NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: PHY 313

COURSE TITLE: Mathematical Methods of Physics II

Course Code	PHY 313
Course Title	Mathematical Methods of Physics II
Course Developer	Dr. OMOLOFE. Babatope Department of Mathematics Faculty of Science Federal University of Technology, Akure, Ondo State.
Content Editor	Dr. Ajibola. S.O **National Open University of Nigeria** Lagos
Programme Leader	Dr. Ajibola S. O. **National Open University of Nigeria** Lagos

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office

5, Dar Es Salaam Street

Off Aminu Kano Crescent

Wuse II, Abuja

Nigeria.

e-mail: centralinfo@nou.edu.ng

URL: www.nou.edu.ng

First Printed 2008

ISBN:

Printed by: For
National Open University of Nigeria

PHY313

MATHEMATICAL METHODS OF PHYSICS II

(3 UNITS)

COURSE CONTENT

Function of complex variables. Analyticity. Complex integration. The residual theorem and its applications. Conformal mapping. The eigenvalue problem for matrices. Diagonalization of matrices. Introduction to tensors. Integral equations. Basic group theory. Application of group theory.

Course Code	PHY 313
Course Title	Mathematical Methods of Physics II
Course Developer	OMOLOFE Babatope Department of Mathematics Faculty of Science Federal University of Technology. Akure Ondo State.

FUNCTIONS OF COMPLEX VARIABLES

MODULE 1

Unit 1 Complex Variables

CONTENTS

1.0 Introduction
2.0 Objectives
3.0 Main Content
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Readings

1.0 INTRODUCTION

Complex number can be defined as an expression of the form , where are real numbers and is a ``number'' such that . The number a is called the real part of z (i.e. a = Re z) and b is the imaginary part (or imaginary coefficient) of z (b = Im z).

The set of all complex numbers is denoted $\mathbb{C}$. We denote $\mathbb{C}^* = \mathbb{C} - 0$.

2.0 OBJECTIVES

1 To know complex variable and its properties.

2 To know operations of complex variable and when applicable

3 To know complex variable as a function

4 To know theorems on limits of functions

3.0 MAIN CONTENT

3.1 Definition: A complex number is an expression of the form , where are real numbers and is a ``number" such that . The number a is called the real part of z (i.e. a = Re z) and b is the imaginary part (or imaginary coefficient) of z (b = Im z).

The set of all complex numbers is denoted $\mathbb{C}$. We denote $\mathbb{C}$* = $\mathbb{C}$- 0.

Properties:

1. Equality: .
2. If .
3. If , z is called pure imaginary.

Example

- $2i$ is pure imaginary.
- 4π is real.

Definition (Operations) Let and .

1. Addition: .
2. Multiplication:

These operations have the same algebraic properties as the corresponding operations in $\mathbb{R}$ (associativity, commutativity, etc.; please prove ...). Thus, the classical formulas (such as Newton's binomial) are also true in $\mathbb{C}$.

$$\forall z_1, z_2 \in \mathbb{C},\ (z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2.$$

$$\forall z_1, z_2 \in \mathbb{C},\ (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2.$$

$$\forall z_1, z_2 \in \mathbb{C},\ (z_1 - z_2)(z_1 + z_2) = z_1^2 - z_2^2.$$

$$\forall z_1, z_2 \in \mathbb{C}, \forall n \in \mathbb{N},\ (z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

.

$$\forall z_1, z_2 \in \mathbb{C}, \forall n \in \mathbb{N},\ z_1^n - z_2^n = \sum_{k=0}^{n-1} z_1^{n-1-k} z_2^k$$

.

<u>Example</u>

$$(2+3i) + (4+7i) = 6 + 10i$$

.

$$(2+3i)(4+7i) = (2 \cdot 4 - 3 \cdot 7) + i(2 \cdot 7 + 3 \cdot 4) = -13 = 26i$$

.

$$(2+3i)^2 = 2^2 + 2 \cdot 2 \cdot 3i + (3i)^2 = 4 + 6i - 9 = -5 + 6i$$

.

$$(5-2i)^3 = 5^3 - 3 \cdot 5^2 \cdot 2i + 3 \cdot 5 \cdot (2i)^2 - (2i)^3 = 125 - 150i - 60 + 8i = 65 - 142i$$

.

$$z^3 + i = z^3 - i^3 = (z-i)(z^2 + iz + i^2 = (z-i)(z^2 + iz - 1)$$

3.2 <u>THEOREMS ON LIMITS OF FUNCTIONS</u>

1. exists, then it is unique
2. and , then
3.
4. — –

Definition: We mean given , such that if where is the domain of then –

In some applications, ∈ S

<u>CONTINUITY</u> – – – **:**

SEQUENCE

If n=1 is an infinite sequence of complex numbers, then we say .

That is, n=1 converges to W if given , ∃ N such that n ≥ N, then –

3.3 CAUCHY SEQUENCE

A sequence { } is Cauchy if given Ɛ>0, ∃ N such that m, such that

Example: Every convergent sequence is Cauchy sequence

By – , we mean as approaches infinity, if for any , M >0 such that

– –

we mean for any N>0,

– – – –

NOTE:

1.

2. is never zero.

3. If x is real, for x >0

for x<0

4. l l = .

Proposition: Given that are two complex variables, then

and

Proof. Denote and , where are real numbers. Then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Thus:

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$$

$$= (x_1 + x_2) - i(y_1 + y_2)$$

$$= (x_1 - iy_1) + (x_2 - iy_2)$$

=

Similarly,

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$, thus:

$$= \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)}$$

$$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1$$

$$= (x_1 - iy_1)(x_2 - iy_2).$$

=

Definition: (Inverse of a complex) If z ≠ 0, then it has a complex inverse z^{-1}. Let , where a and b are real numbers; then we have:

$$z^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

Proof. – ——

Multiply both numerator and denumerator by

$$= \frac{a - ib}{(a + ib)(a - ib)}$$

$$= \frac{a - ib}{a^2 + b^2}$$

$$= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

Theorem: *Let . There exists a real number θ such that* .

Proof. Denote , where and are real numbers. Then if, and only if, , i.e. the image of z in the complex plane is a point on the unit-circle. For each point on the unit-circle, there exist a real number   such that the coordinates of this point are

.

Definition: The number   is called an argument of   and is denoted   .

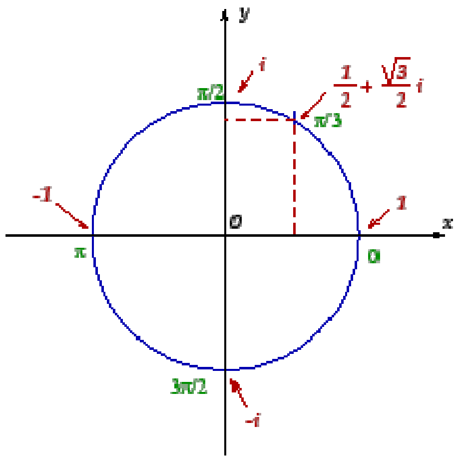
Note that this argument is defined up to an additional   .

Example :

$$- \quad - = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \Longrightarrow \arg\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}.$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Longrightarrow \arg(i) = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}.$$

In Figure below, a value of the argument of the complex number corresponding to a point is diplayed in green.



The unit circle in Cauchy-Argand plane.

Example 1:

Solution: Using the binomial formula,

we obtain

$$= x^4 + 4ix^4y - 6x^2y^2 - 4ixy^3 + y^4$$

$$= x^4 - 6x^2y^2 + y^4 + i(4x^4y - 4xy^3)$$

$$= u(x,y) + iv(x,y)$$

so that $u(x,y) = x^4 - 6x^2y^2 + y^4$ *and* $v(x,y) = (4x^4y - 4xy^3)$

<u>Example 2</u>: Express the function $f(z) = \bar{z}Re(z) + z^2 + Im(z)$ *in the form* $f(z)$ in the form $f(z) = u(x,y) + iv(x,y)$.

Solution. Using the elementary properties of complex numbers, it follows that

$$f(z) = (x - iy)x + (x + iy)^2 + y = (2x^2 - y^2 + y) + i(xy)$$

so that $u(x,y) = 2x^2 - y^2 + y$ *and* $v(x,y) = xy$.

Examples 1 and 2 show how to find u(x,y) and v(x,y) when a rule for computing f is given. Conversely, if u(x,y) and v(x,y) are two real-valued functions of the real variables x and y, they determine a complex-valued function $f(z) = u(x,y) + iv(x,y)$ We can use the formulas

$x = \frac{z+\bar{z}}{2}$ *and* $y = \frac{z-\bar{z}}{2i}$ to find a formula for f involving the variables z *and* $\bar{z}$

<u>Example 3</u>: Express $f(z) = 4x^2 + i4y^2$ by a formula involving the variables z *and* $\bar{z}$.

Solution. Calculation reveals that

$$f(z) = 4\left\{\frac{z+\bar{z}}{2}\right\}^2 + 4\left\{\frac{z-\bar{z}}{2i}\right\}^2$$

$$= z^2 + 2z\bar{z} + \bar{z}^2 - i(z^2 - 2z\bar{z} + \bar{z}^2)$$

$$= (1 - i)z^2 + (2 + 2i)z\bar{z} + (1 - i)\bar{z}^2$$

3.4 GEOMETRIC INTERPRETATION OF A COMPLEX FUNCTION

If D is the domain of real-valued functions $u(x,y)$ *and* $v(x,y)$, the equations

$$u = u(x,y) \text{ and } v = v(x,y)$$

describe a transformation (or mapping) from D in the xy plane into the uv plane, also called the w plane. Therefore, we can also consider the function

$$W = f(z) = u(x,y) + iv(x,y)$$

to be a transformation (or mapping) from the set D in the z plane onto the range R in the w plane. This idea was illustrated in Figure 2.1. In the following paragraphs we present some additional key ideas. They are staples for any kind of function, and you should memorize all the terms in bold.

If A is a subset of the domain D of f, the set $B = \{W = f(z) : z \in A\}$ is called the image of the set A, *and* f is said to map A *onto* B. The image of a single point is a single point, and the image of the entire domain, D, is the range, R. The mapping $W = f(z)$ is said to be from A into S if the image of A is contained in S. Mathematicians use the notation $f\colon A \to S$ to indicate that a function maps A *into* S. Figure below illustrates a function f whose domain is D and whose range is R. The shaded areas depict that the function maps A *onto* B. The function also maps A *into* R, and, of course, it *maps* D *onto* R.

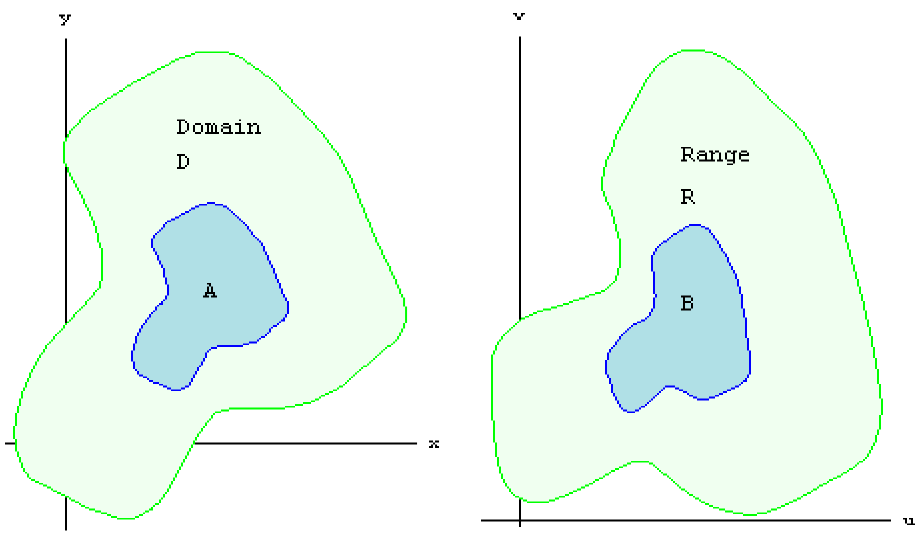


Figure $W = f(z)$ maps A onto B. $W = f(z)$ maps A into R.

The inverse image of a point w is the set of all points z in D such that $W = f(z)$. The inverse image of a point may be one point, several points, or nothing at all. If the last case occurs then the point w is not in the range of f.

Example 4: Express $f(z) = z^5 + 4z^2 - 6$ in polar form.

Solution. We obtain

$$f(z) = f\left(re^{i\theta}\right) = (re^{i\theta})^5 + 4(re^{i\theta})^2 - 6 = ce^{i5\theta} + 4r^2e^{i2\theta} - 6$$

$$= r^5 \cos 5\theta + r^2 \cos 2\theta - 6 + i(r^5 \sin 5\theta + 4\, r^2 \sin 2\theta)$$

$$= U\,(r, \theta) + iV(r, \theta)$$

So that $U\,(r, \theta) = r^5 \cos 5\theta + r^2 \cos 2\theta - 6 \; and \; V(r, \theta) = r^5 \sin 5\theta + 4\, r^2 \sin 2\theta$

Example 5: The ellipse centered at the origin with a horizontal major axis of 4 units and vertical minor axis of 2 units can be represented by the parametric equation

$s(t) = 2\cos t + i \sin t = (2\cos t\,, \sin t)$, for $0 \le t \le 2\pi$

Suppose we wanted to rotate the ellipse by an angle of $\frac{\pi}{6}$ radians and shift the center of the ellipse 2 units to the right and 1 unit up. Using complex arithmetic, we can easily generate a parametric equation $r(t)$ that does so:

$$r(t) = s(t)e^{\frac{i\pi}{6}} + (2 + i) = (2\cos t + i \sin t)\left(\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\right) + (2 + i)$$

$$= \left(2\cos t\, \cos\frac{\pi}{6} - \sin t \sin\frac{\pi}{6}\right) + i\left(2\cos t\, \sin\frac{\pi}{6} + \sin t \cos\frac{\pi}{6}\right) + (2 + i)$$

$$= \left(\sqrt{3}\cos t - \frac{1}{2}\sin t + 2\right) + i(\cos t + \frac{\sqrt{3}}{2}\sin t + 1)$$

$$= (\sqrt{3}\cos t - \frac{1}{2}\sin t + 2, \cos t + \frac{\sqrt{3}}{2}\sin t + 1)$$

for $0 \le t \le 2\pi$, Figures below show parametric plots of these ellipses

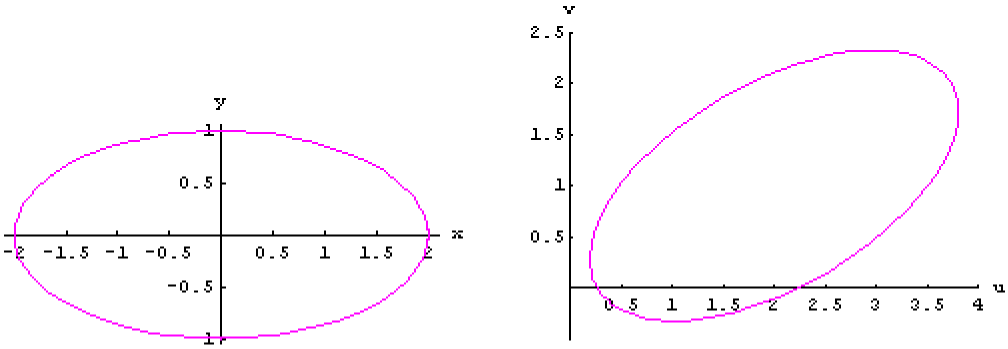


Figure (a) Plot of the original ellipse $s(t) = 2\cos t + i \sin t$ **(b)** Plot of the rotated ellipse $r(t) = s(t)\, e^{i\pi/6} + (2+i)$

Example 6: Show that the image of the right half plane under the linear transformation is the half plane .

Solution: The inverse transformation is given by

which we write as

Substituting into gives which simplifies .

Figure 2.11 illustrates the mapping.

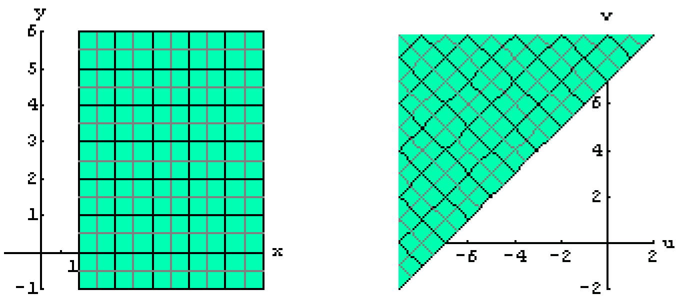


Figure 2: The the linear transformation .

4.0 CONCLUSION

In conclusion Examples 1 and 2 above show how to find u(x,y) and v(x,y) when a rule for computing f is given. Conversely, if u(x,y) and v(x,y) are two real-valued functions of the real variables x and y, they determine a complex-valued function $f(z) = u(x,y) + iv(x,y)$ We can use the formulas

$x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ to find a formula for f involving the variables z and $\bar{z}$

5.0 SUMMARY

If D is the domain of real-valued functions $u(x,y)$ and $v(x,y)$, the equations

$u = u(x,y)$ and $v = v(x,y)$

describe a transformation (or mapping) from D in the xy plane into the uv plane, also called the w plane. Therefore, we can also consider the function

$$W = f(z) = u(x,y) + iv(x,y)$$

to be a transformation (or mapping) from the set D in the z plane onto the range R in the w plane. They are staples for any kind of function, and you should memorize all the terms in bold.

6.0 TMA

1. Express $f(z) = 3x^2 + iy^2$ by a formula involving the variables z and $\bar{z}$.
2. Express $f(z) = z^4 + 2z^2 - 1$ in polar form.
3. Write $f(z) = z^4$ in the form $f(z) = u(x,y) + iv(x,y)$
4. Prove that $\cos z, \sin z, \cosh z,$ and $\sinh z$ are entire functions.
5. What is the idea that led to the Cauchy-Riemann equations?

6 State the Cauchy-Riemann equations from memory.

7 What is an analytic function? Can a function be differentiable at a point z_0 without being analytic at z_0.

7.0 REFRENCES / FURTHER READINGS

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995). 5th ed. *Engineering Maths*. () Palgraw

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 2

ANALYTIC FUNCTION

CONTENTS

1.0 Introduction
2.0 Objectives
3.0 Main Content
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Readings

1.0 INTRODUCTION

A function $f(z)$ is analytic at a point z_0 if its derivative $f'(z)$ exists not only at z_0 but at every point z in a neighborhood of z_0.

One can show that if $f(z)$ is analytic the partial derivatives of u and v of all orders exist and are continuous functions of x and y.

Thus we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

That is
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus both $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation.

2.0 OBJECTIVES

1 To know about analytic functions

2 To know about complex integral

3 To treat theorems and some related examples

3.0 MAIN CONTENT

An analytic function is an infinitely differentiable function such that the Taylor series at any point x_0 in its domain

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

converges to $f(x)$ for x in a neighbourhood of x_0. The set of all real analytic functions on a given set D is often denoted by $C^{\omega}(D)$.

A function f defined on some subset of the real line is said to be real analytic at a point x if there is a neighbourhood D of x on which f is real analytic.

If a complex analytic function is defined in an open ball around a point x_0, its power series expansion at x_0 is convergent in the whole ball. This statement for real analytic functions (with open ball meaning an open interval of the real line rather than an open disk of the complex plane) is not true in general; the function of the example above gives an example for $x_0 = 0$ and a ball of radius exceeding 1, since the power series $1 - x^2 + x^4 - x^6 ...$ diverges for $|x| > 1$.

Any real analytic function on some open set on the real line can be extended to a complex analytic function on some open set of the complex plane. However, not every real analytic function defined on the whole real line can be extended to a complex function defined on the whole complex plane.

Theorem: If the derivative of $f(z)$ exists at a point z, then the partial derivatives of u and v exist at that point and obey the following conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This above equations are called Cauchy Riemann equations

Let *u* and *v* be real and single valued functions of *x* and *y* are called which, together with their partial derivatives of the first order, are continuous at a point. If those partial derivatives satisfy the Cauchy-Riemann conditions at that point, then the derivative of *f* exists at that point.

3.1 Cauchy Riemann Equation:

A necessary condition that $W = f(z) = u(x,y) + v(x,y)$ be analytic in region *R* is that *u* and *v* satisfy Cauchy Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Usually *u*(*x*,*y*) and *v*(*x*,*y*) are called conjugate functions.

3.2 ANALYTIC FUNCTIONS

A function *f*(*z*) is analytic at a point z_0 if its derivative *f'*(*z*) exists not only at z_0 but at every point *z* in a neighborhood of z_0.

One can show that if *f*(*z*) is analytic the partial derivatives of *u* and *v* of all orders exist and are continuous functions of *x* and *y*.

Thus we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

That is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus both *u*(*x*,*y*) and *v*(*x*,*y*) satisfy Laplace's equation.

Example: Show that $f(z): \to \mathbb{C}$ defined by $f(z) = e^z$ is analytic in $\mathbb{C}$ and that $\frac{de^z}{dz} = e^z$.

Solution: Let $z = x + iy$.

Bu definition, $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i\sin y)$

So, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$

To show f is analytic, we verify Cauchy Riemann equations.

$$\frac{\partial u}{\partial x} = e^x \cos y \ , \qquad \frac{\partial v}{\partial x} = e^x \sin y$$
$$\frac{\partial u}{\partial y} = -e^x \sin y \, , \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

So, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied.

Hence, $f(z) = e^z$is analytic. To show $\frac{df}{dz} = e^z$, $f(z) = u(x, y) + iv(x, y) = e^x \cos y + ie^x \sin y$

Since $\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x (\cos y + i\sin y) = e^x e^{iy} = e^{x+iy} = e^z$

Hence, $\frac{de^z}{dz} = e^z$.

3.3 COMPLEX INTEGRAL

Definition (Definite Integral of a Complex Integrand): Let $f(t) = u(t) + iv(t)$ where u(t) and v(t) are real-valued functions of the real variable t for $a \leq t \leq b$. Then

$$\int_a^b f(t)dt = \int_a^b \big(u(t) + iv(t)\big)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt. \qquad \text{(A)}$$

We generally evaluate integrals of this type by finding the anti derivatives of u(t) and v(t) and evaluating the definite integrals on the right side of Equation above. That is, if $U^I(t) = u(t)$ and $V^I(t) = v(t)$, we have

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt = U(b) - U(a) + i(V(b) - V(a)). \qquad \text{(B)}$$

Example: Show that $\int_0^1 (t-1)^3\, dt = -\frac{5}{4}$.

Solution: We write the integrand in terms of its real and imaginary parts,

i.e. $f(t) = (t-1)^3 = t^3 - 3t + i(-3t^2 + 1)$

Here, $u(t) = t^3 - 3t$ and $v(t) = -3t^2 + 1$. The integrals of u(t) and v(t) are

$$\int_0^1 u(t)\, dt = \int_0^1 (t^3 - 3t\ )^3\, dt = \left[\frac{t^4}{4} - \frac{3t^2}{2}\right]_0^1 = -\frac{5}{4}$$

and

$$\int_0^1 v(t)\, dt = \int_0^1 (-3t^2 + 1\ )^3\, dt = \lfloor -t^3 + t \rfloor_0^1 = 0$$

Hence, by Definition (A),

$$\int_0^1 (t-1)^3\, dt = \int_0^1 u(t)dt + i\int_0^1 v(t)dt = -\frac{5}{4} + 0i = -\frac{5}{4}$$

Example: Show that $\int_0^{\frac{\pi}{2}} e^{t+it}dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$.

Solution. We use the method suggested by Definitions (A) and (B) above

$$\int_0^{\frac{\pi}{2}} e^{t+it}dt = \int_0^{\frac{\pi}{2}} e^t e^{it}dt = \int_0^{\frac{\pi}{2}} e^t(\cos t + i\sin t)dt$$

$$= \int_0^{\frac{\pi}{2}} e^t \cos t\, dt + i\int_0^{\frac{\pi}{2}} e^t \sin t\, dt$$

We can evaluate each of the integrals via integration by parts. For example,

$$\int_0^{\frac{\pi}{2}} e^t \cos t\, dt = (e^t \sin t)_{t=0}^{t=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^t \sin t\, dt$$

$$= \left(e^{\frac{\pi}{2}} \sin\frac{\pi}{2} - e^0 \sin 0\right) - \int_0^{\frac{\pi}{2}} e^t \sin t\, dt$$

$$= (e^{\frac{\pi}{2}}.1 - 1.0) - \int_0^{\frac{\pi}{2}} e^t \sin t\, dt$$

$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^t \sin t\, dt$$

$$= e^{\frac{\pi}{2}} - (e^t. -\cos t)_{t=0}^{t=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^t. -\cos t\, dt$$

$$= e^{\frac{\pi}{2}} + (e^t \cos t)_{t=0}^{t=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^t \cos t\, dt$$

$$= e^{\frac{\pi}{2}} + (e^{\frac{\pi}{2}}.0 - 1.1) - \int_0^{\frac{\pi}{2}} e^t \cos t\, dt$$

$$= e^{\frac{\pi}{2}} - 1 - \int_0^{\frac{\pi}{2}} e^t \cos t\, dt$$

Adding $\int_0^{\frac{\pi}{2}} e^t \cos t\, dt$ to both sides of this equation and then dividing by 2 gives $\int_0^{\frac{\pi}{2}} e^t \cos t\, dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$. Likewise, $\int_0^{\frac{\pi}{2}} e^t \sin t\, dt = \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$.

Therefore, $\int_0^{\frac{\pi}{2}} e^{t+it}dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$

Complex integrals have properties that are similar to those of real integrals. We now trace through several commonalities. Let $f(t) = u(t) + iv(t)$ and $g(t) = p(t) + iq(t)$ be continuous on $a \leq t \leq b$.

Using Definition (A), we can easily show that the integral of their sum is the sum of their integrals, that is

$$\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt \qquad \text{(C)}$$

If we divide the interval $a \leq t \leq b$ into $a \leq t \leq c$ and $c \leq t \leq b$ and integrate $f(t)$ over these subintervals by using (A), then we get

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt. \qquad \text{(D)}$$

Similarly, if $\alpha + i\beta$ denotes a complex constant, then

$$\int_a^b (c + id)f(t)dt = (c + id)\int_a^b f(t)dt \quad . \qquad \text{(E)}$$

If the limits of integration are reversed, then

$$\int_a^b f(t)dt = -\int_b^a f(t)dt \quad . \qquad \text{(F)}$$

The integral of the product f(t)g(t) becomes

$$\int_a^b f(t)g(t)dt = \int_a^b (u(t) + iv(t))(\,p(t) + iq(t))dt$$

$$= \int_a^b (u(t)p(t) - v(t)q(t))dt + i\int_a^b (u(t)q(t) - v(t)p(t))dt \qquad \text{(G)}$$

<u>Example:</u> Let us verify property (E). We start by writing

$$(c + id)f(t) = (c + id)\big(u(t) + iv(t)\big)$$

$$= cu(t) - dv(t) + i(cv(t) + du(t))$$

Using Definition (A), we write the left side of Equation (E) as

$\int_a^b (c+id)\big(u(t)+iv(t)\big)dt = \ c\int_a^b u(t)dt - d\int_a^b v(t)dt + ic\int_a^b v(t)dt + \ id\int_a^b u(t)dt$

which is equivalent to

$$\int_a^b (c+id)\big(u(t)+iv(t)\big)dt = \ c\int_a^b u(t)dt + id\int_a^b u(t)dt + ic\int_a^b v(t)dt + idi\int_a^b v(t)dt$$

$$= (c+id)\int_a^b u(t)dt + i(c+id)\int_a^b v(t)dt$$

$$= (c+id)(\int_a^b u(t)dt + \ i\int_a^b v(t)dt)$$

$$Therefore \int_a^b (c+id)f(t)dt = (c+id)\int_a^b f(t)dt$$

It is worthwhile to point out the similarity between equation (B) and its counterpart in calculus. Suppose that U and V are differentiable on $a \le t \le b$ and $F(t) = U(t) + iV(t)$. Since $F^I(t) = \ U^I(t) + iV^I(t) = \ u(t) + iv(t) = f(t)$, equation (B) takes on the familiar form

$$\int_a^b f(t)dt = F(t)]_{t=a}^{t=b} = \ F(b) - F(a). \qquad \text{(H)}$$

where $F^I(t) = f(t)$. We can view Equation (H) as an extension of the fundamental theorem of calculus.

$$\int_a^b f^I(t)dt = \ f(b) - f(a)\ . \qquad \text{(I)}$$

Example: Use Equation (H) to show that $\int_0^{\frac{\pi}{2}} e^{t+it}dt = \ \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \ \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$

Solution: We seek a function F with the property that $F^I(t) = \ e^{(1+i)t}$. We note that $F(t) = \ \frac{1}{1+i}e^{(1+i)t}$ satisfies this requirement, so

$$\int_0^{\frac{\pi}{2}} e^{t+it}dt = \frac{1}{1+i}e^{(1+i)t}]_{t=0}^{t=\frac{\pi}{2}} = \frac{1}{1+i}e^{(1+i)\frac{\pi}{2}} - \frac{1}{1+i}e^0$$

$$= \frac{1}{1+i} e^{\frac{\pi}{2}} e^{i\frac{\pi}{2}} - \frac{1}{1+i} = \frac{1}{1+i} ie^{\frac{\pi}{2}} - \frac{1}{1+i}$$

$$= \frac{1}{1+i}\left(ie^{\frac{\pi}{2}} - 1\right) = \frac{1}{2}(1-i)(-1+ie^{\frac{\pi}{2}})$$

$$= \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$$

4.0 CONCLUSION

Conclusively, A function $f(z)$ is analytic at a point z_0 if its derivative $f'(z)$ exists not only at z_0 but at every point z in a neighborhood of z_0.

5.0 SUMMARY

A necessary condition that $W = f(z) = u(x,y) + v(x,y)$ be analytic in region R is that u and v satisfy Cauchy Riemann equations.

6.0 TMA

1 Use Equation (H) to evaluate $\int_0^{\frac{\pi}{2}} e^{t+it} dt$

2 Evaluate $\int_0^1 (t-1)^5\, dt$.

7.0 REFRENCES / FURTHER READINGS

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995). 5th ed. *Engineering Maths*. () Palgraw

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.

UNIT 3

RESIDUE THEOREM

CONTENTS

1.0 Introduction
2.0 Objectives
3.0 Main Content
4.0 Conclusion
5.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Readings

1.0 INTRODUCTION

The **residue theorem**, sometimes called **Cauchy's Residue Theorem** in complex analysis is a powerful tool to evaluate line integrals of analytic functions over closed curves and can often be used to compute real integrals as well. It generalizes the Cauchy integral theorem and Cauchy's integral formula. From a geometrical perspective, it is a special case of the generalized Stokes' theorem.

2.0 OBJECTIVES

- to be able to determine and explain Residue;
- to be able to use Residue to evaluate integrals; and
- to show that the Residue integration method can be extended to the case of several singular points of $f(z)$ inside C

3.0 MAIN CONTENT

Residue theorem: Suppose U is a simply connected open subset of the complex plane, and $a_1,...,a_n$ are finitely many points of U and f is a function which is defined and holomorphic on

$U \setminus \{a_1,...,a_n\}$. If γ is a rectifiable curve in U which bounds the a_k, but does not meet any and whose start point equals its endpoint, then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k)\mathrm{Res}(f, a_k).$$

If γ is a positively oriented Jordan curve, $\mathrm{I}(\gamma, a_k) = 1$ and so

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, a_k).$$

Here, $\mathrm{Res}(f, a_k)$ denotes the residue of f at a_k, and $\mathrm{I}(\gamma, a_k)$ is the winding number of the curve γ about the point a_k. This winding number is an integer which intuitively measures how many times the curve γ winds around the point a_k; it is positive if γ moves in a counter clockwise ("mathematically positive") manner around a_k and 0 if γ doesn't move around a_k at all.

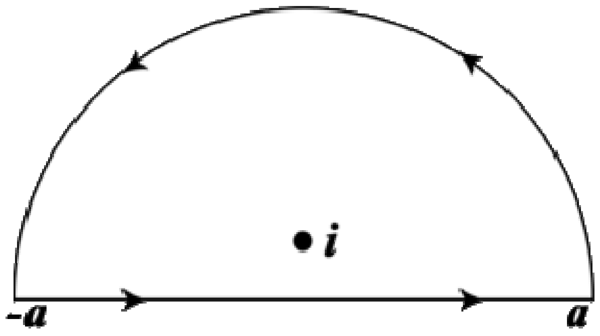
The relationship of the residue theorem to Stokes' theorem is given by the Jordan Curve Theorem. The general plane curve γ must first be reduced to a set of simple closed curves $\{\gamma_i\}$ whose total is equivalent to γ for integration purposes; this reduces the problem to finding the integral of $f\,dz$ along a Jordan curve γ_i with interior V. The requirement that f be holomorphic on $U_0 = U \setminus \{a_k\}$ is equivalent to the statement that the exterior derivative $d(f\,dz) = 0$ on U_0. Thus if two planar regions V and W of U enclose the same subset $\{a_j\}$ of $\{a_k\}$, the regions $V \setminus W$ and $W \setminus V$ lie entirely in U_0, and hence $\int_{V\setminus W} d(f dz) - \int_{W\setminus V} d(f dz)$ is well-defined and equal to zero. Consequently, the contour integral of $f\,dz$ along $\gamma_i = \partial V$ is equal to the sum of a set of integrals along paths λ_j, each enclosing an arbitrarily small region around a single a_j—the residues of f (up to the conventional factor $2\pi i$) at $\{a_j\}$. Summing over $\{\gamma_i\}$, we recover the final expression of the contour integral in terms of the winding numbers $\{\mathrm{I}(\gamma, a_k)\}$.

In order to evaluate real integrals, the residue theorem is used in the following manner: the integrand is extended to the complex plane and its residues are computed (which is usually easy), and a part of the real axis is extended to a closed curve by attaching a half-circle in the upper or lower half-plane. The integral over this curve can then be computed using the residue theorem. Often, the half-circle part of the integral will tend towards zero as the radius of the half-circle grows, leaving only the real-axis part of the integral, the one we were originally interested in.

Example:

The integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$



The contour C. **Fig. 1**

arises in probability theory when calculating the characteristic function of the Cauchy distribution. It resists the techniques of elementary calculus but can be evaluated by expressing it as a limit of contour integrals.

Suppose $t > 0$ and define the contour C that goes along the real line from $-a$ to a and then counterclockwise along a semicircle centered at 0 from a to $-a$. Take a to be greater than 1, so that the imaginary unit i is enclosed within the curve. The contour integral is

$$\int_C f(z)\,dz = \int_C \frac{e^{itz}}{z^2+1}\,dz.$$

Since e^{itz} is an entire function (having no singularities at any point in the complex plane), this function has singularities only where the denominator $z^2 + 1$ is zero. Since $z^2 + 1 = (z + i)(z - i)$, that happens only where $z = i$ or $z = -i$. Only one of those points is in the region bounded by this contour. Because $f(z)$ is

$$\frac{e^{itz}}{z^2+1} = \frac{e^{itz}}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$
$$= \frac{e^{itz}}{2i(z-i)} - \frac{e^{itz}}{2i(z+i)},$$

the residue of $f(z)$ at $z = i$ is

$$\operatorname{Res}_{z=i} f(z) = \frac{e^{-t}}{2i}.$$

According to the residue theorem, then, we have

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}.$$

The contour C may be split into a "straight" part and a curved arc, so that

$$\int_{\text{straight}} f(z)\,dz + \int_{\text{arc}} f(z)\,dz = \pi e^{-t}$$

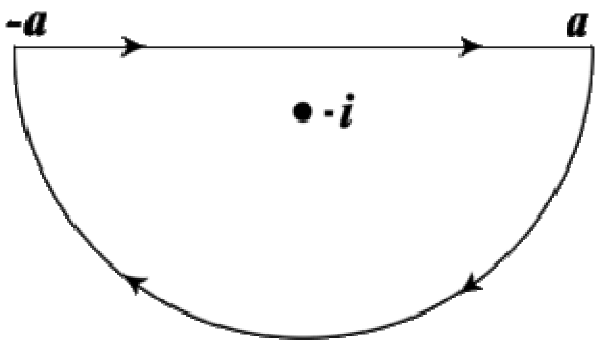
and thus

$$\int_{-a}^{a} f(z)\,dz = \pi e^{-t} - \int_{\text{arc}} f(z)\,dz.$$

Using estimations it can be shown that

Therefore

If $t < 0$ then a similar argument with an arc C' that winds around $-i$ rather than i shows that



The contour C' . **Fig. 2**

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2+1} dz = \pi e^{t}$$

and finally we have

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2+1} dz = \pi e^{-|t|}$$

If $t = 0$ then the integral yields immediately to elementary calculus methods and its value is π.

Example

Show that

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2} dx = \frac{\pi}{k} e^{ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2} dx = 0 \qquad (s > 0, \ \ \mathrm{k} > 0)$$

Solution

In fact, $\frac{e^{isz}}{k^2+x^2}$ has only one pole in the upper plane, namely, a simple pole at $z = ik$, and from (4) we obtain

$$\operatorname*{Re\,s}_{z=ik} \frac{e^{isz}}{k^2+z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \left[\frac{e^{-ks}}{2ik}\right].$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{isz}}{k^2+z^2} dx = 2\pi i \frac{e^{-ks}}{2ik} = \frac{\pi}{k} e^{ks}.$$

Since $e^{isx} = \cos sx + i \sin sx$, this yields the above results

3.2 Types of Real Improper Integrals

Another kind of improper integral is a definite integral

$$\int_A^B f(x)dx$$

whose integral becomes infinite at a point a in the interval of integration,

$$\lim_{x\to a}|f(x)| = \infty$$

Then the integral means

$$\int_A^B f(x)dx = \lim_{\tau\to a}\int_A^{a-\tau} f(x)dx + \lim_{\eta\to 0}\int_{a+\eta}^B f(x)dx$$

where τ and η approaches zero independently and through positive values. It may happen that neither of these limits exists, if $\tau,\eta\to 0$ independently,

but

$$\lim_{\tau\to 0}\left[\int_A^{a-\tau} f(x)dx + \int_{a+\eta}^B f(x)dx\right]$$

exists. This is called the **Cauchy principal value** of the integral. It is written

pv.v. $\int_A^B f(x)dx.$

For example,

$$\text{pv.v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\tau \to 0} \left[\int_{-1}^{-\tau} \frac{dx}{x^3} + \int_{\tau}^{1} \frac{dx}{x^3} \right] = 0$$

the principal value exists although the integral itself has no meaning. The whole situation is quite similar to that discussed in the second part of the previous section.

To evaluate improper integral whose integrands have poles on the real axis, we use a part that avoids these singularities by following small semi-circles at the singular points; the procedure maybe illustrated by the following example.

Example

An Application

Show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

(This is the limit of sine integral Si(x) as $x \to \infty$)

Solution

a. We do not consider $\frac{(\sin z)}{z}$ because this function does not behave suitably at infinity. We consider $\frac{e^{iz}}{z}$, which has a simple pole at z=0, and integrate around the contour in figure below. Since $\frac{e^{iz}}{z}$ is analytic inside and on C Cauchy's integral theorem gives

$$\oint_C \frac{e^{iz}}{z} dz = 0$$

b. We prove that the value of the integral over the large semicircle C_1 approaches R as approaches infinity. Setting $z = R\,e^{i\theta}.dz = iR\,e^{i\theta}d\theta, \frac{dz}{z = id\theta}$ and therefore

$$\left|\int_C \frac{e^{iz}}{z} dz\right| = \left|\int_0^\pi e^{iz} id\theta\right| \le \int_0^\pi \left|e^{iz}\right| d\theta \qquad (z = R\,e^{i\theta})$$

In the integrant on the right,

$$\left|e^{iz}\right| = \left|e^{iR(\cos\theta + i\sin\theta)}\right| = \left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right| = e^{-R\sin\theta}.$$

We insert this, $\sin(\pi - \theta) = \sin\theta$ to get an integral from 0 to $\pi/2$, and then $\varpi \ge 2\theta/\pi$ (when $0 \le \theta \le \pi/2$); to get an integral that we can evaluate:

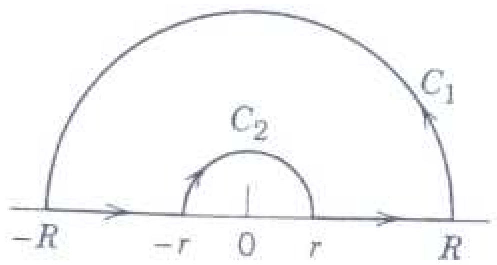


Fig. 3

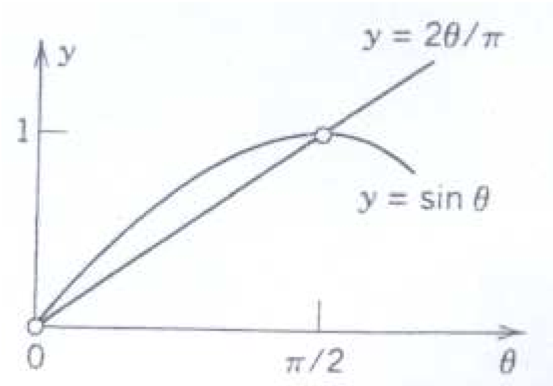


Fig. 4

$$\int_0^{\pi}\left|e^{iz}\right|d\theta=\int_0^{\pi}e^{-R\sin\theta}d\theta=\int_0^{\pi/2}e^{-R\sin\theta d\theta}d\theta$$

$$<2\int_0^{\pi/2}e^{-2R\theta/\pi}d\theta=\frac{\pi}{R}(1-e^{-R})\to 0 \qquad \text{as} \quad \mathrm{R}\to\infty C_1$$

Hence the value of the integral over C_1 approaches as $\mathrm{R}\to\infty$

c. For the integral over small semicircle C_2 in figure above , we have

$$\int_{C_2}\frac{e^{iz}}{z}dz=\int_{C_2}\frac{dz}{z}+\int_{C_2}\frac{e^{iz}-1}{z}dz$$

The first integral on the right equals $-\pi i$. The integral of the second integral is analytic and thus bounded, say, less than some constant M in absolute value for all z on C_2 and between C_2 and the x-axis. Hence by the $ML-$inequality, the absolute value of this integral cannot exceed $M\pi r$. This approaches $r\to 0$. Because of part (b), from (7) we thus obtain

$$\int_{C_2}\frac{e^{iz}}{z}dz=\text{pv.v.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx+\lim_{r\to 0}\int_{C_2}\frac{e^{iz}}{z}dz$$

$$= \text{pv.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i = 0$$

Hence this principal value equals πi ; its real part is 0 and its imaginary part is

$$\text{pv.v.}\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \qquad (8)$$

d. Now the integrand in (8) is not singular at $x = 0$. Furthermore, Since for positive x the function $1/x$ decreases, the area under the curve of the integrand between two consecutive positive zeros decreases in a monotone fashion, that is, the absolute value of the integrals

$$I_n = \int_{n\pi}^{n\pi+\pi} \frac{\sin x}{x} dx \qquad \text{n} = 0,1,\cdots\cdot$$

From a monotone decreasing sequence, $|I_1|, |I_2|, \cdots$ and $I_n \to 0$ as $n \to \infty$. Since these integrals have alternating sign (why?), it follows from the Leibniz test that the infinite series $I_0 + I_1 + I_2 + \cdots$ converges. Clearly, the sum of the series is the integral

$$\int_0^{\infty} \frac{\sin x}{x} dx = \lim_{b \to \infty} \int_0^b \frac{\sin x}{x} dx$$

which therefore exists. Similarly the integral from 0 to $-\infty$ exists. Hence we need not take the principal value in (8), and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

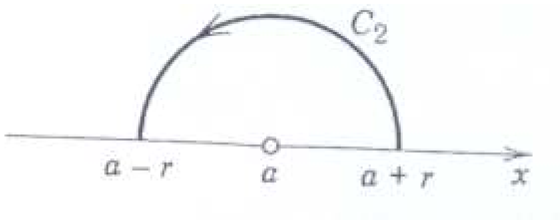
Since the integrand is an even function, the desired result follows.

In part (c) of example 2 we avoided the simple pole by integrating along a small semicircle C_2, and then we let C_2 shrink to a point. This process suggests the following.

3.4.3 Simple Poles on the Real Axis

If (z) *has a simple pole at* $z = a$ *on the real axis, then*

$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z).$$



Theorem 1 Fig. 5

Proof

By the definition of a simple pole the integrand $f(z)$ has at $z=a$ the Laurent series

$$f(z) = \frac{b_1}{z-a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z)$$

where $g(z)$ is analytic on the semicircle of integration

$$C_2 : z = a + re^{i\theta}, \qquad 0 \le \theta = \pi$$

and for all z between C_2 and the x-axis. By integration,

$$\int_{C_2} f(z)dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} ire^{i\theta} d\theta + \int_{C_2} g(z)dz$$

The first integral on the right equals $-b_1\pi i$ .The second cannot exceed $M\pi r$ in absolute value, by the ML-inequality and $M\pi r \to 0$ as $r \to 0$.

We may combine this theorem with (7) or (3) in this section.

Thus,

$$\text{pv.v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Re} sf(z) + \pi i \sum \operatorname{Re} sf(z) \qquad (9)$$

(summation over all poles in the upper half-plane in the first sum, and on the x-axis in the second), valid for rational $f(x) = p(x)/q(x)$ with degree $q \geq \text{degree } p+2$, having simple poles on the x-axis.

This is the end of unit 1, which added another powerful general integration method to the methods discussed in the chapter on integration. Remember that our present residue method is based on Laurent series, which we therefore had to discuss first.

In the next chapter we present a systemic discussion of mapping by analytic functions (**"conformal mapping"**) .Conformal mapping will then be applied to potential theory, our last chapter on complex analysis.

4.0 CONCLUSION

In conclusion, havjng run through this unit we have seen that our simple method have been extended to the case when the integrand has several isolated singularities inside the contour. We also proof the Residue theorem.

5.0 SUMMARY

The **residue** of an analytic function $f(z)$ at a point $z = z_0$ is the coefficient of $\frac{1}{z - z_0}$ the power in the Laurent series

$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$ of $f(z)$ which converges near z_0 (except at z_0 itself). This residue is given by the integral 3.1

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)dz \qquad (1)$$

but can be obtained in various other ways, so that one can use (1) for evaluating integral over closed curves. More generally, the **residue theorem** (sec.3.2) states that if $f(z)$ is analytic in a domain D such except at finitely many points z_j and C is a simple close path in D such that no z_j lies on C and the full interior of C belongs to D, then

$$\oint_{C_j} f(z)dz = \frac{1}{2\pi i}\sum_j \operatorname*{Res}_{z=z_j} f(z) \qquad (2)$$

(summation only over those z_j that lie inside C).

This integration method is elegant and powerful. Formulas for the residue at **poles** are ($m =$ order of the pole)

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right]\right), \qquad m = 1, 2, \cdots \qquad (3)$$

Hence for a simple pole ($m = 1$),

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) \qquad (3^*)$$

Another formula for the case of a simple pole of $f(z) = p(z)/q(z)$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z)}{q'(z)} \qquad (3^{**})$$

Residue integration involves closed curves, but the real interval of integration $0 \le \theta \le 2\pi$ is transformed into the unit circle by setting $z = e^{i\theta}$, so that by residue integration we can integrate **real integrals** of the form (sec. 3.3)

$$\int_0^{2\pi} F(\cos\theta \sin\theta)d\theta$$

where F is a rational function of $\cos\theta$ and $\sin\theta$,such as, for instance, $\frac{\sin^2\theta}{5-4\cos\theta}$, etc.

Another method of integrating *real* integrals by residues is the use of a closed contour consisting of an interval $-R \le x \le R$ of the real axis and a semicircle $|z| = R$. From the residue theorem, if we let $R \to \infty$, we obtain for rational $f(x) = p(x)/q(x)$ (with $q(x) \ne 0$ and $q >$ degree $p+2$)

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) \qquad \text{(sec.3.3)}$$

$$\int_{-\infty}^{\infty} \cos sx dx = -2\pi \sum \operatorname{Im}\operatorname{Res}\left[f(z)e^{isz}\right]$$

$$\int_{-\infty}^{\infty} \sin sx dx = 2\pi \sum \operatorname{Im}\operatorname{Res}\left[f(z)e^{isz}\right] \qquad \text{(sec.3.4)}$$

(sum of all residues at poles in the upper-half plane). In sec.3.4, we also extend this method to real integrals whose integrands become infinite at some point in the interval of integration.

6.0 TUTOR-MARKED ASSIGNMENT

1. Explain the term residues and how it can be used for evaluating integrals
2. Find the residues at the singular points of the following functions;

   (a) $\frac{\cos 2z}{z^4}$ (b) $\tan z$ (c) $\frac{e^z}{(z+\pi i)^6}$

3. Evaluate the following integrals where C is the unit circle (counterclockwise).

   (a) $\oint_C \cot z dz$ (b) $\oint_C \frac{dz}{1-e^z}$ (c) $\oint_C \frac{z^2+1}{z^2-2z}$

4. Show that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2-\cos\theta}} = 2\pi$$

7.0 REFERENCES/FURTHER READINGS

Hildraban, F. B. *Advanced Calculus for Application*.

Murray, R. Spieg (1974). *Schaums Outline Series or Theory and Problem of Advanced Calculus*. Great Britain: McGraw–Hill Inc.

Stephenor, G. (1977). *Mathematical Methods for Science Students*. London: Longman, Group Limited.

Stroud, K.A. (1995). 5th ed. *Engineering Maths*. () Palgraw

Verma, P.D.S. (1995). *Engineering Mathematics*. New Delhi: Vikas Publishing House PVT Ltd.