the 50th fork

$$n + (50 - 1) \times 5 = n + 245$$

Since the frequency of 50th fork is 2n then
n + 245 = 2n
So n = 245 Hz.

3. For the first pipe, the fundamental frequency

$$f_1 = \frac{v_0}{4l_1}$$

where v_0 is the velocity of sound in oxygen and l_1 *is* the length of the 1st pipe. For the second pipe, the fundamental frequency is

$$f_2 = \frac{v_h}{4l2}$$

Where v_h is the velocity of sound in hydrogen and l_2 is the length of the pipe.
Since both the pipes resound to the same frequency, therefore

$$f_1 = f_2 \text{ or } \frac{v_0}{4l_1} = \frac{v_h}{4l_2}$$

$$\therefore \quad l_2 = \frac{1380\, ms^{-1}}{320\, ms^{-1}} \times 25 = 100\, cm$$

4. Group velocity

$$v_g = \frac{d\omega}{dk}$$

and

$$\omega = kv, \text{ we have}$$

$$= C \sin \frac{(kd/2)}{(kd/2)}$$

Now

$$= kv = kC \frac{\sin(kd/2)}{(kd/2)}$$

$$= \frac{2C}{d} \sin(kd/2)$$

or

$$v_g = \frac{d\omega}{dk} = \frac{2C}{d}\left(\cos\frac{kd}{s}\right)\frac{d}{2}$$

or

$$v_g = C\cos(kd/2)$$

UNIT 9 SUPERPOSITION OF WAVES- II

Structure

9.1 INTRODUCTION

In the last unit you have studied the principle of superposition of waves and employed it to study the phenomena of formation of stationary waves, wave groups and beats.

You have also learnt about the superposition of two waves which have the same amplitude and frequency but differ in phase. When such waves superimpose on each other, the phenomenon of interference takes place. For producing interference, the sources of waves must be coherent. That is, they must emit waves with zero or constant difference of phase. In this unit you will study how coherent sources are produced, and how intensity varies in an interference pattern. You will also learn about the appearance of colours in thin films of oil spread over water.

The phenomenon of diffraction which results due to the superposition of many waves of same amplitude and frequency, but differing slightly in phase, is usually referred to as the bending of waves round the corners. Because of this phenomenon, we are at times inclined to think as if waves do not travel in straight lines. There are two classes of diffraction patterns, called Fresnel and Fraunhofer classes of diffraction. You will learn that the distinction between these two types of diffraction is related to the relative separations between the sources of waves and the obstacles (or the apertures) producing the diffraction patterns.

Both interference and diffraction are very important phenomena in physics. They have contributed immensely in justifying the wave nature of light. The difference between the two is quite subtle. Interference arises because of superposition of waves originating from two (or more) narrow sources, derived from the same source. Diffraction arises from superposition of wavelets from different numerous parts of the same wavefront, as will be discussed later in this unit.

In Unit 6, you have studied different kinds of waves like sound and light waves. You have also studied that sound waves are longitudinal while light waves are transverse. Both give rise to the same phenomena when waves superpose on each other. Basically whatever is true for one kind of wave is also true for the other; if light waves show the phenomenon, of interference and diffraction, so do sound waves. Light wave effects have to be observed while sound wave effects have to be heard. Since the wavelength of sound waves is much greater than the wavelength of visible region light waves, the sound wave effects are in general on a larger scale compared to the effects of the light waves.

Objectives

After going through this unit, you will be able to:

- give examples of coherent sources
- derive the condition connecting the path difference between waves from two coherent sources and the wavelength of the waves used for getting maxima and minima of intensity on a screen placed in the path of waves
- outline the variation of intensity in an interference pattern
- explain the principle associated with the appearance of colours in thin films
- explain the phenomenon of diffraction
- explain the diffraction obtained by a single slit, and
- describe the intensity distribution in a diffraction pattern.

9.2 INTERFERENCE

In Unit 8, you have studied the superposition of two waves. You have seen that under certain special condition superposition leads to the phenomenon of interference. We will now study the phenomenon of interference in detail. Let us consider the superposition of the following two waves:

$$y_1 = a\sin(\omega t - kx)$$

and

$$y_2 = a\sin(\omega t - kx + \phi).$$

These two waves have the same angular frequency ω and the same wave vector *k*, and are travelling along the same direction. They have a phase difference ϕ that remains constant with time. Can you determine the energy distribution after these waves have superposed? If you try, you will find that this distribution is not uniform in space. The energy is found to be maximum at certain points and minimum (or probably zero) at others. This type of energy distribution in space is known as an interference pattern.

In Fig. 9.1, the interference pattern of two waves in a shallow water tank is shown. Here S_1 and S_2 are two sources which produce circular waves on the water surface. The sources S_1 and S_2 have to be adjusted in such a way that the waves produced by them on the surface of water are in constant phase. The resultant of these waves will produce an interference pattern. When the crest of one wave falls on the crest of another wave, a stronger crest (i.e., one with larger amplitude) is produced. Similarly, when a trough of one wave falls on the trough of another wave, a shallower trough is produced. However, when a crest of one wave meets a trough of another wave, their effects cancel out. Thus it leads to doubling of the amplitude at some points and its reduction to zero at others. This leads to, what we call, 'an interference pattern.'

Fig 9.1 Interference on the surface of water waves

In most interference experiments performed in the laboratory, the interference pattern is in the form of fringes. The interference fringes are alternately bright and dark bands, as shown in Fig. 9.2.

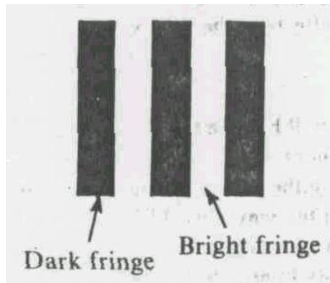


Fig 9.2 Interference fringes.

A bright band appears on the screen wherever the crest of one wave falls on the crest of another, or wherever the trough of one falls on the trough of another. Wherever the crest of one falls on the trough of another, a dark band is produced.

From the above discussion, we can draw the conclusion that for producing an interference pattern, you require basically two sources. Now, the question arises, what should be the nature of these sources?

9.2.1 Coherent Sources

For producing interference we require two coherent sources. Let us now discuss what coherent sources are and what their special properties are.

We find from experience that to have a stable and a well defined interference pattern, the two sources must emit waves either with zero, or with constant difference of phase, say, ϕ. If the sources emit waves with zero or constant difference of phase, they are called coherent sources. How can we obtain such sources? The easiest way to obtain coherent sources is to obtain them from the same original source.

One way to obtain such sources in optics is to put an opaque screen containing two slits in the path of waves emitted by a single source, as shown in Fig. 9.3. The waves originating from the slits have zero, or a constant difference of phase. When these waves overlap, an interference pattern is obtained. This you will study in detail in Subsection 9.3.2.

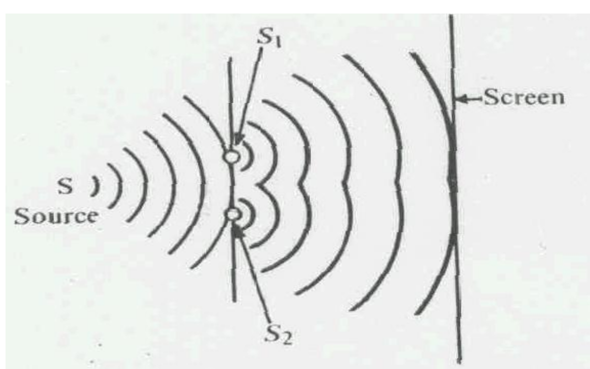


Fig.9.3 Two coherent sources S_1 and S_2 obtained from a single source S in an optics experiment

In sound, two coherent sources may be obtained by dividing the original longitudinal wave into two parts as shown in Fig. 9.4. Here one part goes via path I, while the other part goes via path II. These parts combine again to produce interference. The intensity of sound at any point can easily be noted, though qualitatively, by listening to the sound around this point.

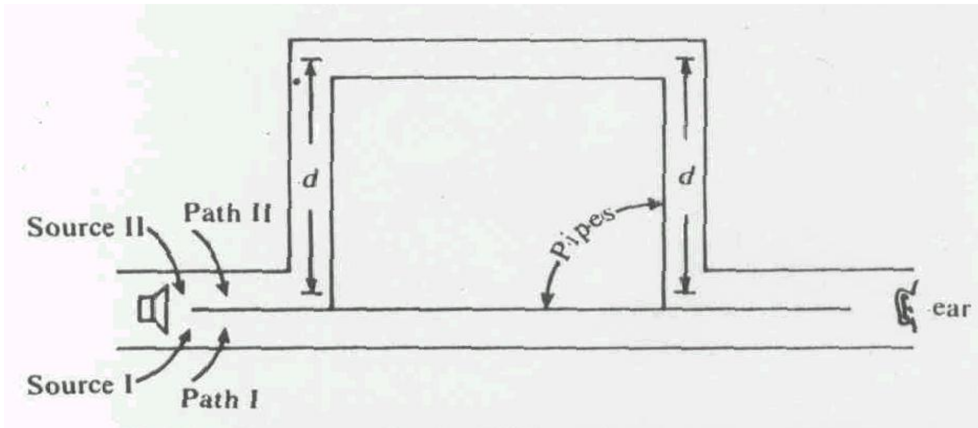


Fig 9.4 Coherent sources. Path I and II for producing interference of sound waves

Let us now pause for a minute and think as to what will happen if instead of two coherent sources we use two independent sources. We find from experimentation that if we use any two independent sources, interference pattern is not produced. This is because with independent sources, the phase difference between the waves changes rapidly and at random, giving rise to fast changing interference patterns. Two independent light sources just give a general illumination on the screen.

SAQ I

Can two small bulbs of 60 W each placed behind two slits form two coherent sources for interference purpose? If not, why?

9.2.2 Interference of Waves from Two Slits

In the last section, you have seen how two coherent sources can be formed from a single source. In the present section you will study the formation of an interference pattern due to such a system. Let S be a point source of the waves. Here (Fig. 9.5) S_1 and S_2 are two narrow slits which are equidistant from the source. A screen *MN* is kept parallel to the plane of the slits. The screen is at a distance *D* from the mid-point of the slits. Since the slits are equidistant from the source, the wavefront reaches the slits S_1 and S_2 at the same time, i.e., with zero phase difference.

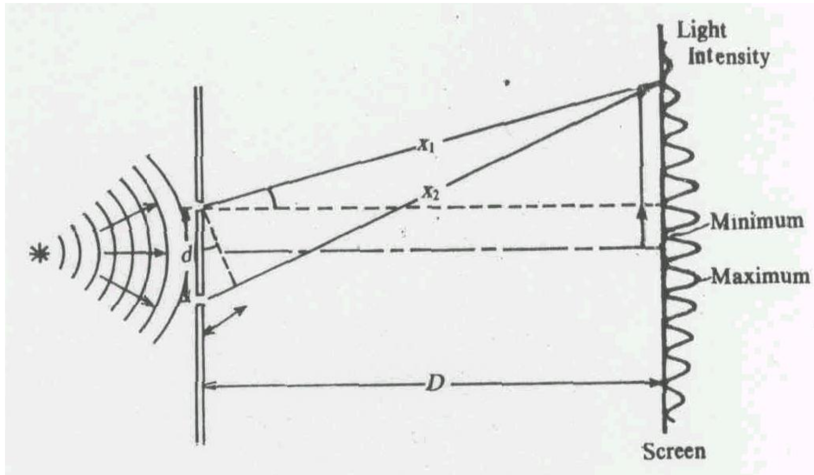


Fig 9.5: Set-up for observing the interference of wave

The waves from slits S_1 and S_2, separated by a distance d, are in phase with each other. Whatever phase difference gets created subsequently between the waves from the two sources is due to their travelling different distances. The slits S_1 and S_2 act like coherent sources and λ waves of angular frequency ω and amplitude *A*. Let us consider a point P at a distance of x_i from *S* and x_2 from x_2. Let these distances be sufficiently large compared to *d*. Let the displacement at P due to the waves from *S*, be

$$y_1 = A\sin(\omega t - kx_1) \tag{9.1}$$

then the displacement at the same point due to source S_2 will be

$$y_2 = A\sin(\omega t - kx_2) \tag{9.2}$$

It is clear from Eqs (9.1) and (9.2) that the path difference (i.e., the difference between the paths covered) between the two waves at *P* is given by $(x_1 - x_2)$. This will lead to a phase difference of

$$\delta = \frac{2\pi}{\lambda}(x_2 - x_1)$$

This is because the phase difference is always associated with the path difference according to the relation

$$\frac{\text{phase difference}}{2\pi} = \frac{\text{path difference}}{\lambda}$$

Due to the superposition of waves at *P* we get

$$y = y_1 + y_2$$

In the expanded form Eq. (9.4) can be written as

$$= A[\sin \omega t \cos kx_1 - \cos \omega t \sin kx_1 + \sin \omega t \cos kx_2 - \cos \omega t \sin kx_2]$$

$$= A(\cos k_1 + \cos k_2) \sin \omega t - A(\sin kx_1 + \sin kx_2) \cos \omega t \qquad (9.5)$$

The terms in parentheses are constant in time. Let us write

$$A(\cos k_1 + \cos k_2) = A_1 \cos \phi \qquad (9.6)$$

and

$$A(\sin k_1 + \sin k_2) = A_1 \sin \phi) \qquad (9.7)$$

so that Eq. (9.5) can be expressed as

$$y = A_1 \sin \omega t \cos \phi (-\cos \omega t \sin \phi)$$

$$= A_1 \sin(\omega t - \phi) \qquad (9.8)$$

Using Eqs. (9.6) and (9.7), we get

$$A_1^2 = A^2 (\cos kx_1 + \cos kx_2)^2 + A^2 (\sin kx_1 + \sin kx_2)^2$$

$$= 2A^2 [1 + \cos kx_1 \cos k_2 + \sin kx_1 \sin kx_2]$$

$$= 2A^2 [1 + \cos k(x_2 - x_1)] = 2A^2 (1 + \cos \delta) \qquad (9.9)$$

where, we have used Eq. (9.3) for the phase difference δ. Dividing Eq. (9.7) by (9.6), and expressing the sum of sine and cosine terms as produced, we get

$$\tan \phi = \frac{\sin kx_1 + \sin kx_2}{\cos kx_1 + \cos kx_2}$$

$$= \frac{2 \sin \frac{k(x_1 + x_2)}{2} \cos \frac{k(x_1 - x_2)}{2}}{2 \cos \frac{k(x_1 + x_2)}{2} \cos \frac{k(x_1 - x_2)}{2}}$$

$$= \tan k \left(\frac{x_1 + x_2}{2} \right)$$

or

$$\phi = \tan^{-1} k \left(\frac{x_1 + x_2}{2} \right) \qquad (9.10)$$

Eq. (9.9) gives us an expression for the intensity of the resultant wave at point P. as

$$I \propto A_1^2 = 2A^2 (1 + \cos \delta) \qquad (9.11)$$

or

$$I \propto 4A^2 \cos^2 \delta / 2$$

Clearly, when $\cos \delta / 2 = \pm 1$,
the intensity $I \propto 4A^2$

which is the maximum intensity and may be denoted by I_{max}.

Now we calculate the position of maxima. Let d be the distance between the centres of the two slits, θ be the angle at which we observe the beams and $(x_2 - x_1)$ the path difference between the two waves.
Then from Fig. 9.6,

$$\sin\theta = \frac{x_2 - x_1}{d} \quad \text{or} \quad d\sin\theta = (x_2 - x_1)$$

Maxima in intensity are obtained whenever this path difference is an integral multiple of λ, the wavelength of the waves used. Thus for maxima,

$$d\sin\theta = n\lambda \text{ with } n = 0, 1, 2, \ldots \tag{9.13}$$

Interference minima occur whenever this path difference becomes an odd integral multiple of $\lambda/2$, i.e.,

$$d\sin\theta = (2n+1)\lambda/2 \text{ with } n = 0, 1, 2, \ldots \tag{9.14}$$

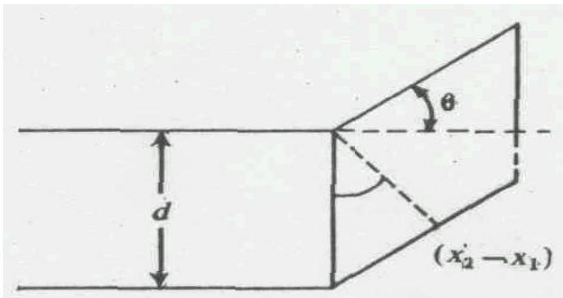


Fig 9.6 : Relation between d and θ

SAQ 2
Light passes through two narrow slits with d = 0.8mm. On a screen 1.6m away the distance of the second order maximum from the axis is 2.5 mm. What is the wavelength of the light used?

In our discussion so far, we have assumed that the two waves have equal amplitudes. If, however, we assume they have amplitudes a_1 and a_2 respectively, then one can show that the resultant amplitude A_1 will be given by

$$A_1^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\delta \tag{9.15}$$

In this case the maxima and minima of intensities (as will be shown in Subsection 9.2.3) become:

$$\left.\begin{aligned} I_{max} &= (a_1 + a_2)^2 \\ I_{min} &= (a_1 - a_2)^2 \end{aligned}\right\} \tag{9.16}$$

with their ratio as

$$\frac{I_{\max}}{I_{\min}} = \frac{(a_1 + a_2)^2}{(a_1 - a_2)^2} \tag{9.17}$$

9.2.3 Intensity Distribution in Interference Pattern

We have seen that the two waves of amplitudes a_1 and a_2 and having a phase difference, δ, are superposed, the resulting intensity I, is given by

$$I = A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos\delta$$

To study the variation of intensity with phase difference, δ, let us plot a graph of I versus δ. This is shown in Fig. 9.7. When the phase difference is 0, 2π, 4π, etc., $\cos\delta = 1$. We then have maximum of intensity, i.e.

$$\begin{aligned} I_{\max} &= A^2 = a_1^2 + a_2^2 + 2a_1a_2 \\ &= (a_1 + a_2)^2 \end{aligned} \tag{9.18}$$

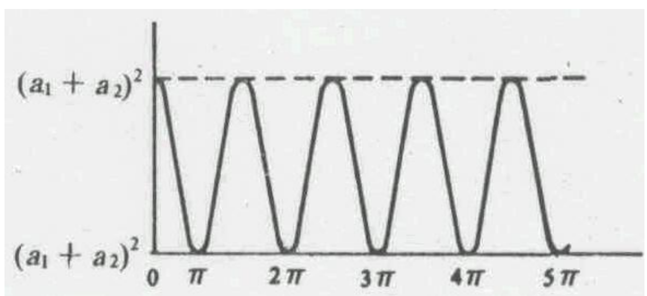


Fig 9.7 Graph between I and δ

On the other hand, whenever the phase differences is π, 3π, 5π, etc., $\cos\delta = -1$. We then have minimum of intensity, i.e.

$$\begin{aligned} I_{\max} &= A^2 = a_1^2 + a_2^2 - 2a_1a_2 \\ &= (a_1 - a_2)^2 \end{aligned} \tag{9.19}$$

For a particular case, when the amplitudes are equal i.e., $a_1 = a_2 = a$, the intensity varies from a maximum of $4a^2$ to a minimum of zero. In the case of light waves from a monochromatic source, one would observe the dark fringes of zero intensity separated by bright fringes.

The intensity distribution curve shows that when the two waves arrive at a point on the screen (exactly) out of phase, they interfere destructively and the resulting intensity (or energy flux) is zero. Whatever amount of energy is lost from a dip in the zero intensity is, by energy conservation, found to be redistributed in the maximum intensity peak.
We have seen earlier that for waves of equal amplitude, the intensity can be written as

$$I = 2a^2(1+\cos\delta) = 4a^2\cos^2\delta/2$$

Since the average value of $\cos^2\delta/2$ is 1/2, the dotted line at $I = 2a^2$ in Fig. 9.7 is the average intensity, which is actually the sum of the separate intensities from each slit.

It can be seen from Fig. 9.5 that the path difference between the waves reaching P from S_1 and S_2 = $(x_2 - x_1) = d\sin\theta$. If θ is very small, and is measured in radians we can use the approximation

$$\sin\theta \approx \tan\theta \approx \theta$$

Using (9.13), we can write for maxima

$$d\sin\theta = d\frac{y_0}{D} = n\lambda$$

giving

$$y_n = \frac{nD\lambda}{d}, \text{ where } n = 0, 1, 2, \dots$$

where y_n is the distance of the nth maxima from the point where the perpendicular bisector or the line joining the two slits meets the screen.

Writing the positions of two adjacent maxima as

$$y_n = \frac{nD\lambda}{d}$$

and

$$y_{n+1} = \frac{(n+1)D\lambda}{d}$$

the separation Δy between any two consecutive maxima (or the fringe widths B) is

$$B = y_{n+1} - y_n = \frac{D\lambda}{d}$$

This shows that as long as θ is very small, the separation between the two consecutive maxima of intensity is independent of *n*, i.e., the maxima are evenly spaced. Similarly, it can be shown that the separation between two adjacent minima is also equal to $\frac{D\lambda}{d}$, and that they too are equally spaced.

SAQ 3

(a) If the two sources S_1 and S_2 in Fig. 9.3 emit waves (i) in phase, and (ii) out of phase by π, discuss the intensity of resultant wave along the perpendicular bisector of S_1 and S_2.

(b) If two waves of amplitude ratio 5:1 interfere, deduce the ratio of intensities at maxima and minima.

SAQ 4

Two loudspeakers connected to a common amplifier are 5m apart. As one walks along a straight path 100m away from the speakers, at what spatial period does the intensity vary? Assume that the wavelength of sound waves = 0.3m.

The interference of light also explains the origin of beautiful colours from oil films on water or soap bubbles. In the next section, we attempt to discuss these in brief.

9.2.4 Interference in Thin Films

You have studied the relations for bright and dark fringes in Subsection 9.2.3. These relations will be used to account for the colours in thin films.

Consider a ray of light *AB* incident on a thin film of uniform thickness *t* and refractive index μ as shown in Fig. 9.8. A part of this is reflected along *BC* while the remaining part is refracted along *BD* into the film. At *D* it is again partly reflected along *DE*. The ray *DE* partly emerges into the air along *EF*, which is parallel to *BC*. The incident ray thus divides at *B* into two beams of different amplitudes, out of which the refracted beam suffers multiple reflections at D, *E*, etc. *EH* is perpendicular from *E* on *BC*.

The path difference between the rays *BC* and *EF* in the reflected system is

$$\mu(BD + DE) - BH$$

This can be shown to be equal to $2\mu t \cos r$, i.e.,

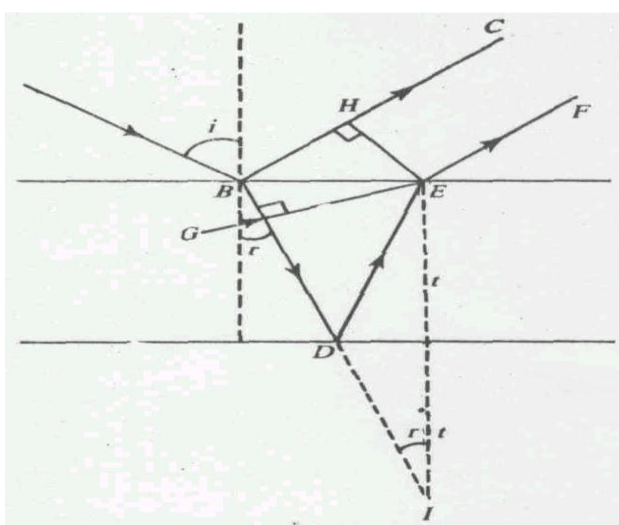


Fig. 9.8 Interference in thin films *BD* is extended to I so that *BD* = *DI*. The path difference between the reflected ray *BC* and *EF* is $\mu(BD \times DE) = \mu GI = 2\mu \cos r$

Path difference = $2\mu t \cos r$

where r is the angle of refraction in the film. We have already learnt that a phase change of π takes place on reflection at a denser medium. This is equivalent to a path difference of $\lambda/2$. The ray *BC* is due to reflection at a denser medium. Hence the net path difference between the reflected rays *BC* and *EF* is given by:

Path difference = $2\mu t \cos r - \lambda/2$

The film appears bright when

$$2\mu t \cos r - \lambda/2 = n\lambda \qquad (9.22)$$

and dark when

$$2\mu t \cos r - \lambda/2 = (2n+1)\lambda/2 \qquad (9.22)$$

We thus see that with monochromatic light alternate bright and dark fringes are obtained. With white light, which is a mixture of several colours, coloured fringes are obtained.

We have seen above that the path difference depends on μ and *r*, apart from *t* and λ. Path difference is different for different colours as μ is different for different colours. Similarly, for different angles of incidence, the angles of refraction r are different. Viewing it from different direction shows different colours. *These* all lead to the appearance of colours in thin films. These arise because of the interference of light.

9.3 DIFFRACTION

It is experimentally observed that when a beam of light passes through a small opening (a small circular hole or a narrow slit) it spreads to some extent into the region of the geometrical shadow. This is known as diffraction of light.

Consider a point source of monochromatic light S as shown in Fig. 9.9. Place an obstacle, say a penny or a sharp razor blade, halfway between the source and the screen. Following the rules of geometrical optics, we expect to see a well-defined and distinct shadow as shown in Fig. 9.9. Now you carefully examine the shadow. If the experiment is performed in a dark room and the wavelength of the light used is of the order of the size of the edges of the obstacle, you will find that the edges of the shadow are not sharp. Inside the shadow, near the edges, the intensity of light gradually decreases. Outside the shadow it gradually increases, forming alternatively bright and dark fringes as shown for a penny in Fig. 9.10.

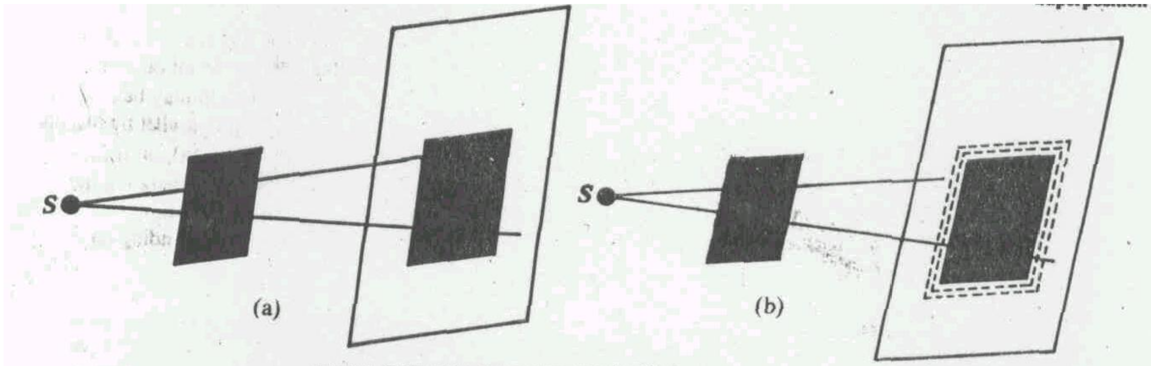


Fig 9.9 Diffraction due to a rectangular block

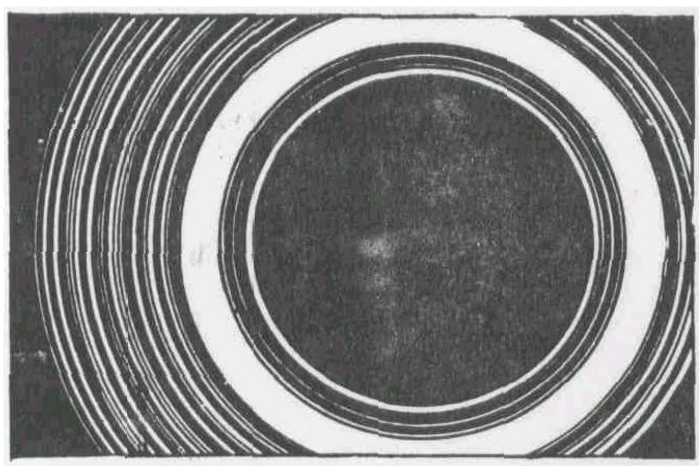


Fig 9.10 Photographs of diffraction pattern in the shadow of a penny

Consider water waves in a ripple tank. Suppose you generate plane waves on the surface of water in a ripple by giving a periodic up and down motion to a straight vibrator such as a ruler. Consider an obstacle, such as a slit *AB* placed in the path of the waves which are travelling as shown in Fig. 9.11a. As long as the opening *AB* is large, the plane waves passing through it appear nearly plane waves. The edges of the emergent plane waves roughly correspond to the edges of the slit *AB*. However, when the width of the slit is made narrow, say comparable to the wavelength of the water waves, then the parallel plane waves entering into the small opening spread out in the form of approximately circular concentric arcs as shown in Fig. 9.11b.

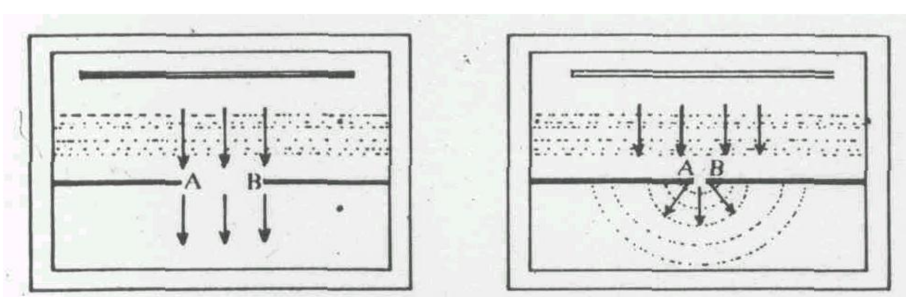


Fig 9.11 Diffraction pattern obtained by the bending of waves round corners

Consequently, these circular waves not only travel in the straight direction but also around the boundaries of the opening. This phenomenon of bending of waves around the edges of the aperture (or of the obstacle) is known as diffraction.

This property of bending of waves is distinctly observed when the wavelength of the waves involved is comparable to the size of the opening (or of the obstacle) through which the waves pass. For sound waves of frequency 500 Hz the wavelength in air is 0.6m, while that for yellow light it is 6×10^{-7} m. Clearly, in the case of sound if the entrance to the room is about 1 m, we may observe diffraction of sound. However, in the case of light, the dimension of the opening must be of the order of 10^{-6} m for a similar diffraction of light to be observed. Thus it is far more difficult to observe diffraction of light than of water waves or sound waves. Anyway diffraction of light can be observed in specially devised experiments and we will discuss these now.

9.3.1 Different Types of Diffraction: Fraunhofer and Fresnel

Diffraction of light is usually classified into two types: Fraunhofer and Fresnel diffractions. In Fraunhofer diffraction (or the far-field diffraction), the diffracting system (i.e. an obstacle, or an aperture) is so far away from the source that the waves generating the pattern may be regarded as plane. This can be achieved in the laboratory by making the rays of light parallel by placing the source at the focus of a convex lens. In Fresnel (or the near-field diffraction), on the other hand, the source of waves is so close to the diffracting system that the waves generating the pattern still retain their curved characteristics. This means that in Fresnel diffraction the convex lens is not used, and the wavefront remains spherical or cylindrical depending on the nature of the source.

Whatever may be the class of diffraction, the resultant distribution of energy in space, or on a screen, is obtained due to the superposition of waves from different parts of the same wave front. In the Fraunhofer class, the wavefront considered is plane, while it is spherical or cylindrical in the Fresnel class. In interference, we have two or more wave sources; while in diffraction, we have many, almost tending to infinity.

In the discussion to follow, we consider two cases in optics, one of diffraction due to a narrow single slit, and the other of diffraction at a straight edge. The former belongs to the Fraunhofer class, while the latter to the Fresnel class.

9.3.2 Fraunhofer Diffraction by a Single Slit

Let us analyse the diffraction pattern produced by plane waves passing through a single slit. We may note that a given slit or an aperture, howsoever small or narrow it may be, has a finite size. According to Huygen's principle, every point in it acts as a source of secondary wavelets. This fact gives rise to an interference between waves from various regions of the same slit.

Fig. 9.12 represents an enlarged diagram of a narrow slit of width, *d*. Let us assume that as the plane wavefront reaches the slit, all points in it emit the secondary wavelets in the same phase. Thus if the disturbance is observed at point P on the far side of the slit at angle θ to the normal, then there is a net path difference of $d\sin\theta$ between the waves from the two edges of the slit AB. According to Eq. (9.3), this corresponds to a phase difference of $2\pi d\sin\theta/\lambda$.

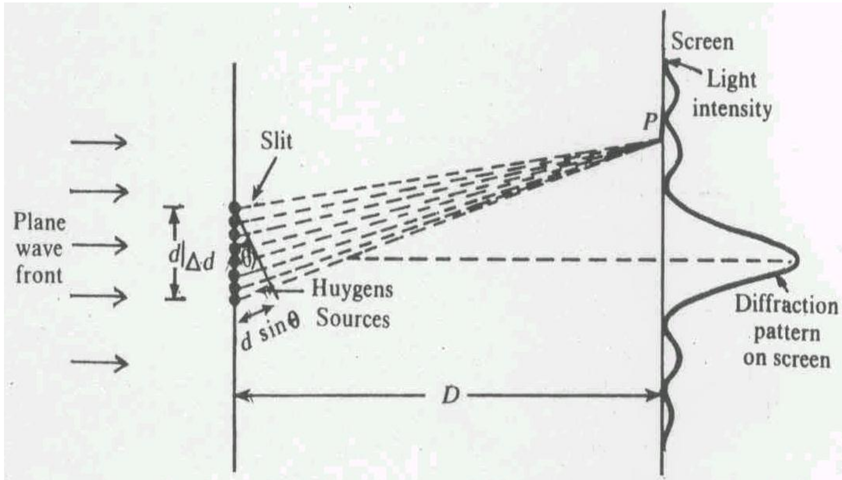


Fig 9.12a Diffraction due to a single slit. Note that light is not travelling in a straight line.

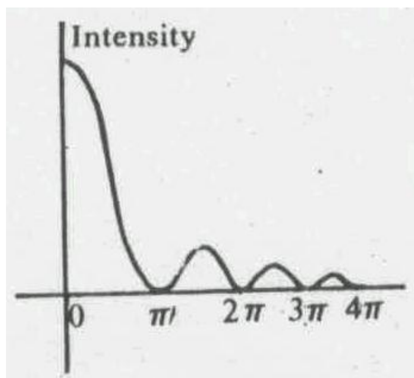


Fig 9.12b Graph of sin $\frac{\sin\alpha}{\alpha}$ versus α

Now imagine the slit *AB* is divided into a large number of strips of equal width, Δd. Each of these strips sends secondary wavelets and the path difference between the waves arriving at a point P from two adjacent strips is equal to $\Delta d \sin\theta$. The corresponding phase difference δ is given by

$$\delta = \frac{2\pi\,\Delta d \sin\theta}{\lambda} \tag{9.24}$$

If we divide the slit *AS* into a total number of *N* strips, then clearly $d = N\Delta d$, and the total phase difference

$$\frac{2\,\Delta d \sin\theta}{\lambda} = \frac{2\pi}{\lambda} N\,\Delta d \sin\theta = N\delta \tag{9.25}$$

Suppose that the amplitude of the secondary wave from each strip is denoted by A_s. Then the resultant disturbance at P is obtained by the superposition of the waves from all these strips. In other words,

$$Y = A_s \sin(\omega t - \phi) + A_s \sin(\omega t - \phi - \delta) + A_s \sin(\omega t - \phi - 2\delta) + \ldots \text{ (up to } N \text{ terms)} \tag{9.26}$$

where $\phi = 2\pi r / \lambda$ is the phase difference corresponding to the distance r from the first slit to point P.

You would recall that we have already discussed this problem of superposition in detail in Unit 2. It was shown there that the amplitude A of the resultant is obtained by the vector sum of N vectors of length A_s, each of which makes an angle δ with its adjacent vector (see Fig. 9.13).

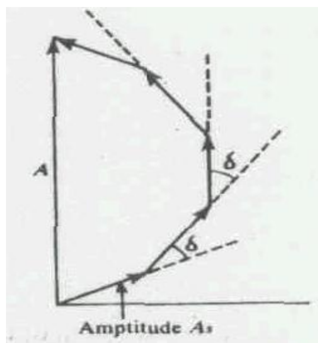


Fig 9.13 Vectorial addition of contributions from adjacent sources lo give the resultant amplitude.

In this case the resultant amplitude is given by

$$A = A_s \frac{\sin(N\delta/2)}{\sin(\delta/2)} \tag{9.27}$$

We must, however, remember that this subdivision of a slit into a finite number of sub-slits (or strips) is artificial. We take the limit as $N \to \infty$, $d \to 0$. In this case we have a continuous variation of phase. The vector diagram given in Fig. 9.13 then becomes a smooth circular arc of radius R. The resultant amplitude is then

$$A = A_0 \frac{\sin\alpha}{\alpha} \tag{9.28}$$

with $A_0 = RN\delta$, and $\alpha = \dfrac{\pi d \sin\theta}{\lambda}$.

From Eq. (9.28), the intensity I_0 of light at any angle θ with respect to the incident direction is given by:

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2} \tag{9.29}$$

The plot of intensity on screen is given in Fig. (9.12).

For a single slit Fraunhofer diffraction pattern, the minima of intensity are observed at angles θ_n from the incident direction, where

$$n\lambda = d \sin \theta_n \tag{9.30}$$

Here n = 1, 2, 3, etc. is the number of the diffraction dark band, starting from the central maximum.

From Eq. (9.29), you may also note that in the limit as $\alpha \to 0$, I_0 approaches I_0. This becomes the intensity of the central maximum. This is because $\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$.

For values of α, for which $\sin \alpha = 0$, $I_0 = 0$. This gives us the positions of various minima which appear for values of α equal to $n\pi$. The corresponding values of θ_n may be calculated using the earlier given relation, i.e.,

$$\alpha = \pi d \sin \theta_n / \lambda$$

For finding the positions of various maxima lying in-between the different minima, we have to differentiate the function $\frac{\sin \alpha}{\alpha}$ with respect to α and equate that to zero. An elaborate calculation shows that these maxima, also called the secondary maxima, appear at values of $\alpha = 1.429\pi$, 2.459π, etc. (The details of this calculation will be given in our course on optics). The heights of these secondary maxima are 1/21, 1/61, 1/120 respectively of the central maxima. This gives us an idea of the intensity distribution in a single slit diffraction pattern which is shown in Fig. 9.12b.

The angular spread of the intensity curve is given by

$$\sin \theta = \frac{\lambda}{d}$$

This shows that as the wavelength λ increases, or the width of the slit decreases, the angular spread increases. That is, the narrower the slit, the wider the diffraction pattern. Similarly, the greater the wavelength, the more widely spread is the pattern.

In terms of the distance D between the slit and the screen the width of the central maximum, Δy, on the screen is given by

$$\Delta y = \frac{D\lambda}{d} \tag{9.31}$$

The central peak in the intensity curve is called the primary maximum, while the other peaks are called secondary maxima. The height of the primary maximum is much more than any of the secondary maxima.

SAQ5

Calculate the angular spread of the central maximum for light of wavelength 6000Å when the width of the slit is (i) 10^{-2} m and (ii) 2×10^{-5} m.

9.3.3 Diffraction at a Straight Edge

If we put an obstacle in the path of waves, like a shaving blade in the path of light from a tiny source, we find that the image of the edge of the blade is not sharp. Instead, the intensity of light on the screen shows a pattern as shown in Fig. 9.14. The light is also found in the region which otherwise should have been a shadow region.

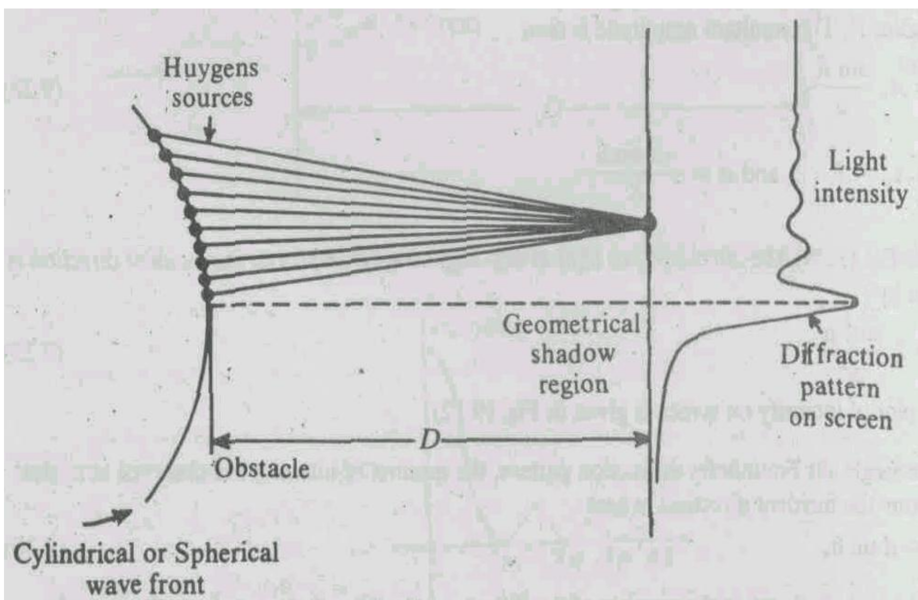


Fig 9.14 Diffraction due to an obstacle placed in the path of tight waves. Note that there is light intensity in the Geometrical shadow region, showing that light is not travelling in a straight line.

The intensity curve of a diffraction pattern is always quite different from that of an interference pattern. In the latter, the heights and widths of peaks are always equal (see Fig. 9.7) meaning thereby that all the maxima (or minima) are of the same intensity, and are equally spaced. This, however, is not the case in a diffraction pattern. In a diffraction pattern, the maxima (or minima) are not of same intensity, and are not equally spaced (see Figs. 9.12 and 9.14).

SAQ6

Sound waves of frequency 1650 Hz fall normally on an opening of width 0.6m. A listener walks parallel to the opening at a distance of 3 m, starting from a point on the perpendicular bisector of the opening. Find the positions at which he will observe a minima of sound.

Take the speed of second in air to be 330 $m\,s^{-1}$.

9.4 SUMMARY

1. Two sources are said to be coherent if they emit waves with no or constant difference of phase.

2. As a result of the superposition of waves from two coherent sources, the distribution of energy in space is not uniform. It is found to alternately pass through maxima and minima. Such a distribution of energy is called an interference pattern.

3. If two waves of the same frequency and of amplitudes a_1 and a_2, and phases ϕ_1 and ϕ_2 are acting on a particle, then according to the superposition principle, the amplitude *A* of the resultant wave is represented by

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2)$$

4. The phase difference between two interfering waves (from two coherent sources), if they travel in different paths, is calculated by using the relation:

phase difference = $2\pi / \lambda$ (path difference)

Maxima in intensity are observed where the path difference is an integral multiple of λ, the wavelength of light used.

5. The distance between any two adjacent maxima or minima in an interference pattern is given by:

$$\beta = \frac{D\lambda}{d}$$

where β is called the fringe width, λ is the wavelength of the light used, d is the distance between the two coherent sources, and *D* is the distance between the sources and the screen.

6. Diffraction refers to the bending of waves around corners. There are two classes of diffraction patterns, named as the Fraunhofer and the Fresnel diffractions.

7. Fresnel diffraction phenomena are observed when the source and the screen for observing the diffraction pattern are at a finite distance from the diffracting aperture or the obstacle.

8. In the Fraunhofer diffraction the source and the screen are at infinite distance from the aperture causing the diffraction.

9. In a single slit diffraction pattern, the minima in intensity are observed at angles θ_n given by: $n\lambda = d \sin\theta_n$.

9.5 TERMINAL QUESTIONS

1. What will be the path difference between the light waves from two coherent sources to produce the third dark fringe? It is given that the wavelength of the light is 5896Å.

2. Young's experiment is performed with the light of the green mercury line. If the fringes are measured with a micrometer eye-piece 80 cm behind the double slit, it is found that 20 of them occupy a distance of 10.92 mm. Find the distance between two slits, given that the wavelength of green mercury line is 5460 Å.

3. Light of wavelength 5000Å is incident normally on a slit. The first minimum of the diffraction pattern is observed to lie at a distance of 5 mm from the central maximum on the screen placed at a distance of 2 m from the slit. Calculate the width of the slit.

9.6 SOLUTIONS

SAQs

SAQ 1

The light waves emitted by these bulbs are neither in the same phase, nor are they with a constant difference of phase.

SAQ 2

For small angles, $\sin\theta$ can be set equal to *y/D*, where *y* is the distance of a given maximum from the axis and *D* is the distance from the slits to the screen. For a maximum of second order, we can write:

$$d\sin\theta = n\lambda \text{ with } n = 2$$

which gives:

$$d\left(\frac{y}{D}\right) = n\lambda$$

or $$\lambda = \frac{dy}{nD}$$

where y = 2.5 mm, D = 1.6 m and d = 0.8 mm

$$\lambda = \frac{(0.8mm)(2.5mm)(1.25\times10^{-3}mm)}{1.8m} = 1.25\times10^{-3}mm$$

SAQ 3

(a) Path difference between waves along the perpendicular bisector is zero. If sources emit waves in phase, there will be maximum intensity along the perpendicular bisector. In the second case, there will be minimum of intensity along the perpendicular bisector.

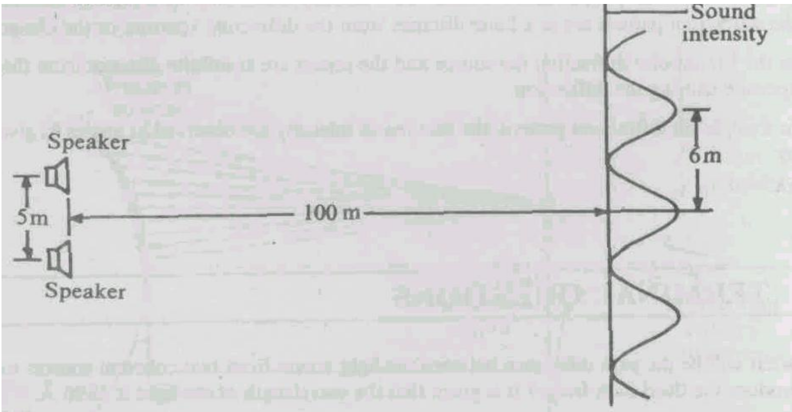
(b) $$\beta = \sqrt{\frac{a_1^2}{a_2^2}} = \sqrt{\frac{25a_2^2}{a_2^2}} = 5$$

$$\frac{I_{\max}}{I_{\min}} = \frac{(5+1)^2}{(5-1)^2} = \frac{36}{16} = 2.25$$

SAQ 4

$$y = \frac{D\lambda}{d} = \frac{0.3\times100m^2}{5.0m}$$

= 6.0 m



SAQ 5

(i) Using Eq. 9.30

$$\sin\theta = \frac{\lambda}{d} = \frac{6000\times10^{-10}}{10^{-2}\,m} = 6000\times10^{-8}\,\text{m}$$

For such a small value of $\sin\theta$,

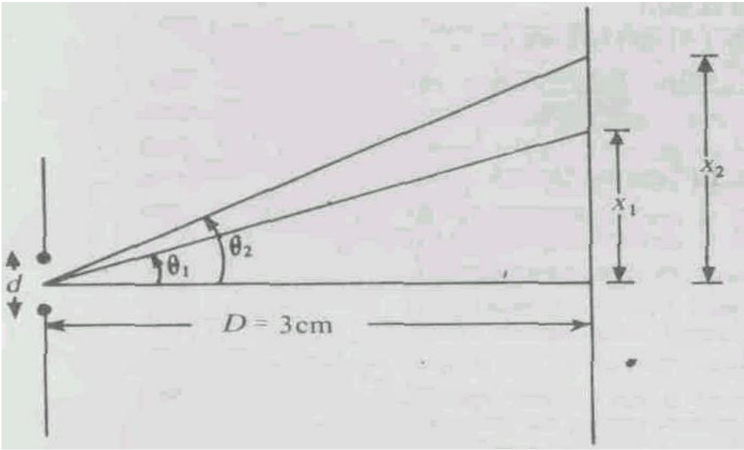
$$\sin\theta \approx \theta = 6000\times10^{-8} = 6\times10^{-5} = \frac{1}{230}\text{ degrees}$$

(ii) $$\sin\theta = \frac{6000\times10^{-10}}{2\times10^{-5}} = 3000\times10^{-5}$$

= 0.03 radians

$= \dfrac{0.03\times180}{3.1416}$

= 1.8 degrees



SAQ 6

$$\lambda = \frac{\text{speed}}{\text{frequency}} = \frac{330}{1650} = 0.2\,\text{m and}$$

$d = 0.6\,\text{m}$

Positions of minima appear where

$d\sin\theta = n\lambda$, with $n = 1, 2, 3, \ldots$

Positions of minima thus lie along the directions $\theta_n = \sin^{-1}\dfrac{n\lambda}{d}$, i.e.,

$$\theta_1 = \sin^{-1}\frac{0.2}{0.6} = \sin^{-1}\frac{1}{3} = 19^0$$

$$\theta_2 = \sin^{-1}\frac{2\times 0.2}{0.6} = \sin^{-1}\frac{2}{3} = 42^0$$

The observer will find minima at

$x_1 = D\tan\theta_1 = 3\tan 19^0 = 3\times 0.344$

$= 1.03$ m

$x_2 = D\tan\theta_2 = 3\tan 42^0 = 3\times 0.9$

$= 2.7$ m

(Note that the positions of minima are not equidistant.)

TQs

1. Let δ be the path length between waves for the 3rd dark fringe. Then,

$\delta = (2n+1)\lambda/2$ with $n = 2$

$\lambda = 5896$Å $= 5896\times 10^{-10}$ m

$\therefore \quad \delta = (5\times 5896\times 10^{-10})/2 = 1.4740\times 10^{-6}$ m

2. The fringe width β in Young's experiment is $\beta = \lambda D/d$. Since 20 fringes occupy a distance of 10.92 mm, the fringe width β is

$\beta = (10.92/20)$ mm $= (10.92\times 10^{-3}/20)$ m

Also,

$D = 80$ cm $= 0.8$ m and $\lambda = 5.460\times 10^{-7}$ m

$$d = \frac{5.460\times 10^{-7}\times 0.8\times 20}{10.92\times 10^{-3}}\text{ m} = 8.0\times 10^{-4}\text{ m}$$

$= 8.0\times 10^{-1}$ mm $= 0.8$ mm

3. The angles of diffraction for minimum intensity due to Fraunhofer diffraction at a single slit are given by:

$d\sin\theta = n\lambda$ with $n = 1, 2, 3, \ldots$

For the first minimum, $n = 1$. We can thus write

$d\sin\theta = \lambda$

If θ is small, then $\sin\theta \approx \theta$ (θ is in radians)

$d\theta = \lambda$ or $\theta = \lambda/d$ radians

Here, $\lambda = 5000$Å $= 5000\times 10^{-8}$ cm and $d = ?$

$\theta = 5000\times 10^{-8}/d$ radians (A)

The distance between the first minimum and the central maximum is 0.5 cm, and the distance of the screen from the slit is 2 m.
i.e., 200 cm. This gives

$\theta \approx 0.5/200$ radians (B)

Equating Eqs. (A) and (B), we get

$$\frac{5000\times10^{-8}\,cm}{d}=\frac{0.5\,cm}{200\,cm}$$

or

$$d=5000\times10^{-8}\,cm\times200/0.5$$
$$= 0.02 \text{ cm}$$