NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: PHY 312

COURSE TITLE: MATHEMATICAL METHODS OF PHYSICS

COURSE GUIDE

PHY312
MATHEMATICAL METHODS OF PHYSICS I

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CONTENTS PAGE

Introduction

The course Mathematical Method of Physics 1- is meant to provide essential methods for solving mathematical problems.

In scientific problems, often times we discover that a factor depends upon several other related factors. For instance, the area of solid depends on its length and breadth. Potential energy of a body depends on gravity, density and height of the body etc. Moreover, the strength of a material depends on temperature, density, isotopy and softness etc.

What You Will Learn in This Course

This is a 3unit course, it is grouped into four (4) modules i.e. module1, 2, 3 and 4. Module 1 has 2units; module 2 also has 2units. Module 3 has only one unit while module 4 has 3units. In summary we have four (4) modules and 8 units.

The course guide gives a brief summary of the total contents contained in the course material. Functions of several variables streamline the relationship between function and variables, the application of Jacobian, down to functional dependence and independence. Also discussed are multiple, line and improper integrals.

Course Aim

The overall aim of this course is to provide you with the essential methods for solving mathematical problems in physics.

Course Objectives

At the end of this course, you should be able to:

- define linear second-order partial differential equation in more than one independent variable
- use the technique of separation of variables in solving important second order linear partial differential equations in physics
- solve the exercises at the end of this unit
- identify whether a given function is even, odd or periodic
- evaluate the Fourier coefficients
- derive and apply Fourier series in forced vibration problems
- use Fourier Integral for treating various problems involving periodic function
- apply half range expansion to solutions of some problems.

Working through This Course

This course involves that you would be required to spend lot of time to read. The content of this material is very dense and require you spending great time to study it. This accounts for the great effort put into its development in the attempt to make it very readable and comprehensible. Nevertheless, the effort required of you is still tremendous.

I would advice that you avail yourself the opportunity of attending the tutorial sessions where you would have the opportunity of comparing knowledge with your peers.

Course Materials

You will be provided with the following materials:

- Course guide
- Study units

In addition, the course comes with a list of recommended textbooks, which though are not compulsory for you to acquire or indeed read, are necessary as supplements to the course material.

Study Units

The following are the study units contained in this course. The units are arranged into four identifiable but related modules.

Module 1 Partial Differential Equations with Applications in Physics

Unit 1 Partial Differential Equations
Unit 2 Fourier Series

Module 2 Application of Fourier to PDEs (Legendre polynomials and Bessel Functions)

Unit 1 Legendre Polynomials
Unit 2 Bessel Functions

Module 3 Application of Fourier to PDEs (Hermite Polynomials and Laguerre Polynomials)

Unit 1 Hermite Polynomials
Unit 2 Laguerre Polynomials

Textbook and References

The following editions of these books are recommended for further reading.

Hildraban, F. B.(nd). *Advanced Calculus for Application*.

Murray, R. S.(1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995). *Engineering Maths* (5th ed.). Palgraw.

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

Assessment

There are two components of assessment for this course. The Tutor-Marked Assignment (TMA) and the end of course examination.

Tutor-Marked Assignment

The (TMA) is the continuous assessment component of your course. It accounts for 30% of the total score. You will be given four (4) TMAs' to answer. Three of these must be answered before you are allowed to sit for the end of course examination. The TMAs' would be given to you by your facilitator and returned after you have done the assignment.

Final Examinations and Grading

This examination concludes the assessment for the course. It constitutes 70% of the whole course. You will be informed of the time of the examination. It may or may not coincide with the University Semester Examination.

Presentation Schedule

Your course materials have important dates for the early and timely completion and submission of your TMAs and attending tutorials. You should remember that you are required to submit all your assignments by the stipulated time and date. You should guard against falling behind in your work.

Course Marking Scheme

Assignment	**Marks**
Assignments 1-4	Four TMAs, best three marks of the four count at 10% each – 30% of course marks.
End of course examination	70% of overall course marks.
Total	100% of course materials.

At the end of each unit, assignments are given to assist you to assess your understanding of the topics that have been discussed.
Course Overview

Each study unit consists of three hours work. Each study unit includes introduction, specific objectives, directions for study, reading materials, conclusions, and summary, Tutor -Marked Assignments (TMAs), references / further reading. The units direct you to work on exercises related to the required readings. In general, these exercises test you on the materials you have just covered or require you to apply it in some way and thereby assist you to evaluate your progress and to reinforce your comprehension of the material. Together with TMAs, these exercises will help you in achieving the stated learning objectives of the individual units and of the course as a whole.

How to Get the Most Out of This Course

Implicit interest and regular culture of reading are of utmost requirements for getting the best out of this course. It is paramount that you should at least purchase one of the textbooks that are recommended for you. More importantly, attending tutorials sessions and completing your assignments on time will certainly assist you to get the best out of this course.

Facilitators/Tutors and Tutorials

There are 16 hours of tutorials provided in support of this course. You will be notified of the dates, times and locations of these tutorials as well as the name and phone number of your facilitator, as soon as you are allocated a tutorial group.

Your facilitator will mark and comment on your assignments, keep a close watch on your progress and any difficulties you might face and provide assistance to you during the course. You are expected to mail your Tutor -Marked Assignment to your facilitator before the scheduled date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not delay to contact your facilitator by telephone or e-mail if you need assistance.

The following might be circumstances in which you would find assistance necessary. You would have to contact your facilitator if:

- you do not understand any part of the study or the assigned readings
- you have difficulty with the self-tests
- you have a question or problem with assignments or with the grading of assignments.

You should endeavour to attend the tutorials. This is the only chance to have face to face contact with your course facilitator and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study.

To gain much benefit from course tutorials prepare a question list before attending them. You will learn a lot from participating actively in discussions.

Summary

It is expected that, going through this course, you have learnt how to use Method of Separation of Variables to Solve Heat Conduction Equation and Wave Equation respectively.

The use of Fourier transforms to solve some differential Equation, Boundary values problems etc. You should also have learnt the use of Laplace transformation to solve some initial and Boundary value problems, which are difficult to handle in addition to the application of convolution theory in solving problems.

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CONTENTS **PAGE**

MODULE 1 PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN PHYSICS

UNIT 1 PARTIAL DIFFERENTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall study some elementary methods of solving partial differential equations which occur frequently in physics and in engineering. In general, the solution of the partial differential equation presents a much more difficult problem than the solution of ordinary differential equations.

We are therefore going to limit ourselves to a few solvable partial differential equations that are of physical interest.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define linear second-order partial differential equation in more than one independent variable
- use the technique of separation of variables in solving important second order linear partial differential equations in physics
- solve the exercises at the end of this unit.

3.0 MAIN CONTENT

3.1 Definition

An equation involving one or more partial derivatives of (unknown) functions of two or more independent variables is called a **partial differential equation.** The ***order*** of a PDE is the highest order partial derivative or derivatives which appear in the equation. For example,

$$U\frac{\partial U}{\partial z}\frac{\partial^3 U}{\partial y^3}+\left(\frac{\partial^2 U}{\partial y^2}\frac{\partial^2 U}{\partial z^2}\right)=e^z \quad (1)$$

is a third order PDE since the highest order term is given by

$$\frac{\partial^3 U}{\partial y^3}$$

A PDE is said to be ***linear*** if it is of the first degree, i.e. not having exponent greater than 1 in the dependent variable or its partial derivatives and does not contain product of such terms in the equation. Partial derivatives with respect to an independent variable are written for brevity as a subscript; thus

$$U_{tt}=\frac{\partial^2 U}{\partial t^2} \quad \text{and} \quad U_{xy}=\frac{\partial^2 U}{\partial x \partial y}$$

The PDE

$$\frac{1}{c^2}U_{tt}=U_{xx}+U_{yy}+U_{zz} \quad (2)$$

(Where c is a constant) is linear and is of the second order while eq. (1) is an example of a nonlinear PDE.

Example 1: Important linear partial differential equations of second order

(1)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional wave equation}$$

(2)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional heat equation}$$

(3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textit{Two-dimensional Laplace equation}$$

(4)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \textit{Two-dimensional poisson equation}$$

(5)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \textit{Three-dimensional Laplace equation}$$

3.2 Linear Second-Order Partial Differential Equations

Many important PDEs occurring in science and engineering are second order linear PDEs. A general form of a second order linear PDE in two independent variables x and y can be expressed as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (3)$$

where A, B, C……,G may be dependent on variables x and y. If $G = 0$, then eq. (3) is called ***homogeneous;*** otherwise it is said to be a ***non-homogeneous.***

The homogeneous form of Eq. (3) resembles the equation of a general conic:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

We thus say that eq. (3) is of

$$\left.\begin{matrix} \textit{elliptic} \\ \textit{hyperbolic} \\ \textit{parabolic} \end{matrix}\right\} \textit{type} \qquad \textit{when} \begin{cases} B^2 - 4AC < 0 \\ B^2 - 4AC > 0 \\ B^2 - 4AC = 0 \end{cases}$$

For example, according to this classification the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is of elliptic type $(A = C = 1, B = D = E = G = 0)$, and the equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad (\alpha \text{ is a real constant})$$

is of hyperbolic type. Similarly, the equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial y} = 0 \quad (\alpha \text{ is a real constant})$$

is of parabola type.

Some important linear second-order partial differential equations that are of physical interest are listed below.

Example 2

Eliminate A and P from the function $Z = Ae^{pt} \sin px$

Solution Let $\frac{\partial Z}{\partial t} = pAe^{pt} \sin px$ and $\frac{\partial^2 Z}{\partial t^2} = p^2 Ae^{pt} \sin px$

also $\frac{\partial Z}{\partial x} = pAe^{pt} \cos px$ and $\frac{\partial^2 Z}{\partial x^2} = -p^2 Ae^{pt} \sin px$

$$\frac{\partial^2 Z}{\partial t^2} + \frac{\partial^2 Z}{\partial x^2} = 0$$

i.e. $p^2 Ae^{pt} \sin px - p^2 Ae^{pt} \sin px = 0$

Example 3

Solve the equation

$$\frac{\partial^2 u}{\partial x^2} - 7\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} = 0$$

Solution: Let $u(x, y) = f(y + m_1 x) + g(y + m_2 x)$

So that $m^2 - 7m + 6 = 0$

This implies that $m = 1 \text{ or } 6$

Hence $u(x, y) = H(y + x) + G(y + 6x)$

3.2.1 Laplace‘s Equation

$$\nabla^2 u = 0 \tag{4}$$

Where ∇^2 is the Laplacian operator $\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\right)$. The function u may be the electrostatic potential in a charge-free region or gravitational potential in a region containing no matter.

3.2.2 Wave Equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{5}$$

Where u represents the displacement associated with the wave and v, the velocity of the wave.

3.2.3 Heat Conduction Equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u \tag{6}$$

Where u is the temperature in a solid at time t. The constant α is called the diffusivity and is related to the thermal conductivity, the specific heat capacity, and the mass density of the object.

3.2.4 Poisson’s Equation

$$\nabla^2 u = \rho(x, y, z) \tag{7}$$

Where the function $\rho(x, y, z)$ is called the source density. For example, if u represents the electrostatic potential in a region containing charges, then ρ is proportional to the electric charge density.

Example 4

Laplace’s equation arises in almost all branches of analysis. A simple example can be found from the motion of an incompressible fluid. Its velocity $\mathbf{v}(x, y, z, t)$ and the fluid density $\rho(x, y, z, t)$ must satisfy the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0$$

If ρ is constant we then have

$$\nabla \bullet \mathbf{v} = 0$$

If furthermore, the motion is irrotational, the velocity vector can be expressed as the gradient of a scalar function V:

$$\mathbf{v} = -\nabla V$$

and the continuity becomes Laplace's equation:

$$\nabla.\mathbf{v} = \nabla.(-\nabla V) = 0, \qquad \text{or} \qquad \nabla^2 V = 0$$

The scalar function V is called the velocity potential

3.3 Method of Separation of Variables

The technique of separation of variables is widely used for solving many of the important second order linear PDEs.
The basic approach of this method in attempting to solve a differential equation (say, two independent variables x and y) is to write the dependent variable $u(x, y)$ as a product of functions of the separate variables $u(x,t) = X(x)T(t)$. In many cases the partial differential equation reduces to ordinary equations for X and T.

3.3.1 Application to Wave Equation

Let us consider the vibration of an elastic string governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{8}$$

where u(x, y) is the deflection of the string. Since the string is fixed at the ends x = 0 and $x = l$, we have the two **boundary conditions**

$$u(0,t) = 0, \qquad u(l,t) = 0 \qquad \text{for all t} \tag{9}$$

The form of the motion of the string will depend on the initial deflection (deflection at t = 0) and on the initial velocity (velocity at t = 0). Denoting the initial deflection by f(x) and the initial velocity by g(x), the two **initial conditions** are

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \tag{10}$$

This method expresses the solution of $u(x,t)$ as the product of two functions with their variables separated, i.e.

$$U(x,t) = X(x)T(t) \tag{11}$$

where X and T are functions of x and t respectively.

Substituting eq. (11) in eq. (8), we obtain

$$X\,T'' = c^2\,X''T$$

or

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} \tag{12}$$

In other words

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \tag{13}$$

The original PDE is then separated into two ODEs, viz.

$$X''(x) - \lambda X(x) = 0 \tag{14}$$

and

$$T''(t) - \lambda c^2\,T(t) = 0 \tag{15}$$

The boundary conditions given by eq. (9) imply

$$X(0)\,T(t) = 0$$

and

$$X(l)\,T(t) = 0$$

Since T(t) is not identically zero, the following conditions are satisfied

$$X(0) = 0 \quad \text{and} \quad X(l) = 0 \tag{16}$$

Thus eq. (14) is to be solved subject to conditions given by eq. (16).

There are 3 cases to be considered.

Case 1 $\lambda > 0$

The solution to eq. (14) yields

$$X(x) = Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x} \tag{17}$$

To satisfy the boundary condition given by eq. (16), we must have

$$Ae^{-\sqrt{\lambda}\,l} + Be^{\sqrt{\lambda}\,l} = 0$$

Since the determinant formed by the coefficients of A and B is non-zero, the only solution is A = B = 0. This yields the trivial solution X(x) = 0.

Case 2 $\lambda = 0$

The solution to eq. (14) yields

$$X(x) = A + Bx$$

To satisfy the boundary condition given by eq. (16), we must have

$$A = 0$$

and

$$A + Bl = 0$$

implying

$$A = 0, \qquad B = 0$$

Again for this case, a trivial solution is obtained

Case 3 $\lambda < 0$

Let $\lambda = -k^2$. The solution to eq. (14) yields

$$X(x) = A\cos kx + B\sin kx \tag{18}$$

To satisfy the boundary condition given by eq. (16), we must have

$$A = 0$$

and

$$B\sin kl = 0$$

To obtain a solution where $B \neq 0$, we must have

$$kl = n\pi \qquad n = 1, 2, \ldots$$

Thus

$$\lambda = -k^2 = -\left(\frac{n\pi}{l}\right)^2 \tag{19}$$

(n=0 corresponds to the trivial solution). The specific values of λ are known as the eigenvalues of eq. (14) and the corresponding solutions, viz, $\sin\left(\frac{n\pi}{l}x\right)$ are called the *eigenfunctions*. Since there are many possible solutions, each is subscripted by n. Thus

$$X_n(x) = B_n \sin\left(\frac{n\pi}{l}x\right) \qquad n = 1, 2, 3, \ldots \tag{20}$$

The solution to Eq. (15) with λ given by Eq. (19) is

$$T_n(t) = E_n \cos\left(\frac{n\pi}{l}ct\right) + F_n \sin\left(\frac{n\pi}{l}ct\right) \qquad n = 1, 2, 3, \ldots \tag{21}$$

Where E_n and F_n are arbitrary constants. There are thus many solutions for eq. (8) which is given by

$$U_n(x,t) = X_n(x)T_n(t)$$
$$\left[a_n \cos\left(\frac{n\pi}{l}ct\right) + b_n \sin\left(\frac{n\pi}{l}ct\right)\right]\sin\frac{n\pi}{l}x \tag{22}$$

Where $a_n = B_n E_n$ and $b_n = B_n F_n$. Since eq. (8) is linear and homogeneous, the general solution is obtained as the linear superposition of all the solutions given by eq. (22), i.e.

$$U(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi}{l}x \tag{23}$$

Differentiating with respect to t, we have

$$U_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}\left(-a_n \sin\frac{n\pi c}{l}t + b_n \cos\frac{n\pi c}{l}t\right)\sin\frac{n\pi}{l}x \tag{24}$$

The coefficients a_n and b_n are obtained by applying the initial conditions in eq. (10). Thus,

$$U(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x \tag{25}$$

$$U(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi}{l}c\right)\sin\frac{n\pi}{l}x \tag{26}$$

In order to determine a_n and b_n we use the orthogonality properties of $\sin\frac{n\pi}{l}x$ in the range $0 \le x \le l$, i.e.

$$\int_0^l \sin\frac{m\pi}{l}x \sin\frac{n\pi}{l}x\, dx = \frac{l}{2}\delta_{m\,n} \tag{27}$$

Where $\delta_{m\,n}$ is the Kronecker delta function having the property

$$\delta_{m\,n} = \begin{cases} 0 & n \ne m \\ 1 & n = m \end{cases} \tag{28}$$

Multiply eq. ((25) by $\sin\frac{m\pi}{l}x$ and integrating between the limits x = 0 and x = l, we get

$$\int_0^l f(x)\sin\frac{m\pi}{l}x\,dx = \sum_{n=1}^{\infty}\int_0^l a_n \sin\frac{m\pi}{l}x\sin\frac{n\pi}{l}x\,dx$$

$$= a_m \frac{l}{2} \tag{29}$$

i.e. $$a_m = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi}{l}x\,dx$$

Similarly multiplying eq. (26) by $\sin\frac{m\pi}{l}x$ and integrating between the limits x = 0 and $x = l$, we get

$$\int_0^l g(x)\sin\frac{m\pi}{l}x\,dx = \sum_{n=1}^{\infty}\int_0^l b_n\left(\frac{n\pi}{l}c\right)\sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\,dx$$

$$= b_m\left(\frac{m\pi}{l}c\right)\frac{l}{2} \tag{30}$$

i.e. $$b_m = \frac{2}{m\pi c}\int_0^l g(x)\sin\frac{m\pi}{l}x\,dx$$

With a_m and b_m obtained for $m = 1, \ldots \infty$, eq. (23) is the solution to PDE given by eq. (8) subject to the initial conditions and the boundary conditions.

3.3.2 Application to Heat Conduction Equation

The one-dimensional heat flow in a rod bounded by the planes x = 0 and x = a is of practical interest. The solution applies to the case where the y and z dimensions extend to infinity. The temperature distribution is determined by solving the one-dimensional heat conduction equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{v}\frac{\partial\theta}{\partial t} \tag{31}$$

Where θ represents the temperature and

$$v = \frac{k}{C\rho} \tag{32}$$

k, C and ρ are the thermal conductivity, specific heat and density of the material respectively. We shall treat the case where the boundary conditions are given by

$$\theta(x = 0, t) = 0 \tag{33}$$

$$\theta(x = a, t) = 0 \qquad (34)$$

The initial temperature distribution is given by

$$\theta(x, t = 0) = f(x) \qquad (35)$$

Solution: Using the method of separation of variables, the x-dependence and t-dependence are separated out as expressed by

$$\theta(x, t) = X(x)T(t) \qquad (36)$$

Substituting eq. (36) into eq. (31) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v}\frac{1}{T}\frac{dT}{dt} = \alpha \qquad (37)$$

We shall now consider three cases corresponding to different values of the constant α.

Case 1 $\lambda = 0$

The separated ODE for $X(x)$ becomes

$$\frac{d^2X}{dx^2} = 0 \qquad (38)$$

i.e. $X(x) = Ax + B$

The boundary conditions expressed by eqs. (33) and (34) are respectively

$$X(x = 0) = 0 \quad \text{and} \quad X(x = a) = 0 \qquad (39)$$

Since T(t) should not be identically zero. Thus for eq. (38) to satisfy the boundary conditions given by eq. (39), we must have A = 0, B = 0. This gives the steady-state solution where temperature in the rod is everywhere zero.

Case 2 $\lambda > 0$

Let $\alpha = k^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = k^2 X \qquad (40)$$

Therefore $X(x) = Ae^{kx} + Be^{-kx}$

Applying the boundary conditions given in eq. (39), we get

$0 = A + B$
$0 = Ae^{ka} + Be^{-ka}$

Again we have A = B = 0

Case 3 $\lambda < 0$

Let $\alpha = -\lambda^2$. The ODE for $X(x)$ becomes

$$\frac{d^2X}{dx^2} = -\lambda^2 X \tag{41}$$

Thus $X(x) = A\cos\lambda x + B\sin\lambda x$

The boundary conditions require

$A = 0$
$$B\sin\lambda a = 0 \tag{42}$$
i.e. $$\lambda a = n\pi \qquad n = 1, 2, \ldots \tag{43}$$

Since there are multiple solutions, each λ is designated by a subscript n as λ_n. The solution of the ODE for T(t) is readily obtained as

$$T(t) = Ce^{-\lambda_n^2 vt} \tag{44}$$

Thus the general solution which is a superposition of all admissible solution is given by

$$\theta(x,t) = \sum_{n=1}^{\infty} D_n e^{-\lambda_n^2 vt} \sin\frac{n\pi}{a}x \tag{45}$$
$$= \sum_{n=1}^{\infty} D_n \exp\left(-\frac{n^2\pi^2}{a^2}vt\right)\sin\frac{n\pi}{a}x \tag{46}$$

To complete the solution D_n must be determined from the remaining initial condition

i.e. $$f(x) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{a}x \tag{47}$$

In order to determine D_n, we multiply eq. (47) by $\sin\frac{m\pi}{a}x$ and integrate the limits x=0 and x = a to obtain

$$\int_0^a f(x)\sin\frac{m\pi}{a}x\,dx = \sum_{n=1}^{\infty} D_n \int_0^a a_n \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x\,dx = D_m\frac{a}{2}$$

Thus

$$D_m = \frac{2}{a}\int_0^a f(x)\sin\frac{m\pi}{a}x\,dx \tag{48}$$

For the specific case where $f(x) = \theta_0$ (constant), the solution is given by

$$\theta(x,t) = \frac{4\theta_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{a}\exp\left[-\frac{v(2n+1)^2\pi^2}{a^2}\right]$$

$$0 < x < a \tag{49}$$

From eq. (49), it can be deduced that a rectangular pulse of height θ_0 for $0 < x < a$ has the Fourier series expansion given by

$$\frac{4\theta_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{a}$$

Also if $f(x) = \gamma x$, then

$$\theta(x,t) = \frac{2a\gamma}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(\frac{n\pi}{a}x\right)\exp\left[-\frac{vn^2\pi^2}{a^2}t\right] \tag{50}$$

If the end boundaries are maintain at different temperature i.e.

$$\begin{aligned}\theta(x=0,t) &= \theta_1\\ \theta(x=a,t) &= \theta_2\end{aligned} \tag{51}$$

Then case 1 of the solution where $\alpha = 0$, would yield the steady-state solution given by $\theta_1 + \frac{x}{a}(\theta_2 - \theta_1)$. The general solution is given by

$$\theta(x,t) = \phi(x,t) + \theta_1 + \frac{x}{a}(\theta_2 - \theta_1) \tag{52}$$

Where $\phi(x,t)$ is the transient solution.

The boundary conditions for $\phi(x,t)$ are obtained as follows:

at x = 0: $\theta(x=0,t)=\theta_1=\phi(x=0,t)+\theta_1 \Rightarrow \phi(x=0,t)=0$
at x = a: $\theta(x=a,t)=\theta_2=\phi(x=a,t)+\theta_2 \Rightarrow \phi(x=a,t)=0$
$\phi(x,t)$ is obtained under case 3.

SELF-ASSESSMENT EXERCISE 1

1. State the nature of each of the following equations (that is, whether elliptic, parabolic or hyperbolic)

(a) $\frac{\partial^2 y}{\partial t^2}+\alpha\frac{\partial^2 y}{\partial x^2}=0$

(b) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+3y^2\frac{\partial u}{\partial x}$

2(a) Show that $y(x,t)=F(2x+5t)+G(2x-5t)$ is a general solution of

$$4\frac{\partial^2 y}{\partial t^2}=25\frac{\partial^2 y}{\partial x^2}$$

(b) Find a particular solution satisfying the conditions
$y(0,t)=y(\pi,t)=0, \quad y(x,0)=\sin 2x, \quad y'(x,0)=0.$

3. Solve the following PDEs

(a) $\frac{\partial^2 u}{\partial x^2}=8xy^2+1$

(b) $\frac{\partial^2 u}{\partial xy}-\frac{\partial u}{\partial y}=6xe^x$

3.4 Laplace Transform Solutions of Boundary-Value Problems

Laplace and Fourier transforms are useful in solving a variety of partial differential equations; the choice of the appropriate transforms depends on the type of boundary conditions imposed on the problem. Laplace transforms can be used in solving boundary-value problems of partial differential equation.

Example 5

Solve the problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \tag{53}$$

$$u(0,t) = u(3,t) = 0, \qquad u(x,0) = 10\sin 2\pi x - 6\sin 4\pi x \tag{54}$$

Solution: Taking the Laplace transform L of Eq. (53) with respect to t gives

$$L\left[\frac{\partial u}{\partial t}\right] = 2L\left[\frac{\partial^2 u}{\partial x^2}\right]$$

Now

$$L\left[\frac{\partial u}{\partial t}\right] = pL(u) - u(x,0)$$

and

$$L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-pt}u(x,t)dt = \frac{\partial^2}{\partial x^2}L[u]$$

Here $\partial^2/\partial x^2$ and $\int_0^\infty ...dt$ are interchangeable because x and t are independent.

For convenience, let

$$U = U(x,p) = L[u(x,t)] = \int_0^\infty e^{-pt}u(x,t)dt$$

We then have

$$pU - u(x,0) = 2L\frac{\partial^2 U}{\partial x^2}$$

from which we obtain, using the given conditions (54),

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{2}pU = 3\sin 4\pi x - 5\sin 2\pi x. \tag{55}$$

Then taking the Laplace transform of the given conditions $u(0,t) = u(3,t) = 0$, we have

$$L[u(0,t)] = 0, \qquad L[u(3,t)] = 0$$

Or

$$U(0,p) = 0, \qquad U(3,p) = 0.$$

These are the boundary conditions on $U(x,p)$. Solving eq. (55) subject to these conditions we find

$$U(x,p) = \frac{5\sin 2\pi x}{p + 16\pi^2} - \frac{3\sin 4\pi x}{p + 64\pi^2}$$

The solution to eq. (55) can now be obtained by taking the inverse Laplace transform

$$u(x,t) = L^{-1}[U(x,p)] = 5e^{16\pi^2 t}\sin 2\pi x - 36e^{64\pi^2 t}\sin 4\pi x.$$

SELF-ASSESSMENT EXERCISE 2

1. Differentiate between ordinary differential equation and partial differential equation.

2. Derive the PDE that give rise to the function

$$Z = a(x+y) + b(x-y) + abt + c = 0$$

3. Use the method of separation of variable to find the solution of the boundary value problem

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$y(0,t) = 0 \qquad t > 0$

$y(1,t) = 0 \qquad t > 0$

$y(x,0) = \sin 2x$

$y'(x,0) = 0 \qquad 0 \le x < \infty$

4.0 CONCLUSION

In this unit, we have studied the notion of a solution of partial differential equation. Also some elementary methods of solving linear partial differential equations which occur frequently in physics and engineering were dealt with.

5.0 SUMMARY

Here in this unit you have learnt about second order partial differential equation. The classical method of separation of variables was extensively studied along with the Laplace transform solutions of boundary-value problems.

6.0 TUTOR- MARKED ASSIGNMENT

1. Form the PDEs whose general solutions are as follow:

   (a) $z = Ae^{-p^2 t}\cos px$

   (b) $z = f\left(\frac{y}{x}\right)$

2. Solve the equation

$$2\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial^2 u}{\partial y^2} = 0$$

3. Find the solution of the differential equation

$$\alpha^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

Where

$$\begin{array}{ll} y(0,t) = 0 & 0 < t < \infty \\ y(0,t) = 0 & 0 \le t < \infty \\ y(x,0) = f(x) & 0 \le x \le L \\ y(x,0) = g(x) & 0 \le x < L \end{array}$$

4. Solve by Laplace transforms the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \textit{for} \quad x > 0,\ t > 0$$

given that $u = u_0$ (a constant) on $x = 0$ for $t > 0$, and $u = 0$ for $x > 0, \ t = 0$

7.0 REFERENCES/ FURTHER READING

Erwin, K. (1991). *Advanced Engineering Mathematics*. John Wiley & Sons, Inc.

Pinsky, M.A. (1991).*Partial Differential Equations and Boundary-Value Problems with Applications*. New York: McGraw-Hill.

UNIT 2 FOURIER SERIES

CONTENTS

1.0 INTRODUCTION

In this unit, we shall discuss basic concepts, facts and techniques in connection with Fourier series. Illustrative examples and some important applications of Fourier series to Partial differential equations will be studied.

We will also study the concept of periodic functions, even and odd functions and the conditions for Fourier expansion.

2.0 OBJECTIVES

At the end of this unit, you should able to:

- identify whether a given function is even, odd or periodic
- evaluate the Fourier coefficients
- derive and apply Fourier series in forced vibration problems
- use Fourier Integral for treating various problems involving periodic function
- apply half range expansion to solutions of some problems.

3.0 MAIN CONTENT

3.1 Periodic Functions

A function $f(x)$ is said to be **periodic** if it defined for all real x and if there is some positive number T such that

$$f(x+T) = f(x) \qquad (1)$$

This number T is then called a **period** of $f(x)$.

Periodic functions occur very frequently in many application of mathematics to various branches of science. Many phenomena in nature such as propagation of water waves, light waves, electromagnetic waves, etc are periodic and we need periodic functions to describe them. Familiar examples of periodic functions are the sine and cosine functions.

Example 1

Find the period of $Tan\,x$.

Solution: Suppose T is its period

$$f(x+T) = Tan(x+T) = Tanx$$

so that

$$Tan(x+T) - Tanx = 0$$

using trigonometric identity, we have

$$\frac{Tan\,T(1-Tan^2 x)}{1-TanxTanT} = 0$$

This implies that

$Tan\,T = 0$ If and only if $1 - Tan^2 x \neq 0$

$T = Tan^{-1} 0$

Hence $T = \pi$

3.2 Even and Odd Functions

A function $f(x)$ defined on interval $[a, b]$ is said to be a even function if

$$f(-x) = f(x) \qquad (2)$$

It is odd otherwise, that is

$$f(-x) = -f(x) \quad (3)$$

Example 2

Let $f(x) = \sin x$

Then $f(-x) = -f(x)$ i.e. $\sin(-x) - = -\sin x$

Thus it is obvious that sine function is always an odd function while cosine function is an even function.

3.3 Fourier Theorem

According to the Fourier theorem, any finite, single valued periodic function $f(x)$ which is either continuous or possess only a finite number of discontinuities (of slope or magnitude), can be represented as the sum of the harmonic terms as

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \quad (4)$$

3.4 Evaluation of Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π which can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad (5)$$

Given such a function $f(x)$ we want to determine the coefficients of a_n and b_n in the corresponding series in eq. (5).

We first determine a_0. Integrating on both sides of eq. (4) from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

If term-by-term integration of the series is allowed, then we obtain

$$\int_{-\pi}^{\pi} f(x)dx = a_0\int_{-\pi}^{\pi}dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx dx + b_n\int_{-\pi}^{\pi} sinnxdx\right)$$

The first term on the right equals $2\pi a_0$. All other integrals on the right are zero, as can be readily seen by performing the integration. Hence our first result is

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (6)$$

We now determine $a_1, a_2, \ldots$ by a similar procedure. We multiply Eq. (5) by $\cos mx$, where m is any fixed positive integer, and then integrate from $-\pi$ to π,

$$\int_{-\pi}^{\pi} f(x)\cos mxdx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mxdx \qquad (7)$$

Integrating term-by-term, we see that the right-hand side becomes

$$a_0\int_{-\pi}^{\pi}\cos mxdx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cos mxdx + b_n\int_{-\pi}^{\pi}\sin nx\cos mxdx\right]$$

The first integration is zero. By applying trigonometric identity, we obtain

$$\int_{-\pi}^{\pi}\cos nx\cos mxdx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx$$
$$\int_{-\pi}^{\pi}\sin nx\cos mxdx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)xdx.$$

Integration shows that the four terms on the right are zero, except for the last term in the first line which equals π when $n = m$. since in eq. (7) this term is multiplied by a_m, the right-hand side in eq. (7) is equal to $a_m\pi$, and our second result is

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mxdx \qquad m = 1, 2\ldots \qquad (8)$$

We finally determine $b_{1,} b_{2,}$ ...... in eq.(5) by $\sin mx$, where m is any fixed positive integer, and the integrate from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin mxdx \qquad (9)$$

Integrating term-by-term, we see that the right-hand side becomes

$$a_0\int_{-\pi}^{\pi}\sin mxdx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\sin mxdx+b_n\int_{-\pi}^{\pi}\sin nx\sin mxdx\right]$$

The first integral is zero. The next integral is of the type considered before, and we know that it is zero for all n = 1, 2,…. For the integral we obtain

$$\int_{-\pi}^{\pi}\sin nx\sin mxdx=\frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx-\frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)xdx$$

The last term is zero. The first term on the right is zero when $n\neq m$ and is π when $n=m$. Since in eq. (9) this term is multiplied by b_m, the right-hand side in eq. (6) is equal to $b_m\pi$, and our last result is

$$b_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin mxdx \qquad\qquad m=1,\ 2\ldots$$

Writing n in place of m, we altogether have the so-called **Euler formulas**

(a) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

(b) $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx \qquad n=1,\ 2\ldots$ (10)

(c) $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$

Example 3 Square wave

Find the Fourier coefficients of the periodic function

$$f(x)=\begin{cases}-k & when \quad -\pi<x<0\\ k & when \quad 0<x<\pi\end{cases} \qquad and \qquad f(x+2\pi)=f(x)$$

Functions of this type may occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc

Solution: From eq. (10a) we obtain $a_0=0$. This can also be seen without integration since the area under curve of f(x) between $-\pi$ and π is zero. From eq. (10b)

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nxdx+\int_{0}^{\pi}k\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi} \right] = 0$$

Because $\sin nx = 0 \ at \ -\pi,\ 0$ and π for all n = 1, 2….Similarly, from Eq. (10c) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi} k\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\cos nx}{n}\Big|_{-\pi}^{0} + k\frac{\cos nx}{n}\Big|_{0}^{\pi} \right] = 0$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi)$$

Now, $\cos\pi = -1, \cos 2\pi = 1, \cos 3\pi = -1$ etc, in general

$$\cos n\pi = \begin{cases} -1 & for \quad odd\ n, \\ 1 & for \quad even\ n, \end{cases}$$ and thus

$$1 - \cos n\pi = \begin{cases} 2 & for \quad odd\ n, \\ 0 & for \quad even\ n, \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}$$

and since the a_n are zero, the corresponding Fourier series is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right) \tag{11}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad etc,$$

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + ...\right)$$

or $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - .... = \frac{\pi}{4}$$

SELF-ASSESSMENT EXERCISE 1

1. Define the periodic function. Give five examples.

2. Find the smallest positive period T of the following functions.
   a. $\sin x$

3. Are the following functions odd, even, or neither odd nor even?
   a. e^x
   b. $x \sin x$
4. Find the Fourier series of the following functions which are assumed to have the

a. period 2π
b. $f(x) = x^2/4 \qquad -\pi < x < \pi$
c. $f(x) = |\sin x| \qquad -\pi < x < \pi$

3.5 Application of Fourier Series in Forced Vibrations

We now consider an important application of Fourier series in solving a differential equation of the type

$$m\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + kx(t) = F(t) \qquad (12)$$

For example, the above equation would represent the forced vibrations of a damped oscillator with Γ representing the damping constant, F(t) the external force and m and k representing the mass of the particle and the force constant respectively. We write eq. (12) in the form

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x(t) = G(t) \qquad (13)$$

Where $K = \frac{\Gamma}{2m}$, $\omega_0^2 = \frac{k}{m}$ and $G(t) = \frac{F(t)}{m}$. The solution of the homogeneous part of eq. (13) can be readily obtained and is given by

$$x(t) = A_1 e^{=Kt}\cos\left[\sqrt{\left(\omega_0^2 - K^2\right)}t + \theta\right] \qquad \text{for} \qquad \omega_0^2 > K^2 \qquad (14)$$

$$x(t) = (A_2 t + B)e^{-Kt} \qquad \text{for} \qquad \omega_0^2 > K^2 \qquad (15)$$

In order to obtain the solution of the inhomogeneous part of eq. (13), we first assume F(t) to be a sine or cosine function; for definiteness we assume

$$G(t) = b \sin \omega t \tag{16}$$

The particular solution of eq. (13) can be written in the form

$$x(t) = C \sin \omega t + D \cos \omega t \tag{17}$$

The values of C and D can readily be obtained by substituting eq. (17) in eq. (13),
and comparing coefficients of $\sin \omega t$ and $\cos \omega t$ we obtain

$$\begin{aligned} D &= -\frac{2\omega K}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 K^2} b \\ C &= -\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 K^2} b \end{aligned} \tag{18}$$

Now, if G(t) is not a sine or cosine function, a general solution of eq. (13) is difficult to obtain. However, if we make a Fourier expansion of G(t) then the general solution of eq.

(13) can easily be written down. As a specific example, we assume

$$G(t) = \alpha t \tag{19}$$

The Fourier expansion of $G(t)$ can readily be obtained as

$$G(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{20}$$

Proceeding in a manner similar to that described above we obtained the following solution for the inhomogeneous part of eq. (13)

$$x(t) = \sum_{n=1}^{\infty} \left[C_n \sin n\omega t + D_n \cos n\omega t\right] \tag{21}$$

Where

$$\begin{aligned} D_n &= -\frac{2n\omega K}{\left(\omega_0^2 - n^2\omega^2\right)^2 + 4n^2\omega^2 K^2} b_n \\ C_n &= -\frac{(\omega_0^2 - n^2\omega^2)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + 4n^2\omega^2 K^2} b_n \end{aligned} \tag{22}$$

thus, if G(t) is a periodic function with period T then eq. (21) will be valid for all values
of t.

3.6 Half-Range Expansions

In various physical and engineering problems there is a practical need for applying Fourier series to functions $f(t)$ which are defined merely on some finite interval. The function $f(t)$ is defined on an interval $0 \leqq t \leqq l$ and on this interval we want represent $f(t)$ by a Fourier series.

A **half-range Fourier series** for a function $f(x)$ is a series consisting of the sine and cosine terms only.

Such functions are defined on an interval $(0, l)$ and we then obtain a Fourier cosine series which represents an even periodic function $f_1(t)$ of period $T = 2l$ so that

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}t \qquad 0 \leqq t \leqq l \tag{23}$$

and the coefficients are

$$a_0 = \frac{1}{l}\int_0^l f(t)tdt, \qquad a_n = \frac{2}{l}\int_0^l f(t)\cos\frac{n\pi}{l}tdt \qquad n = 1, 2\ldots \tag{24}$$

Then we obtain a Fourier sine series which represents an odd periodic function $f_2(t)$ of period $T = 2l$ so that

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}t \qquad 0 \leqq t \leqq l \tag{25}$$

and the coefficients are

$$b_n = \frac{2}{l}\int_0^l f(t)\sin\frac{n\pi}{l}tdt \qquad n = 1, 2\ldots \tag{26}$$

The series in eqs.(23) and (25) with the coefficients in eqs.(24) and (26) are called **half-range expansions** of the given function $f(t)$

Example 4

Find the half-range expansions of the function

$$f(t) = \begin{cases} \frac{2k}{l}t & when \quad 0 < t < \frac{l}{2} \\ \frac{2k}{l}(l-t) & when \quad \frac{l}{2} < t < l \end{cases}$$

Solution: From eq. (24) we obtain

$$a_0 = \frac{1}{l}\left[\frac{2k}{l}\int_0^{l/2} t\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)\,dt\right] = \frac{k}{2}$$
$$a_n = \frac{2}{l}\left[\frac{2k}{l}\int_0^{l/2} t\cos\frac{n\pi}{l}t\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)\cos\frac{n\pi}{l}t\,dt\right]$$

Now by integration by part

$$\int_0^{l/2} t\cos\frac{n\pi}{l}t\,dt = \frac{lt}{n\pi}\sin\frac{n\pi}{l}t\Big|_0^{l/2} - \frac{1}{n\pi}\int_0^{l/2}\sin\frac{n\pi}{l}t\,dt$$
$$= \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

Similarly,

$$\int_{l/2}^{l}(l-t)\cos\frac{n\pi}{l}t\,dt = -\frac{l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

By inserting these two results we obtain

$$u_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

Thus,

$$a_2 = -16k/2^2\pi^2, \quad a_6 = -16k/6^2\pi^2, \quad a_{10} = -16k/10^2\pi^2, \ldots$$

And $a_n = 0$ when $n \neq 2, 6, 10, 14, \ldots$ Hence the first half-range expansion of $f(t)$ is

$$f(t) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{l}t + \frac{1}{6^2}\cos\frac{6\pi}{l}t + \ldots\right)$$

This series represents the even periodic expansion of the function $f(t)$. Similarly from eq. (26)

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

and the other half-range expansion of $f(t)$ is

$$f(t) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{l}t - \frac{1}{3^2}\sin\frac{3\pi}{l}t + \frac{1}{5^2}\sin\frac{5\pi}{l}t - + \ldots\right)$$

This series represents the odd periodic extension of $f(t)$.

Example 5

Find a Fourier sine series for

$$f(x) = \begin{cases} 0 & x \le 2 \\ 2 & x > 2 \end{cases} \quad \text{on } (0,\ 3).$$

Solution: Since the function is odd, then $a_0 = 0$

Then $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}xdx$

$$= \frac{2}{3}\int_0^3 f(x)\sin\frac{n\pi}{3}xdx$$

$$= \frac{2}{3}\int_0^2 0.\sin\frac{n\pi}{3}xdx + \frac{2}{3}\int_2^3 2\sin\frac{n\pi}{3}xdx$$

Now by integration, we have

$$b_n = \frac{4}{n\pi}\left[\cos\frac{2n\pi}{3} - \cos n\pi\right]$$

The series thus becomes

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\left[\cos\frac{2n\pi}{3} - (-1)^n\right]\sin\frac{n\pi x}{3}$$

So that

$$f(x) = \frac{4}{\pi}\left(\frac{1}{2}\sin\frac{\pi x}{3} - \frac{3}{4}\sin\frac{2\pi x}{3} + \frac{2}{3}\sin\frac{3\pi x}{3} - + ...\right)$$

Example 6

Find the Fourier cosine series for

$$f(x) = e^x \quad \text{on} \quad (0,\ \pi)$$

Solution: Since $f(x)$ is an odd function, then

$$b_0 = \frac{1}{l}\int_0^l e^x dx \ = \frac{1}{\pi}\left(e^{\pi} - 1\right)$$

Also

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos\frac{n\pi x}{\pi}dx = \ \frac{2}{\pi}\left(\frac{1}{1+n^2}\right)\left(e^{\pi}\cos n\pi - 1\right)$$

Thus the series becomes

$$e^x = \ \frac{2}{\pi}\left(e^{\pi} - 1\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{1+n^2}\left[(-1)^n e^{\pi} - 1\right]\cos nx$$

SELF-ASSESSMENT EXERCISE 2

1. Find the Fourier sine series for
   $f(x) = e^x$ on $(0, \pi)$

2. Find the Fourier series for
   $f(x) = x$ on $0 < x < 2$
   consisting of (a) sine series only (b) cosine series only

3.7 Fourier Integral

Fourier series are powerful tools in treating various problems involving periodic functions. When the fundamental period is made infinite, the limiting form of the Fourier series becomes an integral which is called ***Fourier Integral.***

3.7.1 Definition

Let $f(x)$ be defined and single valued in the interval $[-L, L]$. If $f(x)$ satisfies the following conditions:

(i) $f(x)$ is periodic and of period 2L
(ii) $f(x)$ and $f'(x)$ are piecewise continuous
(iii) $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent, then $f(x)$ can be expressed as

$$f(x) = \int_0^{\infty} (A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x) dx$$
(27)
$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$$
(28)
$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$$
(29)

3.8 Fourier Integrals of Even and Odd Functions

It is of practical interest to note that if a function is even or odd and can be represented by a Fourier integral, and then this representation will be simpler than in the case of an arbitrary function. This follows immediately from our previous formulas, as we shall now see.

If $f(x)$ is an even function, then $B(\alpha) = 0$

$$A(\alpha) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos\alpha x dx \qquad (30)$$

and eq. (27) reduces to the simpler form

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x dx \qquad (f\ even) \qquad (31)$$

Similarly, if $f(x)$ is odd, then $A(\alpha) = 0$ in eq. (28), also

$$B(\alpha) = \frac{2}{\pi}\int_0^\infty f(x)\sin\alpha x dx$$

(32)

and

$$f(x) = \int_0^\infty B(\alpha)\sin\alpha x dx \qquad (f\ odd) \qquad (33)$$

These simplifications are quite similar to those in the case of a Fourier series discussed.

Example 7

Find the Fourier Integral of $f(x) = x^2 \qquad -\pi \le x \le \pi$

Solution:

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} x^2 \cos\alpha x dx$$

Using integration by parts, we obtain

$$A(\alpha) = \frac{2}{\pi\alpha}\left[\frac{x}{\alpha}\cos\alpha x - \frac{1}{\alpha^2}\sin\alpha x\right]_{-\pi}^{\pi} = 0$$

Also

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} x^2 \sin\alpha x dx$$

So that

$$B(\alpha) = -\frac{1}{\pi}\left[\frac{x^2}{\alpha}\cos\alpha x - \frac{2}{\alpha^3}\cos\alpha x\right]_{-\pi}^{\pi} = \frac{2\pi}{\alpha}(-1)^\alpha$$

From eq. (27)

$$f(x) = \int_0^\infty \left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right)dx$$

and

$$f(x) = \int_0^\infty \left(0 \bullet \cos\alpha x + \frac{2\pi}{\alpha}(-1)^\alpha \sin\alpha x\right)dx = \frac{2\pi}{\alpha^2}(-1)^\alpha$$

Hence

$$f(x) = x^2 = \frac{2\pi}{\alpha}(-1)^\alpha \int_0^\infty \sin\alpha x dx = \frac{2\pi}{\alpha^2}(-1)^\alpha$$

4.0 CONCLUSION

In this unit, you have studied the concept of periodic functions, representations of functions by Fourier series, involving sine and cosine function are given special attention. We also use the series expansion in the determination of Fourier coefficients and the half-range expansions.

5.0 SUMMARY

In this unit, you have studied:

- Even and odd functions
- Fourier Integral representations and Fourier series expansion.
- Application of Fourier Integral technique in the simplification of even and odd functions.

6.0 TUTOR- MARKED ASSIGNMENT

1. Find the smallest positive period T of the following functions
   a. (i) $\sin 2\pi x$
   b. (ii) $\cos\dfrac{2\pi nx}{k}$

2. Find the Fourier series for
   $f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$ where $f(x)$ has period 10

3. Find the Fourier series for
   $f(x) = x^2$ for $0 < x < 2\pi$

4. Find the Fourier series of function
   $f(x) = x + \pi$ when $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

5. Expand the function

$f(t) = t^2 \qquad -\frac{T}{2} < x < \frac{T}{2}$ in a Fourier series to show that

$$f(t) = t^2 = \frac{T^2}{4\pi^2}\left[\frac{\pi^2}{3} - 4\left(\cos\omega t - \frac{1}{4}\cos 2\omega t + \frac{1}{9}\cos\omega t - ...\right)\right]$$

take $\omega = 2\pi/T$

6. Represent the following functions $f(t)$ by a Fourier cosine series

(a) $f(t) = \sin\frac{\pi}{l}t \qquad (0 < t < l)$

(b) $f(t) = e^t \qquad (0 < t < l)$

7. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & when\ |x| < 1, \\ 0 & when\ |x| > 1. \end{cases}$$

7.0 REFERENCES/FURTHER READING

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MODULE 2 APPLICATION OF FOURIER TO PDES (LEGENDRE POLYNOMIALS AND BESSEL FUNCTIONS)

UNIT 1 LEGENDRE POLYNOMIALS

CONTENTS

1.0 INTRODUCTION

In this unit, you will be introduced to the polynomial solutions of the Legendre equation, the generating function as well as the orthogonality of Legendre polynomials. Also we shall consider some important integrals involving Legendre functions which are of considerable use in many areas of physics.

2.0 OBJECTIVES

At the end of this unit, you should be able:

- derive the polynomial solution of the Legendre equation
- use the generating functions to derive some important identities
- determine the orthogonality of the Legendre polynomials.

3.0 MAIN CONTENT

3.1 Legendre Equation

The equation

$$(1-x^2 y''(x) - 2xy'(x) + n(n+1)y(x) = 0 \qquad (1)$$

where n is a constant is known as the ***Legendre's differential equation.*** In this unit we will discuss the solutions of the above equation in the domain $-1 < x < 1$. We will show that when

$$n = 0, 1, 2, 3, \ldots$$

one of the solutions of eq. (1) becomes a polynomial. These polynomial solutions are known as the ***Legendre polynomials,*** which appear in many diverse areas of physics and engineering.

3.2 The Polynomial Solution of the Legendre's Equation

If we compare eq. (1) with homogeneous, linear differential equations of the type

$$y''(x) + U(x)y'(x) + V(x)y(x) = 0 \qquad (2)$$

we find that the coefficients

$$U(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad V(x) = \frac{n(n+1)}{1-x^2} \qquad (3)$$

are analytical at the origin. Thus the point x = 0 is an ordinary point and a series solution of eq. (1) using Frobenius method should be possible. Such that

$$y(x) = C_0 S_n(x) + C_1 T_n(x)$$

where

$$S_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots \qquad (4a)$$

And

$$T_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \ldots (4b)$$

If $n \neq 0.\ 1,\ 2, \ldots$ both eqs. (4a) and (4b) are infinite series and converge only if $|x| < 1$.

It may be readily seen that when

$$n = 0.\ 2,\ 4, \ldots$$

The even series becomes a polynomial and the odd series remains an infinite series. Similarly for

$$n = 1,\ 3,\ 5, \ldots$$

the odd series becomes a polynomial and the even series remains an infinite series.
Thus when

$$n = 0,\ 1,\ 2,\ 3, \ldots$$

one of the solutions becomes a polynomial. The Legendre polynomial, or the Legendre function of the first kind is denoted by $P_n(x)$ and is defined in terms of the terminating series as below:

$$P_n(x) = \begin{cases} \dfrac{S_n(x)}{S_n(1)} & for\ \ n = 0,\ 2,\ 4,\ 6, \ldots \\ \dfrac{T_n(x)}{T_n(1)} & for\ \ n = 1,\ 3,\ 5,\ 7, \ldots \end{cases} \tag{5}$$

Thus,

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \\ &P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \quad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \\ &P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right), \ldots \end{aligned} \tag{6}$$

Obviously, $\quad P_n(1) = 1 \qquad (7)$

Higher order Legendre polynomials can easily be obtained by using the recurrence relation

$$nP_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x)$$

Since for even values of n the polynomials $P_n(x)$ contain only even powers of x and for odd values of n the polynomials contain only odd powers of x, we readily have

$$P_n(-x) = (-1)^n P_n(x) \qquad \text{and obviously} \tag{8}$$

$$P_n(-1) = (-1)^n \tag{9}$$

3.3 The Generating Function

The generating function for the Legendre polynomials is given by

$$G(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n;\ -1 \le x \le 1,\ t < 1 \tag{10}$$

Let us assume that

$$G(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} K_n(x)t^n \qquad (11)$$

Where $K_n(x)$ is a polynomial of degree n. Putting x = 1 in eq. (11), we obtain

$$\sum_{n=0}^{\infty} K_n(x)t^n = (1-2t+t^2)^{-1/2}$$
$$= (1-t)^{-1}$$
$$= 1+t+t^2+t^3+\ldots+t^n+\ \ldots$$

Equating the coefficients of t^n from both sides, we have

$$K_n(1) = 1 \qquad (12)$$

Now, if we can show that $K_n(x)$ satisfies eq. (1), then $K_n(x)$ will be identical to $P_n(x)$. Differentiating G(x, t) with respect to x and t, we obtain

$$(1-2xt+t^2)\frac{\partial G}{\partial t} = (x-t)G(x,t) \qquad (13)$$

and

$$t\frac{\partial G}{\partial t} = (x-t)\frac{\partial G}{\partial x} \qquad (14)$$

Using eqs. (11), (13) and (14), we have

$$(1-2t+t^2)\sum_{n=0}^{\infty} nK_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} K_n(x)t^n \qquad (15)$$

and

$$t\sum_{n=0}^{\infty} nK_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} K_n'(x)t^n \qquad (16)$$

Equating the coefficient of t^{n-1} on both sides of eqs. (15) and (16), we get

$$nK_n(x) - (2n-1)xK_{n-1}(x) + (n-1)K_{n-2}(x) = 0 \qquad (17)$$

and

$$xK_{n-1}'(x) - K_{n-2}'(x) = (n-1)K_{n-1}(x) \qquad (18)$$

Replacing n by n+1 in Eq. (18), we obtain

$$xK_n'(x) - K_{n-1}'(x) = nK_n(x) \qquad (19)$$

We next differentiate Eq. (17) with respect to x and eliminate K_{n-2}' with help of Eq. (18) to obtain

$$K'_n(x) - xK'_{n-1}(x) - nK_{n-1}(x) = 0 \qquad (20)$$

If we multiply eq. (19) by x and subtract it from eq. (20), we would get

$$(1-x^2)K'_n - n(K_{n-1} - xK_n) = 0 \qquad (21)$$

Differentiating the above equation with respect to x, we have

$$(1-x^2)K''_n - 2xK'_n - n(K'_{n-1} - xK'_n - K_n) = 0 \qquad (22)$$

Using eqs. (19) and (22), we obtain

$$(1-x^2)K''_n(x) - 2xK'_n(x) - n(n+1)K_n(x) = 0 \qquad (23)$$

which shows that $K_n(x)$ is a solution of Legendre equation. In view of eqs. (7) and (12) and the fact that $K_n(x)$ is a polynomial in x of degree n, it follows that $K_n(x)$ is nothing but $P_n(x)$. eq. (17) gives the recurrence relation for $P_n(x)$

$$nP_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x) \qquad (24)$$

3.4 Rodrigues' Formula

Let

$$\phi(x) = (x^2-1)^n \qquad (25)$$

Differentiating eq. (25), we get

$$\frac{d\phi}{dx} = 2nx(x^2-1)^{n-1}$$

or

$$(1-x^2)\frac{d^2\phi}{dx^2} + 2x(n-1)\frac{d\phi}{dx} + 2n\phi = 0$$

Differentiating the above equation n times with respect to x, we would get

$$(1-x^2)\frac{d^2\phi_n}{dx^2} + 2x\frac{d\phi_n}{dx} + n(n+1)\phi_n = 0 \qquad (26)$$

where

$$\phi_n = \frac{d^n\phi}{dx^n} = \frac{d^n}{dx^n}\left[(x^2-1)^n\right] \qquad (27)$$

This shows that $\phi_n(x)$ is a solution of the Legendre's equation. Further, it is obvious from eq. (27) that $\phi_n(x)$ is a polynomial of degree n in x. Hence $\phi_n(x)$ should be a constant multiple of $P_n(x)$, i.e.

$$\frac{d^n\left[\left(x^2-1\right)^n\right]}{dx^n} = CP_n(x) \tag{28}$$

$$\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right] = \frac{d^n}{dx^n}\left[\left(x+1\right)^n\left(x-1\right)^n\right]$$

$$= n!(x-1)^n + n\frac{n!}{1!}(x+1)n\left(x-1\right)^{n-1}$$

$$+\frac{n(n-1)}{2!}\frac{n!}{2!}\left(x+1\right)^2 n(n-1)(x-1)^{n-2} + \ldots + (x+1)^n n! \tag{29}$$

It may be seen that all terms on the right hand side of eq. (29) contain a factor (x-1) except for the last term. Hence

$$\left.\frac{d^n}{dx^n}\left(x^2-1\right)^n\right|_{x=1} = 2^n n! \tag{30}$$

Using Eqs. (7), (28) and (29), we obtain

$$C = 2^n n! \tag{31}$$

Therefore

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \tag{32}$$

This is known as the **R*odrigues formula*** for the Legendre polynomials.

For example

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(x^2-1\right)^2$$

$$= \frac{1}{2}\left(3x^2-1\right)$$

Which is consistent with eq. (6)

3.5 Orthogonality of the Legendre Polynomials

Since the Legendre's differential equation is of the Sturm-Liouville form in the interval $-1 \le x \le 1$, with $P_n(x)$ satisfying the appropriate boundary conditions at $x = \pm 1$. The Legendre polynomials form an orthogonal set of functions in the interval $-1 \le x \le 1$, i.e

$$\int_{-1}^{1} p_n(x)\,p_m(x)dx = 0 \qquad m \neq n \tag{33}$$

The Orthogonality of the Legendre polynomials can be proved as follows: $P_n(x)$ satisfies eq. (1) which can be written in the Sturm-Liouville form as

$$\frac{d}{dx}\left[\left(x^2-1\right)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0 \tag{34}$$

Similarly

$$\frac{d}{dx}\left[\left(x^2-1\right)\frac{dP_m(x)}{dx}\right] + m(m+1)P_m(x) = 0 \tag{35}$$

Multiply eq. (34) by $P_m(x)$ and eq. (35) by $P_n(x)$ and subtracting eq. (35) from eq. (34), we get

$$\frac{d}{dx}\left[(1-x^2)\left(P_n'(x)P_m(x) - P_m'(x)P_n(x)\right)\right]$$

$$= (m-n)(n+m+1)P_n(x)P_m(x)$$

Integrating the above equation from x = -1 to x = 1, we get

$$(1-x^2)\left[\left(P_n'(x)P_m(x) - P_m'(x)P_n(x)\right)\right]_{-1}^{+1}$$
$$= (m-n)(n+m+1)\int_{-1}^{1} P_n(x)P_m(x)dx$$

Because of the factor $(1-x^2)$ the left hand side of the above equation vanishes; hence

$$\int_{-1}^{1} P_n(x)P_m(x)dx \qquad \textit{for} \quad m \neq n$$

To determine the value of the integral

$$\int_{-1}^{1} P_m^2(x)dx$$

we square both sides of eq. (10) and obtain

$$(1-2xt+t^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)t^{m+n} \tag{36}$$

Integrating both sides of the above equation with respect to x from -1 to +1 and using eq. (33), we get

$$\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x)dx = \int_{-1}^{1} \frac{1}{1-2xt+t^2}dx = \frac{1}{t}\ln\frac{1+t}{1-t}$$
$$= 2\left(1+\frac{1}{3}t^2+\frac{1}{5}t^4+\ldots+\frac{1}{2n+1}t^{2n}+\ldots\right)$$

Equating the coefficients of t^{2n} on both sides of the above equation, we have

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1} \qquad n = 0, 1, 2, 3, ... \qquad (37)$$

Thus we may write

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$$

where

$$\delta_{nm} = \begin{cases} 0 & if \;\; n \neq m \\ 1 & if \;\; n = m \end{cases}$$

Example

We consider the function $\cos \pi x/2$ and expand it in a series (in the domain $-1 < x < 1$) up to the second power of x:

$$\cos\frac{\pi x}{2} = \sum_{n=0}^{2} C_n P_n(x)$$

Now

$$C_n = \frac{2n+1}{2}\int_{-1}^{+1}\cos\frac{\pi x}{2}P_n(x)dx$$

Substituting for $P_n(x)$ from eq. (6) and carrying out brute force integration, we readily get

$$C_0 = \frac{2}{\pi}; \quad C_1 = 0; \quad C_2 = \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)$$

Thus

$$\cos\frac{\pi x}{2} = \frac{2}{\pi} + \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)\left(\frac{3x^2-1}{2}\right)$$

3.6 The Angular Momentum Problem in Quantum Mechanics

In electrostatics the potential Φ satisfies the Laplace equation

$$\nabla^2\Phi = 0 \tag{38}$$

We wish to solve the above equation for a perfectly conducting sphere (of radius a), place in an electric field which is in the absence of the sphere was of uniform magnitude E_0 along z-direction. We assume the origin of our coordinate system to be at the centre of the sphere. Because the sphere is a perfect conductor, the potential on its surface will be constant which, without any loss of generality, may be assumed to be zero. Thus, eq. (35) is said to be solved subject to the boundary condition

$$\Phi(r = a) = 0 \tag{39}$$

At a large distance from the sphere the field should remain unchanged and thus

$$E(r \to \infty) = E_0 \hat{z}$$

Since

$$E = -\nabla\Phi$$

we have

$$\begin{aligned}\Phi(r \to \infty) &= -E_0 z + C \\ &= -E_0 r\cos\theta + C\end{aligned} \tag{40}$$

Where C is a constant. Obviously, we should use the spherical system of coordinates so that

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{41}$$

From the symmetry of the problem it is obvious that Φ would be independent of the azimuthal coordinate ϕ so that eq. (41) simplifies to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0 \tag{42}$$

Separation of variables

$$\Phi = R(r)\Theta(\theta)$$

will yield

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=a\ constant\ (=\lambda) \qquad (43)$$

Changing the independent variable from θ to μ by the relation

$$\mu=\cos\theta$$

In the angular equation, we get

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\lambda\Theta=0 \qquad (44)$$

In order that the solution of eq. (44) does not diverge at $\mu=\pm1(\theta=0\ and\ \pi)$, we must have

$$\lambda=l(l+1);\ l=0,\ 1,\ 2,\ldots$$

and then

$$\Theta(\theta)=\sqrt{\frac{2l+1}{2}}P_l(\cos\theta) \qquad (45)$$

Thus the radial equation can be written as

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)$$

or

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-l(l+1)R=0 \qquad (46)$$

The above equation is the Cauchy's differential equation and its solution can readily be written as

$$R=A_l r^l+\frac{B_l}{r^{l+1}}$$

Hence the complete solution of eq. (42) is given by

$$\begin{aligned}\Phi(r,\theta)&=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)+\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)\\&=\left[A_oP_o(\cos\theta)+A_1rP_1(\cos\theta)+A_2r^2P_2(\cos\theta)+\ldots\right]\\&+\frac{B_o}{r}P_o(\cos\theta)+\frac{B_1}{r^2}P_1(\cos\theta)+\ldots\end{aligned}$$

Applying the boundary condition given by eq. (40), we get

$$A_0=C,\ A_1=-E_o,\ A_2=A_3=\ldots=0$$

Thus

$$\Phi(r,\theta)=\left(C+\frac{B_0}{r}\right)P_0(\cos\theta)+\left(-E_0r+\frac{B_1}{r^2}\right)P_1(\cos\theta)$$

$$+\frac{B_2}{r^3}P_2(\cos\theta)+\ .\ .\ .$$

Applying the condition at r = a [see eq. (39)], we get

$$\left(C+\frac{B_0}{a}\right)+\left(-E_0a+\frac{B_1}{a^2}\right)P_1(\cos\theta)$$
$$+\frac{B_2}{a^3}P_2(\cos\theta)+\frac{B_2}{a^4}P_3(\cos\theta)+\ .\ .\ .=0$$

Since the above equation has to be satisfied for all values of θ and since $P_n(\cos\theta)$ form a set of orthogonal functions, the coefficients of $P_n(\cos\theta)$ should be zero giving

$$B_0=-aC,\ \ B_1=E_0a^3$$
$$B_2=B_3=B_4\ \ldots=0$$

Thus

$$\Phi(r,\theta)=C\left(1+\frac{a}{r}\right)-E_0\left(1+\frac{a^3}{r^3}\right)r\cos\theta \qquad (47)$$

The $1/r$ potential would correspond to a charged sphere and, therefore, for an uncharged sphere we must have C = 0 giving

$$\Phi(r,\theta)=-E_0r\cos\theta\left(1+\frac{a^3}{r^3}\right) \qquad (48)$$

This is the required solution to the problem. One can easily determine the components of the electric field as:

$$E_r=-\frac{\partial\Phi}{\partial r}=E_0\cos\theta\left(1+2\frac{a^3}{r^3}\right)$$
$$E_\theta=-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}=E_0\sin\theta\left(1-\frac{a^3}{r^3}\right)$$
$$E_\phi=-\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}=0$$

3.7 Important Integrals Involving Legendre Functions

We give below some important integrals involving Legendre functions which are of considerable use in many areas of physics.

$$P_n(x)=\frac{1}{\pi}\int_0^\infty\left[x+(x^2-1)^{1/2}\cos\theta\right]^n d\theta \qquad (49)$$

$$P_n(\cos\phi) = \frac{1}{\pi}\int_0^{\pi}\left(\cos\phi + i\sin\phi\cos\theta\right)^n d\theta \tag{50}$$

$$\int_{-1}^{1}(1-x)^{-1/2}P_n(x)dx = \frac{2^{3/2}}{2n+1} \tag{51}$$

$$\int_{-1}^{1}\frac{1}{(1-x^2)}\left[P_n^m(x)\right]^2 dx = \frac{(n+m)!}{m(n-m)!} \tag{52}$$

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{1}{\left(n+\frac{1}{2}\right)}\frac{(n+m)!}{(n-m)!} \tag{53}$$

SELF-ASSESSMENT EXERCISE

1. Show that $(n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$
2. Using the Rodrigue's formula show that
   $P'_n(x) = \frac{1}{2}n(n+1)$

4.0 CONCLUSION

The concept of generating function for the Legendre polynomials allows us to readily derive some important identities.

We have also established in this unit, relationship between Orthogonality of the Legendre polynomials and the generating function.

5.0 SUMMARY

This unit deals with Legendre functions and its applications to physical problems especially in quantum mechanics.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$
$$= -\frac{n(n+1)}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

2. Determine the coefficients
   $C_0, \; C_1, \; C_2, \; C_3$, in the expansion

$$\sin\left(\frac{nx}{2}\right) = \sum_{n=0}^{3} C_n P_n(x) \qquad -1 < x < 1$$

3. Consider the function

$$f(x) = \begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 < x \le 1 \end{cases}$$

Show that

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}\left[P_{n+1}(0) - P_{n-1}(0) - \right]P_n(x) \qquad -1 < x < 1$$

4. Show that the generating function

$$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n$$

Hint: Start from the binomial expansion of $1/\sqrt{1-v}$, set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

7.0 REFERENCES/FURTHER READING

Ghatak, A.K.; Goyal, I.C. & Chua, S.J. (1995). *Mathematical Physics*. Macmillan India Ltd.

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UNIT 2 BESSEL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit we shall consider the series solution as well as Bessel functions of the first and second kinds of order n.
We will also be introduced to some integrals which are useful in obtaining solutions of some problems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive the solution of Bessel function of the first kind
- prove a relationship between the recurrence relation and the generating functions
- derive the solution of Bessel function of the second kind.

3.0 MAIN CONTENT

3.1 Bessel Differential Equation

The equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx}(x^2 - n^2)y(x) = 0 \tag{1}$$

Where n is a constant known as ***Bessel's differential equation.***
Since n^2 appears in eq. (1), we will assume, without any loss of generality, that n is either zero or a positive number. The two linearly independent solutions of eq. (1) are

$J_n(x)$ and $J_{-n}(x)$

Where $J_n(x)$ is defined by the infinite series

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \tag{2}$$

or

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \dots\right] \tag{3}$$

where $\Gamma(n+r+1)$ represents the gamma function.

3.2 Series Solution and Bessel Function of the First Kind

If we use eq. (1) with the homogeneous, linear differential equation of the type

$$y''(x) + U(x)y'(x) + V(x)y(x) = R(x) \tag{4}$$

we find the coefficients

$U(x) = \frac{1}{x}$ and $V(x) = 1 - \frac{n^2}{x^2}$

are singular at x = 0. However, x = 0 is a regular singular point of the differential equation and a series solution of eq. (1) in ascending powers of x. Indeed, one of the solutions of eq. (1) is given by

$$J_n(x) = C_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \dots\right] \tag{5}$$

and where C_0 is an arbitrary constant. This solution is analytic at x = 0 for $n \geq 0$ and converges for all finite values of x. If we choose

$$C_0 = 2^n \Gamma(n+1) \tag{6}$$

then the eq. (5) is denoted by $J_n(x)$ and is known as the ***Bessel function of the first kind of order n.***

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$
$$= \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^{n} - \frac{1}{1!\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4} - \dots \quad (7)$$

In particular

$$J_0(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \dots \quad (8)$$

$$J_{1/2}(x) = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{1!\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2!\Gamma(7/2)} + \dots$$
$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]$$
$$= \sqrt{\frac{2}{\pi x}} \sin x \quad (9)$$

It follows immediately from eqs. (7) and (8) that

$$J_n(0) = 0 \quad \textit{for} \quad n > 0$$

and

$$J_n(0) = 1$$

If $n \neq 0, 1, 2, 3, \dots$ then

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)} \quad (10)$$

Example 1

In this example we will determine the value of $J_{-1/2}(x)$ from eq. (10).

Thus

$$J_{-1/2}(x) = \frac{(x/2)^{-1/2}}{\Gamma(1/2)} - \frac{(x/2)^{3/2}}{1!\Gamma(3/2)} + \frac{(x/2)^{7/2}}{2!\Gamma(5/2)} + \dots$$
$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right]$$
$$= \sqrt{\frac{2}{\pi x}} \cos x$$

Which is linearly independent of $J_{1/2}(x)$ [see eq. (9)] and it can be verified that $J_{-1/2}(x)$ does in fact satisfy eq. (1) for n = ½. Thus

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad (11)$$

and

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \qquad (12)$$

Using the above two equations and the recurrence relation [see Eq. (21)]

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad (13)$$

We can readily obtain closed form expression for $J_{\pm 3/2}(x)$, $J_{\pm 5/2}(x), J_{\pm 7/2}(x), \ldots$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \qquad (14)$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{1}{x} \cos x - \sin x \right) \qquad (15)$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{(3 - x^2)}{x^2} \sin x - \frac{3}{x} \cos x \right) \qquad (16)$$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{(3 - x^2)}{x^2} \cos x - \frac{3}{x} \sin x \right) \qquad (17)$$

etc.

Next, we will examine eq.(10) when n is a positive integer. To be specific we assume n = 4; then the first, second, third and fourth terms in the series given by eq. (10) will contain the terms

$$\frac{1}{\Gamma(-3)}, \ \frac{1}{\Gamma(-2)}, \ \frac{1}{\Gamma(-1)}, \text{ and } \frac{1}{\Gamma(0)}$$

respectively and all these terms are zero. In general the first n terms of the series would vanish giving

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)} \qquad (18)$$

If we put r = k+n, we would obtain

$$\begin{aligned} J_{-n}(x) &= \sum_{k=0}^{\infty} (-1)^{k+n} \frac{(x/2)^{n+2k}}{(k+n)!\Gamma(k+1)} \\ &= \sum_{k=0}^{\infty} (-1)^{k} \frac{(x/2)^{n+2k}}{k!\Gamma(k+n+1)} \\ &= (-1)^n J_n(x) \qquad (19) \end{aligned}$$

Thus for $n = 0, 1, 2, 3, \ldots, J_{-n}(x)$ does not represent the second independent solution of eq. (1). The second independent solution will be discussed later.

3.3 Recurrence Relations

The following are some very useful relations involving $J_n(x)$:

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \text{(20a)}$$

$$= xJ_{n-1}(x) - nJ_n(x) \quad \text{(20b)}$$

Thus

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad (21)$$

Also

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \quad (22)$$

In order to prove eq. (20a) w.r.t x to obtain

$$xJ_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2} x \quad (23)$$

or

$$xJ_n'(x) = n\sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ x\sum_{r=0}^{\infty} (-1)^r \frac{1}{(r-1)!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n(x) - x\sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \quad (24)$$

or

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad (25)$$

Which proves eq. (20a). eq. (23) can also be written as

$$xJ_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{2(n+r)(x/2)^{n+2r-1}}{r!\Gamma(n+r+1)} \frac{x}{2}$$

$$- n\sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{r!\Gamma(n+r+1)}$$

$$= x\sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n-1+2r}}{r!\Gamma(n+r)} - nJ_n(x)$$

or

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x) \quad (26)$$

Which proves eq. (20b). From eq. (25) we readily obtain

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = x^{-n}J_{n+1}(x) \tag{27}$$

Further, adding eqs. (25) and (26) we get

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \tag{28}$$

Using eq. (21) we may write

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \tag{29}$$

$$\begin{aligned} J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) \\ &= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \end{aligned} \tag{30}$$

$$\begin{aligned} J_4(x) &= \frac{6}{x}J_3(x) - J_2(x) \\ &= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} + 1\right)J_0(x) \end{aligned} \tag{31}$$

etc.

The proof of eq. (22) is simple

$$\begin{aligned} \frac{d}{dx}\left[x^n J_n(x)\right] &= x^n J_n'(x) + nx^{n-1}J_n(x) \\ &= x^n\left[J_{n-1}(x) - \frac{n}{x}J_n(x)\right] + nx^{n-1}J_n(x) \\ &= x^n J_{n-1}(x) \qquad \text{[Using eq. (20b)]} \end{aligned} \tag{32}$$

Now using eq. (20a)

$$J_0'(x) = -J_1(x) \tag{33}$$

Therefore

$$\int J_1(x)dx = -J_0(x) + \text{Constant} \tag{34}$$

or

$$\int_0^\infty J_1(x)dx = 1 \qquad \text{[Because } J_0(0) = 1\text{]} \tag{35}$$

Equation (32) gives us

$$\int x^n J_{n-1}(x)dx = x^n J_n(x) \tag{36}$$

Example 2

In this example we will evaluate the integral

$$\int x^4 J_1(x)dx$$

in terms of $J_0(x)$ and $J_1(x)$. Since

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \qquad \text{[see eq. (22)]}$$

we have

$$\int x^p J_{p-1}(x)dx = x^p J_p(x)$$

Thus

$$\begin{aligned}\int x^4 J_1(x)dx &= \int x^2\left[x^2 J_1(x)\right]dx \\ &= x^2\left[x^2 J_2(x)\right] - \int 2x^3 J_2(x)dx \\ &= x^4 J_2(x) - 2x^3 J_3(x) \\ &= x^4 J_2(x) - 2x^3\left[\frac{4}{x}J_2(x) - J_1(x)\right] \\ &= (x^4 - 8x^2)\left(\frac{2}{x}J_1(x) - J_0(x)\right) + 2x^3 J_1(x) \\ &= \left(4x^3 16x\right)J_1(x) - (x^4 - 8x^2)J_0(x)\end{aligned}$$

plus, of course, a constant of integration.

3.4 The Generating Function

Bessel functions are often *defined* through the generating function G(z,t) which is given by the following equation

$$G(z,t) = \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] \qquad (37)$$

For every finite value of z, the function G(z,t) is a regular function of t for all (real or complex) values of t except at point t = 0. Thus it can be expanded in a Laurent series

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(z) \qquad (38)$$

In the above equation, the coefficient of t^n is defined as $J_n(z)$; we will presently show that this definition is consistent with series given by eq. (3). Now, for any finite value of z and for $0 < |t| < \infty$ we may write

$$\exp\left[\frac{zt}{2}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{zt}{2}\right)^n$$
$$=1+\frac{z}{2}\frac{1}{1!}+\left(\frac{z}{2}\right)^2\frac{t^2}{2!}+\left(\frac{z}{2}\right)^3\frac{t^3}{3!}+\ldots \qquad (39)$$

and

$$\exp\left[-\frac{z}{2t}\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{z}{2t}\right)^n$$
$$=1-\frac{z}{2}\frac{1}{t}+\left(\frac{z}{2}\right)^2\frac{1}{2!t^2}-\left(\frac{z}{2}\right)^3\frac{1}{3!t^3}+\ldots \qquad (40)$$

Thus the generating function can be expressed as a series of the form

$$G(z,t)=\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{+\infty}A_n(z)t^n \qquad (41)$$

or

$$\sum_{n=-\infty}^{+\infty}A_n(z)t^n=\left[1+\frac{z}{2}\frac{1}{1!}+\left(\frac{z}{2}\right)^2\frac{t^2}{2!}+\left(\frac{z}{2}\right)^3\frac{t^3}{3!}+\ldots\right]$$
$$\times\left[1-\frac{z}{2}\frac{1}{1!t}+\left(\frac{z}{2}\right)^2\frac{1}{2!t^2}-\left(\frac{z}{2}\right)^3\frac{1}{3!t^3}+\ldots\right] \qquad (42)$$

On the other hand, the coefficient of t^0 will be given by

$$A_0(z)=1-\left(\frac{z}{2}\right)^2\frac{1}{(1!)^2}+\left(\frac{z}{2}\right)^4\frac{1}{(2!)^2}-\left(\frac{z}{2}\right)^6\frac{1}{(3!)^2}+\ldots \qquad (43)$$

Comparing the above equation with eq. (8), we find

$$A_0(z)=J_0(z)$$

Similarly, the coefficient of t^n on the right hand side of eq. (42) will be given by

$$A_n(z)=\left(\frac{z}{2}\right)^n\frac{1}{n!}+\left(\frac{z}{2}\right)^{n+2}\frac{1}{(n+1)!1!}+\left(\frac{z}{2}\right)^{n+4}\frac{1}{(n+2)!}-\ldots$$

which when compared with eq. (7) gives us

$$A_n(z)=J_n(z)$$

Proving

$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{+\infty}t^nJ_n(z)$$

In the above equation, if we replace t by -1/y, we obtain

$$\exp\left[\frac{z}{2}\left(y-\frac{1}{y}\right)\right]=\sum_{n=-\infty}^{+\infty}(-1)^n y^{-n}J_n(z)=\sum_{n=-\infty}^{+\infty}y^nJ_n(z)$$

Thus

$$J_n(z) = (-1)^n J_{-n}(z)$$

3.4.1 Derivation of the Recurrence Relations from the Generating Function

Differentiating eq. (38) w.r.t z, we obtain

$$\frac{1}{2}\left(t - \frac{1}{t}\right)\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n'(z) \qquad (44)$$

Thus

$$\sum_{n=-\infty}^{+\infty} t^{n+1} J_n(z) - \sum_{n=-\infty}^{+\infty} t^{n-1} J_n(z) = \sum_{n=-\infty}^{+\infty} t^n 2J_n'(z)$$

Comparing the coefficients of t^n, we obtain

$$J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z)$$

Similarly, if we differentiate eq. (38) w.r.t t we will obtain

$$\frac{z}{2}\left(1 + \frac{1}{t^2}\right)\sum_{n=-\infty}^{+\infty} t^n J_n(z) = \sum_{n=-\infty}^{+\infty} nt^{n-1} J_n(z)$$

Comparing the coefficients of t^{n-1}, we get

$$z\left[J_{n-1}(z) - J_{n+1}(z)\right] = 2nJ_n(z)$$

3.5 Some Useful Integrals

Using $J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos[x\sin\theta - n\theta] d\theta$

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta) d\theta \qquad (45)$$

Thus

$$\int_0^{\infty} e^{-\alpha x} J_0(x) dx = \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-\alpha x}\, \frac{e^{ix\sin\theta} + e^{-ix\sin\theta}}{2} dx\right] d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left[\frac{1}{\alpha - i\sin\theta} + \frac{1}{\alpha + i\sin\theta}\right] d\theta$$

$$= \frac{2\alpha}{\pi}\int_0^{\pi/2} \frac{d\theta}{\alpha^2 + \sin^2\theta} \qquad (46)$$

or

$$\int_0^{\infty} e^{-\alpha x} J_0(x) dx = \frac{1}{\sqrt{1 + \alpha^2}} \qquad (47)$$

where in evaluating the integral on the right hand side of eq. (46), we have used the substitution $y = \alpha \cot\theta$. By making $\alpha \to 0$, we get

$$\int_0^\infty J_0(x)dx = 1 \qquad (48)$$

From eq. (28), we have

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Thus

$$2\int_0^\infty J_n'(x) = \int_0^\infty J_{n-1}(x) - \int_0^\infty J_{n+1}(x)$$

But

$$\int_0^\infty J_n'(x) = J_n(x)\Big|_0^\infty$$
$$= 0 \quad for \quad n > 0$$

Thus

$$\int_0^\infty J_{n+1}(x) = \int_0^\infty J_{n-1}(x) \qquad n > 0 \qquad (49)$$

Since

$$\int_0^\infty J_1(x) = 1 \qquad \text{[see eq. (35)]}$$

and

$$\int_0^\infty J_0(x) = 1 \qquad \text{[see eq. (48)]}$$

Using eq. (49), we have

$$\int_0^\infty J_n(x) = 1 \qquad n = 0, 1, 2, 3, \ldots \qquad (50)$$

Replacing α by $\alpha + i\beta$ in eq. (47), we get

$$\int_0^\infty e^{-(\alpha+i\beta)x} J_0(x)dx = \frac{1}{\sqrt{(\alpha+i\beta)^2 + 1}} \qquad (51)$$

which in the limit of $\alpha \to 0$ becomes

$$\int_0^\infty e^{-i\beta x} J_0(x)dx = \frac{1}{\sqrt{1-\beta^2}} \qquad (52)$$

For $\beta < 1$, the right hand side is real and we have

$$\int_0^\infty J_0(x)\cos\beta x dx = \frac{1}{\sqrt{1-\beta^2}} \qquad (53)$$

and

$$\int_0^\infty J_0(x)\sin\beta x dx = 0$$

Similarly, $\beta > 1$, the right hand side of Eq. (52) is imaginary and we have

$$\int_0^\infty J_0(x)\cos\beta x dx = 0$$

$$\int_0^\infty J_0(x)\sin\beta x dx = \frac{1}{\sqrt{1-\beta^2}} \qquad (54)$$

3.6 Spherical Bessel Functions

We start with the Bessel equation eq. (1)] with $n = l + \frac{1}{2}$, i.e.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}\left[x - \left(l+\frac{1}{2}\right)^2\right]y(x) = 0 \qquad (55)$$

where

$$l = 0,\ 1,\ 2, \ldots$$

The solutions of eq. (55) are

$$J_{l+\frac{1}{2}}(x) \text{ and } J_{-l-\frac{1}{2}}(x)$$

If we make the transformation

$$f(x) = \frac{1}{\sqrt{x}}y(x) \qquad (56)$$

we would readily obtain

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{df}{dx}\right) + \left[1 - \frac{l(l+1)}{x^2}\right]f(x) = 0 \qquad (57)$$

The above equation represents the *spherical Bessel* equation. From eqs. (55) and (56) it readily follows that the two independent solutions of eq.(57) are

$$\frac{1}{\sqrt{x}}J_{l+\frac{1}{2}}(x) \text{ and } \frac{1}{\sqrt{x}}J_{-l-\frac{1}{2}}(x)$$

The spherical Bessel functions are defined through the equations

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x) \qquad (58)$$

and

$$n_l(x) = (-1)^l\sqrt{\frac{\pi}{2x}}J_{-l-\frac{1}{2}}(x) \qquad (59)$$

and represent the two independent solutions of eq. (57). Now, if we define the function

u(x) = xf(x)

then eq. (57) takes the form

$$\frac{d^2u}{dx^2} + \left[1 - \frac{l(l+1)}{x^2}\right]u(x) = 0 \qquad (60)$$

The above equation also appears at many places and the general solution is given by

$$u(x) = c_1[xJ_l(x)] + c_2[xn_l(x)] \quad (61)$$

which also be written in the form

$$u(x) = A_1\left[\sqrt{x}J_{l+\frac{1}{2}}(x)\right] + A_2\left[\sqrt{x}J_{-l-\frac{1}{2}}(x)\right] \quad (62)$$

For $l = 0$, the solutions of eq. (60) are

$\sin x$ and $\cos x$

Thus, for $l = 0$ the two independent solutions of eq.(57) are

$\frac{\sin x}{x}$ and $\frac{\cos x}{x}$

Indeed if we use the definitions of $j_l(x)$ and $n_l(x)$ given eqs. (58) and (59) respectively, we would readily obtain

$$j_0(x) = \frac{\sin x}{x} \quad (63)$$

$$n_0(x) = \frac{\cos x}{x} \quad (64)$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad (65)$$

$$n_1(x) = \frac{\cos x}{x2} - \frac{\sin x}{x} \quad \text{etc} \quad (66)$$

Further, if we multiply the recurrence relation [Eq. (21)]

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

by $\sqrt{\frac{\pi}{2x}}$ and assume $n = l - \frac{l}{2}$, we would get

$$j_l(x) = \frac{(2l-1)}{x}n_{l-1}(x) - n_{l-2}(x) \quad (67)$$

using which we can readily obtain analytic expression for $j_2(x), j_3(x), \ldots$ etc.

Similarly,

$$n_l(x) = \frac{(2l-1)}{x}n_{l-1}(x) - n_{l-2}(x) \quad (68)$$

3.7 Bessel Functions of the Second Kind: Y_n

The Bessel functions of the second kind, denoted by $Y_n(x)$**,** are solutions of the Bessel differential equation. They have a singularity at the origin (x = 0). $Y_n(x)$ is sometimes also called the **Neumann function.** For non-integer n, it is related to $J_n(x)$ by:

$$Y_n(x) = \frac{J_n(x)\cos\mu\pi - J_{-n}(x)}{\sin\mu\pi} \qquad (69)$$

or

$$Y_n(x) = \frac{1}{n}\left[\frac{\partial}{\partial\mu} J_\mu(x) - (-)^n \frac{\partial}{\partial\mu} J_{-\mu}(x)\right]_{\mu=n} \qquad (70)$$

We need to show now that $Y_n(x)$ defined by eq.(70) satisfies Eq.(1) where n is either zero or an integer. We know that

$$J''_\mu(x) + \frac{1}{x} J'_\mu(x) + \left(1 - \frac{\mu^2}{x^2}\right) J_\mu(x) = 0 \qquad (71)$$

for any value of μ. Differentiating the above equation with respect to μ, we get

$$\frac{d^2}{dx^2}\frac{\partial J_\mu(x)}{\partial\mu} + \frac{1}{x}\frac{d}{dx}\frac{\partial J_\mu(x)}{\partial\mu} + \left(1 - \frac{\mu^2}{x^2}\right)\frac{\partial J_\mu(x)}{\partial\mu} = \frac{2\mu}{x^2} J_\mu(x) \qquad (72)$$

Similarly

$$\frac{d^2}{dx^2}\frac{\partial J_{-\mu}(x)}{\partial\mu} + \frac{1}{x}\frac{d}{dx}\frac{\partial J_{-\mu}(x)}{\partial\mu} + \left(1 - \frac{\mu^2}{x^2}\right)\frac{\partial J_{-\mu}(x)}{\partial\mu} = \frac{2\mu}{x^2} J_{-\mu}(x) \qquad (73)$$

From eqs. (72) and (73), it is easy to show that

$$\frac{d^2}{dx^2} S_\mu(x) + \frac{1}{x}\frac{d}{dx} S_\mu(x) + \left(1 - \frac{\mu^2}{x^2}\right) S_\mu(x)$$

$$= \frac{2\mu}{x^2}\left[J_\mu(x) - (-1)^n J_{-\mu}(x)\right] \qquad (74)$$

where

$$S_\mu(x) = \frac{\partial}{\partial\mu} J_\mu(x) - (-1)^n \frac{\partial}{\partial\mu} J_{-\mu}(x) \qquad (75)$$

Thus $Y_n(x)$ is the second solution of Bessel's equation for all real values of n and is known as the Bessel function of the second kind of order n. The general solution of eq.(1) can, therefore, be written as

$$y = C_1 J_n(x) + C_2 Y_n(x) \qquad (76)$$

where C_1 and C_2 are arbitrary constants.

The expression for $Y_n(x)$ for $n = 0,\ 1,\ 2, \ldots$ can be obtained by using eqs. (2) and (70) and is given below

$$Y_n(x) = \frac{2}{\pi}\left(\ln(x/2)+\gamma\right)J_n(x) - \frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x^2}{4}\right)^r$$
$$-\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{r=0}^{\infty}(-1)^r\frac{(x^2/4)^r}{r!(n+r)!}\left[\varphi(r)+\varphi(r+n)\right] \qquad (77)$$

Where $\varphi(r) = \sum_{s=1}^{m} s^{-1}; \quad \varphi(0) = 0$

and $\gamma = \underset{n\to\infty}{Lim}\left[\varphi(n) - \ln n\right]$

Example 3

In this example we will solve the radial part of the Schrodinger equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\frac{2\mu E}{\hbar} - \frac{l(l+1)}{r^2}\right)R(x) = 0; \quad l = 0, 1, 2, \qquad (78)$$

in the region $0 < r < a$ subject to the following boundary conditions that

$$R(a) = 0 \qquad (79)$$

and $R(r)$ is finite in the region $0 < r < a$. Equation (78) can be conveniently written in the form

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left(1 - \frac{l(l+1)}{\rho^2}\right)R(\rho) = 0$$

Where

$\rho = kr; \quad k = (2\mu E/\hbar^2)^{1/2}$

Thus the general solution of the above equation is given by

$$R(\rho) = Aj_l(\rho) + Bn_l(\rho) \qquad (80)$$

But $n_l(\rho)$ diverges at $\rho = 0$, therefore, we must choose B=0. The boundary condition R(a)=0 leads to the transcendental equation

$$j_l(ka) = 0 \qquad (81)$$

Thus, for $l = 0$, we have

$$ka = n\pi; \; n = 1, \; 2, ... \qquad (82)$$

Which will give allowed values of k. Similarly, for $l = 1$, we get

$$\tan ka = ka \qquad (83)$$

3.8 Modified Bessel Functions

If we replace x by ix in eq. (1), we obtain

$$x^2 y'' + xy' - (x^2 + n^2)y = 0 \qquad (84)$$

The two solutions of the above equation will obviously be
$J_n(ix)$ and $Y_n(ix)$

As these functions are real for all values of n, let us define a real function as

$$I_n(x) = i^{-n} J_n(ix) \tag{85}$$

or

$$I_n(x) = \sum_{r=0}^{\infty} \frac{(x/4)^{n+2r}}{r!(n+r+1)!} \tag{86}$$

This function will be the solution of eq. (84) and is known as the Modified Bessel function of the first kind. For very large values of x

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{87}$$

The other solution known as the Modified Bessel function of the second kind is defined as

$$K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi} \tag{88}$$

For non-integer values of n, I_n and I_{-n} are linearly independent and as such $K_n(x)$ is a linear combination of these functions [compare with eq. (69) which gives the definition of $Y_n(x)$]. When n is an integer, it can be shown [see eq. (86)] that

$$I_{-n} = I_n \tag{89}$$

and therefore $K_n(x)$ becomes indeterminate for n = 0 or an integer. As in the case of $Y_n(x)$ for n = 0 or an integer, we define $K_n(x)$ as

$$K_n(x) = \lim_{\mu \to n} \left[\frac{\pi}{2} \frac{I_{-\mu}(x) - I_\mu(x)}{\sin \mu\pi} \right] \tag{90}$$

or

$$K_n(x) = \frac{(-1)^n}{2} \left[\frac{\partial I_{-\mu}(x)}{\partial \mu} - \frac{\partial I_\mu(x)}{\partial \mu} \right]_{\mu=n} \tag{91}$$

For x very large

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \tag{92}$$

From eq. (88) it follows that

$$K_{-n}(x) = K_n(x) \tag{93}$$

Which is true for all values of n. recurrence relations for I_n can be derived from those of $J_n(x)$ and Eq. (85). They are

$xI_n'(x) = xI_{n-1}(x) - nI_n(x)$

(94) $xI_n'(x) = nI_n(x) + xI_{n+1}(x)$

95) $I_{n-1}(x) + I_{n+1}(x) = 2I_n'(x)$ (96)

and similarly

$xK_n' = (nK_n - xK_{n-1})$ (97)

$xK_n' = nK_n + xK_{n+1}$ (98)

$K_{n-1} + K_{n+1} = -2K_n'$ (99)

Example 4

In this example we will consider the solutions of the equation

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[\left(k_0^2 n^2(r) - \beta^2\right)r^2 - l^2\right]R(r) = 0 \qquad l = 0,\ 1, \ldots \quad (100)$$

Where $n(r) = n_1 \quad 0 < r < a$ (101)

$\quad = n_2 \quad r > a$

and $n_2 < n_{1;}\ k_0(\omega / c)$ represents the free space wave number. The quantity β represents the propagation constant and for guided modes β^2 takes discrete values in the domain

$k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2$ (102)

Thus, in the regions $0 < r < a$ and $r > a$, eq. (100) can be written in the form

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[U^2 \frac{r^2}{a^2} - l^2\right]R(r) = 0 \qquad 0 < r < a. \quad (103)$$

and

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[W^2 \frac{r^2}{a^2} + l^2\right]R(r) = 0 \qquad r > a. \quad (104)$$

where

$U^2 = a^2\left[k_0^2 n_1^2 - \beta^2\right]$ (105)

and

$W^2 = a^2\left[\beta^2 - k_0^2 n_2^2\right]$ (106)

so that

$V^2 = U^2 + W^2 = a^2 k_0^2 (n_1^2 - n_2^2)$ (107)

is a constant. The solutions of Eq. (103) are

$J_l\left(U\frac{r}{a}\right)$ and $Y_l\left(U\frac{r}{a}\right)$ (108)

and the latter solution has to be rejected as it diverges at r = 0. Similarly, the solutions of eq. (104) are

$$K_l\left(W\frac{r}{a}\right) \text{ and } I_l\left(W\frac{r}{a}\right)$$

and the second solution has to be rejected because it diverges as $r \to \infty$. Thus

$$\text{and } R(r) = \begin{cases} \dfrac{A}{J_l(U)} & J_l\left(U\dfrac{r}{a}\right) & 0 < r < a \\ \dfrac{A}{K_l(W)} & K_l\left(W\dfrac{r}{a}\right) & r > a \end{cases} \tag{109}$$

where the constants have been so chosen and R(r) is continuous at r = a. Continuity of dR/dr at r = a gives us

$$U\frac{J_l'(U)}{J_l(U)} = WU\frac{K_l'(U)}{K_l(U)} \tag{110}$$

which is the fundamental equation determining the eigenvalues β / k_0.

SELF-ASSESSMENT EXERCISE

1. Using

   $J_0(2) = 0.22389,\quad J_1(2) = 0.57672,\quad \textit{calculate } J_2(2),\quad J_3(2),\quad \textit{and } J_4(2).$

   Hint: Use Eq. (21)

2. Show that

   $$\int_0^a J_n^2(x)xdx = \frac{1}{2}a^2 J_n^2(a)\left[1 - \frac{J_{n-1}(a)J_{n+1}(a)}{J_n^2(a)}\right]$$

4.0 CONCLUSION

In this unit, we have considered Bessel function and spherical Bessel function.

We have also established in this unit, relationship between the recurrence relation and the generating function.

5.0 SUMMARY

This unit is on Bessel functions. It has a lot of application that arises in numerous diverse areas of applied mathematics. This unit will be of significant importance in the subsequent course in quantum mechanics.

6.0 TUTOR- MARKED ASSIGNMENT

1. Using

$$J_1(2) = 0.57672,\ J_2(2) = 0.35283 \quad calculate\ J_3(2),\quad J_4(2),\quad and\ \ J_5(2).$$

Hint: Use Eq. (21)

2. Using the integral

$$\int_0^1 (1-x^2)^m x^{2n+2r+1} dx = \frac{\Gamma(n+r+1)\Gamma(m+1)}{2\Gamma(m+n+r+2)}; \quad m > -1,\ n > -1$$

Prove that

$$J_{n+m+1}(x) = \frac{2}{\Gamma(m+1)}\left(\frac{x}{2}\right)^{m+1} \int_0^1 (1-y^2)^m y^{n+1} J_n(xy) dy$$

3. *Hint:* Use the expansion given by eq. (2) and integrate term by term.

In problem 2 assume $m = n = -\frac{1}{2}$, and use eq. (12) to deduce

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xy}{\sqrt{1-y^2}} dy$$

4. Show that the solution of the differential equation

$$y''(x) + (ae^x - b)y(x) = 0$$

is given by $y(x) = AJ_\mu(\xi) + BJ_\mu(\xi); \qquad \xi = 2\sqrt{a}e^{x/2}; \ \ \mu = 2\sqrt{b}$

7.0 REFERENCES/FURTHER READING

Erwin, Kreyszig (1991). *Advanced Engineering Mathematics*. John Wiley & Sons, Inc.

Arfken, G. (1990). *Mathematical Methods for Physicists*. New York: Academic Press

MODULE 3 APPLICATION OF FOURIER TO PDES (HERMITE POLYNOMIALS AND LAGUERRE POLYNOMIALS)

UNIT 1 HERMITE POLYNOMIALS

CONTENTS

1.0 INTRODUCTION

In this unit, we shall consider certain boundary value problems whose solutions form orthogonal set of functions. It can also be seen in this unit how the generating function can readily be used to derive the Rodrigues' formula.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define Hermite polynomials as the polynomial solutions of the Hermite differential equation
- prove the Orthogonality of Hermite polynomials
- derive the Rodrigues' formula which can be used to obtain explicit expressions for Hermite polynomials
- solve the exercises at the end of this unit.

3.0 MAIN CONTENT

3.1 Hermite Differential Equation

The equation

$$y''(x) - 2xy'(x) + (\lambda - 1)y(x) = 0 \qquad (1)$$

where λ is a constant is known as the ***Hermite*** differential equation. When λ is an odd integer, i.e. when

$$\lambda = 2n + 1; \qquad n = 0,\ 1,\ \ 2,\ .\ .\ . \qquad (2)$$

One of the solutions of eq. (1) becomes a polynomial. These polynomial solutions are called ***Hermite polynomials.*** Hermite polynomials appear in many diverse areas, the most important being the harmonic oscillator problem in quantum mechanics.

Using Frobenius method to solve eq.(1), and following the various steps, we have

Step1: We substitute the power series

$$y(x) = \sum_{r=0}^{\infty} C_r x^{p+r} \qquad (3)$$

in eq. (1) and obtain the identity

$$C_0 p(p-1) + C_1(p+1)px + \sum_{r=2}^{\infty}[C_r(p+r)(p+r-1) - C_{r-2}(2p+2r-3-\lambda)]x^r = 0$$

Step 2: Equating to zero the coefficients of various powers of x, we obtain

(i) $p = 0$ or $p = 1$ (4a)

(ii) $p(p+1)C_1 = 0$ (4b)

(iii) $C_r = \dfrac{2p+2r-3-\lambda}{(p+r)(p+r-1)} C_{r-2} \quad for\ r \geq 2$ (4c)

When $p = 0$, C_1 becomes indeterminate; hence p = 0 will yield both the linearly independent solutions of eq. (1). Thus, we get

$$C_r = \frac{2r-3-\lambda}{r(r-1)} C_{r-2} \qquad for\ r \geq 2 \qquad (5)$$

which gives

$$C_2 = \frac{1-\lambda}{2!} C_0$$

$$C_3 = \frac{3-\lambda}{3!} C_1,\ .\ .$$

$$C_4 = \frac{(1-\lambda)(5-\lambda)}{4!} C_0$$
$$C_5 = \frac{(3-\lambda)(7-\lambda)}{5!} C_1, \ . \ . \ . \text{etc}$$

Because $C_2, C_4, . \ . \ .$ are related to C_0 and $C_3, C_5, . \ . \ .$ are related to C_1, we can split the solution into even and odd series. Thus, we may write

$$\begin{aligned} y(x) &= (C_0 + C_2 x^2 + C_4 x^4 + . \ . \ .) + (C_1 x + C_3 x^3 + + . \ . \ .) \\ &= C_0 \left[1 + \frac{1-\lambda}{2!} x^2 + \frac{(1-\lambda)(1-\lambda)}{4!} x^4 + . \ . \ . \right] \\ &+ C_1 \left[x + \frac{(3-\lambda)}{3!} x^3 + \frac{(3-\lambda)(7-\lambda)}{5!} x^5 + . \ . \ . \right] \qquad (6) \end{aligned}$$

It may be readily seen that when

$$\lambda = 1, \ 5, \ 9, \ . \ . \ .$$

the even series becomes a polynomial and the odd series remains an infinite series. Similarly, for

$$\lambda = 3, \ 7, \ 11, \ . \ . \ .$$

the odd series becomes a polynomial and the even series remains an infinite series. Thus, when

$$\lambda = 2n + 1; \qquad n = 0, \ 1, \ 2, \ . \ . \ .$$

One of the solutions becomes a polynomial. If the multiplication constant C_0 or C_1 is chosen that the coefficient of the highest power of x in the polynomial becomes 2^n, then these polynomials are known as Hermite polynomials of order n and are denoted by $H_n(x)$. For example, for $\lambda = 9 \ (n = 4)$, the polynomial solution

$$y(x) = C_0 \left[1 - 4x + \frac{4}{3} x^4 \right]$$

If we choose

$$C_0 = 12$$

the coefficient of x^4 becomes 2^4 and, therefore

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Similarly,

for $\lambda = 7 \ (n = 3)$, the polynomial solution is given by

$$y(x) = C_1 \left[x - \frac{2}{3} x^3 \right]$$

Choosing

$$C_1 = -12$$

we get

$$H_3(x) = 8x^3 - 12x$$

In general

$$H_n(x) = \sum_{r=0}^{N} \frac{n!(2x)^{n-2r}}{r!(n-2r)!} \tag{7}$$

where

$$N = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}$$

Using eq. (7) one can obtain Hermite polynomials of various orders, the first few are given below:

$$\left.\begin{array}{l} H_0(x) = 1; \quad H_1(x) = 2x; \quad H_2(x) = 4x^2 - 2; \\ H_3(x) = 8x^3 - 12x; \quad H_4(x) = 16x^4 - 48x^2 + 12 \end{array}\right\} \tag{8}$$

Higher order Hermite polynomials can easily be obtained either by using eq. (7) or by using the recurrence relation (see eq. 20)

3.2 The Generating Function

The generating function for Hermite polynomials is given by

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n \tag{9}$$

Expanding e^{-t^2} and e^{2xt} in power series, we have

$$e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \ldots$$

$$e^{2xt} = 1 + (2x)t + \frac{(2x)^2}{2!}t^2 + \frac{(2x)^3}{3!}t^3 + \ldots$$

Multiplying the above two series, we shall obtain a power series in t with

Coefficient of $t^0 = 1 \qquad = \frac{1}{0!}H_0(x)$

" " $t = 2 \qquad = \frac{1}{1!}H_1(x)$

" " $t^2 = 2x^2 - 1 \qquad = \frac{1}{2!}H_2(x)$ etc

It is also evident that the coefficient of t^2 in the multiplication of the two series will be a polynomial of degree n and will contain odd powers when n is odd and even powers when n is even. In this polynomial, the coefficient of x^n can easily be seen to be $(2^n / n!)$. We then assume that

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty}\frac{1}{n!}K_n(x)t^n \qquad (10)$$

Where $K_n(x)$ is a polynomial of degree n. Differentiating eq. (10) with respect to t, we get

$$(2x-2t)e^{-t^2+2xt} = \sum_{n=0}^{\infty}\frac{n}{n!}K_n(x)t^{n-1} = \sum_{n=0}^{\infty}\frac{1}{(n-1)!}K_n(x)t^{n-1}$$

or

$$2(x-t)\sum_{n=0}^{\infty}\frac{1}{n!}K_n(x)t^n = \sum_{n=0}^{\infty}\frac{1}{n!}K_{n+1}(x)t^n \qquad (11)$$

Comparing the coefficients of t^n on both sides of eq. (11), we obtain

$$2xK_n(x) - 2nK_{n-1}(x) = K_{n+1}(x) \qquad (12)$$

We next differentiate eq.(10) with respect to x to obtain

$$2t\sum_{n=0}^{\infty}\frac{1}{n!}K_n(x)t^n = \sum_{n=0}^{\infty}\frac{1}{n!}K_n'(x)t^n \qquad (13)$$

Comparing the coefficients of t^n on both sides of eq. (11), we get

$$K_n'(x) = 2nK_{n-1}(x) \qquad (14)$$

If we replace n by (n+1) in eq.(14), we would get

$$K_{n+1}'(x) = 2(n+1)K_n(x) \qquad (15)$$

Differentiating eqs.(14) and (12) with respect to x, we obtain respectively

$$K_n''(x) = 2nK_{n-1}'(x) \qquad (16)$$

and

$$2xK'_n(x) + 2K_n(x) - 2nK'_{n-1}(x) = K'_{n+1}(x) \qquad (17)$$

Subtracting eqs. (17) and (16) and using (15), we get

$$K''_n(x) - 2xK'_n(x) + 2nK_n(x) = 0 \qquad (18)$$

which shows that $K_n(x)$ is a solution of the Hermite equation (1) with $\lambda = 2n+1$, i.e. of the equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0 \qquad (19)$$

Since, as discussed before, $K_n(x)$ is also a polynomial of degree n (with coefficient of x^n equal to 2^n), $K_n(x)$ is, therefore, nothing but $H_n(x)$. Equations (12) and (14), thus, give recurrence relations for $H_n(x)$

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \qquad (20)$$

and

$$H'_n(x) = 2nH_{n-1}(x) \qquad (21)$$

3.3 Rodrigues Formula

In the preceding section we have shown that

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n \qquad (22)$$

One can rewrite the generating function G(x, t) in the form

$$G(x,t) = e^{x^2} e^{-(t+x)^2}$$

It may be easily seen that

$$\frac{\partial^n G}{\partial t^n} = e^{x^2} (-1)^n \frac{\partial^n}{\partial x^n} e^{-(t-x)^2} \qquad (23)$$

From eq. (22) it follows that

$$\left. \frac{\partial^n G}{\partial t^n} \right|_{t=0} = H_n(x) \qquad (24)$$

Using eqs (23) and (24), we obtain

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (25)$$

which is known as **Rodrigues formula** for Hermite polynomials. For example,

$$\begin{aligned} H_2(x) &= e^{x^2} \frac{d^2}{dx^2} e^{-x^2} = e^{x^2} \frac{d}{dx}(-2xe^{-x^2}) \\ &= e^{x^2} \left[-2e^{-x^2} + 4x^2 e^{-x^2} \right] \\ &= 4x^2 - 2 \end{aligned}$$

Which is consistent with eq. (8). Similarly, we can determine other Hermite polynomials by elementary differentiation of eq. (25).

3.4 Orthogonality of Hermite Polynomials

The Hermite polynomials satisfy eq.(1) for $\lambda = 2n+1$. Thus, we have

$$\frac{d^2 H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n(x) = 0 \qquad (26)$$

In order to derive the Orthogonality condition we transform eq. (26) to the Sturm-Liouville form by multiplying it by

$$\exp\left[-\int 2x dx\right] = e^{-x^2} \qquad (27)$$

to obtain

$$\frac{d}{dx}\left[e^{-x^2}\frac{dH_n}{dx}\right] = -2n\left[e^{-x^2}H_n(x)\right] \qquad (28)$$

Similarly

$$\frac{d}{dx}\left[e^{-x^2}\frac{dH_m}{dx}\right] = -2m\left[e^{-x^2}H_m(x)\right] \qquad (29)$$

We multiply eq.(28) by $H_m(x)$ and eq.(29) by $H_n(x)$, subtract them and integrate the resulting equation with respect to x from $-\infty$ to ∞ to obtain

$$\int_{-\infty}^{+\infty}\left\{H_m(x)\frac{d}{dx}\left[e^{-x^2}\frac{dH_n}{dx}\right] - H_n(x)\frac{d}{dx}\left[e^{-x^2}\frac{dH_m}{dx}\right]\right\}dx$$

$$= 2(m-n)\int_{-\infty}^{+\infty} e^{-x^2}H_m(x)H_n(x)dx$$

Now

$$\text{LHS} = \int_{-\infty}^{+\infty}\frac{d}{dx}\left\{H_m(x)\left[e^{-x^2}\frac{dH_n}{dx}\right] - H_n(x)\left[e^{-x^2}\frac{dH_m}{dx}\right]\right\}dx$$

$$= \left[H_m(x)e^{-x^2}\frac{dH_n}{dx} - H_n(x)e^{-x^2}\frac{dH_m}{dx}\right]_{-\infty}^{+\infty}$$

$$= 0$$

Thus

$$\int_{-\infty}^{+\infty} e^{-x^2}H_m(x)H_n(x)dx = 0; \qquad m \neq n \qquad (30)$$

which shows that the Hermite polynomials are Orthogonal with respect to the weight function e^{-x^2}. Thus if we define the functions

$$\phi_n(x) = N_n e^{-x^2/2}H_n(x); \qquad n = 0,\ 1,\ 2, \ldots \qquad (31)$$

then eq. (30) assumes the form

$$\int_{-\infty}^{+\infty}\phi_m(x)\phi_n(x)dx = 0; \qquad m \neq n \qquad (32)$$

3.5 The Integral Representation of the Hermite Polynomials

The integral representation of the Hermite polynomial is given by

$$H_n(x) = \frac{2^n(-i)^n}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} t^n e^{-t^2+2ixt} dt \qquad (33)$$

In order to prove the above relation we start with the relation

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} e^{-t^2+2ixt} dt$$

which can easily be obtained from the well known formula

$$\int_{-\infty}^{+\infty} e^{-\alpha t^2+\beta t} dt = \sqrt{\frac{\pi}{\alpha}} \exp\left[\frac{\beta^2}{4\alpha}\right]$$

by assuming $\alpha = 1$and $\beta = 2ix$. Now according to the Rodrigues formula

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \\ &= (-1)^n e^{x^2} \frac{1}{\sqrt{\pi}} \frac{d^n}{dx^n} \int_{-\infty}^{+\infty} e^{-t^2+2ixt} dt \\ &= (-1)^n \frac{1}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} (2i)^n t^n e^{-t^2+2ixt} dt \end{aligned}$$

from which eq. (50) readily follows.

3.6 Fourier Transform of Hermite-Gauss Functions

In this section we will show that

$$e^{-x^2/2} H_n(x) = \frac{1}{i^n\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[e^{k^2/2} H_n(k)\right] e^{ikx} dk \qquad (34)$$

Implying that the Fourier transform of the Hermite-Gauss function is a Hermite-Guass function. In order to prove eq. (34) we start with the generating function

$$G(x,t) = e^{2kt-t^2} = \sum_{n=0,1,\ldots}^{\infty} \frac{1}{n!} H_n(k) t^n$$

We multiply the above by $\left(ikx - \frac{1}{2}k^2\right)$ and integrate over k to obtain

$$e^{-t^2} \int_{-\infty}^{+\infty} e\left[-\frac{1}{2}k^2 + (2t+ix)k\right] dk = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{+\infty} H_n(x) e^{-k^2/2} e^{ikx} dk \qquad (35)$$

Now

$$\text{LHS} = e^{-t^2}\sqrt{2\pi}\exp\left[\frac{(2t+ix)^2}{2}\right]$$
$$= \sqrt{2\pi}e^{t^2+2ixt}e^{-x^2/2}$$
$$= \sqrt{2\pi}e^{-x^2/2}\sum_n \frac{H_n(x)}{n!}(it)^n$$

Comparing coefficients of t^n on both sides of eq. (35), we get eq. (34).

3.7 Some Important Formulae Involving Hermite Polynomials

$$H_n(x+y) = 2^{-n/2}\sum_p^n \frac{n!}{p!(n-p)!}H_{n-p}(x\sqrt{2})H_p(y\sqrt{2}) \qquad (36)$$

$$H_n(x) \underset{n\to\infty}{\rightarrow} \sqrt{2}\left(\frac{2n}{e}\right)^{n/2} e^{x^2/2}\cos\left(\sqrt{(2n+1)}x - \frac{n\pi}{2}\right) \qquad (37)$$

$$\left.\begin{aligned} x^{2s} &= \frac{(2s)!}{2^{2s}}\sum_{n=0,1,\ldots}^{s}\frac{H_{2s-2n}(x)}{n!(2s-2n)!} \\ x^{2s+1} &= \frac{(2s+1)!}{2^{2s+1}}\sum_{n=0,1,\ldots}^{s}\frac{H_{2s+1-2n}(x)}{n!(2s+1-2n)!}\end{aligned}\right\} \quad s = 0,\ 1,\ 2,\ldots \qquad (38)$$

SELF-ASSESSMENT EXERCISE

Using the generating function for $H_n(x)$, show that

(a) $\frac{1}{e}\cosh 2x = \sum_{n=0,1,\ldots}^{\infty}\frac{1}{(2n)!}H_{2n}(x)$

(b) $\frac{1}{e}\sinh 2x = \sum_{n=0,1,\ldots}^{\infty}\frac{1}{(2n+1)!}H_{2n+1}(x)$

(c) $e\cos 2x = \sum_{n=0,1,\ldots}^{\infty}(-1)^n\frac{1}{(2n)!}H_{2n}(x)$

(d) $e\sin 2x = \sum_{n=0,1,\ldots}^{\infty}(-1)^n\frac{1}{(2n+1)!}H_{2n+1}(x)$

Hint: To obtain (a) and (b) substitute t = 1 and t = -1in eq. (9) add and subtract the resulting equations. Similarly for (c) and (d), substitute t = I and equate real and imaginary parts.

Prove that

$$\int_{-\infty}^{+\infty}e^{-x^2}H_{2n}(\alpha,x)dx = \sqrt{n}\,\frac{(2n)!}{n!}(\alpha^2-1)^n$$

Hint: Replace x by αy in eq.(9), multiply the resulting equation by e^{-y^2} and integrate with respect to y.

4.0 CONCLUSION

Here, in this unit, we have dealt with the Hermite polynomials which are Orthogonal with respect to the weight function e^{-x^2} . We have also established that the Fourier transform of the Hermite-Gauss function is a Hermite-Guass function.

5.0 SUMMARY

This unit was on the Hermite polynomials. It has a lot of application in linear harmonic oscillator problem in quantum mechanics. The unit will be of immense importance in the subsequent course in classical mechanics.

6.0 TUTOR- MARKED ASSIGNMENT

1. If two operators are defined as

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx} \right)$$

$$\bar{a} = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx} \right)$$

Show that

$$a\phi_n(x) = \sqrt{n}\phi_{n-1}(x)$$

$$\bar{a}\,\phi_n(x) = \sqrt{n}\phi_{n-1}(x)$$

2. Prove that

$$\int_{-\infty}^{+\infty} H_n\left(x + \frac{1}{2}x_0 \right) e^{-x^2/2} dx = \sqrt{n}\, x_0^n$$

Hint: Multiply Eq. (9) by $\left[-\left(x + \frac{1}{2}x_0 \right)^2 \right]$ and integrate over x.

7.0 REFERENCES/FURTHER READING

Erwin, K. (1991). *Advanced Engineering Mathematics.* John Wiley & Sons, Inc.

Arfken, G. (1990). *Mathematical Methods for Physicists.* New York: Academic Press.

UNIT 2 LAGUERRE POLYNOMIALS

CONTENTS

1.0 INTRODUCTION

In the previous unit, you came across solutions of orthogonal set of functions. This unit which is the last one in this book will examine critically how a Laguerre differential equation can be transformed to Sturm-Liouville form.

It shows that Laguerre polynomials and the associated functions arise in many branches of physics, e.g. in the hydrogen atom problem in quantum mechanics, in optical fibers characterised by parabolic variation of refractive index, etc.

We also show that Laguerre polynomials are orthogonal in the interval $0 \le x \le \infty$ with respect to the weight function e^{-x}.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use Frobenius method to obtain the polynomial solution of the Laguerre differential equations
- determine the Orthogonality of the Laguerre polynomials
- derive the Rodrigues formula
- derive the second solution of the Laguerre differential equation.

3.0 MAIN CONTENT

3.1 Laguerre Differential Equation

The equation

$$xy''(x) - (1-x)y'(x) + ny(x) = 0 \qquad (1)$$

where n is a constant known as the ***Laguerre*** differential equation. When $n = 0,\ 1,\ 2, \ldots$ (2)

One of the solutions of eq. (1) becomes a polynomial. These polynomial solutions are known as the ***Laguerre polynomials.***

Using Frobenius method to solve eq.(1), and following the various steps, we have

Step We substitute the power series

$$y(x) = \sum_{r=0}^{\infty} C_r x^{p+r}, \qquad C_0 \neq 0$$

Eq. (1) and obtain the identity

$$\sum_{r=0}^{\infty} C_r (p+r)^2 x^{p+r-1} - \sum_{r=0}^{\infty} C_r (p+r-n) x^{p+r} = 0$$

or

$$C_0 p^2 x^{p-1} - \sum_{r=1}^{\infty} \left[C_r (p+r)^2 - C_{r-1}(p+r-n-1) \right] x^{p+r-1} = 0 \qquad (3)$$

Step 2 Equating to zero the coefficients of various powers of x in the identity (3), we obtain

(i) $p^2 = 0$ INDICIAL EQUATION (4)

(ii) $C_r = \dfrac{p+r-n-1}{(p+r)^2} C_{r-1}$ $r \geq 1$ RECURRENCE RELATION (5)

Substituting $p = 0$ in eq. (5), we get

$$C_r = \frac{r-n-1}{r^2} C_{r-1} \qquad r \geq 1$$

which gives

$$C_1 = -\frac{n}{(1!)^2} C_0 \qquad C_2 = \frac{n(n-1)}{(2!)^2} C_0$$

$$C_3 = \frac{n(n-1)(n-2)}{(3!)^2} C_0 \qquad \text{etc}$$

$$C_n = (-1)^n \frac{n!}{(n!)^2} = \frac{(-1)^n}{n!}$$

and

$$C_{n+1} = C_{n+2} = 0. \ldots = 0$$

Therefore one of the solutions of eq. (1) can be written as

$$y(x) = C_0 \left\{ 1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 - \ldots + (-1)^n \frac{x^n}{n!} \right\} \tag{6}$$

which is a polynomial of degree n. If the multiplication constant C_0 is chosen to be unity so that the constant term becomes unity, the polynomial solution given by eq. (6) is known as **Laguerre Polynomial** of degree n and denoted by $L_n(x)$. Thus

$$L_n(x) = 1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 - \ldots + (-1)^n \frac{x^n}{n!}$$

or

$$L_n(x) = \sum_{r=0}^{n} (-1)^n \frac{n!}{(n-r)!(r!)^2} x^r \tag{7}$$

with

$$L_n(0) = 1 \tag{8}$$

The first four Laguerre polynomials can be written as:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 1 - x, \\ L_2(x) &= 1 - 2x + \frac{1}{2} x^2, \\ L_3(x) &= 1 - 3x + \frac{3}{2} x^2 - \frac{1}{6} x^3, \ldots \end{aligned} \tag{9}$$

Higher order polynomials can easily be obtained either by using eq.(7) or by using the recurrence relation [see eq. (20)].

3.2 The Generating Function

The generating function for Laguerre polynomials is given by

$$G(x,t) = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n(x) t^n; \quad |t| < 1 \tag{10}$$

We expand the left hand side of eq. (10) to obtain

$$(1-t)^{-1} \exp\left[-\frac{xt}{1-t} \right]$$

$$= (1-t)^{-1} - xt(1-t)^{-2} + \frac{x^2 t^2 (1-t)^3}{2!} - \ldots$$

$$= (1 + t + t^2 + \ldots) - xt(1 + 2t + 3t^2 + \ldots) + \frac{x^2 t^2}{2!}(1 + 3t + 6t^2 + \ldots) - \ldots \tag{11}$$

The right hand side of eq.(11) can be written as a power series in t with
Coefficient of $t^0 = 1 \qquad = L_0(x)$
“ “ $t = 1 - x \qquad = L_1(x)$
“ “ $t^2 = 1 - 2x + x^2/2 \qquad = L_2(x)$

etc. It is also evident that the coefficient of t^2 on the right hand side of eq.(11) will be a polynomial of degree n and that the constant term in this polynomial will be unity. We can then assume that

$$G(x,t) = \frac{1}{1-t}\exp\left\{-\frac{xt}{(1-t)}\right\} = \sum_{n=0}^{\infty} K_n(x)t^n \tag{12}$$

where $K_n(x)$ is a polynomial of degree n. Differentiating eq. (12) with respect to t, we get

$$\frac{(1-x-t)}{(1-t)^3}\exp\left\{-\frac{xt}{(1-t)}\right\} = \sum_{n=0}^{\infty} nK_n(x)t^{n-1}$$

or

$$(1-x-t)\sum_{n=0}^{\infty} K_n(x)t^n = (1-2t+t^2)\sum_{n=1}^{\infty} nK_n(x)t^{n-1}$$

Comparing the coefficients of t^n on both sides of the above equation, we get

$$(n+1)K_{n+1}(x) - (2n+1-x)K_n(x) + nK_{n-1}(x) = 0; \qquad n \geq 1 \tag{13}$$

We next differentiate eq.(12) with respect to x to obtain

$$-t\sum_{n=0}^{\infty} K_n(x)t^n = (1-t)\sum_{n=0}^{\infty} K_n'(x)t^n \tag{14}$$

Comparing the coefficients of t^n on both sides of the above equation, we get

$$K_n'(x) - K_{n-1}'(x) = -K_{n-1}(x) \tag{15}$$

If we replace n by (n+1) in the above equation, we would get

$$K_{n+1}'(x) = K_n'(x) - K_n(x) \tag{16}$$

Differentiating eq. (13) with respect to x, we obtain

$$(n+1)K_{n+1}'(x) - (2n+1-x)K_n'(x) + K_n(x) + nK_{n-1}'(x) = 0 \tag{17}$$

Substituting $K_{n-1}'(x)$ and $K_{n+1}'(x)$ from eqs. (15) and (16) respectively in eq. (17), we get

$$xK_n'(x) = nK_n(x) - nK_{n-1}(x) \tag{18}$$

Differentiating the above equation with respect to x and using eq. (15), we have

$$xK_n''(x) + K_n'(x) = -nK_{n-1}(x) \qquad (19)$$

Subtracting eq. (18) from eq. (19), we get

$$xK_n''(x) + (1-x)K_n'(x) + nK_n(x) = 0$$

Showing that $K_n(x)$ is a solution of the Laguerre equation, i.e. of the equation

$$xy''(x) - (1-x)y'(x) + ny(x) = 0$$

Hence $K_n(x)$ nothing but $L_n(x)$. Equations (13) and (18) give the following recurrence relations respectively:

$$(n+1)L_{n+1}(x) = (2n+1-1)L_n(x) - nL_{n-1}(x) \qquad (20)$$

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x) \qquad (21)$$

We also have

$$L_n(x) = L_n'(x) - L_{n+1}'(x) \qquad (22)$$

3.3 Rodrigues Formula

In the preceding section we have shown that

$$G(x,t) = \frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n$$

We can write the above equation as

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{1}{1-t}\exp\left[-\frac{x(1-t-1)}{1-t}\right]$$

or

$$\sum_{n=0}^{\infty} L_n(x)t^n = e^x\left[\frac{1}{1-t}\exp\left(-\frac{x}{1-t}\right)\right] \qquad (23)$$

Differentiating eq. (23) n times with respect to t and then putting t = 0, we will have

$$n!L_n(x) = e^x\left[\frac{\partial^n}{\partial t^n}\left[\frac{1}{1-t}\exp\left(-\frac{x}{1-t}\right)\right]\right]\Bigg|_{t=0}$$

$$= e^x\left[\frac{\partial^n}{\partial t^n}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{(1-t)^{r+1} r!}\right\}\right]\Bigg|_{t=0}$$

$$= e^x \left[\sum_{r=0}^{\infty} (-1)^r \frac{(r+1)(r+2)\ldots(r+n)}{(1-t)^{r+n+1} r!} \right]\Bigg|_{t=0}$$

$$= e^x \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)!}{(r!)^2} x^r$$

or

$$L_n(x) = \frac{e^x}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)!}{(r!)^2} x^2 \qquad (24)$$

$$\frac{d^n}{dx^n}(x^n e^{-x}) = \frac{d^2}{dx^2}\left[x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^r}{r!} \right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)(n+r-1)\ldots(r+1)}{r!} x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)!}{(r!)^2} x^r$$

Thus

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) \qquad (25)$$

This is known as **Rodrigues formula** for the Laguerre polynomials. For example, putting n = 2 in the Rodrigues' formula, we have

$$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2}(x^2 e^{-x})$$

$$= \frac{e^x}{2!} \frac{d}{dx}(2x^2 e^{-x} - x^2 e^{-x})$$

$$= \frac{e^x}{2!} \frac{d}{dx}(2e^{-x} - 4xe^{-x} + x^2 e^{-x})$$

$$= 1 - 2x + \frac{x^2}{2}$$

Which is consistent with eq. (9). Similarly, we can determine other Laguerre polynomials by elementary differentiation of the result expressed by eq. (25).

3.4 Orthogonality of Hermite Polynomials

As Laguerre differential equation is not of the form of Sturm-Liouville differential equation, its solutions $L_n(x)$, therefore, do not by themselves form an Orthogonal set. However, in order to transform Laguerre differential equation to the Sturm-Liouville form, we may write eq. (1) as

$$y''(x) - \frac{(1-x)}{x} y'(x) + \frac{n}{x} y(x) = 0$$

Multiplying the above equation by

$$p(x) = \exp\left[\int \frac{1-x}{x}\right] = xe^{-x^2} \qquad (26)$$

We obtain

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + ne^{-x}y(x) = 0 \qquad (27)$$

Thus for Laguerre polynomials, the Sturm-Liouville form is given by

$$\frac{d}{dx}\left[p(x)\frac{dL_n(x)}{dx}\right] = -ne^{-x}L_n(x) \qquad (28)$$

Similarly

$$\frac{d}{dx}\left[p(x)\frac{dL_m(x)}{dx}\right] = -me^{-x}L_m(x) \qquad (29)$$

Multiply eq.(28) by $L_m(x)$ and eq.(29) by $L_n(x)$ and subtracting the resulting equations, we obtain

$$L_m(x)\frac{d}{dx}\left[p(x)\frac{dL_n(x)}{dx}\right] - L_m(x)\frac{d}{dx}\left[p(x)\frac{dL_m(x)}{dx}\right]$$
$$= (m-n)L_m(x)L_n(x) \qquad (30)$$

The left hand side of eq.(30) is simply

$$\frac{d}{dx}\left[L_m(x)p(x)\frac{dL_n(x)}{dx} - L_m(x)p(x)\frac{dL_m(x)}{dx}\right] \qquad (31)$$

Integrating eq.(30) and using eq.(31), we get

$$(m-n)\int_0^\infty e^{-x}L_m(x)L_n(x)dx = \left[p(x)\left\{L_m(x)\frac{dL_n(x)}{dx} - L_m(x)\frac{dL_m(x)}{dx}\right\}\right]_0^\infty$$

Since p(x) = 0 at x = 0 and at $x = \infty$, the right hand side vanishes and we readily obtain

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = 0 \qquad for\ m \neq n \qquad (32)$$

The above equation shows that the Laguerre polynomials are Orthogonal in the interval $0 \leq x \leq \infty$ with respect to the weight function e^{-x}. We now define the functions

$$\phi_n(x) = N_n L_n(x) e^{-x^2/2} \qquad (33)$$

The constant N_n is chosen so that the functions $\phi_n(x)$ are normalised, i.e.

$$\int_0^{+\infty} \phi_n^2(x)dx = 1 \qquad for \quad m = n \qquad (32)$$

3.5 The Integral Representation of the Laguerre Polynomials

The integral representation of the Laguerre polynomial is given by

$$L_n(x) = \frac{e^x}{n!}\int_0^{+\infty} e^{-t} t^n dt\, J_0\left[2(xt)^{1/2}\right]dt \qquad (33)$$

In order to prove the above relation we start with the relation

$$\int_0^{+\infty} e^{-t} t^n dt\, J_0\left[2(xt)^{1/2}\right]dt$$

$$= \int_0^{+\infty} e^{-t} t^n \sum_{r=0}^{\infty} \frac{(-1)^r (tx)^r}{(r!)^2} dt$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (x)^r}{(r!)^2} \int_0^{+\infty} e^{-t} t^{n+r} dt$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^r \Gamma(n+r+1)}{(r!)^2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)!}{(r!)^2} x^r \qquad (34)$$

Using eqs. (24) and (33), we get

$$\int_0^{+\infty} e^{-t} t^n dt\, J_0\left[2(xt)^{1/2}\right]dt = e^{-x} n! L_n(x) \qquad (35)$$

from which eq. (33) readily follows.

3.6 Some Important Results Involving Laguerre Polynomials

We give some important results involving Laguerre polynomials which can be readily derived:

$$\int_0^x L_n(x)dx = L_n(x) - L_{n+1}(x) \qquad \text{[Use Eq. (22)]} \qquad (36)$$

$$\sum_{n=0}^{\infty} \frac{y^n L_n(x)}{n!} = e^y J_0\left[2(xy)^{1/2}\right] \qquad (37)$$

$$\int_0^{+\infty} x^m e^{-x} L_n(x) dx = \begin{cases} 0 & if\ m < n \\ (-1)^n n! & if\ m = n \end{cases} \qquad (38)$$

$$\sum_{n=0}^{N} L_n(x) L_n(y) = \frac{(N+1)}{x-y}\left[L_N(x)L_{N+1}(y) - L_{N+1}(x)L_N(y)\right]_0 \qquad (39)$$

from which eq. (50) readily follows.

3.7 The Second Solution of the Laguerre Differential Equation

Since the indicial equation [eq. (4)] has two equal roots, the two independent solutions of eq. (1)

$$(y)_{p=0} \quad and \quad \left(\frac{\partial y}{\partial p}\right)_{p=0}$$

Now

$$y(x,p) = x^p\left\{1 + \frac{p-n}{(p+1)^2}x + \frac{(p-n)(p-n+1)}{(p+1)^2(p+2)^2}x^2 + \right.$$
$$\left. \frac{(p-n)(p-n+1)(p-n+2)}{(p+1)^2(p+2)^2(p+3)^3}x^3 + \ldots\right\} \qquad (40)$$

Thus,

$$y_1(x) = y(x,\ p=0)$$
$$= 1 - nx + \frac{n(n-1)}{(2!)^2}x^2 - \frac{n(n-1)(n-2)}{(3!)^2}x^3 \ .\ . \qquad (41)$$

and

$$y_2(x) = \left.\frac{\partial y}{\partial p}\right| = \left[x^p \ln x\left\{1 + \frac{p-n}{(p+1)^2}x + \frac{(p-n)(p-n+1)}{(p+1)^2(p+2)^2}x^2 + \ldots\right\}\right.$$

$$+ x^p\left\{\left(\frac{1}{p-n} - \frac{2}{p+1}\right)\frac{p-n}{(p+1)^2}x + \left(\frac{1}{p-n} - \frac{2}{p-n+1} - \frac{2}{p+1} - \frac{2}{p+2}\right)\right.$$
$$\left.\left.\times\left\{\frac{(p-n)(p-n+1)}{(p+1)^2(p+2)^2}x^2 + \ldots\right\}\right]_{p=0}$$

$$= y_1(x)\ln x + \left\{(2n+1)x - \frac{3n^2 - n - 1}{(2!)^2}x^2 + \ldots\right\} \qquad (42)$$

For example, for $n = 0$

$$y_1(x) = 1 = L_0(x)$$

and

$$y_2(x) = \ln x + x + \frac{x^2}{(2!)^2} + \frac{2!x^3}{(3!)^2} + \frac{3!x^4}{(4!)^2} + \ldots \qquad (43)$$

Similarly, for $n = 1$

$$y_1(x) = 1 - x = L_1(x)$$

and

$$y_2(x) = (1-x)\ln x + 3x - \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} - \ldots \qquad (44)$$

3.8 Associated Laguerre Polynomials

Replace n by $(n+k)$ in eq. (1), it is obvious that $L_{n+k}(x)$ will be a solution of the following differential equation.

$$xy'' - (1-x)y' + (n+k)y = 0 \quad (45)$$

Differentiating the above equation k times, it can easily be shown that

$$y = \frac{d^k}{dx^k}\left[L_{n+k}(x)\right] \quad (46)$$

or a constant multiple of it is a solution of the differential equation

$$xy'' - (k+1-x)y' + ny = 0 \quad (47)$$

Where n and k are positive integers or zero. The above equation is known as the **Associated Laguerre Equation.** Its polynomial solutions [see eq.(45)] are denoted by $L_n^k(x)$ and are defined by

$$L_n(x) = (-1)^n \frac{d^k}{dx^k}\left[L_{n+k}(x)\right] \quad (48)$$

This is known as the **Associated Laguerre Polynomials.** It is obvious from eq. (48) that
$L_n^k(x)$ is polynomial of degree n in x and that

$$L_n^0(x) = L_n(x) \quad (49)$$

Using eqs. (7) and (48), it follows that

$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{(n+k)!}{(n-r)!(r+k)!r!} x^r \quad (50)$$

We will define $L_n^k(x)$ for non-integer values of k, we may, therefore, write the above equation as

$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{\Gamma(n+k+1)}{(n-r)!\Gamma(r+k)\ r!} x^r \quad (51)$$

Using the above equation, the first three polynomials can easily be written as:

$$L_0^k(x) = 1$$
$$L_1^k(x) = k+1-x \quad (52)$$
$$L_2^k(x) = \frac{1}{2}(k+2)(k+1) - (k+2)x + \frac{1}{2}x^2$$

Differentiating the Laguerre generating function [eq. (10)] k times with respect to x, one can easily obtain the generating function for the associated Laguerre polynomials. Thus

$$g(x,t) \equiv \frac{1}{(1-t)^{k+1}} \exp\left\{-\frac{xt}{1-t}\right\} \sum_{n=0}^{\infty} L_n^k(x) t^n \quad (53)$$

Furthermore, from eq.(51)

$$L_n^k(0) = \frac{\Gamma(n+k+1)}{n!\,\Gamma(k+1)} \tag{54}$$

SELF-ASSESSMENT EXERCISE

1. Show that

$$x^4 = \Gamma(5+k)\ 4!\sum_{r=0}^{4}\frac{(-1)^r L_r^k(x)}{\Gamma(r+k+1)(4-r)!}$$

2. ***Hint:*** Use eq. (51 Show that

$$L_n(0) = 1$$
$$L_n'(0) = -n$$
$$L_n''(0) = \frac{1}{2}n(n-1)$$

Hint: Use Eq. (7).

4.0 CONCLUSION

In this unit, we have established the relationship between Laguerre and associated Laguerre polynomials. The generating function and some important results involving Laguerre polynomials were also dealt with.

5.0 SUMMARY

This unit deals with Laguerre functions and its applications to physical problems especially in Quantum mechanics.

6.0 TUTOR-MARKED ASSIGNMENT

1. Show that

$$L_n^{\gamma+k+1}(x+y) = \sum_{r=0}^{n} L_r^{\gamma}(x) L_{n-r}^{k}(y), \qquad n = 0, 1, 2, \ldots$$

Hint: Use the generating function.

2. Show that

$$L_n^{1/2}(x) = \frac{(-1)^n}{2^{2n+1}n!}\frac{H_{2n+1}(x^{1/2})}{x^{1/2}}$$
$$L_n^{-1/2}(x) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(x^{1/2})$$

Hint: Use the integral representation of $L_n^k(x)$ and $H_n(x)$.

3. Using eq. (53), prove the identity

$$(1-t)\frac{\partial g}{\partial t}+\left[x-(1-t)(1+k)\right]g=0$$

and then derive the recurrence relation [eq. (56)].

4. Using eq.(53), prove the identity

$$(1-t)\frac{\partial g}{\partial x}+tg(x,t)=0$$

and hence derive the following relation

$$\frac{dL_n^k(x)}{dx}-\frac{dL_{n-1}^k(x)}{dx}+dL_{n-1}^k(x)=0$$

$$n=1,\ 2,.\ .\ .$$

5. Show that

$$\int_0^x L_n(t)dt=L_n(x)-L_{n+1}(x)$$

Hint: Use the relation derived in problem 4.

7.0 REFERENCES/FURTHER READING

Erwin, K. (1991). *Advanced Engineering Mathematics*. John Wiley & Sons, Inc.

Arfken, G. (1990). *Mathematical Methods for Physicists*. New York: Academic Press.

MODULE 1 PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN PHYSICS

UNIT 1 PARTIAL DIFFERENTIAL EQUATIONS

CONTENTS

7.0 INTRODUCTION

In this unit, we shall study some elementary methods of solving partial differential equations which occur frequently in physics and in engineering. In general, the solution of the partial differential equation presents a much more difficult problem than the solution of ordinary differential equations.

We are therefore going to limit ourselves to a few solvable partial differential equations that are of physical interest.

8.0 OBJECTIVES

At the end of this unit, you should be able to:

- define linear second-order partial differential equation in more than one independent variable
- use the technique of separation of variables in solving important second order linear partial differential equations in physics
- solve the exercises at the end of this unit.

9.0 MAIN CONTENT

3.9 Definition

An equation involving one or more partial derivatives of (unknown) functions of two or more independent variables is called a **partial differential equation.** The ***order*** of a PDE is the highest order partial derivative or derivatives which appear in the equation. For example,

$$U\frac{\partial U}{\partial z}\frac{\partial^3 U}{\partial y^3}+\left(\frac{\partial^2 U}{\partial y^2}\frac{\partial^2 U}{\partial z^2}\right)=e^z \qquad (1)$$

is a third order PDE since the highest order term is given by

$$\frac{\partial^3 U}{\partial y^3}$$

A PDE is said to be ***linear*** if it is of the first degree, i.e. not having exponent greater than 1 in the dependent variable or its partial derivatives and does not contain product of such terms in the equation. Partial derivatives with respect to an independent variable are written for brevity as a subscript; thus

$$U_{tt}=\frac{\partial^2 U}{\partial t^2} \quad \text{and} \quad U_{xy}=\frac{\partial^2 U}{\partial x \partial y}$$

The PDE

$$\frac{1}{c^2}U_{tt}=U_{xx}+U_{yy}+U_{zz} \qquad (2)$$

(Where c is a constant) is linear and is of the second order while eq. (1) is an example of a nonlinear PDE.

Example 1: Important linear partial differential equations of second order

(1)
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional wave equation}$$

(2)
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional heat equation}$$

(3)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textit{Two-dimensional Laplace equation}$$

(4)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \textit{Two-dimensional poisson equation}$$

(5)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \textit{Three-dimensional Laplace equation}$$

3.10 Linear Second-Order Partial Differential Equations

Many important PDEs occurring in science and engineering are second order linear PDEs. A general form of a second order linear PDE in two independent variables x and y can be expressed as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (3)$$

where A, B, C……,G may be dependent on variables x and y. If $G = 0$, then eq. (3) is called ***homogeneous;*** otherwise it is said to be a ***non-homogeneous.***

The homogeneous form of Eq. (3) resembles the equation of a general conic:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

We thus say that eq. (3) is of

$$\left.\begin{matrix} \textit{elliptic} \\ \textit{hyperbolic} \\ \textit{parabolic} \end{matrix}\right\} \textit{type} \qquad \textit{when} \begin{cases} B^2 - 4AC < 0 \\ B^2 - 4AC > 0 \\ B^2 - 4AC = 0 \end{cases}$$

For example, according to this classification the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is of elliptic type $(A = C = 1, B = D = E = G = 0)$, and the equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad (\alpha \text{ is a real constant})$$

is of hyperbolic type. Similarly, the equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial y} = 0 \quad (\alpha \text{ is a real constant})$$

is of parabola type.

Some important linear second-order partial differential equations that are of physical interest are listed below.

Example 2

Eliminate A and P from the function $Z = Ae^{pt} \sin px$

Solution Let $\frac{\partial Z}{\partial t} = pAe^{pt} \sin px$ and $\frac{\partial^2 Z}{\partial t^2} = p^2 Ae^{pt} \sin px$

also $\frac{\partial Z}{\partial x} = pAe^{pt} \cos px$ and $\frac{\partial^2 Z}{\partial x^2} = -p^2 Ae^{pt} \sin px$

$$\frac{\partial^2 Z}{\partial t^2} + \frac{\partial^2 Z}{\partial x^2} = 0$$

i.e. $p^2 Ae^{pt} \sin px - p^2 Ae^{pt} \sin px = 0$

Example 3

Solve the equation

$$\frac{\partial^2 u}{\partial x^2} - 7\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} = 0$$

Solution: Let $u(x, y) = f(y + m_1 x) + g(y + m_2 x)$

So that $m^2 - 7m + 6 = 0$

This implies that $m = 1 \ or \ 6$

Hence $u(x, y) = H(y + x) + G(y + 6x)$

3.2.1 Laplace‘s Equation

$$\nabla^2 u = 0 \qquad (4)$$

Where ∇^2 is the Laplacian operator $\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\right)$. The function u may be the electrostatic potential in a charge-free region or gravitational potential in a region containing no matter.

3.2.2 Wave Equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad (5)$$

Where u represents the displacement associated with the wave and v, the velocity of the wave.

3.4.3 Heat Conduction Equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u \qquad (6)$$

Where u is the temperature in a solid at time t. The constant α is called the diffusivity and is related to the thermal conductivity, the specific heat capacity, and the mass density of the object.

3.4.4 Poisson’s Equation

$$\nabla^2 u = \rho(x, y, z) \qquad (7)$$

Where the function $\rho(x, y, z)$ is called the source density. For example, if u represents the electrostatic potential in a region containing charges, then ρ is proportional to the electric charge density.

Example 4

Laplace’s equation arises in almost all branches of analysis. A simple example can be found from the motion of an incompressible fluid. Its velocity $\mathbf{v}(x, y, z, t)$ and the fluid density $\rho(x, y, z, t)$ must satisfy the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0$$

If ρ is constant we then have

$$\nabla \bullet \mathbf{v} = 0$$

If furthermore, the motion is irrotational, the velocity vector can be expressed as the gradient of a scalar function V:

$$\mathbf{v} = -\nabla V$$

and the continuity becomes Laplace's equation:

$$\nabla.\mathbf{v} = \nabla.(-\nabla V) = 0, \quad \text{or} \quad \nabla^2 V = 0$$

The scalar function V is called the velocity potential

3.5 Method of Separation of Variables

The technique of separation of variables is widely used for solving many of the important second order linear PDEs.
The basic approach of this method in attempting to solve a differential equation (say, two independent variables x and y) is to write the dependent variable $u(x, y)$ as a product of functions of the separate variables $u(x,t) = X(x)T(t)$. In many cases the partial differential equation reduces to ordinary equations for X and T.

3.3.1 Application to Wave Equation

Let us consider the vibration of an elastic string governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (8)$$

where u(x, y) is the deflection of the string. Since the string is fixed at the ends x = 0 and $x = l$, we have the two **boundary conditions**

$$u(0,t) = 0, \qquad u(l,t) = 0 \qquad \text{for all t} \qquad (9)$$

The form of the motion of the string will depend on the initial deflection (deflection at t = 0) and on the initial velocity (velocity at t = 0). Denoting the initial deflection by f(x) and the initial velocity by g(x), the two **initial conditions** are

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \qquad (10)$$

This method expresses the solution of $u(x,t)$ as the product of two functions with their variables separated, i.e.

$$U(x,t) = X(x)T(t) \qquad (11)$$

where X and T are functions of x and t respectively.

Substituting eq. (11) in eq. (8), we obtain

$$X\,T'' = c^2\,X''\,T$$

or

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} \qquad (12)$$

In other words

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda \qquad (13)$$

The original PDE is then separated into two ODEs, viz.

$$X''(x) - \lambda X(x) = 0 \qquad (14)$$

and

$$T''(t) - \lambda c^2\,T(t) = 0 \qquad (15)$$

The boundary conditions given by eq. (9) imply

$$X(0)\,T(t) = 0$$

and

$$X(l)\,T(t) = 0$$

Since T(t) is not identically zero, the following conditions are satisfied

$$X(0) = 0 \qquad \text{and} \qquad X(l) = 0 \qquad (16)$$

Thus eq. (14) is to be solved subject to conditions given by eq. (16).

There are 3 cases to be considered.

Case 1 $\lambda > 0$

The solution to eq. (14) yields

$$X(x) = Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x} \qquad (17)$$

To satisfy the boundary condition given by eq. (16), we must have

$$Ae^{-\sqrt{\lambda}\,l} + Be^{\sqrt{\lambda}\,l} = 0$$

Since the determinant formed by the coefficients of A and B is non-zero, the only solution is A = B = 0. This yields the trivial solution X(x) = 0.

Case 2 $\lambda = 0$

The solution to eq. (14) yields

$$X(x) = A + Bx$$

To satisfy the boundary condition given by eq. (16), we must have

$$A = 0$$

and

$$A + Bl = 0$$

implying

$$A = 0, \qquad B = 0$$

Again for this case, a trivial solution is obtained

Case 3 $\lambda < 0$

Let $\lambda = -k^2$. The solution to eq. (14) yields

$$X(x) = A\cos kx + B\sin kx \qquad (18)$$

To satisfy the boundary condition given by eq. (16), we must have

$$A = 0$$

and

$$B\sin kl = 0$$

To obtain a solution where $B \neq 0$, we must have

$$kl = n\pi \qquad n = 1, 2, ...$$

Thus

$$\lambda = -k^2 = -\left(\frac{n\pi}{l}\right)^2 \qquad (19)$$

(n=0 corresponds to the trivial solution). The specific values of λ are known as the eigenvalues of eq. (14) and the corresponding solutions, viz, $\sin\left(\frac{n\pi}{l}x\right)$ are called the *eigenfunctions*. Since there are many possible solutions, each is subscripted by n. Thus

$$X_n(x) = B_n \sin\left(\frac{n\pi}{l}x\right) \qquad n = 1, 2, 3, ... \qquad (20)$$

The solution to Eq. (15) with λ given by Eq. (19) is

$$T_n(t) = E_n \cos\left(\frac{n\pi}{l}ct\right) + F_n \sin\left(\frac{n\pi}{l}ct\right) \qquad n = 1, 2, 3, ... \qquad (21)$$

Where E_n and F_n are arbitrary constants. There are thus many solutions for eq. (8) which is given by

$$U_n(x,t) = X_n(x)T_n(t)$$
$$\left[a_n \cos\left(\frac{n\pi}{l}ct\right) + b_n \sin\left(\frac{n\pi}{l}ct\right)\right]\sin\frac{n\pi}{l}x \qquad (22)$$

Where $a_n = B_n E_n$ and $b_n = B_n F_n$. Since eq. (8) is linear and homogeneous, the general solution is obtained as the linear superposition of all the solutions given by eq. (22), i.e.

$$U(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi}{l}x \qquad (23)$$

Differentiating with respect to t, we have

$$U_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}\left(-a_n \sin\frac{n\pi c}{l}t + b_n \cos\frac{n\pi c}{l}t\right)\sin\frac{n\pi}{l}x \qquad (24)$$

The coefficients a_n and b_n are obtained by applying the initial conditions in eq. (10). Thus,

$$U(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x \qquad (25)$$
$$U(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi}{l}c\right)\sin\frac{n\pi}{l}x \qquad (26)$$

In order to determine a_n and b_n we use the orthogonality properties of $\sin\frac{n\pi}{l}x$ in the range $0 \le x \le l$, i.e.

$$\int_0^l \sin\frac{m\pi}{l}x \sin\frac{n\pi}{l}x\,dx = \frac{l}{2}\delta_{m\,n} \qquad (27)$$

Where $\delta_{m\,n}$ is the Kronecker delta function having the property

$$\delta_{m\,n} = \begin{cases} 0 & n \ne m \\ 1 & n = m \end{cases} \qquad (28)$$

Multiply eq. ((25) by $\sin\frac{m\pi}{l}x$ and integrating between the limits x = 0 and x = l, we get

$$\int_0^l f(x)\sin\frac{m\pi}{l}x\,dx = \sum_{n=1}^{\infty}\int_0^l a_n \sin\frac{m\pi}{l}x\sin\frac{n\pi}{l}x\,dx$$

$$= a_m\frac{l}{2} \qquad (29)$$

i.e. $a_m = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi}{l}x\,dx$

Similarly multiplying eq. (26) by $\sin\frac{m\pi}{l}x$ and integrating between the limits x = 0 and $x = l$, we get

$$\int_0^l g(x)\sin\frac{m\pi}{l}x\,dx = \sum_{n=1}^{\infty}\int_0^l b_n\left(\frac{n\pi}{l}c\right)\sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\,dx$$

$$= b_m\left(\frac{m\pi}{l}c\right)\frac{l}{2} \qquad (30)$$

i.e. $b_m = \frac{2}{m\pi c}\int_0^l g(x)\sin\frac{m\pi}{l}x\,dx$

With a_m and b_m obtained for $m=1,\ldots\ \infty$, eq. (23) is the solution to PDE given by eq. (8) subject to the initial conditions and the boundary conditions.

3.3.2 Application to Heat Conduction Equation

The one-dimensional heat flow in a rod bounded by the planes x = 0 and x = a is of practical interest. The solution applies to the case where the y and z dimensions extend to infinity. The temperature distribution is determined by solving the one-dimensional heat conduction equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{v}\frac{\partial\theta}{\partial t} \qquad (31)$$

Where θ represents the temperature and

$$v = \frac{k}{C\rho} \qquad (32)$$

k, C and ρ are the thermal conductivity, specific heat and density of the material respectively. We shall treat the case where the boundary conditions are given by

$$\theta(x=0,t) = 0 \qquad (33)$$

$$\theta(x = a, t) = 0 \qquad (34)$$

The initial temperature distribution is given by

$$\theta(x, t = 0) = f(x) \qquad (35)$$

Solution: Using the method of separation of variables, the x-dependence and t-dependence are separated out as expressed by

$$\theta(x, t) = X(x)T(t) \qquad (36)$$

Substituting eq. (36) into eq. (31) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v}\frac{1}{T}\frac{dT}{dt} = \alpha \qquad (37)$$

We shall now consider three cases corresponding to different values of the constant α.

Case 1 $\lambda = 0$

The separated ODE for $X(x)$ becomes

$$\frac{d^2X}{dx^2} = 0 \qquad (38)$$

i.e. $X(x) = Ax + B$

The boundary conditions expressed by eqs. (33) and (34) are respectively

$$X(x = 0) = 0 \quad \text{and} \quad X(x = a) = 0 \qquad (39)$$

Since T(t) should not be identically zero. Thus for eq. (38) to satisfy the boundary conditions given by eq. (39), we must have A = 0, B = 0. This gives the steady-state solution where temperature in the rod is everywhere zero.

Case 2 $\lambda > 0$

Let $\alpha = k^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = k^2X \qquad (40)$$

Therefore $X(x) = Ae^{kx} + Be^{-kx}$

Applying the boundary conditions given in eq. (39), we get

$$0 = A + B$$
$$0 = Ae^{ka} + Be^{-ka}$$

Again we have A = B = 0

Case 3 $\lambda < 0$

Let $\alpha = -\lambda^2$. The ODE for $X(x)$ becomes

$$\frac{d^2X}{dx^2} = -\lambda^2 X \tag{41}$$

Thus $X(x) = A\cos\lambda x + B\sin\lambda x$

The boundary conditions require

$$A = 0$$
$$B\sin\lambda a = 0 \tag{42}$$

i.e. $\lambda a = n\pi \qquad n = 1, 2, \ldots$ (43)

Since there are multiple solutions, each λ is designated by a subscript n as λ_n. The solution of the ODE for T(t) is readily obtained as

$$T(t) = Ce^{-\lambda_n^2 vt} \tag{44}$$

Thus the general solution which is a superposition of all admissible solution is given by

$$\theta(x,t) = \sum_{n=1}^{\infty} D_n e^{-\lambda_n^2 vt} \sin\frac{n\pi}{a}x \tag{45}$$

$$= \sum_{n=1}^{\infty} D_n \exp\left(-\frac{n^2\pi^2}{a^2}vt\right)\sin\frac{n\pi}{a}x \tag{46}$$

To complete the solution D_n must be determined from the remaining initial condition

i.e. $f(x) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{a}x$ (47)

In order to determine D_n, we multiply eq. (47) by $\sin\frac{m\pi}{a}x$ and integrate the limits x=0 and x = a to obtain

$$\int_0^a f(x)\sin\frac{m\pi}{a}x\,dx = \sum_{n=1}^{\infty} D_n \int_0^a a_n \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x\,dx = D_m\frac{a}{2}$$

Thus

$$D_m = \frac{2}{a}\int_0^a f(x)\sin\frac{m\pi}{a}x\,dx \tag{48}$$

For the specific case where $f(x) = \theta_0$ (constant), the solution is given by

$$\theta(x,t) = \frac{4\theta_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{a}\exp\left[-\frac{v(2n+1)^2\pi^2}{a^2}\right]$$

$$0 < x < a \tag{49}$$

From eq. (49), it can be deduced that a rectangular pulse of height θ_0 for $0 < x < a$ has the Fourier series expansion given by

$$\frac{4\theta_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{a}$$

Also if $f(x) = \gamma x$, then

$$\theta(x,t) = \frac{2a\gamma}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(\frac{n\pi}{a}x\right)\exp\left[-\frac{vn^2\pi^2}{a^2}t\right] \tag{50}$$

If the end boundaries are maintain at different temperature i.e.

$$\begin{aligned}\theta(x=0,t) &= \theta_1\\ \theta(x=a,t) &= \theta_2\end{aligned} \tag{51}$$

Then case 1 of the solution where $\alpha = 0$, would yield the steady-state solution given by $\theta_1 + \frac{x}{a}(\theta_2 - \theta_1)$. The general solution is given by

$$\theta(x,t) = \phi(x,t) + \theta_1 + \frac{x}{a}(\theta_2 - \theta_1) \tag{52}$$

Where $\phi(x,t)$ is the transient solution.

The boundary conditions for $\phi(x,t)$ are obtained as follows:

at x = 0: $\theta(x=0,t)=\theta_1=\phi(x=0,t)+\theta_1 \Rightarrow \phi(x=0,t)=0$
at x = a: $\theta(x=a,t)=\theta_2=\phi(x=a,t)+\theta_2 \Rightarrow \phi(x=a,t)=0$
$\phi(x,t)$ is obtained under case 3.

SELF-ASSESSMENT EXERCISE 1

8. State the nature of each of the following equations (that is, whether elliptic, parabolic or hyperbolic)

(a) $\dfrac{\partial^2 y}{\partial t^2}+\alpha\dfrac{\partial^2 y}{\partial x^2}=0$

(b) $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial y^2}+3y^2\dfrac{\partial u}{\partial x}$

2(a) Show that $y(x,t)=F(2x+5t)+G(2x-5t)$ is a general solution of

$$4\frac{\partial^2 y}{\partial t^2}=25\frac{\partial^2 y}{\partial x^2}$$

(b) Find a particular solution satisfying the conditions
$y(0,t)=y(\pi,t)=0,\quad y(x,0)=\sin 2x,\quad y'(x,0)=0.$

3. Solve the following PDEs

(a) $\dfrac{\partial^2 u}{\partial x^2}=8xy^2+1$

(b) $\dfrac{\partial^2 u}{\partial xy}-\dfrac{\partial u}{\partial y}=6xe^x$

3.6 Laplace Transform Solutions of Boundary-Value Problems

Laplace and Fourier transforms are useful in solving a variety of partial differential equations; the choice of the appropriate transforms depends on the type of boundary conditions imposed on the problem. Laplace transforms can be used in solving boundary-value problems of partial differential equation.

Example 5

Solve the problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \tag{53}$$

$$u(0,t) = u(3,t) = 0, \qquad u(x,0) = 10\sin 2\pi x - 6\sin 4\pi x \tag{54}$$

Solution: Taking the Laplace transform L of Eq. (53) with respect to t gives

$$L\left[\frac{\partial u}{\partial t}\right] = 2L\left[\frac{\partial^2 u}{\partial x^2}\right]$$

Now

$$L\left[\frac{\partial u}{\partial t}\right] = pL(u) - u(x,0)$$

and

$$L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-pt}u(x,t)dt = \frac{\partial^2}{\partial x^2}L[u]$$

Here $\partial^2/\partial x^2$ and $\int_0^\infty ...dt$ are interchangeable because x and t are independent.

For convenience, let

$$U = U(x,p) = L[u(x,t)] = \int_0^\infty e^{-pt}u(x,t)dt$$

We then have

$$pU - u(x,0) = 2L\frac{\partial^2 U}{\partial x^2}$$

from which we obtain, using the given conditions (54),

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{2}pU = 3\sin 4\pi x - 5\sin 2\pi x. \tag{55}$$

Then taking the Laplace transform of the given conditions $u(0,t) = u(3,t) = 0$, we have

$$L[u(0,t)] = 0, \qquad L[u(3,t)] = 0$$

Or

$$U(0,p) = 0, \qquad U(3,p) = 0.$$

These are the boundary conditions on $U(x,p)$. Solving eq. (55) subject to these conditions we find

$$U(x,p) = \frac{5\sin 2\pi x}{p + 16\pi^2} - \frac{3\sin 4\pi x}{p + 64\pi^2}$$

The solution to eq. (55) can now be obtained by taking the inverse Laplace transform

$$u(x,t) = L^{-1}[U(x,p)] = 5e^{16\pi^2 t}\sin 2\pi x - 36e^{64\pi^2 t}\sin 4\pi x.$$

SELF-ASSESSMENT EXERCISE 2

4. Differentiate between ordinary differential equation and partial differential equation.

5. Derive the PDE that give rise to the function

$$Z = a(x+y) + b(x-y) + abt + c = 0$$

6. Use the method of separation of variable to find the solution of the boundary value problem

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$y(0,t) = 0 \qquad t > 0$

$y(1,t) = 0 \qquad t > 0$

$y(x,0) = \sin 2x$

$y'(x,0) = 0 \qquad 0 \le x < \infty$

7.0 CONCLUSION

In this unit, we have studied the notion of a solution of partial differential equation. Also some elementary methods of solving linear partial differential equations which occur frequently in physics and engineering were dealt with.

8.0 SUMMARY

Here in this unit you have learnt about second order partial differential equation. The classical method of separation of variables was extensively studied along with the Laplace transform solutions of boundary-value problems.

9.0 TUTOR- MARKED ASSIGNMENT

1. Form the PDEs whose general solutions are as follow:

   (a) $z = Ae^{-p^2 t}\cos px$

   (b) $z = f\left(\frac{y}{x}\right)$

2. Solve the equation

$$2\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial^2 u}{\partial y^2} = 0$$

3. Find the solution of the differential equation

$$\alpha^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

Where
$y(0,t) = 0 \qquad 0 < t < \infty$
$y(0,t) = 0 \qquad 0 \le t < \infty$
$y(x,0) = f(x) \qquad 0 \le x \le L$
$y(x,0) = g(x) \qquad 0 \le x < L$

4. Solve by Laplace transforms the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \textit{for} \quad x > 0,\ t > 0$$

given that $u = u_0$ (a constant) on $x = 0$ for $t > 0$, and $u = 0$ for $x > 0,\ \ t = 0$

7.0 REFERENCES/ FURTHER READING

Erwin, K. (1991). *Advanced Engineering Mathematics.* John Wiley & Sons, Inc.

Pinsky, M.A. (1991).*Partial Differential Equations and Boundary-Value Problems with Applications*. New York: McGraw-Hill.

UNIT 2 FOURIER SERIES

CONTENTS

8.0 INTRODUCTION

In this unit, we shall discuss basic concepts, facts and techniques in connection with Fourier series. Illustrative examples and some important applications of Fourier series to Partial differential equations will be studied.
We will also study the concept of periodic functions, even and odd functions and the conditions for Fourier expansion.

9.0 OBJECTIVES

At the end of this unit, you should able to:

- identify whether a given function is even, odd or periodic
- evaluate the Fourier coefficients
- derive and apply Fourier series in forced vibration problems
- use Fourier Integral for treating various problems involving periodic function
- apply half range expansion to solutions of some problems.

10.0 MAIN CONTENT

3.1 Periodic Functions

A function $f(x)$ is said to be **periodic** if it defined for all real x and if there is some positive number T such that

$$f(x+T) = f(x) \qquad (1)$$

This number T is then called a **period** of $f(x)$.

Periodic functions occur very frequently in many application of mathematics to various branches of science. Many phenomena in nature such as propagation of water waves, light waves, electromagnetic waves, etc are periodic and we need periodic functions to describe them. Familiar examples of periodic functions are the sine and cosine functions.

Example 1

Find the period of $Tan\,x$.

Solution: Suppose T is its period

$$f(x+T) = Tan(x+T) = Tanx$$

so that

$$Tan(x+T) - Tanx = 0$$

using trigonometric identity, we have

$$\frac{Tan\,T(1-Tan^2 x)}{1-TanxTanT} = 0$$

This implies that

$Tan\,T = 0$ If and only if $1-Tan^2 x \neq 0$

$T = Tan^{-1} 0$

Hence $T = \pi$

3.2 Even and Odd Functions

A function $f(x)$ defined on interval $[a, b]$ is said to be a even function if

$$f(-x) = f(x) \qquad (2)$$

It is odd otherwise, that is

$$f(-x) = -f(x) \qquad (3)$$

Example 2

Let $f(x) = \sin x$

Then $f(-x) = -f(x)$ i.e. $\sin(-x) - = -\sin x$

Thus it is obvious that sine function is always an odd function while cosine function is an even function.

3.11 Fourier Theorem

According to the Fourier theorem, any finite, single valued periodic function $f(x)$ which is either continuous or possess only a finite number of discontinuities (of slope or magnitude), can be represented as the sum of the harmonic terms as

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx$$

$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (4)$$

3.12 Evaluation of Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π which can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (5)$$

Given such a function $f(x)$ we want to determine the coefficients of a_n and b_n in the corresponding series in eq. (5).

We first determine a_0. Integrating on both sides of eq. (4) from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx$$

If term-by-term integration of the series is allowed, then we obtain

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right)$$

The first term on the right equals $2\pi a_0$. All other integrals on the right are zero, as can be readily seen by performing the integration. Hence our first result is

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{6}$$

We now determine $a_1, a_2, \ldots$ by a similar procedure. We multiply Eq. (5) by $\cos mx$, where m is any fixed positive integer, and then integrate from $-\pi$ to π,

$$\int_{-\pi}^{\pi} f(x)\cos mxdx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mxdx \tag{7}$$

Integrating term-by-term, we see that the right-hand side becomes

$$a_0\int_{-\pi}^{\pi}\cos mxdx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cos mxdx + b_n\int_{-\pi}^{\pi}\sin nx\cos mxdx\right]$$

The first integration is zero. By applying trigonometric identity, we obtain

$$\int_{-\pi}^{\pi}\cos nx\cos mxdx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx$$
$$\int_{-\pi}^{\pi}\sin nx\cos mxdx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)xdx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)xdx.$$

Integration shows that the four terms on the right are zero, except for the last term in the first line which equals π when $n = m$. since in eq. (7) this term is multiplied by a_m, the right-hand side in eq. (7) is equal to $a_m\pi$, and our second result is

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mxdx \qquad m = 1, 2\ldots \tag{8}$$

We finally determine $b_{1,} b_{2,}$ ...... in eq.(5) by $\sin mx$, where m is any fixed positive integer, and the integrate from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin mxdx \tag{9}$$

Integrating term-by-term, we see that the right-hand side becomes

$$a_0\int_{-\pi}^{\pi}\sin mxdx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\sin mxdx+b_n\int_{-\pi}^{\pi}\sin nx\sin mxdx\right]$$

The first integral is zero. The next integral is of the type considered before, and we know that it is zero for all n = 1, 2,…. For the integral we obtain

$$\int_{-\pi}^{\pi}\sin nx\sin mxdx=\frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)xdx-\frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)xdx$$

The last term is zero. The first term on the right is zero when $n\neq m$ and is π when $n=m$. Since in eq. (9) this term is multiplied by b_m, the right-hand side in eq. (6) is equal to $b_m\pi$, and our last result is

$$b_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin mxdx \qquad m=1,2\ldots$$

Writing n in place of m, we altogether have the so-called **Euler formulas**

(a) $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

(b) $$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx \qquad n=1,2\ldots \qquad (10)$$

(c) $$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

Example 3 Square wave

Find the Fourier coefficients of the periodic function

$$f(x)=\begin{cases}-k & when \quad -\pi<x<0\\ k & when \quad 0<x<\pi\end{cases} \qquad and \qquad f(x+2\pi)=f(x)$$

Functions of this type may occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc

Solution: From eq. (10a) we obtain $a_0=0$. This can also be seen without integration since the area under curve of f(x) between $-\pi$ and π is zero. From eq. (10b)

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nxdx+\int_{0}^{\pi}k\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\bigg|_{0}^{\pi} \right] = 0$$

Because $\sin nx = 0\ at\ -\pi,\ 0$ and π for all n = 1, 2….Similarly, from Eq. (10c) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi} k\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\cos nx}{n}\bigg|_{-\pi}^{0} + k\frac{\cos nx}{n}\bigg|_{0}^{\pi} \right] = 0$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1-\cos n\pi)$$

Now, $\cos\pi = -1,\ \cos 2\pi = 1,\ \cos 3\pi = -1$ etc, in general

$$\cos n\pi = \begin{cases} -1 & for\quad odd\ n, \\ 1 & for\quad even\ n, \end{cases}$$ and thus

$$1-\cos n\pi = \begin{cases} 2 & for\quad odd\ n, \\ 0 & for\quad even\ n, \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi},\qquad b_2 = 0,\qquad b_3 = \frac{4k}{3\pi},\qquad b_4 = 0,\qquad b_5 = \frac{4k}{5\pi}$$

and since the a_n are zero, the corresponding Fourier series is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right) \tag{11}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x,\qquad\qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right),\qquad etc,$$

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + ...\right)$$

or $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - .... = \frac{\pi}{4}$$

SELF-ASSESSMENT EXERCISE 1

5. Define the periodic function. Give five examples.

6. Find the smallest positive period T of the following functions.
   a. $\sin x$

7. Are the following functions odd, even, or neither odd nor even?
   a. e^x
   b. $x \sin x$
8. Find the Fourier series of the following functions which are assumed to have the

d. period 2π
e. $f(x) = x^2/4 \qquad -\pi < x < \pi$
f. $f(x) = |\sin x| \qquad -\pi < x < \pi$

3.13 Application of Fourier Series in Forced Vibrations

We now consider an important application of Fourier series in solving a differential equation of the type

$$m\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + kx(t) = F(t) \tag{12}$$

For example, the above equation would represent the forced vibrations of a damped oscillator with Γ representing the damping constant, F(t) the external force and m and k representing the mass of the particle and the force constant respectively. We write eq. (12) in the form

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x(t) = G(t) \tag{13}$$

Where $K = \frac{\Gamma}{2m}$, $\omega_0^2 = \frac{k}{m}$ and $G(t) = \frac{F(t)}{m}$. The solution of the homogeneous part of eq. (13) can be readily obtained and is given by

$$x(t) = A_1 e^{=Kt}\cos\left[\sqrt{(\omega_0^2 - K^2)}t + \theta\right] \qquad \text{for} \qquad \omega_0^2 > K^2 \tag{14}$$

$$x(t) = (A_2 t + B)e^{-Kt} \qquad \text{for} \qquad \omega_0^2 > K^2 \tag{15}$$

In order to obtain the solution of the inhomogeneous part of eq. (13), we first assume F(t) to be a sine or cosine function; for definiteness we assume

$$G(t) = b \sin \omega t \tag{16}$$

The particular solution of eq. (13) can be written in the form

$$x(t) = C \sin \omega t + D \cos \omega t \tag{17}$$

The values of C and D can readily be obtained by substituting eq. (17) in eq. (13),
and comparing coefficients of $\sin \omega t$ and $\cos \omega t$ we obtain

$$\begin{aligned} D &= -\frac{2\omega K}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 K^2} b \\ C &= -\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 K^2} b \end{aligned} \tag{18}$$

Now, if G(t) is not a sine or cosine function, a general solution of eq. (13) is difficult to obtain. However, if we make a Fourier expansion of G(t) then the general solution of eq.

(13) can easily be written down. As a specific example, we assume

$$G(t) = \alpha t \tag{19}$$

The Fourier expansion of $G(t)$ can readily be obtained as

$$G(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{20}$$

Proceeding in a manner similar to that described above we obtained the following solution for the inhomogeneous part of eq. (13)

$$x(t) = \sum_{n=1}^{\infty} \left[C_n \sin n\omega t + D_n \cos n\omega t\right] \tag{21}$$

Where

$$\begin{aligned} D_n &= -\frac{2n\omega K}{\left(\omega_0^2 - n^2\omega^2\right)^2 + 4n^2\omega^2 K^2} b_n \\ C_n &= -\frac{(\omega_0^2 - n^2\omega^2)}{\left(\omega_0^2 - n^2\omega^2\right)^2 + 4n^2\omega^2 K^2} b_n \end{aligned} \tag{22}$$

thus, if G(t) is a periodic function with period T then eq. (21) will be valid for all values
of t.

3.14 Half-Range Expansions

In various physical and engineering problems there is a practical need for applying Fourier series to functions $f(t)$ which are defined merely on some finite interval. The function $f(t)$ is defined on an interval $0 \leqq t \leqq l$ and on this interval we want represent $f(t)$ by a Fourier series.

A **half-range Fourier series** for a function $f(x)$ is a series consisting of the sine and cosine terms only.

Such functions are defined on an interval $(0,l)$ and we then obtain a Fourier cosine series which represents an even periodic function $f_1(t)$ of period $T = 2l$ so that

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}t \qquad 0 \leqq t \leqq l$$

(23)

and the coefficients are

$$a_0 = \frac{1}{l}\int_0^l f(t)tdt, \qquad a_n = \frac{2}{l}\int_0^l f(t)\cos\frac{n\pi}{l}tdt \qquad n = 1, 2...$$

(24)

Then we obtain a Fourier sine series which represents an odd periodic function $f_2(t)$ of period $T = 2l$ so that

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}t \qquad 0 \leqq t \leqq l$$

(25)

and the coefficients are

$$b_n = \frac{2}{l}\int_0^l f(t)\sin\frac{n\pi}{l}tdt \qquad n = 1, 2...$$

(26)

The series in eqs.(23) and (25) with the coefficients in eqs.(24) and (26) are called **half-range expansions** of the given function $f(t)$

Example 4

Find the half-range expansions of the function

$$f(t) = \begin{cases} \frac{2k}{l}t & when \quad 0 < t < \frac{l}{2} \\ \frac{2k}{l}(l-t) & when \quad \frac{l}{2} < t < l \end{cases}$$

Solution: From eq. (24) we obtain

$$a_0 = \frac{1}{l}\left[\frac{2k}{l}\int_0^{l/2} t\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)dt\right] = \frac{k}{2}$$
$$a_n = \frac{2}{l}\left[\frac{2k}{l}\int_0^{l/2} t\cos\frac{n\pi}{l}t\,dt + \frac{2k}{l}\int_{l/2}^{l}(l-t)\cos\frac{n\pi}{l}t\,dt\right]$$

Now by integration by part

$$\int_0^{l/2} t\cos\frac{n\pi}{l}t\,dt = \frac{lt}{n\pi}\sin\frac{n\pi}{l}t\Big|_0^{l/2} - \frac{1}{n\pi}\int_0^{l/2}\sin\frac{n\pi}{l}t\,dt$$
$$= \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

Similarly,

$$\int_{l/2}^{l}(l-t)\cos\frac{n\pi}{l}t\,dt = -\frac{l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

By inserting these two results we obtain

$$u_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

Thus,

$$a_2 = -16k/2^2\pi^2, \quad a_6 = -16k/6^2\pi^2, \quad a_{10} = -16k/10^2\pi^2, \ldots$$

And $a_n = 0$ when $n \neq 2, 6, 10, 14, \ldots$ Hence the first half-range expansion of $f(t)$ is

$$f(t) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{l}t + \frac{1}{6^2}\cos\frac{6\pi}{l}t + \ldots\right)$$

This series represents the even periodic expansion of the function $f(t)$. Similarly from eq. (26)

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

and the other half-range expansion of $f(t)$ is

$$f(t) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{l}t - \frac{1}{3^2}\sin\frac{3\pi}{l}t + \frac{1}{5^2}\sin\frac{5\pi}{l}t - + \ldots\right)$$

This series represents the odd periodic extension of $f(t)$.

Example 5

Find a Fourier sine series for

$$f(x)=\begin{cases}0 & x\le 2\\ 2 & x>2\end{cases} \quad \text{on } (0,\ 3).$$

Solution: Since the function is odd, then $a_0 = 0$

Then $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}xdx$

$$= \frac{2}{3}\int_0^3 f(x)\sin\frac{n\pi}{3}xdx$$

$$= \frac{2}{3}\int_0^2 0.\sin\frac{n\pi}{3}xdx + \frac{2}{3}\int_2^3 2\sin\frac{n\pi}{3}xdx$$

Now by integration, we have

$$b_n = \frac{4}{n\pi}\left[\cos\frac{2n\pi}{3} - \cos n\pi\right]$$

The series thus becomes

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\left[\cos\frac{2n\pi}{3} - (-1)^n\right]\sin\frac{n\pi x}{3}$$

So that

$$f(x) = \frac{4}{\pi}\left(\frac{1}{2}\sin\frac{\pi x}{3} - \frac{3}{4}\sin\frac{2\pi x}{3} + \frac{2}{3}\sin\frac{3\pi x}{3} - + \ldots\right)$$

Example 6

Find the Fourier cosine series for

$$f(x) = e^x \quad \text{on} \quad (0,\ \pi)$$

Solution: Since $f(x)$ is an odd function, then

$$b_0 = \frac{1}{l}\int_0^l e^x dx = \frac{1}{\pi}\left(e^{\pi} - 1\right)$$

Also

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^x \cos\frac{n\pi x}{\pi}dx = \frac{2}{\pi}\left(\frac{1}{1+n^2}\right)\left(e^{\pi}\cos n\pi - 1\right)$$

Thus the series becomes

$$e^x = \frac{2}{\pi}\left(e^{\pi} - 1\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{1+n^2}\left[(-1)^n e^{\pi} - 1\right]\cos nx$$

SELF-ASSESSMENT EXERCISE 2

1. Find the Fourier sine series for
   $f(x) = e^x$ on $(0, \pi)$

9. Find the Fourier series for
   $f(x) = x$ on $0 < x < 2$
   consisting of (a) sine series only (b) cosine series only

3.15 Fourier Integral

Fourier series are powerful tools in treating various problems involving periodic functions. When the fundamental period is made infinite, the limiting form of the Fourier series becomes an integral which is called ***Fourier Integral.***

3.15.1 Definition

Let $f(x)$ be defined and single valued in the interval $[-L, L]$. If $f(x)$ satisfies the following conditions:

(i) $f(x)$ is periodic and of period 2L

(ii) $f(x)$ and $f'(x)$ are piecewise continuous

(iii) $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent, then $f(x)$ can be expressed as

$$f(x) = \int_0^{\infty} \left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right) dx$$

(27)

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$$

(28)

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$$

(29)

3.16 Fourier Integrals of Even and Odd Functions

It is of practical interest to note that if a function is even or odd and can be represented by a Fourier integral, and then this representation will be simpler than in the case of an arbitrary function. This follows immediately from our previous formulas, as we shall now see.

If $f(x)$ is an even function, then $B(\alpha) = 0$

$$A(\alpha) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos\alpha x dx \qquad (30)$$

and eq. (27) reduces to the simpler form

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x dx \qquad (f\ even) \qquad (31)$$

Similarly, if $f(x)$ is odd, then $A(\alpha) = 0$ in eq. (28), also

$$B(\alpha) = \frac{2}{\pi}\int_0^\infty f(x)\sin\alpha x dx$$

(32)

and

$$f(x) = \int_0^\infty B(\alpha)\sin\alpha x dx \qquad (f\ odd) \qquad (33)$$

These simplifications are quite similar to those in the case of a Fourier series discussed.

Example 7

Find the Fourier Integral of $f(x) = x^2 \qquad -\pi \le x \le \pi$

Solution:

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos\alpha x dx$$
$$= \frac{1}{\pi}\int_{-\infty}^\infty x^2\cos\alpha x dx$$

Using integration by parts, we obtain

$$A(\alpha) = \frac{2}{\pi\alpha}\left[\frac{x}{\alpha}\cos\alpha x - \frac{1}{\alpha^2}\sin\alpha x\right]\Bigg|_{-\pi}^{\pi} = 0$$

Also

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin\alpha x dx$$
$$= \frac{1}{\pi}\int_{-\infty}^\infty x^2\sin\alpha x dx$$

So that

$$B(\alpha) = -\frac{1}{\pi}\left[\frac{x^2}{\alpha}\cos\alpha x - \frac{2}{\alpha^3}\cos\alpha x\right]\Bigg|_{-\pi}^{\pi} = \frac{2\pi}{\alpha}(-1)^\alpha$$

From eq. (27)

$$f(x) = \int_0^\infty \left(A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right)dx$$

and

$$f(x) = \int_0^\infty \left(0 \bullet \cos\alpha x + \frac{2\pi}{\alpha}(-1)^\alpha \sin\alpha x\right)dx = \frac{2\pi}{\alpha^2}(-1)^\alpha$$

Hence

$$f(x) = x^2 = \frac{2\pi}{\alpha}(-1)^\alpha \int_0^\infty \sin\alpha x dx = \frac{2\pi}{\alpha^2}(-1)^\alpha$$

11.0 CONCLUSION

In this unit, you have studied the concept of periodic functions, representations of functions by Fourier series, involving sine and cosine function are given special attention. We also use the series expansion in the determination of Fourier coefficients and the half-range expansions.

12.0 SUMMARY

In this unit, you have studied:

- Even and odd functions
- Fourier Integral representations and Fourier series expansion.
- Application of Fourier Integral technique in the simplification of even and odd functions.

13.0 TUTOR- MARKED ASSIGNMENT

1. Find the smallest positive period T of the following functions

   c. (i) $\sin 2\pi x$

   d. (ii) $\cos\frac{2\pi nx}{k}$

2. Find the Fourier series for

   $f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$ where $f(x)$ has period 10

10. Find the Fourier series for

    $f(x) = x^2$ for $0 < x < 2\pi$

11. Find the Fourier series of function

    $f(x) = x + \pi$ when $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

12. Expand the function

$f(t) = t^2 \qquad -\frac{T}{2} < x < \frac{T}{2}$ in a Fourier series to show that

$$f(t) = t^2 = \frac{T^2}{4\pi^2}\left[\frac{\pi^2}{3} - 4\left(\cos\omega t - \frac{1}{4}\cos 2\omega t + \frac{1}{9}\cos\omega t - ...\right)\right]$$

take $\omega = 2\pi/T$

13. Represent the following functions $f(t)$ by a Fourier cosine series

(a) $f(t) = \sin\frac{\pi}{l}t \qquad (0 < t < l)$

(b) $f(t) = e^t \qquad (0 < t < l)$

14. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & when\ |x| < 1, \\ 0 & when\ |x| > 1. \end{cases}$$

14.0 REFERENCES/FURTHER READING

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MODULE 2 APPLICATION OF FOURIER TO PDES (LEGENDRE POLYNOMIALS AND BESSEL FUNCTIONS)

UNIT 1 LEGENDRE POLYNOMIALS

CONTENTS

2.0 INTRODUCTION

In this unit, you will be introduced to the polynomial solutions of the Legendre equation, the generating function as well as the orthogonality of Legendre polynomials. Also we shall consider some important integrals involving Legendre functions which are of considerable use in many areas of physics.

7.0 OBJECTIVES

At the end of this unit, you should be able:

- derive the polynomial solution of the Legendre equation
- use the generating functions to derive some important identities
- determine the orthogonality of the Legendre polynomials.

8.0 MAIN CONTENT

3.1 Legendre Equation

The equation

$$(1-x^2 y''(x)-2xy'(x)+n(n+1)y(x)=0 \qquad (1)$$

where n is a constant is known as the ***Legendre's differential equation.*** In this unit we will discuss the solutions of the above equation in the domain $-1<x<1$. We will show that when

$$n=0, 1, 2, 3, \ldots$$

one of the solutions of eq. (1) becomes a polynomial. These polynomial solutions are known as the ***Legendre polynomials,*** which appear in many diverse areas of physics and engineering.

3.6 The Polynomial Solution of the Legendre's Equation

If we compare eq. (1) with homogeneous, linear differential equations of the type

$$y''(x)+U(x)y'(x)+V(x)y(x)=0 \qquad (2)$$

we find that the coefficients

$$U(x)=-\frac{2x}{1-x^2} \quad \text{and} \quad V(x)=\frac{n(n+1)}{1-x^2} \qquad (3)$$

are analytical at the origin. Thus the point x = 0 is an ordinary point and a series solution of eq. (1) using Frobenius method should be possible. Such that

$$y(x)=C_0 S_n(x)+C_1 T_n(x)$$

where

$$S_n(x)=1-\frac{n(n+1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4-\ldots \qquad (4a)$$

And

$$T_n(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5-\ldots \qquad (4b)$$

If $n \neq 0.\ 1,\ 2, \ldots$ both eqs. (4a) and (4b) are infinite series and converge only if $|x|<1$.

It may be readily seen that when

$$n=0.\ 2,\ 4, \ldots$$

The even series becomes a polynomial and the odd series remains an infinite series. Similarly for

$$n = 1,\ \ 3,\ \ 5, \ldots$$

the odd series becomes a polynomial and the even series remains an infinite series.
Thus when

$$n = 0,\ \ 1,\ \ 2,\ \ 3, \ldots$$

one of the solutions becomes a polynomial. The Legendre polynomial, or the Legendre function of the first kind is denoted by $P_n(x)$ and is defined in terms of the terminating series as below:

$$P_n(x) = \begin{cases} \dfrac{S_n(x)}{S_n(1)} & for\ \ n = 0,\ 2,\ \ 4,\ \ 6, \ldots \\ \dfrac{T_n(x)}{T_n(1)} & for\ \ n = 1,\ 3,\ \ 5,\ 7, \ldots \end{cases} \tag{5}$$

Thus,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right),$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \quad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \tag{6}$$
$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right), \ldots$$

Obviously, $\quad P_n(1) = 1 \quad$ (7)

Higher order Legendre polynomials can easily be obtained by using the recurrence relation

$$nP_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x)$$

Since for even values of n the polynomials $P_n(x)$ contain only even powers of x and for odd values of n the polynomials contain only odd powers of x, we readily have

$$P_n(-x) = (-1)^n P_n(x) \qquad \text{and obviously} \tag{8}$$
$$P_n(-1) = (-1)^n \tag{9}$$

3.7 The Generating Function

The generating function for the Legendre polynomials is given by

$$G(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n;\ -1 \le x \le 1,\ t < 1 \tag{10}$$

Let us assume that

$$G(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} K_n(x)t^n \qquad (11)$$

Where $K_n(x)$ is a polynomial of degree n. Putting x = 1 in eq. (11), we obtain

$$\sum_{n=0}^{\infty} K_n(x)t^n = (1-2t+t^2)^{-1/2}$$
$$= (1-t)^{-1}$$
$$= 1+t+t^2+t^3+\ldots+t^n+\ \ldots$$

Equating the coefficients of t^n from both sides, we have

$$K_n(1) = 1 \qquad (12)$$

Now, if we can show that $K_n(x)$ satisfies eq. (1), then $K_n(x)$ will be identical to $P_n(x)$. Differentiating G(x, t) with respect to x and t, we obtain

$$(1-2xt+t^2)\frac{\partial G}{\partial t} = (x-t)G(x,t) \qquad (13)$$

and

$$t\frac{\partial G}{\partial t} = (x-t)\frac{\partial G}{\partial x} \qquad (14)$$

Using eqs. (11), (13) and (14), we have

$$(1-2t+t^2)\sum_{n=0}^{\infty} nK_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} K_n(x)t^n \qquad (15)$$

and

$$t\sum_{n=0}^{\infty} nK_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} K'_n(x)t^n \qquad (16)$$

Equating the coefficient of t^{n-1} on both sides of eqs. (15) and (16), we get

$$nK_n(x) - (2n-1)xK_{n-1}(x) + (n-1)K_{n-2}(x) = 0 \qquad (17)$$

and

$$xK'_{n-1}(x) - K'_{n-2}(x) = (n-1)K_{n-1}(x) \qquad (18)$$

Replacing n by n+1 in Eq. (18), we obtain

$$xK'_n(x) - K'_{n-1}(x) = nK_n(x) \qquad (19)$$

We next differentiate Eq. (17) with respect to x and eliminate K'_{n-2} with help of Eq. (18) to obtain

$$K_n'(x) - xK_{n-1}'(x) - nK_{n-1}(x) = 0 \quad (20)$$

If we multiply eq. (19) by x and subtract it from eq. (20), we would get

$$(1-x^2)K_n' - n(K_{n-1} - xK_n) = 0 \quad (21)$$

Differentiating the above equation with respect to x, we have

$$(1-x^2)K_n'' - 2xK_n' - n(K_{n-1}' - xK_n' - K_n) = 0 \quad (22)$$

Using eqs. (19) and (22), we obtain

$$(1-x^2)K_n''(x) - 2xK_n'(x) - n(n+1)K_n(x) = 0 \quad (23)$$

which shows that $K_n(x)$ is a solution of Legendre equation. In view of eqs. (7) and (12) and the fact that $K_n(x)$ is a polynomial in x of degree n, it follows that $K_n(x)$ is nothing but $P_n(x)$. eq. (17) gives the recurrence relation for $P_n(x)$

$$nP_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x) \quad (24)$$

3.8 Rodrigues' Formula

Let

$$\phi(x) = (x^2-1)^n \quad (25)$$

Differentiating eq. (25), we get

$$\frac{d\phi}{dx} = 2nx(x^2-1)^{n-1}$$

or

$$(1-x^2)\frac{d^2\phi}{dx^2} + 2x(n-1)\frac{d\phi}{dx} + 2n\phi = 0$$

Differentiating the above equation n times with respect to x, we would get

$$(1-x^2)\frac{d^2\phi_n}{dx^2} + 2x\frac{d\phi_n}{dx} + n(n+1)\phi_n = 0 \quad (26)$$

where

$$\phi_n = \frac{d^n\phi}{dx^n} = \frac{d^n}{dx^n}\left[(x^2-1)^n\right] \quad (27)$$

This shows that $\phi_n(x)$ is a solution of the Legendre's equation. Further, it is obvious from eq. (27) that $\phi_n(x)$ is a polynomial of degree n in x. Hence $\phi_n(x)$ should be a constant multiple of $P_n(x)$, i.e.

$$\frac{d^n\left[\left(x^2-1\right)^n\right]}{dx^n} = CP_n(x) \tag{28}$$

$$\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right] = \frac{d^n}{dx^n}\left[(x+1)^n(x-1)^n\right]$$

$$= n!(x-1)^n + n\frac{n!}{1!}(x+1)n(x-1)^{n-1}$$

$$+\frac{n(n-1)}{2!}\frac{n!}{2!}(x+1)^2 n(n-1)(x-1)^{n-2} + \ldots + (x+1)^n n! \tag{29}$$

It may be seen that all terms on the right hand side of eq. (29) contain a factor (x-1) except for the last term. Hence

$$\left.\frac{d^n}{dx^n}\left(x^2-1\right)^n\right|_{x=1} = 2^n n! \tag{30}$$

Using Eqs. (7), (28) and (29), we obtain

$$C = 2^n n! \tag{31}$$

Therefore

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \tag{32}$$

This is known as the ***Rodrigues formula*** for the Legendre polynomials.

For example

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(x^2-1\right)^2$$
$$= \frac{1}{2}\left(3x^2-1\right)$$

Which is consistent with eq. (6)

3.9 Orthogonality of the Legendre Polynomials

Since the Legendre's differential equation is of the Sturm-Liouville form in the interval $-1 \le x \le 1$, with $P_n(x)$ satisfying the appropriate boundary conditions at $x = \pm 1$. The Legendre polynomials form an orthogonal set of functions in the interval $-1 \le x \le 1$, i.e

$$\int_{-1}^{1} p_n(x)\, p_m(x) dx = 0 \quad m \neq n \tag{33}$$

The Orthogonality of the Legendre polynomials can be proved as follows: $P_n(x)$ satisfies eq. (1) which can be written in the Sturm-Liouville form as

$$\frac{d}{dx}\left[\left(x^2-1\right)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0 \tag{34}$$

Similarly

$$\frac{d}{dx}\left[\left(x^2-1\right)\frac{dP_m(x)}{dx}\right] + m(m+1)P_m(x) = 0 \tag{35}$$

Multiply eq. (34) by $P_m(x)$ and eq. (35) by $P_n(x)$ and subtracting eq. (35) from eq. (34), we get

$$\frac{d}{dx}\left[(1-x^2)\left(P_n'(x)P_m(x) - P_m'(x)P_n(x)\right)\right]$$

$$= (m-n)(n+m+1)P_n(x)P_m(x)$$

Integrating the above equation from x = -1 to x = 1, we get

$$(1-x^2)\left[\left(P_n'(x)P_m(x) - P_m'(x)P_n(x)\right)\right]_{-1}^{+1}$$
$$= (m-n)(n+m+1)\int_{-1}^{1} P_n(x)P_m(x)dx$$

Because of the factor $(1-x^2)$ the left hand side of the above equation vanishes; hence

$$\int_{-1}^{1} P_n(x)P_m(x)dx \qquad for \quad m \neq n$$

To determine the value of the integral

$$\int_{-1}^{1} P_m^2(x)dx$$

we square both sides of eq. (10) and obtain

$$(1-2xt+t^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)t^{m+n} \tag{36}$$

Integrating both sides of the above equation with respect to x from -1 to +1 and using eq. (33), we get

$$\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x)dx = \int_{-1}^{1} \frac{1}{1-2xt+t^2} dx = \frac{1}{t}\ln\frac{1+t}{1-t}$$
$$= 2\left(1+\frac{1}{3}t^2+\frac{1}{5}t^4+\ldots+\frac{1}{2n+1}t^{2n}+\ldots\right)$$

Equating the coefficients of t^{2n} on both sides of the above equation, we have

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1} \qquad n = 0, 1, 2, 3, \ldots. \qquad (37)$$

Thus we may write

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$$

where

$$\delta_{nm} = \begin{cases} 0 & if \;\; n \neq m \\ 1 & if \;\; n = m \end{cases}$$

Example

We consider the function $\cos \pi x/2$ and expand it in a series (in the domain $-1 < x < 1$) up to the second power of x:

$$\cos\frac{\pi x}{2} = \sum_{n=0}^{2} C_n P_n(x)$$

Now

$$C_n = \frac{2n+1}{2}\int_{-1}^{+1} \cos\frac{\pi x}{2} P_n(x)dx$$

Substituting for $P_n(x)$ from eq. (6) and carrying out brute force integration, we readily get

$$C_0 = \frac{2}{\pi}; \quad C_1 = 0; \quad C_2 = \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)$$

Thus

$$\cos\frac{\pi x}{2} = \frac{2}{\pi} + \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)\left(\frac{3x^2-1}{2}\right)$$

3.6 The Angular Momentum Problem in Quantum Mechanics

In electrostatics the potential Φ satisfies the Laplace equation

$$\nabla^2\Phi = 0 \qquad (38)$$

We wish to solve the above equation for a perfectly conducting sphere (of radius a), place in an electric field which is in the absence of the sphere was of uniform magnitude E_0 along z-direction. We assume the origin of our coordinate system to be at the centre of the sphere. Because the sphere is a perfect conductor, the potential on its surface will be constant which, without any loss of generality, may be assumed to be zero. Thus, eq. (35) is said to be solved subject to the boundary condition

$$\Phi(r = a) = 0 \qquad (39)$$

At a large distance from the sphere the field should remain unchanged and thus

$$E(r \to \infty) = E_0\hat{z}$$

Since

$$E = -\nabla\Phi$$

we have

$$\begin{aligned}\Phi(r \to \infty) &= -E_0 z + C \\ &= -E_0 r\cos\theta + C \qquad (40)\end{aligned}$$

Where C is a constant. Obviously, we should use the spherical system of coordinates so that

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \qquad (41)$$

From the symmetry of the problem it is obvious that Φ would be independent of the azimuthal coordinate ϕ so that eq. (41) simplifies to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0 \qquad (42)$$

Separation of variables

$$\Phi = R(r)\Theta(\theta)$$

will yield

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=a\ constant\ (=\lambda) \qquad (43)$$

Changing the independent variable from θ to μ by the relation

$$\mu=\cos\theta$$

In the angular equation, we get

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\lambda\Theta=0 \qquad (44)$$

In order that the solution of eq. (44) does not diverge at $\mu=\pm1(\theta=0\ and\ \pi)$, we must have

$$\lambda=l(l+1);\ l=0,\ 1,\ 2,\ldots$$

and then

$$\Theta(\theta)=\sqrt{\frac{2l+1}{2}}P_l(\cos\theta) \qquad (45)$$

Thus the radial equation can be written as

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)$$

or

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-l(l+1)R=0 \qquad (46)$$

The above equation is the Cauchy's differential equation and its solution can readily be written as

$$R=A_l r^l+\frac{B_l}{r^{l+1}}$$

Hence the complete solution of eq. (42) is given by

$$\begin{aligned}\Phi(r,\theta)&=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)+\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)\\&=\left[A_oP_o(\cos\theta)+A_1rP_1(\cos\theta)+A_2r^2P_2(\cos\theta)+\ldots\right]\\&+\frac{B_o}{r}P_o(\cos\theta)+\frac{B_1}{r^2}P_1(\cos\theta)+\ldots\end{aligned}$$

Applying the boundary condition given by eq. (40), we get

$$A_0=C,\ A_1=-E_o,\ A_2=A_3=\ldots=0$$

Thus

$$\Phi(r,\theta)=\left(C+\frac{B_0}{r}\right)P_0(\cos\theta)+\left(-E_0r+\frac{B_1}{r^2}\right)P_1(\cos\theta)$$

$$+\frac{B_2}{r^3}P_2(\cos\theta)+\ .\ .\ .$$

Applying the condition at r = a [see eq. (39)], we get

$$\left(C+\frac{B_0}{a}\right)+\left(-E_0a+\frac{B_1}{a^2}\right)P_1(\cos\theta)$$
$$+\frac{B_2}{a^3}P_2(\cos\theta)+\frac{B_2}{a^4}P_3(\cos\theta)+\ .\ .\ .=0$$

Since the above equation has to be satisfied for all values of θ and since $P_n(\cos\theta)$ form a set of orthogonal functions, the coefficients of $P_n(\cos\theta)$ should be zero giving

$$B_0=-aC,\ \ B_1=E_0a^3$$
$$B_2=B_3=B_4\ .\ .\ .=0$$

Thus

$$\Phi(r,\theta)=C\left(1+\frac{a}{r}\right)-E_0\left(1+\frac{a^3}{r^3}\right)r\cos\theta \qquad (47)$$

The $1/r$ potential would correspond to a charged sphere and, therefore, for an uncharged sphere we must have C = 0 giving

$$\Phi(r,\theta)=-E_0r\cos\theta\left(1+\frac{a^3}{r^3}\right) \qquad (48)$$

This is the required solution to the problem. One can easily determine the components of the electric field as:

$$E_r=-\frac{\partial\Phi}{\partial r}=E_0\cos\theta\left(1+2\frac{a^3}{r^3}\right)$$
$$E_\theta=-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}=E_0\sin\theta\left(1-\frac{a^3}{r^3}\right)$$
$$E_\phi=-\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}=0$$

3.7 Important Integrals Involving Legendre Functions

We give below some important integrals involving Legendre functions which are of considerable use in many areas of physics.

$$P_n(x)=\frac{1}{\pi}\int_0^\infty\left[x+(x^2-1)^{1/2}\cos\theta\right]^n d\theta \qquad (49)$$

$$P_n(\cos\phi) = \frac{1}{\pi}\int_0^{\pi}\left(\cos\phi + i\sin\phi\cos\theta\right)^n d\theta \qquad (50)$$

$$\int_{-1}^{1}(1-x)^{-1/2}P_n(x)dx = \frac{2^{3/2}}{2n+1} \qquad (51)$$

$$\int_{-1}^{1}\frac{1}{(1-x^2)}\left[P_n^m(x)\right]^2 dx = \frac{(n+m)!}{m(n-m)!} \qquad (52)$$

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \frac{1}{\left(n+\frac{1}{2}\right)}\frac{(n+m)!}{(n-m)!} \qquad (53)$$

SELF-ASSESSMENT EXERCISE

3. Show that $(n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$
4. Using the Rodrigue's formula show that

$$P'_n(x) = \frac{1}{2}n(n+1)$$

9.0 CONCLUSION

The concept of generating function for the Legendre polynomials allows us to readily derive some important identities.

We have also established in this unit, relationship between Orthogonality of the Legendre polynomials and the generating function.

10.0 SUMMARY

This unit deals with Legendre functions and its applications to physical problems especially in quantum mechanics.

11.0 TUTOR-MARKED ASSIGNMENT

4. Show that

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$

$$= -\frac{n(n+1)}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

5. Determine the coefficients

C_0, C_1, C_2, C_3, in the expansion

$$\sin\left(\frac{nx}{2}\right) = \sum_{n=0}^{3} C_n P_n(x) \qquad -1 < x < 1$$

6. Consider the function

$$f(x) = \begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 < x \le 1 \end{cases}$$

Show that

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}\left[P_{n+1}(0) - P_{n-1}(0) - \right]P_n(x) \qquad -1 < x < 1$$

4. Show that the generating function

$$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n$$

Hint: Start from the binomial expansion of $1/\sqrt{1-v}$, set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

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UNIT 2 BESSEL FUNCTIONS

CONTENTS

1.0 INTRODUCTION

In this unit we shall consider the series solution as well as Bessel functions of the first and second kinds of order n.
We will also be introduced to some integrals which are useful in obtaining solutions of some problems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- derive the solution of Bessel function of the first kind
- prove a relationship between the recurrence relation and the generating functions
- derive the solution of Bessel function of the second kind.

3.0 MAIN CONTENT

3.1 Bessel Differential Equation

The equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx}(x^2 - n^2)y(x) = 0 \qquad (1)$$

Where n is a constant known as ***Bessel's differential equation.***
Since n^2 appears in eq. (1), we will assume, without any loss of generality, that n is either zero or a positive number. The two linearly independent solutions of eq. (1) are

$J_n(x)$ and $J_{-n}(x)$

Where $J_n(x)$ is defined by the infinite series

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \qquad (2)$$

or

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right] \qquad (3)$$

where $\Gamma(n+r+1)$ represents the gamma function.

3.4 Series Solution and Bessel Function of the First Kind

If we use eq. (1) with the homogeneous, linear differential equation of the type

$$y''(x) + U(x)y'(x) + V(x)y(x) = R(x) \qquad (4)$$

we find the coefficients

$$U(x) = \frac{1}{x} \quad \text{and} \quad V(x) = 1 - \frac{n^2}{x^2}$$

are singular at x = 0. However, x = 0 is a regular singular point of the differential equation and a series solution of eq. (1) in ascending powers of x. Indeed, one of the solutions of eq. (1) is given by

$$J_n(x) = C_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right] \qquad (5)$$

and where C_0 is an arbitrary constant. This solution is analytic at x = 0 for $n \geq 0$ and converges for all finite values of x. If we choose

$$C_0 = 2^n \Gamma(n+1) \qquad (6)$$

then the eq. (5) is denoted by $J_n(x)$ and is known as the ***Bessel function of the first kind of order n.***

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^{n} - \frac{1}{1!\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4} - \ldots \qquad (7)$$

In particular

$$J_0(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \ . \ . \ . \qquad (8)$$

$$J_{1/2}(x) = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{1!\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2!\Gamma(7/2)} + . \ . \ .$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - . \ . \ .\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x \qquad (9)$$

It follows immediately from eqs. (7) and (8) that

$$J_n(0) = 0 \quad \textit{for} \quad n > 0$$

and

$$J_n(0) = 1$$

If $n \neq 0, 1, 2, 3, . \ . \ .$ then

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)} \qquad (10)$$

Example 1

In this example we will determine the value of $J_{-1/2}(x)$ from eq. (10).

Thus

$$J_{-1/2}(x) = \frac{(x/2)^{-1/2}}{\Gamma(1/2)} - \frac{(x/2)^{3/2}}{1!\Gamma(3/2)} + \frac{(x/2)^{7/2}}{2!\Gamma(5/2)} + . \ . \ .$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - . \ . \ .\right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$

Which is linearly independent of $J_{1/2}(x)$ [see eq. (9)] and it can be verified that $J_{-1/2}(x)$ does in fact satisfy eq. (1) for n = ½. Thus

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \qquad (11)$$

and

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \qquad (12)$$

Using the above two equations and the recurrence relation [see Eq. (21)]

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \qquad (13)$$

We can readily obtain closed form expression for $J_{\pm 3/2}(x)$, $J_{\pm 5/2}(x), J_{\pm 7/2}(x)$, …

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \qquad (14)$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{1}{x}\cos x - \sin x\right) \qquad (15)$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{(3-x^2)}{x^2}\sin x - \frac{3}{x}\cos x\right) \qquad (16)$$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{(3-x^2)}{x^2}\cos x - \frac{3}{x}\sin x\right) \qquad (17)$$

etc.

Next, we will examine eq.(10) when n is a positive integer. To be specific we assume n = 4; then the first, second, third and fourth terms in the series given by eq. (10) will contain the terms

$$\frac{1}{\Gamma(-3)},\ \frac{1}{\Gamma(-2)},\ \frac{1}{\Gamma(-1)},\ \text{and}\ \frac{1}{\Gamma(0)}$$

respectively and all these terms are zero. In general the first n terms of the series would vanish giving

$$J_{-n}(x) = \sum_{r=n}^{\infty}\frac{(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)} \qquad (18)$$

If we put r = k+n, we would obtain

$$\begin{aligned} J_{-n}(x) &= \sum_{k=0}^{\infty}(-1)^{k+n}\frac{(x/2)^{n+2k}}{(k+n)!\Gamma(k+1)} \\ &= \sum_{k=0}^{\infty}(-1)^{k}\frac{(x/2)^{n+2k}}{k!\Gamma(k+n+1)} \\ &= (-1)^n J_n(x) \end{aligned} \qquad (19)$$

Thus for $n = 0, 1, 2, 3, \ldots, J_{-n}(x)$ does not represent the second independent solution of eq. (1). The second independent solution will be discussed later.

3.5 Recurrence Relations

The following are some very useful relations involving $J_n(x)$:

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \tag{20a}$$

$$= xJ_{n-1}(x) - nJ_n(x) \tag{20b}$$

Thus

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{21}$$

Also

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \tag{22}$$

In order to prove eq. (20a) w.r.t x to obtain

$$xJ_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2} x \tag{23}$$

or

$$xJ_n'(x) = n\sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + x\sum_{r=0}^{\infty} (-1)^r \frac{1}{(r-1)!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n(x) - x\sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \tag{24}$$

or

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \tag{25}$$

Which proves eq. (20a). eq. (23) can also be written as

$$xJ_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{2(n+r)(x/2)^{n+2r-1}}{r!\Gamma(n+r+1)} \frac{x}{2} - n\sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{r!\Gamma(n+r+1)}$$

$$= x\sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n-1+2r}}{r!\Gamma(n+r)} - nJ_n(x)$$

or

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x) \tag{26}$$

Which proves eq. (20b). From eq. (25) we readily obtain

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = x^{-n}J_{n+1}(x) \qquad (27)$$

Further, adding eqs. (25) and (26) we get

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \qquad (28)$$

Using eq. (21) we may write

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \qquad (29)$$

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \qquad (30)$$

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} + 1\right)J_0(x) \qquad (31)$$

etc.

The proof of eq. (22) is simple

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_n'(x) + nx^{n-1}J_n(x)$$

$$= x^n\left[J_{n-1}(x) - \frac{n}{x}J_n(x)\right] + nx^{n-1}J_n(x)$$

$$= x^n J_{n-1}(x) \qquad \text{[Using eq. (20b)]} \qquad (32)$$

Now using eq. (20a)

$$J_0'(x) = -J_1(x) \qquad (33)$$

Therefore

$$\int J_1(x)dx = -J_0(x) + \text{Constant} \qquad (34)$$

or

$$\int_0^\infty J_1(x)dx = 1 \qquad [\text{Because } J_0(0) = 1] \qquad (35)$$

Equation (32) gives us

$$\int x^n J_{n-1}(x)dx = x^n J_n(x) \qquad (36)$$

Example 2

In this example we will evaluate the integral

$$\int x^4 J_1(x)dx$$

in terms of $J_0(x)$ and $J_1(x)$. Since

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \qquad \text{[see eq. (22)]}$$

we have

$$\int x^p J_{p-1}(x)dx = x^p J_p(x)$$

Thus

$$\begin{aligned}\int x^4 J_1(x)dx &= \int x^2\left[x^2 J_1(x)\right]dx \\ &= x^2\left[x^2 J_2(x)\right] - \int 2x^3 J_2(x)dx \\ &= x^4 J_2(x) - 2x^3 J_3(x) \\ &= x^4 J_2(x) - 2x^3\left[\frac{4}{x}J_2(x) - J_1(x)\right] \\ &= (x^4 - 8x^2)\left(\frac{2}{x}J_1(x) - J_0(x)\right) + 2x^3 J_1(x) \\ &= \left(4x^3 16x\right)J_1(x) - (x^4 - 8x^2)J_0(x)\end{aligned}$$

plus, of course, a constant of integration.

3.4 The Generating Function

Bessel functions are often *defined* through the generating function G(z,t) which is given by the following equation

$$G(z,t) = \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] \tag{37}$$

For every finite value of z, the function G(z,t) is a regular function of t for all (real or complex) values of t except at point t = 0. Thus it can be expanded in a Laurent series

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(z) \tag{38}$$

In the above equation, the coefficient of t^n is defined as $J_n(z)$; we will presently show that this definition is consistent with series given by eq. (3). Now, for any finite value of z and for $0 < |t| < \infty$ we may write

$$\exp\left[\frac{zt}{2}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{zt}{2}\right)^n$$
$$= 1 + \frac{z}{2}\frac{1}{1!} + \left(\frac{z}{2}\right)^2\frac{t^2}{2!} + \left(\frac{z}{2}\right)^3\frac{t^3}{3!} + \ldots \quad (39)$$

and

$$\exp\left[-\frac{z}{2t}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{z}{2t}\right)^n$$
$$= 1 - \frac{z}{2}\frac{1}{t} + \left(\frac{z}{2}\right)^2\frac{1}{2!t^2} - \left(\frac{z}{2}\right)^3\frac{1}{3!t^3} + \ldots \quad (40)$$

Thus the generating function can be expressed as a series of the form

$$G(z,t) = \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} A_n(z) t^n \quad (41)$$

or

$$\sum_{n=-\infty}^{+\infty} A_n(z)t^n = \left[1 + \frac{z}{2}\frac{1}{1!} + \left(\frac{z}{2}\right)^2\frac{t^2}{2!} + \left(\frac{z}{2}\right)^3\frac{t^3}{3!} + \ldots\right]$$
$$\times\left[1 - \frac{z}{2}\frac{1}{1!t} + \left(\frac{z}{2}\right)^2\frac{1}{2!t^2} - \left(\frac{z}{2}\right)^3\frac{1}{3!t^3} + \ldots\right] \quad (42)$$

On the other hand, the coefficient of t^0 will be given by

$$A_0(z) = 1 - \left(\frac{z}{2}\right)^2\frac{1}{(1!)^2} + \left(\frac{z}{2}\right)^4\frac{1}{(2!)^2} - \left(\frac{z}{2}\right)^6\frac{1}{(3!)^2} + \ldots \quad (43)$$

Comparing the above equation with eq. (8), we find

$$A_0(z) = J_0(z)$$

Similarly, the coefficient of t^n on the right hand side of eq. (42) will be given by

$$A_n(z) = \left(\frac{z}{2}\right)^n\frac{1}{n!} + \left(\frac{z}{2}\right)^{n+2}\frac{1}{(n+1)!1!} + \left(\frac{z}{2}\right)^{n+4}\frac{1}{(n+2)!} - \ldots$$

which when compared with eq. (7) gives us

$$A_n(z) = J_n(z)$$

Proving

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(z)$$

In the above equation, if we replace t by -1/y, we obtain

$$\exp\left[\frac{z}{2}\left(y - \frac{1}{y}\right)\right] = \sum_{n=-\infty}^{+\infty}(-1)^n y^{-n} J_n(z) = \sum_{n=-\infty}^{+\infty} y^n J_n(z)$$

Thus

$$J_n(z) = (-1)^n J_{-n}(z)$$

3.8.1 Derivation of the Recurrence Relations from the Generating Function

Differentiating eq. (38) w.r.t z, we obtain

$$\frac{1}{2}\left(t-\frac{1}{t}\right)\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n'(z) \tag{44}$$

Thus

$$\sum_{n=-\infty}^{+\infty} t^{n+1} J_n(z) - \sum_{n=-\infty}^{+\infty} t^{n-1} J_n(z) = \sum_{n=-\infty}^{+\infty} t^n 2J_n'(z)$$

Comparing the coefficients of t^n, we obtain

$$J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z)$$

Similarly, if we differentiate eq. (38) w.r.t t we will obtain

$$\frac{z}{2}\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{+\infty} t^n J_n(z) = \sum_{n=-\infty}^{+\infty} nt^{n-1} J_n(z)$$

Comparing the coefficients of t^{n-1}, we get

$$z\left[J_{n-1}(z) - J_{n+1}(z)\right] = 2nJ_n(z)$$

3.9 Some Useful Integrals

Using $J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos\left[x\sin\theta - n\theta\right] d\theta$

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta) d\theta \tag{45}$$

Thus

$$\int_0^{\infty} e^{-\alpha x} J_0(x) dx = \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-\alpha x}\frac{e^{ix\sin\theta} + e^{-ix\sin\theta}}{2} dx\right] d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left[\frac{1}{\alpha - i\sin\theta} + \frac{1}{\alpha + i\sin\theta}\right] d\theta$$

$$= \frac{2\alpha}{\pi}\int_0^{\pi/2}\frac{d\theta}{\alpha^2 + \sin^2\theta} \tag{46}$$

or

$$\int_0^{\infty} e^{-\alpha x} J_0(x) dx = \frac{1}{\sqrt{1+\alpha^2}} \tag{47}$$

where in evaluating the integral on the right hand side of eq. (46), we have used the substitution $y = \alpha \cot\theta$. By making $\alpha \to 0$, we get

$$\int_0^\infty J_0(x)dx = 1 \qquad (48)$$

From eq. (28), we have

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Thus

$$2\int_0^\infty J_n'(x) = \int_0^\infty J_{n-1}(x) - \int_0^\infty J_{n+1}(x)$$

But

$$\int_0^\infty J_n'(x) = J_n(x)\Big|_0^\infty$$
$$= 0 \quad for \quad n > 0$$

Thus

$$\int_0^\infty J_{n+1}(x) = \int_0^\infty J_{n-1}(x) \qquad n > 0 \qquad (49)$$

Since

$$\int_0^\infty J_1(x) = 1 \qquad \text{[see eq. (35)]}$$

and

$$\int_0^\infty J_0(x) = 1 \qquad \text{[see eq. (48)]}$$

Using eq. (49), we have

$$\int_0^\infty J_n(x) = 1 \qquad n = 0, 1, 2, 3, \ldots \qquad (50)$$

Replacing α by $\alpha + i\beta$ in eq. (47), we get

$$\int_0^\infty e^{-(\alpha+i\beta)x} J_0(x)dx = \frac{1}{\sqrt{(\alpha+i\beta)^2 + 1}} \qquad (51)$$

which in the limit of $\alpha \to 0$ becomes

$$\int_0^\infty e^{-i\beta x} J_0(x)dx = \frac{1}{\sqrt{1-\beta^2}} \qquad (52)$$

For $\beta < 1$, the right hand side is real and we have

$$\int_0^\infty J_0(x)\cos\beta x dx = \frac{1}{\sqrt{1-\beta^2}} \qquad (53)$$

and

$$\int_0^\infty J_0(x)\sin\beta x dx = 0$$

Similarly, $\beta > 1$, the right hand side of Eq. (52) is imaginary and we have

$$\int_0^\infty J_0(x)\cos\beta x dx = 0$$

$$\int_0^\infty J_0(x)\sin\beta x dx = \frac{1}{\sqrt{1-\beta^2}} \tag{54}$$

3.10 Spherical Bessel Functions

We start with the Bessel equation eq. (1)] with $n = l + \frac{1}{2}$, i.e.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}\left[x - \left(l+\frac{1}{2}\right)^2\right]y(x) = 0 \tag{55}$$

where

$$l = 0,\ 1,\ 2, \ldots$$

The solutions of eq. (55) are

$$J_{l+\frac{1}{2}}(x) \text{ and } J_{-l-\frac{1}{2}}(x)$$

If we make the transformation

$$f(x) = \frac{1}{\sqrt{x}}y(x) \tag{56}$$

we would readily obtain

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{df}{dx}\right) + \left[1 - \frac{l(l+1)}{x^2}\right]f(x) = 0 \tag{57}$$

The above equation represents the *spherical Bessel* equation. From eqs. (55) and (56) it readily follows that the two independent solutions of eq.(57) are

$$\frac{1}{\sqrt{x}}J_{l+\frac{1}{2}}(x) \text{ and } \frac{1}{\sqrt{x}}J_{-l-\frac{1}{2}}(x)$$

The spherical Bessel functions are defined through the equations

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x) \tag{58}$$

and

$$n_l(x) = (-1)^l\sqrt{\frac{\pi}{2x}}J_{-l-\frac{1}{2}}(x) \tag{59}$$

and represent the two independent solutions of eq. (57). Now, if we define the function

u(x) = xf(x)

then eq. (57) takes the form

$$\frac{d^2u}{dx^2} + \left[1 - \frac{l(l+1)}{x^2}\right]u(x) = 0 \tag{60}$$

The above equation also appears at many places and the general solution is given by

$$u(x) = c_1[xJ_l(x)] + c_2[xn_l(x)] \qquad (61)$$

which also be written in the form

$$u(x) = A_1\left[\sqrt{x}J_{l+\frac{1}{2}}(x)\right] + A_2\left[\sqrt{x}J_{-l-\frac{1}{2}}(x)\right] \qquad (62)$$

For $l = 0$, the solutions of eq. (60) are

$\sin x$ and $\cos x$

Thus, for $l = 0$ the two independent solutions of eq.(57) are

$\frac{\sin x}{x}$ and $\frac{\cos x}{x}$

Indeed if we use the definitions of $j_l(x)$ and $n_l(x)$ given eqs. (58) and (59) respectively, we would readily obtain

$$j_0(x) = \frac{\sin x}{x} \qquad (63)$$

$$n_0(x) = \frac{\cos x}{x} \qquad (64)$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad (65)$$

$$n_1(x) = \frac{\cos x}{x2} - \frac{\sin x}{x} \quad \text{etc} \qquad (66)$$

Further, if we multiply the recurrence relation [Eq. (21)]

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

by $\sqrt{\frac{\pi}{2x}}$ and assume $n = l - \frac{l}{2}$, we would get

$$j_l(x) = \frac{(2l-1)}{x}n_{l-1}(x) - n_{l-2}(x) \qquad (67)$$

using which we can readily obtain analytic expression for $j_2(x), j_3(x), \ldots$ etc.

Similarly,

$$n_l(x) = \frac{(2l-1)}{x}n_{l-1}(x) - n_{l-2}(x) \qquad (68)$$

3.11 Bessel Functions of the Second Kind: Y_n

The Bessel functions of the second kind, denoted by $Y_n(x)$**,** are solutions of the Bessel differential equation. They have a singularity at the origin (x = 0). $Y_n(x)$ is sometimes also called the **Neumann function.** For non-integer n, it is related to $J_n(x)$ by:

$$Y_n(x) = \frac{J_n(x)\cos\mu\pi - J_{-n}(x)}{\sin\mu\pi} \tag{69}$$

or

$$Y_n(x) = \frac{1}{n}\left[\frac{\partial}{\partial\mu}J_\mu(x) - (-)^n\frac{\partial}{\partial\mu}J_{-\mu}(x)\right]_{\mu=n} \tag{70}$$

We need to show now that $Y_n(x)$ defined by eq.(70) satisfies Eq.(1) where n is either zero or an integer. We know that

$$J''_\mu(x) + \frac{1}{x}J'_\mu(x) + \left(1 - \frac{\mu^2}{x^2}\right)J_\mu(x) = 0 \tag{71}$$

for any value of μ. Differentiating the above equation with respect to μ, we get

$$\frac{d^2}{dx^2}\frac{\partial J_\mu(x)}{\partial\mu} + \frac{1}{x}\frac{d}{dx}\frac{\partial J_\mu(x)}{\partial\mu} + \left(1 - \frac{\mu^2}{x^2}\right)\frac{\partial J_\mu(x)}{\partial\mu} = \frac{2\mu}{x^2}J_\mu(x) \tag{72}$$

Similarly

$$\frac{d^2}{dx^2}\frac{\partial J_{-\mu}(x)}{\partial\mu} + \frac{1}{x}\frac{d}{dx}\frac{\partial J_{-\mu}(x)}{\partial\mu} + \left(1 - \frac{\mu^2}{x^2}\right)\frac{\partial J_{-\mu}(x)}{\partial\mu} = \frac{2\mu}{x^2}J_{-\mu}(x) \tag{73}$$

From eqs. (72) and (73), it is easy to show that

$$\frac{d^2}{dx^2}S_\mu(x) + \frac{1}{x}\frac{d}{dx}S_\mu(x) + \left(1 - \frac{\mu^2}{x^2}\right)S_\mu(x)$$
$$= \frac{2\mu}{x^2}\left[J_\mu(x) - (-1)^n J_{-\mu}(x)\right] \tag{74}$$

where

$$S_\mu(x) = \frac{\partial}{\partial\mu}J_\mu(x) - (-1)^n\frac{\partial}{\partial\mu}J_{-\mu}(x) \tag{75}$$

Thus $Y_n(x)$ is the second solution of Bessel's equation for all real values of n and is known as the Bessel function of the second kind of order n. The general solution of eq.(1) can, therefore, be written as

$$y = C_1 J_n(x) + C_2 Y_n(x) \tag{76}$$

where C_1 and C_2 are arbitrary constants.

The expression for $Y_n(x)$ for $n = 0,\ 1,\ 2, \ldots$ can be obtained by using eqs. (2) and (70) and is given below

$$Y_n(x) = \frac{2}{\pi}\left(\ln(x/2)+\gamma\right)J_n(x) - \frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x^2}{4}\right)^r$$

$$-\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{r=0}^{\infty}(-1)^r\frac{(x^2/4)^r}{r!(n+r)!}\left[\varphi(r)+\varphi(r+n)\right] \qquad (77)$$

Where $\varphi(r) = \sum_{s=1}^{m} s^{-1}; \quad \varphi(0) = 0$

and $\gamma = \lim_{n\to\infty}\left[\varphi(n) - \ln n\right]$

Example 3

In this example we will solve the radial part of the Schrodinger equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\frac{2\mu E}{\hbar} - \frac{l(l+1)}{r^2}\right)R(x) = 0; \quad l = 0, 1, 2, \qquad (78)$$

in the region $0 < r < a$ subject to the following boundary conditions that

$$R(a) = 0 \qquad (79)$$

and $R(r)$ is finite in the region $0 < r < a$. Equation (78) can be conveniently written in the form

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left(1 - \frac{l(l+1)}{\rho^2}\right)R(\rho) = 0$$

Where

$\rho = kr; \quad k = (2\mu E/\hbar^2)^{1/2}$

Thus the general solution of the above equation is given by

$$R(\rho) = Aj_l(\rho) + Bn_l(\rho) \qquad (80)$$

But $n_l(\rho)$ diverges at $\rho = 0$, therefore, we must choose B=0. The boundary condition R(a)=0 leads to the transcendental equation

$$j_l(ka) = 0 \qquad (81)$$

Thus, for $l = 0$, we have

$$ka = n\pi; \; n = 1, \; 2, \; . \; . \; . \qquad (82)$$

Which will give allowed values of k. Similarly, for $l = 1$, we get

$$\tan ka = ka \qquad (83)$$

3.12 Modified Bessel Functions

If we replace x by ix in eq. (1), we obtain

$$x^2 y'' + xy' - (x^2 + n^2)y = 0 \qquad (84)$$

The two solutions of the above equation will obviously be
$J_n(ix)$ and $Y_n(ix)$

As these functions are real for all values of n, let us define a real function as

$$I_n(x) = i^{-n} J_n(ix) \tag{85}$$

or

$$I_n(x) = \sum_{r=0}^{\infty} \frac{(x/4)^{n+2r}}{r!(n+r+1)!} \tag{86}$$

This function will be the solution of eq. (84) and is known as the Modified Bessel function of the first kind. For very large values of x

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \tag{87}$$

The other solution known as the Modified Bessel function of the second kind is defined as

$$K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi} \tag{88}$$

For non-integer values of n, I_n and I_{-n} are linearly independent and as such $K_n(x)$ is a linear combination of these functions [compare with eq. (69) which gives the definition of $Y_n(x)$]. When n is an integer, it can be shown [see eq. (86)] that

$$I_{-n} = I_n \tag{89}$$

and therefore $K_n(x)$ becomes indeterminate for n = 0 or an integer. As in the case of $Y_n(x)$ for n = 0 or an integer, we define $K_n(x)$ as

$$K_n(x) = \lim_{\mu \to n} \left[\frac{\pi}{2} \frac{I_{-\mu}(x) - I_\mu(x)}{\sin \mu\pi} \right] \tag{90}$$

or

$$K_n(x) = \frac{(-1)^n}{2} \left[\frac{\partial I_{-\mu}(x)}{\partial \mu} - \frac{\partial I_\mu(x)}{\partial \mu} \right]_{\mu=n} \tag{91}$$

For x very large

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \tag{92}$$

From eq. (88) it follows that

$$K_{-n}(x) = K_n(x) \tag{93}$$

Which is true for all values of n. recurrence relations for I_n can be derived from those of $J_n(x)$ and Eq. (85). They are

$xI'_n(x) = xI_{n-1}(x) - nI_n(x)$

(94) $xI'_n(x) = nI_n(x) + xI_{n+1}(x)$

95) $I_{n-1}(x) + I_{n+1}(x) = 2I'_n(x)$ (96)

and similarly

$$xK'_n = (nK_n - xK_{n-1}) \quad (97)$$

$$xK'_n = nK_n + xK_{n+1} \quad (98)$$

$$K_{n-1} + K_{n+1} = -2K'_n \quad (99)$$

Example 4

In this example we will consider the solutions of the equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[\left(k_0^2 n^2(r) - \beta^2\right)r^2 - l^2\right]R(r) = 0 \qquad l = 0,\ 1, \ldots \quad (100)$$

Where
$$n(r) = n_1 \qquad 0 < r < a$$
$$= n_2 \qquad r > a \quad (101)$$

and $n_2 < n_{1;}\ k_0(\omega / c)$ represents the free space wave number. The quantity β represents the propagation constant and for guided modes β^2 takes discrete values in the domain

$$k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2 \quad (102)$$

Thus, in the regions $0 < r < a$ and $r > a$, eq. (100) can be written in the form

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[U^2\frac{r^2}{a^2} - l^2\right]R(r) = 0 \qquad 0 < r < a. \quad (103)$$

and

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[W^2\frac{r^2}{a^2} + l^2\right]R(r) = 0 \qquad r > a. \quad (104)$$

where

$$U^2 = a^2\left[k_0^2 n_1^2 - \beta^2\right] \quad (105)$$

and

$$W^2 = a^2\left[\beta^2 - k_0^2 n_2^2\right] \quad (106)$$

so that

$$V^2 = U^2 + W^2 = a^2 k_0^2(n_1^2 - n_2^2) \quad (107)$$

is a constant. The solutions of Eq. (103) are

$$J_l\left(U\frac{r}{a}\right) \text{ and } Y_l\left(U\frac{r}{a}\right) \quad (108)$$

and the latter solution has to be rejected as it diverges at r = 0. Similarly, the solutions of eq. (104) are

$$K_l\left(W\frac{r}{a}\right) \text{ and } I_l\left(W\frac{r}{a}\right)$$

and the second solution has to be rejected because it diverges as $r \to \infty$. Thus

$$\text{and } R(r) = \begin{cases} \dfrac{A}{J_l(U)} & J_l\left(U\dfrac{r}{a}\right) \quad 0 < r < a \\ \dfrac{A}{K_l(W)} & K_l\left(W\dfrac{r}{a}\right) \quad r > a \end{cases} \qquad (109)$$

where the constants have been so chosen and R(r) is continuous at r = a. Continuity of dR/dr at r = a gives us

$$U\frac{J_l'(U)}{J_l(U)} = WU\frac{K_l'(U)}{K_l(U)} \qquad (110)$$

which is the fundamental equation determining the eigenvalues β / k_0.

SELF-ASSESSMENT EXERCISE

3. Using
   $J_0(2) = 0.22389,\quad J_1(2) = 0.57672,\quad \textit{calculate } J_2(2),\quad J_3(2),\quad \textit{and } J_4(2).$

   Hint: Use Eq. (21)

4. Show that
   $$\int_0^a J_n^2(x)xdx = \frac{1}{2}a^2J_n^2(a)\left[1 - \frac{J_{n-1}(a)J_{n+1}(a)}{J_n^2(a)}\right]$$

4.0 CONCLUSION

In this unit, we have considered Bessel function and spherical Bessel function.

We have also established in this unit, relationship between the recurrence relation and the generating function.

5.0 SUMMARY

This unit is on Bessel functions. It has a lot of application that arises in numerous diverse areas of applied mathematics. This unit will be of significant importance in the subsequent course in quantum mechanics.

6.0 TUTOR- MARKED ASSIGNMENT

1. Using

$$J_1(2) = 0.57672,\ J_2(2) = 0.35283 \quad \textit{calculate}\ J_3(2),\quad J_4(2),\quad \textit{and}\ \ J_5(2).$$

Hint: Use Eq. (21)

2. Using the integral

$$\int_0^1 (1-x^2)^m x^{2n+2r+1} dx = \frac{\Gamma(n+r+1)\Gamma(m+1)}{2\Gamma(m+n+r+2)}; \qquad m > -1,\ n > -1$$

Prove that

$$J_{n+m+1}(x) = \frac{2}{\Gamma(m+1)}\left(\frac{x}{2}\right)^{m+1} \int_0^1 (1-y^2)^m y^{n+1} J_n(xy) dy$$

3. *Hint:* Use the expansion given by eq. (2) and integrate term by term.

In problem 2 assume $m = n = -\frac{1}{2}$, and use eq. (12) to deduce

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xy}{\sqrt{1-y^2}} dy$$

4. Show that the solution of the differential equation

$$y''(x) + (ae^x - b)y(x) = 0$$

is given by $y(x) = AJ_\mu(\xi) + BJ_\mu(\xi);\qquad \xi = 2\sqrt{a}e^{x/2};\ \ \mu = 2\sqrt{b}$

7.0 REFERENCES/FURTHER READING

Erwin, Kreyszig (1991). *Advanced Engineering Mathematics*. John Wiley & Sons, Inc.

Arfken, G. (1990). *Mathematical Methods for Physicists*. New York: Academic Press

MODULE 3 APPLICATION OF FOURIER TO PDES (HERMITE POLYNOMIALS AND LAGUERRE POLYNOMIALS)

UNIT 1 HERMITE POLYNOMIALS

CONTENTS

4.0 INTRODUCTION

In this unit, we shall consider certain boundary value problems whose solutions form orthogonal set of functions. It can also be seen in this unit how the generating function can readily be used to derive the Rodrigues' formula.

5.0 OBJECTIVES

At the end of this unit, you should be able to:

- define Hermite polynomials as the polynomial solutions of the Hermite differential equation
- prove the Orthogonality of Hermite polynomials
- derive the Rodrigues' formula which can be used to obtain explicit expressions for Hermite polynomials
- solve the exercises at the end of this unit.

6.0 MAIN CONTENT

3.1 Hermite Differential Equation

The equation

$$y''(x) - 2xy'(x) + (\lambda - 1)y(x) = 0 \qquad (1)$$

where λ is a constant is known as the ***Hermite*** differential equation. When λ is an odd integer, i.e. when

$$\lambda = 2n + 1; \qquad n = 0,\ 1,\ \ 2,\ .\ .\ . \qquad (2)$$

One of the solutions of eq. (1) becomes a polynomial. These polynomial solutions are called ***Hermite polynomials.*** Hermite polynomials appear in many diverse areas, the most important being the harmonic oscillator problem in quantum mechanics.

Using Frobenius method to solve eq.(1), and following the various steps, we have

Step1: We substitute the power series

$$y(x) = \sum_{r=0}^{\infty} C_r x^{p+r} \qquad (3)$$

in eq. (1) and obtain the identity

$$C_0 p(p-1) + C_1(p+1)px + \sum_{r=2}^{\infty}[C_r(p+r)(p+r-1) - C_{r-2}(2p+2r-3-\lambda)]x^r = 0$$

Step 2: Equating to zero the coefficients of various powers of x, we obtain

(i) $p = 0 \quad \text{or} \quad p = 1$ (4a)

(ii) $p(p+1)C_1 = 0$ (4b)

(iii) $C_r = \dfrac{2p + 2r - 3 - \lambda}{(p+r)(p+r-1)} C_{r-2} \qquad \textit{for } r \geq 2$ (4c)

When $p = 0, C_1$ becomes indeterminate; hence p = 0 will yield both the linearly independent solutions of eq. (1). Thus, we get

$$C_r = \frac{2r - 3 - \lambda}{r(r-1)} C_{r-2} \qquad \textit{for } r \geq 2 \qquad (5)$$

which gives

$$C_2 = \frac{1-\lambda}{2!} C_0 \qquad .$$

$$C_3 = \frac{3-\lambda}{3!} C_1,\ .\ .$$

$$C_4 = \frac{(1-\lambda)(5-\lambda)}{4!} C_0$$
$$C_5 = \frac{(3-\lambda)(7-\lambda)}{5!} C_1, \ . \ . \ . \text{etc}$$

Because $C_2, C_4, . \ . \ .$ are related to C_0 and $C_3, C_5, . \ . \ .$ are related to C_1, we can split the solution into even and odd series. Thus, we may write

$$\begin{aligned} y(x) &= (C_0 + C_2x^2 + C_4x^4 + . \ . \ .) + (C_1x + C_3x^3 + + . \ . \ .) \\ &= C_0\left[1 + \frac{1-\lambda}{2!}x^2 + \frac{(1-\lambda)(1-\lambda)}{4!}x^4 + . \ . \ .\right] \\ &+ C_1\left[x + \frac{(3-\lambda)}{3!}x^3 + \frac{(3-\lambda)(7-\lambda)}{5!}x^5 + . \ . \ .\right] \end{aligned} \quad (6)$$

It may be readily seen that when

$$\lambda = 1, \ 5, \ 9, \ . \ . \ .$$

the even series becomes a polynomial and the odd series remains an infinite series. Similarly, for

$$\lambda = 3, \ 7, \ 11, \ . \ . \ .$$

the odd series becomes a polynomial and the even series remains an infinite series. Thus, when

$$\lambda = 2n+1; \qquad n = 0, \ 1, \ 2, \ . \ . \ .$$

One of the solutions becomes a polynomial. If the multiplication constant C_0 or C_1 is chosen that the coefficient of the highest power of x in the polynomial becomes 2^n, then these polynomials are known as Hermite polynomials of order n and are denoted by $H_n(x)$. For example, for $\lambda = 9 \ (n = 4)$, the polynomial solution

$$y(x) = C_0\left[1 - 4x + \frac{4}{3}x^4\right]$$

If we choose

$$C_0 = 12$$

the coefficient of x^4 becomes 2^4 and, therefore

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Similarly,

for $\lambda = 7 \ (n = 3)$, the polynomial solution is given by

$$y(x) = C_1\left[x - \frac{2}{3}x^3\right]$$

Choosing

$$C_1 = -12$$

we get

$$H_3(x) = 8x^3 - 12x$$

In general

$$H_n(x) = \sum_{r=0}^{N} \frac{n!(2x)^{n-2r}}{r!(n-2r)!} \tag{7}$$

where

$$N = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Using eq. (7) one can obtain Hermite polynomials of various orders, the first few are given below:

$$\left.\begin{array}{l} H_0(x) = 1; \quad H_1(x) = 2x; \quad H_2(x) = 4x^2 - 2; \\ H_3(x) = 8x^3 - 12x; \quad H_4(x) = 16x^4 - 48x^2 + 12 \end{array}\right\} \tag{8}$$

Higher order Hermite polynomials can easily be obtained either by using eq. (7) or by using the recurrence relation (see eq. 20)

3.5 The Generating Function

The generating function for Hermite polynomials is given by

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n \tag{9}$$

Expanding e^{-t^2} and e^{2xt} in power series, we have

$$e^{-t^2} = 1 - t^2 + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \ldots$$

$$e^{2xt} = 1 + (2x)t + \frac{(2x)^2}{2!}t^2 + \frac{(2x)^3}{3!}t^3 + \ldots$$

Multiplying the above two series, we shall obtain a power series in t with

Coefficient of $t^0 = 1 \qquad = \frac{1}{0!}H_0(x)$

" " $t = 2 \qquad = \frac{1}{1!}H_1(x)$

" " $t^2 = 2x^2 - 1 \qquad = \frac{1}{2!}H_2(x)$ etc

It is also evident that the coefficient of t^2 in the multiplication of the two series will be a polynomial of degree n and will contain odd powers when n is odd and even powers when n is even. In this polynomial, the coefficient of x^n can easily be seen to be $(2^n / n!)$. We then assume that

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty}\frac{1}{n!}K_n(x)t^n \qquad (10)$$

Where $K_n(x)$ is a polynomial of degree n. Differentiating eq. (10) with respect to t, we get

$$(2x-2t)e^{-t^2+2xt} = \sum_{n=0}^{\infty}\frac{n}{n!}K_n(x)t^{n-1} = \sum_{n=0}^{\infty}\frac{1}{(n-1)!}K_n(x)t^{n-1}$$

or

$$2(x-t)\sum_{n=0}^{\infty}\frac{1}{n!}K_n(x)t^n = \sum_{n=0}^{\infty}\frac{1}{n!}K_{n+1}(x)t^n \qquad (11)$$

Comparing the coefficients of t^n on both sides of eq. (11), we obtain

$$2xK_n(x) - 2nK_{n-1}(x) = K_{n+1}(x) \qquad (12)$$

We next differentiate eq.(10) with respect to x to obtain

$$2t\sum_{n=0}^{\infty}\frac{1}{n!}K_n(x)t^n = \sum_{n=0}^{\infty}\frac{1}{n!}K'_n(x)t^n \qquad (13)$$

Comparing the coefficients of t^n on both sides of eq. (11), we get

$$K'_n(x) = 2nK_{n-1}(x) \qquad (14)$$

If we replace n by (n+1) in eq.(14), we would get

$$K'_{n+1}(x) = 2(n+1)K_n(x) \qquad (15)$$

Differentiating eqs.(14) and (12) with respect to x, we obtain respectively

$$K''_n(x) = 2nK'_{n-1}(x) \qquad (16)$$

and

$$2xK_n'(x) + 2K_n(x) - 2nK_{n-1}'(x) = K_{n+1}'(x) \qquad (17)$$

Subtracting eqs. (17) and (16) and using (15), we get

$$K_n''(x) - 2xK_n'(x) + 2nK_n(x) = 0 \qquad (18)$$

which shows that $K_n(x)$ is a solution of the Hermite equation (1) with $\lambda = 2n+1$, i.e. of the equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0 \qquad (19)$$

Since, as discussed before, $K_n(x)$ is also a polynomial of degree n (with coefficient of x^n equal to 2^n), $K_n(x)$ is, therefore, nothing but $H_n(x)$. Equations (12) and (14), thus, give recurrence relations for $H_n(x)$

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \qquad (20)$$

and

$$H_n'(x) = 2nH_{n-1}(x) \qquad (21)$$

3.6 Rodrigues Formula

In the preceding section we have shown that

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n \qquad (22)$$

One can rewrite the generating function G(x, t) in the form

$$G(x,t) = e^{x^2} e^{-(t+x)^2}$$

It may be easily seen that

$$\frac{\partial^n G}{\partial t^n} = e^{x^2} (-1)^n \frac{\partial^n}{\partial x^n} e^{-(t-x)^2} \qquad (23)$$

From eq. (22) it follows that

$$\left.\frac{\partial^n G}{\partial t^n}\right|_{t=0} = H_n(x) \qquad (24)$$

Using eqs (23) and (24), we obtain

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (25)$$

which is known as **Rodrigues formula** for Hermite polynomials. For example,

$$\begin{aligned} H_2(x) &= e^{x^2} \frac{d^2}{dx^2} e^{-x^2} = e^{x^2} \frac{d}{dx}(-2xe^{-x^2}) \\ &= e^{x^2}\left[-2e^{-x^2} + 4x^2 e^{-x^2}\right] \\ &= 4x^2 - 2 \end{aligned}$$

Which is consistent with eq. (8). Similarly, we can determine other Hermite polynomials by elementary differentiation of eq. (25).

3.7 Orthogonality of Hermite Polynomials

The Hermite polynomials satisfy eq.(1) for $\lambda = 2n+1$. Thus, we have

$$\frac{d^2 H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n(x) = 0 \tag{26}$$

In order to derive the Orthogonality condition we transform eq. (26) to the Sturm-Liouville form by multiplying it by

$$\exp\left[-\int 2x dx\right] = e^{-x^2} \tag{27}$$

to obtain

$$\frac{d}{dx}\left[e^{-x^2}\frac{dH_n}{dx}\right] = -2n\left[e^{-x^2}H_n(x)\right] \tag{28}$$

Similarly

$$\frac{d}{dx}\left[e^{-x^2}\frac{dH_m}{dx}\right] = -2m\left[e^{-x^2}H_m(x)\right] \tag{29}$$

We multiply eq.(28) by $H_m(x)$ and eq.(29) by $H_n(x)$, subtract them and integrate the resulting equation with respect to x from $-\infty$ to ∞ to obtain

$$\int_{-\infty}^{+\infty}\left\{H_m(x)\frac{d}{dx}\left[e^{-x^2}\frac{dH_n}{dx}\right] - H_n(x)\frac{d}{dx}\left[e^{-x^2}\frac{dH_m}{dx}\right]\right\}dx$$

$$= 2(m-n)\int_{-\infty}^{+\infty} e^{-x^2}H_m(x)H_n(x)dx$$

Now

$$\text{LHS} = \int_{-\infty}^{+\infty}\frac{d}{dx}\left\{H_m(x)\left[e^{-x^2}\frac{dH_n}{dx}\right] - H_n(x)\left[e^{-x^2}\frac{dH_m}{dx}\right]\right\}dx$$

$$= \left[H_m(x)e^{-x^2}\frac{dH_n}{dx} - H_n(x)e^{-x^2}\frac{dH_m}{dx}\right]_{-\infty}^{+\infty}$$

$$= 0$$

Thus

$$\int_{-\infty}^{+\infty} e^{-x^2}H_m(x)H_n(x)dx = 0; \qquad m \neq n \tag{30}$$

which shows that the Hermite polynomials are Orthogonal with respect to the weight function e^{-x^2}. Thus if we define the functions

$$\phi_n(x) = N_n e^{-x^2/2}H_n(x); \qquad n = 0, \ 1, \ 2, \ldots \tag{31}$$

then eq. (30) assumes the form

$$\int_{-\infty}^{+\infty}\phi_m(x)\phi_n(x)dx = 0; \qquad m \neq n \tag{32}$$

3.5 The Integral Representation of the Hermite Polynomials

The integral representation of the Hermite polynomial is given by

$$H_n(x) = \frac{2^n(-i)^n}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} t^n e^{-t^2+2ixt} dt \qquad (33)$$

In order to prove the above relation we start with the relation

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} e^{-t^2+2ixt} dt$$

which can easily be obtained from the well known formula

$$\int_{-\infty}^{+\infty} e^{-\alpha t^2+\beta t} dt = \sqrt{\frac{\pi}{\alpha}} \exp\left[\frac{\beta^2}{4\alpha}\right]$$

by assuming $\alpha = 1$and $\beta = 2ix$. Now according to the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$= (-1)^n e^{x^2} \frac{1}{\sqrt{\pi}} \frac{d^n}{dx^n} \int_{-\infty}^{+\infty} e^{-t^2+2ixt} dt$$

$$= (-1)^n \frac{1}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{+\infty} (2i)^n t^n e^{-t^2+2ixt} dt$$

from which eq. (50) readily follows.

3.6 Fourier Transform of Hermite-Gauss Functions

In this section we will show that

$$e^{-x^2/2} H_n(x) = \frac{1}{i^n \sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[e^{k^2/2} H_n(k)\right] e^{ikx} dk \qquad (34)$$

Implying that the Fourier transform of the Hermite-Gauss function is a Hermite-Guass function. In order to prove eq. (34) we start with the generating function

$$G(x,t) = e^{2kt-t^2} = \sum_{n=0,1,\ldots}^{\infty} \frac{1}{n!} H_n(k) t^n$$

We multiply the above by $\left(ikx - \frac{1}{2}k^2\right)$ and integrate over k to obtain

$$e^{-t^2} \int_{-\infty}^{+\infty} e\left[-\frac{1}{2}k^2 + (2t+ix)k\right] dk = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{+\infty} H_n(x) e^{-k^2/2} e^{ikx} dk \qquad (35)$$

Now

$$\text{LHS} = e^{-t^2}\sqrt{2\pi}\exp\left[\frac{(2t+ix)^2}{2}\right]$$

$$= \sqrt{2\pi}e^{t^2+2ixt}e^{-x^2/2}$$

$$= \sqrt{2\pi}e^{-x^2/2}\sum_n \frac{H_n(x)}{n!}(it)^n$$

Comparing coefficients of t^n on both sides of eq. (35), we get eq. (34).

3.7 Some Important Formulae Involving Hermite Polynomials

$$H_n(x+y) = 2^{-n/2}\sum_p^n \frac{n!}{p!(n-p)!}H_{n-p}(x\sqrt{2})H_p(y\sqrt{2}) \qquad (36)$$

$$H_n(x) \underset{n\to\infty}{\rightarrow} \sqrt{2}\left(\frac{2n}{e}\right)^{n/2} e^{x^2/2}\cos\left(\sqrt{(2n+1)}x - \frac{n\pi}{2}\right) \qquad (37)$$

$$\left.\begin{aligned} x^{2s} &= \frac{(2s)!}{2^{2s}}\sum_{n=0,1,\ldots}^{s}\frac{H_{2s-2n}(x)}{n!(2s-2n)!} \\ x^{2s+1} &= \frac{(2s+1)!}{2^{2s+1}}\sum_{n=0,1,\ldots}^{s}\frac{H_{2s+1-2n}(x)}{n!(2s+1-2n)!}\end{aligned}\right\} \quad s = 0,\ 1,\ 2,\ldots \qquad (38)$$

SELF-ASSESSMENT EXERCISE

Using the generating function for $H_n(x)$, show that

(a) $\frac{1}{e}\cosh 2x = \sum_{n=0,1,\ldots}^{\infty}\frac{1}{(2n)!}H_{2n}(x)$

(b) $\frac{1}{e}\sinh 2x = \sum_{n=0,1,\ldots}^{\infty}\frac{1}{(2n+1)!}H_{2n+1}(x)$

(c) $e\cos 2x = \sum_{n=0,1,\ldots}^{\infty}(-1)^n\frac{1}{(2n)!}H_{2n}(x)$

(d) $e\sin 2x = \sum_{n=0,1,\ldots}^{\infty}(-1)^n\frac{1}{(2n+1)!}H_{2n+1}(x)$

Hint: To obtain (a) and (b) substitute t = 1 and t = -1in eq. (9) add and subtract the resulting equations. Similarly for (c) and (d), substitute t = I and equate real and imaginary parts.

Prove that

$$\int_{-\infty}^{+\infty} e^{-x^2}H_{2n}(\alpha, x)dx = \sqrt{n}\frac{(2n)!}{n!}(\alpha^2 - 1)^n$$

Hint: Replace x by αy in eq.(9), multiply the resulting equation by e^{-y^2} and integrate with respect to y.

10.0 CONCLUSION

Here, in this unit, we have dealt with the Hermite polynomials which are Orthogonal with respect to the weight function e^{-x^2} . We have also established that the Fourier transform of the Hermite-Gauss function is a Hermite-Guass function.

11.0 SUMMARY

This unit was on the Hermite polynomials. It has a lot of application in linear harmonic oscillator problem in quantum mechanics. The unit will be of immense importance in the subsequent course in classical mechanics.

12.0 TUTOR- MARKED ASSIGNMENT

1. If two operators are defined as

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$$

$$\bar{a} = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

Show that

$$a\phi_n(x) = \sqrt{n}\phi_{n-1}(x)$$

$$\bar{a}\phi_n(x) = \sqrt{n}\phi_{n-1}(x)$$

2. Prove that

$$\int_{-\infty}^{+\infty} H_n\left(x + \frac{1}{2}x_0\right)e^{-x^2/2}dx = \sqrt{n}\, x_0^n$$

Hint: Multiply Eq. (9) by $\left[-\left(x + \frac{1}{2}x_0\right)^2\right]$ and integrate over x.

7.0 REFERENCES/FURTHER READING

Erwin, K. (1991). *Advanced Engineering Mathematics.* John Wiley & Sons, Inc.

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UNIT 2 LAGUERRE POLYNOMIALS

CONTENTS

1.0 INTRODUCTION

In the previous unit, you came across solutions of orthogonal set of functions. This unit which is the last one in this book will examine critically how a Laguerre differential equation can be transformed to Sturm-Liouville form.

It shows that Laguerre polynomials and the associated functions arise in many branches of physics, e.g. in the hydrogen atom problem in quantum mechanics, in optical fibers characterised by parabolic variation of refractive index, etc.

We also show that Laguerre polynomials are orthogonal in the interval $0 \le x \le \infty$ with respect to the weight function e^{-x}.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- use Frobenius method to obtain the polynomial solution of the Laguerre differential equations
- determine the Orthogonality of the Laguerre polynomials
- derive the Rodrigues formula
- derive the second solution of the Laguerre differential equation.

3.0 MAIN CONTENT

3.1 Laguerre Differential Equation

The equation

$$xy''(x) - (1-x)y'(x) + ny(x) = 0 \qquad (1)$$

where n is a constant known as the ***Laguerre*** differential equation. When $n = 0,\ 1,\ 2, . . .$ (2)

One of the solutions of eq. (1) becomes a polynomial. These polynomial solutions are known as the ***Laguerre polynomials.***

Using Frobenius method to solve eq.(1), and following the various steps, we have

Step We substitute the power series

$$y(x) = \sum_{r=0}^{\infty} C_r x^{p+r}, \qquad C_0 \neq 0$$

Eq. (1) and obtain the identity

$$\sum_{r=0}^{\infty} C_r (p+r)^2 x^{p+r-1} - \sum_{r=0}^{\infty} C_r (p+r-n) x^{p+r} = 0$$

or

$$C_0 p^2 x^{p-1} - \sum_{r=1}^{\infty} \left[C_r (p+r)^2 - C_{r-1}(p+r-n-1)\right] x^{p+r-1} = 0 \qquad (3)$$

Step 2 Equating to zero the coefficients of various powers of x in the identity (3), we obtain

(i) $p^2 = 0$ INDICIAL EQUATION (4)

(ii) $C_r = \dfrac{p+r-n-1}{(p+r)^2} C_{r-1}$ $r \geq 1$ RECURRENCE RELATION (5)

Substituting $p = 0$ in eq. (5), we get

$$C_r = \frac{r-n-1}{r^2} C_{r-1} \qquad r \geq 1$$

which gives

$$C_1 = -\frac{n}{(1!)^2} C_0 \qquad C_2 = \frac{n(n-1)}{(2!)^2} C_0$$

$$C_3 = \frac{n(n-1)(n-2)}{(3!)^2} C_0 \qquad \text{etc}$$

$$C_n = (-1)^n \frac{n!}{(n!)^2} = \frac{(-1)^n}{n!}$$

and

$$C_{n+1} = C_{n+2} = 0. . . . = 0$$

Therefore one of the solutions of eq. (1) can be written as

$$y(x) = C_0 \left\{ 1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 - \ldots + (-1)^n \frac{x^n}{n!} \right\} \tag{6}$$

which is a polynomial of degree n. If the multiplication constant C_0 is chosen to be unity so that the constant term becomes unity, the polynomial solution given by eq. (6) is known as **Laguerre Polynomial** of degree n and denoted by $L_n(x)$. Thus

$$L_n(x) = 1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 - \ldots + (-1)^n \frac{x^n}{n!}$$

or

$$L_n(x) = \sum_{r=0}^{n} (-1)^n \frac{n!}{(n-r)!(r!)^2} x^r \tag{7}$$

with

$$L_n(0) = 1 \tag{8}$$

The first four Laguerre polynomials can be written as:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 1 - x, \\ L_2(x) &= 1 - 2x + \frac{1}{2} x^2, \\ L_3(x) &= 1 - 3x + \frac{3}{2} x^2 - \frac{1}{6} x^3, \ldots \end{aligned} \tag{9}$$

Higher order polynomials can easily be obtained either by using eq.(7) or by using the recurrence relation [see eq. (20)].

3.2 The Generating Function

The generating function for Laguerre polynomials is given by

$$G(x,t) = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n(x) t^n; \quad |t| < 1 \tag{10}$$

We expand the left hand side of eq. (10) to obtain

$$(1-t)^{-1} \exp\left[-\frac{xt}{1-t} \right]$$

$$= (1-t)^{-1} - xt(1-t)^{-2} + \frac{x^2 t^2 (1-t)^3}{2!} - \ldots$$

$$= (1 + t + t^2 + \ldots) - xt(1 + 2t + 3t^2 + \ldots) + \frac{x^2 t^2}{2!}(1 + 3t + 6t^2 + \ldots) - \ldots \tag{11}$$

The right hand side of eq.(11) can be written as a power series in t with
Coefficient of $t^0 = 1 \qquad = L_0(x)$

" " $t = 1 - x \qquad = L_1(x)$

" " $t^2 = 1 - 2x + x^2/2 \qquad = L_2(x)$

etc. It is also evident that the coefficient of t^2 on the right hand side of eq.(11) will be a polynomial of degree n and that the constant term in this polynomial will be unity. We can then assume that

$$G(x,t) = \frac{1}{1-t}\exp\left\{-\frac{xt}{(1-t)}\right\} = \sum_{n=0}^{\infty} K_n(x)t^n \qquad (12)$$

where $K_n(x)$ is a polynomial of degree n. Differentiating eq. (12) with respect to t, we get

$$\frac{(1-x-t)}{(1-t)^3}\exp\left\{-\frac{xt}{(1-t)}\right\} = \sum_{n=0}^{\infty} nK_n(x)t^{n-1}$$

or

$$(1-x-t)\sum_{n=0}^{\infty} K_n(x)t^n = (1-2t+t^2)\sum_{n=1}^{\infty} nK_n(x)t^{n-1}$$

Comparing the coefficients of t^n on both sides of the above equation, we get

$$(n+1)K_{n+1}(x) - (2n+1-x)K_n(x) + nK_{n-1}(x) = 0; \qquad n \geq 1 \qquad (13)$$

We next differentiate eq.(12) with respect to x to obtain

$$-t\sum_{n=0}^{\infty} K_n(x)t^n = (1-t)\sum_{n=0}^{\infty} K_n'(x)t^n \qquad (14)$$

Comparing the coefficients of t^n on both sides of the above equation, we get

$$K_n'(x) - K_{n-1}'(x) = -K_{n-1}(x) \qquad (15)$$

If we replace n by (n+1) in the above equation, we would get

$$K_{n+1}'(x) = K_n'(x) - K_n(x) \qquad (16)$$

Differentiating eq. (13) with respect to x, we obtain

$$(n+1)K_{n+1}'(x) - (2n+1-x)K_n'(x) + K_n(x) + nK_{n-1}'(x) = 0 \qquad (17)$$

Substituting $K_{n-1}'(x)$ and $K_{n+1}'(x)$ from eqs. (15) and (16) respectively in eq. (17), we get

$$xK_n'(x) = nK_n(x) - nK_{n-1}(x) \qquad (18)$$

Differentiating the above equation with respect to x and using eq. (15), we have

$$xK_n''(x) + K_n'(x) = -nK_{n-1}(x) \qquad (19)$$

Subtracting eq. (18) from eq. (19), we get

$$xK_n''(x) + (1-x)K_n'(x) + nK_n(x) = 0$$

Showing that $K_n(x)$ is a solution of the Laguerre equation, i.e. of the equation

$$xy''(x) - (1-x)y'(x) + ny(x) = 0$$

Hence $K_n(x)$ nothing but $L_n(x)$. Equations (13) and (18) give the following recurrence relations respectively:

$$(n+1)L_{n+1}(x) = (2n+1-1)L_n(x) - nL_{n-1}(x) \qquad (20)$$

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x) \qquad (21)$$

We also have

$$L_n(x) = L_n'(x) - L_{n+1}'(x) \qquad (22)$$

3.3 Rodrigues Formula

In the preceding section we have shown that

$$G(x,t) = \frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n$$

We can write the above equation as

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{1}{1-t}\exp\left[-\frac{x(1-t-1)}{1-t}\right]$$

or

$$\sum_{n=0}^{\infty} L_n(x)t^n = e^x\left[\frac{1}{1-t}\exp\left(-\frac{x}{1-t}\right)\right] \qquad (23)$$

Differentiating eq. (23) n times with respect to t and then putting t = 0, we will have

$$n!L_n(x) = e^x\left[\frac{\partial^n}{\partial t^n}\left[\frac{1}{1-t}\exp\left(-\frac{x}{1-t}\right)\right]\right]\Bigg|_{t=0}$$

$$= e^x\left[\frac{\partial^n}{\partial t^n}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{(1-t)^{r+1} r!}\right\}\right]\Bigg|_{t=0}$$

$$= e^x \left[\sum_{r=0}^{\infty} (-1)^r \frac{(r+1)(r+2)\ldots(r+n)}{(1-t)^{r+n+1} r!} \right]\Bigg|_{t=0}$$

$$= e^x \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)!}{(r!)^2} x^r$$

or

$$L_n(x) = \frac{e^x}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)!}{(r!)^2} x^2 \qquad (24)$$

$$\frac{d^n}{dx^n}(x^n e^{-x}) = \frac{d^2}{dx^2}\left[x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^r}{r!} \right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)(n+r-1)\ldots(r+1)}{r!} x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)!}{(r!)^2} x^r$$

Thus

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) \qquad (25)$$

This is known as **Rodrigues formula** for the Laguerre polynomials. For example, putting n = 2 in the Rodrigues' formula, we have

$$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2}(x^2 e^{-x})$$

$$= \frac{e^x}{2!} \frac{d}{dx}(2x^2 e^{-x} - x^2 e^{-x})$$

$$= \frac{e^x}{2!} \frac{d}{dx}(2e^{-x} - 4xe^{-x} + x^2 e^{-x})$$

$$= 1 - 2x + \frac{x^2}{2}$$

Which is consistent with eq. (9). Similarly, we can determine other Laguerre polynomials by elementary differentiation of the result expressed by eq. (25).

3.4 Orthogonality of Hermite Polynomials

As Laguerre differential equation is not of the form of Sturm-Liouville differential equation, its solutions $L_n(x)$, therefore, do not by themselves form an Orthogonal set. However, in order to transform Laguerre differential equation to the Sturm-Liouville form, we may write eq. (1) as

$$y''(x) - \frac{(1-x)}{x} y'(x) + \frac{n}{x} y(x) = 0$$

Multiplying the above equation by

$$p(x) = \exp\left[\int \frac{1-x}{x}\right] = xe^{-x^2} \tag{26}$$

We obtain

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + ne^{-x}y(x) = 0 \tag{27}$$

Thus for Laguerre polynomials, the Sturm-Liouville form is given by

$$\frac{d}{dx}\left[p(x)\frac{dL_n(x)}{dx}\right] = -ne^{-x}L_n(x) \tag{28}$$

Similarly

$$\frac{d}{dx}\left[p(x)\frac{dL_m(x)}{dx}\right] = -me^{-x}L_m(x) \tag{29}$$

Multiply eq.(28) by $L_m(x)$ and eq.(29) by $L_n(x)$ and subtracting the resulting equations, we obtain

$$L_m(x)\frac{d}{dx}\left[p(x)\frac{dL_n(x)}{dx}\right] - L_m(x)\frac{d}{dx}\left[p(x)\frac{dL_m(x)}{dx}\right]$$
$$= (m-n)L_m(x)L_n(x) \tag{30}$$

The left hand side of eq.(30) is simply

$$\frac{d}{dx}\left[L_m(x)p(x)\frac{dL_n(x)}{dx} - L_m(x)p(x)\frac{dL_m(x)}{dx}\right] \tag{31}$$

Integrating eq.(30) and using eq.(31), we get

$$(m-n)\int_0^{\infty} e^{-x}L_m(x)L_n(x)dx = \left[p(x)\left\{L_m(x)\frac{dL_n(x)}{dx} - L_m(x)\frac{dL_m(x)}{dx}\right\}\right]_0^{\infty}$$

Since p(x) = 0 at x = 0 and at $x = \infty$, the right hand side vanishes and we readily obtain

$$\int_0^{\infty} e^{-x}L_m(x)L_n(x)dx = 0 \qquad for\ m \neq n \tag{32}$$

The above equation shows that the Laguerre polynomials are Orthogonal in the interval $0 \leq x \leq \infty$ with respect to the weight function e^{-x}. We now define the functions

$$\phi_n(x) = N_n L_n(x)e^{-x^2/2} \tag{33}$$

The constant N_n is chosen so that the functions $\phi_n(x)$ are normalised, i.e.

$$\int_0^{+\infty} \phi_n^2(x)dx = 1 \qquad for \quad m = n \tag{32}$$

3.5 The Integral Representation of the Laguerre Polynomials

The integral representation of the Laguerre polynomial is given by

$$L_n(x) = \frac{e^x}{n!}\int_0^{+\infty} e^{-t}t^n dt\, J_0\left[2(xt)^{1/2}\right]dt \tag{33}$$

In order to prove the above relation we start with the relation

$$\int_0^{+\infty} e^{-t}t^n dt\, J_0\left[2(xt)^{1/2}\right]dt$$

$$= \int_0^{+\infty} e^{-t}t^n \sum_{r=0}^{\infty}\frac{(-1)^r (tx)^r}{(r!)^2}dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r (x)^r}{(r!)^2}\int_0^{+\infty} e^{-t}t^{n+r}dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r x^r \Gamma(n+r+1)}{(r!)^2}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r (n+r)!}{(r!)^2}x^r \tag{34}$$

Using eqs. (24) and (33), we get

$$\int_0^{+\infty} e^{-t}t^n dt\, J_0\left[2(xt)^{1/2}\right]dt = e^{-x}n!L_n(x) \tag{35}$$

from which eq. (33) readily follows.

3.6 Some Important Results Involving Laguerre Polynomials

We give some important results involving Laguerre polynomials which can be readily derived:

$$\int_0^x L_n(x)dx = L_n(x) - L_{n+1}(x) \qquad \text{[Use Eq. (22)]} \tag{36}$$

$$\sum_{n=0}^{\infty}\frac{y^n L_n(x)}{n!} = e^y J_0\left[2(xy)^{1/2}\right] \tag{37}$$

$$\int_0^{+\infty} x^m e^{-x}L_n(x)dx = \begin{cases} 0 & if\ m < n \\ (-1)^n n! & if\ m = n \end{cases} \tag{38}$$

$$\sum_{n=0}^{N} L_n(x)L_n(y) = \frac{(N+1)}{x-y}\left[L_N(x)L_{N+1}(y) - L_{N+1}(x)L_N(y)\right]_0 \tag{39}$$

from which eq. (50) readily follows.

3.9 The Second Solution of the Laguerre Differential Equation

Since the indicial equation [eq. (4)] has two equal roots, the two independent solutions of eq. (1)

$$(y)_{p=0} \quad and \quad \left(\frac{\partial y}{\partial p}\right)_{p=0}$$

Now

$$y(x,p) = x^p\left\{1+\frac{p-n}{(p+1)^2}x+\frac{(p-n)(p-n+1)}{(p+1)^2(p+2)^2}x^2+\right.$$
$$\left.\frac{(p-n)(p-n+1)(p-n+2)}{(p+1)^2(p+2)^2(p+3)^3}x^3+\ .\ .\ .\right\} \qquad (40)$$

Thus,

$$y_1(x) = y(x,\ p=0)$$
$$= 1-nx+\frac{n(n-1)}{(2!)^2}x^2-\frac{n(n-1)(n-2)}{(3!)^2}x^3\ .\ . \qquad (41)$$

and

$$y_2(x) = \frac{\partial y}{\partial p}\bigg| = \left[x^p \ln x\left\{1+\frac{p-n}{(p+1)^2}x+\frac{(p-n)(p-n+1)}{(p+1)^2(p+2)^2}x^2+\ldots\right\}\right.$$

$$+x^p\left\{\left(\frac{1}{p-n}-\frac{2}{p+1}\right)\frac{p-n}{(p+1)^2}x+\left(\frac{1}{p-n}-\frac{2}{p-n+1}-\frac{2}{p+1}-\frac{2}{p+2}\right)\right.$$
$$\left.\left.\times\left\{\frac{(p-n)(p-n+1)}{(p+1)^2(p+2)^2}x^2+\ .\ .\ .\right\}\right]_{p=0}$$

$$= y_1(x)\ln x+\left\{(2n+1)x-\frac{3n^2-n-1}{(2!)^2}x^2+\ldots\right\} \qquad (42)$$

For example, for $n=0$

$$y_1(x) = 1 = L_0(x)$$

and

$$y_2(x) = \ln x + x + \frac{x^2}{(2!)^2}+\frac{2!x^3}{(3!)^2}+\frac{3!x^4}{(4!)^2}+\ldots \qquad (43)$$

Similarly, for $n=1$

$$y_1(x) = 1-x = L_1(x)$$

and

$$y_2(x) = (1-x)\ln x + 3x - \frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}-\ldots \qquad (44)$$

3.10 Associated Laguerre Polynomials

Replace n by $(n+k)$ in eq. (1), it is obvious that $L_{n+k}(x)$ will be a solution of the following differential equation.

$$xy'' - (1-x)y' + (n+k)y = 0 \quad (45)$$

Differentiating the above equation k times, it can easily be shown that

$$y = \frac{d^k}{dx^k}\left[L_{n+k}(x)\right] \quad (46)$$

or a constant multiple of it is a solution of the differential equation

$$xy'' - (k+1-x)y' + ny = 0 \quad (47)$$

Where n and k are positive integers or zero. The above equation is known as the **Associated Laguerre Equation.** Its polynomial solutions [see eq.(45)] are denoted by $L_n^k(x)$ and are defined by

$$L_n(x) = (-1)^n \frac{d^k}{dx^k}\left[L_{n+k}(x)\right] \quad (48)$$

This is known as the **Associated Laguerre Polynomials.** It is obvious from eq. (48) that
$L_n^k(x)$ is polynomial of degree n in x and that

$$L_n^0(x) = L_n(x) \quad (49)$$

Using eqs. (7) and (48), it follows that

$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{(n+k)!}{(n-r)!(r+k)!r!} x^r \quad (50)$$

We will define $L_n^k(x)$ for non-integer values of k, we may, therefore, write the above equation as

$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{\Gamma(n+k+1)}{(n-r)!\Gamma(r+k)\ r!} x^r \quad (51)$$

Using the above equation, the first three polynomials can easily be written as:

$$L_0^k(x) = 1$$
$$L_1^k(x) = k+1-x \quad (52)$$
$$L_2^k(x) = \frac{1}{2}(k+2)(k+1) - (k+2)x + \frac{1}{2}x^2$$

Differentiating the Laguerre generating function [eq. (10)] k times with respect to x, one can easily obtain the generating function for the associated Laguerre polynomials. Thus

$$g(x,t) \equiv \frac{1}{(1-t)^{k+1}} \exp\left\{-\frac{xt}{1-t}\right\} \sum_{n=0}^{\infty} L_n^k(x) t^n \quad (53)$$

Furthermore, from eq.(51)

$$L_n^k(0) = \frac{\Gamma(n+k+1)}{n!\Gamma(k+1)} \qquad (54)$$

SELF-ASSESSMENT EXERCISE

1. Show that

$$x^4 = \Gamma(5+k)\ 4!\sum_{r=0}^{4}\frac{(-1)^r L_r^k(x)}{\Gamma(r+k+1)(4-r)!}$$

2. ***Hint:*** Use eq. (51 Show that

$$L_n(0) = 1$$
$$L_n'(0) = -n$$
$$L_n''(0) = \frac{1}{2}n(n-1)$$

Hint: Use Eq. (7).

4.0 CONCLUSION

In this unit, we have established the relationship between Laguerre and associated Laguerre polynomials. The generating function and some important results involving Laguerre polynomials were also dealt with.

5.0 SUMMARY

This unit deals with Laguerre functions and its applications to physical problems especially in Quantum mechanics.

7.0 TUTOR-MARKED ASSIGNMENT

1. Show that

$$L_n^{\gamma+k+1}(x+y) = \sum_{r=0}^{n} L_r^{\gamma}(x) L_{n-r}^{k}(y), \qquad n = 0, 1, 2, \ldots$$

Hint: Use the generating function.

2. Show that

$$L_n^{1/2}(x) = \frac{(-1)^n}{2^{2n+1} n!}\frac{H_{2n+1}(x^{1/2})}{x^{1/2}}$$
$$L_n^{-1/2}(x) = \frac{(-1)^n}{2^{2n} n!} H_{2n}(x^{1/2})$$

Hint: Use the integral representation of $L_n^k(x)$ and $H_n(x)$.

3. Using eq. (53), prove the identity

$$(1-t)\frac{\partial g}{\partial t}+\left[x-(1-t)(1+k)\right]g=0$$

and then derive the recurrence relation [eq. (56)].

4. Using eq.(53), prove the identity

$$(1-t)\frac{\partial g}{\partial x}+tg(x,t)=0$$

and hence derive the following relation

$$\frac{dL_n^k(x)}{dx}-\frac{dL_{n-1}^k(x)}{dx}+dL_{n-1}^k(x)=0$$

$$n=1,\ 2,\ldots$$

5. Show that

$$\int_0^x L_n(t)dt=L_n(x)-L_{n+1}(x)$$

Hint: Use the relation derived in problem 4.

7.0 REFERENCES/FURTHER READING

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